Study on Inscribed Ellipse of Triangle

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Abstract

In this paper, the authors deal with the properties of inscribed ellipse of triangle, using tools of projective transformation, analytical geometry and complex plane, and lead to several conclusions on the center, foci and major/minor axes, including the locus of the center of inscribed ellipse, the maximum sum of major axis and minor axis, and several other geometric inequalities. To some extent, this paper enriches the knowledge about the inscribed ellipse of triangle. And by using algebraic methods the authors reveal some beautiful geometric characteristics.

Key words: Geometry, Triangle, Inscribed Ellipse.

Contents

Ⅰ Introduction

There have been sufficient studies on the incircle of triangle. From Euler's Formula ($R^2=d^2+2Rr$) to Feuerbach's Theorem (the nine-point circle of any triangle is tangent internally to the incircle and externally to the three excircles), there have been various conclusions. However, far less has been done with ellipses inscribed in a triangle and little is known about it. In this paper the authors tried to make up this lack of knowledge by exploring triangle's inscribed ellipse.

To be convenient, a circle shall be treated as a special case of an ellipse, but a line segment shall not. Also, when complex numbers and the complex plane are mentioned in this paper, it is always assumed that the origin lies at the circumcenter of ΔABC.

Let's inspect the meaning of an ellipse inscribed in a triangle as the first step.

Given two points P and Q inside ΔABC, and an ellipse with foci P and Q is tangent to the sides BC, CA and AB of ΔABC at points D, E and F respectively. (Refer to Fig. 1) For a random point X on the plane, with an ellipse's essential nature we know the following facts, $PX + QX > 2a$, if X lies outside the ellipse, $PX + QX = 2a$, if X lies on the ellipse, or $PX + QX < 2a$, if X lies inside the ellipse, where and thereinafter "a" stands for the semi-major axis length of the ellipse.

Thus we know that out of all points on side BC, D is the one with the shortest sum distance to P and Q. It is also true for point E with respect to side CA, and F with respect to side AB. Furthermore, we have

$PD + QD = PE + QE = PF + QF$

On the other hand, for two points P and Q inside ΔABC, if their minimum sum distances to the points on side BC, CA and AB equal to each other, then there exists an ellipse inscribed in ΔABC with P, Q as its focal points.

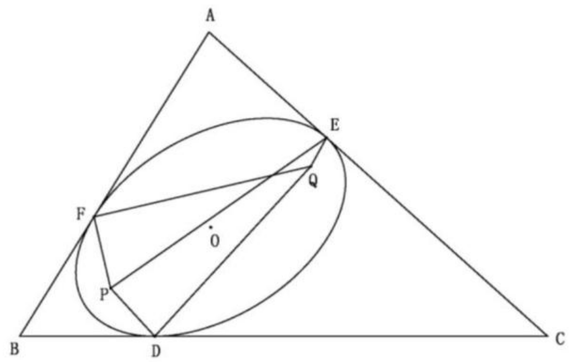


Fig. 1

The above statements lead to the following lemma.

[Lemma 1] For any given point P inside ΔABC, there exists an ellipse tangent to ΔABC's three sides with P as one of its foci; moreover, the other focal point of the ellipse is the isogonal conjugate point of P inside the triangle. (For more about isogonal conjugate point please refer to Reference [1])

To prove it we just need to demonstrate that there is another point Q inside ΔABC suffices that the minimum sum distances of P and Q to the points on the three sides of ΔABC equal to each other.

Denote by P_A, P_B and P_C the symmetry points of point P with respect to BC, CA, and AB respectively; denote by Q the circumcenter of $\Delta P_AP_BP_C$. (Refer to Fig. 2)

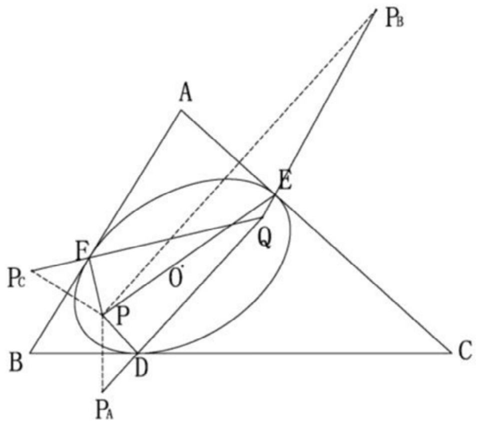


Fig. 2

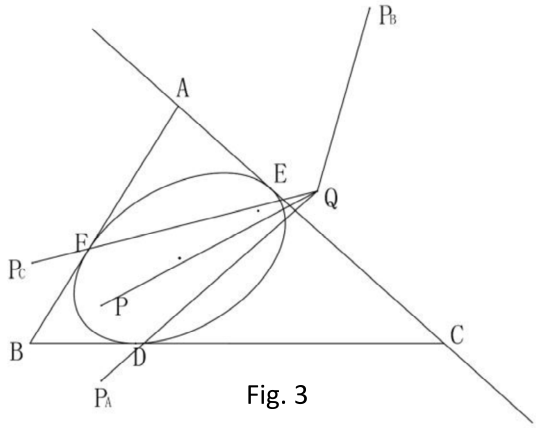


Fig. 3

Firstly, point Q must be inside ΔABC. As a matter of fact, if Q is outside ΔABC, (Refer to Fig. 3) we may assume that Q lies at the other side of line AC as to P (and also point B, P_B and P_C). Since P and P_B are symmetrical with respect to AC, we know $QP_B<QP$. Also, point Q must be either at the other side of BC as to P_A or at the other side of AB as to P_C, or both; assuming for instance that Q is at the other side of BC as to P_A, we know that Q locates at the same side of BC as to P, and thus $QP<QP_A$. With the above two inequalities we attain $QP_B<PQ<QP_A$, which contradicts the fact that Q is the circumcenter of $\Delta P_AP_BP_C$. The assumption of Q lies on one of the three sides of ΔABC shall also lead to contradiction. Thus it is proven that Q must be inside ΔABC.

Denote by points D, E, F the intersections of line QP_A and BC, QP_B and CA, QP_C and AB, respectively. (Refer to Fig. 4) Then D, E, F are the points on line BC, CA and AB respectively with the minimum sum distance to P and Q, and

$$PD + QD = P_AD + QD = P_AQ = P_BQ = PE + QE = PF + QF$$

Thus it has been proven that there exists an ellipse internally tangent to ΔABC's three sides at point D, E, F with P, Q being its focal points.

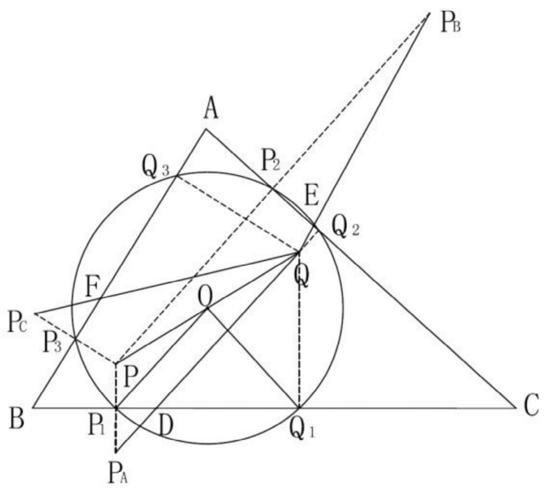


Fig. 4

Denote by O the midpoint of line segment PQ, and then O is the center point of the ellipse mentioned above. Draw perpendiculars from P and Q to line BC, CA and AB respectively with feet at P_1, P_2, P_3 and Q_1, Q_2, Q_3. Noticing that points P_1, P_2, P_3 and O are midpoints of line segments PP_A, PP_B, PP_C and PQ, we know that under a homothetic transformation with center at P and ratio of 1/2, P_A, P_B, P_C and Q shall be converted to P_1, P_2, P_3 and O respectively. We also know that O is the circumcenter of $\Delta P_1P_2P_3$ from the fact that Q is the circumcenter of $\Delta P_AP_BP_C$. Since PP_1Q_1Q is actually a trapezoid (or a rectangle), and O is the midpoint of its oblique side, we know $OP_1=OQ_1$, or in other words Q_1 is also on the circumcircle of $\Delta P_1P_2P_3$. Similarly Q_2 and Q_3 are on the same circle. Hence all the six points P_1, P_2, P_3, Q_1, Q_2 and Q_3 are on a circle centered at O.

Since $QQ_1\perp CQ_1$, $QQ_2\perp CQ_2$, we know $\angle QCQ_2=90°-\angle CQQ_2$; and the four points C, Q_1, Q, and Q_2 are concyclic, then $\angle CQQ_2=\angle CQ_1Q_2$, and thus $\angle QCQ_2=90°-\angle CQ_1Q_2$. (Refer to Fig. 5) Likewise, $\angle PCP_1=90°-\angle CP_2P_1$. On the other hand, since points P_1, Q_1, Q_2, P_2 are concyclic, we have $\angle CQ_1Q_2=\angle CP_2P_1$. Hence

$$\angle QCQ_2=90°-\angle CQ_1Q_2=90°-\angle CP_2P_1=\angle PCP_1$$

which means $\angle PCB=\angle QCA$. For the same reason we get $\angle PBC=\angle QBA$ and $\angle PAB=\angle QAC$. This is equivalent to that P and Q are isogonal conjugate points. So if an ellipse is inscribed in ΔABC, its two focal points are isogonal conjugate points; on the other hand, for any given point inside ΔABC there exists an ellipse internally tangent to the triangle with its foci at this given point and its isogonal conjugate point.

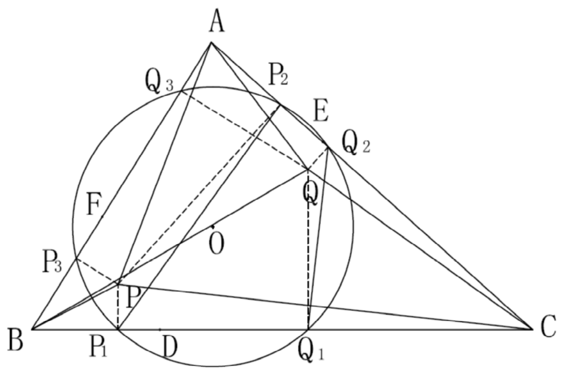


Fig. 5

So far, Lemma 1 has been proven.

Ⅱ Center Point of Ellipse Inscribed in Triangle

With the proof of Lemma 1 we have already discovered some characteristics of the center point O of a triangle's inscribed ellipse. In this chapter we will investigate the locus of point O. We discovered that the set of all possible points O equals to the area inside the medial triangle of ΔABC. In other words we have the following theorem.

[Theorem 1] There exists an ellipse inscribed in ΔABC centered at point O iff point O is inside the medial triangle of ΔABC.

In the proof of this Theorem parallel projection is used as a tool, which is common when dealing with problems regarding an ellipse [2]. Parallel projection has some useful properties, such as the tangency of a curve and a line remains, and so do the ratio of line segments and the relative positions of points.

The proof consists of two parts, necessity and sufficiency.

1. Necessity

Firstly, we will prove that for each ellipse inscribed in ΔABC, its center shall be inside ΔABC's medial triangle $A_0B_0C_0$ (Refer to Fig. 6 where A_0, B_0, C_0 are the midpoints of BC, CA, AB respectively.)

