The distribution of the first digit in polynomials with positive integer coefficients

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Abstract

In this paper, some properties concerning the number of the first digit in a given polynomial $p(x)$ with positive integer coefficients under the radix-l are investigated. Denotes $\lambda_{m,n,l}(p)$ the count of the numbers of $p(x)$, whose first digit equals to m and value in $1, 2, \ldots, n$ and N_0 a sufficiently large positive integer, we demonstrate that $\forall l \in N_*, l \geq 3$

$$\lambda_{1,n,l}(p) > \lambda_{2,n,l}(p) > \cdots > \lambda_{l-1,n,l}(p),$$

where $n > N_0$.

Keywords: Number theory; Natural number power; Polynomial; The first digit

1 Introduction

In the elementary number theory, the law of distribution of the last digit or the last k ones and their calculation have been investigated deeply and many well properties or conclusions are proposed. However, ones pay less attention to the law of distribution of the first digit. In fact, it also has some well properties.

For example, when investigating the distribution of the first digit $1, 4, 9, 1, 2, 3, 4, 6, 8, 1, 1, \ldots$ in the sequence of the perfect square numbers $1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, \ldots$ under the radix-10, we can see that the count of the numbers of which the first digit equals 1 is larger than the count of 2 and so on in accordance with the increasing of the perfect square number.

In this literature, we investigate the distribution of the first digit in the sequence of the perfect square numbers and the distribution of the arbitrary positive integer power of the positive integer under the radix-10, then extend to the radix of the arbitrary positive integer.

For convenient, all the alphabets in this paper represent integer except described specially.

2 The distribution of the first digit of the perfect square numbers under the radix-10

***Definition* 1.** *For $n \in \mathbb{N}^*$, denotes δ_n the digit of the number n, i.e., $\delta_n = \lfloor \log_{10} n \rfloor + 1$.*

***Definition* 2.** *For $m \in \{1, 2, \ldots, 9\}, n \in \mathbb{N}^*$, denotes $\lambda_{m,n}$ the count of the perfect square numbers, whose first digit equals to m and value in $1 \sim n$.*

***Definition* 3.** *For $m \in \{1, 2, \ldots, 9\}, n \in \mathbb{N}^*$, denotes $\lambda'_{m,n} = \lambda_{m,n} - \lambda_{m,10^{\delta_n - 1} - 1}$.*

***Definition* 4.** *For $m \in \{1, 2, \ldots, 9\}, t \in \mathbb{N}^*$, denotes $\sigma_{m,t}$ the count of the perfect square numbers between $m * 10^t$ and $(m+1) * 10^t - 1$.*

***Definition* 5.**

$$F(m) = \left(2 * \sqrt{m+1} - \sqrt{m} - \sqrt{m+2}\right), m \in \mathbb{N}^*.$$

***Lemma* 1.** *For $\forall p \geq 0$, denotes $n = 10^{p+1} - 1$, then*

$$\lambda_{m,n} = \lambda_{m,10^{p+1}-1} = \sum_{t=0}^{p} \sigma_{m,t}$$

(This can be easily induced by the definitons of 2 and 4)

***Lemma* 2.** *When $n = 999999$, for $\forall u, v \in \{1, 2, \ldots, 9\}, u < v$,*

$$\lambda_{u,n} > \lambda_{v,n}.$$

***Proof*.** When $t \leq 5$, one can obtain all $\sigma_{m,t}$ by computer programme for every m and t, further the sum $\lambda_{m,999999}$ by the Lemma 1, the results as follows:

$$\lambda 1, 999999 = 193,$$
$$\lambda 2, 999999 = 146,$$
$$\lambda 3, 999999 = 123,$$
$$\lambda 4, 999999 = 111,$$
$$\lambda 5, 999999 = 97,$$
$$\lambda 6, 999999 = 91,$$
$$\lambda 7, 999999 = 84,$$
$$\lambda 8, 999999 = 78,$$
$$\lambda 9, 999999 = 76.$$

From these, we can know that

$$\lambda_{1,n} > \lambda_{2,n} > \lambda_{3,n} > \lambda_{4,n} > \lambda_{5,n} > \lambda_{6,n} > \lambda_{7,n} > \lambda_{8,n} > \lambda_{9,n} \quad (1)$$

when $n = 10^6 - 1$ and Lemma 2 is proved. □

***Lemma* 3.** *$F(m)$ is monotone decreasing.*

Proof. From the derivation of $F(m)$ on m,

$$\begin{aligned}F'(m) &= \frac{1}{\sqrt{m+1}} - \frac{1}{2\sqrt{m}} - \frac{1}{2\sqrt{m+2}} \\ &= \frac{1}{2}\left[\left(\frac{1}{\sqrt{m+1}} - \frac{1}{\sqrt{m}}\right) + \left(\frac{1}{\sqrt{m+1}} - \frac{1}{\sqrt{m+2}}\right)\right] \\ &= \frac{1}{2}\left(\frac{\sqrt{m}-\sqrt{m+1}}{\sqrt{m(m+1)}} + \frac{\sqrt{m+2}-\sqrt{m+1}}{\sqrt{(m+2)(m+1)}}\right) \\ &= -\frac{1}{2}\left(\frac{1}{\sqrt{m(m+1)}\left(\sqrt{m}+\sqrt{m+1}\right)} - \frac{1}{\sqrt{(m+2)(m+1)}\left(\sqrt{m+2}+\sqrt{m+1}\right)}\right)\end{aligned} \tag{2}$$

we can see

$$\sqrt{m(m+1)} < \sqrt{(m+2)(m+1)},$$
$$\sqrt{m} + \sqrt{m+1} < \sqrt{m+2} + \sqrt{m+1}$$

and $F'(m) < 0$。 □

***Lemma* 4.** $\forall t \geq 6, \forall m \in \{1, 2, \ldots, 8\}$,

$$\left(\sqrt{(m+1)*10^t - 1} - \sqrt{m*10^t}\right) - \left(\sqrt{(m+2)*10^t - 1} - \sqrt{(m+1)*10^t}\right) > 2.$$

Proof. Rewrite this inequality as

$$\begin{aligned}&\left(\sqrt{(m+1)*10^t} - \sqrt{m*10^t}\right) - \left(\sqrt{(m+2)*10^t} - \sqrt{(m+1)*10^t}\right) \\ &- \left(\sqrt{(m+1)*10^t} - \sqrt{(m+1)*10^t - 1}\right) \\ &+ \left(\sqrt{(m+2)*10^t} - \sqrt{(m+2)*10^t - 1}\right) > 2.\end{aligned} \tag{3}$$

