

A Research On a Kind of Special Points Inside Convex

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Abstract

In this paper, we study the point set consisting of the centers of all inscribed central-symmetric convex polygons of a convex. We prove that for any convex, the area of the point set consisting of the centers of its inscribed central-symmetric convex polygons is not greater than $\frac{1}{4}$ of the area of the convex (the equality holds if and only if the convex is a triangle). This conclusion can provide us a method to measure the extent of central symmetry of a planar figure.

Key words: plane convex set, central-symmetric figure

1 Introduction

For a convex, every point inside the convex is the midpoint of a certain chord. What about the points that are the common midpoint of at least two chords? Such points are the centers of inscribed central-symmetric convex polygons of the convex. In this paper, we will study the properties of the point set of all these centers.

Definition 1.1 *For two arbitrary points A, B of point set M , we call M a convex set if all points on segment AB belong to M .*

Definition 1.2 *We call M a convex if convex set M is a bounded closed set.*

Let Ω be a convex in the plane. Let T be the set consisting of the centers of all inscribed central-symmetric convex polygons of Ω . Denote the area of T by $S(T)$.

We have the following theorem:

Theorem 1.3 *Let Ω be a convex in the plane, then*

$$0 \leq S(T) \leq S(\Omega),$$

the left equality holds if and only if Ω is a central-symmetric figure. The right equality holds if and only if Ω is a triangle.

In the following of the paper, if there is no special illustration, a convex or a central-symmetric figure only refers to a figure in the plane. A convex or a central-symmetric figure cannot be a line or part of a line.

2 Preliminary discussion

2.1 Definitions and Notations

Definition 2.1 *Let M be a convex. We shall call l a support line of convex M if:*

- (i) the line l has at least one point in common with convex M .*
- (ii) all points of convex M lie either on one side of line l or on the line l .*

Thus, when given a certain direction, convex M has two support lines both parallel to the direction; when given a certain point, there must exist at least one support line of convex M passing through the point.

Definition 2.2 For convex M , define the mid-parallel line of M as following:

For triangle ABC , let D, E, F be the midpoint of BC, CA, AB . Define segment DE, EF, DF as the mid-parallel line of triangle ABC .

For parallelogram $ABDC$ ($AB \parallel CD, AD \parallel BC$), let E, F, G, H be the midpoint of AB, BC, CD, DA . Define segment EG, FH as the mid-parallel line of triangle $ABCD$.

For trapezoid $ABCD$ ($AD \parallel BC$), let E, F be the midpoints of AB, CD . Define segment EF as the mid-parallel line of triangle ABC .

Here are some notations which will appear later in this paper:

For point set M , we use ∂M to denote the boundary of M .

For line l and point A (or point (x, y)), we use $d(l, A)$ (or $d(l, (x, y))$) to denote the distance from point A to line l (or point (x, y)).

For convex M in the plane, since M is a closed set, M is measurable.

When the Lebesgue measure of M is not zero, we call the area of M as the Lebesgue measure of M . When M is a Lebesgue zero measure set, we may say that the area of M is zero, namely, $S(M) = 0$.

For two points A, B on the boundary of convex M , we call AB a chord of M .

For two points A, B on the boundary of convex M , we call AB a pseudo-diameter of M if there exist two support lines l, m of M passing through A, B respectively such that l parallel to m .

2.2 Properties of convex

Here are some properties of convex which will be used in the paper.

Lemma 2.3 For convex Ω , the length of $\partial\Omega$ or any arc on $\partial\Omega$ is measurable.

Proof. Place the convex Ω in the plane rectangular coordinate system. Construct support lines l_1, l_2 of convex Ω parallel to x-axis and support lines l_3, l_4 parallel to y-axis. Denote the rectangular enclosed by $l_1, l_2; l_3, l_4$ by $ABCD$. For n arbitrary division points $x_1, x_2 \dots x_n$ on $\partial\Omega$, we have

$$\sum_{k=1}^n x_i x_{i+1} \leq AB + BC + CD + DA, (x_{n+1} = x_1)$$

Thus,

$$\sup \sum_{k=1}^n x_k x_{k+1}$$

exists, that is, the length of $\partial\Omega$ is measurable. Similarly, the length of any arc of $\partial\Omega$ is measurable. ■

Lemma 2.4 *If there are three collinear points A, B, C on $\partial\Omega$, then line ABC must be a support line of convex Ω .*

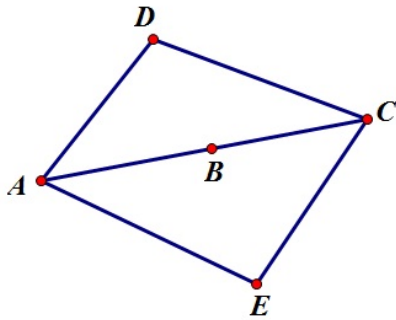


Figure 1

Proof. For simplicity we may assume that A, B, C are arranged successively on line ABC . (See Figure 1) If there exist two points D, E that are on the opposite side of line ABC , then point B is inside the quadrilateral $ADCE$, namely, point B is inside the convex Ω . This is contradictory to that B is on the boundary $\partial\Omega$ of convex Ω . Consequently, line ABC is a support line of convex Ω . ■

Lemma 2.5 *Any convex Ω has inscribed parallelogram.*

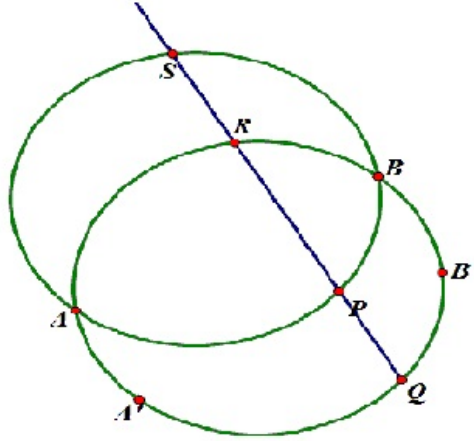


Figure 2

Proof. See Figure 2. Take a point P inside a inscribed triangle of convex Ω . (Since the points of Ω do not lie on a single line, such inscribed triangle must exist.) Thus, P is inside the convex Ω . Take an arbitrary point Q on the boundary of convex Ω . Translate the boundary $\partial\Omega$ along vector \overrightarrow{QP} to get a closed curve noted as C . Extend PQ to intersect with $\partial\Omega$, C at R , S respectively.