It is well known that for an ellipse there is exactly one parallel projection transforming this ellipse into a circle. Denote ΔA'B'C' as the image of ΔABC under this transformation, and I the image of point O. Since the tangency between a curve and a line remains under the transformation, the ellipse inscribed in ΔABC is transformed to the incircle of ΔA'B'C' centered at point I.

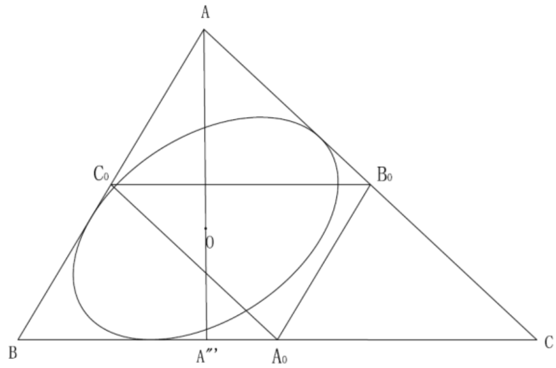


Fig. 6

Denote by A'' the intersection of A'I and B'C'. We will prove A'I>A''I. (Refer to Fig. 7)

Since B'I bisects ∠A'B'A'', we know $\frac{A'I}{A''I}=\frac{A'B'}{A''B'}$; furthermore, A'A'' bisects ∠B'A'C', hence $\frac{A'B'}{A'C'}=\frac{A''B'}{A''C'}$, or $A''B'=A'B'\cdot\frac{B'C'}{A'B'+A'C'}<A'B'$, hence $A''I<A'I$.

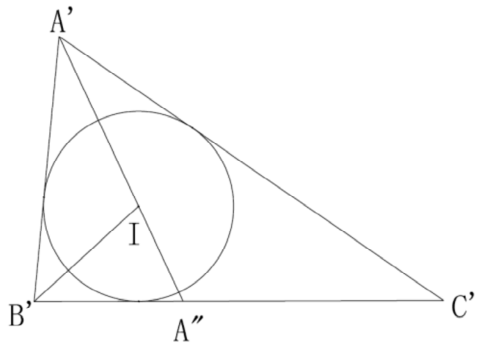


Fig. 7

Now come back to the original drawing. Suppose AO intersects BC at A''', then from $\frac{AO}{A'''O}=\frac{A'I}{A''I}$ we know AO>A'''O, which indicates that the distance from point O to line BC is less than half the distance from point A to line BC, or in other words that point O and point A_0 are at the same side of median B_0C_0. Similarly, points O and B_0 are at the same side of line C_0A_0, and points O and C_0 are at the same side of line A_0B_0. Hence point O is inside $ΔA_0B_0C_0$. Thus we've proven that for any ellipse inscribed in ΔABC, its center lies inside ΔABC's medial triangle.

2. Sufficiency

Next we will prove that, for any given point O inside $A_0B_0C_0$, the medial triangle of ΔABC, there exists an ellipse inscribed in ΔABC and centered at O.

Our approach is to find out a projection and a triangle; with this projection the triangle we find transforms to the given ΔABC, and the triangle's incenter becomes the given point O which is inside $ΔA_0B_0C_0$. Here we need another lemma.

[Lemma 2] For any given ΔABC and ΔXYZ, there exists one triangle similar to ΔXYZ and one parallel projection transformation under which the image of this triangle is exactly ΔABC.

The following is the proof of Lemma 2.

Denote by α the plane on which ΔABC exists. Consider perpendiculars ℓ_1 and ℓ_2 of α through point A and B respectively. We will find points Y_0 and Z_0 on ℓ_1 and ℓ_2 respectively, so that $\frac{AY_0}{AZ_0}=\frac{XY}{XZ}$ and $\angle Y_0AZ_0=\angle YXZ$. This is done with analytic geometry.

Symbolize $\lambda=\frac{XY}{XZ}$, and $\theta=\angle YXZ$. Set a space rectangular coordinate system with point A as its origin, α as plane X-Y, and $\overrightarrow{AC}$ as the positive direction of X axis. Thus we know A(0,0,0). Assume $B(a_1,b_1,0)$, $C(a_2, 0, 0)$, $Y_0(a_1, b_1, x)$, $Z_0(a_2, 0, y)$, where $b_1\neq 0, a_2\neq 0$. Variables x, y satisfy the following equation

$$\frac{\sqrt{a_1^2+b_1^2+x^2}}{\sqrt{a_2^2+y^2}}=\lambda \tag{1}$$

and

$$\frac{a_1a_2+xy}{\sqrt{a_1^2+b_1^2+x^2}\sqrt{a_2^2+y^2}}=\cos\theta \tag{2}$$

From equation (1) we know $\sqrt{a_1^2+b_1^2+x^2}=\lambda\sqrt{a_2^2+y^2}$. Substituting it into Equation (2) leads to

$$\frac{a_1a_2+xy}{\lambda(a_2^2+y^2)}=\cos\theta\ ,$$

i.e. (2) is equivalent to

$$xy=\lambda(a_2^2+y^2)\cos\theta-a_1a_2 \tag{3}$$

At the same time (1) is equivalent to

$$a_1^2+b_1^2+x^2=\lambda^2(a_2^2+y^2) \tag{4}$$

Next we will verify that there exist real numbers x, y satisfying both (3) and (4). There are two different cases.

i. If $\lambda\neq\frac{a_1}{a_2\cos\theta}$

Squaring and reorganizing (3) we get

$$x^2y^2=\lambda^2\cos^2\theta\cdot y^4+2\lambda\cos\theta\cdot(\lambda a_2^2\cos\theta-a_1a_2)y^2+(\lambda a_2^2\cos\theta-a_1a_2)^2 \tag{5}$$

Multiply equation (4) with y^2 and substitute (5) into it,

$$(a_1^2+b_1^2)y^2+\lambda^2\cos^2\theta\cdot y^4+2\lambda\cos\theta\cdot(\lambda a_2^2\cos\theta-a_1a_2)y^2+(\lambda a_2^2\cos\theta-a_1a_2)^2$$
$$=\lambda^2a_2^2y^2+\lambda^2y^4$$

Reorganize it to be a quadratic equation of y^2,

$$\lambda^2 \sin^2\theta\cdot(y^2)^2 - [a_1^2 + b_1^2 - \lambda^2 a_2^2 + 2\lambda\cos\theta\cdot(\lambda a_2^2\cos\theta - a_1 a_2)]y^2 - (\lambda a_2^2\cos\theta - a_1 a_2)^2 = 0$$

From the above assumption we know that the coefficient of the quadratic term of this quadratic equation is positive, while its constant term is negative, so this equation must have both positive and negative roots. Take the positive one and extract its square roots, the positive square root is a possible non-zero value for y. By substituting it back into (5) we get $x = \dfrac{\lambda\cos\theta(a_2^2 + y^2) - a_1 a_2}{y}$, and then we get x, y satisfying both equations (3) and(4).

ii. If $\lambda = \dfrac{a_1}{a_2\cos\theta}$

Then with $|\cos\theta| < 1$ we get $|\lambda| > \left|\dfrac{a_1}{a_2}\right|$。

Equation (3) can be simplified to

$$xy = \frac{a_1}{a_2} y^2 \tag{6}$$

while equation (4) represented as

$$a_1^2 + b_1^2 + x^2 = \lambda^2 a_2^2 + \lambda^2 y^2 \tag{7}$$

If $a_1^2 + b_1^2 \le \lambda^2 a_2^2$, then $x = \sqrt{\lambda^2 a_2^2 - a_1^2 - b_1^2}, y = 0$ satisfy both equations (6) and (7), and they are also a pair of real numbers satisfying equations (3) and (4).

On the other hand, if $a_1^2 + b_1^2 > \lambda^2 a_2^2$, we dictate $x = \dfrac{a_1}{a_2} y$, then substitute it into (7) and represent (7) as

$$a_1^2 + b_1^2 + \frac{a_1^2}{a_2^2} y^2 = \lambda^2 a_2^2 + \lambda^2 y^2$$

or

$$y^2 = \frac{a_1^2 + b_1^2 - \lambda^2 a_2^2}{\lambda^2 - \dfrac{a_1^2}{a_2^2}} > 0 \quad \text{(notice that } |\lambda| > \left|\frac{a_1}{a_2}\right|\text{)}$$

That means that there exists fitting y, substituting which back into $x = \frac{a_1}{a_2} y$

we get real numbers x, y that satisfy both (3) and (4).

The combination of the two cases i and ii leads to the conclusion that there exist x, y satisfying (3) and (4) simultaneously, and it also shows that there exist fitting points Y_0 and Z_0 we need. Denote by β the plane on which ΔAY_0Z_0 locates, then under the parallel projective transformation from β to α, ΔAY_0Z_0 transforms to ΔABC. Moreover, with $\frac{AY_0}{AZ_0} = \frac{XY}{XZ}$ and $\angle Y_0AZ_0 = \angle YXZ$ we know $\Delta XYZ \sim \Delta AY_0Z_0$. Hence ΔAY_0Z_0 and the projective transformation from β to α are what are supposed to be found for the proof.

Thus Lemma 2 is proven.

Now we continue the proof of Theorem 1on its sufficiency.

Suppose the lines AO, BO and CO intersect with the opposite sides at points A_1, B_1 and C_1 respectively. (Refer to Fig. 8) Since point O lies inside $A_0B_0C_0$ which is the medial triangle of ΔABC, we get

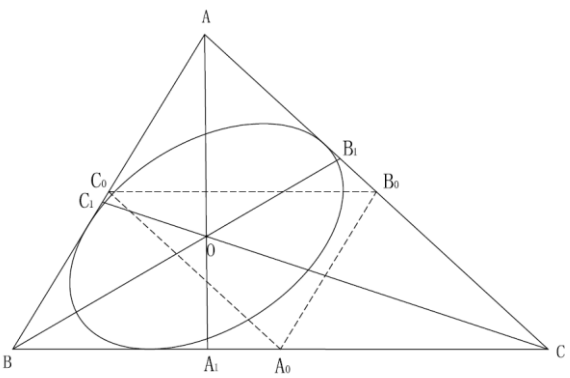


Fig. 8

$$S_{\Delta OBC} < \frac{1}{2} S_{\Delta ABC}$$

$$S_{\Delta OCA} < \frac{1}{2} S_{\Delta ABC}$$

$$S_{\Delta OAB} < \frac{1}{2} S_{\Delta ABC}$$

meaning that there exists ΔXYZ so that

$$YZ : ZX : XY = S_{\Delta OBC} : S_{\Delta OCA} : S_{\Delta OAB}$$

Denote by I the incenter of ΔXYZ, X_1, Y_1 and Z_1 the intersections of XI, YI and ZI and their opposite sides respectively (Refer to Fig. 9). Hence

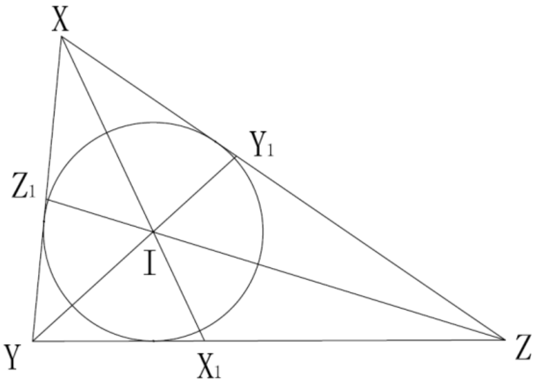


Fig. 9

$$\frac{YX_1}{ZX_1} = \frac{YX}{ZX} = \frac{S_{\Delta OAB}}{S_{\Delta OCA}} = \frac{BA_1}{CA_1}$$

Similarly

$$\frac{ZY_1}{XY_1}=\frac{CB_1}{AB_1},\frac{XZ_1}{YZ_1}=\frac{AC_1}{BC_1}$$

According to Lemma 2, there exists one parallel projective transformation and one triangle similar to ΔXYZ, so that the image of this triangle under this transformation is ΔABC. Since $\frac{YX_1}{ZX_1}=\frac{BA_1}{CA_1}$, thus the image point of X_1 is A_1. Similarly, the image of Y_1 is B_1, and the image of Z_1 is C_1. Accordingly, the image of point I which is the intersection of XX_1, YY_1 and ZZ_1, is point O, the intersection point of AA_1, BB_1 and CC_1. Also, the image of the incircle of ΔXYZ is the inscribed ellipse of ΔABC, so the center of this inscribed ellipse is point O , the image of the original incenter I. Thus we can conclude that there exists an ellipse centered at O and inscribed in ΔABC.