Since

$$\begin{aligned}&\left(\sqrt{(m+1)*10^t} - \sqrt{m*10^t}\right) - \left(\sqrt{(m+2)*10^t} - \sqrt{(m+1)*10^t}\right) \\ &= \left(2*\sqrt{m+1} - \sqrt{m} - \sqrt{m+2}\right) * \sqrt{10^t}\end{aligned}$$

and $m \in \{1, 2, \ldots, 8\}$, one can get $F(m) \geq F(8)$ due to Lemma 3. Further, from $t \geq 6$,

$$\begin{aligned}&\left(2*\sqrt{m+1} - \sqrt{m} - \sqrt{m+2}\right) * \sqrt{10^t} \\ &\geq \left(2*\sqrt{9} - \sqrt{8} - \sqrt{10}\right) * \sqrt{10^6} \\ &\approx 9.2952 > 3\end{aligned} \tag{4}$$

is right. Obviously,

$$0 < \left|\sqrt{(m+1)*10^t} - \sqrt{(m+1)*10^t - 1}\right| < 1, \tag{5}$$

$$0 < \left|\sqrt{(m+2)*10^t} - \sqrt{(m+2)*10^t - 1}\right| < 1. \tag{6}$$

From (4),(5) and (6), we can get:

$$\left(\sqrt{(m+1)*10^t}-\sqrt{m*10^t}\right)-\left(\sqrt{(m+2)*10^t}-\sqrt{(m+1)*10^t}\right)$$
$$-\left(\sqrt{(m+1)*10^t}-\sqrt{(m+1)*10^t-1}\right)$$
$$+\left(\sqrt{(m+2)*10^t}-\sqrt{(m+2)*10^t-1}\right)>3-1+0=2. \quad (7)$$

We can see (7) equals to (3), then Lemma 4 is proved. □

***Lemma* 5.** *For* $\forall t\geq 6, \forall m\in\{1,2,\ldots,8\}$, *then*

$$\sigma_{m,t}>\sigma_{m+1,t}.$$

Proof. According to the properties of the perfect square number, we can get

$$\sigma_{m,t}=\lfloor\sqrt{(m+1)*10^t-1}\rfloor-\lceil\sqrt{m*10^t}\rceil+1. \quad (8)$$

From the properties of Gauss function, we know that

$$\lfloor\sqrt{(m+1)*10^t-1}\rfloor-\lceil\sqrt{m*10^t}\rceil\leq\sqrt{(m+1)*10^t-1}-\sqrt{m*10^t} \quad (9)$$

and

$$\lfloor\sqrt{(m+1)*10^t-1}\rfloor-\lceil\sqrt{m*10^t}\rceil\geq\sqrt{(m+1)*10^t-1}-\sqrt{m*10^t}-2 \quad (10)$$

are right. From (9)(10) and the Lemma 4, we can see that:

$$\left(\lfloor\sqrt{(m+1)*10^t-1}\rfloor-\lceil\sqrt{m*10^t}\rceil+1\right)$$
$$-\left(\lfloor\sqrt{(m+2)*10^t-1}\rfloor-\lceil\sqrt{(m+1)*10^t}\rceil+1\right)$$
$$\geq\left(\sqrt{(m+1)*10^t-1}-\sqrt{m*10^t}\right)$$
$$-\left(\sqrt{(m+2)*10^t-1}-\sqrt{(m+1)*10^t}\right)-2$$
$$>2-2=0. \quad (11)$$

Expand $\sigma_{m,t}$ and $\sigma_{m+1,t}$ by (8) and then from (11), the lemma is proved. □

***Lemma* 6.** *Let* P *is nonnegative integer, for sufficiently large positive integer* N_0, *and* $\forall n=10^{p+1}-1>N_0$ $\forall u,v\in\{1,2,\ldots,9\}, u<v$, *then*

$$\lambda_{u,n}>\lambda_{v,n}.$$

Proof. Since N_0 is sufficiently large, we can write $\lambda_{m,n}$ as

$$\lambda_{m,n}=\lambda_{m,999999}+\sum_{t=6}^{p}\sigma_{m,t}. \quad (12)$$

For $\forall u,v\in\{1,2,\ldots,9\}, u<v$, the Lemma (5) tells:

$$\sigma_{u,t}>\sigma_{v,t}. \quad (13)$$

After summing the inequality, we can get

$$\sum_{t=6}^{p} \sigma_{u,t} > \sum_{t=6}^{p} \sigma_{v,t}. \tag{14}$$

According to the Lemma 2:

$$\lambda_{u,999999} > \lambda_{v,999999} \tag{15}$$

holds. Then from (14) and (15), the Lemma is proved. □

***Theorem* 1.** *For sufficiently large positive integer* N_0, $\forall n > N_0$, $\forall u, v \in \{1, 2, \ldots, 9\}, u < v$,

$$\lambda_{u,n} > \lambda_{v,n}$$

***Proof*.** From the definitions of (3) and (4), we can get

$$0 \leq \lambda'_{m,n} \leq \sigma_{m,\delta_n - 1}. \tag{16}$$

Denote ϕ_n the first digit of n.
If $u < \phi_n$, then $\lambda'_{u,n} = \sigma_{u,\delta_n-1}$. From (5) and (16), we can get

$$\lambda'_{u,n} = \sigma_{u,\delta_n-1} \geq \sigma_{v,\delta_n-1} \geq \lambda'_{v,n};$$

If $v > \phi_n$, then $\lambda'_{v,n} = 0$. From (16), we can get

$$\lambda'_{u,n} \geq 0 = \lambda'_{v,n}.$$

Since $u < v$, one of the above two cases must be right, i.e,.

$$\lambda'_{u,n} \geq \lambda'_{v,n}. \tag{17}$$

Denote N_0 in Lemma (6) as N'_0 and $n' = 10^{\delta_n - 1} - 1$. Due to N_0 is sufficiently large, we can take $n' > N'_0$ when $n > N_0$. The condition in Lemma (6) is satisfied and we can see that

$$\lambda_{u,10^{\delta_n-1}-1} > \lambda_{v,10^{\delta_n-1}-1}. \tag{18}$$

From the definition of (3) and (17) and (18), we can see the theorem is right. □

3 The distribution of the first digit of the arbitrary positive integer power of the positive integer under the radix-10

***Definition* 6.** *For* $m \in \{1, 2, \ldots, 9\}, n \in \mathbb{N}^*, k \in \mathbb{N}^*$, *denotes* $\lambda_{m,n,k}$ *the count of the numbers of* k *power, whose first digit equals to* m *and value in* $1, 2, \ldots, n$.

***Definition* 7.** *For* $m \in \{1, 2, \ldots, 9\}, n \in \mathbb{N}^*, k \in \mathbb{N}^*$, *let* $\lambda'_{m,n,k} = \lambda_{m,n,k} - \lambda_{m,10^{\delta_n-1}-1,k}$.