Thus, S is outside convex Ω . (Otherwise, as P is inside the convex Ω and R is inside PS , we can infer that R is inside the convex Ω , which is in contradiction with that R is on the boundary of convex Ω .) Now P , S divide C into two arcs which do not intersect inward. According to the Jordan Curve Theorem, either arc has an intersection point with $\partial\Omega$ inward, and the two intersection points do not coincide. Denote these two intersection points by A, B .

Translate A , B respectively along vector \overrightarrow{PQ} to get point A', B' . Since $A, B \in C$, A', B' are both on $\partial\Omega$. Since AB intersects with QS , $AA' \parallel QS$, AA', B are not on the same line consequently. Thus, quadrilateral $ABB'A'$ is a parallelogram. ■

Lemma 2.6 *For any point P inside convex Ω , there must exist a chord l of Ω passing through P such that P is the midpoint of l .*

Proof. Place convex Ω into the plane rectangular coordinate system. For inclination angle $\theta, 0 \leq \theta < \pi$, let \overrightarrow{MN} be the chord whose inclination angle is θ passing through P. Define function $f(\theta) = |MP| - |NP|$ and $f(\pi) = -f(0)$. Since the boundary $\partial\Omega$ of convex Ω is continuous, f is a continuous function on $(0, \pi)$. Besides,

$$\lim_{\theta \rightarrow \pi^-} f(\theta) = -f(0)$$

thus, f is a continuous function on $[0, \pi]$. According to $f(\pi) = -f(0)$ and the intermediate value theorem of continuous function, there exists certain θ_0 such that $f(\theta_0) = 0$. Let l be a chord of convex Ω whose inclination angle is θ_0 . Thus, P is the midpoint of l . ■

3 Special Cases

Here are two special cases of the theorem.

Proposition 3.1 *For convex Ω , $S(T) = 0$ if and only if Ω is a central-symmetric figure.*

Proof. We will prove the proposition from the following aspects.

a: When convex Ω is centrally symmetric, $S(T) = 0$.

Let O be the center of Ω

When convex Ω is a central-symmetric figure, if there exist two different points A,B on $\partial\Omega$ such that segment AB is included in $\partial\Omega$, construct a segment $A'B'$ passing through O such that

1: $A'B'$ is both parallel and equal to AB

2: O is the midpoint of $A'B'$.

Let T' be the union of center O and all segments like $A'B'$. For an arbitrary point P on $\partial\Omega$, there are at most two segments passing through P such that either segment is the boundary of $\partial\Omega$. Thus, the sum of the lengths of all segments in T' is not greater than twice of the circumference of $\partial\Omega$. Besides, as all segments included in T' pass through point O, for any $\varepsilon > 0$, there must exist a figure with an area not larger than $2\varepsilon C_\Omega$ covering T' . This illustrates that the area of point set T' is zero.

For an arbitrary point P in convex Ω , if P does not belong to T' , and there exist two chords AB, CD of Ω such that P is the midpoint of AC, BD , then $ABCD$ must be a parallelogram. Construct two mid-parallel lines l_1, l_2 of parallelogram $ABCD$. (As Figure 3 shows.)

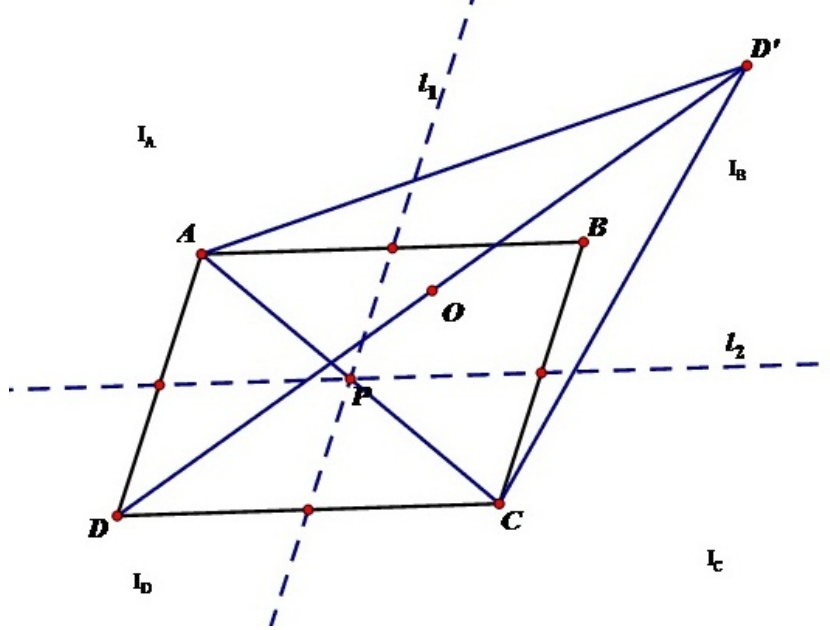


Figure 3

Thus, l_1, l_2 divide the plane (in which $ABCD$ exists) into four parts I_A, I_B, I_C, I_D , among which A, B, C, D are inside I_A, I_B, I_C, I_D respectively. If O is not on l_1 or l_2 , just suppose $O \in I_B$. Let D' be the reflected point of D about O . Thus, D' is on the boundary of Ω . Since B is inside triangle ACD' , B is inside convex Ω . However, this is contradictory to that B is on the boundary of Ω .

Thus, either $O \in l_1$ or $O \in l_2$ is true. Without loss of generality, suppose $O \in l_1$. Let C' be the reflected point of C about O . Then A, D, C' are collinear. Since A, D, C' are all on the boundary of Ω , segment AD is included in $\partial\Omega$. Similarly, segment BC is included in $\partial\Omega$. As P is the center of parallelogram $ABCD$, thus $P \in T'$, which contradicts $P \notin T'$.

This illustrates that for an arbitrary point P , if $P \in T$, then $P \in T'$, namely, $T \subseteq T'$. That is,

$$S(T) = 0$$

b: When $S(T) = 0$, Ω must be a central-symmetric figure.

When $S(T) = 0$, we construct an inscribed parallelogram ABCD of Ω (See Figure 4), and there exist two support lines of Ω passing through point C, D that are not parallel to each other. (Otherwise, we could extend segment CD in the direction of \overrightarrow{AD} to satisfy the condition.)

