

ANALYTIC TORSION FOR TWISTED DE RHAM COMPLEXES

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Abstract

We define analytic torsion $\tau(X, \mathcal{E}, H) \in \det H^\bullet(X, \mathcal{E}, H)$ for the twisted de Rham complex, consisting of the spaces of differential forms on a compact oriented Riemannian manifold X valued in a flat vector bundle \mathcal{E} , with a differential given by $\nabla^\mathcal{E} + H \wedge \cdot$, where $\nabla^\mathcal{E}$ is a flat connection on \mathcal{E} , H is an odd-degree closed differential form on X , and $H^\bullet(X, \mathcal{E}, H)$ denotes the cohomology of this \mathbb{Z}_2 -graded complex. The definition uses pseudodifferential operators and residue traces. We show that when $\dim X$ is odd, $\tau(X, \mathcal{E}, H)$ is independent of the choice of metrics on X and \mathcal{E} and of the representative H in the cohomology class $[H]$. We define twisted analytic torsion in the context of generalized geometry and show that when H is a 3-form, the deformation $H \mapsto H - dB$, where B is a 2-form on X , is equivalent to deforming a usual metric g to a generalized metric (g, B) . We demonstrate some basic functorial properties. When H is a top-degree form, we compute the torsion, define its simplicial counterpart, and prove an analogue of the Cheeger-Müller Theorem. We also study the twisted analytic torsion for T -dual circle bundles with integral 3-form fluxes.

Introduction

Let X be a compact oriented smooth manifold (without boundary) and let $\rho: \pi_1(X) \rightarrow \mathrm{GL}(E)$ be an orthogonal or unitary representation of the fundamental group $\pi_1(X)$ on a vector space E . The Reidemeister-Franz torsion, or R -torsion, of ρ is defined in terms of a triangulation of X . In [46, 47], Ray and Singer introduced its analytic counterpart, an alternating product of the regularized determinants of Laplacians, and conjectured that the latter is equal to the R -torsion. (For lens spaces, the equality of the two torsions was established in [45].) The Ray-Singer conjecture was proved independently by Cheeger [19] and Müller [40] for orthogonal or unitary representations of the fundamental group and was extended to unimodular representations by Müller [41]. Another proof of the Cheeger-Müller theorem, as well as an extension of it to

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arbitrary flat bundles, is due to Bismut and Zhang [8], who used the Witten deformation technique.

In this paper we generalize the classical construction of the Ray-Singer torsion to the twisted de Rham complex with an odd-degree differential form as flux and with coefficients in a flat vector bundle. The twisted de Rham complex was first defined for 3-form fluxes by Rohm and Witten in the appendix of [49] and has played an important role in string theory [11, 1], for the Ramond-Ramond fields (and their charges) in type II string theories lie in the twisted cohomology of spacetime. T -duality of the type II string theories on compactified spacetime gives rise to a duality isomorphism of twisted cohomology groups [12].

Let H be a closed differential form of odd degree on X and let $\rho: \pi_1(X) \rightarrow \mathrm{GL}(E)$ be a representation of $\pi_1(X)$ on a finite dimensional vector space E . Denote by \mathcal{E} the corresponding flat bundle over X with the canonical flat connection $\nabla^{\mathcal{E}}$. The twisted de Rham complex is the \mathbb{Z}_2 -graded complex $(\Omega^\bullet(X, \mathcal{E}), \nabla^{\mathcal{E}} + H \wedge \cdot)$. Its cohomology, denoted by $H^\bullet(X, \mathcal{E}, H)$, is called the twisted de Rham cohomology. We show that the twisted cohomology groups are invariant under scalings of H provided its degree is at least 3 and under smooth homotopy equivalences that match the cohomology classes of the flux forms. We establish Poincaré duality and Künneth isomorphism for twisted cohomology groups. We define analytic torsion of the twisted de Rham complex $\tau(X, \mathcal{E}, H) \in \det H^\bullet(X, \mathcal{E}, H)$ as a ratio of zeta-function regularized determinants of partial Laplacians, multiplied by the ratio of volume elements of the cohomology groups. While the de Rham complex has a \mathbb{Z} -grading, the twisted de Rham complex is only \mathbb{Z}_2 -graded. As a result, analytic techniques used to establish the basic properties in the classical case have to be generalized accordingly. These regularized determinants turn out to be more complicated to define, as they require properties of pseudodifferential projections. The definition is based on the fact that the non-commutative residues or the Guillemin-Wodzicki residue traces [55, 30] of such projections vanish [55, 15, 25]. We show that when $\dim X$ is odd, $\tau(X, \mathcal{E}, H)$ is independent of the choice of the Riemannian metric on X and the Hermitian metric on \mathcal{E} . The torsion $\tau(X, \mathcal{E}, H)$ is also invariant (under a natural identification) if H is deformed within its cohomology class. The comparison of the deformations of the metrics and of the flux leads naturally to the concept of generalized metric [29]. We define twisted analytic torsion in the context of generalized geometry and show that when H is a 3-form, the deformation $H \mapsto H - dB$, where B is a 2-form on X , is equivalent to deforming a usual metric g to a generalized metric (g, B) . We establish some basic functorial properties of this torsion. We then compute the torsion for odd-dimensional manifolds with a top-degree flux form, which is especially useful for 3-manifolds. When the degree of H is sufficiently high, we introduce a combinatorial counterpart of $\tau(X, \mathcal{E}, H)$ and show that they are equal

when H is a top-degree form. Finally, if (X, H) and $(\widehat{X}, \widehat{H})$ are T -dual circle bundles with background fluxes, then the T -duality isomorphism identifies the determinant lines $\det H^\bullet(X, H) \cong (\det H^\bullet(\widehat{X}, \widehat{H}))^{-1}$. Under this identification, we relate the twisted torsions for 3-dimensional T -dual circle bundles with integral 3-form fluxes.

The outline of the paper is as follows. In §1, we set up the notation in the paper and review the twisted de Rham complex and its cohomology [49, 11] with an odd-degree closed differential form as flux and with coefficients in a flat vector bundle associated to a representation of the fundamental group. In §2, we introduce the key definition of the analytic torsion of the twisted de Rham complex using the vanishing of non-commutative residues of pseudodifferential projections. In §3, we show that the twisted analytic torsion is independent of the metrics on the manifold and on the flat bundle. We also show that it depends on the flux only through its cohomology class. The relation to generalized geometry is then explored. In §4, we establish the basic functorial properties of the analytic torsion for the twisted de Rham complex. §5 contains calculations of analytic torsion for the twisted de Rham complex and a simplicial version of it under certain restrictions. In this special case, the analogue of the Cheeger-Müller theorem is established. Finally, we study the behavior of the twisted analytic torsion under T -duality for circle bundles with a closed 3-form as flux.

There is an extensive literature on the torsion of \mathbb{Z} -graded complexes. Analytic torsion has been studied for manifolds with boundary [34, 35, 54, 20, 16], for the Dolbeault complex [48, 3, 6], in the equivariant setting [34, 35, 9, 17, 4], and for fibrations [21, 36, 38], where torsion forms [3, 5, 7, 37, 38] appear. The analytic torsion was also identified as the partition function of certain topological field theories and was studied for arbitrary (\mathbb{Z} -graded) elliptic complexes [50]. Recently, refined and complex-valued analytic torsions have been introduced and studied [13, 14, 18, 53]. It is tempting to extend these developments to \mathbb{Z}_2 -graded complexes. Until recently, there has appeared to be no simplicial analogue of the twisted de Rham complex, except in a special case in §5.2, since the cup product is in general not graded commutative at the level of the cochain complex.

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1. Twisted de Rham complexes

To set up the notation in the paper, we review the twisted de Rham cohomology [49, 11] with an odd-degree flux form and with coefficients in a flat vector bundle. We show that the twisted cohomology does not

change under the scalings of the flux form when its degree is at least 3. We also establish homotopy invariance, Poincaré duality, and Künneth isomorphism for these cohomology groups.

1.1. Flat vector bundles, representations, and Hermitian metrics.

Let X be a connected, compact, oriented smooth manifold. Let $\rho: \pi_1(X) \rightarrow \mathrm{GL}(E)$ be a representation of the fundamental group $\pi_1(X)$ on a vector space E . The associated vector bundle $p: \mathcal{E} \rightarrow X$ is given by $\mathcal{E} = (E \times \tilde{X}) / \sim$, where \tilde{X} denotes the universal covering of X and $(v, x\gamma) \sim (\rho(\gamma)v, x)$ for all $\gamma \in \pi_1(X)$, $x \in \tilde{X}$ and $v \in E$. If the representation ρ is real or complex, so is the bundle \mathcal{E} , respectively. A smooth section s of \mathcal{E} can be uniquely represented by a smooth equivariant map $\phi: \tilde{X} \rightarrow E$, satisfying $\phi(x\gamma) = \rho(\gamma)^{-1}\phi(x)$ for all $\gamma \in \pi_1(X)$ and $x \in \tilde{X}$.

Given any vector bundle $p: \mathcal{E} \rightarrow X$ over X , denote by $\Omega^i(X, \mathcal{E})$ the space of smooth differential i -forms on X with values in \mathcal{E} . A *flat connection* on \mathcal{E} is a linear map

$$\nabla^{\mathcal{E}}: \Omega^i(X, \mathcal{E}) \rightarrow \Omega^{i+1}(X, \mathcal{E})$$

such that

$$\nabla^{\mathcal{E}}(f\omega) = df \wedge \omega + f \nabla^{\mathcal{E}}\omega \quad \text{and} \quad (\nabla^{\mathcal{E}})^2 = 0$$

for any smooth function f on X and any $\omega \in \Omega^i(X, \mathcal{E})$. If the vector bundle \mathcal{E} is associated with a representation ρ as in the previous paragraph, an element of $\Omega^\bullet(X, \mathcal{E})$ can be uniquely represented as a $\pi_1(X)$ -invariant element in $E \otimes \Omega^\bullet(\tilde{X})$. If $\omega \in \Omega^\bullet(\tilde{X})$ and $v \in E$, then $v \otimes \omega$ is said to be $\pi_1(X)$ -invariant if $\rho(\gamma)v \otimes \gamma^*\omega = v \otimes \omega$ for all $\gamma \in \pi_1(X)$. On such a vector bundle, there is a *canonical flat connection* $\nabla^{\mathcal{E}}$ given by, under the above identification, $\nabla^{\mathcal{E}}(v \otimes \omega) = v \otimes d\omega$, where d is the exterior derivative on forms.

The usual wedge product on differential forms can be extended to

$$\wedge: \Omega^i(X) \otimes \Omega^j(X, \mathcal{E}) \rightarrow \Omega^{i+j}(X, \mathcal{E}).$$

Together with the evaluation map $\mathcal{E} \otimes \mathcal{E}^* \rightarrow \mathbb{C}$, we have another product

$$\wedge: \Omega^i(X, \mathcal{E}) \otimes \Omega^j(X, \mathcal{E}^*) \rightarrow \Omega^{i+j}(X).$$

A Riemannian metric g_X defines the Hodge star operator

$$*: \Omega^i(X, \mathcal{E}) \rightarrow \Omega^{n-i}(X, \mathcal{E}),$$

where $n = \dim X$. A Euclidean or Hermitian metric $g_{\mathcal{E}}$ on \mathcal{E} determines an \mathbb{R} -linear bundle isomorphism $\sharp: \mathcal{E} \rightarrow \mathcal{E}^*$, which extends to an \mathbb{R} -linear isomorphism

$$\sharp: \Omega^i(X, \mathcal{E}) \rightarrow \Omega^i(X, \mathcal{E}^*);$$

when \mathcal{E} is complex, the isomorphism is conjugate linear. One sets $\Gamma = * \sharp = \sharp *$ and for any $\omega, \omega' \in \Omega^i(X, \mathcal{E})$, let

$$(\omega, \omega') = \int_X \omega \wedge \Gamma \omega'.$$

This makes each $\Omega^i(X, \mathcal{E})$, $0 \leq i \leq n$, a pre-Hilbert space.

When \mathcal{E} is associated to an orthogonal or unitary representation ρ of $\pi_1(X)$, $g_{\mathcal{E}}$ can be chosen to be compatible with the canonical flat connection. This is not possible in general. We will not assume that ρ is unimodular except in §5, where we calculate the torsion and establish an simplicial analogue under special conditions.

1.2. Twisted de Rham cohomology. Given a flat vector bundle $p: \mathcal{E} \rightarrow X$ and an odd-degree closed differential form H on X , we set $\Omega^{\bar{0}}(X, \mathcal{E}) := \Omega^{\text{even}}(X, \mathcal{E})$, $\Omega^{\bar{1}}(X, \mathcal{E}) := \Omega^{\text{odd}}(X, \mathcal{E})$ and $\nabla^{\mathcal{E}, H} := \nabla^{\mathcal{E}} + H \wedge \cdot$. We are primarily interested in the case when H does not contain a 1-form component, which can be absorbed in the flat connection $\nabla^{\mathcal{E}}$. We define the *twisted de Rham cohomology groups of \mathcal{E}* as the quotients

$$H^{\bar{k}}(X, \mathcal{E}, H) = \frac{\ker(\nabla^{\mathcal{E}, H}: \Omega^{\bar{k}}(X, \mathcal{E}) \rightarrow \Omega^{\overline{k+1}}(X, \mathcal{E}))}{\text{im}(\nabla^{\mathcal{E}, H}: \Omega^{\overline{k+1}}(X, \mathcal{E}) \rightarrow \Omega^{\bar{k}}(X, \mathcal{E}))}, \quad k = 0, 1.$$

Here and below, the bar over an integer means taking the value modulo 2. The groups $H^{\bar{k}}(X, \mathcal{E}, H)$ ($k = 0, 1$) are manifestly independent of the choice of the Riemannian metric on X or the Hermitian metric on \mathcal{E} . The corresponding *twisted Betti numbers* are denoted by

$$b_{\bar{k}} = b_{\bar{k}}(X, \mathcal{E}, H) := \dim H^{\bar{k}}(X, \mathcal{E}, H), \quad k = 0, 1.$$

Suppose H is replaced by $H' = H - dB$ for some $B \in \Omega^{\bar{0}}(X)$; then there is an isomorphism $\varepsilon_B := e^B \wedge \cdot: \Omega^{\bullet}(X, \mathcal{E}) \rightarrow \Omega^{\bullet}(X, \mathcal{E})$ satisfying

$$\varepsilon_B \circ \nabla^{\mathcal{E}, H} = \nabla^{\mathcal{E}, H'} \circ \varepsilon_B.$$

Therefore the Poincaré lemma holds for the twisted differential when the space is contractible. In general, ε_B induces an isomorphism (denoted by the same)

$$(1) \quad \varepsilon_B: H^{\bullet}(X, \mathcal{E}, H) \rightarrow H^{\bullet}(X, \mathcal{E}, H')$$

on the twisted de Rham cohomology. So the twisted Betti numbers depend only on the de Rham cohomology class of H . If they are finite, the Euler characteristic

$$\chi(X, \mathcal{E}, H) := \sum_{k=0,1} (-1)^k b_{\bar{k}}(X, \mathcal{E}, H) = \chi(X, \mathcal{E}) = \chi(X) \text{rk } \mathcal{E}$$

is independent of H and depends on \mathcal{E} only through its rank. If X is odd-dimensional, then $\chi(X, \mathcal{E}, H) = \chi(X, \mathcal{E}) = \chi(X) = 0$.