***Definition* 8.** *For $m \in \{1, 2, \ldots, 9\}, t \in \mathbb{N}^*, k \in \mathbb{N}^*$, denotes $\sigma_{m,t,k}$ the count of the numbers which is just the k power of a positive integer between $m * 10^t$ and $(m+1) * 10^t - 1$.*

***Definition* 9.**

$$G(m, k) = \sqrt[k]{m}, m, k \in \mathbb{N}^*.$$

***Lemma* 7.** *For $\forall p \geq 0$, let $n = 10^{p+1} - 1$, then*

$$\lambda_{m,n,k} = \lambda_{m,10^{p+1}-1,k} = \sum_{t=0}^{p} \sigma_{m,t,k}.$$

(This can be easily proved by the definitions of (6) ana (8))

***Lemma* 8.** *For $\forall m \in \mathbb{N}^*, \forall k \in \mathbb{N}^*, k \geq 2$, then*

$$2\sqrt[k]{m+1} > \sqrt[k]{m} + \sqrt[k]{m+2}$$

***Proof*.** Since the second order derivation of $G(m, k)$ satisfies

$$G''(m, k) = \frac{1}{k}\left(\frac{1}{k} - 1\right) m^{\frac{1}{k}-2} < 0,$$

we can see $G(m, k)$ is a convex function and get the inequality by *Jensen* inequality as:

$$\frac{\sqrt[k]{m} + \sqrt[k]{m+2}}{2} \leq \sqrt[k]{\frac{m + (m+2)}{2}} = \sqrt[k]{m+1}. \tag{19}$$

Since $m \neq m + 2$, so the equal sign can't access in (19) and the lemma is proved. □

***Lemma* 9.** *For $\forall m \in \{1, 2, \ldots, 8\}, \forall k \in \mathbb{N}^*, k \geq 2$, we can see there is $t_0 \in \mathbb{N}^*$ when $t \geq t_0$ and*

$$\left(\sqrt[k]{(m+1) * 10^t - 1} - \sqrt[k]{m * 10^t}\right) - \left(\sqrt[k]{(m+2) * 10^t - 1} - \sqrt[k]{(m+1) * 10^t}\right) > 2$$

holds.

***Proof*.** Similar to the proof of (4), we rewrite the inequality as

$$\begin{aligned}&\left(\sqrt[k]{(m+1) * 10^t} - \sqrt[k]{m * 10^t}\right) - \left(\sqrt[k]{(m+2) * 10^t} - \sqrt[k]{(m+1) * 10^t}\right)\\&- \left(\sqrt[k]{(m+1) * 10^t} - \sqrt[k]{(m+1) * 10^t - 1}\right)\\&+ \left(\sqrt[k]{(m+2) * 10^t} - \sqrt[k]{(m+2) * 10^t - 1}\right) > 2.\end{aligned} \tag{20}$$

Similar to (5) and (6), we can see:

$$0 < \left|\sqrt[k]{(m+1) * 10^t} - \sqrt[k]{(m+1) * 10^t - 1}\right| < 1, \tag{21}$$

$$0 < \left|\sqrt[k]{(m+2) * 10^t} - \sqrt[k]{(m+2) * 10^t - 1}\right| < 1. \tag{22}$$

Define $T_{m,k}$ to satisfy:

$$\left(\sqrt[k]{(m+1)*10^{T_{m,k}}}-\sqrt[k]{m*10^{T_{m,k}}}\right)$$
$$-\left(\sqrt[k]{(m+2)*10^{T_{m,k}}}-\sqrt[k]{(m+1)*10^{T_{m,k}}}\right)=3.$$

i.e,.

$$T_{m,k}=\log_{10}\left(\frac{3}{2\sqrt[k]{m+1}-\sqrt[k]{m}-\sqrt[k]{m+2}}\right)^k.$$

Due to Lemma (8), the $T_{m,k}$ above really exists.
Taking $t_0=\max\{\lceil T_{m,k}\rceil\}$, then when $t\geq t_0$, we can get

$$\left(\sqrt[k]{(m+1)*10^t}-\sqrt[k]{m*10^t}\right)-\left(\sqrt[k]{(m+2)*10^t}-\sqrt[k]{(m+1)*10^t}\right)\geq 3. \tag{23}$$

From (21),(22) and (23),

$$\begin{aligned}&\left(\sqrt[k]{(m+1)*10^t}-\sqrt[k]{m*10^t}\right)-\left(\sqrt[k]{(m+2)*10^t}-\sqrt[k]{(m+1)*10^t}\right)\\&-\left(\sqrt[k]{(m+1)*10^t}-\sqrt[k]{(m+1)*10^t-1}\right)\\&+\left(\sqrt[k]{(m+2)*10^t}-\sqrt[k]{(m+2)*10^t-1}\right)>3-1+0=2\end{aligned} \tag{24}$$

holds. Since (24) equals to (20), the Lemma is proved. □

***Lemma* 10.** *For $\forall m\in\{1,2,\ldots,8\}and\forall k\in\mathbb{N}^*$, we can see there is $t_0\in\mathbb{N}^*$ when $t\geq t_0$ and*

$$\sigma_{m,t,k}\geq\sigma_{m+1,t,k}+1$$

holds.

***Proof*.** Similar to the proof of (5), by the definition of (8), we can get

$$\sigma_{m,t,k}=\lfloor\sqrt[k]{(m+1)*10^t-1}\rfloor-\lceil\sqrt[k]{m*10^t}\rceil+1. \tag{25}$$

According to the properties of the *Gauss* function,

$$\lfloor\sqrt[k]{(m+1)*10^t-1}\rfloor-\lceil\sqrt[k]{m*10^t}\rceil\leq\sqrt[k]{(m+1)*10^t-1}-\sqrt[k]{m*10^t}, \tag{26}$$

and

$$\lfloor\sqrt[k]{(m+1)*10^t-1}\rfloor-\lceil\sqrt[k]{m*10^t}\rceil\geq\sqrt[k]{(m+1)*10^t-1}-\sqrt[k]{m*10^t}-2 \tag{27}$$

hold and then we can see that

$$\begin{aligned}&\left(\lfloor\sqrt[k]{(m+1)*10^t-1}\rfloor-\lceil\sqrt[k]{m*10^t}\rceil+1\right)\\&-\left(\lfloor\sqrt[k]{(m+2)*10^t-1}\rfloor-\lceil\sqrt[k]{(m+1)*10^t}\rceil+1\right)\\&\geq\left(\sqrt[k]{(m+1)*10^t-1}-\sqrt[k]{m*10^t}\right)\\&-\left(\sqrt[k]{(m+2)*10^t-1}-\sqrt[k]{(m+1)*10^t}\right)-2.\end{aligned} \tag{28}$$