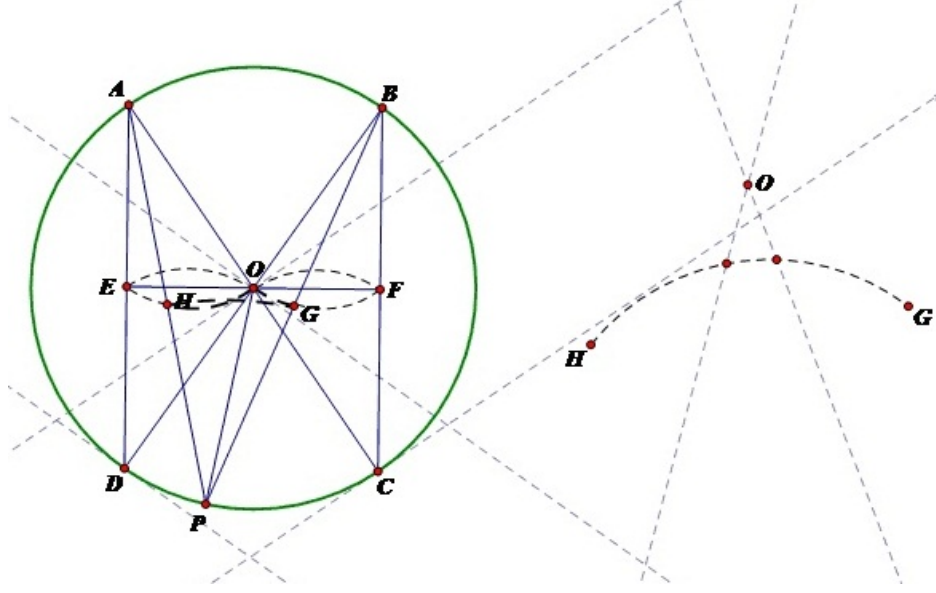


Figure 4

Let O be the center of parallelogram ABCD. For every point P on arc \widehat{CD} , extend PO to intersect with $\partial\Omega$ at P' . If $PO \neq P'O$, just suppose that $PO > P'O$.

Construct $T(AB, CP)$ (as shown in Figure 4, it' s the figure enclosed by arc $\widehat{EO}, \widehat{OG}, \widehat{GH}, \widehat{HE}$) and $T(AB, CP)$ (it' s the figure enclosed by arc $\widehat{OH}, \widehat{HG}, \widehat{GF}, \widehat{FO}$).

Construct two support lines l_C, l_D of Ω passing through C, D which are not parallel to each other. Construct two lines l'_C, l'_D passing through O that are parallel to l_C, l_D respectively. Thus, arc \widehat{OH} and l'_D are on the opposite side of l'_C , arc \widehat{OG} and l'_C are on the opposite side of l'_D . As $PO > P'O$, according to the construction of arc \widehat{GH} , O and P are on the opposite side of arc \widehat{GH} . Thus l'_C, l'_D , segment OH , OG , and arc \widehat{GH} encircle a domain with a certain area. Denote the domain by Λ . Thus, $\Lambda \subseteq T(AB, CP), \Lambda \subseteq T(AB, PD)$.

This illustrates that $\Lambda \subseteq T, S(T) > S(\Lambda) > 0$, which is contradictory to $S(T) = 0$

This illustrates that for every point P on arc \widehat{CD} , extend PO to intersect $\partial\Omega$ at P' , we have $PO = P'O$. Similarly, for every point P on $\partial\Omega$, extend PO to intersect $\partial\Omega$ at P' , we have $PO = P'O$. Namely, Ω is a central-symmetric figure. ■

Corollary 3.2 *For convex Ω , if the curvature of the boundary $\partial\Omega$ of Ω is permanently not zero, then point O is the center of any inscribed central-symmetric convex polygon of convex Ω .*

Proof. When the curvature of the boundary $\partial\Omega$ of Ω is not zero permanently, $T' = \{O\}$, the center of any inscribed convex polygon of Ω is O . ■

Proposition 3.3 *For convex Ω , if Ω is a triangle, then*

$$S(T) = S(DEF) = \frac{1}{4}S(\Omega)$$

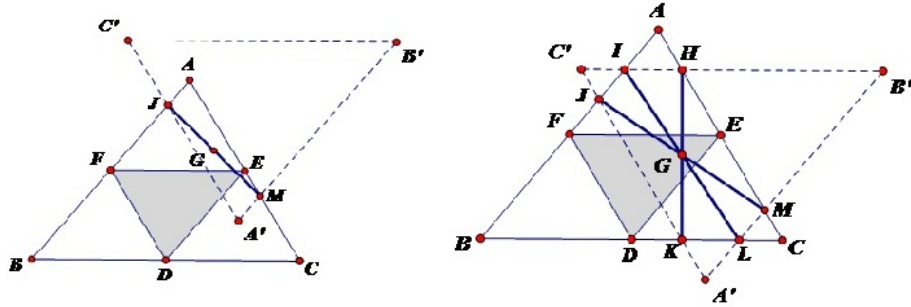


Figure 5

Proof. As shown in figure 5, when Ω is a triangle, denote the vertices of Ω by A, B, C , and DEF as the midpoint triangle of triangle ABC .

In the following part, we will prove that T refers to triangle DEF except for its vertices D, E, F .

When point G is inside triangle DEF , let A', B', C' be the reflected point of A, B, C respectively. Since G is inside triangle DEF , segment $B'C'$ intersects segment AB, AC at I, H respectively. Segment $A'C'$ intersects

segment AB, BC at J, K respectively. Segment $A'B'$ intersects segment BC, AC at L, M respectively. Thus, G is the midpoint of KH, IL, JM, also the center of parallelogram KLHI.

When G is on the boundary of triangle DEF (with the exception of point D, E, F), similarly, there exists a parallelogram whose center is G.

When G is on the boundary of triangle ABC, it is obvious that G cannot be the center of a certain inscribed convex polygon of triangle ABC. Thus, G does not belong to T .

When G is not inside DEF or on the boundary of DEF, we may assume that G is inside AEF. Let A', B', C' be the reflected point of A, B, C about G respectively. Thus, triangle $A'B'C'$ is center-symmetric with triangle ABC about point G. Since any chord l of triangle ABC with midpoint G is also a chord of triangle $A'B'C'$, the end points of l must be the intersection points of triangle $A'B'C'$ and triangle ABC. As triangle $A'B'C'$ and triangle ABC have only two intersection points J, M, thus, inside the triangle ABC there is only one chord JM whose midpoint is G. Thus, there does not exist an inscribed parallelogram of Ω with its center at G, namely, G does not belong to T .