When H is a 1-form, $H^{\bullet}(X, \mathcal{E}, H)$ has a \mathbb{Z} -grading but the dimension can jump when H is rescaled by a non-zero number [42, 43]. The behavior is qualitatively different when the degree of H is at least 3.

Proposition 1.1. *Let \mathcal{E} be a flat vector bundle over X , and let H be an odd-degree closed form on X . Suppose $H = \sum_{i \geq 1} H_{2i+1}$, where each H_{2i+1} is a $(2i+1)$ -form. For any $\lambda \in \mathbb{R}$, let $H^{(\lambda)} = \sum_{i \geq 1} \lambda^i H_{2i+1}$. Then $H^\bullet(X, \mathcal{E}, H) \cong H^\bullet(X, \mathcal{E}, H^{(\lambda)})$ if $\lambda \neq 0$.*

Proof. For any λ , let c_λ act on $\Omega^\bullet(X, \mathcal{E})$ by multiplying $\lambda^{\lfloor \frac{i}{2} \rfloor}$ on i -forms. Then $H^{(\lambda)} = c_\lambda(H)$ and $c_\lambda \circ \nabla^{\mathcal{E}, H} = \lambda^k \nabla^{\mathcal{E}, H^{(\lambda)}} \circ c_\lambda$ on $\Omega^k(X, \mathcal{E})$ for $k = 0, 1$. If $\lambda \neq 0$, then c_λ induces the desired isomorphism on twisted cohomology groups. q.e.d.

Although the twisted differential $\nabla^{\mathcal{E}, H}$ does not preserve the \mathbb{Z} -grading of the de Rham complex, it does respect a filtration F given by [49, 1]

$$F^p \Omega^k(X, \mathcal{E}) = \bigoplus_{i \geq p, i \equiv k \pmod{2}} \Omega^i(X, \mathcal{E}).$$

This filtration gives rise to a spectral sequence $\{E_r^{pq}, \delta_r\}$ converging to the twisted cohomology $H^\bullet(X, \mathcal{E}, H)$. Without loss of generality, we assume that H contains no component of 1-form, which can be absorbed in the flat connection. That is, $H = H_3 + H_5 + \dots$, where H_i is an i -form ($i = 3, 5, \dots$). Then

$$E_2^{pq} = \begin{cases} H^p(X, \mathcal{E}), & \text{if } q = 0, \\ 0, & \text{if } q = 1. \end{cases}$$

As usual, E_{r+1}^\bullet is computed from a complex (E_r^\bullet, δ_r) for $r \geq 2$. We have $\delta_2 = \delta_4 = \dots = 0$, while $\delta_3, \delta_5, \dots$ are given by the cup products with $[H_3], [H_5], \dots$ and by the higher Massey products with them [49, 1]. Proposition 1.1 can also be derived by using this spectral sequence.

1.3. Homotopy invariance of twisted de Rham cohomology.

Given X, \mathcal{E} , and H as above, any smooth map $f: Y \rightarrow X$ (where Y is another smooth manifold) induces a homomorphism

$$f^*: H^\bullet(X, \mathcal{E}, H) \rightarrow H^\bullet(Y, f^*\mathcal{E}, f^*H).$$

We will show that this map depends only on the homotopy class of f . For simplicity, we assume that \mathcal{E} is a trivial line bundle with the trivial connection and denote $\nabla^{\mathcal{E}, H}$ by d^H in this case. Let I be the unit interval.

Lemma 1.2. *Let $\pi: X \times I \rightarrow X$ denote the projection onto X , and let $s: X \rightarrow X \times I$ denote the map $x \mapsto (x, 0)$ for $x \in X$. Then the maps $\pi^*: H^\bullet(X, H) \rightarrow H^\bullet(X \times I, \pi^*H)$ and $s^*: H^\bullet(X \times I, \pi^*H) \rightarrow H^\bullet(X, H)$ are inverses to each other.*

Proof. Clearly, $s^* \circ \pi^*$ is the identity map on $H^\bullet(X, H)$. By the homotopy invariance of de Rham cohomology, there is a chain homotopy operator $K: \Omega^i(X \times I) \rightarrow \Omega^{i-1}(X \times I)$ defined by (cf. §I.4 of [10]), $K(\pi^*\alpha f(x, t)) = 0$ and $K(\pi^*\alpha \wedge f(x, t)dt) = (-1)^i \pi^*\alpha \int_0^t f(x, t')dt'$,

where $\alpha \in \Omega^i(X)$, $f \in C^\infty(X \times I)$ and $x \in X$, $t \in I$. For any $\omega \in \Omega^\bullet(X \times I)$, we have

$$\omega - \pi^* s^* \omega = dK\omega + Kd\omega.$$

Since $K(\pi^* H \wedge \omega) = -\pi^* H \wedge K(\omega)$, we have

$$\omega - \pi^* s^* \omega = d^{\pi^* H} K\omega + Kd^{\pi^* H} \omega.$$

Therefore, $\pi^* \circ s^*$ is the identity map on $H^\bullet(X \times I, \pi^* H)$. q.e.d.

Proposition 1.3. *Let $f_0, f_1: Y \rightarrow X$ be two smooth maps that are homotopic. Then there exists $B \in \Omega^{\bar{0}}(Y)$ such that $f_1^* H = f_0^* H - dB$ and the following diagram commutes*

$$\begin{array}{ccc} & H^\bullet(X, H) & \\ f_0^* \swarrow & & \searrow f_1^* \\ H^\bullet(Y, f_0^* H) & \xrightarrow{\varepsilon_B} & H^\bullet(Y, f_1^* H). \end{array}$$

Proof. Let $\pi: Y \times I \rightarrow Y$ be the projection onto Y . Define the smooth maps $s_j: Y \rightarrow Y \times I$ ($j = 0, 1$) by $s_j(y) = (y, j)$, where $y \in Y$. Then a homotopy between f_0 and f_1 is a smooth map $F: Y \times I \rightarrow X$ such that $f_j = F \circ s_j$ for $j = 0, 1$. There exists $\tilde{B} \in \Omega^{\bar{0}}(Y \times I)$ such that $F^* H = \pi^* f_0^* H - d\tilde{B}$ and $s_0^* \tilde{B} = 0$. Let $B = s_1^* \tilde{B} \in \Omega^{\bar{0}}(Y)$ and $\tilde{B}' = \tilde{B} - \pi^* B$. Then $f_1^* H - f_0^* H = -dB$, $F^* H = \pi^* f_1^* H - d\tilde{B}'$, and $s_1^* \tilde{B}' = 0$. There is a commutative diagram

$$\begin{array}{ccccc} & & H^\bullet(Y \times I, f_0^* H) & \xleftarrow{\pi^*} & H^\bullet(Y, f_0^* H) \\ & \varepsilon_{\tilde{B}} \swarrow & \downarrow \varepsilon_{\pi^* B} & & \downarrow \varepsilon_B \\ H^\bullet(X, H) & \xrightarrow{F^*} & H^\bullet(Y \times I, F^* H) & & \\ & \varepsilon_{\tilde{B}'} \swarrow & \downarrow & \xleftarrow{\pi^*} & H^\bullet(Y, f_1^* H) \\ & & H^\bullet(Y \times I, f_1^* H) & & \end{array}$$

By Lemma 1.2, $s_0 = (\pi^*)^{-1}: H^\bullet(Y \times I, \pi^* f_0^* H) \rightarrow H^\bullet(Y, f_0^* H)$. Since $s_0^* \tilde{B} = 0$, $(\pi^*)^{-1} \circ \varepsilon_{\tilde{B}}^{-1} = s_0^*: H^\bullet(Y \times I, F^* H) \rightarrow H^\bullet(Y, f_0^* H)$. Similarly, $(\pi^*)^{-1} \circ \varepsilon_{\tilde{B}'}^{-1} = s_1^*: H^\bullet(Y \times I, F^* H) \rightarrow H^\bullet(Y, f_1^* H)$. The result follows since $f_j^* = s_j^* \circ F^*$, $j = 0, 1$. q.e.d.

It is clear from the proof that in addition to $f_1^* H = f_0^* H - dB$, B has to come from the homotopy. In fact, $B = -K(F^* H)$, where K is the homotopy chain map in the proof of Lemma 1.2. If H is fixed in the homotopy process, then $f_0^* = f_1^*$.

Corollary 1.4. *Suppose X, X' are smooth manifolds and H, H' are closed odd-degree forms on X, X' , respectively. If there is a smooth homotopy equivalence $f: X \rightarrow X'$ such that $[f^* H'] = [H]$, then $H^\bullet(X, H) \cong H^\bullet(X', H')$.*

1.4. Products and Poincaré duality. Unlike the de Rham cohomology group $H^\bullet(X)$, the twisted de Rham cohomology group $H^\bullet(X, H)$ is not a ring for a fixed $H \neq 0$. Instead, the flux H adds in the cup product.

Lemma 1.5. *Let $\mathcal{E}, \mathcal{E}'$ be flat vector bundles and let H, H' be closed odd-degree differential forms on a smooth manifold X . Then the wedge product*

$$\Omega^{\bar{k}}(X, \mathcal{E}) \otimes \Omega^{\bar{l}}(X, \mathcal{E}') \xrightarrow{\wedge} \Omega^{\overline{k+l}}(X, \mathcal{E} \otimes \mathcal{E}')$$

induces a natural cup product

$$H^{\bar{k}}(X, \mathcal{E}, H) \otimes H^{\bar{l}}(X, \mathcal{E}', H') \xrightarrow{\cup} H^{\overline{k+l}}(X, \mathcal{E} \otimes \mathcal{E}', H + H'),$$

where $k, l = 0, 1$. In particular, there is a natural cup product

$$H^{\bar{k}}(X, \mathcal{E}, H) \otimes H^{\bar{l}}(X, \mathcal{E}^*, -H) \xrightarrow{\cup} H^{\overline{k+l}}(X).$$

Proof. The existence of the cup product follows from the formula

$$\nabla^{\mathcal{E} \otimes \mathcal{E}', H+H'}(\omega \wedge \omega') = \nabla^{\mathcal{E}, H} \omega \wedge \omega' + (-1)^{\bar{k}} \omega \wedge \nabla^{\mathcal{E}', H'} \omega',$$

where $\omega \in \Omega^{\bar{k}}(X, \mathcal{E})$, $\omega' \in \Omega^{\bar{l}}(X, \mathcal{E}')$. When $\mathcal{E}' = \mathcal{E}^*$, $H' = -H$, the cup product, composed with the pairing between \mathcal{E} and \mathcal{E}^* , takes values in $H^{\overline{k+l}}(X)$. q.e.d.

It is possible to define, following §1.2, the twisted de Rham cohomology groups of compact support, denoted by $H_c^{\bar{k}}(X, \mathcal{E}, H)$, by using the space of differential forms $\Omega_c^{\bar{k}}(X, \mathcal{E})$ of compact support. (There is no restriction on the support of H .) As in Lemma 1.5, the wedge product

$$\Omega_c^{\bar{k}}(X, \mathcal{E}) \otimes \Omega_c^{\bar{l}}(X, \mathcal{E}') \xrightarrow{\wedge} \Omega_c^{\overline{k+l}}(X, \mathcal{E} \otimes \mathcal{E}')$$

induces cup products

$$H_c^{\bar{k}}(X, \mathcal{E}, H) \otimes H_c^{\bar{l}}(X, \mathcal{E}', H') \xrightarrow{\cup} H_c^{\overline{k+l}}(X, \mathcal{E} \otimes \mathcal{E}', H + H')$$

and

$$H_c^{\bar{k}}(X, \mathcal{E}, H) \otimes H_c^{\bar{l}}(X, \mathcal{E}^*, -H) \xrightarrow{\cup} H_c^{\overline{k+l}}(X).$$

Proposition 1.6 (Poincaré duality). *Let X be an oriented manifold of dimension n with a finite good cover. Let \mathcal{E} be a flat vector bundle on X . Suppose H is a closed odd-degree differential form on X . Then, for $k = 0, 1$, there is a natural isomorphism*

$$H_c^{\bar{k}}(X, \mathcal{E}, H) \cong (H_c^{\overline{n-k}}(X, \mathcal{E}^*, -H))^*.$$

Proof. Since X is orientable, we have $H_c^n(X) \cong \mathbb{R}$. There are pairings between $\Omega_c^{\bar{k}}(X, \mathcal{E})$ and $\Omega_c^{\overline{n-k}}(X, \mathcal{E}^*)$ and between $H_c^{\bar{k}}(X, \mathcal{E}, H)$ and

$H_c^{\overline{n-k}}(X, \mathcal{E}^*, -H)$; the former is given by integration on X and the latter is given by the cup product to $H_c^{\overline{n}}(X)$ followed by a projection to $H_c^{\overline{n}}(X)$. Thus we have linear maps

$$\Omega^{\overline{k}}(X, \mathcal{E}) \rightarrow (\Omega_c^{\overline{n-k}}(X, \mathcal{E}^*))^* \quad \text{and} \quad H^{\overline{k}}(X, \mathcal{E}, H) \rightarrow (H_c^{\overline{n-k}}(X, \mathcal{E}^*, -H))^*.$$

Here $(\Omega_c^\bullet(X, \mathcal{E}^*))^*$ is the linear dual of $\Omega_c^\bullet(X, \mathcal{E}^*)$. To show that the latter is an isomorphism, we observe that it is so if X is contractible, as the two cohomology groups can be calculated explicitly. For any two open subsets U, V in X , we have a morphism of short exact sequences of \mathbb{Z}_2 -graded cochain complexes

$$\begin{array}{ccccccc} 0 & \rightarrow & \Omega^\bullet(U \cup V, \mathcal{E}) & \rightarrow & \Omega^\bullet(U, \mathcal{E}) \oplus \Omega^\bullet(V, \mathcal{E}) & \rightarrow & \Omega^\bullet(U \cap V, \mathcal{E}) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \Omega_c^{\overline{n-\bullet}}(U \cup V, \mathcal{E}^*)^* & \rightarrow & \Omega_c^{\overline{n-\bullet}}(U, \mathcal{E}^*)^* \oplus \Omega_c^{\overline{n-\bullet}}(V, \mathcal{E}^*)^* & \rightarrow & \Omega_c^{\overline{n-\bullet}}(U \cap V, \mathcal{E}^*)^* \rightarrow 0. \end{array}$$

This induces a morphism of the Mayer-Vietoris sequences (which are six-term long exact sequences). Following §I.5 of [10], the result can be proved by an induction on the number of open sets in the finite good cover and by using the five lemma. q.e.d.