From (28) and Lemma (9), we know there is $t_0 \in \mathbb{N}^*$ when $t \geq t_0$ such that

$$\begin{aligned} &\left(\lfloor \sqrt[k]{(m+1)*10^t - 1} \rfloor - \lceil \sqrt[k]{m*10^t} \rceil + 1\right) \\ &- \left(\lfloor \sqrt[k]{(m+1)*10^t - 1} \rfloor - \lceil \sqrt[k]{m*10^t} \rceil + 1\right) > 0. \end{aligned} \tag{29}$$

Expanding $\sigma_{m,t,k}$ and $\sigma_{m+1,t,k}$ according to (25) and substitute into (29), we know there is $t_0 \in \mathbb{N}^*$ when$t \geq t_0$ such that

$$\sigma_{m,t,k} > \sigma_{m+1,t,k}. \tag{30}$$

From the definitions of $\sigma_{m,t,k}$ and $\sigma_{m+1,t,k}$, we know they are integer and then (30) equivalent to this Lemma. □

Lemma 11. *Let p be positive integer and for sufficiently large positive integer N_0, for $\forall n = 10^{p+1} - 1 > N_0$, $\forall k \in \mathbb{N}^*$ and $\forall u, v \in \{1, 2, \ldots, 9\}, u < v$, the inequality*

$$\lambda_{u,n,k} > \lambda_{v,n,k}$$

holds.

Proof. For $\forall k \in \mathbb{N}^*$, $\forall u, v \in \{1, 2, \ldots, 9\}, u < v$ and from Lemma (10), we can see there is $t_0 \in \mathbb{N}^*$ when $t \geq t_0$, and

$$\sigma_{u,t,k} \geq \sigma_{v,t,k} + 1 \tag{31}$$

holds. For these t_0, denotes $\Delta = \max\{\lambda_{v,10^{t_0}-1,k} - \lambda_{u,10^{t_0}-1,k}, 0\}$. Since N_0 is sufficiently large, we can choose $p \geq t_0 + \Delta$, i.e., $n \geq 10^{t_0+\Delta+1} - 1$, and get

$$\lambda_{m,n,k} = \lambda_{m,10^{t_0}-1,k} + \sum_{t=t_0}^{t_0+\Delta} \sigma_{m,t,k} + \sum_{t=t_0+\Delta+1}^{p} \sigma_{m,t,k} \tag{32}$$

by (7) and

$$\begin{aligned} &\left(\lambda_{u,10^{t_0}-1,k} + \sum_{t=t_0}^{t_0+\Delta} \sigma_{u,t,k}\right) - \left(\lambda_{v,10^{t_0}-1,k} + \sum_{t=t_0}^{t_0+\Delta} \sigma_{v,t,k}\right) \\ &= \left(\lambda_{u,10^{t_0}-1,k} - \lambda_{v,10^{t_0}-1,k}\right) + \left(\sum_{t=t_0}^{t_0+\Delta} \sigma_{u,t,k} - \sum_{t=t_0}^{t_0+\Delta} \sigma_{v,t,k}\right) \\ &\geq -\Delta + (\Delta + 1) = 1 > 0 \end{aligned} \tag{33}$$

by (31). Similarly by (31), we can also get

$$\sum_{t=t_0+\Delta+1}^{p} \sigma_{u,t,k} \geq \sum_{t=t_0+\Delta+1}^{p} \sigma_{v,t,k}. \tag{34}$$

From (32), (33) and (34), we can see that the Lemma is right. □

***Theorem* 2.** *For sufficiently large positive integer N_0, $\forall n > N_0$, $\forall k \in \mathbb{N}^*$, $\forall u, v \in \{1, 2, \ldots, 9\}, u < v$,*

$$\lambda_{u,n,k} > \lambda_{v,n,k}$$

holds.

Proof. Similarly to the proof of the Theorem (1), from the definitions of (7) and (8), we can get

$$0 \leq \lambda'_{m,n,k} \leq \sigma_{m,\delta_n-1,k}. \tag{35}$$

Takeing the first digit of n and denoting it ϕ_n.
If $u < \phi_n$, then $\lambda'_{u,n,k} = \sigma_{u,\delta_n-1,k}$ and from (10) and (35), we can get

$$\lambda'_{u,n,k} = \sigma_{u,\delta_n-1,k} \geq \sigma_{v,\delta_n-1,k} \geq \lambda'_{v,n,k}.$$

If $v > \phi_n$, then$\lambda'_{v,n,k} = 0$ and from (35), we can get

$$\lambda'_{u,n,k} \geq 0 = \lambda'_{v,n,k}.$$

Since $u < v$, one of the above two cases must be held, i.e.,

$$\lambda'_{u,n,k} \geq \lambda'_{v,n,k} \tag{36}$$

Denoting N_0 in Lemma 11 as N'_0 and let $n' = 10^{\delta_n-1} - 1$. Since N_0 is sufficiently large, we can choose $n > N_0$ such that $n' > N'_0$. So the conditions in Lemma (11) is satisfied and from it, the inequality

$$\lambda_{u,10^{\delta_n-1}-1,k} > \lambda_{v,10^{\delta_n-1}-1,k} \tag{37}$$

holds. From (7), (36) and (37), this Lemma is proved. □

4 The distribution of the first digit of the arbitrary positive integer power of the positive integer under the radix of the arbitrary positive integer.

This section is the extension of the section (3) and the procedures of these proofs are similar.

***Definition* 10.** *For $n \in \mathbb{N}^*, l \in \mathbb{N}^*, l \geq 3$, denotes $\delta_{n,l}$ the digit of n under the radix-l, i.e., $\delta_{n,l} = \lfloor \log_l n \rfloor + 1$*

***Definition* 11.** *For $l \in \mathbb{N}^*, l \geq 3, m \in \{1, 2, \ldots, l-1\}, n \in \mathbb{N}^*, k \in \mathbb{N}^*$, denotes $\lambda_{m,n,k,l}$ the count of the numbers of k power, whose first digit equals to m and value in $1, 2, \ldots, n$ under radix-l.*

***Definition* 12.** *For $l \in \mathbb{N}^*, l \geq 3, m \in \{1, 2, \ldots, l-1\}, n \in \mathbb{N}^*, k \in \mathbb{N}^*$, let $\lambda'_{m,n,k,l} = \lambda_{m,n,k,l} - \lambda_{m,l^{\delta_{n,l}-1}-1,k,l}$*

***Definition* 13.** *For $l \in \mathbb{N}^*, l \geq 3, m \in \{1, 2, \ldots, l-1\}, t \in \mathbb{N}^*, k \in \mathbb{N}^*$, denotes $\sigma_{m,t,k,l}$ the count of the numbers which is just the k power of a positive integer between $m * l^t$ and $(m+1) * l^t - 1$.*

***Lemma* 12.** *For $\forall p \geq 0, \forall l \geq 3$, let $n = l^{p+1} - 1$, then*

$$\lambda_{m,n,k,l} = \lambda_{m,l^{p+1}-1,k,l} = \sum_{t=0}^{p} \sigma_{m,t,k,l}.$$