To sum up, T refers to triangle DEF except for points D, E, F.

Thus, when Ω is a triangle, we have

$$S(T) = S(DEF) = \frac{1}{4}S(\Omega)$$

■

4 Proof of the right inequality

4.1 Preparations

Here are some preparations for the proof.

For convex Ω and an arbitrary ε , there exist n division points $A_1, A_2 \dots A_n$ arranged successively on $\partial\Omega$ and support lines $l_i, i = 1, 2 \dots n, l_{n+1} = l_1$ passing through A_i such that $\widehat{A_i A_{i+1}} \leq \varepsilon$, and the angle between l_i and l_{i+1} is not larger than $2\pi\varepsilon$ (A_i, A_{i+1} may be the same point). Besides, except for arc $\widehat{A_n A_1}$, if $\widehat{A_i A_{i+1}} \leq \varepsilon$ then the angle between l_i and l_{i+1} is equals to $2\pi\varepsilon$.

Then, we could have an estimation for the range of n :

$$\begin{aligned} n &\leq \sum_{\widehat{A_i A_{i+1}} = \varepsilon} 1 + \sum_{\angle l_i l_{i+1} = 2\pi\varepsilon} 1 + 1 \\ &\leq \frac{2c + 1}{\varepsilon} \end{aligned}$$

Among which $\angle l_i l_{i+1}$ refers to the angle between l_i and l_{i+1} , c refers to the perimeter of the convex Ω . For a given convex Ω , there exists ε_0 . When $0 \leq \varepsilon \leq \varepsilon_0$, for any division of $\partial\Omega$ that satisfies the conditions above, we have $\widehat{A_i A_{i+1}}$ (inferior arc) does not include any pseudo-diameter of Ω . Besides, for an arbitrary pseudo-diameter MN of Ω , we have $MN \geq \varepsilon$. (Otherwise, Ω has no area, which is in contradiction.)

For any given ε that satisfies $0 \leq \varepsilon \leq \varepsilon_0$, note an arbitrary group of division points $A_1, A_2 \dots A_n$ on $\partial\Omega$ that satisfies the conditions above and the corresponding support line $l_i, i = 1, 2 \dots n$ passing through A_i as a ε -**division** of $\partial\Omega$.

We take a ε -division of Ω . For arc $\widehat{A_i A_{i+1}}$ and arc $\widehat{A_j A_{j+1}}$ ($i, j \in 1, 2 \dots n, A_{n+1} = A_1$. Here we consider that every arc include its endpoint, and some arcs might degenerate to a single point.) Consider the locus of the midpoints of the segments joining two points respectively from arc $\widehat{A_i A_{i+1}}$ and arc $\widehat{A_j A_{j+1}}$, and denote it by $T(A_i A_{i+1}, A_j A_{j+1})$.

If there do not exist $M \in \widehat{A_i A_{i+1}}, N \in \widehat{A_j A_{j+1}}$ that make MN be a pseudo-diameter of Ω , then we call $T(A_i A_{i+1}, A_j A_{j+1})$ as RegionI.

If there exist $M \in \widehat{A_i A_{i+1}}, N \in \widehat{A_j A_{j+1}}$ that make MN be a pseudo-diameter of Ω , then we call $T(A_i A_{i+1}, A_j A_{j+1})$ as RegionII.

4.2 RegionI

RegionI takes the majority of $T(A_i A_{i+1}, A_j A_{j+1})$. In addition, its shape is quite regular, and its area is easy to be figured out.

In the following part, we will discuss about the shape of RegionI.

For RegionI $T(AB, CD)$ (among which exist $A_i = A, A_{i+1} = B, A_j = C, A_{j+1} = D$), we have

Lemma 4.1 *For any arbitrary point P inside RegionI $T(AB, CD)$, there is the only pair of $M \in \widehat{AB}, N \in \widehat{CD}$ such that P is the midpoint of MN .*

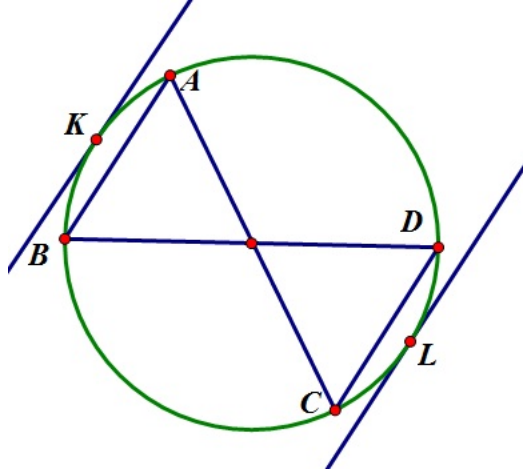


Figure 6

Proof. Proof by contradiction. See Figure 6. If there exist $M, R \in \widehat{AB}$, $N, S \in \widehat{CD}$ that make P the midpoint of MN, RS , then $MR \parallel NS$. Therefore, there exist a point K on \widehat{MR} and a point L on \widehat{NS} such that there exist support lines of Ω parallel to MR passing through K, L respectively. This is contradictory to that $T(AB, CD)$ is Region I. ■

Lemma 4.2 For Region I $T(AB, CD)$, denote the midpoint of AC, AD, BC, BD by E, F, G, H respectively.

Let A, B be the homothetic centers, $\frac{1}{2}$ be the homothetic ratio, construct the image of arc \widehat{CD} , namely, arc $\widehat{EG}, \widehat{FH}$;

Let C, D be the homothetic centers, $\frac{1}{2}$ be the homothetic ratio, construct the image of arc \widehat{AB} , namely, arc $\widehat{EF}, \widehat{GH}$.

Thus, arc $\widehat{EG}, \widehat{GH}, \widehat{HF}, \widehat{FE}$ together form a closed curve C that does not intersect inward. Then C and the interior of C refers to $T(AB, CD)$.

