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A DEFORMATION OF PENNER'S SIMPLICIAL COORDINATE

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Abstract

We find a one-parameter family of coordinates $\{\Psi_h\}_{h\in\mathbb{R}}$ which is a deformation of Penner's simplicial coordinate of the decorated Teichmüller space of an ideally triangulated punctured surface (S, T) of negative Euler characteristic. If $h \geqslant 0$, the decorated Teichmüller space in the Ψ_h coordinate becomes an explicit convex polytope $P(T)$ independent of h, and if $h < 0$, the decorated Teichmüller space becomes an explicit bounded convex polytope $P_h(T)$ so that $P_h(T) \subset P_{h'}(T)$ if $h < h'$. As a consequence, Bowditch-Epstein and Penner's cell decomposition of the decorated Teichmüller space is reproduced.

1. Introduction

Decorated Teichmüller space of a punctured surface was introduced by Penner in [15] as a fiber bundle over the Teichmüller space of complete hyperbolic metrics with cusp ends. He also gave a cell decomposition of the decorated Teichmüller space invariant under the mapping class group action. To give the cell decomposition, Penner used the convex hull construction and introduced the simplicial coordinate Ψ in which the cells can be easily described. In [4], Bowditch-Epstein obtained the same cell decomposition using the Delaunay construction. The corresponding results for the Teichmüller space of a surface with geodesic boundary have also been obtained. Using Penner's convex hull construction, Ushijima [19] found a mapping class group invariant cell decomposition, and following the approach of Bowditch-Epstein [4], Hazel [10] obtained a natural cell decomposition of the Teichmüller space of a surface with fixed geodesic boundary lengths. As a counterpart of Penner's simplical coordinate Ψ, Luo [12] introduced a coordinate Ψ_0 on the Teichmüller space of an ideally triangulated surface with geodesic boundary, and Mondello [14] pointed out that the Ψ_0 coordinate gave a natural cell decomposition of the Teichmüller space.

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In [13], Luo deformed the Ψ_0 coordinate to a one-parameter family of coordinates $\{\Psi_h\}_{h\in\mathbb{R}}$ of the Teichmüller space of a surface with geodesic boundary, and proved that, for $h \geqslant 0$, the image of Ψ_h is an explicit open convex polytope independent of h. For $h < 0$, Guo [6] proved that the image of Ψ_h is an explicit bounded open polytope. It is then a natural question to ask if there is a corresponding deformation of Penner's simplicial coordinate Ψ. The purpose of this paper is to provide an affirmative answer to this question. We give a one-parameter family of coordinates $\{\Psi_h\}_{h\in\mathbb{R}}$ of the decorated Teichmüller space of an ideally triangulated punctured surface so that Ψ_0 coincides with Penner's simplicial coordinate Ψ (Theorem 1.1). We also describe the image of Ψ_h (Theorem 1.2) and show that Ψ_h is the unique possible deformation of Ψ (Theorem 5.1). As an application, Bowditch-Epstein and Penner's cell decomposition of the decorated Teichmüller space is reproduced using the Ψ_h coordinate (Corollary 1.4). The main results of this paper can be considered as a counterpart of the work of [6], [13] and [8].

To be precise, let $\overline{T}$ be a triangulation of a closed surface $\overline{S}$ and let V, E and F be the set of vertices, edges and triangles of $\overline{T}$ respectively. We call $T = \{\sigma - V \mid \sigma \in F\}$ an ideal triangulation of the punctured surface $S = \overline{S} - V$, and V the set of ideal vertices (or cusps) of S. As a convention in this paper, S is assumed to have negative Euler characteristic. Let $T_c(S)$ be the Teichmüller space of complete hyperbolic metrics with cusp ends on S. According to Penner [15], a *decorated hyperbolic metric* $(d, r) \in T_c(S) \times \mathbb{R}^V_{>0}$ on S is the equivalence class of a hyperbolic metric d in $T_c(S)$ such that each cusp v is associated with a horodisk B_v centered at v so that the length of ∂B_v is r_v. The space of decorated hyperbolic metrics $T_c(S) \times \mathbb{R}^V_{>0}$ is the *decorated Teichmüller space.*

Let us recall Penner's simplicial coordinate Ψ. Let $(d, r) \in T_c(S) \times \mathbb{R}^V_{>0}$ be a decorated hyperbolic metric and let e be an edge of T. If a and a' are the generalized angles (see Section 2) facing e, and b, b', c and c' are the generalized angles adjacent to e, then Penner's simplicial coordinate $\Psi\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to \mathbb{R}^E$ is defined by

$$\Psi(d, r)(e) = \frac{b + c - a}{2} + \frac{b' + c' - a'}{2}.$$

An edge path $(t_0, e_1, t_1, \ldots, e_n, t_n)$ in a triangulation T is an alternating sequence of edges e_i with $e_i \neq e_{i+1}$ for $i = 1, \ldots, n-1$ and triangles t_i so that adjacent triangles t_{i-1} and t_i share the same edge e_i for any $i = 1, \ldots, n$. An *edge loop* is an edge path with $t_n = t_0$. A *fundamental edge path* is an edge path so that each edge in the triangulation appears at most once, and a *fundamental edge loop* is an edge loop so that each edge in the triangulation appears at most twice. In [15], Penner proved

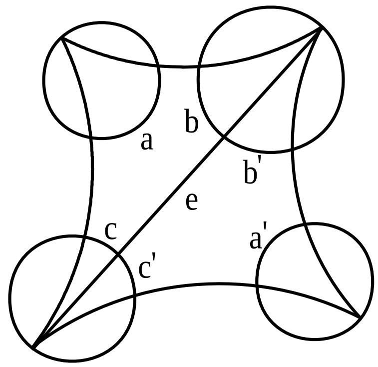

Figure 1. Penner's simplicial coordinate.

that for any vector $z \in \mathbb{R}^E_{\geqslant 0}$ such that $\sum_{i=1}^k z(e_i) > 0$ for any fundamental edge loop $(e_1, t_1, \ldots, e_k, t_k)$, there exists a unique decorated complete hyperbolic metric (d, r) on S so that $\Psi(d, r) = z$. By using a variational principle on decorated ideal triangles, Guo and Luo [7] were able to prove that Penner's simplicial coordinate $\Psi\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to \mathbb{R}^E$ is a smooth embedding with image the convex polytope

$$P(T) = \Big\{ z \in \mathbb{R}^E \mid \sum_{i=1}^{k} z(e_i) > 0 \\ \text{for any fundamental edge loop } (e_1, t_1, \ldots, e_k, t_k) \Big\}.$$

Let (S, T) be an ideally triangulated punctured surface. To deform Penner's simplicial coordinate, we define for each $h \in \mathbb{R}$ a map $\Psi_h\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to \mathbb{R}^E$ by

$$\Psi_h(d, r)(e) = \int_0^{\frac{b+c-a}{2}} e^{ht^2}\, dt + \int_0^{\frac{b'+c'-a'}{2}} e^{ht^2}\, dt,$$

where a and a' are the generalized angles facing e, and b, b', c and c' are the generalized angles adjacent to e as in Figure 1. The main theorems of this paper are the following