(This can be easily proved by the definitions of (11) ana (13))

***Lemma* 13.** *For $\forall l \in \mathbb{N}^*, l \geq 3, \forall m \in \{1, 2, \ldots, l-2\}, \forall k \in \mathbb{N}^*$, $\exists t_0 \in \mathbb{N}^*$ and when $t \geq t_0$, then*

$$\left(\sqrt[k]{(m+1) * l^t - 1} - \sqrt[k]{m * l^t}\right) - \left(\sqrt[k]{(m+2) * l^t - 1} - \sqrt[k]{(m+1) * l^t}\right) > 2.$$

***Proof*.** Similar to the proof of (9) and we rewrite this inequality as :

$$\begin{aligned}&\left(\sqrt[k]{(m+1) * l^t} - \sqrt[k]{m * l^t}\right) - \left(\sqrt[k]{(m+2) * l^t} - \sqrt[k]{(m+1) * l^t}\right)\\ &- \left(\sqrt[k]{(m+1) * l^t} - \sqrt[k]{(m+1) * l^t - 1}\right)\\ &+ \left(\sqrt[k]{(m+2) * l^t} - \sqrt[k]{(m+2) * l^t - 1}\right) > 2. \qquad (38)\end{aligned}$$

and then

$$0 < \left|\sqrt[k]{(m+1) * l^t} - \sqrt[k]{(m+1) * l^t - 1}\right| < 1, \qquad (39)$$

$$0 < \left|\sqrt[k]{(m+2) * l^t} - \sqrt[k]{(m+2) * l^t - 1}\right| < 1. \qquad (40)$$

Define $T_{m,k,l}$ to satisfy:

$$\begin{aligned}&\left(\sqrt[k]{(m+1) * l^{T_{m,k,l}}} - \sqrt[k]{m * l^{T_{m,k,l}}}\right)\\ &- \left(\sqrt[k]{(m+2) * l^{T_{m,k,l}}} - \sqrt[k]{(m+1) * l^{T_{m,k,l}}}\right) = 3\end{aligned}$$

i.e.,

$$T_{m,k,l} = \log_l \left(\frac{3}{2\sqrt[k]{m+1} - \sqrt[k]{m} - \sqrt[k]{m+2}}\right)^k.$$

From Lemma 8, we can know $T_{m,k,l}$ above really exists.
Taking $t_0 = \max\{\lceil T_{m,k,l} \rceil\}$ and when $t \geq t_0$, we can see that

$$\left(\sqrt[k]{(m+1) * l^t} - \sqrt[k]{m * l^t}\right) - \left(\sqrt[k]{(m+2) * l^t} - \sqrt[k]{(m+1) * l^t}\right) \geq 3. \qquad (41)$$

From (39),(40) and (41),

$$\begin{aligned}&\left(\sqrt[k]{(m+1) * l^t} - \sqrt[k]{m * l^t}\right) - \left(\sqrt[k]{(m+2) * l^t} - \sqrt[k]{(m+1) * l^t}\right)\\ &- \left(\sqrt[k]{(m+1) * l^t} - \sqrt[k]{(m+1) * l^t - 1}\right)\\ &+ \left(\sqrt[k]{(m+2) * l^t} - \sqrt[k]{(m+2) * l^t - 1}\right) > 3 - 1 + 0 = 2 \qquad (42)\end{aligned}$$

holds. Since (42) equals to (38), this Lemma is proved. □

***Lemma* 14.** *For* $\forall l \in \mathbb{N}^*, l \geq 3, \forall m \in \{1, 2, \dots, l-2\}, \forall k \in \mathbb{N}^*$, *we can see there is* $t_0 \in \mathbb{N}^*$ *when* $t \geq t_0$ *and*

$$\sigma_{m,t,k,l} \geq \sigma_{m+1,t,k,l} + 1$$

holds.

***Proof*.** Similar to the proof of Lemma (10), by the definition of (13), we can see that

$$\sigma_{m,t,k,l} = \lfloor \sqrt[k]{(m+1) * l^t - 1} \rfloor - \lceil \sqrt[k]{m * l^t} \rceil + 1. \tag{43}$$

According to the properties of the *Gauss* function,

$$\lfloor \sqrt[k]{(m+1) * l^t - 1} \rfloor - \lceil \sqrt[k]{m * l^t} \rceil \leq \sqrt[k]{(m+1) * l^t - 1} - \sqrt[k]{m * l^t} \tag{44}$$

and

$$\lfloor \sqrt[k]{(m+1) * l^t - 1} \rfloor - \lceil \sqrt[k]{m * l^t} \rceil \geq \sqrt[k]{(m+1) * l^t - 1} - \sqrt[k]{m * l^t} - 2 \tag{45}$$

hold and we can see that

$$\begin{aligned} &\Big(\lfloor \sqrt[k]{(m+1) * l^t - 1} \rfloor - \lceil \sqrt[k]{m * l^t} \rceil + 1 \Big) \\ &- \Big(\lfloor \sqrt[k]{(m+2) * l^t - 1} \rfloor - \lceil \sqrt[k]{(m+1) * l^t} \rceil + 1 \Big) \\ &\geq \Big(\sqrt[k]{(m+1) * l^t - 1} - \sqrt[k]{m * l^t} \Big) \\ &- \Big(\sqrt[k]{(m+2) * l^t - 1} - \sqrt[k]{(m+1) * l^t} \Big) - 2. \end{aligned} \tag{46}$$

From (46) and Lemma (13), we know there is $t_0 \in \mathbb{N}^*$ when $t \geq t_0$ and

$$\begin{aligned} &\Big(\lfloor \sqrt[k]{(m+1) * l^t - 1} \rfloor - \lceil \sqrt[k]{m * l^t} \rceil + 1 \Big) \\ &- \Big(\lfloor \sqrt[k]{(m+2) * l^t - 1} \rfloor - \lceil \sqrt[k]{(m+1) * l^t} \rceil + 1 \Big) > 0 \end{aligned} \tag{47}$$

Expanding $\sigma_{m,t,k,l}$ and $\sigma_{m+1,t,k,l}$ according to (43) and substitute into (47), we know there is $t_0 \in \mathbb{N}^*$ when $t \geq t_0$ such that

$$\sigma_{m,t,k,l} > \sigma_{m+1,t,k,l}. \tag{48}$$

From the definitions of $\sigma_{m,t,k,l}$ and $\sigma_{m+1,t,k,l}$, we know they are integer and then (48) equivalent to this Lemma. □

***Lemma* 15.** *Let* p *is positive integer, for sufficiently large positive integer* N_0, $\forall l \in \mathbb{N}^*, l \geq 3$, $\forall n = l^{p+1} - 1 > N_0$, $\forall k \in \mathbb{N}^*$ *and* $\forall u, v \in \{1, 2, \dots, l-1\}, u < v$,

$$\lambda_{u,n,k,l} > \lambda_{v,n,k,l}$$

holds.