So far, Theorem 1has been proven.

As a matter of fact, with the tool of parallel projective transformation we may also prove that the largest inscribed ellipse of ΔABC is tangent to the three sides at their midpoints, and the center of this ellipse happens to be the centroid of ΔABC [4]. Please refer to Appendix 1 for details.

III Foci of the Ellipse Inscribed in a Triangle

In this chapter we will use the tool of complex number to study the properties of the foci of the ellipse inscribed in a triangle, and based on the results we will also introduce an estimation of the total length of its major and minor axes.

Denote by z_1, z_2, z_3 the corresponding complex numbers of points A, B and C on complex plane. It makes no difference if we assume that ΔABC is inscribed in the unit circle, which means $|z_1|=|z_2|=|z_3|=1$. Then

$$\overline{z_1}=\frac{1}{z_1},\quad \overline{z_2}=\frac{1}{z_2},\quad \overline{z_3}=\frac{1}{z_3} \tag{8}$$

Furthermore, we assume that a given point P inside ΔABC corresponds to a complex number x_1, and its isogonal conjugate point Q corresponds to another complex number x_2. Then with Lemma 1 we know there exists an ellipse inscribed in ΔABC with its foci at points P and Q. Regarding the condition to be met by x_1 and x_2 we have Lemma 3 as follows.

[Lemma 3] If there exists an ellipse inscribed in a triangle with its foci at points P and Q, then

$$x_1 + x_2 + z_1 z_2 z_3 \overline{x_1 x_2} = z_1 + z_2 + z_3$$

Here is the proof for Lemma 3.

By Lemma 1, P and Q are isogonal conjugate points, meaning that the bisectors of ∠BAC and ∠PAQ coincide. This is equivalent to

$$\frac{(x_1 - z_1)(x_2 - z_1)}{(z_2 - z_1)(z_3 - z_1)} \in \mathbb{R}$$

i.e.

$$\frac{(x_1 - z_1)(x_2 - z_1)}{(z_2 - z_1)(z_3 - z_1)} = \frac{(\overline{x_1} - \overline{z_1})(\overline{x_2} - \overline{z_1})}{(\overline{z_2} - \overline{z_1})(\overline{z_3} - \overline{z_1})}$$

Substitute (8) into the above equation and simplify it we get

$$(x_1 - z_1)(x_2 - z_1) = z_2 z_3 (z_1 \overline{x_1} - 1)(z_1 \overline{x_2} - 1)$$

i.e.

$$x_1 x_2 - (x_1 + x_2) z_1 + z_1^2 = z_1^2 z_2 z_3 \overline{x_1 x_2} - z_1 z_2 z_3 (\overline{x_1} + \overline{x_2}) + z_2 z_3 \tag{9}$$

In the same way we may also get

$$x_1 x_2 - (x_1 + x_2) z_2 + z_2^2 = z_2^2 z_3 z_1 \overline{x_1 x_2} - z_1 z_2 z_3 (\overline{x_1} + \overline{x_2}) + z_3 z_1 \tag{10}$$

and

$$x_1 x_2 - (x_1 + x_2) z_3 + z_3^2 = z_3^2 z_1 z_2 \overline{x_1 x_2} - z_1 z_2 z_3 (\overline{x_1} + \overline{x_2}) + z_1 z_2 \tag{11}$$

Subtracting (10) from (9) leads to

$$(z_2 - z_1)(x_1 + x_2) + z_1^2 - z_2^2 = (z_1 - z_2) z_1 z_2 z_3 \overline{x_1 x_2} + (z_2 - z_1) z_3$$

Since $z_2 \neq z_1$, thus

$$x_1 + x_2 + z_1 z_2 z_3 \overline{x_1 x_2} = z_1 + z_2 + z_3 \tag{12}$$

When equation (12) is substituted back into (9), (10) and (11), it turns out that (12) is equivalent to the other three. This means that equation (12) is the necessary and sufficient condition for P and Q to be isogonal conjugate points.

Thus Lemma 3 has been proven.

Generally with equation (12) we can express x_2 in terms of x_1. Performing conjugation on the both sides of equation (12) we get $\overline{x_1}+\overline{x_2}+\overline{z_1z_2z_3}x_1x_2=\overline{z_1}+\overline{z_2}+\overline{z_3}$, which can also be expressed as

$$x_1x_2+z_1z_2z_3(\overline{x_1}+\overline{x_2})=z_1z_2+z_2z_3+z_3z_1 \qquad (13)$$

Take (12) and (13) as a linear system of binary equations about x_2 and $\overline{x_2}$. When $|x_1|\neq 1$ we have

$$x_2=\frac{z_1z_2z_3\overline{x_1}^2-(z_1z_2+z_2z_3+z_3z_1)\overline{x_1}-x_1+z_1+z_2+z_3}{1-|x_1|^2} \qquad (14)$$

Notice that the focus P of the inscribed ellipse lies inside ΔABC, that is, $|x_1|<1$. With equation (14) we can prove the following Lemma 4.

[Lemma 4] For any ellipse inscribed in ΔABC with points P and Q as its foci, the length of its major axis $\mathbf{2a=\frac{|(x_1-z_1)(x_1-z_2)(x_1-z_3)|}{1-|x_1|^2}}$.

The proof of Lemma 4 is shown as follows.

From the discussion in Introduction we know that the length of the ellipse's major axis equals to the distance between the two points P_A and Q, where P_A stands for the symmetric point of P with respect to line BC. Suppose point P_A corresponds to a complex number x_0. Next we will find x_0. (Refer to Fig. 10)

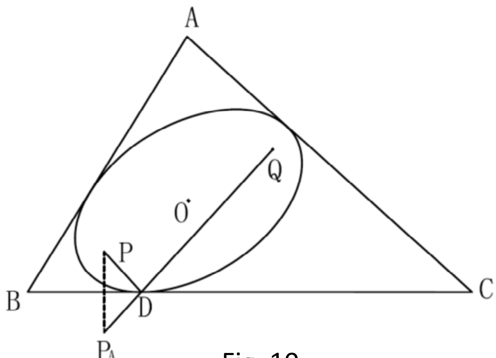


Fig. 10

Since points P and P_A are symmetric to each other with respect to line BC, thus $PB=P_AB$, $PC=P_AC$. So $|x_0-z_2|=|x_1-z_2|$

That is

$$(x_0-z_2)(\overline{x_0}-\overline{z_2})=(x_1-z_2)(\overline{x_1}-\overline{z_2})$$

By expanding both sides we get

$$x_0\overline{x_0}-z_2\overline{x_0}-\overline{z_2}x_0=x_1\overline{x_1}-z_2\overline{x_1}-\overline{z_2}x_1 \qquad (15)$$

Similarly we have

$$x_0\overline{x_0}-z_3\overline{x_0}-\overline{z_3}x_0=x_1\overline{x_1}-z_3\overline{x_1}-\overline{z_3}x_1 \tag{16}$$

Subtracting (16) from (15) leads to

$$(z_3-z_2)(\overline{x_0}-\overline{x_1})+(\overline{z_3}-\overline{z_2})(x_0-x_1)=0$$

By substituting (8) into it and reorganize it, we get $z_2z_3(\overline{x_0}-\overline{x_1})=(x_0-x_1)$

(notice that $z_3-z_2\neq 0$) that is

$$\overline{x_0}=\frac{x_0-x_1}{z_2z_3}+\overline{x_1} \tag{17}$$

Substitute (17) into (15) and eliminate $\overline{x_0}$, then reorganize the resulting equation in the form of quadric equation about x_0 such that

$$x_0^2+(z_2z_3\overline{x_1}-z_2-z_3-x_1)x_0+(z_2+z_3-z_2z_3\overline{x_1})x_1=0$$

which can be converted to

$$(x_0-x_1)(x_0+z_2z_3\overline{x_1}-z_2-z_3)=0$$

Cast out the root that makes point P coincide with point P_A, we get

$$x_0=z_2+z_3-z_2z_3\overline{x_1} \tag{18}$$

With (14) and (18) we can deduce that the length of the ellipse's major axis

$$\begin{aligned}2a&=|x_0-x_2|\\&=\left|z_2+z_3-z_2z_3\overline{x_1}-\frac{z_1z_2z_3\overline{x_1}^2-(z_1z_2+z_2z_3+z_3z_1)\overline{x_1}-x_1+z_1+z_2+z_3}{1-|x_1|^2}\right|\\&=\left|\frac{-(z_2+z_3)x_1\overline{x_1}+z_2z_3x_1\overline{x_1}^2-z_1z_2z_3\overline{x_1}^2+(z_1z_2+z_3z_1)\overline{x_1}+x_1-z_1}{1-|x_1|^2}\right|\end{aligned}$$

Noticing that the numerator in the last fractional expression is factorable,

$$-(z_2+z_3)x_1\overline{x_1}+z_2z_3x_1\overline{x_1}^2-z_1z_2z_3\overline{x_1}^2+(z_1z_2+z_3z_1)\overline{x_1}+x_1-z_1=(x_1-z_1)(z_2\overline{x_1}-1)(z_3\overline{x_1}-1)$$

Then we have the following expression (notice that $|x_1|<1$, $|z_2|=|z_3|=1$ and $\overline{z_2}=\frac{1}{z_2},\overline{z_3}=\frac{1}{z_3}$)

$$2a=\frac{|(x_1-z_1)(z_2\overline{x_1}-1)(z_3\overline{x_1}-1)|}{1-|x_1|^2}=\frac{|(x_1-z_1)(\overline{z_2}x_1-1)(\overline{z_3}x_1-1)|}{1-|x_1|^2}=\frac{|(x_1-z_1)(x_1-z_2)(x_1-z_3)|}{1-|x_1|^2}$$

Hereto Lemma 4 has been proven.

With the help of Lemma 4 we discovered the following theorem concerning the sum of major and minor axes of inscribed ellipse of triangle.

[Theorem 2] For ΔABC, the sum of major and minor axes of its inscribed ellipse is no more than the diameter of its circumcircle, or

$$\mathbf{2a+2b\le 2R} \tag{19}$$

and, if ΔABC is an acute triangle there exists an inscribed ellipse that makes the equality hold.

Below is the proof of Theorem 2, during which the previous symbols and assumptions remain the same.