Theorem 1.1. *Suppose that (S, T) is an ideally triangulated punctured surface. Then for all $h \in \mathbb{R}$, the map $\Psi_h\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to \mathbb{R}^E$ is a smooth embedding.*

Theorem 1.2. *For $h \in \mathbb{R}$ and an ideally triangulated punctured surface (S, T), let $P_h(T)$ be the set of points $z \in \mathbb{R}^E$ such that*

(a) $z(e) < 2 \int_0^{+\infty} e^{ht^2} dt$ for each edge $e \in E$,

(b) $\sum_{i=1}^n z(e_i) > -2 \int_0^{+\infty} e^{ht^2} dt$ for each fundamental edge path $(t_0, e_1, t_1, \ldots, e_n, t_n)$,

(c) $\sum_{i=1}^{n} z(e_i) > 0$ *for each fundamental edge loop* $(e_1, t_1, \ldots, e_n, t_n)$.
Then we have $\Psi_h(T_c(S) \times \mathbb{R}^V_{>0}) = P_h(T)$. *Furthermore, if* $h \geqslant 0$, *then conditions (a) and (b) become trivial, and the image of* Ψ_h *is the open convex polytope* $P(T)$, *hence independent of* h; *and if* $h < 0$, *then the image* $P_h(T)$ *is a bounded open convex polytope so that* $P_h(T) \subset P_{h'}(T)$ *if* $h < h'$.

Clearly Ψ_0 coincides with Penner's simplicial coordinate Ψ and Ψ_h is a deformation of Ψ. Theorem 1.1 is proved in Section 2 using the strategy of Guo-Luo [7]. We set up a variational principle from the derivative cosine law of decorated ideal triangles whose energy function V_h is strictly concave. For $i = 1, \ldots, |E|$, each variable of V_h is a smooth monotonic function of the edge length l_i in the decorated hyperbolic metric (d, r), and Ψ_h is the gradient of V_h, hence is a smooth embedding. We study various degenerations of decorated ideal triangles in Section 3 with which we will prove Theorem 1.2 in Section 4. We will also prove that $\{\Psi_h\}_{h \in \mathbb{R}}$ is the unique possible deformation of Penner's simplicial coordinate by using a variational principle (Theorem 5.1).

The Delaunay cell decomposition of a decorated hyperbolic surface will be reviewed in Section 6 and we will prove the following

Theorem 1.3. *Suppose* (S, T) *is an ideally triangulated punctured surface, and* $(d, r) \in T_c(S) \times \mathbb{R}^V_{>0}$ *is a decorated hyperbolic metric so that the horodisks associated to the ideal vertices do not intersect. Then for all* $h \in \mathbb{R}$, *the corresponding Delaunay decomposition* $\Sigma_{d,r}$ *coincides with the ideal triangulation* T *if and only if* $\Psi_h(d, r)(e) > 0$ *for each* $e \in E$.

Bowditch-Epstein [4] and Penner [15] showed that there is a natural cell decomposition of the decorated Teichmüller space $T_c(S) \times \mathbb{R}^V_{>0}$ invariant under the mapping class group action. One interesting consequence of Theorems 1.1, 1.2 and 1.3 is the following. Let $A(S) - A_\infty(S)$ be the fillable arc complex as in [9], and let $|A(S) - A_\infty(S)|$ be its underlying space. Penner [15] provided a mapping class group equivariant homeomorphism

$$\Pi\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

so that the restriction of Π to each simplex of maximum dimension is given by the simplicial coordinate Ψ. Using Penner's method, we have the following

Corollary 1.4. *Suppose that* S *is a punctured surface of negative Euler characteristic.*

(a) For all $h > 0$, *there is a homeomorphism*

$$\Pi_h\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

equivariant under the mapping class group action so that the restriction of Π_h *to each simplex of maximum dimension is given by the* Ψ_h *coordinate.*

(b) *The cell structures for various* $h > 0$ *are the same as Penner's.*

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2. A variational principle on decorated ideal triangles

Let (S, T) be an ideally triangulated punctured surface with a set of ideal vertices V and a set of edges E. We assume that S has negative Euler characteristic. The proof of Theorem 1.1 goes as follows. By Penner [15], there is a smooth parametrization of the decorated Teichmüller space $T_c(S) \times \mathbb{R}^V_{>0}$ by $\mathbb{R}^E$ using the edge lengths. From the cosine law of decorated ideal triangles [15], we construct for each $h \in \mathbb{R}$ a smooth strictly convex function V_h on a convex subset of $\mathbb{R}^E$ so that its gradient is Ψ_h. By a variational principle, for each $h \in \mathbb{R}$, the map $\Psi\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to R^E$ is a smooth embedding. This variational principle, whose proof is elementary, is: *If* X *is an open convex set in* $\mathbb{R}^n$ *and* $f\colon\ X \to \mathbb{R}$ *is smooth strictly concave, then the gradient* $\nabla f\colon\ X \to \mathbb{R}^n$ *is injective. Furthermore, if the Hessian of* f *is negative definite for all* $x \in X$*, then* ∇f *is a smooth embedding.*

A *decorated ideal triangle* Δ in the hyperbolic plane $\mathbb{H}^2$ is an ideal triangle such that each ideal vertex v is associated with a horodisk B_v centered at v. If e_1 and e_2 are two edges adjacent to an ideal vertex v of Δ, then the *generalized angle* of Δ at v is defined to be the length of the intersection of ∂B_v and the cusp region enclosed by e_1 and e_2. (In [15], Penner called the generalized angles the h-lengths of a decorated ideal triangle, and in [7], Guo and Luo defined the generalized angle to be twice of the generalized angle defined here.) If e is an edge of Δ with ideal vertices u and v, then the *generalized edge length* (or *edge length* for simplicity) of e in Δ is the signed hyperbolic distance between the intersection of e and ∂B_u and the intersection of e and ∂B_v (Figure 2 (a)). Note that if $B_u \cap B_v \neq \emptyset$, then the generalized edge length of e is either zero or negative (Figure 2 (b)). In a decorated hyperbolic metric $(d, r) \in T_c(S) \times \mathbb{R}^V_{>0}$, each triangle σ in T is isometric to an ideal triangle and the decoration $r \in \mathbb{R}^V_{>0}$ induces a decoration on σ. If $e \in E$ is an edge and σ is an ideal triangle adjacent to e, then the *generalized* *edge length* $l_{d,r}(e)$ of e is defined to be the generalized edge length of e in σ. It is clear that $l_{d,r}(e)$ does not depend on the choice of σ.