Proposition 1.7 (Künneth isomorphism). *For $i = 1, 2$, let X_i be smooth manifolds. Suppose \mathcal{E}_i are flat vector bundles over X_i , and H_i are closed odd-degree forms on X_i , respectively. Let $\pi_i: X_1 \times X_2 \rightarrow X_i$ be the projections. Set $\mathcal{E}_1 \boxtimes \mathcal{E}_2 = \pi_1^* \mathcal{E}_1 \otimes \pi_2^* \mathcal{E}_2$ and $H_1 \boxplus H_2 = \pi_1^* H_1 + \pi_2^* H_2$. If either X_1 or X_2 has a finite good cover, then, for each $k = 0, 1$, there is a natural isomorphism*

$$H^{\overline{k}}(X_1 \times X_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2, H_1 \boxplus H_2) \cong \bigoplus_{l=0,1} H^{\overline{l}}(X_1, \mathcal{E}_1, H_1) \otimes H^{\overline{k-l}}(X_2, \mathcal{E}_2, H_2).$$

Proof. Suppose X_1 has a finite good cover. If X_1 is contractible, then \mathcal{E}_1 is trivial and H_1 is exact on X_1 . So are $\pi^* \mathcal{E}_1$ and $\pi_1^* H_1$ on $X_1 \times X_2$. By the isomorphism (1) and by homotopy invariance (Corollary 1.4), we get a natural isomorphism

$$\bigoplus_{l=0,1} H^{\overline{k}}(X_1, \mathcal{E}_1, H_1) \otimes H^{\overline{k-l}}(X_2, \mathcal{E}_2, H_2) \xrightarrow{\cong} H^{\overline{k}}(X_1 \times X_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2, H_1 \boxplus H_2)$$

induced by the map $\omega_1 \otimes \omega_2 \in \Omega^\bullet(X_1, \mathcal{E}_1) \otimes \Omega^\bullet(X_2, \mathcal{E}_2) \mapsto \pi_1^* \omega_1 \wedge \pi_2^* \omega_2 \in \Omega^\bullet(X_1 \times X_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2)$. For any two open subsets U, V of X_1 , there is a morphism of short exact sequences of \mathbb{Z}_2 -graded cochain complexes

$$\begin{array}{ccccccc} 0 & \rightarrow & \bigoplus_{l=0,1} \Omega^{\overline{l}}(U \cup V) \otimes \Omega^{\bullet-l}(X_2) & \rightarrow & \bigoplus_{l=0,1} (\Omega^{\overline{l}}(U) \oplus \Omega^{\overline{l}}(V)) \otimes \Omega^{\bullet-l}(X_2) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \Omega^\bullet((U \cup V) \times X_2) & \rightarrow & \Omega^\bullet(U \times X_2) \oplus \Omega^\bullet(V \times X_2) & & \\ & & & & \rightarrow \bigoplus_{l=0,1} \Omega^{\overline{l}}(U \cap V) \otimes \Omega^{\bullet-l}(X_2) & \rightarrow & 0 \\ & & & & \downarrow & & \\ & & & & \Omega^\bullet((U \cap V) \times X_2) & \rightarrow & 0. \end{array}$$

Here the obvious dependence on the bundles is suppressed for brevity. This induces a morphism of the Mayer-Vietoris sequences (which are six-term long exact sequences). Following §I.5 of [10] again, the result can be proved by an induction on the number of open sets in the finite good cover and by using the five lemma. q.e.d.

2. Analytic torsion of twisted de Rham complexes

In this section, we define analytic torsion $\tau(X, \mathcal{E}, H) \in \det H^\bullet(X, \mathcal{E}, H)$ of the twisted de Rham complexes introduced in §1.2. Since these complexes are only \mathbb{Z}_2 -graded, the twisted analytic torsion is more complicated to define and to study than its classical counterpart. For simplicity, assume that X is orientable.

2.1. The construction of analytic torsion. To simplify notation, let $C^{\bar{k}} := \Omega^{\bar{k}}(X, \mathcal{E})$ and let $d_{\bar{k}} = d_{\bar{k}}^{\mathcal{E}, H}$ be the operator $\nabla^{\mathcal{E}, H}$ acting on $C^{\bar{k}}$ ($k = 0, 1$). Then $d_{\bar{1}}d_{\bar{0}} = d_{\bar{0}}d_{\bar{1}} = 0$ and we have a complex

$$(2) \quad \dots \xrightarrow{d_{\bar{1}}} C^{\bar{0}} \xrightarrow{d_{\bar{0}}} C^{\bar{1}} \xrightarrow{d_{\bar{1}}} C^{\bar{0}} \xrightarrow{d_{\bar{0}}} \dots$$

Denote by $d_{\bar{k}}^\dagger$ the adjoint of $d_{\bar{k}}$ with respect to the scalar product of §1.1. Then the twisted Laplacians

$$\Delta_{\bar{k}} = \Delta_{\bar{k}}^{\mathcal{E}, H} := d_{\bar{k}}^\dagger d_{\bar{k}} + d_{\bar{k}+1} d_{\bar{k}+1}^\dagger, \quad k = 0, 1$$

are elliptic operators and therefore the complex (2) is elliptic. By Hodge theory, the natural map $\ker(\Delta_{\bar{k}}) \rightarrow H^{\bar{k}}(X, \mathcal{E}, H)$ taking each twisted harmonic form to its cohomology class is an isomorphism. Ellipticity of $\Delta_{\bar{k}}$ ensures that the twisted Betti numbers $b_{\bar{k}}(X, \mathcal{E}, H)$ ($k = 0, 1$) are finite.

We establish the relation of Hodge star with adjoint and harmonic form in the twisted case.

Lemma 2.1. *In the above notations, we have*

$$(d_{\bar{k}}^{\mathcal{E}, H})^\dagger = (-1)^{k+1} \Gamma^{-1} \circ d_{\bar{n}+1-\bar{k}}^{\mathcal{E}^*, -H} \circ \Gamma, \quad \Gamma \circ \Delta_{\bar{k}}^{\mathcal{E}, H} = \Delta_{\bar{n}-\bar{k}}^{\mathcal{E}^*, -H} \circ \Gamma.$$

Proof. For $\omega \in C^{\bar{k}}, \omega' \in C^{\overline{k+1}}$, we have

$$\begin{aligned} (d_{\bar{k}}^{\mathcal{E}, H} \omega, \omega') &= \int_X (\nabla^{\mathcal{E}} \omega + H \wedge \omega) \wedge \Gamma \omega' = (-1)^k \int_X \omega \wedge (-\nabla^{\mathcal{E}^*} \Gamma \omega' + H \wedge \Gamma \omega') \\ &= (-1)^{k+1} (\omega, \Gamma^{-1} d_{\bar{n}+1-\bar{k}}^{\mathcal{E}^*, -H} \Gamma \omega'). \end{aligned}$$

Therefore, $(d_{\bar{k}}^{\mathcal{E}, H})^\dagger = (-1)^{k+1} \Gamma^{-1} d_{\bar{n}+1-\bar{k}}^{\mathcal{E}^*, -H} \Gamma$. Replacing \mathcal{E} by \mathcal{E}^* , H by $-H$, and using the identity $\Gamma^2 = (-1)^{k(n-k)}$ on $C^{\bar{k}}$, we get $d_{\bar{k}}^{\mathcal{E}, H} = (-1)^{k+1} \Gamma^{-1} (d_{\bar{n}+1-\bar{k}}^{\mathcal{E}^*, -H})^\dagger \Gamma$. Therefore, $\Delta_{\bar{k}}^{\mathcal{E}, H} = \Gamma^{-1} \Delta_{\bar{n}-\bar{k}}^{\mathcal{E}^*, -H} \Gamma$. q.e.d.

Consequently, the two twisted Laplacians $\Delta_k^{\mathcal{E},H}$ and $\Delta_{n-k}^{\mathcal{E}^*,-H}$ have the same spectrum. In particular, if ω is harmonic with respect to $\Delta_k^{\mathcal{E},H}$, then $\Gamma\omega$ is harmonic with respect to $\Delta_k^{\mathcal{E}^*,-H}$. This provides another proof of Poincaré duality (Proposition 1.6) when X is compact.

The scalar product on $C^{\bar{k}}$ restricts to one on the space of twisted harmonic forms $\ker(\Delta_{\bar{k}}) \cong H^{\bar{k}}(X, \mathcal{E}, H)$. Let $\{\nu_{\bar{k},i}\}_{i=1}^{b_{\bar{k}}}$ be an oriented orthonormal basis of $H^{\bar{k}}(X, \mathcal{E}, H)$ and let $\eta_{\bar{k}} = \eta_{\bar{k}}^{\mathcal{E},H} := \nu_{\bar{k},1} \wedge \cdots \wedge \nu_{\bar{k},b_{\bar{k}}}$ be the unit volume element. Then $\eta_{\bar{0}} \otimes \eta_{\bar{1}}^{-1} \in \det H^\bullet(X, \mathcal{E}, H)$. The *analytic torsion of the twisted de Rham complex* is defined to be

$$(3) \quad \tau(X, \mathcal{E}, H) := (\text{Det}' d_0^\dagger d_0)^{1/2} (\text{Det}' d_1^\dagger d_1)^{-1/2} \eta_{\bar{0}} \otimes \eta_{\bar{1}}^{-1} \in \det H^\bullet(X, \mathcal{E}, H),$$

where $\text{Det}' d_k^\dagger d_k$ denotes the zeta-function regularized determinant of $d_k^\dagger d_k$ on the orthogonal complement of its kernel. The next subsection is devoted to showing that these determinants make sense. When \mathcal{E} is the trivial line bundle over X with the trivial connection, we set $\tau(X, H) = \tau(X, \mathcal{E}, H)$.

We explain the motivation for definition (3) by considering the case $H = 0$. Then $C^{\bar{k}} = \bigoplus_{i=k \bmod 2} C^i$ and $d_{\bar{k}} = \sum_{i=k \bmod 2} d_i$ ($k = 0, 1$), where $C^i = \Omega^i(X, \mathcal{E})$ ($0 \leq i \leq n$) and the differentials $d_i = d_i^{\mathcal{E}}$ ($0 \leq i \leq n - 1$) form the \mathbb{Z} -graded de Rham complex

$$0 \rightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^n \rightarrow 0$$

with $d_i d_{i-1} = 0$ ($1 \leq i \leq n$). By spectral theory, $\text{Det}' d_i^\dagger d_i$ ($0 \leq i \leq n - 1$) can be defined and is equal to $\prod_{j=0}^{n-i-1} (\text{Det}' \Delta_{i-j})^{(-1)^j}$, where $\Delta_i = d_i^\dagger d_i + d_{i-1} d_{i-1}^\dagger$ (with $d_{-1} = d_n = 0$) is the Laplacian on C^i . Thus the determinant factor in (3) is

$$\begin{aligned} (\text{Det}' d_0^\dagger d_0)^{1/2} (\text{Det}' d_1^\dagger d_1)^{-1/2} &= \prod_{i=0}^{n-1} (\text{Det}' d_i^\dagger d_i)^{(-1)^i/2} \\ &= \prod_{i=0}^n (\text{Det}' \Delta_i)^{(-1)^{i+1}i/2}, \end{aligned}$$

which yields the Ray-Singer torsion $\tau(X, \mathcal{E})$ [46]. When \mathcal{E} is the trivial line bundle over X , we set $\tau(X) = \tau(X, \mathcal{E})$.

We wish to point out that the classical signature or Dirac complex, despite being a 2-term complex, is not of the form (2) because it does not satisfy $d_1 d_0 = d_0 d_1 = 0$. Therefore, no torsion is defined in these cases.

If \mathcal{E} is a complex vector bundle, the torsion is only defined up to a phase due to the ambiguity in the choice of the unit volume elements $\eta_{\bar{k}}$. Therefore, an equality of torsions means that they are equal up to a

phase or that the volume elements can be chosen so that they are equal. More intrinsic is the norm on the determinant line (cf. [44, 3, 8] for the \mathbb{Z} -graded case) given by

$$\|\cdot\| = (\text{Det}' d_0^\dagger d_0)^{1/2} (\text{Det}' d_1^\dagger d_1)^{-1/2} |\cdot|,$$

where $|\cdot|$ is the norm induced by the scalar products on $\ker(\Delta_{\bar{k}}) \cong H^{\bar{k}}(X, \mathcal{E}, H)$, $k = 0, 1$. However, to facilitate the presentation, we will still regard torsions as (equivalent classes of) elements in the determinant lines. Recently, refined and complex-valued analytic torsions were introduced as well-defined elements of the determinant line [13, 14, 18].

2.2. The zeta-function regularized determinants. Given a semi-positive definite self-adjoint operator A , the *zeta-function* of A (whenever it is defined) is

$$\zeta(s, A) := \text{Tr}' A^{-s},$$

where Tr' stands for the trace restricted to the subspace orthogonal to $\ker(A)$. If $\zeta(s, A)$ can be extended meromorphically in s so that it is holomorphic at $s = 0$, then the *zeta-function regularized determinant* of A is defined as

$$\text{Det}' A = e^{-\zeta'(0, A)}.$$

If A is an elliptic differential operator of order m on a compact manifold of dimension n , then $\zeta(s, A)$ is holomorphic when $\Re(s) > n/m$ and can be extended meromorphically to the entire complex plane with possible simple poles at $\{\frac{n-l}{m}, l = 0, 1, 2, \dots\}$ only [51] (cf. [52]). Moreover, the extended function is holomorphic at $s = 0$ and therefore the determinant $\text{Det}' A$ is defined for such an operator. Examples are the Laplacians $\Delta_i^\mathcal{E}$ acting on i -forms on a compact Riemannian manifold X with values in a vector bundle \mathcal{E} with a Hermitian structure; their determinants $\text{Det}' \Delta_i^\mathcal{E}$ enter the Ray-Singer analytic torsion for the de Rham complex [46, 47]. For the twisted de Rham complex (2), the Laplacians $\Delta_{\bar{k}} = \Delta_{\bar{k}}^{\mathcal{E}, H}$ ($k = 0, 1$) acting on even/odd-degree forms are also elliptic, and therefore the determinants $\text{Det}' \Delta_{\bar{k}}$ ($k = 0, 1$) still make sense (and are in fact equal). However, what appear in the twisted analytic torsion (3) are not these determinants, but $\text{Det}' d_{\bar{k}}^\dagger d_{\bar{k}}$, which are much harder to define.