If the semi major axis of the ellipse $a\le\frac{1}{2}$, since its semi minor axis $b\le a\le\frac{1}{2}$, thus the sum of its major and minor axes $2a+2b\le 2=2R$, which means that under such conditions the equation (19) is true.

Now we consider the case of $a>\frac{1}{2}$, that is $2a-1>0$. With Lemma 4 we know, if $x_1=0$ then $2a=|z_1z_2z_3|=1$, which means $a=\frac{1}{2}$ and causes contradiction. Hence it could only be $x_1\ne 0$.

Consider the focal length $2c=|x_1-x_2|$. By substituting (14) in and with awareness of $|x_1|^2=x_1\overline{x_1}$, we know

$$2c=\left|\frac{-x_1^2\overline{x_1}-z_1z_2z_3\overline{x_1^2}+(z_1z_2+z_2z_3+z_3z_1)\overline{x_1}+2x_1-z_1-z_2-z_3}{1-|x_1|^2}\right|$$

$$=\left|\frac{-x_1^2\overline{x_1^2}-z_1z_2z_3\overline{x_1^3}+(z_1z_2+z_2z_3+z_3z_1)\overline{x_1^2}+2x_1\overline{x_1}-(z_1+z_2+z_3)\overline{x_1}}{(1-|x_1|^2)\overline{x_1}}\right|$$

$$=\left|\frac{-(x_1^2\overline{x_1^2}-2x_1\overline{x_1}+1)+1-(z_1+z_2+z_3)\overline{x_1}+(z_1z_2+z_2z_3+z_3z_1)\overline{x_1^2}-z_1z_2z_3\overline{x_1^3}}{(1-|x_1|^2)\overline{x_1}}\right|$$

$$=\left|\frac{-(1-|x_1|^2)^2+(1-z_1\overline{x_1})(1-z_2\overline{x_1})(1-z_3\overline{x_1})}{(1-|x_1|^2)\overline{x_1}}\right|$$

Thus

$$2c\geq\left|\frac{(1-|x_1|^2)^2-|(1-z_1\overline{x_1})(1-z_2\overline{x_1})(1-z_3\overline{x_1})|}{(1-|x_1|^2)\overline{x_1}}\right|=\frac{\left|1-|x_1|^2-\frac{|(z_1-x_1)(z_2-x_1)(z_3-x_1)|}{1-|x_1|^2}\right|}{|x_1|}$$

With the application of Lemma 4, we get (AM-GM inequality is used at the last step)

$$2c\geq\frac{\left|1-|x_1|^2-2a\right|}{|x_1|}=\frac{(2a-1)+|x_1|^2}{|x_1|}\geq2\sqrt{2a-1}$$

That is

$$c\geq\sqrt{2a-1} \tag{20}$$

Square on the two sides of the inequality and substitute it into $c^2=a^2-b^2$,we get

$a^2-b^2\geq2a-1$

or

$$(1-a)^2\geq b^2 \tag{21}$$

Since the ellipse is inscribed in ΔABC, thus the ellipse lies within ΔABC or on its sides, and also lies within ΔABC's circumcircle or on the circumcircle. Furthermore, the major axis of the ellipse is no longer than the circumradius, that is, $a\leq1$.Thus with (21) we get $1-a\geq b$, or $a+b\leq1$.

Thus $2a+2b\leq2=2R$, which means that equation (19) is true.

All these facts lead to the conclusion that the sum of major and minor axes of ΔABC's inscribed ellipse is no more than the diameter of ΔABC's circumcircle.

From now on we will discuss the condition of holding equality in (19).

Obviously, if $a\leq\frac{1}{2}$, the equality holds only on $a=b=\frac{1}{2}$, and in this case the inscribed ellipse degenerates into the incircle of ΔABC with its circumradius $r=\frac{1}{2}$;

with the Euler's Formula for triangles [1], $R^2 = d^2 + 2Rr$ (where d is the distance between the incenter and circumcenter of ΔABC), we know $1 = R^2 \geq 2Rr = 2r$, or $\frac{1}{2} \geq r$, where equality holds iff $d = 0$ meaning that the incenter and circumcenter of ΔABC coincide, or that ΔABC is equilateral. In other words, if $a \leq \frac{1}{2}$, equality holds for Inequality (19) iff ΔABC is equilateral, and the inscribed ellipse coincides with the incircle.

Next we will discuss the condition of holding equality in Inequality (19) in the case of $a > \frac{1}{2}$, which is equivalent to Inequality (20). In the proof above we use magnification method twice; the first is

$$\left|\frac{-(1-|x_1|^2)^2 + (1-z_1\overline{x_1})(1-z_2\overline{x_1})(1-z_3\overline{x_1})}{(1-|x_1|^2)\overline{x_1}}\right| \geq \left|\frac{(1-|x_1|^2)^2 - |(1-z_1\overline{x_1})(1-z_2\overline{x_1})(1-z_3\overline{x_1})|}{(1-|x_1|^2)\overline{x_1}}\right|$$

and the second is the mean-value inequality $\frac{(2a-1)+|x_1|^2}{|x_1|} \geq 2\sqrt{2a-1}$. For the first one, equality holds if the corresponding vectors of complex numbers $-(1-|x_1|^2)^2$ and $(1-z_1\overline{x_1})(1-z_2\overline{x_1})(1-z_3\overline{x_1})$ are opposite directions (or one of them equals to zero). Since $-(1-|x_1|^2)^2$ happens to be a negative real number (noticing that $|x_1| < 1$), thus this is equivalent to that $(1-z_1\overline{x_1})(1-z_2\overline{x_1})(1-z_3\overline{x_1})$ is a non-negative real number, or

$$\overline{(1-z_1\overline{x_1})(1-z_2\overline{x_1})(1-z_3\overline{x_1})} = (1-\frac{x_1}{z_1})(1-\frac{x_1}{z_2})(1-\frac{x_1}{z_3})$$

is a non-negative real number.

As for the second one the equality holds iff $|x_1|^2 = 2a-1$ or

$|x_1|^2 = \frac{|(x_1-z_1)(x_1-z_2)(x_1-z_3)|}{1-|x_1|^2} - 1$, which can be reorganized as

$1-|x_1|^4 = |(x_1-z_1)(x_1-z_2)(x_1-z_3)|$

Aware that the inscribed ellipse with a focal point at P and the one with a focal point at Q are actually the same, we obtain the other necessary-sufficient condition for inequality (20) to hold equality by substituting x_1 in the above equation with x_2; in particular, when (20) holds equality, $|x_2|^2 = 2a-1$ must also be true. So we get $|x_1|^2 = 2a-1 = |x_2|^2$, or $|x_1| = |x_2| = \sqrt{2a-1}$. Moreover, (20) holding equality means $c = \sqrt{2a-1}$, and thus

$$|x_1 - x_2| = 2c = 2\sqrt{2a-1} = |x_1| + |x_2|$$

Which means that the vectors corresponding to x_1 and $-x_2$ are in the same direction; noticing that $|x_1| = |x_2|$, we know the origin is the midpoint of the points corresponding to x_1 and x_2, or in other words $x_1 + x_2 = 0$.

Notice that x_1 and x_2 are the corresponding complex numbers of the foci of the inscribed ellipse of ΔABC, which should satisfy (12), or

$$x_1 + x_2 + z_1 z_2 z_3 \overline{x_1 x_2} = z_1 + z_2 + z_3$$

Substituting $x_2 = -x_1$ into it we get

$$-\overline{x_1^2} = \frac{1}{z_1 z_2} + \frac{1}{z_2 z_3} + \frac{1}{z_3 z_1}$$

Thus

$$x_1^2 = -\overline{\frac{1}{z_1 z_2} + \frac{1}{z_2 z_3} + \frac{1}{z_3 z_1}} = -(z_1 z_2 + z_2 z_3 + z_3 z_1)$$

On the other hand, it can be proven that if ΔABC is not equilateral, then $z_1 z_2 + z_2 z_3 + z_3 z_1 \neq 0$. Actually, since $|z_1| = |z_2| = |z_3| = 1$, we know that at least one of the three values $<\vec{z_1}, \vec{z_2}>, <\vec{z_2}, \vec{z_3}>, <\vec{z_3}, \vec{z_1}>$ is not equal to $\frac{2\pi}{3}$. We may suppose $<\vec{z_1}, \vec{z_2}> \neq \frac{2\pi}{3}$, then $|z_1 + z_2| = 2\cos\frac{<\vec{z_1}, \vec{z_2}>}{2} \neq 1$, or $\frac{|z_1 z_2|}{|z_1 + z_2|} \neq 1$. Thus $-\frac{z_1 z_2}{z_1 + z_2} \neq z_3$, and that is to say $z_1 z_2 + z_2 z_3 + z_3 z_1 \neq 0$.

Hence $x_1 \neq 0$.

Next we will verify if this x_1 meets the two conditions for (20) to hold equality.

Firstly, since $x_1^2 = -(z_1z_2 + z_2z_3 + z_3z_1)$, we know

$$
\begin{aligned}
&(1-\frac{x_1}{z_1})(1-\frac{x_1}{z_2})(1-\frac{x_1}{z_3}) \\
&= 1 + \frac{z_1+z_2+z_3}{z_1z_2z_3}x_1^2 - \frac{z_1z_2+z_2z_3+z_3z_1}{z_1z_2z_3}x_1 - \frac{1}{z_1z_2z_3}x_1^3 \\
&= 1 - \frac{z_1+z_2+z_3}{z_1z_2z_3}(z_1z_2+z_2z_3+z_3z_1) - \frac{z_1z_2+z_2z_3+z_3z_1}{z_1z_2z_3}x_1 + \frac{1}{z_1z_2z_3}x_1 \cdot (z_1z_2+z_2z_3+z_3z_1) \\
&= 1 - (z_1+z_2+z_3)(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}) \\
&= 1 - |z_1+z_2+z_3|^2 \in \mathbb{R}
\end{aligned}
$$

Thus the first condition for equation (20) to hold equality is met iff $|z_1+z_2+z_3| \leq 1$.

Secondly, since

$$
|x_1^2| = |z_1z_2+z_2z_3+z_3z_1| = |\overline{z_1z_2+z_2z_3+z_3z_1}| = \left|\frac{z_1+z_2+z_3}{z_1z_2z_3}\right| = |z_1+z_2+z_3|
$$

we know

$$
|(x_1-z_1)(x_1-z_2)(x_1-z_3)| = \left|(1-\frac{x_1}{z_1})(1-\frac{x_1}{z_2})(1-\frac{x_1}{z_3})\right| = 1-|z_1+z_2+z_3|^2 = 1-|x_1|^4
$$

Thus the second condition holds true for inequality (20) to hold equality.

To sum up, the inequality (20) holds equality iff $|z_1+z_2+z_3| \leq 1$ and ΔABC is not equilateral, and the condition for equality is $x_1 = \pm\sqrt{-(z_1z_2+z_2z_3+z_3z_1)}$, and $x_2 = -x_1$.