Proof. Similar to the proof of (11), for $\forall k \in \mathbb{N}^*, \forall l \in \mathbb{N}^*, l \geq 3$ and $\forall u, v \in \{1, 2, \ldots, l-1\}, u < v$, from 14, we know there is $t_0 \in \mathbb{N}^*$ when $t \geq t_0$ and

$$\sigma_{u,t,k,l} \geq \sigma_{v,t,k,l} + 1. \tag{49}$$

For these t_0, denoting $\Delta = \max\{\lambda_{v,l^{t_0}-1,k,l} - \lambda_{u,l^{t_0}-1,k,l}, 0\}$, since N_0 is sufficiently large, we can choose $p \geq t_0 + \Delta$, i.e., $n \geq l^{t_0+\Delta+1} - 1$ and get

$$\lambda_{m,n,k,l} = \lambda_{m,l^{t_0}-1,k,l} + \sum_{t=t_0}^{t_0+\Delta} \sigma_{m,t,k,l} + \sum_{t=t_0+\Delta+1}^{p} \sigma_{m,t,k,l} \tag{50}$$

by Lemma (12) and

$$\begin{aligned} &\left(\lambda_{u,l^{t_0}-1,k,l} + \sum_{t=t_0}^{t_0+\Delta} \sigma_{u,t,k,l}\right) - \left(\lambda_{v,l^{t_0}-1,k,l} + \sum_{t=t_0}^{t_0+\Delta} \sigma_{v,t,k,l}\right) \\ &= \left(\lambda_{u,l^{t_0}-1,k,l} - \lambda_{v,l^{t_0}-1,k,l}\right) + \left(\sum_{t=t_0}^{t_0+\Delta} \sigma_{u,t,k,l} - \sum_{t=t_0}^{t_0+\Delta} \sigma_{v,t,k,l}\right) \\ &\geq -\Delta + (\Delta + 1) = 1 > 0 \end{aligned} \tag{51}$$

by (49). Similarly by (49), we can also get

$$\sum_{t=t_0+\Delta+1}^{p} \sigma_{u,t,k,l} \geq \sum_{t=t_0+\Delta+1}^{p} \sigma_{v,t,k,l} \tag{52}$$

From (50), (51) and (52), we can see that the Lemma is right. □

Theorem 3. *For sufficiently large positive integer N_0, $\forall n > N_0$, $\forall l \in \mathbb{N}^*, l \geq 3$, $\forall k \in \mathbb{N}^*$, and $\forall u, v \in \{1, 2, \ldots, l-1\}, u < v$,*

$$\lambda_{u,n,k,l} > \lambda_{v,n,k,l}$$

holds.

Proof. Similarly to the proof of the Theorem (2), from the definitions of (12) and (13), we can get

$$0 \leq \lambda'_{m,n,k,l} \leq \sigma_{m,\delta_{n,l}-1,k,l} \tag{53}$$

Taking the first digit of n under the radix-l and denotes it $\phi_n \in \{1, 2, \ldots, l-1\}$
If $u < \phi_n$, then $\lambda'_{u,n,k,l} = \sigma_{u,\delta_{n,l}-1,k,l}$, and from Lemma (14) and (53), we can get

$$\lambda'_{u,n,k,l} = \sigma_{u,\delta_{n,l}-1,k,l} \geq \sigma_{v,\delta_{n,l}-1,k,l} \geq \lambda'_{v,n,k,l};$$

If $v > \phi_n$, then $\lambda'_{v,n,k,l} = 0$, and from (53) we can get

$$\lambda'_{u,n,k,l} \geq 0 = \lambda'_{v,n,k,l}.$$

Since $u < v$, one of the above two cases must be held, i.e.,

$$\lambda'_{u,n,k,l} \geq \lambda'_{v,n,k,l}. \tag{54}$$

Denoting N_0 in Lemma (15) as N_0' and let $n' = 10^{\delta_n - 1} - 1$. Since N_0 is sufficiently large, we can choose $n > N_0$ such that $n' > N_0'$. So the conditions in Lemma (15) is satisfied and from it, the inequality

$$\lambda_{u,l^{\delta_{n,l}-1}-1,k,l} > \lambda_{v,l^{\delta_{n,l}-1}-1,k,l} \tag{55}$$

holds. From (12), (54) and (55), this Lemma is proved. □

5 The distribution of the first digit of the polynomials with positive integer coefficients

***Definition* 14.** *Denotes P the set of the polynomial with positive integer coefficient, i.e.,*

$$P = \{p(x) = \sum_{k=0}^{n} a_k x^k : n \in \mathbb{N}^*, \forall k, a_k \in \mathbb{N}, a_n > 0\}.$$

***Definition* 15.** *For $\forall p(x) \in P$, $l \in \mathbb{N}^*, l \geq 3, m \in \{1, 2, \ldots, l-1\}, n \in \mathbb{N}^*$, denoting $\lambda_{m,n,l}(p)$ the count of the numbers of $p(x)$, whose first digit equals to m and value in $1, 2, \ldots, n$ under the radix-l.*

***Definition* 16.** *For $\forall p(x) \in P$, $l \in \mathbb{N}^*, l \geq 3, m \in \{1, 2, \ldots, l-1\}, n \in \mathbb{N}^*$, let $\lambda'_{m,n,l}(p) = \lambda_{m,n,l}(p) - \lambda_{m,l^{\delta_{n,l}-1}-1,l}(p)$.*

***Definition* 17.** *For $\forall p(x) \in P$, $l \in \mathbb{N}^*, l \geq 3, m \in \{1, 2, \ldots, l-1\}, t \in \mathbb{N}^*$, denoting $\sigma_{m,t,l}(p)$ the count of the solution of the inequality $m * l^t \leq p(x) \leq (m+1) * l^t - 1$ in $\mathbb{Z}$.*

***Lemma* 16.** *For $\forall p(x) \in P$, $p(x)$ is invertible and its inverse $p^{-1}(x)$ monotone increasing.*

***Proof*.** □

***Lemma* 17.** *For $\forall s \geq 0, \forall l \geq 3$, let $n = l^{s+1} - 1$, then*

$$\lambda_{m,n,l}(p) = \lambda_{m,l^{s+1}-1,l}(p) = \sum_{t=0}^{s} \sigma_{m,t,l}(p)$$

holds.
(This can be easily proved by the definitions of (15) ana (17))

***Lemma* 18.** *For $\forall p \in P, \forall l \in \mathbb{N}^*, l \geq 3, \forall m \in \{1, 2, \ldots, l-2\}, \forall k \in \mathbb{N}^*$, we can see that there is $t_0 \in \mathbb{N}^*$ when $t \geq t_0$ and*

$$\big(p^{-1}((m+1) * l^t - 1) - p^{-1}(m * l^t)\big) - \big(p^{-1}(m+2) * l^t - 1) - p^{-1}((m+1) * l^t)\big) > 2$$

holds.