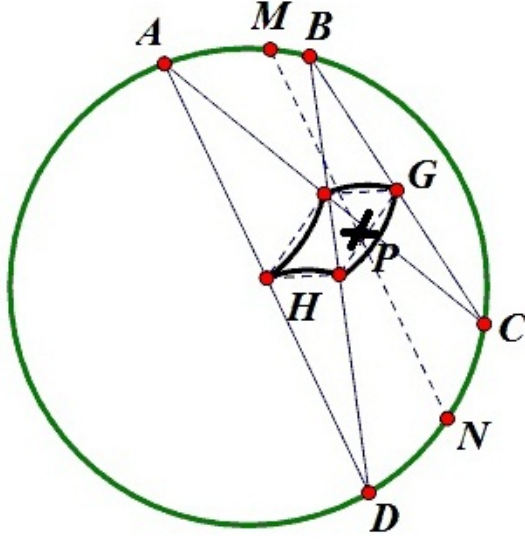


Figure 7

Proof. According to Lemma 4.1, C does not intersect inward.

As $T(AB, CD)$ is a closed set, we only have to prove that the boundary of $T(AB, CD)$ is C . Let P be a point inside $T(AB, CD)$, then there exist $M \in AB, N \in CD$ that make P the midpoint of MN (See Figure 7).

Since $T(AB, CD)$ is Region I, as shown in Figure 7, two arbitrary support lines of Ω passing through M, N are not parallel.

If M, N are not the end points A, B, C, D of $\widehat{AB}, \widehat{CD}$, according to Lemma 4.2, P does not belong to C . Besides, when M', N' are in a certain neighborhood of M, N , the locus of P' is the neighborhood of P , and every point inside the neighborhood belongs to $T(AB, CD)$.

If there is at least one point between M, N that refers to the end points A, B, C, D of $\widehat{AB}, \widehat{CD}$, suppose $M = A$, then $P \in C$.

For every neighborhood of P on Ω , there must be points inside the neighborhood that do not belong to $T(AB, CD)$.

To sum up, the boundary of $T(AB, CD)$ is C , namely, C and the interior of C together form $T(AB, CD)$. ■

For the area of Region I, we have

Lemma 4.3 For region I $T(AB, CD)$, its area is $S(AB, CD)$, then,

$$S(AB, CD) = \frac{1}{4} |AB| \times |CD| \sin \angle(\overrightarrow{AB}, \overrightarrow{CD})$$

Proof. When $T(AB, CD)$ forms the RegionI, the area of the figure enclosed by arc \widehat{EG} and segment EG is equal to that of the figure enclosed by arc \widehat{FH} and segment FH , and the area of the figure enclosed by arc \widehat{EF} and segment EF is equal to that of the figure enclosed by arc \widehat{GH} and segment GH . Thus, the area of $T(AB, CD)$ is equal to that of parallelogram $EFHG$.

As the area of parallelogram $EFHG$ is

$$|EF| \times |EG| \times \sin \angle(\overrightarrow{EF}, \overrightarrow{EG})$$

and EF , EG are the mid-parallel lines of triangle ADC , ADB , thus, $EF \parallel AB$, $EG \parallel CD$, $EF = \frac{1}{2}AB$, $EG = \frac{1}{2}CD$.

Namely,

$$S_{EFGH} = \frac{1}{4} |AB| \times |CD| \times \sin \angle(\overrightarrow{AB}, \overrightarrow{CD})$$

Namely,

$$S(AB, CD) = \frac{1}{4} |AB| \times |CD| \times \sin \angle(\overrightarrow{AB}, \overrightarrow{CD})$$

■

For a ε - division of Ω , let I_1 be the set consisting of all number pairs (i, j) ($i, j = 1, 2, 3 \dots n$) such that $T(A_i A_{i+1}, A_j A_{j+1})$ is RegionI, S_{I_1} is the sum of the area of RegionI. Denote by d_i the distance between two support lines of convex Ω parallel to $A_i A_{i+1}$. Thus, we have,

Lemma 4.4

$$\frac{1}{4} \sum_{i=1}^n A_i A_{i+1} \times d_i - \frac{1}{2} c\varepsilon \leq S_{I_1} \leq \frac{1}{4} \sum_{i=1}^n A_i A_{i+1} \times d_i$$

Proof. Let the length of arc $\widehat{A_i A_{i+1}}$ be x_i . Thus, $x_i \leq \varepsilon$

For every $\widehat{A_i A_{i+1}}$, there exists a certain k_i such that

$$d(A_i A_{i+1}, A_{i+2}) \leq d(A_i A_{i+1}, A_{i+2}) \leq \dots d(A_i A_{i+1}, A_{k_i})$$

$$d(A_i A_{i+1}, A_{k_i}) \geq d(A_i A_{i+1}, A_{k_i+1}) \leq \dots d(A_i A_{i+1}, A_{i-1})$$

(the suffix is figured as the remainder of module n) and

$$d(A_i A_{i+1}, A_{k_i}) \leq d_i \leq d(A_i A_{i+1}, A_{k_i}) + 2\varepsilon$$

According to Lemma 4.3, we have

$$\begin{aligned}
& S_{I_1} \\
&= \frac{1}{8} \sum_{i=1}^n A_i A_{i+1} \sum_{j=1}^n A_j A_{j+1} \sin \angle(A_i A_{i+1}, A_j A_{j+1}) \\
&= \frac{1}{8} \sum_{i=1}^n A_i A_{i+1} \sum_{j=1}^n d(A_i A_{i+1}, A_j) \\
&\geq \frac{1}{8} \sum_{i=1}^n A_i A_{i+1} \times (2d_i - 4\varepsilon) \\
&= \frac{1}{4} \sum_{i=1}^n A_i A_{i+1} \times d_i - \frac{1}{2} c\varepsilon
\end{aligned}$$

and $S_{I_1} \leq \frac{1}{4} \sum_{i=1}^n A_i A_{i+1} \times d_i$

By combining the two equations above, we have

$$\frac{1}{4} \sum_{i=1}^n A_i A_{i+1} \times d_i - \frac{1}{2} c\varepsilon \leq S_{I_1} \leq \frac{1}{4} \sum_{i=1}^n A_i A_{i+1} \times d_i$$

■

4.3 RegionII

RegionII takes the minority of $T(A_i A_{i+1}, A_j A_{j+1})$. In addition, its shape is quite irregular, and its area is hard to figure out. Thus, we have the following estimation for its area:

Lemma 4.5 *For RegionII $T(AB, CD)$ (among which there are $A_i = A, A_{i+1} = B, A_j = C, A_{j+1} = D$), then there exist a figure with its area not larger than $2\pi \times \varepsilon^3$ covering $T(AB, CD)$.*

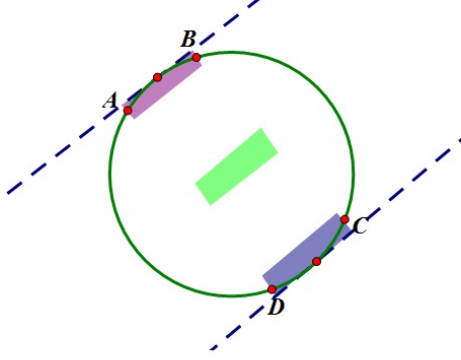


Figure 8

Proof. As shown in Figure 8, note RegionII as $T(AB, CD)$ and suppose $l_A = l_i, l_B = l_{i+1}, l_C = l_j, l_D = l_{j+1}$. According to the definition, there exist $M \in \widehat{AB}$, $N \in \widehat{CD}$ such that there are two support lines l, l' passing through M, N that satisfy $l // l'$.