It is well known that, since ΔABC inscribed in a unit circle, the complex number $z_1+z_2+z_3$ corresponds to the orthocenter of ΔABC [1]. Thus if ΔABC is an acute-angled triangle, then its orthocenter lies within it, which leads to $|z_1+z_2+z_3| \leq 1$. So if ΔABC is acute-angled but not equilateral, inequality (20) holds equality iff $x_1 = \pm\sqrt{-(z_1z_2+z_2z_3+z_3z_1)}$ and $x_2 = -x_1$. Furthermore, since $x_1 + x_2 = 0$, we know the center of the ellipse, or the midpoint of PQ that makes (20) hold equality is the circumcenter of ΔABC. It is also well known that the circumcenter of ΔABC is also the orthocenter of its medial triangle which is similar to ΔABC and thus is also acute angled. [1] That is to say that this midpoint lies inside the medial triangle of ΔABC, so, with Theorem 1 we know there exists one inscribed ellipse with foci at P and Q. This ellipse is the only ellipse inscribed in ΔABC making inequality (20), and also (19) hold equality.

Summing up the two cases $a \le \frac{1}{2}$ and $a > \frac{1}{2}$, we conclude that inequality (19) does hold equality if ΔABC is acute angled.

So far we finished the proof of Theorem 2.

Furthermore, we can also explore the cases of ΔABC being right angled and obtuse angled. (Refer to Fig. 11 and Fig. 12)

If ΔABC is right angled, the situation is similar to that of an acute triangle, but the inscribed ellipse that makes inequality (19) holds equality degenerates to a line segment, the hypotenuse of ΔABC. So, all the non-degenerated ellipses inscribed in ΔABC cannot make inequality (19) holds equality; the sum of their major and minor axes is strictly less than, yet could infinitely approach, the diameter of circumcircle.

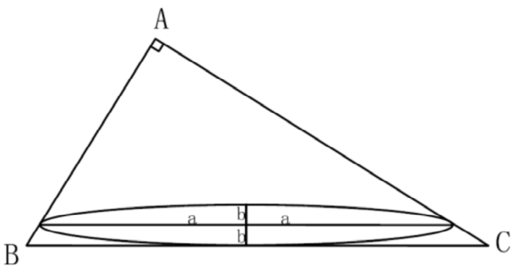


Fig. 11

If ΔABC is obtuse angled, for convenience we suppose ∠A is the obtuse angle. Draw CT⊥AB intersecting at T. Assume Ω is one of the inscribed ellipse of ΔABC, then Ω is enclosed within ΔTBC. Apply homothetic transformation to Ω with homothetic center at B, so that Ω', the image of Ω, is tangent to TC. Thus Ω' is an inscribed ellipse of the right angled ΔTBC, and the homothetic ratio $k \ge 1$, that is, the sum of major and minor axes of Ω' is no less than that of Ω. By previous discussion we know, that the sum of major and minor axes of Ω' is strictly less than the diameter of ΔTBC's circumcircle or BC, thus the sum of major and minor axes of Ω is also strictly less than the length of BC. It is also true that as Ω approaches BC infinitely, the sum of its major and minor axes also approaches the length of BC infinitely.

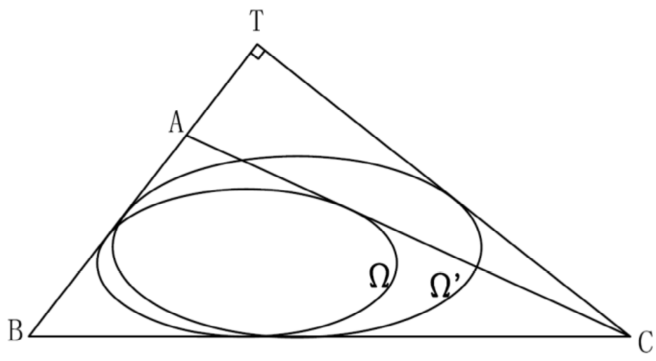


Fig. 12

With the two parts of discussion we get the conclusion that, for a non-acute angled triangle given, the sum of major and minor axes of its inscribed ellipse is less than its longest side, and could approach the longest side infinitely.

IV Some Other Conclusions

During the study we found some other conclusions about inscribed ellipse of triangle, they are all geometric inequalities.

1 Distances from the focal point of inscribed ellipse to the triangle's sides

We discovered that, with respect to the distances from the points P and Q to any one side of ΔABC, at least one of the two is no more than r, the inradius of ΔABC. (Refer to Fig. 13)

We will keep the previous symbols, and denote by I the incenter of ΔABC; through point O draw perpendiculars of BC, CA and AB, and denote by O_1, O_2 and O_3 the feet respectively. By symmetry we only need to prove the case of $\min\{PP_1, QQ_1\} \le r$. This can be done by proving another conclusion $PP_1 \cdot QQ_1 \le r^2$, which is slightly stronger.

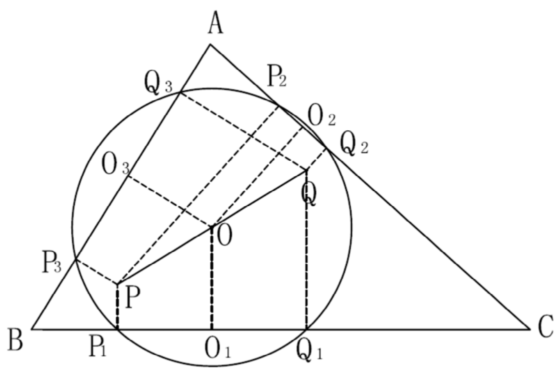


Fig. 13

It is well known that, regarding isogonal conjugate points P and Q there are the following relations:

$PP_1 \cdot QQ_1 = PP_2 \cdot QQ_2 = PP_3 \cdot QQ_3$[1]

Since O is the midpoint of PQ, so $PP_1 + QQ_1 = 2OO_1$.By AM-GM inequality we know that $PP_1 \cdot QQ_1 \le OO_1^2$; similarly $PP_2 \cdot QQ_2 \le OO_2^2$, $PP_3 \cdot QQ_3 \le OO_3^2$.

Thus if we manage to prove $\min\{OO_1, OO_2, OO_3\} \le r$, then

$$PP_1 \cdot QQ_1 = PP_2 \cdot QQ_2 = PP_3 \cdot QQ_3 \le \min\{OO_1^2, OO_2^2, OO_3^2\} \le r^2,$$

which leads to the conclusion we need.

Below we use proof by contradiction to show $\min\{OO_1, OO_2, OO_3\} \le r$.

Suppose $\min\{OO_1, OO_2, OO_3\} > r$. Consider $S_{\Delta ABC}$.

On the one hand,

$$S_{\Delta ABC} = S_{\Delta IBC} + S_{\Delta ICA} + S_{\Delta IAB} = \frac{1}{2} r(AB + BC + CA)$$

On the other hand,

$$S_{\Delta ABC} = S_{\Delta OAB} + S_{\Delta OBC} + S_{\Delta OCA} = \frac{1}{2} OO_1 \cdot BC + \frac{1}{2} OO_2 \cdot CA + \frac{1}{2} OO_3 \cdot AB > \frac{r}{2}(BC + CA + AB)$$

Contradiction happens! It indicates that the supposition is not true, or in other words, $\min\{\mathrm{OO}_1, \mathrm{OO}_2, \mathrm{OO}_3\} \le r$.

Thus we have proven $\mathrm{PP}_1 \cdot \mathrm{QQ}_1 \le r^2$, which leads to $\min\{PP_1, QQ_1\} \le r$. In a similar way, we can prove $\min\{PP_2, QQ_2\} \le r$ and $\min\{PP_3, QQ_3\} \le r$.

As a matter of fact, r is the ultimate value for this inequality, because all the distances from P and Q to the three sides equal to r if P and Q coincide with incenter I. This means that r is the smallest upper bound for the inequality.

2 The Distance between Foci of the Largest Inscribed Ellipse and Triangle's Vertices

Next we will discuss the property of W_1 and W_2, the foci of ΔABC's largest inscribed ellipse. (Refer to Fig. 14)

We may suppose as well that ΔABC is inscribed in the unit circle for convenience. We discovered that, for any one vertex of ΔABC, among its distances to the two foci W_1 and W_2, at least one of them is no more than 1.

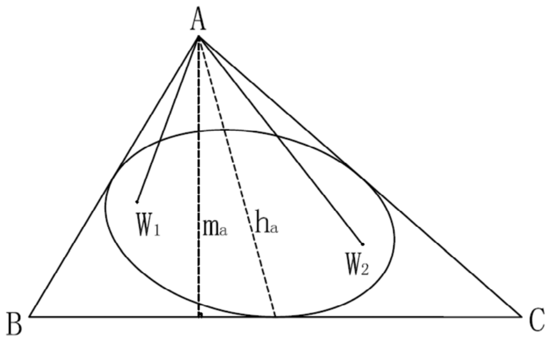


Fig. 14

We have mentioned previously that, the largest inscribed ellipse of ΔABC centered at the centroid G of ΔABC [4]. Denote by w_1 and w_2 the complex numbers corresponding to points W_1 and W_2; then they satisfy

$$w_1 + w_2 = \frac{2(z_1 + z_2 + z_3)}{3} \tag{22}$$

By Lemma3 we have

$$w_1 + w_2 + z_1 z_2 z_3 \overline{w_1 w_2} = z_1 + z_2 + z_3 \qquad (23)$$

By substituting (22) into (23) and reorganizing it we get

$$z_1 z_2 z_3 \overline{w_1 w_2} = \frac{z_1 + z_2 + z_3}{3}$$

Applying conjugation to both sides of the above equation and substituting $\overline{z_1} = \frac{1}{z_1}, \overline{z_2} = \frac{1}{z_2}, \overline{z_3} = \frac{1}{z_3}$ into it, we have

$$w_1 w_2 = \frac{z_1 z_2 + z_2 z_3 + z_3 z_1}{3} \qquad (24)$$

So w_1 and w_2 are the two roots of the following quadric equation for w

$$w^2 - \frac{2(z_1 + z_2 + z_3)}{3} w + \frac{z_1 z_2 + z_2 z_3 + z_3 z_1}{3} = 0$$

They are also the two roots of the following equation

$$(w - z_1)(w - z_2) + (w - z_2)(w - z_3) + (w - z_3)(w - z_1) = 0 \qquad (25)$$

Now let's consider the conclusion we want. By symmetry we only need to prove that one of AW_1 and AW_2 is no more than 1, or $\min\{|w_1 - z_1|, |w_2 - z_1|\} \le 1$.

Since w_1 and w_2 are the two roots of equation (25), so for any complex number w we have

$$(w - z_1)(w - z_2) + (w - z_2)(w - z_3) + (w - z_3)(w - z_1) = 3(w - w_1)(w - w_2)$$

Dictate $w = z_1$ we get

$$(z_1 - z_2)(z_1 - z_3) = 3(z_1 - w_1)(z_1 - w_2) \qquad (26)$$

Furthermore, $w_1 + w_2 = \frac{2(z_1 + z_2 + z_3)}{3}$, thus

$$(z_1 - w_1) + (z_1 - w_2) = 2z_1 - \frac{2(z_1 + z_2 + z_3)}{3} = \frac{2(2z_1 - z_2 - z_3)}{3}$$

Suppose h_a is the height of ΔABC with respect to side BC, and m_a is the length of the median line of ΔABC with respect to side BC. Obviously $m_a \geq h_a$; and with the well-known fact $AB \cdot AC = 2R \cdot h_a = 2h_a \leq 2m_a$, or $|z_1 - z_2| \cdot |z_1 - z_3| \leq |2z_1 - z_2 - z_3|$, we get

$$|(z_1 - w_1) + (z_1 - w_2)| = \frac{2}{3}|2z_1 - z_2 - z_3| \geq \frac{2}{3}|z_1 - z_2| \cdot |z_1 - z_3| = 2|z_1 - w_1| \cdot |z_1 - w_2|$$

Then we know

$\left|\frac{1}{z_1 - w_1} + \frac{1}{z_1 - w_2}\right| \geq 2$, and thus

$$\max\{\left|\frac{1}{z_1 - w_1}\right|, \left|\frac{1}{z_1 - w_2}\right|\} \geq \frac{1}{2}(\left|\frac{1}{z_1 - w_1}\right| + \left|\frac{1}{z_1 - w_2}\right|) \geq \frac{1}{2}\left|\frac{1}{z_1 - w_1} + \frac{1}{z_1 - w_2}\right| \geq 1.$$

That is equivalent to $\min\{|w_1 - z_1|, |w_2 - z_1|\} \leq 1$.