Proof. Let $r_1 = p^{-1}((m+1)*l^t - 1)$, $l_1 = p^{-1}(m*l^t)$,
$r_2 = p^{-1}((m+2)*l^t - 1)$, $l_2 = p^{-1}((m+1)*l^t)$, i.e., they satisfy:

$$(r_1 - l_1) - (r_2 - l_2) > 2. \tag{56}$$

From the properties of the inverse function, we know that if

$$p(r_1) - p(l_1) = p(r_2) - p(l_2) = l^t - 1, \tag{57}$$

i.e.,

$$\begin{aligned} \sum_{k=0}^{n} a_k(r_1^k - l_1^k) = l^t - 1, \\ \sum_{k=0}^{n} a_k(r_2^k - l_2^k) = l^t - 1, \end{aligned} \tag{58}$$

then

$$\begin{aligned} r_1 - l_1 = \frac{l^t - 1}{\sum_{k=1}^{n}(a_k \sum_{i=0}^{k-1} l_1^i r_1^{k-1-i})}, \\ r_2 - l_2 = \frac{l^t - 1}{\sum_{k=1}^{n}(a_k \sum_{i=0}^{k-1} l_2^i r_2^{k-1-i})}. \end{aligned} \tag{59}$$

hold. Subtracted the two equalities, we can get

$$\begin{aligned} &(r_1 - l_1) - (r_2 - l_2) \\ &= \frac{l^t - 1}{\sum_{k=1}^{n}(a_k \sum_{i=0}^{k-1} l_1^i r_1^{k-1-i})} - \frac{l^t - 1}{\sum_{k=1}^{n}(a_k \sum_{i=0}^{k-1} l_2^i r_2^{k-1-i})} \\ &= (l^t - 1)(\frac{1}{\sum_{k=1}^{n}(a_k \sum_{i=0}^{k-1} l_1^i r_1^{k-1-i})} - \frac{1}{\sum_{k=1}^{n}(a_k \sum_{i=0}^{k-1} l_2^i r_2^{k-1-i})}) \\ &= (l^t - 1)\frac{\sum_{k=1}^{n}(a_k \sum_{i=0}^{k-1} l_2^i r_2^{k-1-i}) - \sum_{k=1}^{n}(a_k \sum_{i=0}^{k-1} l_1^i r_1^{k-1-i})}{\sum_{k=1}^{n}(a_k \sum_{i=0}^{k-1} l_1^i r_1^{k-1-i}) \sum_{k=1}^{n}(a_k \sum_{i=0}^{k-1} l_2^i r_2^{k-1-i})} \\ &= (l^t - 1)\frac{\sum_{k=1}^{n} a_k \sum_{i=0}^{k-1}(l_2^i r_2^{k-1-i} - l_1^i r_1^{k-1-i})}{\sum_{k=1}^{n}(a_k \sum_{i=0}^{k-1} l_1^i r_1^{k-1-i}) \sum_{k=1}^{n}(a_k \sum_{i=0}^{k-1} l_2^i r_2^{k-1-i})}. \end{aligned} \tag{60}$$

Since $l2 > l1, r2 > r1, a_k > 0$, so $(r_1 - l_1) - (r_2 - l_2)$ is monotone increasing on t.

Define $T_{m,l}(p)$ to satisfy:

$$(l^{T_{m,l}(p)}-1)\frac{\sum_{k=1}^{n}(a_k\sum_{i=0}^{k-1}l_2^i r_2^{k-1-i})-\sum_{k=1}^{n}(a_k\sum_{i=0}^{k-1}l_1^i r_1^{k-1-i})}{\sum_{k=1}^{n}(a_k\sum_{i=0}^{k-1}l_1^i r_1^{k-1-i})\sum_{k=1}^{n}(a_k\sum_{i=0}^{k-1}l_2^i r_2^{k-1-i})}=2, \tag{61}$$

i.e.,

$$T_{m,l}(p)=\log_l\left(\frac{2\sum_{k=1}^{n}(a_k\sum_{i=0}^{k-1}l_1^i r_1^{k-1-i})\sum_{k=1}^{n}(a_k\sum_{i=0}^{k-1}l_2^i r_2^{k-1-i})}{\sum_{k=1}^{n}(a_k\sum_{i=0}^{k-1}l_2^i r_2^{k-1-i})-\sum_{k=1}^{n}(a_k\sum_{i=0}^{k-1}l_1^i r_1^{k-1-i})}+1\right).$$

Take $t_0=\max\{\lceil T_{m,l}(p)\rceil\}$ and when $t>t_0$, we can get $(r_1-l_1)-(r_2-l_2)>2$, i.e.,

$$\begin{aligned}&\big(p^{-1}((m+1)*l^t-1)-p^{-1}(m*l^t)\big)\\&-\big(p^{-1}(m+2)*l^t-1)-p^{-1}((m+1)*l^t)\big)>2\end{aligned} \tag{62}$$

□

***Lemma* 19.** *For* $\forall p\in P,\forall l\in\mathbb{N}^*,l\geq 3,\forall m\in\{1,2,\ldots,l-2\}$*, we can see there is* $t_0\in\mathbb{N}^*$ *when* $t\geq t_0$ *and*

$$\sigma_{m,t,l}(p)\geq\sigma_{m+1,t,l}(p)+1$$

holds.

Proof. Similar to the proof of the Lemma (14), by the definition of (17) and Lemma (16), we can get:

$$c->q\sigma_{m,t,l}(p)=\lfloor p^{-1}((m+1)*l^t-1)\rfloor-\lceil p^{-1}(m*l^t)\rceil+1. \tag{63}$$

According to the properties of the *Gauss* function,

$$\lfloor p^{-1}((m+1)*l^t-1)\rfloor-\lceil p^{-1}(m*l^t)\rceil\leq p^{-1}((m+1)*l^t-1)-p^{-1}(m*l^t) \tag{64}$$

and

$$\lfloor p^{-1}((m+1)*l^t-1)\rfloor-\lceil p^{-1}(m*l^t)\rceil\geq p^{-1}((m+1)*l^t-1)-p^{-1}(m*l^t)-2. \tag{65}$$

From (64) and (65), we can get

$$\begin{aligned}&\big(\lfloor p^{-1}((m+1)*l^t-1)\rfloor-\lceil p^{-1}(m*l^t)\rceil+1\big)\\&-\big(\lfloor p^{-1}((m+2)*l^t-1)\rfloor-\lceil p^{-1}((m+1)*l^t)\rceil+1\big)\\&\geq\big(p^{-1}((m+1)*l^t-1)-p^{-1}(m*l^t)\big)\\&-\big(p^{-1}((m+2)*l^t-1)-p^{-1}((m+1)*l^t)\big)-2.\end{aligned} \tag{66}$$