Let $\text{spec}(A)$ ($\text{spec}'(A)$, respectively) be the set of eigenvalues (positive eigenvalues, respectively) of A . For any $\lambda \in \text{spec}(A)$, let $m(\lambda, A)$ be its multiplicity. Then

$$\zeta(s, A) = \sum_{\lambda \in \text{spec}'(A)} \frac{m(\lambda, A)}{\lambda^s}.$$

Given a flat vector bundle \mathcal{E} over a manifold X and a closed odd-degree form H on X , set $\text{spec}_I(\Delta_{\bar{0}}) := \text{spec}(\Delta_{\bar{0}}|_{\text{im}(d_0^\dagger)}) = \text{spec}'(d_0^\dagger d_0)$, $m_I(\lambda, \Delta_{\bar{0}}) := m(\lambda, \Delta_{\bar{0}}|_{\text{im}(d_0^\dagger)})$ and $\text{spec}_{II}(\Delta_{\bar{0}}) := \text{spec}(\Delta_{\bar{0}}|_{\text{im}(d_1)}) =$

$\text{spec}'(d_1^\dagger d_1)$, $m_{\text{II}}(\lambda, \Delta_0) := m(\lambda, \Delta_0|_{\text{im}(d_1)})$. Since Δ_0 is diagonal with respect to the decomposition $C^0 = \text{im}(d_0^\dagger) \oplus \text{im}(d_1) \oplus \ker(\Delta_0)$, we have $\text{spec}'(\Delta_0) = \text{spec}_\text{I}(\Delta_0) \cup \text{spec}_\text{II}(\Delta_0)$ and $m(\lambda, \Delta_0) = m_\text{I}(\lambda, \Delta_0) + m_{\text{II}}(\lambda, \Delta_0)$ if $\lambda > 0$. Therefore,

$$(4) \quad \zeta(s, d_0^\dagger d_0) = \sum_{\lambda \in \text{spec}_\text{I}(\Delta_0)} \frac{m_\text{I}(\lambda, \Delta_0)}{\lambda^s}, \quad \zeta(s, d_1^\dagger d_1) = \sum_{\lambda \in \text{spec}_\text{II}(\Delta_0)} \frac{m_{\text{II}}(\lambda, \Delta_0)}{\lambda^s}.$$

The sum of the two zeta-functions is

$$(5) \quad \sum_{k=0,1} \zeta(s, d_k^\dagger d_k) = \zeta(s, \Delta_0) = \zeta(s, \Delta_1).$$

However, what we need for (3) is their difference.

Theorem 2.2. *For $k = 0, 1$, $\zeta(s, d_k^\dagger d_k)$ is holomorphic in the half plane $\Re(s) > n/2$ and extends meromorphically to \mathbb{C} with possible simple poles at $\{\frac{n-l}{2}, l = 0, 1, 2, \dots\}$ and possible double poles at negative integers only, and is holomorphic at $s = 0$.*

Proof. Let P_k ($k = 0, 1$) be the orthogonal projection onto the closure of the subspace $\text{im}(d_k^\dagger)$. As $d_k^\dagger d_k$ and Δ_{k+1} are equal and invertible on (the closure of) $\text{im}(d_k^\dagger)$, we have

$$P_k = d_k^\dagger (d_k^\dagger d_k)^{-1} d_k = d_k^\dagger (\Delta_{k+1})^{-1} d_k,$$

which implies that P_k is a pseudodifferential operator of order 0. Moreover,

$$\zeta(s, d_k^\dagger d_k) = \text{Tr}(P_k \Delta_k^{-s}).$$

By general theory [27, 26], $\zeta(s, d_k^\dagger d_k)$ is holomorphic in the half plane $\Re(s) > n/2$ and extends meromorphically to \mathbb{C} with possible simple poles at $\{\frac{n-l}{2}, l = 0, 1, 2, \dots\}$ and possible double poles at negative integers only. The Laurent series of $\zeta(s, d_k^\dagger d_k)$ at $s = 0$ is

$$\text{Tr}(P_k \Delta_k^{-s}) = \frac{c_{-1}(P_k, \Delta_k)}{s} + c_0(P_k, \Delta_k) + \sum_{l=1}^\infty c_l(P_k, \Delta_k) s^l.$$

Here $c_{-1}(P_k, \Delta_k) = \frac{1}{2} \text{res}(P_k)$, where $\text{res}(P_k)$ is known as the non-commutative residue or the Guillemin-Wodzicki residue trace of P_k [55, 30]. Since P_k is a projection, $\text{res}(P_k) = 0$ [55, 15, 25]. Therefore, $\zeta(s, d_k^\dagger d_k)$ is regular at $s = 0$. q.e.d.

Theorem 2.2 justifies the definition of the twisted analytic torsion in (3). The constant term $\zeta(0, d_k^\dagger d_k) = c_0(P_k, \Delta_k)$ of the above Laurent series is related to the Kontsevich-Vishik trace [33, 26]. It can nevertheless be studied by standard heat kernel techniques (Lemma 2.3,

Corollaries 3.2 and 3.6 below). Recall the notion of twisted Betti numbers $b_{\bar{k}}(X, \mathcal{E}, H)$ ($k = 0, 1$) from §1.2 and the fact that the zeta-function is related to the heat kernel by a Mellin transform

$$\zeta(s, A) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}' e^{-tA} dt.$$

Lemma 2.3. *If $\dim X$ is odd, then*

$$\sum_{k=0,1} \zeta(0, d_{\bar{k}}^\dagger d_{\bar{k}}) = -b_{\bar{0}}(X, \mathcal{E}, H) = -b_{\bar{1}}(X, \mathcal{E}, H).$$

Proof. When $n = \dim X$ is odd, $b_{\bar{0}}(X, \mathcal{E}, H) = b_{\bar{1}}(X, \mathcal{E}, H)$ as $\chi(X, \mathcal{E}, H) = 0$. By (5), it suffices to show that $\zeta(0, \Delta_{\bar{k}}) = -b_{\bar{k}}(X, \mathcal{E}, H)$, $k = 0, 1$. By the asymptotic expansion of the heat kernel (cf. [24, 2]),

$$\text{Tr} e^{-t\Delta_{\bar{k}}} \sim \sum_{l=0}^{\infty} c_{\bar{k},l} t^{-n/2+l}$$

as $t \downarrow 0$, where $c_{\bar{k},l} \in \mathbb{R}$. We have, for $\Re(s) > n/2$,

$$\begin{aligned} \zeta(s, \Delta_{\bar{k}}) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (\text{Tr} e^{-t\Delta_{\bar{k}}} - b_{\bar{k}}) dt \\ &= \frac{1}{\Gamma(s)} \left(-\frac{b_{\bar{k}}}{s} + \sum_{l=0}^N \frac{c_{\bar{k},l}}{s - n/2 + l} + R_N(s) \right), \end{aligned}$$

where N is a sufficiently large integer. Here $R_N(s)$ is holomorphic when $\Re(s) > n/2 - N$. Since n is odd and since the Gamma function $\Gamma(s)$ has a simple pole at $s = 0$, the result follows. q.e.d.

3. Twisted analytic torsion under metric and flux deformations

3.1. Variation of analytic torsion with respect to the metrics.

We assume that X is a compact oriented manifold of odd dimension. Let g_X be a Riemannian metric on X and let $g_{\mathcal{E}}$ be a Hermitian metric on \mathcal{E} . Let $Q_{\bar{k}}$ ($k = 0, 1$) be the orthogonal projection from (the completion of) $C^{\bar{k}}$ to $\ker(\Delta_{\bar{k}})$. Suppose that the pair $(g_X, g_{\mathcal{E}})$ is deformed smoothly along a one-parameter family with parameter $u \in \mathbb{R}$; then the operators $*$, \sharp and $\Gamma = *\sharp = \sharp*$ (see §1.1) all depend smoothly on u . Let

$$\alpha = \Gamma^{-1} \frac{\partial \Gamma}{\partial u}.$$

We show the invariance of the analytic torsion (3) by showing in the next two lemmas that the variation of the regularized determinants cancels that of the volume elements.

Lemma 3.1. *Under the above assumptions,*

$$\frac{\partial}{\partial u} \log[\text{Det}' d_0^\dagger d_0 (\text{Det}' d_1^\dagger d_1)^{-1}] = \sum_{k=0,1} (-1)^k \text{Tr}(\alpha Q_{\bar{k}}).$$

Proof. While $d_{\bar{k}}$ is independent of u , we have

$$\frac{\partial d_{\bar{k}}^\dagger}{\partial u} = -[\alpha, d_{\bar{k}}^\dagger],$$

which follows easily from Lemma 2.1. Using $P_{\bar{k}} d_{\bar{k}}^\dagger = d_{\bar{k}}^\dagger$, $d_{\bar{k}} P_{\bar{k}} = d_{\bar{k}}$, and $P_{\bar{k}}^2 = P_{\bar{k}}$, we get $d_{\bar{k}}^\dagger d_{\bar{k}} P_{\bar{k}} = P_{\bar{k}} d_{\bar{k}}^\dagger d_{\bar{k}} = d_{\bar{k}}^\dagger d_{\bar{k}}$ and

$$\frac{\partial P_{\bar{k}}}{\partial u} = \frac{\partial P_{\bar{k}}}{\partial u} P_{\bar{k}}, \quad P_{\bar{k}} \frac{\partial P_{\bar{k}}}{\partial u} = 0.$$

Following the \mathbb{Z} -graded case [46, 47], we set

$$\begin{aligned} f(s, u) &= \sum_{k=0,1} (-1)^k \int_0^\infty t^{s-1} \text{Tr}(e^{-td_{\bar{k}}^\dagger d_{\bar{k}}} P_{\bar{k}}) dt \\ &= \Gamma(s) \sum_{k=0,1} (-1)^k \zeta(s, d_{\bar{k}}^\dagger d_{\bar{k}}). \end{aligned}$$

Using the above identities on $P_{\bar{k}}$, the trace property, and by an application of Duhamel's principle, we get

$$\begin{aligned} \frac{\partial f}{\partial u} &= \sum_{k=0,1} (-1)^k \int_0^\infty t^{s-1} \text{Tr} \left(t[\alpha, d_{\bar{k}}^\dagger] d_{\bar{k}} e^{-td_{\bar{k}}^\dagger d_{\bar{k}}} + e^{-td_{\bar{k}}^\dagger d_{\bar{k}}} \frac{\partial P_{\bar{k}}}{\partial u} P_{\bar{k}} \right) dt \\ &= \sum_{k=0,1} (-1)^k \int_0^\infty t^{s-1} \text{Tr} \left(t\alpha [d_{\bar{k}}^\dagger, d_{\bar{k}}] e^{-td_{\bar{k}}^\dagger d_{\bar{k}}} + P_{\bar{k}} e^{-td_{\bar{k}}^\dagger d_{\bar{k}}} \frac{\partial P_{\bar{k}}}{\partial u} \right) dt \\ &= \sum_{k=0,1} (-1)^k \int_0^\infty t^{s-1} \text{Tr} \\ &\quad \left(t\alpha (e^{-td_{\bar{k}}^\dagger d_{\bar{k}}} d_{\bar{k}}^\dagger d_{\bar{k}} - e^{-td_{\bar{k}} d_{\bar{k}}^\dagger} d_{\bar{k}}^\dagger) + e^{-td_{\bar{k}}^\dagger d_{\bar{k}}} P_{\bar{k}} \frac{\partial P_{\bar{k}}}{\partial u} \right) dt \\ &= \sum_{k=0,1} (-1)^k \int_0^\infty t^s \text{Tr}(\alpha e^{-t\Delta_{\bar{k}}} \Delta_{\bar{k}}) dt \\ &= - \sum_{k=0,1} (-1)^k \int_0^\infty t^s \frac{\partial}{\partial t} \text{Tr}(\alpha (e^{-t\Delta_{\bar{k}}} - Q_{\bar{k}})) dt. \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} \frac{\partial f}{\partial u} &= s \sum_{k=0,1} (-1)^k \int_0^\infty t^{s-1} \text{Tr}(\alpha (e^{-t\Delta_{\bar{k}}} - Q_{\bar{k}})) dt \\ &= s \sum_{k=0,1} (-1)^k \left(\int_0^1 + \int_1^\infty \right) t^{s-1} \text{Tr}(\alpha (e^{-t\Delta_{\bar{k}}} - Q_{\bar{k}})) dt. \end{aligned}$$

Since α is a smooth tensor and n is odd, the asymptotic expansion as $t \downarrow 0$ for $\text{Tr}(\alpha e^{-t\Delta_{\bar{k}}})$ does not contain a constant term (see for example [24], lemma 1.7.4). Therefore, $\int_0^1 t^{s-1} \text{Tr}(\alpha e^{-t\Delta_{\bar{k}}}) dt$ does not have a pole at $s = 0$. On the other hand, because of the exponential decay of $\text{Tr}(\alpha(e^{-t\Delta_{\bar{k}}} - Q_{\bar{k}}))$ for large t , the function $\int_1^\infty t^{s-1} \text{Tr}(\alpha(e^{-t\Delta_{\bar{k}}} - Q_{\bar{k}})) dt$ is entire in s . So

$$(6) \quad \left. \frac{\partial f}{\partial u} \right|_{s=0} = -s \sum_{k=0,1} (-1)^k \int_0^1 t^{s-1} \text{Tr}(\alpha Q_{\bar{k}}) dt \Big|_{s=0} = - \sum_{k=0,1} (-1)^k \text{Tr}(\alpha Q_{\bar{k}})$$

and hence

$$(7) \quad \frac{\partial}{\partial u} \sum_{k=0,1} (-1)^k \zeta(0, d_{\bar{k}}^\dagger d_{\bar{k}}) = 0.$$

Finally, the result follows from (6), (7), and

$$\log[\text{Det}' d_0^\dagger d_0 (\text{Det}' d_1^\dagger d_1)^{-1}] = - \lim_{s \rightarrow 0} \left[f(s, u) - \frac{1}{s} \sum_{k=0,1} (-1)^k \zeta(0, d_{\bar{k}}^\dagger d_{\bar{k}}) \right].$$

q.e.d.

Corollary 3.2. *Under the above deformation, each $\zeta(0, d_{\bar{k}}^\dagger d_{\bar{k}})$ ($k = 0, 1$) is invariant.*

Proof. By (7), their difference is invariant. By Lemma 2.3, their sum is also invariant since $b_0(X, \mathcal{E}, H)$ is defined without using the metrics. q.e.d.

Lemma 3.3. *Under the same assumptions, along any one-parameter deformation of $(g_X, g_{\mathcal{E}})$, the volume elements $\eta_{\bar{0}}, \eta_{\bar{1}}$ can be chosen so that*

$$\frac{\partial}{\partial u} (\eta_{\bar{0}} \otimes \eta_{\bar{1}}^{-1}) = -\frac{1}{2} \sum_{k=0,1} (-1)^k \text{Tr}(\alpha Q_{\bar{k}}) \eta_{\bar{0}} \otimes \eta_{\bar{1}}^{-1}.$$

Proof. Recall that $\eta_{\bar{k}} = \nu_{\bar{k},1} \wedge \cdots \wedge \nu_{\bar{k},b_{\bar{k}}}$, where $\{\nu_{\bar{k},i}\}_{i=1}^{b_{\bar{k}}}$ is an orthonormal basis of $H^{\bar{k}}(X, \mathcal{E}, H)$, $k = 0, 1$. Since $(\nu_{\bar{k},i}, \nu_{\bar{k},j}) = \int_X \nu_{\bar{k},i} \wedge \Gamma \nu_{\bar{k},j} = \delta_{ij}$, we get, by taking the derivative with respect to u ,

$$\Re \left(\frac{\partial \nu_{\bar{k},i}}{\partial u}, \nu_{\bar{k},i} \right) = -\frac{1}{2} (\nu_{\bar{k},i}, \alpha \nu_{\bar{k},i}).$$

We can adjust the phase of $\nu_{\bar{k},i}$ so that $(\frac{\partial \nu_{\bar{k},i}}{\partial u}, \nu_{\bar{k},i})$ is real. Since we identify $\det \ker(\Delta_{\bar{k}})$ with $\det H^{\bar{k}}(X, \mathcal{E}, H)$ along the deformation, we

have

$$\begin{aligned} \frac{\partial \eta_{\bar{k}}}{\partial u} &= \sum_{i=1}^{b_{\bar{k}}} \nu_{\bar{k},1} \wedge \cdots \wedge \frac{\partial \nu_{\bar{k},i}}{\partial u} \wedge \cdots \wedge \nu_{\bar{k},b_{\bar{k}}} \\ &= -\frac{1}{2} \sum_{i=1}^{b_{\bar{k}}} (\nu_{\bar{k},i}, \alpha \nu_{\bar{k},i}) \eta_{\bar{k}} = -\frac{1}{2} \text{Tr}(\alpha Q_{\bar{k}}) \eta_{\bar{k}}. \end{aligned}$$

The result follows. q.e.d.