So far we have proven this conclusion.

3 The Minimum Value of an Algebraic Expression

In exploring the minimum length of the major axis of inscribed ellipse in ΔABC we revealed that, this minimum is closely related to the maximum value of one function with respect to a moving point inside the triangle.

Suppose that ΔABC is inscribed in a unit circle, and point P is one point inside the triangle, and denote by z_1, z_2, z_3 and x_1 the complex numbers corresponding to points A, B, C and P. From Lemma 4 we derive that $2a = \frac{|(x_1 - z_1)(x_1 - z_2)(x_1 - z_3)|}{1 - |x_1|^2}$, where 2a stands for the length of major axis of the inscribed ellipse.

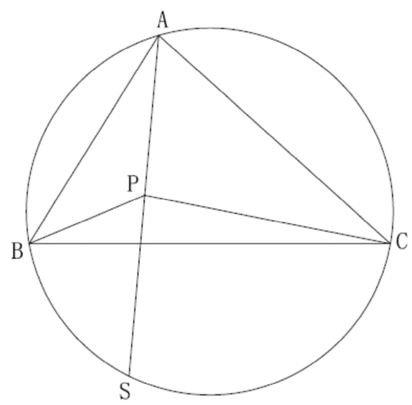


Fig. 15

Connect AP, BP and CP, and suppose line AP intersects with ΔABC's circumcircle at point S. (Refer to Fig. 15) Thus we have $|(x_1 - z_1)(x_1 - z_2)(x_1 - z_3)| = AP \cdot BP \cdot CP$. Furthermore, we know $1-|x_1|^2 = AP\cdot PS$, so $2a=\dfrac{BP\cdot CP}{SP}$. That is to say, in order to determine the minimum length of major axis of inscribed ellipse, we only need to find the minimum value of $f(P)=\dfrac{BP\cdot CP}{SP}$ while point P moves inside ΔABC.

But it is not so easy to find the minimum of function $f(P)$, by contrast it is easier to prove with geometric method that, the length of major axis of the inscribed ellipse is no less than $2r$, where r stands for the inradius of the triangle.

Denote by Q the other focal point of the ellipse, and O the midpoint of PQ; suppose the inscribed ellipse contacts with side BC at point D. (Refer to Fig. 16) And suppose the projections of points P and Q on BC are P_1 and Q_1 respectively, and those of point O on BC, CA and AB are O_1, O_2 and O_3 respectively. Thus $2a = PD + QD \geq PP_1 + QQ_1 = 2OO_1$,

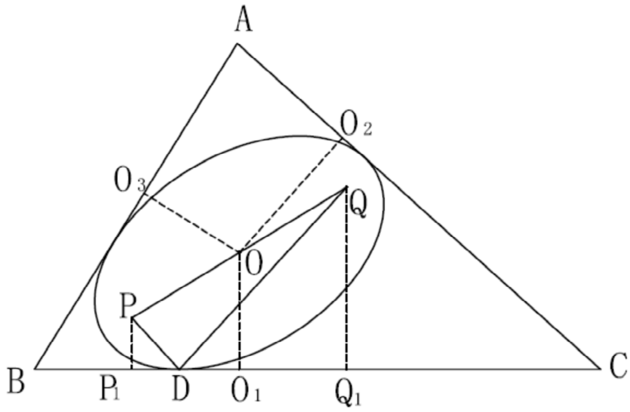


Fig. 16

then $OO_1 \leq a$; similarly we know $OO_2 \leq a, OO_3 \leq a$.

Let's consider the area of ΔABC, $S_{\Delta ABC}$. On the one hand, $S_{\Delta ABC}=\frac{r}{2}(BC+CA+AB)$; on the other hand,

$$S_{\Delta ABC}=S_{\Delta OAB}+S_{\Delta OBC}+S_{\Delta OCA}=\frac{1}{2}OO_1\cdot BC+\frac{1}{2}OO_2\cdot CA+\frac{1}{2}OO_3\cdot AB\leq\frac{a}{2}(BC+CA+AB).$$

Combine the above two equations we get $r \leq a$, so the length of major axis of the inscribed ellipse $2a \geq 2r$.

Furthermore, $2r$ could actually be obtained when the inscribed ellipse coincides with the incircle of ΔABC.

To sum up, the minimum length of major axis of ΔABC's inscribed ellipse is $2r$, and this indirectly shows that the minimum value of $f(P)$ is also $2r$.

Later on we managed to find a way to find the minimum value of $f(P)$ without the help of inscribed ellipse of triangle, but it is much more awkward. Please refer to Appendix 2 for detailed proof.

Ⅴ Postscript

During the study, we have found quite a few interesting characteristics regarding inscribed ellipse in triangle, but we have also left some questions that we cannot answer.

The first one is about the relationship between the center and the foci of an inscribed ellipse. In Chapter II we proved that for any given point O within the medial triangle of ΔABC, there exists an ellipse centered at O inscribed in ΔABC. Yet up till now we cannot determine the two foci directly by O, nor do we know much about the relationship of P, Q and O.

The second one is about the fact that the largest inscribed ellipse of ΔABC is tangent to each side of ΔABC at the midpoint. Reference [1] provides an ingenious proof based on projective transformation, but we haven't found a direct proof based on merely Euclidean geometry.

The last one involves complex function. We know that the complex numbers corresponding to the two foci of the largest inscribed ellipse of ΔABC are the two roots of the equation $(w-z_1)(w-z_2)+(w-z_2)(w-z_3)+(w-z_3)(w-z_1)=0$, or the two zeroes of the derivative of the complex function $f(w)=(w-z_1)(w-z_2)(w-z_3)$. So quite naturally we will ask whether there is any connection between these facts, or whether we can extend it to more points. Unfortunately, limited by our knowledge, we cannot delve further into this question now.

There is another thing that we need to mention. Though the whole study is original, a small part of the results has already been published and we were not aware of it. One of the judges, Professor Pan Jianzhong from Chinese Academy of Sciences, told us that he had found a paper *On Inscribed and Escribed Ellipses of a Triangle* (Keisuke MATSUMOTO, Kazunori FUJITA and Hiroo FUKAISHI, Mem. Fac. Educ., Kagawa Univ. II, 59(2008), 1-10), part of which coincides with our Lemma 1. So wes hereby explain the case.

Supplement for the Study on Inscribed Ellipse of Triangle

November 2010

After the submission of the research paper we continued to work on another problem, the lower bound for the sum of major and minor axes of inscribed ellipse of triangle. Fortunately we have done it recently, which leads to Theorem 3 as follows.

[Theorem 3] For ΔABC, the sum of major and minor axes of its inscribed ellipse is greater than the least altitude of the triangle, or

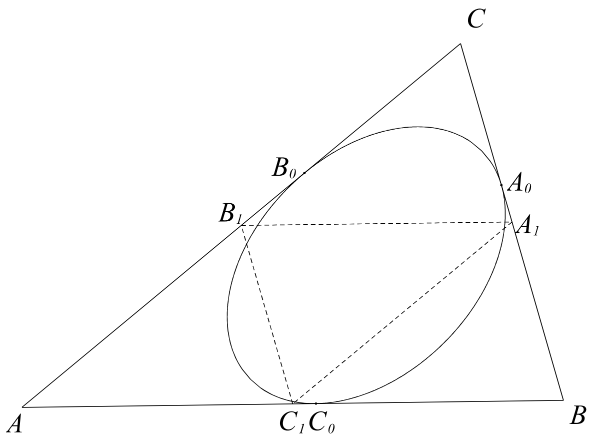
$$2a + 2b > h_{\min}$$

and, while the inscribed ellipse approaches the least altitude of the triangle, the sum of its major and minor axes approximate the least height infinitely.

Here is the proof of Theorem 3.

Denote by A_0, B_0 and C_0 the tangent points of the inscribed ellipse with the three sides of ΔABC, and A_1, B_1 and C_1 the midpoints of the three sides (refer to the drawing below).

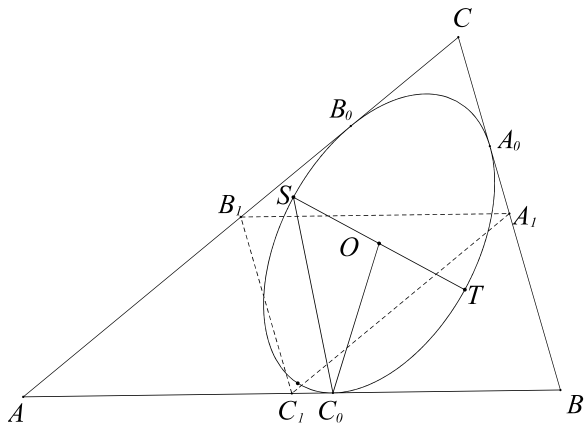
According to the proof of necessity of Theorem 1, with the help of parallel projective transformation, it is easy to prove that AA_0, BB_0 and CC_0 are concurrent. Thus by Ceva's Theorem we know that



$$\frac{AC_0}{C_0B}\cdot\frac{BA_0}{A_0C}\cdot\frac{CB_0}{B_0A}=1$$

which means that the three ratios $\frac{AC_0}{C_0B}$, $\frac{BA_0}{A_0C}$ and $\frac{CB_0}{B_0A}$ cannot be greater than 1 at the same time, neither can they be smaller than 1 simultaneously. Thus one of the three cases $\frac{BA_0}{A_0C}\geq 1$ while $\frac{CB_0}{B_0A}\leq 1$, $\frac{CB_0}{B_0A}\geq 1$ while $\frac{AC_0}{C_0B}\leq 1$, and $\frac{AC_0}{C_0B}\geq 1$ while $\frac{BA_0}{A_0C}\leq 1$ must be true. So we may assume that $\frac{BA_0}{A_0C} \geq 1$ while $\frac{CB_0}{B_0A} \leq 1$,which means B_0 lies on CB_1, and A_0 lies on CA_1 (including end points). It will make no difference by further assuming that $\angle A \leq \angle B$.

