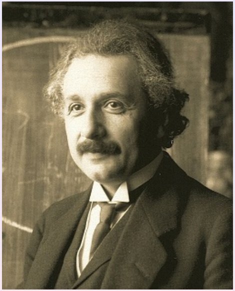
My Personal Journey on the Geometric Aspect of General Relativity

Shing-Tung Yau
Harvard University

The first annual meeting of ICCM 2017
Sun Yat-sen University, Guangzhou
December 28, 2017

This talk is based on joint work with Mu-Tao Wang and Po-Ning Chen.

About 100 years ago, Einstein accomplished one of the most spectacular work in physics and radically changed the view of space and time in the history of mankind.



Einstein

The foundation laid by Isaac Newton on the theory of gravity was completely changed by the theory of general relativity. In the very successful theory of Newton, space is static and time is independent of space.

Newton

By 1905, when Einstein established special relativity along with Poincaré and others, it was realized that space and time are linked and that the very foundation of special relativity, and that information cannot travel faster than light, is in contradiction with Newtonian gravity where action at a distance was used.

Poincaré

Einstein learnt from his teacher in 1908 that special relativity is best described as the geometry of the Minkowski spacetime. He realized gravitational potential cannot be described by a scalar function. It should be described by a tensor.

Minkowski

After tremendous helps from his two friends in mathematics: Grossmann and Hilbert, Einstein finally wrote down the famous Einstein equation:

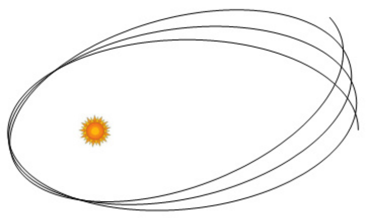
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}.$$

Grossmann

Hilbert

Note that Hilbert was the first one that write down the action principle of gravity, which plays the most important role in any attempts to quantize general relativity. The action is given by the total scalar curvature of the metric tensor which is considered to be the gravitational potential. If gravity is coupled with other matter, we simply add the matter Lagrangian.

The field equation was used by Einstein to calculate the perihelion of mercury and the light bending predicted by the Schwarzschild solution of the Einstein equation, which was found shortly. This was of course a great triumph of the Einstein theory of general relativity.

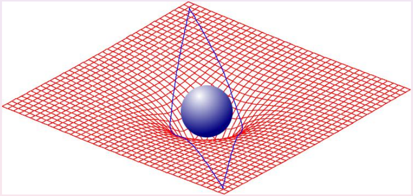


Perihelion of mercury

However, since the theory is highly nonlinear and the geometry of spacetime is dynamical, the actual understanding of Einstein equation was very difficult: even to Einstein himself.

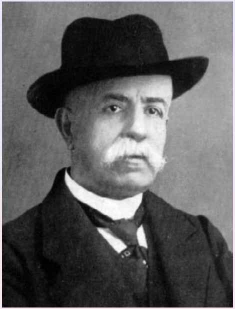
Einstein thought that the equation determined gravity completely. But that is actually not true as we cannot tell what is the initial condition for the field equation and we have difficulty to find the boundary condition.

Most of the problems in general relativity can be formulated in terms of geometry. This is natural because that was the original intention of Einstein. Gravity is described by the geometry of spacetime.



Local information of spacetime is pretty much governed by the full curvature tensor of the spacetime. Matter tensor is described by only part of the full curvature tensor: namely the Ricci tensor which is basically the trace of the full curvature tensor.

The dynamical equation for spacetime is described by the Einstein equation which says exactly that the matter stress tensor is described by the Ricci tensor. This is probably the most fascinating equation in geometry. It has influenced the development of geometry in the past 100 years!



Ricci

I was very puzzled by the Einstein equation when I learnt about it 47 years ago. I was intrigued by the beauty of the equation. But at the same time, I wonder what happens when there is no matter in the universe, it seems to me that the spacetime may still be nontrivial. (Trivial spacetime is the Minkowski spacetime where curvature tensor is identically zero.)

There is certainly possibility that Ricci tensor of a spacetime is identically zero while its full curvature tensor is not zero. I was wondering whether we can find a global spacetime which exhibit the following phenomenon: it is geodesically complete and nontrivial while the Ricci curvature is identically zero.

The problem of finding and classifying geodesic complete Ricci flat metric spaces is a problem that I have always been excited about. When I was a graduate student, there is not a single space of this sort that has no singularity. It was a fascinating question that drove me to look into Kähler geometry.