Combining Lemma 3.1 and Lemma 3.3, we have

Theorem 3.4 (metric independence of analytic torsion). *Let X be a compact oriented manifold of odd dimension, let \mathcal{E} be a flat vector bundle over X , and let H be a closed differential form on X of odd degree. Then the analytic torsion $\tau(X, \mathcal{E}, H)$ of the twisted de Rham complex does not depend on the choice of the Riemannian metric on X or the Hermitian metric on \mathcal{E} .*

3.2. Variation of analytic torsion with respect to the flux in a cohomology class. We continue to assume that $\dim X$ is odd and use the same notation as above. Suppose the (real) flux form H is deformed smoothly along a one-parameter family with parameter $v \in \mathbb{R}$ in such a way that the cohomology class $[H] \in H^1(X, \mathbb{R})$ is fixed. Then $\frac{\partial H}{\partial v} = -dB$ for some form $B \in \Omega^0(X)$ that depends smoothly on v ; let

$$\beta = B \wedge \cdot .$$

As before, we show in the next two lemmas that the variation of the regularized determinants cancels that of the volume elements.

Lemma 3.5. *Under the above assumptions,*

$$\frac{\partial}{\partial v} \log[\text{Det}' d_0^\dagger d_0 (\text{Det}' d_1^\dagger d_1)^{-1}] = 2 \sum_{k=0,1} (-1)^k \text{Tr}(\beta Q_k).$$

Proof. As in the proof of Lemma 3.1, we set

$$f(s, v) = \sum_{k=0,1} (-1)^k \int_0^\infty t^{s-1} \text{Tr}(e^{-td_k^\dagger d_k} P_k) dt.$$

We note that B , hence β , is real. Using

$$\begin{aligned} \frac{\partial d_{\bar{k}}}{\partial v} &= [\beta, d_{\bar{k}}], & \frac{\partial d_{\bar{k}}^\dagger}{\partial v} &= -[\beta^\dagger, d_{\bar{k}}^\dagger], \\ P_{\bar{k}}^2 &= P_{\bar{k}} = P_{\bar{k}}^\dagger, & P_{\bar{k}} \frac{\partial P_{\bar{k}}}{\partial v} P_{\bar{k}} &= 0 \end{aligned}$$

and by Dumahel's principle, we get

$$\begin{aligned}
\frac{\partial f}{\partial v} &= \sum_{k=0,1} (-1)^k \int_0^\infty t^{s-1} \operatorname{Tr} \\
&\quad \left(t([\beta^\dagger, d_{\bar{k}}^\dagger] d_{\bar{k}} - d_{\bar{k}}^\dagger [\beta, d_{\bar{k}}]) e^{-td_{\bar{k}}^\dagger d_{\bar{k}}} + e^{-td_{\bar{k}}^\dagger d_{\bar{k}}} \left\{ \frac{\partial P_{\bar{k}}}{\partial v}, P_{\bar{k}} \right\} \right) dt \\
&= 2 \sum_{k=0,1} (-1)^k \int_0^\infty t^{s-1} \operatorname{Tr} \\
&\quad \left(t\beta (e^{-td_{\bar{k}}^\dagger d_{\bar{k}}} d_{\bar{k}}^\dagger d_{\bar{k}} - e^{-td_{\bar{k}}^\dagger d_{\bar{k}}} d_{\bar{k}}^\dagger d_{\bar{k}}^\dagger) + e^{-td_{\bar{k}}^\dagger d_{\bar{k}}} P_{\bar{k}} \frac{\partial P_{\bar{k}}}{\partial v} P_{\bar{k}} \right) dt \\
&= 2 \sum_{k=0,1} (-1)^k \int_0^\infty t^s \operatorname{Tr} (\beta e^{-t\Delta_{\bar{k}}} \Delta_{\bar{k}}) dt \\
&= -2 \sum_{k=0,1} (-1)^k \int_0^\infty t^s \frac{\partial}{\partial t} \operatorname{Tr} (\beta (e^{-t\Delta_{\bar{k}}} - Q_{\bar{k}})) dt.
\end{aligned}$$

The rest is similar to the proof of Lemma 3.1.

q.e.d.

Corollary 3.6. *Under the above deformation, each $\zeta(0, d_{\bar{k}}^\dagger d_{\bar{k}})$ ($k = 0, 1$) is invariant.*

Proof. We follow the proof of Corollary 3.2, using the fact that $b_{\bar{0}}(X, \mathcal{E}, H)$ depends only on the cohomology class of H . q.e.d.

If n is odd and $H = 0$, then $\zeta(0, \Delta_i^\mathcal{E}) = -b_i(X, \mathcal{E})$ and

$$\begin{aligned}
\zeta(0, d_{\bar{0}}^\dagger d_{\bar{0}}) &= \sum_{i=1}^{(n+1)/2} i (b_{2i}(X, \mathcal{E}) - b_{2i-1}(X, \mathcal{E})), \\
\zeta(0, d_{\bar{1}}^\dagger d_{\bar{1}}) &= \sum_{i=1}^{(n-1)/2} i (b_{2i+1}(X, \mathcal{E}) - b_{2i}(X, \mathcal{E})).
\end{aligned}$$

We hope that when n is odd but $H \neq 0$, $\zeta(0, d_{\bar{k}}^\dagger d_{\bar{k}})$ can also be expressed in terms of topological numbers that are invariant under homotopy equivalences preserving $[H]$.

Lemma 3.7. *Under the same assumptions, along any one-parameter deformation of H that fixes the cohomology class $[H]$, the volume elements $\eta_{\bar{0}}, \eta_{\bar{1}}$ can be chosen so that*

$$\frac{\partial}{\partial v} (\eta_{\bar{0}} \otimes \eta_{\bar{1}}^{-1}) = - \sum_{k=0,1} (-1)^k \operatorname{Tr} (\beta Q_{\bar{k}}) \eta_{\bar{0}} \otimes \eta_{\bar{1}}^{-1},$$

where we identify $\det H^\bullet(X, \mathcal{E}, H)$ along the deformation using (1).

Proof. Fix a reference point, say $v = 0$, and let $H^{(0)}, \eta_{\bar{k}}^{(0)}$ be the values of $H, \eta_{\bar{k}}$, respectively, at $v = 0$. To compare the volume elements $\eta_{\bar{k}} \in \det H^k(X, \mathcal{E}, H)$ at different values of v , we map them to

$\det H^{\bar{k}}(X, \mathcal{E}, H^{(0)})$ by the inverse of the isomorphism

$$\det \varepsilon_B: \det H^\bullet(X, \mathcal{E}, H^{(0)}) \rightarrow \det H^\bullet(X, \mathcal{E}, H)$$

induced by (1). Since $\varepsilon_B = e^\beta$ on $\Omega^\bullet(X, \mathcal{E})$, we have, for $k = 0, 1$,

$$\frac{\partial}{\partial v} (\det \varepsilon_B)^{-1} \eta_{\bar{k}} = -\text{Tr}(\beta Q_{\bar{k}}) (\det \varepsilon_B)^{-1} \eta_{\bar{k}}.$$

The result follows. q.e.d.

Combining Lemma 3.5 and Lemma 3.7, we have

Theorem 3.8 (flux representative independence of analytic torsion). *Let X be a compact oriented manifold of odd dimension, and let \mathcal{E} be a flat vector bundle over X . Suppose H and H' are closed differential forms on X of odd degrees representing the same de Rham cohomology class, and let B be an even form so that $H' = H - dB$. Then the analytic torsion $\tau(X, \mathcal{E}, H') = (\det \varepsilon_B) \tau(X, \mathcal{E}, H)$.*

3.3. Relation to generalized geometry. Recall that in generalized geometry [31, 28], the bundle $TX \oplus T^*X$ has a bilinear form of signature (n, n) given by

$$\langle \xi_1 + W_1, \xi_2 + W_2 \rangle := (\xi_1(W_2) + \xi_2(W_1))/2,$$

where for $i = 1, 2$, ξ_i are 1-forms and W_i are vector fields on X . A *generalized metric* on X is a reduction of the structure group $O(n, n)$ to $O(n) \times O(n)$. Equivalently, a generalized metric is a splitting of $TX \oplus T^*X$ to a direct sum of two sub-bundles of rank n so that the bilinear form is positive on one and negative on the other. The positive sub-bundle is the graph of $g + B \in \Gamma(\text{Hom}(TX, T^*X))$, where $g = g_X$ is a usual Riemannian metric on X and B is a 2-form on X .

A generalized metric on X defines as follows a scalar product, called the Born-Infeld metric [29], on $\Omega^\bullet(X)$. Let σ be the isomorphism from $\Omega^\bullet(X)$ to itself so that if ω is the wedge product of 1-forms, then $\sigma(\omega)$ is the product with the order of 1-forms reversed. Thus $\sigma(\omega \wedge \omega') = \sigma(\omega') \wedge \sigma(\omega)$ for any forms ω, ω' and $\sigma(B) = -B$ if B is a 2-form. Choose a (local) orthonormal frame $\{e_i, i = 1, \dots, n\}$ on X with respect to g and let $\hat{e}_i := \iota_{e_i} + \iota_{e_i}(g + B) \wedge \cdot$ ($i = 1, \dots, n$) be operators acting on forms. Define a new star operation by

$$\star_{(g,B)} \omega = \sigma(\hat{e}_n \cdots \hat{e}_2 \hat{e}_1 \omega).$$

When $B = 0$, $\star_{(g,B)}$ is the usual Hodge star \star_g given by g . The *Born-Infeld metric* (scalar product) on $\Omega^\bullet(X)^\mathbb{C}$ is [29]

$$(\omega, \omega')_{(g,B)} := \int_X \omega \wedge \star_{(g,B)} \overline{\omega'}$$

for $\omega, \omega' \in \Omega^\bullet(X)^\mathbb{C}$.

We show that the isomorphism ε_B intertwines the Born-Infeld metric $(\cdot, \cdot)_{(g,B)}$ and the usual scalar product $(\cdot, \cdot)_g$ defined by the Riemannian metric g .

Lemma 3.9. *For any $\omega, \omega' \in \Omega^\bullet(X)^\mathbb{C}$, we have*

$$(\omega, \omega')_{(g,B)} = (\varepsilon_B(\omega), \varepsilon_B(\omega'))_g.$$

Proof. Since

$$(\varepsilon_B^{-1}(\omega), \varepsilon_B^{-1}(\omega'))_{(g,B)} = \int_X \omega \wedge \sigma(\varepsilon_B \hat{e}_n \cdots \hat{e}_1 \varepsilon_B^{-1} \overline{\omega'}),$$

it suffices to check that $\varepsilon_B \hat{e}_n \cdots \hat{e}_1 \varepsilon_B^{-1}$ is independent of B . We replace B by vB , where $v \in \mathbb{R}$. Then, since $\frac{\partial \hat{e}_i}{\partial v} = \iota_{e_i} B \wedge \cdot = -[\beta, \hat{e}_i]$, we get

$$\frac{\partial}{\partial v} (\varepsilon_{vB} \hat{e}_n \cdots \hat{e}_1 \varepsilon_{vB}^{-1}) = \varepsilon_{vB} \left([\beta, \hat{e}_n \cdots \hat{e}_1] + \sum_{i=1}^n \hat{e}_n \cdots \frac{\partial \hat{e}_i}{\partial v} \cdots \hat{e}_1 \right) \varepsilon_{vB}^{-1} = 0$$

and the result follows. q.e.d.

For simplicity, we take \mathcal{E} as the trivial line bundle over X . Let $(d_k^H)^\dagger_{(g,B)}$ be the adjoint of $d_k^H = d + H \wedge \cdot$ acting on $\Omega^k(X)$ ($k = 0, 1$) with respect to the Born-Infeld metric [29]. Let $(\Delta_k^H)_{(g,B)} = (d_k^H)^\dagger_{(g,B)} d_k^H + d_{k+1}^H (d_{k+1}^H)^\dagger_{(g,B)}$ be the corresponding Laplacians. Generalizing (3) in §2.1, we define the *twisted analytic torsion under a generalized metric* (g, B) as

$$\begin{aligned} \tau_{(g,B)}(X, H) &:= (\text{Det}'(d_0^H)^\dagger_{(g,B)} d_0^H)^{1/2} (\text{Det}'(d_1^H)^\dagger_{(g,B)} d_1^H)^{-1/2} (\eta_0^H)_{(g,B)} \\ &\quad \otimes (\eta_1^H)_{(g,B)}^{-1} \in \det H^\bullet(X, H), \end{aligned}$$

where, for $k = 0, 1$, the determinant of $(d_k^H)^\dagger_{(g,B)} d_k^H$ is defined as before due to the vanishing of the non-commutative residue of the pseudodifferential projection onto $\text{im}(d_k^H)^\dagger_{(g,B)}$ and $(\eta_k^H)_{(g,B)}$ is the unit volume element of $\ker(\Delta_k^H)_{(g,B)} \cong H^k(X, H)$ with respect to the Born-Infeld metric. When $B = 0$, the torsion is $\tau_g(X, H)$ defined in (3).

We specialize to the interesting case when H is a 3-form.

Proposition 3.10. *Let H be a closed 3-form on X and $H' = H - dB$, where B is the 2-form in the generalized metric. Then*

$$(d^{H'})^\dagger_g = \varepsilon_B \circ (d^H)^\dagger_{(g,B)} \circ \varepsilon_B^{-1}, \quad \Delta_g^{H'} = \varepsilon_B \circ \Delta_{(g,B)}^H \circ \varepsilon_B^{-1}.$$

Proof. We have $d^{H'} = \varepsilon_B \circ d^H \circ \varepsilon_B^{-1}$ from §1.2. By Lemma 3.9, for any $\omega, \omega' \in \Omega^\bullet(X)^\mathbb{C}$, we get

$$\begin{aligned} (\omega, (d^H)^\dagger_{(g,B)} \omega')_{(g,B)} &= (d^H \omega, \omega')_{(g,B)} = (\varepsilon_B d^H \omega, \varepsilon_B \omega')_g \\ &= (d^{H'} \varepsilon_B \omega, \varepsilon_B \omega')_g = (\varepsilon_B \omega, (d^{H'})^\dagger_g \varepsilon_B \omega')_g \\ &= (\omega, \varepsilon_B^{-1} (d^{H'})^\dagger_g \varepsilon_B \omega')_{(g,B)}. \end{aligned}$$

Therefore, the first equality holds and the second one follows. q.e.d.