If A, M, B are the same point or C, N, D are the same point, according to the definition of $T(AB, CD)$, $T(AB, CD)$ is an arc. Thus, there exists a figure with its area not larger than $2\pi \times \varepsilon^3$ covering $T(AB, CD)$.

If neither A, M, B nor C, N, D are not the same point, then since the angle between l_A and l_B is not larger than $2\pi\varepsilon$, we have the angle between l_A and l , the angle between l and l_B are neither larger than $2\pi\varepsilon$. Similarly, the angle between l_C and l' and the angle between l' and l_D are neither larger than $2\pi\varepsilon$.

Thus, there exist two rectangles E_1, E_2 . Either of their lengths is ε , and either of their widths is $\varepsilon \sin 2\pi\varepsilon$. Besides, one length of E_1, E_2 lies respectively on l, l' . Consequently, E_1, E_2 respectively covers the arc $\widehat{AB}, \widehat{CD}$. We respectively take one point inside E_1, E_2 , and the midpoint of the segment joining the two points are in a rectangular with the same size as E_1, E_2 .

Denote this rectangle by E_3 . Thus, we have $T(AB, CD) \subseteq E_3$. Then

$$S(E_3) = \varepsilon^2 \sin 2\pi\varepsilon \leq 2\pi\varepsilon^3$$

Thus, this rectangle with its area not larger than $2\pi \times \varepsilon^3$ can cover $T(AB, CD)$. ■

For a ε -division of convex Ω , note I_2 as the sets that make S_{I_2} as the sum of the areas of all RegionII.

Lemma 4.6 *There exists a constant W such that for any arbitrary ε division of Ω , there exists a figure with its area not larger than $W\varepsilon$ covering I_2 .*

Proof. According to the estimation of n in the preparation, $n \leq \frac{2c+1}{\varepsilon}$,

$$\begin{aligned} S_{I_2} &= \sum_{(i,j) \in I_2} S(A_i A_{i+1}, A_j A_{j+1}) \\ &\leq \sum_{(i,j) \in I_2} 2\pi \times \varepsilon^3 \\ &\leq n^2 \times 2\pi \times \varepsilon^3 \\ &\leq (2c+1)^2 \times 2\pi \varepsilon \end{aligned}$$

Let $W = 2\pi(2c+1)^2$ and there exists a figure with its area not larger than $W\varepsilon$ covering I_2 . ■

4.4 Two Propositions

Here are two further conclusions about convex Ω that are necessary for the proof:

Proposition 4.7 *For convex Ω with its barycenter at G and an arbitrary $\theta \in [0, 2\pi]$, let $d_G(\theta)$ be the distance from G to a support line of Ω with the inclination angle of θ , and $h(\theta)$ be the distance between two support lines of Ω whose inclination angles are both θ . Then, $3d_G(\theta) \geq h(\theta)$, the equality holds only if Ω is a triangle.*

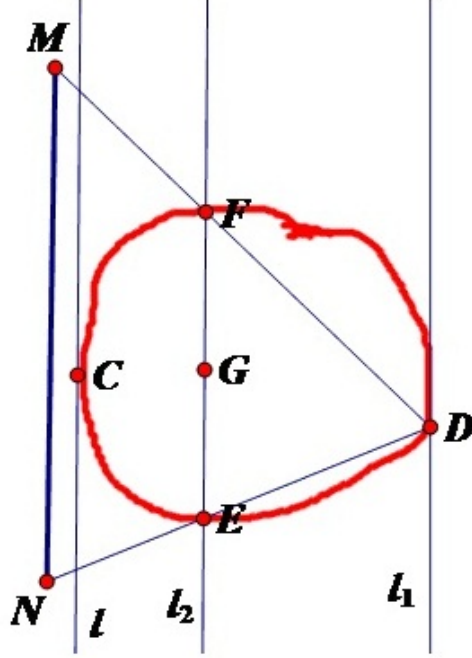


Figure 9

Proof. As shown in Figure 9, let l be a support line of Ω passing through C (point A , B are not shown in Figure 9), l_1 be another support line of Ω parallel to l . l_1 intersects Ω at D . Construct l_2 parallel to l passing through G . l_2 divides Ω into two parts noted as Ω_1 and Ω_2 , among which $C \in \Omega_1$ $D \in \Omega_2$. According to the definition, $h(\theta)$ represents the distance from l to l_1 , and $d_G(\theta)$ represents the distance from l to l_2 .

As Ω is a convex, G is inside Ω . Let l_2 and Ω intersect at points E and F . Take point M, N respectively on ray DE , DF such that

$$DM = \frac{3}{2}DF, DN = \frac{3}{2}DE$$

Thus, $MN \parallel l_2$.

Let d_1 be the distance from l to l_1 , d_2 be the distance from l to l_2 , and d_3 be the distance from MN to l_2 . Thus,

$$2d_3 = d_2 - d_1$$

According to the definition of M,N, l_2 passes through the barycenter of triangle DMN.