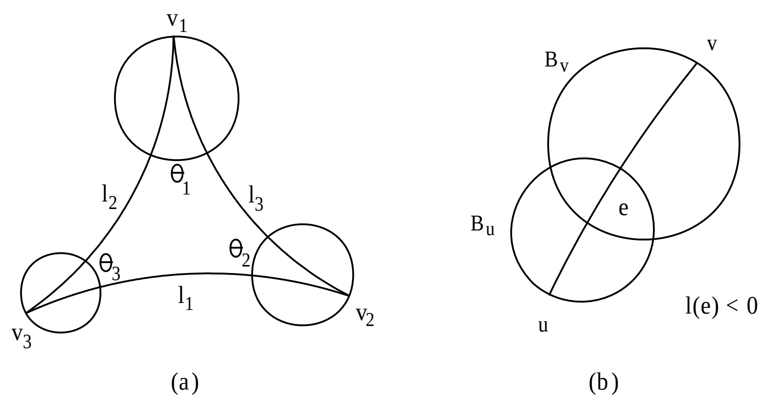

Figure 2. Generalized angles and edge lengths.

Penner [15] defined the length parametrization

$$L\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to \mathbb{R}^E$$
$$(d,r) \mapsto l_{d,r}$$

and showed that L is a diffeomorphism. (The exponential of half of the generalized edge length, which is called the λ-length in [15], is sometimes called Penner's coordinate in the literature.) Penner also proved the following cosine law of decorated ideal triangles. Suppose that Δ is a decorated ideal triangle with edge lengths l_1, l_2 and l_3 and opposite generalized angles θ_1, θ_2 and θ_3. For $i,j,k = 1,2,3$,

$$\theta_i = e^{\frac{l_i - l_j - l_k}{2}} \quad \text{and} \quad e^{l_i} = \frac{1}{\theta_j \theta_k}. \tag{1}$$

As a consequence, there is the sine law of decorated triangles:

$$\frac{\theta_1}{e^{l_1}} = \frac{\theta_2}{e^{l_2}} = \frac{\theta_3}{e^{l_3}}. \tag{2}$$

For $i,j,k = 1,2,3$ and $x_i = \frac{\theta_j + \theta_k - \theta_i}{2}$, let $\mu(x_i) = \int_0^{x_i} e^{ht^2} dt$ and $u_i = \int_0^{l_i} e^{-he^{-t}} dt$. Denote by $U \subset \mathbb{R}^3$ the set of all possible values of $u = (u_1, u_2, u_3)$.

Lemma 2.1. *For each $h \in \mathbb{R}$, the differential 1-form $\omega_h = \sum_{i=1}^3 \mu(x_i) du_i$ is closed in U and the function F_h defined by the integral $F_h(u) = \int_0^u \omega_h$ is strictly concave in U. Furthermore,*

$$\frac{\partial F_h}{\partial u_i} = \int_0^{x_i} e^{ht^2} dt.$$

Proof. Consider the matrix $H = [\frac{\partial \mu(x_i)}{\partial u_j}]_{3\times 3}$. The closedness of ω_h is equivalent to that H is symmetric, and the strict concavity of F_h will follow from the negative definiteness of H. It follows from the partial derivatives of (1) that $\frac{\partial x_i}{\partial l_i} = -\frac{x_i+x_j+x_k}{2}$ and $\frac{\partial x_i}{\partial l_j} = \frac{x_k}{2}$. We have

$$\frac{\partial \mu(x_i)}{\partial u_i} = \frac{e^{hx_i^2}}{e^{-he^{-l_i}}}\frac{\partial x_i}{\partial l_i} = -\frac{x_i+x_j+x_k}{2}e^{h\left(\frac{\theta_i^2+\theta_j^2+\theta_k^2}{4}+\frac{3\theta_j\theta_k-\theta_i\theta_k-\theta_i\theta_j}{2}\right)},$$

and for $i \neq j$, we have

$$\frac{\partial \mu(x_i)}{\partial u_j} = \frac{e^{hx_i^2}}{e^{-he^{-l_j}}}\frac{\partial x_i}{\partial l_j} = \frac{x_k}{2}e^{h\left(\frac{\theta_i^2+\theta_j^2+\theta_k^2}{4}+\frac{\theta_j\theta_k+\theta_i\theta_k-\theta_i\theta_j}{2}\right)},$$

from which we see that H is symmetric. Let

$$c = \frac{1}{2}e^{h\left(\frac{\theta_i^2+\theta_j^2+\theta_k^2}{4}-\frac{\theta_j\theta_k+\theta_i\theta_k+\theta_i\theta_j}{2}\right)} > 0$$

and let D be the diagonal matrix whose (i,i)-th entry is $e^{h\theta_j\theta_k}$. The matrix H can be written as $cDMD$, where

$$M = \begin{bmatrix} -(x_1+x_2+x_3) & x_3 & x_2 \\ x_3 & -(x_1+x_2+x_3) & x_1 \\ x_2 & x_1 & -(x_1+x_2+x_3) \end{bmatrix}.$$

The negative definiteness of H is equivalent to that of M, i.e., the positive definiteness of $-M$. This follows from the direct calculation that each leading principal minor is positive using Sylvester's criterion. q.e.d

Proof of Theorem 1.1. For a decorated hyperbolic metric $(d,r) \in T_c(S) \times \mathbb{R}^V_{>0}$, let $l_{d,r} \in \mathbb{R}^E$ be its length parameter. The integral $u(e) = \int_0^{l_{d,r}(e)} e^{-he^{-t}}dt$ is a smooth monotonic function of $l_{d,r}(e)$, and the possible values of u form an open convex cube U in $\mathbb{R}^E$. With $u_i = u(e_i)$, the energy function $V_h\colon U \to \mathbb{R}$ is defined by

$$V_h(u) = \sum_{\{e_i,e_j,e_k\}} F_h(u_i,u_j,u_k),$$

in which the summation is taken over all of the decorated ideal triangles. By Lemma 2.1, V_h is smooth and strictly concave in U and

$$\frac{\partial V_h}{\partial u_i} = \Psi_h(e_i),$$

i.e., $\nabla V_h = \Psi_h$. By the variational principle, the map $\Psi_h = \nabla V_h\colon U \to \mathbb{R}^E$ is a smooth embedding. q.e.d

3. Degenerations of decorated ideal triangles

To describe the image of Ψ_h, we study degenerations of decorated ideal triangles. Suppose Δ is a decorated ideal triangle with edge lengths l_1, l_2 and l_3 and opposite generalized angles θ_1, θ_2 and θ_3.

Lemma 3.1.

(I) If $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, c_2, c_3)$ with $c_2, c_3 \in (-\infty, +\infty]$, then $\{\theta_1\}$ converges to 0, and we can take a subsequence so that at least one of $\{\theta_2\}$ and $\{\theta_3\}$ converges to $+\infty$.

(II) If $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, -\infty, c_3)$ with $c_3 \in (-\infty, +\infty]$, then $\{\theta_3\}$ converges to $+\infty$, and we can take a subsequence so that at least one of $\{\theta_1\}$ and $\{\theta_2\}$ converges to a finite number.