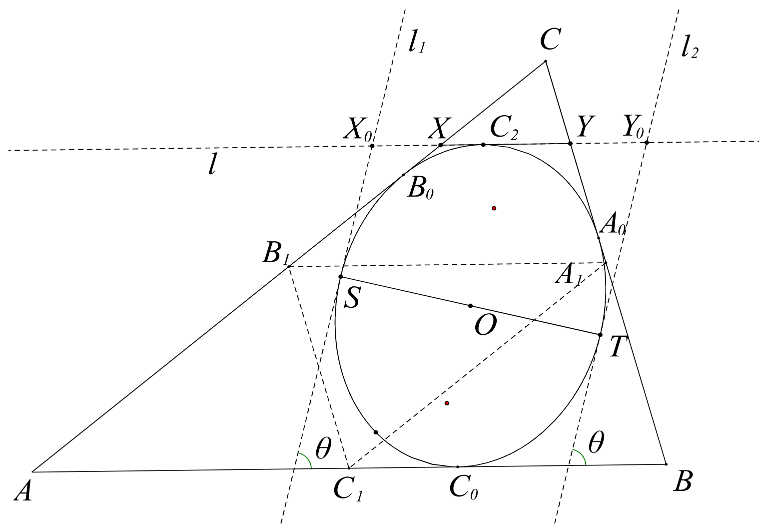
Denote by S and T the two end points of the inscribed ellipse's minor axis.



(1) If one of points S and T lies above or on the line segment A_1B_1 (supposing S is the one will make no difference), then the length of the minor axis $b = SO$, and $a \geq OC_0$. Thus $SC_0 \leq SO + OC_0 \leq a + b$

Since S is not below A_1B_1, thus $SC_0 \geq \frac{1}{2} h_c$

(where h_c is the altitude on side AB), hence $2a + 2b \geq 2SC_0 \geq h_c$. Furthermore, if all the inequalities hold equality, it will lead to the fact that OC_0 is the semi-major axis of the ellipse and S, O and C_0 are collinear points. But the major and minor axes of the ellipse are perpendicular to each other! Contradiction happens. Thus $2a + 2b > h_c \geq h_{\min}$ has been proven.

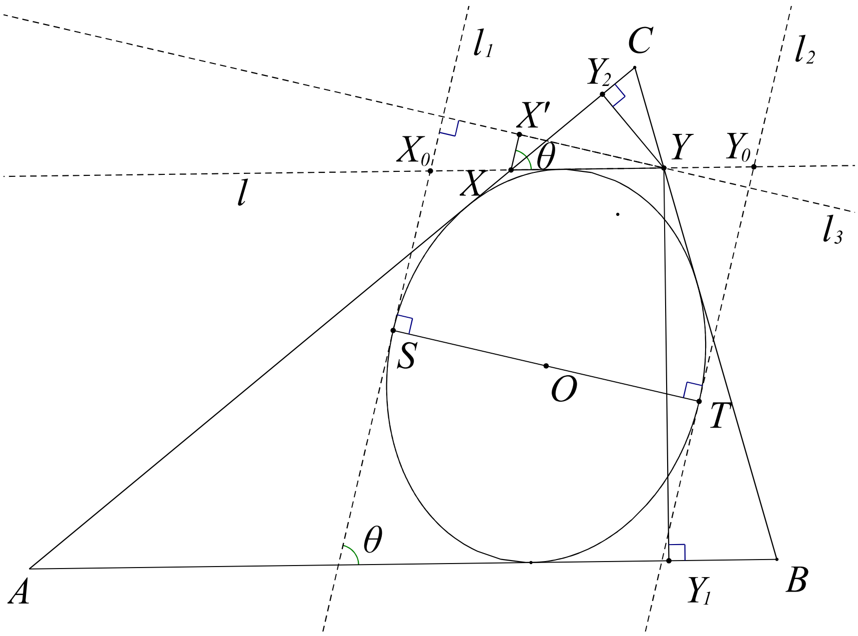


(2) Next we will assume that both S and T lie below A_1B_1, and denote by C_2 the symmetric point of C_0 with respect to point O. Draw line l through C_2 parallel to AB, and l is tangent to the ellipse at C_2. Then draw line l_1 through S and line l_2 through T, both tangent to the ellipse. Denote by X_0 and Y_0 the intersection points of l_1 and l_2 with l, X and Y the intersection points of AC and BC with l, respectively. Denote by θ the separation angle between AB and the two parallel lines l_1

and l_2 (counterclockwise).

As the first step we will prove $\angle A < \theta < \pi - \angle B$.

As a matter of fact, the inscribed ellipse is surrounded by two pairs of parallel lines: l_1 and l_2, l and AB, that is, the ellipse is within a parallelogram enclosed by the above mentioned two pairs of parallel lines. With the fact that the ellipse is tangent to line segments CA_1 and CB_1 respectively we know that the two line segments have their parts within the parallelogram; in other words, the line segment XY is part of line segment X_0Y_0. Aware of that both points S and T are below A_1B_1, and point C is above A_1B_1, so if X_0 is on the left of point X then $\theta > \angle A$ must be true, and if Y_0 is on the right of point Y it must be $\theta < \pi - \angle B$.



Summing up we get $\angle A < \theta < \pi - \angle B$, and thus $\sin\theta > \min\{\sin A, \sin B\} = \sin A$ (please notice that both $\angle A$ and $\angle B$ are ΔABC's interior angles and $\angle A \le \angle B$).

Draw line l_3 through Y perpendicular to lines l_1 and l_2, denote by X' the projective shadow of point X on line l_3, Y_1 the shadow of point Y on AB, Y_2 its shadow on AC.

Since both points X and Y lie between the parallel lines l_1 and l_2, and ST is the distance between l_1 and l_2, we know $X'Y \le ST$. Furthermore

$$X'Y = XY \cdot \sin\theta > XY \cdot \sin A = XY \cdot \sin\angle YXY_2 = YY_2$$

To sum up

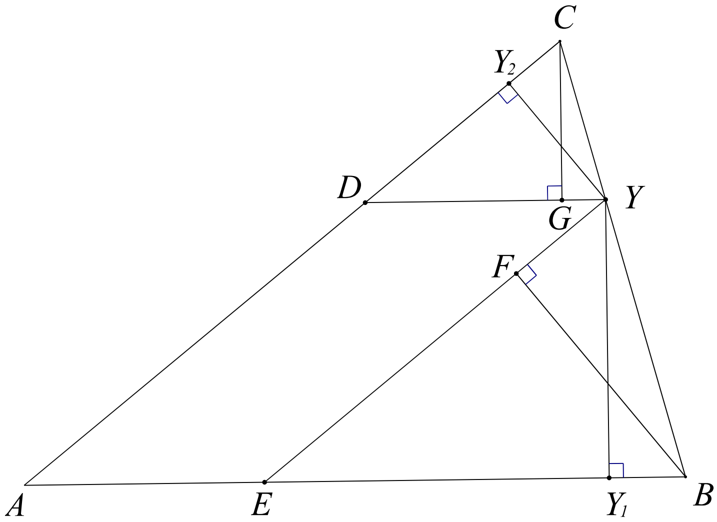
$$2b = ST > YY_2$$

On the other hand, the length of the ellipse's major axis $2a \ge YY_1$ which, together with the above conclusions, leads to

$$2a + 2b > YY_1 + YY_2$$

To finish the proof we need only to show $YY_1 + YY_2 \ge h_{\min}$.

Draw a line through point parallel to AB and AC respectively, intersecting with AC and AB at points D and E respectively. Suppose $BF \perp EY$ at point F, $CG \perp DY$ at point G. Next we will do it in two situations, $\angle B \ge \angle C$ and $\angle B < \angle C$.



If $\angle B \ge \angle C$, from $\angle A \le \angle B$ we know $\angle B$ is the greatest interior angle of ΔABC, which means that the attitude h_b from B is the least attitude. By $\Delta EBY \sim \Delta ABC$ we know that BF is the least attitude of ΔEBF, and thus $YY_1 \ge BF$. Hence

$$YY_1 + YY_2 \ge BF + YY_2 = h_b = h_{\min}$$

If $\angle B < \angle C$, and since $\angle A \le \angle B$, we know $\angle C$ is the greatest interior angle of ΔABC, and h_c is its least attitude. By $\Delta DYC \sim \Delta ABC$ we know that CG is the least attitude of ΔDYC, thus $YY_2 \ge CG$. Hence $YY_1 + YY_2 \ge CG + YY_1 = h_C = h_{\min}$.

That is to say that $YY_1 + YY_2 \ge h_{\min}$ is always true. Thus $2a + 2b > YY_1 + YY_2 \ge h_{\min}$ is also true.

With the reasoning in the two situations we have proven Theorem 3.

References

[1] Roger A., Johnson, Modern Geometry, Shanghai Educational Press, 1999.

[2] прасолов в.в, Russian Collection of Plane Geometry Problems, 6th edition, Harbin Institute of Technology Press, 2009.

[3] Heinrich Dorrie, 100 Great Problems of Elementary Mathematics - Their History and Solution, Shanghai Scientific and Technical Publishers, 1982.

98 Steiner's Ellipse Problem

Of all the ellipses that can be circumscribed about (inscribed in) a given triangle, which one has the smallest (largest) area?

"Dans le plan, la question des polygones d'aire maximum ou minimum inscrits ou circonscrits à une ellipse ne présente aucune difficulté. Il suffit de projeter l'ellipse de telle manière qu'elle devienne un cercle, et l'on est ramené à une question bien connue de géométrie élémentaire"* (Darboux, *Principes de Géométrie analytique*, p. 287).

* Translation: "In a plane the question of polygons of maximum or minimum area inscribed in or circumscribed about an ellipse offers no difficulty. All that is necessary is to project the ellipse in such manner that it is transformed into a circle, and the problem is reduced to a well-known question of elementary geometry".

The solution of the problem is based on the two auxiliary theorems:

I. *Of all the triangles inscribed in a circle the one possessing the maximum area is the equilateral.*

II. *Of all the triangles that can be circumscribed about a circle the one possessing the minimum area is the equilateral.*

Proof of I. We call the circle diameter d, the sides and angles of an inscribed triangle p, q, r and α, β, γ, respectively, the area of the triangle J. Then

$$J = \tfrac{1}{2}pq\sin\gamma$$

and

$$p = d\sin\alpha, \qquad q = d\sin\beta,$$

and consequently,

$$J = \tfrac{1}{2}d^2\cdot\sin\alpha\sin\beta\sin\gamma.$$

According to No. 92, the product of the sines $\sin\alpha\sin\beta\sin\gamma$ of the three angles α, β, γ of constant sum (180°) is at a maximum when

$$\alpha = \beta = \gamma (= 60°),$$

i.e., when the triangle is equilateral. The area of this maximal triangle is $\frac{3}{16}\sqrt{3}d^2$, thus $\sqrt{27}/4\pi$ of the area of the circle.

Proof of II. If we designate the sides of an arbitrary circumscribed triangle PQR as p, q, r, then the tangents to the circle from the vertexes P, Q, R are $x = s - p$, $y = s - q$, $z = s - r$, where s

represents half the perimeter of the triangle

$$\left(s = \frac{p + q + r}{2} = x + y + z\right).$$

The area J of the triangle and the radius ρ of the inscribed circle are given by the well-known formulas

$$J = \rho s \quad \text{and} \quad J = \sqrt{xyzs} \quad \text{(Hero of Alexandria)}.$$

These give us

$$s\rho^2 = xyz.$$

Making use of the formula $J = \rho s$, we write this equation in the following two ways:

$$\frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy} = \frac{1}{\rho^2}, \tag{1}$$

$$\frac{1}{yz}\cdot\frac{1}{zx}\cdot\frac{1}{xy} = \frac{1}{J^2\rho^2}. \tag{2}$$

We now introduce the new unknowns

$$u = \frac{1}{yz}, \qquad v = \frac{1}{zx}, \qquad w = \frac{1}{xy}$$

and obtain

$$u + v + w = \frac{1}{\rho^2}, \qquad uvw = \frac{1}{J^2\rho^2}.$$

Since J is supposed to be a minimum and ρ is constant, uvw must attain a maximum.