At the same time, from (66) and Lemma (18), we know that there is $t_0 \in \mathbb{N}^*$ when $t \geq t_0$,

$$\begin{aligned}&\left(\lfloor p^{-1}((m+1)*l^t-1)\rfloor - \lceil p^{-1}(m*l^t)\rceil + 1\right)\\&-\left(\lfloor p^{-1}((m+1)*l^t-1)\rfloor - \lceil p^{-1}(m*l^t)\rceil + 1\right) > 0 \qquad (67)\end{aligned}$$

holds. Expanded $\sigma_{m,t,l}(p)$ and $\sigma_{m+1,t,l}(p)$ on (63) and substitute them into (67), we can know that there is $t_0 \in \mathbb{N}^*$ when $t \geq t_0$ such that

$$\sigma_{m,t,l}(p) > \sigma_{m+1,t,l}(p). \qquad (68)$$

Due to the definitions of $\sigma_{m,t,l}(p)$ and $\sigma_{m+1,t,l}(p)$, they are integer. So (68) equivalent to this Lemma. □

Lemma 20. *Let s be positive integer and for sufficiently large positive integer N_0, for $\forall p \in P$, $\forall l \in \mathbb{N}^*, l \geq 3$, $\forall n = l^{s+1} - 1 > N_0$ and $\forall u, v \in \{1, 2, \ldots, l-1\}, u < v$, the inequality*

$$\lambda_{u,n,l}(p) > \lambda_{v,n,l}(p)$$

holds.

***Proof*.** Similarly to the proof of the Theorem (15), for $\forall l \in \mathbb{N}^*, l \geq 3$ and $\forall u, v \in \{1, 2, \ldots, l-1\}, u < v$ and from Lemma 19, we can get there is $t_0 \in \mathbb{N}^*$ when $t \geq t_0$ such that

$$\sigma_{u,t,l}(p) \geq \sigma_{v,t,l}(p) + 1. \qquad (69)$$

For these t_0, let $\Delta = \max\{\lambda_{v,l^{t_0}-1,l}(p) - \lambda_{u,l^{t_0}-1,l}(p), 0\}$. Since N_0 is sufficiently large, we can choose $s \geq t_0 + \Delta$, i.e., $n \geq l^{t_0+\Delta+1} - 1$. Then we can get

$$\lambda_{m,n,l}(p) = \lambda_{m,l^{t_0}-1,l}(p) + \sum_{t=t_0}^{t_0+\Delta} \sigma_{m,t,l}(p) + \sum_{t=t_0+\Delta+1}^{s} \sigma_{m,t,l}(p). \qquad (70)$$

by the Lemma (17) and

$$\begin{aligned}&\left(\lambda_{u,l^{t_0}-1,l}(p) + \sum_{t=t_0}^{t_0+\Delta} \sigma_{u,t,l}(p)\right) - \left(\lambda_{v,l^{t_0}-1,l}(p) + \sum_{t=t_0}^{t_0+\Delta} \sigma_{v,t,l}(p)\right)\\&= \left(\lambda_{u,l^{t_0}-1,l}(p) - \lambda_{v,l^{t_0}-1,l}(p)\right) + \left(\sum_{t=t_0}^{t_0+\Delta} \sigma_{u,t,l}(p) - \sum_{t=t_0}^{t_0+\Delta} \sigma_{v,t,l}(p)\right)\\&\geq -\Delta + (\Delta+1) = 1 > 0 \qquad (71)\end{aligned}$$

by (69). Similarly by (69), we can also get

$$\sum_{t=t_0+\Delta+1}^{s} \sigma_{u,t,l}(p) \geq \sum_{t=t_0+\Delta+1}^{s} \sigma_{v,t,l}(p). \qquad (72)$$

From (70), (71) and (72)we can see that the Lemma is right. □

***Theorem* 4.** *For sufficiently large positive integer N_0, $\forall p \in P$, $\forall n > N_0$, $\forall l \in \mathbb{N}^*, l \geq 3$, and $\forall u, v \in \{1, 2, \ldots, l-1\}, u < v$,*

$$\lambda_{u,n,l}(p) > \lambda_{v,n,l}(p)$$

holds.

***Proof*.** Similar to the proof of the Theorem 3 and from the definitions of (16) and (17), we can get

$$0 \leq \lambda'_{m,n,l}(p) \leq \sigma_{m,\delta_{n,l}-1,l}(p) \tag{73}$$

Taking the first digit of n under the radix-l and denoting it $\phi_n \in \{1, 2, \ldots, l-1\}$.
If $u < \phi_n$, then $\lambda'_{u,n,l}(p) = \sigma_{u,\delta_{n,l}-1,l}(p)$ and from Lemma 19 and (73), we can get

$$\lambda'_{u,n,l}(p) = \sigma_{u,\delta_{n,l}-1,l}(p) \geq \sigma_{v,\delta_{n,l}-1,l}(p) \geq \lambda'_{v,n,l}(p).$$

If $v > \phi_n$, then $\lambda'_{v,n,l}(p) = 0$ and from (73), we can get

$$\lambda'_{u,n,l}(p) \geq 0 = \lambda'_{v,n,l}(p).$$

Since $u < v$, one of the above two cases must be held, i.e.,

$$\lambda'_{u,n,l}(p) \geq \lambda'_{v,n,l}(p) \tag{74}$$

Denoting N_0 in Lemma 20 as N'_0 and let $n' = l^{\delta_{n,l}-1} - 1$. Since N_0 is sufficiently large, we can choose $n > N_0$ such that $n' > N'_0$. So the conditions in Lemma 20 are satisfied and from it, the inequality

$$\lambda_{u,l^{\delta_{n,l}-1}-1,l}(p) > \lambda_{v,l^{\delta_{n,l}-1}-1,l}(p) \tag{75}$$

holds. From (16), (74) and (75), this Lemma is proved. □

6 Conjecture and Prospect

***Conjecture* 1.** *For these integer polynomials whose first coefficients are positive, the similar conclusions hold, i.e.,*
for $P = \{p(x) = \sum_{k=0}^{n} a_k x^k : n \in \mathbb{N}^*, \forall k, a_k \in \mathbb{Z}, a_n > 0\}$ and $u < v$, then

$$\lambda_{u,n,l}(p) > \lambda_{v,n,l}(p)$$

holds.

***Conjecture* 2.** *For multivariate Polynomial $P[X_1, X_2, \ldots, X_n]$, $\exists N_1, N_2, \ldots, N_n \in \mathbb{N}^*$ and when $x_1 > N_1, x_2 > N_2, \ldots, x_n > N_n$, the similar conclusions hold.*

Bibliography

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