(III) If $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, -\infty, -\infty)$, then we can take a subsequence such that at least two of $\{\theta_1\}$, $\{\theta_2\}$ and $\{\theta_3\}$ converge to $+\infty$.

Proof. For (I), if $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, c_2, c_3)$, then $\{\frac{l_1-l_2-l_3}{2}\}$ converges to $-\infty$. By cosine law (1), $\{\theta_1\} = \{e^{\frac{l_1-l_2-l_3}{2}}\}$ converges to 0. Let $a_2 = \frac{l_2-l_1-l_3}{2}$ and $a_3 = \frac{l_3-l_1-l_2}{2}$, so $\{a_2 + a_3\} = \{-l_1\}$ converges to $+\infty$. Thus, by taking a subsequence if necessary, at least one of $\{a_2\}$ and $\{a_3\}$, say $\{a_2\}$, converges to $+\infty$, and $\{\theta_2\} = \{e^{a_2}\}$ converges to $+\infty$. For (II), if $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, -\infty, c_3)$, then $\{\frac{l_3-l_1-l_2}{2}\}$ converges to $+\infty$, and $\{\theta_3\} = \{e^{\frac{l_3-l_1-l_2}{2}}\}$ converges to $+\infty$. Letting $a_1 = \frac{l_1-l_2-l_3}{2}$ and $a_2 = \frac{l_2-l_1-l_3}{2}$, we have $\{a_1 + a_2\} = \{-l_3\}$ converges to $-c_3$. Thus, either both $\{a_1\}$ and $\{a_2\}$ converge to a finite number, or by taking a subsequence if necessary, at least one of $\{a_1\}$ and $\{a_2\}$, say $\{a_1\}$, converges to $-\infty$. In the former case, both $\{\theta_1\} = \{e^{a_1}\}$ and $\{\theta_2\} = \{e^{a_2}\}$ converge to a finite number, and in the latter case, $\{\theta_1\} = \{e^{a_1}\}$ converges to 0. For (III), we have by cosine law (1) that $\{\theta_1\theta_2\} = \{e^{-l_3}\}$ converges to $+\infty$. Thus, by taking a subsequence if necessary, at least one of $\{\theta_1\}$ and $\{\theta_2\}$, say $\{\theta_1\}$, converges to $+\infty$. Since $\{\theta_2\theta_3\} = \{e^{-l_1}\}$ converges to $+\infty$ as well, by taking a subsequence, at least one of $\{\theta_2\}$ and $\{\theta_3\}$ converges to $+\infty$. q.e.d

We call a converging sequence of decorated ideal triangles in (I), (II) and (III) of Lemma 3.1 a *degenerated decorated ideal triangle of type I, II and III* respectively. If a is the generalized angle facing an edge e in a decorated triangle Δ, and b and c are the generalized angles adjacent to e, then we call $x(e) = \frac{b+c-a}{2}$ the *x-invariant* of e in Δ.

Corollary 3.2. *If Δ is a degenerated decorated ideal triangle of type I, II or III, then by taking a subsequence if necessary, there is an edge e of Δ such that $\{l(e)\}$ converges to $-\infty$ and $\{x(e)\}$ converges to $+\infty$.*

Proof. If Δ is of type I and $\{l_1\}$ converges to $-\infty$, then by Lemma 3.1 (I), $\{x_1\} = \{\frac{\theta_2+\theta_3-\theta_1}{2}\}$ converges to $+\infty$. If Δ is of type II and $\{(l_1, l_2, l_3)\}$ converges to $(-\infty, -\infty, c_3)$, then by Lemma 3.1 and taking a subsequence if necessary, at least one of $\{\theta_1\}$ and $\{\theta_2\}$, say $\{\theta_1\}$, converges to a finite number, and $\{\theta_3\}$ converges to $+\infty$. Thus, $\{l_1\}$ converges to $-\infty$ and $\{x_1\} = \{\frac{\theta_2+\theta_3-\theta_1}{2}\}$ converges to $+\infty$. If Δ is of type III, then there are at least two of $\{\theta_1\}$, $\{\theta_2\}$ and $\{\theta_3\}$ that converge to $+\infty$. Suppose $\{\theta_3\}$ is one of the two that converge to $+\infty$. Since $\{x_1+x_2\} = \{\theta_3\}$ converges to $+\infty$, by taking a subsequence if necessary, at least one of $\{x_1\}$ and $\{x_2\}$, say $\{x_1\}$, converges to $+\infty$. Thus, $\{l_1\}$ converges to $-\infty$ and $\{x_1\}$ converges to $+\infty$. q.e.d

We call an edge e as in Corollary 3.2 where $l(e) \to -\infty$ and $x(e) \to +\infty$ a *bad edge* of Δ, and otherwise, e is a *good edge*. Note that there may be more than one bad edge in a degenerated ideal triangle.

Lemma 3.3. *Let $\{\Delta^{(m)}\}$ be a sequence of decorated ideal triangles that converges to a degenerated decorated ideal triangle Δ of type I, II or III. Then we can take a subsequence so that for m sufficiently large, the length of each bad edge of $\Delta^{(m)}$ is strictly less than the length of each good edge.*

Proof. If Δ is of type I, then by Lemma 3.1, the length of the only bad edge converges to $-\infty$ and the length of other two edges converge to a finite number. For m sufficiently large, the length of the bad edge is less than the lengths of the good edges.

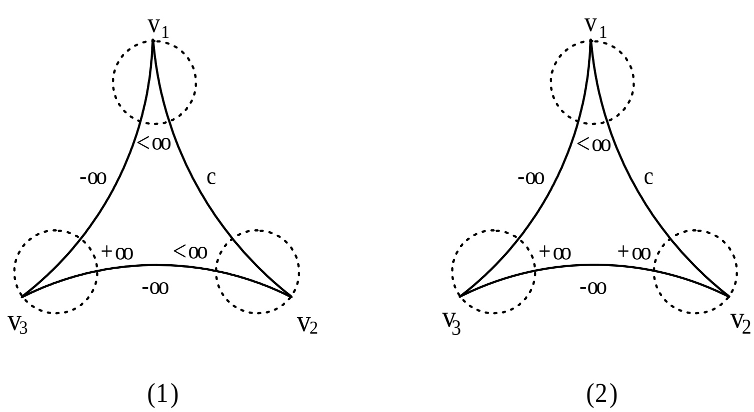

Figure 3. Type II.

If Δ is of type II, we may assume that $\{(l_1^{(m)}, l_2^{(m)}, l_3^{(m)})\}$ converges to $(-\infty, -\infty, c)$ with $c \in (-\infty, +\infty]$. By Lemma 3.1, there are two cases to be considered (Figure 3).

Case 1. Suppose that $\theta_3^{(m)}$ converges to $+\infty$ and both $\theta_1^{(m)}$ and $\theta_2^{(m)}$ converge to a finite number. In this case, both l_1 and l_2 are bad and converge to $-\infty$. The only good edge length l_3 converges to $c \in (-\infty, +\infty]$. Hence for m sufficiently large, $l_1^{(m)} < l_3^{(m)}$ and $l_2^{(m)} < l_3^{(m)}$.