Proof by contradiction. Suppose $3d_G(\theta) < h(\theta)$, namely, $3d_2 < d_1$, thus, $d_3 > d_2$. Since Ω is a convex, thus $\Omega_1 \subseteq$ trapezoid EFMN, triangle DEF $\subseteq \Omega_2$. Place Ω into the plane rectangular coordinate system. Let (0,0) be the central coordinate. According to the properties of barycenter, we have

$$\begin{aligned}\iint_{EFMN} d(l_2, (x, y)) dx dy &= \iint_{DEF} d(l_2, (x, y)) dx dy \\ \iint_{\Omega_1} d(l_2, (x, y)) dx dy &= \iint_{\Omega_2} d(l_2, (x, y)) dx dy\end{aligned}$$

Let the two side dot multiply the unit normal vector of l_2 , we have

$$\begin{aligned}\iint_{EFMN} d(l_2, (x, y)) dx dy &= \iint_{DEF} d(l_2, (x, y)) dx dy \\ \iint_{\Omega_1} d(l_2, (x, y)) dx dy &= \iint_{\Omega_2} d(l_2, (x, y)) dx dy\end{aligned}$$

Since $d(l_2, (x, y))$ is non-negative, thus,

$$\begin{aligned}& \iint_{EFMN} d(l_2, (x, y)) dx dy \\ &= \iint_{DEF} d(l_2, (x, y)) dx dy \\ &\leq \iint_{\Omega_2} d(l_2, (x, y)) dx dy \\ &= \iint_{\Omega_1} d(l_2, (x, y)) dx dy \\ &\leq \iint_{EFMN} d(l_2, (x, y)) dx dy\end{aligned}$$

This illustrates that the equalities should hold for all the inequalities above. This requires that $d(l_2, (x, y)) = 0$ should be true for all $(x, y) \in EFMN/\Omega_1$. That is, the point set $EFMN/\Omega_1$ is on l_2 . This requires Ω_1 =trapezoid EFMN, which is in contradiction with $d_3 > d_2$.

Thus, $3d_G(\theta) \geq h(\theta)$ is true for every angle θ of $\partial\Omega$. When there exists a certain angle θ such that $3d_G(\theta) = h(\theta)$, similarly, we can infer that Ω_1 =trapezoid EFMN. In like manner, we can infer that Ω_2 =triangle DEF. That is, Ω is a triangle. ■

Proposition 4.8 *For a point A inside convex Ω , if A can be exactly represented as the midpoint of two chords of Ω , then there exist two support lines of Ω that are symmetrical about A .*

Proof. Let Λ be the point set consisting of the points that can be exactly represented as the midpoint of two chords of Ω , Λ' be the point set consisting of the midpoints of pseudo-diameters of Ω

Place convex Ω into the plane rectangular coordinate system. For an arbitrary point O inside Λ and angle θ , when angle $\theta \in [0, \pi)$, let \overrightarrow{MN} be the chord passing through O with the inclination angle θ . Define function $f(\theta) = |MO| - |NO|$, $\theta \in [0, \pi)$ and $f(\theta + \pi) = -f(\theta)$, ($\theta \in (-\infty, 0) \cup [\pi, +\infty)$). Thus, f is a continuous function.

Since $O \in \Lambda$, there exists angle θ_0 such that $f(\theta_0) \neq 0$. We assume that $f(\theta_0) > 0$, thus, there exactly exist two $\theta \in (\theta_0, \theta_0 + \pi)$ such that $f(\theta) = 0$. Denote them by θ_1, θ_2 .

According to the continuity of f , we have $f(\theta) > 0$ ($\theta \in (\theta_0, \theta_1)$), $f(\theta) < 0$ ($\theta \in (\theta_2, \theta_0 + \pi)$), and the signs of f remain constant in the interval (θ_1, θ_2) . Suppose $f(\theta) > 0$ ($\theta \in (\theta_1, \theta_2)$), according to the continuity of f , there exists a certain deleted neighborhood of θ_1 noted as $\dot{U}(\theta_1, \eta)$ (namely, the interval $(\theta_1 - \eta, \theta_1) \cup (\theta_1, \theta_1 + \eta)$) such that

$$f(\theta) > 0, \forall \theta \in \dot{U}(\theta_1, \eta)$$

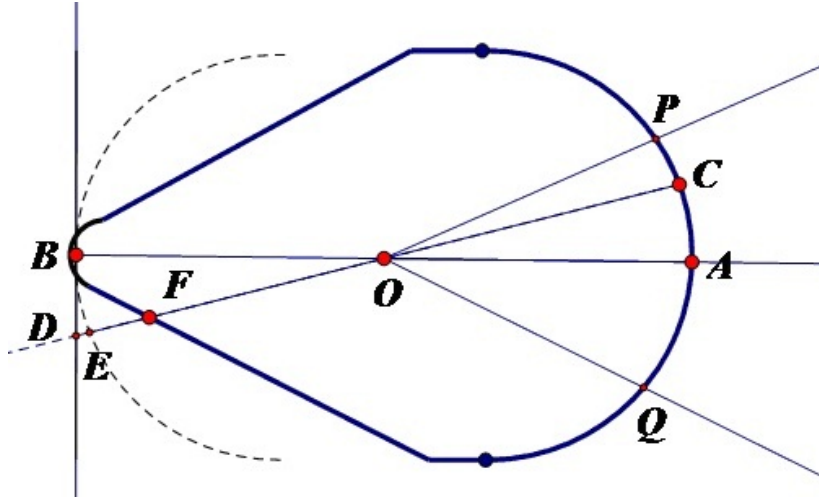


Figure 10

As Figure 10 shows, let AB be the directed chord passing through O with the inclination angle θ_1 . $\angle POA = \angle AOQ = \eta$

Construct support line l of Ω passing through A. Construct support line l' of Ω passing through B. l' and l are Symmetric about point O.

Construct arc $\widehat{P'Q'}$ such that $\widehat{P'Q'}$ and \widehat{PQ} are symmetric about point O. Thus, l' is a support line of $\widehat{P'Q'}$.

For point C on arc \widehat{PQ} , extend CO so as to intersect the boundary of Ω at F. Then the inclination angle of the directed chord CF belongs to $\dot{U}(\theta_1, \eta)$. Thus, $CO > OF$. Take point E on the extended line of CO such that $CO = OE$. Thus, $OE > OF$; while E is on $\widehat{P'Q'}$.

Extend CO to intersect l' at D. Thus, $OD > OE > OF$. Since the selection of C on arc \widehat{PQ} is arbitrary, according to the definition, l' is also a support line of Ω . Also, as l' passes through B, O is the midpoint of the two support lines of Ω parallel to l . Thus, $O \in \Lambda'$. ■

Corollary 4.9 *In convex Ω , under the former symbols, $m^*(\Lambda) = 0$, among which $m^*(E)$ is the exterior measure of E .*

Proof. According to Proposition 4.8, $\Lambda \subseteq \Lambda'$, we only have to prove that $S(\Lambda') = 0$.