Corollary 3.11. *Under the above notations, we have*

$$\tau_g(X, H') = (\det \varepsilon_B) (\tau_{(g,B)}(X, H)) \in \det H^\bullet(X, H').$$

Proof. By Proposition 3.10, for $k = 0, 1$, the operators $(d_k^{H'})^\dagger_g d_k^{H'}$ and $(d_k^H)^\dagger_{(g,B)} d_k^H$ have the same spectrum and hence the same regularized determinants. On the other hand, together with Lemma 3.9, we conclude that $\varepsilon_B: \ker(\Delta_k^H)_{(g,B)} \rightarrow \ker(\Delta_k^{H'})_g$ is an isometry and hence we can choose $(\eta_k^{H'})_g = (\det \varepsilon_B) (\eta_k^H)_{(g,B)}$. The result follows. q.e.d.

We thus conclude that deformation of H by a B -field is equivalent to deforming the usual metric to a generalized metric. In this way, deformations of the usual metric and that of the flux by a B -field are unified to a deformation of generalized metric. Theorems 3.4 and 3.8 state that the torsion is invariant under such a deformation.

4. Functorial properties of analytic torsion

In this section, we state the basic functorial properties of analytic torsion for the twisted de Rham complex. These can be established by a generalization of the proofs of the corresponding results for the usual analytic torsion [46, 19, 41] to the \mathbb{Z}_2 -graded case. We write $d_k^\mathcal{E} = d_k^{\mathcal{E}, H}$, $\Delta_k^\mathcal{E} = \Delta_k^{\mathcal{E}, H}$ and $\eta_k^\mathcal{E} = \eta_k^{\mathcal{E}, H}$ since the dependence on the flux form H is clear.

Proposition 4.1. *Let X be a compact oriented Riemannian manifold and let $\mathcal{E}_1, \mathcal{E}_2$ be flat Hermitian vector bundles on X . Suppose H is a closed odd-degree form on X . Then*

$$\tau(X, \mathcal{E}_1 \oplus \mathcal{E}_2, H) = \tau(X, \mathcal{E}_1, H) \otimes \tau(X, \mathcal{E}_2, H)$$

under the canonical identification

$$\det H^\bullet(X, \mathcal{E}_1 \oplus \mathcal{E}_2, H) \cong \det H^\bullet(X, \mathcal{E}_1, H) \otimes \det H^\bullet(X, \mathcal{E}_2, H)$$

induced by the isomorphism $H^\bullet(X, \mathcal{E}_1 \oplus \mathcal{E}_2, H) \cong H^\bullet(X, \mathcal{E}_1, H) \oplus H^\bullet(X, \mathcal{E}_2, H)$.

Proof. On $\Omega^\bullet(X, \mathcal{E}_1 \oplus \mathcal{E}_2) \cong \Omega^\bullet(X, \mathcal{E}_1) \oplus \Omega^\bullet(X, \mathcal{E}_2)$, the operator $d_k^{\mathcal{E}_1 \oplus \mathcal{E}_2} = d_k^{\mathcal{E}_1} \oplus d_k^{\mathcal{E}_2}$ is block-diagonal. Thus the determinant factorizes: $\text{Det}'((d_k^{\mathcal{E}_1 \oplus \mathcal{E}_2})^\dagger d_k^{\mathcal{E}_1 \oplus \mathcal{E}_2}) = \text{Det}'((d_k^{\mathcal{E}_1})^\dagger d_k^{\mathcal{E}_1}) \text{Det}'((d_k^{\mathcal{E}_2})^\dagger d_k^{\mathcal{E}_2})$. Under the above identification, we can choose the volume elements such that $\eta_k^{\mathcal{E}_1 \oplus \mathcal{E}_2} = \eta_k^{\mathcal{E}_1} \otimes \eta_k^{\mathcal{E}_2}$. Hence the result. q.e.d.

Proposition 4.2. *Let X be a compact oriented manifold of dimension n and let \mathcal{E} be a flat vector bundle on X . Suppose H is a closed odd-degree form on X . Then*

$$\tau(X, \mathcal{E}, H) = \tau(X, \mathcal{E}^*, -H)^{(-1)^{n+1}}$$

under the canonical identification

$$\det H^\bullet(X, \mathcal{E}, H) \cong \det H^\bullet(X, \mathcal{E}^*, -H)^{(-1)^{n+1}}$$

induced by Poincaré duality in Proposition 1.6.

Proof. By Lemma 2.1, $(d_k^{\mathcal{E}, H})^\dagger d_k^{\mathcal{E}, H} = \Gamma^{-1} d_{n+1-k}^{\mathcal{E}^*, -H} (d_{n+1-k}^{\mathcal{E}^*, -H})^\dagger \Gamma$. So the non-zero spectrum of $(d_k^{\mathcal{E}, H})^\dagger d_k^{\mathcal{E}, H}$, counting multiplicity, is identical to that of $(d_{n+1-k}^{\mathcal{E}^*, -H})^\dagger d_{n+1-k}^{\mathcal{E}^*, -H}$, and so is the regularized determinant. The isometry Γ induces Poincaré duality, under which the volume elements $\eta_k^{\mathcal{E}, H} = \eta_{n+1-k}^{\mathcal{E}^*, -H}$. The result then follows from the definition of the twisted torsion. q.e.d.

Proposition 4.3. *Let X_1, X_2 be two compact oriented manifolds with the same universal covering manifold. Suppose the fundamental group $\pi_1(X_1)$ is a subgroup of $\pi_1(X_2)$. Let ρ_1 be a representation of $\pi_1(X_1)$ and let ρ_2 be the induced representation of $\pi_1(X_2)$. Denote the flat vector bundles associated with ρ_1, ρ_2 by $\mathcal{E}_1, \mathcal{E}_2$, respectively. Suppose the closed odd-degree forms H_1 on X_1 and H_2 on X_2 pull-back to the same form on the universal covering. Then*

$$\tau(X_1, \mathcal{E}_1, H_1) = \tau(X_2, \mathcal{E}_2, H_2)$$

under the canonical identification

$$\det H^\bullet(X_1, \mathcal{E}_1, H_1) \cong \det H^\bullet(X_2, \mathcal{E}_2, H_2)$$

induced by the isomorphism $H^\bullet(X_1, \mathcal{E}_1, H_1) \cong H^\bullet(X_2, \mathcal{E}_2, H_2)$.

Proof. By Theorem 3.4, we can choose the Riemannian metrics on X_1 and X_2 so that they pull-back to the same metric on the universal covering and the Hermitian metrics on \mathcal{E}_1 and \mathcal{E}_2 associated to the metric on the space of representation ρ_1 and the induced metric on the space of representation ρ_2 . Then the canonical isomorphism $\Omega^\bullet(X_1, \mathcal{E}_1, H_1) \cong \Omega^\bullet(X_2, \mathcal{E}_2, H_2)$ is an isometry. Following the proof of Theorem 2.6 in [46], we deduce that the spectrums, and hence the regularized determinants of $(d_k^{\mathcal{E}_1})^\dagger d_k^{\mathcal{E}_1}$ and $(d_k^{\mathcal{E}_2})^\dagger d_k^{\mathcal{E}_2}$, coincide. The volume

elements of $H^\bullet(X_1, \mathcal{E}_1, H_1)$ and $H^\bullet(X_2, \mathcal{E}_2, H_2)$ also coincide under the isomorphism. q.e.d.

Proposition 4.4. *In addition to the conditions of Proposition 1.7, assume that both X_1 and X_2 are compact manifolds. Then under the canonical identification*

$$\det H^\bullet(X_1 \times X_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2, H_1 \boxplus H_2) \cong (\det H^\bullet(X_1, \mathcal{E}_1, H_1))^{\otimes \chi(X_2, \mathcal{E}_2)} \otimes (\det H^\bullet(X_2, \mathcal{E}_2, H_2))^{\otimes \chi(X_1, \mathcal{E}_1)}$$

induced by the Künneth isomorphism, we have

$$\tau(X_1 \times X_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2, H_1 \boxplus H_2) = \tau(X_1, \mathcal{E}_1, H_1)^{\otimes \chi(X_2, \mathcal{E}_2)} \otimes \tau(X_2, \mathcal{E}_2, H_2)^{\otimes \chi(X_1, \mathcal{E}_1)}.$$

Proof. For $k = 0, 1$, the space $\Omega^{\bar{k}}(X_1 \times X_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2)$ has a dense subspace that is isomorphic to $\bigoplus_{l=0,1} \Omega^{\bar{l}}(X_1, \mathcal{E}_1) \otimes \Omega^{\bar{k-l}}(X_2, \mathcal{E}_2)$. Under this identification, the operators $d^{\mathcal{E}_1 \boxtimes \mathcal{E}_2} = d^{\mathcal{E}_1} \otimes 1 + 1 \otimes d^{\mathcal{E}_2}$, $(d^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})^\dagger = (d^{\mathcal{E}_1})^\dagger \otimes 1 + 1 \otimes (d^{\mathcal{E}_2})^\dagger$, $\Delta^{\mathcal{E}_1 \boxtimes \mathcal{E}_2} = \Delta^{\mathcal{E}_1} \otimes 1 + 1 \otimes \Delta^{\mathcal{E}_2}$ on $\Omega^\bullet(X_1 \times X_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2)$. (We omit the superscript H of the operators.) Therefore, an eigenvalue of $\Delta^{\mathcal{E}_1 \boxtimes \mathcal{E}_2}$ is the sum of those of $\Delta^{\mathcal{E}_1}$ and $\Delta^{\mathcal{E}_2}$, and the corresponding eigenspace is a tensor product of those of the latter. (All the eigenspaces are finite dimensional by the compactness assumption.) In particular, $\ker(\Delta^{\mathcal{E}_1 \boxtimes \mathcal{E}_2}) = \ker(\Delta^{\mathcal{E}_1}) \otimes \ker(\Delta^{\mathcal{E}_2})$; this is another proof of the Künneth isomorphism (Proposition 1.7) when X_1, X_2 are both compact. In addition, we have

$$\text{spec}'(\Delta_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2}) = \{\lambda_1 + \lambda_2 > 0 \mid \lambda_i \in \text{spec}(\Delta_0^{\mathcal{E}_i}) \text{ or } \lambda_i \in \text{spec}(\Delta_1^{\mathcal{E}_i}), i = 1, 2\}$$

and therefore by (4) we get

$$(8) \quad \sum_{k=0,1} (-1)^k \zeta(s, (d_k^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})^\dagger d_k^{\mathcal{E}_1 \boxtimes \mathcal{E}_2}) = \sum_{k=0,1} \left(\sum_{\substack{\lambda_i \in \text{spec}(\Delta_k^{\mathcal{E}_i}), i=1,2 \\ \lambda_1 + \lambda_2 \in \text{spec}_I(\Delta_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})}} \frac{m_I(\lambda_1 + \lambda_2, \Delta_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})}{(\lambda_1 + \lambda_2)^s} - \sum_{\substack{\lambda_i \in \text{spec}(\Delta_k^{\mathcal{E}_i}), i=1,2 \\ \lambda_1 + \lambda_2 \in \text{spec}_{II}(\Delta_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})}} \frac{m_{II}(\lambda_1 + \lambda_2, \Delta_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})}{(\lambda_1 + \lambda_2)^s} \right).$$

We now show that total contribution to the sum (8) from (λ_1, λ_2) vanishes if both $\lambda_1, \lambda_2 > 0$. We consider four cases.

1. If both $\lambda_i \in \text{spec}_I(\Delta_0^{\mathcal{E}_i})$, then there is a non-zero $\omega_i \in \text{im}(d_0^{\mathcal{E}_i})^\dagger$ such that $\Delta_0^{\mathcal{E}_i} \omega_i = \lambda_i \omega_i$ for each $i = 1, 2$. It is easy to see that $\omega_1 \otimes \omega_2 \in \text{im}(d_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})^\dagger$ is an eigenvector of $\Delta_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2}$ with eigenvalue $\lambda_1 + \lambda_2 \in \text{spec}_I(\Delta_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})$. On the other hand, $d_0^{\mathcal{E}_i} \omega_i$ is a (non-zero) eigenvector of $\Delta_1^{\mathcal{E}_i}$ with eigenvalue λ_i . Hence $\lambda_i \in \text{spec}'(\Delta_1^{\mathcal{E}_i})$. Since $d_0^{\mathcal{E}_1} \omega_1 \otimes d_0^{\mathcal{E}_2} \omega_2 \in \text{im}(d_1^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})$, we also have $\lambda_1 + \lambda_2 \in \text{spec}_{II}(\Delta_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})$. So the contribution of $(\lambda_1, \lambda_2) \in \text{spec}'(\Delta_0^{\mathcal{E}_1}) \times \text{spec}'(\Delta_0^{\mathcal{E}_2})$ cancels that of $(\lambda_1, \lambda_2) \in \text{spec}'(\Delta_1^{\mathcal{E}_1}) \times \text{spec}'(\Delta_1^{\mathcal{E}_2})$.
2. Similarly, if both $\lambda_i \in \text{spec}_{II}(\Delta_0^{\mathcal{E}_i})$ ($i = 1, 2$), the corresponding contribution is also canceled.
3. If $\lambda_1 \in \text{spec}_I(\Delta_0^{\mathcal{E}_1})$ but $\lambda_2 \in \text{spec}_{II}(\Delta_0^{\mathcal{E}_2})$, let ω_i ($i = 1, 2$) be the corresponding eigenvectors of $\Delta_0^{\mathcal{E}_i}$. Then $\omega_1 \otimes \omega_2$ and $d_0^{\mathcal{E}_1} \omega_1 \otimes (d_1^{\mathcal{E}_2})^\dagger \omega_2$ are linearly independent eigenvectors of $\Delta_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2}$ with the same eigenvalue $\lambda_1 + \lambda_2$. It is easy to see that one linear combination $\omega_1 \otimes \omega_2 - \lambda_1^{-1} d_0^{\mathcal{E}_1} \omega_1 \otimes (d_1^{\mathcal{E}_2})^\dagger \omega_2$ is in $\text{im}(d_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})^\dagger$, yielding $\lambda_1 + \lambda_2 \in \text{spec}_I(\Delta_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})$, while another (independent) combination $\omega_1 \otimes \omega_2 + \lambda_2^{-1} d_0^{\mathcal{E}_1} \omega_1 \otimes (d_1^{\mathcal{E}_2})^\dagger \omega_2$ is in $\text{im}(d_1^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})$, yielding $\lambda_1 + \lambda_2 \in \text{spec}_{II}(\Delta_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})$. So the contributions of (λ_1, λ_2) also cancel in this case.
4. The case $\lambda_1 \in \text{spec}_{II}(\Delta_0^{\mathcal{E}_1})$, $\lambda_2 \in \text{spec}_I(\Delta_0^{\mathcal{E}_2})$ is similar.