Case 2. Suppose that $\theta_3^{(m)}$ converges to $+\infty$, and one of $\theta_1^{(m)}$ and $\theta_2^{(m)}$, say $\theta_2^{(m)}$, converges to $+\infty$ and $\theta_1^{(m)}$ converges to a finite number. In this case l_1 is bad. If l_2 is also bad, then both l_1 and l_2 converge to $-\infty$, and l_3 converges to $c \in (-\infty, +\infty]$. Hence for m sufficiently large, $l_1^{(m)} < l_3^{(m)}$ and $l_2^{(m)} < l_3^{(m)}$. If l_2 is good, then $\theta_1^{(m)} < \theta_2^{(m)}$ for m sufficiently large, since $\theta_1^{(m)}$ converges to a finite number and $\theta_2^{(m)}$ converges to $+\infty$. By sine law (2), $l_1^{(m)} < l_2^{(m)}$.

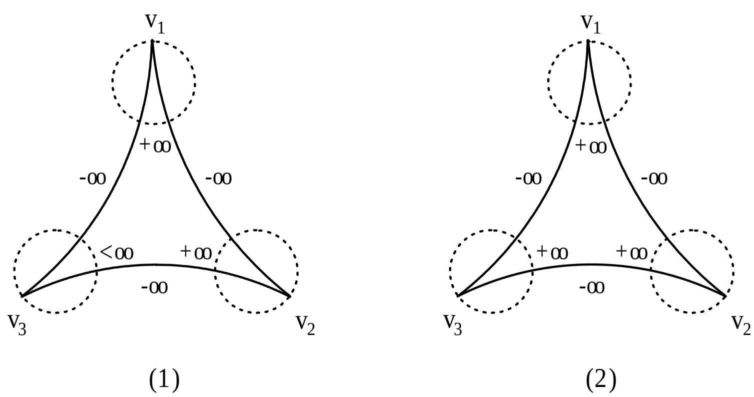

Figure 4. Type III.

If Δ is of type III, then by Lemma 3.1, we also consider two cases (Figure 4).

Case 1. Two of $\theta_1^{(m)}$, $\theta_2^{(m)}$ and $\theta_3^{(m)}$, say $\theta_1^{(m)}$ and $\theta_2^{(m)}$ converge to $+\infty$, and $\theta_3^{(m)}$ converges to a finite number. In this case, l_3 is bad. Since $\theta_3^{(m)} < \theta_1^{(m)}$ and $\theta_3^{(m)} < \theta_2^{(m)}$ for m sufficiently large, by sine law (2), $l_3^{(m)} < l_1^{(m)}$ and $l_3^{(m)} < l_2^{(m)}$. If one of l_1 and l_2, say l_2, is also bad, then $x_2^{(m)} = \frac{\theta_1^{(m)} + \theta_3^{(m)} - \theta_2^{(m)}}{2}$ converges to $+\infty$. Since $\theta_3^{(m)}$ converges to a finite number, $\theta_2^{(m)} < \theta_1^{(m)}$ for m sufficiently large. By sine law (2), $l_2^{(m)} < l_1^{(m)}$.

Case 2. All of $\theta_1^{(m)}$, $\theta_2^{(m)}$ and $\theta_3^{(m)}$ converge to $+\infty$. In this case, since $x_i^{(m)} + x_j^{(m)} = \theta_k^{(m)}$ converges to $+\infty$, by taking a subsequence if necessary, at least two of $x_1^{(m)}$, $x_2^{(m)}$ and $x_3^{(m)}$, say $x_1^{(m)}$ and $x_2^{(m)}$, converge to $+\infty$. Therefore, l_3 is the only possible good edge length, and $x_3^{(m)}$ converges to a finite number. For m sufficiently large, $\theta_1^{(m)} = x_2^{(m)}+x_3^{(m)} < x_1^{(m)}+x_2^{(m)} = \theta_3^{(m)}$ and $\theta_2^{(m)} = x_1^{(m)}+x_3^{(m)} < x_1^{(m)}+x_2^{(m)} = \theta_3^{(m)}$. By sine law (2), $l_1^{(m)} < l_3^{(m)}$ and $l_2^{(m)} < l_3^{(m)}$. q.e.d

Lemma 3.4.

(a) If $\{(l_1, l_2, l_3)\}$ *converges to* $(+\infty, f_2, f_3)$ *with* $f_2, f_3 \in \mathbb{R}$*, then* $\{(\theta_1, \theta_2, \theta_3)\}$ *converges to* $(+\infty, 0, 0)$*.*

(b) If $\{(l_1, l_2, l_3)\}$ *converges to* $(+\infty, +\infty, f_3)$ *with* $f_3 \in \mathbb{R}$*, then* $\{\theta_3\}$ *converges to* 0*.*

(c) If $\{(l_1, l_2, l_3)\}$ *converges to* $(+\infty, +\infty, +\infty)$*, then we can take a subsequence such that at least two of* $\{\theta_1\}$*,* $\{\theta_2\}$ *and* $\{\theta_3\}$ *converge to* 0*.*

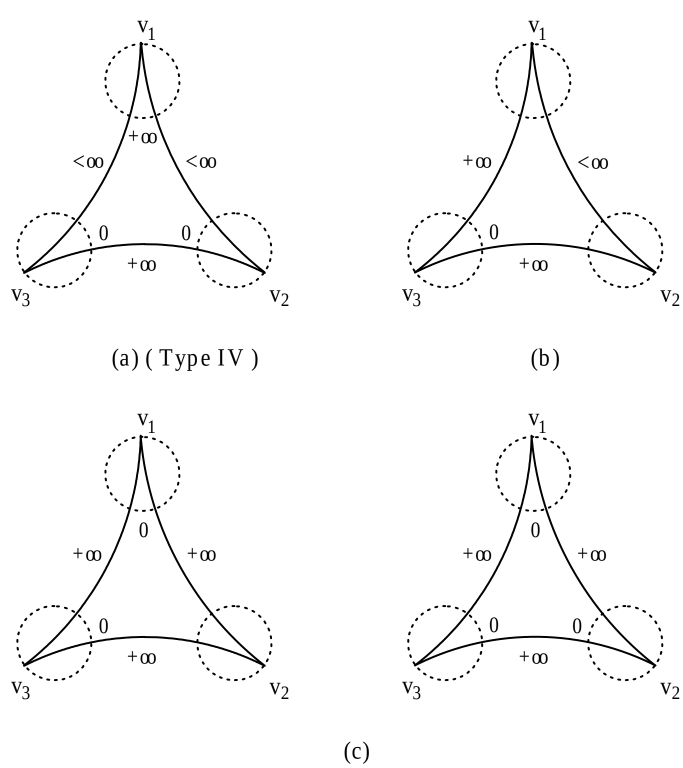

Figure 5. Type IV and other types.