Kähler

Even before the birth of general relativity in 1915, many mathematicians have been communicating with Einstein. The most notable one was his friend Grossmann, but there was also Levi Civita and also Hilbert. In fact the equation describing vacuum was already correctly written by the joint paper of Einstein and Grossmann in 1912.

Herman Weyl, Kaluza and Élie Cartan followed Einstein on trying to unify gravity with other theory in the framework of general relativity.

Weyl

Cartan

Two very major works were created. One is gauge theory developed by Herman Weyl from 1918 to 1928. The gauge group that proposed by Weyl was noncompact at the beginning and was criticized by Einstein.

The birth of quantum mechanics gave the hint of importance of phase and Weyl changed the group to be $U(1)$. This is a spectacular achievement of Weyl which has deep influence in fundamental physics after it was realized by Yang and Mills that it is more fruitful to replace $U(1)$ by $SU(n)$.

The other major work was due to another mathematician Kaluza and later followed by a physicist Klein, and is called Kaluza-Klein model. They want to take a model of gravity with spacetime to be five-dimensional. And they want the spacetime to have a circle symmetry.

Kaluza

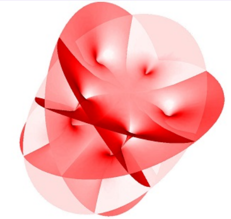
Klein

They look at the Einstein equation with no matter in the five spacetime dimension. The reduction to four-dimensional spacetime by the circle symmetry become an Einstein equation coupling gravity with electromagnetism. That worked out like magic and Einstein was very much impressed by it. Unfortunately the theory also created an extra scalar particle which was not observed. Although the theory was abandoned at the time, it never died.

In fact, when it was found sixty years later that in order for gravity and quantum mechanics to be consistent on a spacetime with a symmetry called supersymmetry, the spacetime has to be ten-dimensional.

String theorists proposed that some six-dimensional space can curl up to be so tiny that the ten-dimensional spacetime looks like four-dimensional. The simplest way to curl up is that the ten-dimensional spacetime is the product of a four-dimensional spacetime with another six-dimensional space with some kind of supersymmetry.

It was observed by Candelas-Horowitz-Strominger-Witten that this six-dimensional space is a Kähler manifold with zero Ricci curvature. They called this six-dimensional space Calabi-Yau space in honor of what I did in 1976 in proving the Calabi conjecture.



Calabi-Yau space

I was strongly motivated to prove the Calabi conjecture starting from the time when I was a graduate student in Berkeley. This was the story I mentioned earlier: I was fascinated by the question of existence of a non singular metric which describes vacuum, but not the trivial spacetime where curvature is identically zero.

I found later that Eugene Calabi proposed a way to construct such a metric many years ago using the framework of Kähler geometry. It depends on an ansatz which requires to solve some highly nonlinear equation. It was in 1976 that I managed to solve the equation with a great deal of hard works.

The very important character of this Ricci flat metric has extra quality that it enjoys a symmetry called supersymmetry and that was why the above four authors were so happy with them when they knew such spacetime exists.

Recently I found out that Kähler who was the founder of Kähler geometry wrote about the ansatz twenty years before Calabi. He called them Kähler-Einstein metric and I believe Kähler was influenced by the strong desire to understand Einstein equation right after general relativity was created.

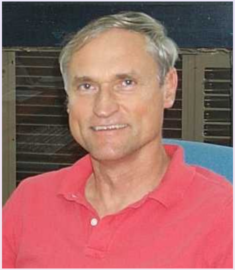
The constructions of those Kähler-Einstein metrics were used by me to solve some old problems in algebraic geometry eight years before they were recognized by my friends in physics community. But those metrics are so beautiful that I have never doubt that should appear in nature.

Extra dimension theory may still be weird for many classical physicists. But following Kalusa-Klein theory, the topology of Calabi-Yau manifold can be used to compute particle contents of the universe if those Calabi-Yau spaces are actually the right models of the physical universe. Nobody has really said that the universe cannot be modeled by the Calabi-Yau spaces. The only problem is that we do not have a good selection principle to choose the right model among billions of such models.

In any case ,while I was spending a lot of time on this very rich subject of Calabi-Yau spaces, I was also very interested in the classical four-dimensional spacetime where galaxies and black holes were modeled. In order to model such isolated physical system , we assume the spacetime metric is asymptotically trivial and in mathematically terms, the four-spacetime looks like Minkowski spacetime near infinity.