The non-zero contributions to (8) are thus from the subspaces $\text{im}(d_k^{\mathcal{E}_1})^\dagger \otimes \ker(\Delta_l^{\mathcal{E}_2})$, $\ker(\Delta_l^{\mathcal{E}_1}) \otimes \text{im}(d_k^{\mathcal{E}_2})^\dagger \subset \text{im}(d_{k+l}^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})^\dagger$, $k, l = 0, 1$. Since $\dim \ker(\Delta_l^{\mathcal{E}_i}) = b_l(X_i, \mathcal{E}_i, H_i)$, we have

$$\begin{aligned}
& \sum_{k=0,1} (-1)^k \zeta(s, (d_k^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})^\dagger d_k^{\mathcal{E}_1 \boxtimes \mathcal{E}_2}) \\
= & \sum_{k,l=0,1} (-1)^{k+l} \left(\sum_{\lambda_1 \in \text{spec}'((d_k^{\mathcal{E}_1})^\dagger d_k^{\mathcal{E}_1})} \frac{m(\lambda_1, (d_k^{\mathcal{E}_1})^\dagger d_k^{\mathcal{E}_1}) b_l(X_2, \mathcal{E}_2, H_2)}{\lambda_1^s} \right. \\
& \quad \left. + \sum_{\lambda_2 \in \text{spec}'((d_k^{\mathcal{E}_2})^\dagger d_k^{\mathcal{E}_2})} \frac{m(\lambda_2, (d_k^{\mathcal{E}_2})^\dagger d_k^{\mathcal{E}_2}) b_l(X_1, \mathcal{E}_1, H_1)}{\lambda_2^s} \right) \\
= & \chi(X_2, \mathcal{E}_2) \sum_{k=0,1} (-1)^k \zeta(s, (d_k^{\mathcal{E}_1})^\dagger d_k^{\mathcal{E}_1}) + \chi(X_1, \mathcal{E}_1) \sum_{k=0,1} (-1)^k \zeta(s, (d_k^{\mathcal{E}_2})^\dagger d_k^{\mathcal{E}_2})
\end{aligned}$$

and therefore

$$\frac{\text{Det}'(d_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2} \dagger d_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})}{\text{Det}'(d_1^{\mathcal{E}_1 \boxtimes \mathcal{E}_2} \dagger d_1^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})} = \left(\frac{\text{Det}'(d_0^{\mathcal{E}_1} \dagger d_0^{\mathcal{E}_1})}{\text{Det}'(d_1^{\mathcal{E}_1} \dagger d_1^{\mathcal{E}_1})} \right)^{\chi(X_2, \mathcal{E}_2)} \left(\frac{\text{Det}'(d_0^{\mathcal{E}_2} \dagger d_0^{\mathcal{E}_2})}{\text{Det}'(d_1^{\mathcal{E}_2} \dagger d_1^{\mathcal{E}_2})} \right)^{\chi(X_1, \mathcal{E}_1)}.$$

For the volume elements, we can choose

$$\eta_k^{\mathcal{E}_1 \boxtimes \mathcal{E}_2} = \bigotimes_{l=0,1} (\eta_l^{\mathcal{E}_1})^{\otimes b_{k-1}(X_2, \mathcal{E}_2, H_2)} \otimes (\eta_l^{\mathcal{E}_2})^{\otimes b_{k-1}(X_1, \mathcal{E}_1, H_1)}$$

and hence

$$\eta_0^{\mathcal{E}_1 \boxtimes \mathcal{E}_2} \otimes (\eta_1^{\mathcal{E}_1 \boxtimes \mathcal{E}_2})^{-1} = (\eta_0^{\mathcal{E}_1} \otimes (\eta_1^{\mathcal{E}_1})^{-1})^{\otimes \chi(X_2, \mathcal{E}_2)} \otimes (\eta_0^{\mathcal{E}_2} \otimes (\eta_1^{\mathcal{E}_2})^{-1})^{\otimes \chi(X_1, \mathcal{E}_1)}.$$

q.e.d.

It would be interesting to establish the behavior of the analytic torsion (form) under a general smooth fibration, analogous to [7, 21, 36], for the twisted de Rham or other \mathbb{Z}_2 -graded complexes.

5. Calculations of analytic torsion and the simplicial analogue

5.1. Analytic torsion when the flux is a top-degree form. Recall that the twisted cohomology groups can be computed by the spectral sequence in §1.2. In this process, each complex (E_r^\bullet, δ_r) is finite dimensional for $r \geq 2$ when X is compact. The Knudsen-Mumford isomorphisms [32] $\det E_r^\bullet \cong \det E_{r+1}^\bullet$ for $r \geq 2$ yield an isomorphism

$$(9) \quad \kappa: \det H^\bullet(X, \mathcal{E}) \rightarrow \det H^\bullet(X, \mathcal{E}, H)$$

since the spectral sequence converges to the twisted cohomology.

Proposition 5.1. *Suppose X is a compact oriented manifold of odd dimension and \mathcal{E} is a flat vector bundle associated to an orthogonal or unitary representation of $\pi_1(X)$. Assume $n = \dim X > 1$ and let H be an n -form on X . Then*

$$\tau(X, \mathcal{E}, H) = \kappa(\tau(X, \mathcal{E})).$$

Proof. By Theorem 3.4, we can choose a Riemannian metric on X so that $\text{vol}(X) = 1$; let $\nu = *1$ be the volume form on X . By Theorem 3.8, we can also assume that $H = [H]\nu$, where $[H] \in H^n(X, \mathbb{R}) \cong \mathbb{R}$ is a real number. If $[H] = 0$, then the statement is trivial; we assume that $[H] \neq 0$. Since \mathcal{E} is a flat vector bundle associated to an orthogonal or unitary representation, we have $H^0(X, \mathcal{E}) \cong \overline{H^n(X, \mathcal{E})}^*$; let $b_0 := \dim H^0(X, \mathcal{E}) = \dim H^n(X, \mathcal{E})$. Let η_i be the unit volume element of $H^i(X, \mathcal{E})$ for $0 \leq i \leq n$. The metric-independent isomorphism (9) is given by

$$\kappa: \bigotimes_{i=0}^n \eta_i^{(-1)^i} \mapsto |[H]|^{b_0} \eta_0 \otimes \eta_1^{-1}.$$

Let d_i ($0 \leq i \leq n - 1$) be the differential on $C^i = \Omega^i(X, \mathcal{E})$. Then d_k is equal to $\begin{pmatrix} d_0 & 0 \\ H & d_{n-1} \end{pmatrix}$ on $C^0 \oplus C^{n-1}$, d_i on C^i for $i \leq i \leq n - 2$, and 0 on C^n . Here H also stands for taking wedge product with H . The Ray-Singer torsion is

$$\tau(X, \mathcal{E}) = \prod_{i=0}^{n-1} (\text{Det}' d_i^\dagger d_i)^{(-1)^i/2} \bigotimes_{i=0}^n \eta_i^{(-1)^i}$$

while the torsion for the twisted de Rham complex is

$$\tau(X, \mathcal{E}, H) = \text{Det}' \begin{pmatrix} d_0^\dagger d_0 + H^\dagger H & H^\dagger d_{n-1} \\ d_{n-1}^\dagger H & d_{n-1}^\dagger d_{n-1} \end{pmatrix}^{1/2} \prod_{i=1}^{n-2} (\text{Det}' d_i^\dagger d_i)^{(-1)^i/2} \eta_0 \otimes \eta_1^{-1}.$$

The result follows from the following lemma. q.e.d.

Lemma 5.2. *Under the above assumptions, we have*

$$\text{Det}' \begin{pmatrix} d_0^\dagger d_0 + H^\dagger H & H^\dagger d_{n-1} \\ d_{n-1}^\dagger H & d_{n-1}^\dagger d_{n-1} \end{pmatrix} = [H]^{2b_0} \text{Det}' d_0^\dagger d_0 \text{Det}' d_{n-1}^\dagger d_{n-1}.$$

Proof. Let Q_i be the orthogonal projection from (the completion of) C^i onto $\ker(\Delta_i)$, $0 \leq i \leq n$. Set $\tilde{\Delta}_{n-1} = \Delta_{n-1} + Q_{n-1}$. Then

$$\begin{aligned} \text{Det}' \begin{pmatrix} d_0^\dagger d_0 + H^\dagger H & H^\dagger d_{n-1} \\ d_{n-1}^\dagger H & d_{n-1}^\dagger d_{n-1} \end{pmatrix} &= \text{Det}' \begin{pmatrix} \Delta_0 + [H]^2 & H^\dagger d_{n-1} \\ d_{n-1}^\dagger H & \tilde{\Delta}_{n-1} \end{pmatrix} \\ &(\text{Det}' d_{n-2}^\dagger d_{n-2})^{-1}. \end{aligned}$$

Since $d_{n-1} \tilde{\Delta}_{n-1}^{-1} d_{n-1}^\dagger = 1 - Q_n$ and hence $H^\dagger d_{n-1} \tilde{\Delta}_{n-1}^{-1} d_{n-1}^\dagger H = [H]^2(1 - Q_0)$, we have

$$\begin{aligned} \begin{pmatrix} \Delta_0 + [H]^2 & H^\dagger d_{n-1} \\ d_{n-1}^\dagger H & \tilde{\Delta}_{n-1} \end{pmatrix} &= \begin{pmatrix} \Delta_0 + [H]^2 Q_0 & H^\dagger d_{n-1} \\ 0 & \tilde{\Delta}_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tilde{\Delta}_{n-1}^{-1} d_{n-1}^\dagger H & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & H^\dagger d_{n-1} \tilde{\Delta}_{n-1}^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta_0 + [H]^2 Q_0 & 0 \\ 0 & \tilde{\Delta}_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tilde{\Delta}_{n-1}^{-1} d_{n-1}^\dagger H & 1 \end{pmatrix}. \end{aligned}$$

We note that $n = \dim X$ is odd. Since the elliptic pseudo-differential operators

$$\begin{pmatrix} \Delta_0 + [H]^2 & H^\dagger d_{n-1} \\ d_{n-1}^\dagger H & \tilde{\Delta}_{n-1} \end{pmatrix}, \quad \begin{pmatrix} \Delta_0 + [H]^2 Q_0 & H^\dagger d_{n-1} \\ 0 & \tilde{\Delta}_{n-1} \end{pmatrix}, \quad \begin{pmatrix} \Delta_0 + [H]^2 Q_0 & 0 \\ 0 & \tilde{\Delta}_{n-1} \end{pmatrix}$$

of order 2 on $C^0 \oplus C^{n-1}$ are odd in the sense of Kontsevich and Vishik [33] and are invertible, the pseudo-differential operators

$$\begin{pmatrix} 1 & H^\dagger d_{n-1} \tilde{\Delta}_{n-1}^{-1} \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ \tilde{\Delta}_{n-1}^{-1} d_{n-1}^\dagger H & 1 \end{pmatrix}$$

of order 0 are also odd and their determinants are defined [33]; we will denote these determinants by \det to distinguish them from zeta-function

regularized determinants Det' . In fact, for any $a > 0$,

$$\text{Det}' \begin{pmatrix} \Delta_0 + a & H^\dagger d_{n-1} \\ 0 & \tilde{\Delta}_{n-1} \end{pmatrix} = \det \begin{pmatrix} 1 & H^\dagger d_{n-1} \tilde{\Delta}_{n-1}^{-1} \\ 0 & 1 \end{pmatrix} \text{Det}' \begin{pmatrix} \Delta_0 + a & 0 \\ 0 & \tilde{\Delta}_{n-1} \end{pmatrix}.$$

Choosing a such that the spectrums of $\Delta_0 + a$ and $\tilde{\Delta}_{n-1}$ are disjoint, the operators

$$\begin{pmatrix} \Delta_0 + a & H^\dagger d_{n-1} \\ 0 & \tilde{\Delta}_{n-1} \end{pmatrix}, \quad \begin{pmatrix} \Delta_0 + a & 0 \\ 0 & \tilde{\Delta}_{n-1} \end{pmatrix}$$

have identical spectrums and hence the same zeta-function regularized determinant. Thus

$$\det \begin{pmatrix} 1 & H^\dagger d_{n-1} \tilde{\Delta}_{n-1}^{-1} \\ 0 & 1 \end{pmatrix} = 1$$

and, similarly,

$$\det \begin{pmatrix} 1 & 0 \\ \tilde{\Delta}_{n-1}^{-1} d_{n-1}^\dagger H & 1 \end{pmatrix} = 1.$$

As determinants factorize for odd pseudo-differential operators of non-negative order on an odd-dimensional manifold [33], we get

$$\begin{aligned} \text{Det}' \begin{pmatrix} \Delta_0 + [H]^2 & H^\dagger d_{n-1} \\ d_{n-1}^\dagger H & \tilde{\Delta}_{n-1} \end{pmatrix} &= \text{Det}' \begin{pmatrix} \Delta_0 + [H]^2 Q_0 & 0 \\ 0 & \tilde{\Delta}_{n-1} \end{pmatrix} \\ &= [H]^{2b_0} \text{Det}' d_0^\dagger d_0 \text{Det}' d_{n-1}^\dagger d_{n-1} \text{Det}' d_{n-2}^\dagger d_{n-2} \end{aligned}$$

and the result follows. q.e.d.

We note that neither Lemma 5.2 nor Proposition 5.1 is valid if $\dim X = 1$ and $[H] \neq 0$. We give a heuristic explanation of Lemma 5.2 when $n > 1$. For any $\lambda \in \text{spec}'(d_0^\dagger d_0)$, let ω_λ be an eigenvector corresponding to λ . Then $*d_0 \omega_\lambda / \sqrt{\lambda}$ is an eigenvector of $d_{n-1}^\dagger d_{n-1}$ with the same eigenvalue. On the subspace spanned by ω_λ and $*d_0 \omega_\lambda / \sqrt{\lambda}$, the operator $d_0^\dagger d_0$ acts as $\begin{pmatrix} \lambda + [H]^2 & [H] \sqrt{\lambda} \\ [H] \sqrt{\lambda} & \lambda \end{pmatrix}$, whose determinant is λ^2 . Notice that

$$C^0 \oplus \text{im}(d_{n-1}^\dagger) = \ker(\Delta_0) \oplus \bigoplus_{\lambda \in \text{spec}'(d_0^\dagger d_0)} \text{span}_{\mathbb{C}} \{ \omega_\lambda, *d_0 \omega_\lambda / \sqrt{\lambda} \}$$

and $\ker(\Delta_0)$ is in the eigenspace of $\Delta_0 + [H]^2 Q_0$ corresponding to the eigenvalue $[H]^2$ (with multiplicity b_0). The “product” of these λ^2 together with $[H]^2$ leads to the result.

Under the assumptions of Proposition 5.1, $\zeta(0, d_1^\dagger d_1)$ for any H is the same as its value when $H = 0$; it would be interesting to find the value of $\zeta(0, d_0^\dagger d_0)$ when $[H] \neq 0$. (See Corollary 3.6 and the discussion that follows.)

In addition to κ in (9), there is another natural isomorphism κ_0 which maps between the alternating products of unit volume elements, i.e.,

$$\kappa_0: \bigotimes_{i=0}^n \eta_i^{(-1)^i} \mapsto \eta_{\bar{0}} \otimes \eta_{\bar{1}}^{-1}.$$

If H is a top-degree form as in Proposition 5.1, then κ_0 is independent of the choice of metrics on X and on \mathcal{E} . The appearance of $||[H]||$ in

$$\tau(X, \mathcal{E}, H) = ||[H]||^{b_0} \kappa_0(\tau(X, \mathcal{E}))$$

is consistent with the metric invariance of both $\tau(X, \mathcal{E})$ and $\tau(X, \mathcal{E}, H)$ and dependence of the latter on the cohomology class $[H]$ only.