We call a converging sequence of decorated ideal triangles in (a) of Lemma 3.4 a *degenerated decorated ideal triangle of type IV* (Figure 5).

Proof. For (a), if $\{(l_1, l_2, l_3)\}$ converges to $(+\infty, f_2, f_3)$, then by cosine law (1), $\{\theta_1\} = \{e^{\frac{l_1-l_2-l_3}{2}}\}$ converges to $+\infty$, $\{\theta_2\} = \{e^{\frac{l_2-l_1-l_3}{2}}\}$ converges to 0, and $\{\theta_3\} = \{e^{\frac{l_3-l_1-l_2}{2}}\}$ converges to 0. For (b), if $\{(l_1, l_2, l_3)\}$ converges to $(+\infty, +\infty, f_3)$, then $\{\frac{l_3-l_1-l_2}{2}\}$ converges to $-\infty$, and $\{\theta_3\} = \{e^{\frac{l_3-l_1-l_2}{2}}\}$ converges to 0. For (c), if $\{(l_1, l_2, l_3)\}$ converges to $(+\infty, +\infty, +\infty)$, then we have by cosine law (1) that $\{\theta_1\theta_2\} = \{e^{-l_3}\}$ converges to 0. Thus, by taking a subsequence if necessary, at least one of $\{\theta_1\}$ and $\{\theta_2\}$, say $\{\theta_1\}$, converges to 0. Since $\{\theta_2\theta_3\} = \{e^{-l_1}\}$ converges to 0 as well, by taking a subsequence, at least one of $\{\theta_2\}$ and $\{\theta_3\}$ converges to 0. q.e.d

4. The image of Ψ_h

The image of Ψ_h is described in Theorem 1.2. The main task of this section is to give a proof of this theorem. To show that the image of Ψ_h is indeed $P_h(T)$, we make use of the following propositions which are proved in this section.

Proposition 4.1. *$\Psi_h(T_c(S) \times \mathbb{R}^V_{>0}) \subset P_h(T)$ for all $h \in \mathbb{R}$.*

Proposition 4.2. *For all $h \in \mathbb{R}$, the image $\Psi_h(T_c(S) \times \mathbb{R}^V_{>0})$ is closed in $P_h(T)$.*

Proof of Theorem 1.2. Let $P(T)$ be defined as in Theorem 1.2. For $h \geqslant 0$, $P(T) = P_h(T)$ is determined by finitely many strict linear inequalities corresponding to the fundamental edge loops and hence is an open convex polytope independent of h. For $h < 0$, $P_h(T)$ is likewise determined by fundamental edge loops and fundamental edge paths. Moreover, since each edge e can be regarded as a fundamental edge path, conditions (a) and (b) imply that $-2\int_0^{+\infty} e^{ht^2}dt < z(e) < 2\int_0^{+\infty} e^{ht^2}dt$ for each $e \in E$. Thus, $P_h(T)$ is bounded. The monotonicity of the function $f(h) = \int_0^{+\infty} e^{ht^2}dt$ implies that $P_h(T) \subset P_{h'}(T)$ if $h < h'$, and the fact that $\lim_{h\to-\infty} f(h) = \lim_{h\to-\infty} \sqrt{\frac{\pi}{-2h}} = 0$ implies that $\bigcap_{h\in\mathbb{R}_{<0}} P_h(T) = \emptyset$. By Theorem 1.1 and the Invariance of Domain Theorem, $\Psi_h(T_c(S) \times \mathbb{R}^V_{>0})$ is open in $P_h(T)$. By Proposition 4.2, $\Psi_h(T_c(S) \times \mathbb{R}^V_{>0})$ is closed in $P_h(T)$. Connectedness of $P_h(T)$ therefore implies that $\Psi_h(T_c(S) \times \mathbb{R}^V_{>0}) = P_h(T)$. q.e.d

The following Lemma 4.3 will be used in the proof of Propositions 4.1 and 4.2.

Lemma 4.3. *If $r \in \mathbb{R}$ and $x > 0$, then* *(a) for each* $h \in \mathbb{R}$,

$$\int_0^{x+r} e^{ht^2} dt + \int_0^{x-r} e^{ht^2} dt > 0,$$

(b) for each $h \geqslant 0$,

$$\int_0^{x+r} e^{ht^2} dt + \int_0^{x-r} e^{ht^2} dt \geqslant 2 \int_0^{x} e^{ht^2} dt.$$

Proof. For (a), let $f(x) = \int_0^{x+r} e^{ht^2} dt + \int_0^{x-r} e^{ht^2} dt$. Since $f'(x) = e^{h(x+r)^2} + e^{h(x-r)^2} > 0$, the function f is strictly increasing, hence $f(x) > f(0) = 0$ for $x > 0$. For (b), let $g(x) = \int_0^{x+r} e^{ht^2} dt + \int_0^{x-r} e^{ht^2} dt - 2\int_0^x e^{ht^2} dt$. We have that $g(0) = 0$ and $g'(x) = e^{h(x+r)^2} + e^{h(x-r)^2} - 2e^{hx^2} \geqslant 0$. The last inequality follows from the convexity of the function $F(t) = e^{ht^2}$ for $h \geqslant 0$. Since g is increasing, $g(x) \geqslant g(0) = 0$ for $x > 0$. q.e.d

Proof of Proposition 4.1. For $h \geqslant 0$, fix a decorated hyperbolic metric $(d, r) \in T_c(S) \times \mathbb{R}^V_{>0}$. For any fundamental edge loop $(e_1, t_1, \ldots, e_k, t_k)$, let a_i be the generalized angle adjacent to e_i and e_{i+1} (where $e_{k+1} = e_1$). Let the generalized angles of t_i facing e_i and e_{i+1} respectively be b_i and c_i. By definition, the contribution of $\sum_{i=1}^k z(e_i)$ from t_i is

$$\int_0^{\frac{a_i+b_i-c_i}{2}} e^{ht^2} dt + \int_0^{\frac{a_i+c_i-b_i}{2}} e^{ht^2} dt,$$

which is strictly larger than 0 from Lemma 4.3 (a) since $a_i > 0$.