For such spacetime, there is asymptotic Poincaré group acting at infinity of the spacetime. With such a group, the concept of total mass and total linear momentum can be defined based on the theory of Noether. This concept was in fact noticed by Einstein himself. And was formulated more precisely by Arnowitt-Deser-Misner many years later.

But it left a very important question that they all wanted to solve: namely, in order for an isolated physical system to be stable, the energy defined in such a way better be positive. This remained to be unsolved for a long time until Richard Schoen and myself solved it in 1978-1979.



Schoen

Our method has initiated a new direction in the interaction between geometry and general relativity. The subsequent argument of Witten, using Dirac equation to reprove the positive mass theorem, provided another important tool for the subject that flourished in the last forty years .

The development allows us to give rigorous treatment for black hole theory. For example, in 1983, Schoen and I were able to demonstrate a folklore statement that when matter density is large in a fixed region of the space, black hole will form. Our argument in fact gives an effective estimate of the density.

In fact, some very important achievements were made way before of this. Schwarzchild and Kerr found the exact model of stationary black hole when there is circle symmetry and there is no matter. The work of Kerr is probably one of the most remarkable achievement in the theory of nonlinear evolution equation. Practically every scholar used this model to build their theory of black holes.

It was also remarkable that in the early 1970s, Israel, Carter and Hawking were able to demonstrate that under some smoothness assumptions on the event horizon of the black hole, stationary black hole with no other matter must be the one constructed by Kerr. This was coined to be no hair theorem by John Wheeler. This theorem has been the foundation for most discussion of black hole theory. On the other hand, whether the smooth assumption is valid should be examined.

Many more important achievements were made in the last forty years . To mention a few, my former student Robert Bartnik was able to construct a nontrivial coupled static nonsingular solution of Einstein-Yang-Mills equation.

Felix Finster, Joel Smoller, and I demonstrated there are infinite (discrete) number of static black holes based on Einstein-Yang-Mills equations. For this natural classical system to exhibit quantum phenomenon is interesting. Their physical significance has not been understood and need to be explored.

Then there was the important work of Christodoulou-Kleinemann on the dynamical stability of Minkowski spacetime, which gave a much more precise understanding of the so called Penrose compactification of spacetime. It also gave rise to the theory of Christodoulou on nonlinear memory effect of gravitational waves.

The theory of nonlinear memory effect was generalized by Lydia Bieri, Po-Ning Chen, David Garfinkle and myself to include many other fields. Hopefully, such memory effect can be tested in the recent LIGO observations.

Many basic nonlinear effects on general relativity are being established using rather deep works developed in geometric analysis. A very important development is the establishment of the important concept of quasilocal physical quantities such as quasilocal mass in spacetime. Many authors worked on this subject. Most notable ones were due to Penrose, Hawking, Robert Bartnik, Brown-York and others.

Personally I am more satisfied with the definition due to Mu-Tao Wang and myself with recent cooperation with Po-Ning Chen. These are important concepts that enable us to discuss Newtonian concepts even when the gravity field is very strong.

The necessity of the concept of quasilocal mass can be seen from the following question: When two black holes are interacting, we know the concept of their total energy, according to Einstein. But what about their binding energy? We need to know the energy of individual black hole before we can talk about its binding energy.

With the quasilocal energy defined, it would be interested to see how binding energy of two interacting black holes behave during collision. It should be related to the gravitational radiation which I believe is still a mysterious quantity, despite the most recent discovery of it by LIGO.

The concept of quasilocal energy can be used to understand the energy radiated by an isolated physical system in the following manner: If we use the Kerr black hole as a model, the formation of the black hole will release gravitational waves. The possible deformations were studied by Chandrasekar using Regge-Wheeler equations.

Po-Ning Chen, Mu-Tao Wang and myself proposed to use quasilocal mass to study gravitational radiation by measuring quasilocal mass enclosed by a sphere of radius one far away from the source. The order of the quasilocal mass turns out to be $(1/d)^2$, where d is the distance to the source.

As a result, take a closed loop and vector field along the loop, we can define the flux of gravitational energy passing within the loop along that vector field by simply computing the energy carried by the hemispherical cell capped by this loop along this vector field.

The flux defined in this way is well defined and it has order $1/d$. In this way, we can measure the gravitational flux in any closed loop along a vector field. By changing the geometry of the loop, we have a good way to confirm the measurement of the gravitational radiation.

Thank you!