Proposition 5.1 applies especially to 3-dimensional manifolds because H is automatically a top-degree form if it contains no 1-form (which can be absorbed in the flat connection). The Ray-Singer torsion has been calculated explicitly, directly or with the help of the Cheeger-Müller theorem, for many 3-manifolds including lens spaces [45, 22] and compact hyperbolic manifolds [23]. As a consequence, we get many non-trivial examples of analytic torsion for the twisted de Rham complexes of 3-manifolds.

5.2. Simplicial analogue of the torsion in a special case. One of the standard ways to compute the classical Ray-Singer torsion is to use the Cheeger-Müller theorem, relating it to the Reidemeister torsion. Although there is difficulty in defining the simplicial counterpart of the twisted analytic torsion in general, we will be able to do so under the condition that the degree of the flux form is sufficiently high.

We first recall the construction of the Reidemeister torsion (cf. [41]). Suppose the manifold X is equipped with a smooth triangulation or a CW complex structure. Let $(C_\bullet(K), \partial)$ be the chain complex of the simplicial or cellular complex K with real coefficients. Choose an embedding of K as a fundamental domain in the corresponding complex \tilde{K} of the universal covering space \tilde{X} . Then each $C_i(\tilde{K})$ ($0 \leq i \leq n$, where $n = \dim X$) is a free module over the group algebra $\mathbb{R}[\pi_1(X)]$ and the i -cells of K form a basis. Given a finite dimensional representation $\rho: \pi_1(X) \rightarrow \text{GL}(E)$, we define a cochain complex

$$C^\bullet(K, E) := \text{Hom}_{\mathbb{R}[\pi_1(X)]}(C_\bullet(\tilde{K}), E)$$

with coboundary map ∂^* , whose cohomology is denoted by $H^\bullet(K, E)$. With a Hermitian form on E , we choose a unit volume element of E . This, together with the basis dual to the i -cells in K , defines a volume element $\mu_i \in \det C^i(K, E)$. We assume that the representation ρ is unimodular. Unimodularity means that $|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(X)$. Then the volume element μ_i is, up to a phase, independent of the choice of the embedding of K in \tilde{K} . The *Reidemeister torsion* or *R-torsion* $\tau(K, E) \in \det H^\bullet(K, E)$ is defined as the image of $\otimes_{i=0}^n \mu_i^{(-1)^i}$ under the

isomorphism $\det C^\bullet(K, E) \cong \det H^\bullet(K, E)$. It is invariant under subdivisions of the complex K . If X is odd-dimensional, the Euler number $\chi(K) = 0$, and $\tau(K, E)$ (up to a phase) does not depend on the choice of the Hermitian form on E . By the de Rham theorem, $H^\bullet(X, \mathcal{E}) \cong H^\bullet(K, E)$ and hence $\det H^\bullet(X, \mathcal{E}) \cong \det H^\bullet(K, E)$. The theorem of Cheeger and Müller [19, 40, 41] states that $\tau(X, \mathcal{E}) = \tau(K, E)$ under the above identification of determinant lines.

Recall that the cup product at the cochain level is associative but not graded commutative. We now assume that each homogeneous component of H is of degree greater than $\dim X/2 = n/2$. Let $h \in C^1(K, E)$ be a representative of $[H] \in H^1(X, \mathcal{E}) \cong H^1(K, E)$. Then since $h \cup h = 0$, we have a \mathbb{Z}_2 -graded cochain complex $(C^\bullet(K, E), \partial_h^*)$, where $\partial_h^* = \partial^* + h \cup \cdot$. Denote its cohomology groups by $H^k(K, E, h)$, $k = 0, 1$. There is then an isomorphism $\det C^\bullet(K, E) \cong \det H^\bullet(K, E, h)$. We define the twisted version of the R -torsion $\tau(K, E, h)$ as the image of $\otimes_{i=0}^n \mu_i^{(-1)^i}$ under the above isomorphism. This will be the simplicial counterpart of the analytic torsion $\tau(X, \mathcal{E}, H)$.

Lemma 5.3. *There is a canonical isomorphism $H^\bullet(X, \mathcal{E}, H) \cong H^\bullet(K, E, h)$.*

Proof. Just as $\Omega^\bullet(X, \mathcal{E})$, the complex $C^\bullet(K, E)$ has a filtration

$$F^p C^{\bar{k}}(K, E) = \bigoplus_{\substack{i \geq p \\ i = k \pmod 2}} C^i(K, E),$$

which yields a spectral sequence $\{E_r^{pq}, \delta_r'\}$ converging to $H^\bullet(K, E, h)$. The cochain map $\Omega^\bullet(X, \mathcal{E}) \rightarrow C^\bullet(K, E)$ that induces the de Rham isomorphism preserves the filtrations. Therefore there is a morphism of the spectral sequences $\{E_r^{pq}, \delta_r\} \rightarrow \{E_r^{pq}, \delta_r'\}$. By the de Rham theorem, this morphism is an isomorphism starting with the E_2 -terms, which implies the result. q.e.d.

We have the following analogue of the Cheeger-Müller theorem when H or h is of top degree.

Theorem 5.4. *With the same assumptions of Proposition 5.1 and under identification given by Lemma 5.3, we have*

$$\tau(X, \mathcal{E}, H) = \tau(K, E, h).$$

Proof. Let

$$\kappa' : \det H^\bullet(K, E) \rightarrow \det H^\bullet(K, E, h)$$

be the isomorphism induced by the Knudsen-Mumford isomorphisms in the spectral sequence $\{E_r^{pq}\}$. The morphism of the two spectral

sequences in the proof of Lemma 5.3 induces a commutative diagram

$$\begin{array}{ccc} \det H^\bullet(X, \mathcal{E}) & \xrightarrow{\kappa} & \det H^\bullet(X, \mathcal{E}, H) \\ \cong \downarrow & & \cong \downarrow \\ \det H^\bullet(K, E) & \xrightarrow{\kappa'} & \det H^\bullet(K, E, h). \end{array}$$

By Proposition 5.1, we have $\tau(X, \mathcal{E}, H) = \kappa(\tau(X, \mathcal{E}))$. On the other hand, it is clear from the definition of $\tau(K, E, h)$ that $\tau(K, E, h) = \kappa'(\tau(K, E))$. The results follow from the Cheeger-Müller theorem $\tau(X, \mathcal{E}) = \tau(K, E)$ since the representation is orthogonal or unitary. q.e.d.

Consider for example the lens space $X = L(1, p)$, $p \in \mathbb{Z}$. It has a cellular structure K with one i -cell e_i for each $i = 0, 1, 2, 3$. On the dual basis e_i^* ($0 \leq i \leq 3$), we have

$$\partial^* e_0^* = 0, \quad \partial^* e_1^* = p e_2^*, \quad \partial^* e_2^* = 0, \quad \partial^* e_3^* = 0.$$

So the Reidemeister torsion is $\tau(K) = |p|^{-1} \eta_0 \otimes \eta_3^{-1}$. If $h = q e_3^*$, then

$$\partial_h^* e_0^* = q e_3^*, \quad \partial_h^* e_1^* = p e_2^*, \quad \partial_h^* e_2^* = 0, \quad \partial_h^* e_3^* = 0,$$

and the twisted torsion is $\tau(K, h) = |qp^{-1}|$.

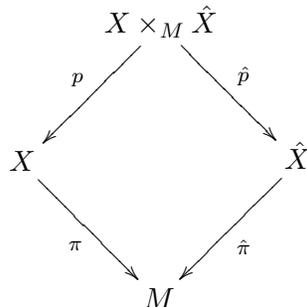
5.3. T -duality for circle bundles and analytic torsion. Let \mathbb{T} be the circle group. Suppose X is a compact oriented manifold and is the total space of a principal \mathbb{T} -bundle

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & X \\ & & \pi \downarrow \\ & & M \end{array}$$

over a compact oriented manifold, M and H is a closed 3-form on X that has integral periods. The flat vector bundle \mathcal{E} is taken to be the trivial real line bundle with the trivial connection. Let $\hat{\mathbb{T}}$ be the dual circle group. Then the T -dual principal circle bundle [12]

$$\begin{array}{ccc} \hat{\mathbb{T}} & \longrightarrow & \hat{X} \\ & & \hat{\pi} \downarrow \\ & & M \end{array}$$

is determined topologically by its first Chern class $c_1(\hat{X}) = \pi_*[H]$. We have the commutative diagram



where $X \times_M \hat{X}$ denotes the correspondence space. The Gysin sequence for \hat{X} enables us to define a T -dual flux $[\hat{H}] \in H^3(\hat{X}, \mathbb{Z})$ satisfying $c_1(X) = \hat{\pi}_*[\hat{H}]$ and $p^*[H] = \hat{p}^*[\hat{H}] \in H^3(X \times_M \hat{X}, \mathbb{Z})$. Thus, T -duality for circle bundles exchanges the H -flux on the one side and the Chern class on the other. It can be shown [12] that $H^\bullet(X, H) \cong H^{\bullet+1}(\hat{X}, \hat{H})$ and, consequently,

$$(10) \quad \det H^\bullet(X, H) \cong (\det H^\bullet(\hat{X}, \hat{H}))^{-1}.$$

We wish to explore the relation between the twisted torsions $\tau(X, H) \in \det H^\bullet(X, H)$ and $\tau(\hat{X}, \hat{H}) \in \det H^\bullet(\hat{X}, \hat{H})$ under the above identification.

We next explain T -duality at the level of differential forms. Choosing connection 1-forms A and \hat{A} on the circle bundles X and \hat{X} , we define the metrics on X and \hat{X} by

$$g_X = \pi^*g_M + A \odot A, \quad g_{\hat{X}} = \hat{\pi}^*g_M + \hat{A} \odot \hat{A},$$

respectively. Since a closed 3-form is cohomologous to a \mathbb{T} -invariant one and the twisted cohomology groups depend only on the cohomology class of the flux H (Theorem 3.8), we can assume, without loss of generality, that H is a \mathbb{T} -invariant 3-form on X . Denote by $F, \hat{F} \in \Omega^2(M)$ the curvature 2-forms of A, \hat{A} , respectively. Since $H - A \wedge \pi^*\hat{F}$ is a basic differential form on X , we have $H = A \wedge \pi^*\hat{F} - \pi^*\Omega$ for some $\Omega \in \Omega^3(M)$. Define the T -dual flux \hat{H} by $\hat{H} = \hat{\pi}^*F \wedge \hat{A} - \hat{\pi}^*\Omega$. Then \hat{H} is closed and $\hat{\mathbb{T}}$ -invariant. We define linear maps $T: \Omega^{\bar{k}}(X) \rightarrow \Omega^{\overline{k+1}}(\hat{X})$ for $k = 0, 1$ by

$$T(\omega) = \int_{\mathbb{T}} e^{p^*A \wedge \hat{p}^*\hat{A}} p^*\omega, \quad \omega \in \Omega^\bullet(X).$$

Lemma 5.5. *Under the above choices of Riemannian metrics and flux forms,*

$$T: \Omega^{\bar{k}}(X)^{\mathbb{T}} \rightarrow \Omega^{\overline{k+1}}(\hat{X})^{\hat{\mathbb{T}}},$$

for $k = 0, 1$, are isometries, inducing isometries on the spaces of twisted harmonic forms and hence on the twisted cohomology groups.

Proof. For any $\omega = \pi^*\omega_1 + A \wedge \pi^*\omega_2 \in \Omega^\bullet(X)^\mathbb{T}$, where $\omega_1, \omega_2 \in \Omega^\bullet(M)$, we have $T(\omega) = \hat{\pi}^*\omega_2 + \hat{A} \wedge \hat{\pi}^*\omega_1$. The isometry of T follows from

$$\int_X \omega \wedge *_X \omega = \int_M \omega_1 \wedge *_M \omega_1 + \int_M \omega_2 \wedge *_M \omega_2.$$

Since $d(p^*A \wedge \hat{p}^*\hat{A}) = -p^*H + \hat{p}^*\hat{H}$, we have $T \circ d^H = d^{\hat{H}} \circ T$. So T acts on the spaces of twisted harmonic forms and on the twisted cohomology groups. q.e.d.

When X is a 3-manifold, Proposition 5.1 relates $\tau(X, H)$ to $\tau(X)$, which can be calculated by the spectral sequence of fibration [21, 22, 36, 38].

Proposition 5.6. *Let X be an oriented 3-manifold with a \mathbb{T} -fibration over a compact oriented surface M , and let H be a flux 3-form on X . Suppose there is a T -dual fibration \hat{X} with flux form \hat{H} . Then under identification (10), we have*

$$\frac{\tau(X, H)}{(2\pi)^{\chi(M)}} = \left[\frac{\tau(\hat{X}, \hat{H})}{(2\pi)^{\chi(M)}} \right]^{-1}.$$

Proof. We can choose the metrics and the flux forms on X, \hat{X} as above. Let $p = c_1(X) \in H^2(M, \mathbb{Z}) \cong \mathbb{Z}$ and $q = [H] \in H^3(X, \mathbb{Z}) \cong \mathbb{Z}$. If $p = 0$, then $X = M \times \mathbb{T}$. If $q = 0$ as well, then $\tau(X) = (2\pi)^{\chi(M)} \eta_0^X \otimes (\eta_1^X)^{-1}$. If $q \neq 0$, then by Proposition 5.1,

$$\tau(X, H) = |[H]| \kappa_0(\tau(X)) = (2\pi)^{\chi(M)} |q| \eta_0^{X,H} \otimes (\eta_1^{X,H})^{-1}.$$

If $p \neq 0$ but $q = 0$, then since the \mathbb{T} -bundle $X \rightarrow M$ is oriented, we can compute $\tau(X)$ by the Gysin sequence of the fibration $X \rightarrow M$ (see for example [36], corollary 0.9) and get $\tau(X) = (2\pi)^{\chi(M)} |p|^{-1} \eta_0^X \otimes (\eta_1^X)^{-1}$. If both $p, q \neq 0$, then again by Proposition 5.1,

$$(11) \quad \tau(X, H) = |[H]| \kappa_0(\tau(X)) = (2\pi)^{\chi(M)} |qp|^{-1} \eta_0^{X,H} \otimes (\eta_1^{X,H})^{-1}.$$

The result follows since T -duality interchanges p and q and since the isometries in Lemma 5.5 identify $\eta_k^{X,H}$ with $\eta_{k+1}^{\hat{X}, \hat{H}}$ for $k = 0, 1$. q.e.d.

We note that (11) is consistent with the simplicial calculation in §5.2 when $X = L(1, p)$, verifying Theorem 5.4 in this case. It can be generalized to the case when X is an S^k -bundle over a compact oriented manifold M of dimension $k + 1$ and H is a top form on X . The behavior of the twisted torsion under T -duality when X is of any dimension and H is a closed 3-form remains an interesting problem. Such a relation will provide a new way of calculating twisted analytic torsions and, in particular, the classical Ray-Singer torsion using T -duality.

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