For $h < 0$, let e be any edge in the ideal triangulation T, and let a and a' be the generalized angles facing e. Let b, c, b' and c' be the generalized angles adjacent to e. Then

$$\Psi_h(d, r)(e) = \int_0^{\frac{b+c-a}{2}} e^{ht^2} dt + \int_0^{\frac{b'+c'-a'}{2}} e^{ht^2} dt < 2 \int_0^{+\infty} e^{ht^2} dt.$$

Thus, condition (a) in the definition of $P_h(T)$ is satisfied. Given a fundamental edge path $(t_0, e_0, t_1, \ldots, e_n, t_n)$, let θ_i be the generalized angle in t_i adjacent to e_i and e_{i+1} for $i = 1, \ldots, n-1$, and let β_i and γ_i respectively be the generalized angles of t_i facing e_i and e_{i+1}. Denote by a_0 the generalized angle of t_0 facing e_0, and by a_n the generalized angle of t_n facing e_n. Let b_0 and c_0 be the generalized angles of t_0 adjacent to e_0, and let b_n and c_n be the generalized angles of t_n adjacent to e_n. We have

$$\begin{aligned}
&\sum_{i=1}^{n}\Psi_h(d,r)(e_i)\\
=&\int_0^{\frac{b_0+c_0-a_0}{2}} e^{ht^2}dt+\sum_{i=1}^{n-1}\left(\int_0^{\frac{\theta_i+\gamma_i-\beta_i}{2}} e^{ht^2}dt+\int_0^{\frac{\theta_i+\beta_i-\gamma_i}{2}} e^{ht^2}dt\right)\\
&+\int_0^{\frac{b_n+c_n-a_n}{2}} e^{ht^2}dt\\
>&\int_0^{\frac{b_0+c_0-a_0}{2}} e^{ht^2}dt+\int_0^{\frac{b_n+c_n-a_n}{2}} e^{ht^2}dt\\
>&-2\int_0^{+\infty} e^{ht^2}dt,
\end{aligned}$$

where the first inequality is by Lemma 4.3 (a). Thus, condition (b) is satisfied. Given a fundamental edge loop $(e_1,t_1,\dots,e_n,t_n)$ with $e_{n+1}=e_1$, let θ_i for $i=1,\dots,n$ be the generalized angle in t_i adjacent to e_i and e_{i+1}, and let β_i (resp. γ_i) be the generalized angle in t_i facing e_i (resp. e_{i+1}). Again by Lemma 4.3 (a),

$$\sum_{i=1}^{n}\Psi_h(d,r)(e_i)=\sum_{i=1}^{n}\left(\int_0^{\frac{\theta_i+\gamma_i-\beta_i}{2}} e^{ht^2}dt+\int_0^{\frac{\theta_i+\beta_i-\gamma_i}{2}} e^{ht^2}dt\right)>0.$$

Thus, condition (c) is satisfied, and $\Psi_h(T_c(S)\times\mathbb{R}^V_{>0})\subset P_h(T)$. q.e.d

To prove Proposition 4.2, we use Penner's length parametrization. For each sequence $\{l^{(m)}\}$ in $\mathbb{R}^E$ such that $\{\Psi_h(l^{(m)})\}$ converges to a point $z\in P(T)$, we claim that $\{l^{(m)}\}$ contains a subsequence converging to a point in $\mathbb{R}^E$. Let $\theta^{(m)}$ be the generalized angles of the decorated ideal triangles in (S,T) in the decorated hyperbolic metric $l^{(m)}$. By taking a subsequence if necessary, we may assume that $\{l^{(m)}\}$ converges in $[-\infty,+\infty]^E$ and that for each generalized angle θ_i, the limit $\lim_{m\to\infty}\theta_i^{(m)}$ exists in $[0,+\infty]$. In the case that $h\geqslant 0$, we need the following

Lemma 4.4. *If $h\geqslant 0$, then $\lim_{m\to\infty}\theta_i^{(m)}\in[0,+\infty)$ for all i.*

Proof. Suppose to the contrary that $\lim_{m\to\infty}\theta_1^{(m)}=+\infty$ for some generalized angle θ_1. Let e_2 and e_3 be the edges adjacent to θ_1 in the triangle t_1, and θ_2 and θ_3 respectively be the generalized angles facing e_2 and e_3. Take a fundamental edge loop $(e_{n_1},t_{n_1},\dots,e_{n_k},t_{n_k})$ containing (e_2,t_1,e_3). By Lemma 4.3, we have

$$\begin{aligned}\sum_{i=1}^{k} z(e_{n_i}) &= \lim_{m\to\infty}\sum_{i=1}^{k}\Psi_h(l^{(m)})(e_{n_i})\\ &\geqslant \lim_{m\to\infty}\left(\int_0^{\frac{\theta_1^{(m)}+\theta_2^{(m)}-\theta_3^{(m)}}{2}} e^{ht^2}dt+\int_0^{\frac{\theta_1^{(m)}+\theta_3^{(m)}-\theta_2^{(m)}}{2}} e^{ht^2}dt\right)\\ &\geqslant \lim_{m\to\infty} 2\int_0^{\frac{\theta_1^{(m)}}{2}} e^{ht^2}dt\\ &= +\infty.\end{aligned}$$

This contradicts the assumption that $z \in P(T)$. q.e.d

Proof of Proposition 4.2. For $h \geqslant 0$, by taking a subsequence of $\{l^{(m)}\}$, we may assume that $\lim_{m\to\infty} l^{(m)} = l \in [-\infty,+\infty]^E$. If l were not in $\mathbb{R}^E$, then there would exist an edge $e \in E$ so that $l(e) = \pm\infty$. Let Δ be a decorated ideal triangle adjacent to e, and let $\theta_1^{(m)}$ and $\theta_2^{(m)}$ be the generalized angles in Δ adjacent to e in the metric $l^{(m)}$. By (1),

$$e^{l^{(m)}(e)} = \frac{1}{\theta_1^{(m)}\theta_2^{(m)}},$$

and $\theta_i^{(m)} \in (0,+\infty)$ for $i = 1, 2$.

Case 1 If $l(e) = -\infty$, then $e^{l(e)} = 0$. By the identity above, one of $\lim_{m\to\infty}\theta_i^{(m)}$ for $i = 1, 2$ must be $+\infty$. This contradicts Lemma 4.4.

Case 2 If $l(e) = +\infty$, then $e^{l(e)} = +\infty$. By the identity above, one of $\lim_{m\to\infty}\theta_i^{(m)}$ for $i = 1, 2$ must be zero. Suppose without loss of generality that $\lim_{m\to\infty}\theta_1^{(m)} = 0$. Let e_1 be the edge in the decorated ideal triangle Δ opposite to θ_2, and let θ_3 be the generalized angle in Δ facing e. By (1), we have

$$e^{l^{(m)}(e_1)} = \frac{1}{\theta_1^{(m)}\theta_3^{(m)}}.$$

By Lemma 4.4, $\theta_3^{(m)}$ is bounded above, hence $l(e_1) = +\infty$. For any decorated ideal triangle Δ adjacent to e with $l(e) = +\infty$, we have an edge e_1 in Δ and a generalized angle θ_1 adjacent to e and e_1 so that $l(e_1) = +\infty$ and $\lim_{m\to\infty}\theta_1^{(m)} = 0$. Applying this logic to e_1 and the decorated ideal triangle Δ_1 adjacent to e_1 other than Δ, we obtain the next angle θ_2 and edge e_2 in Δ_1 so that $l(e_2) = +\infty$ and $\lim_{m\to\infty}\theta_2^{(m)} = 0$. Since there are only finitely many edges and triangles, this yields a fundamental edge loop $(e_k, \Delta_k, \ldots, e_n, \Delta_n)$ in T such that $l(e_i) = +\infty$ for $i = k, \ldots, n$ and $\lim_{m\to\infty} \theta_i^{(m)} = 0$, where θ_i is the generalized angle in Δ_{i-1} adjacent to e_{i-1} and e_i. Denote respectively by β_i and γ_i the generalized angles of Δ_{i-1} facing e_{i-1} and e_i, and let $\bar{\beta}_i = \lim_{m\to\infty} \beta_i^{(m)}$ and $\bar{\gamma}_i = \lim_{m\to\infty} \gamma_i^{(m)}$. By Lemma 4.4, both $\bar{\beta}_i$ and $\bar{\gamma}_i$ are finite real numbers, and we have