For a ε -division of convex Ω , according to the definition of RegionII, Λ' is included in the union of all RegionII. Thus, according to Lemma 4.11, we have

$$S(\Lambda') \leq S_{I_2} \leq W\varepsilon$$

W is a constant that has nothing to do with ε . Let $\varepsilon \rightarrow 0$, then $m^*(\Lambda') = 0$ ■

4.5 Proof of the theorem

For convex Ω , let T_i be the set consisting of points that can be represented as the midpoint of i chords of Ω . $i = 1, 2, 3 \dots$. Thus, we have $T_1 = \Omega \setminus \partial\Omega, T_2 \setminus T_3 = \Lambda$

Firstly, we shall prove the existence of the area of T_2, T_3 . Hence, we come up with the following proposition:

Proposition 4.10 *For convex Ω and point set T_2, T_3 , T_2 and T_3 are measurable.*

Proof. For every $\varepsilon = \frac{1}{N}$, take a ε -division of Ω . For every $\frac{1}{k}$ ($k = N_0, N_0 + 1, \dots, N_0 \varepsilon_0 > 1$) division, (The definition of ε_0 is in 4.1) let D_{k_i} be all Region I, D'_{k_i} be all Region II. Note

$$E_k = \bigcup_{i < j} (D_{k_i} \cap D_{k_j})$$

$$E'_l = \bigcup_{k=N_0}^l E_k (l = N_0, N_0 + 1, \dots)$$

$$F_k = \bigcup_i D'_{k_i}$$

Thus, we have

$$E_k \subseteq E'_k \subseteq T_2 \subseteq E_k \cup F_k \subseteq E'_k \cup F_k \quad (1)$$

And $E'_{N_0} \subseteq E'_{N_0+1} \subseteq E'_{N_0+2} \dots$

Since E_k, E'_k is the union of a finite amount of curved quadrilaterals, and every curved quadrilateral can be measured, thus, E_k, E'_k are all measurable. And then, since

$$\lim_{k \rightarrow +\infty} E'_k$$

is a Borel set, according to the properties of Borel set,

$$\lim_{k \rightarrow +\infty} E'_k$$

is measurable. Let $m^*(E)$ be the exterior measure of E , then we have $\lim_{k \rightarrow \infty} m^*(F_k) = 0$. This illustrates that $T_2 \setminus \lim_{k \rightarrow +\infty} E'_k$ is a zero measure set. Therefore T_2 is measurable.

According to the corollary 4.9, we can draw the conclusion that T_3 is measurable. ■

Lemma 4.11 For a ε -division of convex Ω ,

$$\lim_{\varepsilon \rightarrow 0} \sum_{i=1}^n A_i A_{i+1} \times d_i \geq 4(S(T_1) + S(T_2) + S(T_3))$$

Proof. Take a ε -division of Ω . For every point P inside any RegionI, according to Lemma 4.1, there exists only one chord of convex Ω whose midpoint is P. If P appears in k RegionI, then, $P \in T_k$.

Therefore, for every point P in $T_k (k = 1, 2, 3)$, P appears either in a certain RegionII, or in k RegionI. According to Lemma 4.6, let T_{II} be the figure with an area not larger than $W\varepsilon$ that can cover all RegionII, $D_i, i = 1, 2, \dots$ be all the RegionI, $T'(2) = \bigcup_{i < j} (D_i \cap D_j), T'_3 = \bigcup_{i < j < k} (D_i \cap D_j \cup D_k)$

Thus, $T_1 \subseteq (\bigcap D_i) \cup (T_{II}), T_2 \subseteq (T'_2 \cup T_{II}), T_3 \subseteq (T'_3 \cup T_{II})$

According to including-excluding principle,

$$S_{I_1} \geq S(T'_1) + S(T'_2) + S(T'_3)$$

As all sets above are measurable, thus, we have,

$$S_{I_1} + 3S(II) \geq S(T_1) + S(T_2) + S(T_3)$$

According to Lemma 4.4, let $\varepsilon \rightarrow 0$, we have proved that

$$\lim_{\varepsilon \rightarrow 0} \sum_{i=1}^n A_i A_{i+1} \times d_i \geq 4(S(T_1) + S(T_2) + S(T_3))$$

■

If point O inside convex Ω is the center of an inscribed central-symmetric convex polygon of convex Ω , then O must be the center of a certain inscribed parallelogram of convex Ω . And the converse proposition is also true. Thus, we only need to study the centers of the inscribed parallelograms of convex Ω .

According to the corollary of Proposition 4.8, we have $S(T_2) = S(T_3)$

Let θ_i be the inclination angle of the support line of Ω parallel to $\widehat{A_i A_{i+1}}$.

According to Lemma 4.11, Proposition 4.7 and the corollary 4.9, as shown in Figure 11, we have

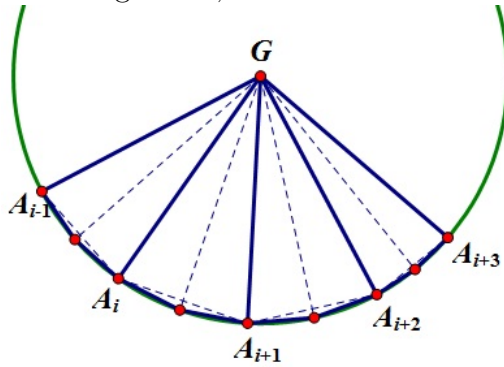


Figure 11

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^n |A_i A_{i+1}| \times d_i \\
& \leq \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^n |A_i A_{i+1}| \times 3d_G(\theta_i) \\
& \leq 6S(\Omega)
\end{aligned}$$

Namely,

$$4(S(T_1) + S(T_2) + S(T_3)) \leq 4S(\Omega) + 8S(T) \leq 6S(\Omega)$$

Namely,

$$S(T) \leq \frac{1}{4}S(\Omega)$$

When the equality holds, there exists at least one angle θ such that $3d_G(\theta) = h(\theta)$. This illustrates that Ω is a triangle.

5 Supplementary

Mostly, point set Λ (which can be exactly represented as the midpoint of two chords) is a curve whose length is measurable, also the boundary ∂T of point set T . But this conclusion still needs to be proved.

In addition, if we regard triangle as the least central-symmetric figure, the ratio of the area of point set T to that of point set Ω of can precisely describe the extent of central symmetry of convex Ω .

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