A product uvw of numbers u, v, w of constant sum ($u + v + w =$ const.) reaches a maximum, however (No. 10), when the numbers are equal to each other: $u = v = w$. The circumscribed triangle therefore becomes smallest when $yz = zx = xy$, i.e., when $x = y = z$, i.e., when $p = q = r$, which proves II.

We find that the area of the smallest circumscribed triangle is four times that of the maximum inscribed triangle, i.e., $\sqrt{27}\,\rho^2$, and for the ratio of this area to the area of the circle we obtain the improper fraction $\sqrt{27}/\pi$.

Now for the *solution of the ellipse problem*! Let $\mathfrak{E}$ be any ellipse circumscribed about (inscribed in) the given triangle *abc*, f its surface area, δ the area of the triangle *abc*. We consider $\mathfrak{E}$ as the normal projection of a circle $\mathfrak{K}$, whose surface area we will call F. In the projection the inscribed (circumscribed) triangle ABC of the circle, possessing an area we will call Δ, corresponds to the inscribed (circumscribed) triangle *abc* of the ellipse. If μ represents the cosine of the angle between the plane of the circle and the plane of the ellipse, then the normal projection of every surface lying in the plane of the circle is the μ-multiple of the surface. This gives us the formulas

$$f = \mu F, \qquad \delta = \mu \Delta.$$

Since δ is constant, f attains a minimum (maximum) when the quotient f/δ or the equal quotient F/Δ reaches a minimum (maximum). The latter quotient, however, according to auxiliary theorem I. (II.) reaches its minimal (maximal) value $4\pi/\sqrt{27}$ $(\pi/\sqrt{27})$ when the triangle ABC is equilateral.

To establish more exactly the ellipse determined by this condition, we make use of the properties of a normal projection: 1. *Parallelism is not annulled by projection.* 2. *The ratio between parallel segments is maintained in projection:* in particular, the ratio of two segments of the same line is not altered.

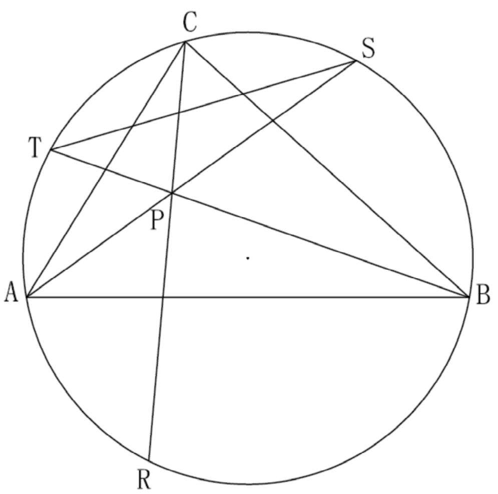
Now, the center M of the circle is the point of intersection of the medians of the equilateral triangle ABC and the diameter through C bisects the chords of the circle parallel to AB. Consequently, the point of intersection of the medians of the triangle *abc* is the center point m of the sought-for ellipse, and the ellipse diameter through c bisects the ellipse chords parallel to the side *ab*, so that *ab* and *mc* are *conjugate* directions of the ellipse. Now, since the circle radius MK parallel to the circle chord (tangent) AB is equal to $1/\sqrt{3}(\sqrt{3}/6)$ of AB, the ellipse half diameter mk parallel to the ellipse chord (tangent) *ab* is also equal to $1/\sqrt{3}(\sqrt{3}/6)$ of *ab*.

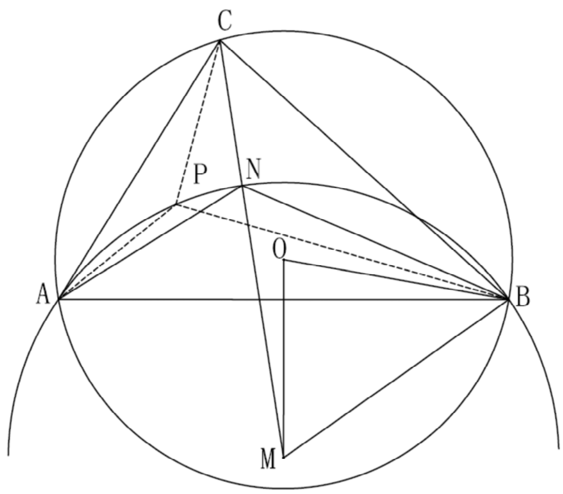
Result. Of all the ellipses that can be circumscribed about (inscribed in) a given triangle *abc*, the one with the smallest (greatest) area is the ellipse whose midpoint m is the point of intersection of the medians of the triangle *abc* and from which the ellipse half diameter to c (to the center of *ab*) and the ellipse half diameter parallel to *ab*, $mk = ab/\sqrt{3}(ab/2\sqrt{3})$, are conjugate half diameters. The area of the ellipse thus characterized—the so-called Steiner ellipse—is

$$\frac{4\pi}{\sqrt{27}} \left(\frac{\pi}{\sqrt{27}}\right) \quad \text{of the area of the triangle.}$$

Appendix 2: The problem about the minimum of $f(P)=\dfrac{BP\cdot CP}{SP}$ mentioned in Part 3, Chapter 4.

Assume that ΔABC is inscribed in $\odot O$, and the inradius and circumradius of ΔABC are r and R, respectively. P is a randomly given point with ΔABC. Connect and extend AP, BP and CP, and assume they intersect with $\odot O$ at points S, T and R. From Part 3, Chapter 4 we can see that $f(P)=\dfrac{BP\cdot CP}{SP}$ is actually symmetrical with respect to A, B and C, so we may assume that $\angle C$ is an acute angle.





Now we're going to find the minimum of $f(P)=\dfrac{BP\cdot CP}{SP}$.

Since A, B, S and T are concyclic, we know $\Delta APB\sim\Delta TPS$, and thus $\dfrac{BP}{SP}=\dfrac{AB}{TS}$.

Accordingly,

$$f(P)=CP\cdot\frac{AB}{ST}=CP\cdot\frac{AB}{2R\sin\angle SAT}=CP\cdot\frac{AB}{2R\sin(\angle APB-\angle ATB)}=CP\cdot\frac{AB}{2R\sin(\angle APB-\angle C)}$$

So if we draw the circumcircle Γ of ΔAPB, and denote by M its center, then OM is the perpendicular bisector of AB, and M lies below O.

It's easy to know that C lies outside of Γ. Denote by N the intersection of CM and Γ, and then we know $CP \geq CN = CM - NM = CM - BM$. Thus

$$f(P) \geq \frac{(CM - BM) \cdot AB}{2R\sin(\angle APB - \angle C)} = \frac{(CM - BM)\sin C}{\sin(\angle APB - \angle C)}.$$

Connect OB, OM and BM. From $\angle BOM = \frac{\angle AOB}{2} = \angle C$, and $\angle OMB = \frac{\angle AMB}{2} = \pi - \angle APB$, we know

$$\angle OBM = \pi - \angle C - (\pi - \angle APB) = \angle APB - \angle C.$$

Accordingly, we know

$$\frac{\sin C}{\sin(\angle APB - \angle C)} = \frac{BM}{OM},$$

thus

$$f(P) \geq \frac{(CM - BM)\sin C}{\sin(\angle APB - \angle C)} = \frac{(CM - BM) \cdot BM}{OM}.$$

Now let's prove that $\frac{(CM - BM) \cdot BM}{OM}$ reaches its minimum iff M lies on $\odot O$.

Set up a rectangular coordinate system with O as its origin and $\overrightarrow{OM}$ as the negative direction of Y axis. Assume OA=OB=OB=1.

Since OM is the perpendicular bisector of AB, we can assume that $A(-\cos\beta, \sin\beta)$, $B(\cos\beta, \sin\beta)$, $C(\cos\alpha, \sin\alpha)$; also, $M(0, -x)$, where $x > 0$.

Thus we know $CM = \sqrt{x^2 + 2x\sin\alpha + 1}$, $BM = \sqrt{x^2 + 2x\sin\beta + 1}$, $OM = x$, and

$$\frac{(CM - BM) \cdot BM}{OM} = \frac{\sqrt{(x^2 + 2x\sin\alpha + 1)(x^2 + 2x\sin\beta + 1)} - (x^2 + 2x\sin\beta + 1)}{x}$$

Accordingly we only need to figure out the minimum of

$$f(x) = \frac{\sqrt{(x^2 + 2x\sin\alpha + 1)(x^2 + 2x\sin\beta + 1)} - (x^2 + 2x\sin\beta + 1)}{x}$$

Denote $L = \sqrt{(x^2 + 2x\sin\alpha + 1)(x^2 + 2x\sin\beta + 1)}$, and then the derivative of $f(x)$ is

$f'(x) =$

$$\frac{\frac{1}{2}x\frac{[(x^2+2x\sin\alpha+1)(2x+2\sin\beta)+(x^2+2x\sin\beta+1)(2x+2\sin\alpha)]}{\sqrt{(x^2+2x\sin\alpha+1)(x^2+2x\sin\beta+1)}}-\sqrt{(x^2+2x\sin\alpha+1)(x^2+2x\sin\beta+1)}-x^2+1}{x^2}$$

$$=\frac{x[2x^3+3(\sin\alpha+\sin\beta)x^2+(4\sin\alpha\sin\beta+2)x+2]-L^2-(x^2-1)L}{Lx^2}$$

$$=\frac{(x^2-1)[x^2+(\sin\alpha+\sin\beta)x+1-L]}{Lx^2}$$

Furthermore, from AM-GM inequality, we know that

$L \le \frac{(x^2+2x\sin\alpha+1)+(x^2+2x\sin\beta+1)}{2} = x^2+(\sin\alpha+\sin\beta)x+1$, and since $\sin\alpha \ne \sin\beta$, the equality cannot hold, meaning $x^2+(\sin\alpha+\sin\beta)x+1-L>0$ is always true.

Thus $f'(x)$ is negative when $0<x<1$, is equal to 0 when $x=1$, and is positive when $x>1$.

So we know on $(0,+\infty)$ $f(x)$ reaches its minimum when $x=1$.

It means that when $\frac{(CM-BM)\cdot BM}{OM}$ reaches its minimum, $OM = OA$, or in other words M lies on $\odot O$. Since OM is the perpendicular bisector of AB, we know that M is the midpoint of the minor arc AB, which means that CM bisects ∠BAC. Thus $BM = 2R\sin\frac{C}{2}, CM = 2R\sin(B+\frac{C}{2}), OM = R$; substitute it into the expression we get

$$\frac{(CM-BM)\cdot BM}{OM} = 4R\sin\frac{C}{2}[\sin(B+\frac{C}{2})-\sin\frac{C}{2}] = 8R\sin\frac{C}{2}\sin\frac{A}{2}\sin\frac{B}{2} = 2r$$

So we have $f(P) \ge \frac{(CM-BM)\cdot BM}{OM} \ge 2r$.

On the other hand, when P coincides with the incenter of ΔABC, with a few steps of calculation we know $f(P) = \frac{BP\cdot CP}{SP} = 2r$. In sum, the minimum of $f(P)$ is $2r$.