$$\begin{aligned}
\sum_{i=k}^{n} z(e_i) &= \lim_{m\to\infty} \sum_{i=k}^{n} \Psi_h(l^{(m)})(e_i) \\
&= \lim_{m\to\infty} \sum_{i=k}^{n} \left(\int_0^{\frac{\theta_i^{(m)}+\beta_i^{(m)}-\gamma_i^{(m)}}{2}} e^{ht^2} dt + \int_0^{\frac{\theta_i^{(m)}+\gamma_i^{(m)}-\beta_i^{(m)}}{2}} e^{ht^2} dt \right) \\
&= \sum_{i=k}^{n} \left(\int_0^{\frac{\bar{\beta}_i-\bar{\gamma}_i}{2}} e^{ht^2} dt + \int_0^{\frac{\bar{\gamma}_i-\bar{\beta}_i}{2}} e^{ht^2} dt \right) \\
&= 0.
\end{aligned}$$

This contradicts the assumption that $z \in P(T)$.

For $h < 0$ and each sequence $\{l^{(m)}\}$ in $\mathbb{R}^E$ so that $\{\Psi_h(l^{(m)})\}$ converges to a point $z \in P_h(T)$, we claim that $\{l^{(m)}\}$ contains a subsequence converging to a point in $\mathbb{R}^E$. By taking a subsequence if necessary, we may assume that $\{l^{(m)}\}$ converges to $l \in [-\infty, +\infty]^E$. If l were not in $\mathbb{R}^E$, there would exist an edge e so that $l(e) = \pm\infty$.

Case 1. If $l(e) = -\infty$ for some $e \in E$, then there is a degenerated decorated ideal triangle Δ of type I, II or III. By Corollary 3.2, there is a bad edge e_1 in Δ. Let Δ_1 be the other decorated ideal triangle adjacent to e_1, and let x_0 and x_1 respectively be the x-invariants of e_1 in Δ and Δ_1. If e_1 is bad in Δ_1, then

$$\begin{aligned}
z(e_1) &= \lim_{m\to\infty} \Psi_h(l^{(m)})(e_1) = \lim_{m\to\infty} \left(\int_0^{x_0^{(m)}} e^{ht^2} dt + \int_0^{x_1^{(m)}} e^{ht^2} dt \right) \\
&= 2 \int_0^{+\infty} e^{ht^2} dt,
\end{aligned}$$

which contradicts the assumption that $z \in P_h(T)$. Therefore e_1 has to be a good edge in Δ_1. Since $l(e_1) = -\infty$, the decorated triangle Δ_1 is degenerated of type I, II or III. By Corollary 3.2, there is a bad edge e_2 in Δ_1. For the same reason, e_2 has to be good in the other decorated ideal triangle Δ_2 adjacent to e_2, and there is a bad edge e_3 in Δ_2. Serially applying this logic and using that there are finitely many edges, we find an edge loop $(e_k, \Delta_k, \ldots, e_n, \Delta_n)$ with $e_{n+1} = e_k$ so that for each $i = k, \ldots, n$ the edge e_i is good in Δ_i and the edge e_{i+1} is bad in Δ_i. By Lemma 3.3, we can take a subsequence so that $l^{(m)}(e_i) > l^{(m)}(e_{i+1})$ for m sufficiently large. Thus, we have $l^{(m)}(e_k) > l^{(m)}(e_{n+1})$, which contradicts that $e_{n+1} = e_k$.

In light of Case 1, we may assume that $l \in (-\infty, +\infty]^E$.

Case 2. If $l(e) = +\infty$ for some $e \in E$, let Δ_1 be a decorated ideal triangle adjacent to e. If Δ_1 is not of type IV, then by Lemma 3.4, there is an edge e_1 of Δ_1 and an generalized angle θ_1 adjacent to e and e_1 so that $l(e_1) = +\infty$ and $\lim_{m\to\infty} \theta_1^{(m)} = 0$ (see Figure 5). The other decorated ideal triangle Δ_2 adjacent to e_1 is either of type IV or contains an edge e_2 and a generalized angle θ_2 adjacent to e_1 and e_2 so that $l(e_2) = +\infty$ and $\lim_{m\to\infty} \theta_2^{(m)} = 0$. Again, the serial application of this procedure terminates with an edge e_p and a decorated ideal triangle Δ_{p+1} adjacent to e_p so that $l(e_p) = +\infty$ and Δ_{p+1} is of type IV, or since there are only finitely many edges, produces a fundamental edge loop $(e_k, \Delta_k, \ldots, e_n, \Delta_n)$ such that $l(e_i) = +\infty$ for $i = k, \ldots, n$ and $\lim_{m\to\infty} \theta_i^{(m)} = 0$, where θ_i is the generalized angle in Δ_i adjacent to e_i and e_{i+1}. If it yields such a fundamental edge loop $(e_k, \Delta_k, \ldots, e_n, \Delta_n)$, denote by β_i (resp. γ_i) the generalized angle in Δ_i facing e_i (resp. e_{i+1}) for $i = k, \ldots, n$. Let $\bar{\beta}_i = \lim_{m\to\infty} \beta_i^{(m)}$ and $\bar{\gamma}_i = \lim_{m\to\infty} \gamma_i^{(m)}$, so that

$$\begin{aligned}
\sum_{i=k}^{n} z(e_i) &= \lim_{m\to\infty} \sum_{i=1}^{k} \Psi_h(l^{(m)})(e_i) \\
&= \lim_{m\to\infty} \sum_{i=1}^{k} \left(\int_0^{\frac{\theta_i^{(m)}+\beta_i^{(m)}-\gamma_i^{(m)}}{2}} e^{ht^2} dt + \int_0^{\frac{\theta_i^{(m)}+\gamma_i^{(m)}-\beta_i^{(m)}}{2}} e^{ht^2} dt \right) \\
&= \sum_{i=1}^{k} \left(\int_0^{\frac{\bar{\beta}_i-\bar{\gamma}_i}{2}} e^{ht^2} dt + \int_0^{\frac{\bar{\gamma}_i-\bar{\beta}_i}{2}} e^{ht^2} dt \right) \\
&= 0,
\end{aligned}$$

which contradicts the assumption that $z \in P_h(T)$. If it terminates with e_p and Δ_{p+1} of type IV, then we consider the other decorated ideal triangle Δ_0 adjacent to e. If Δ_0 is not of type IV, then it contains an edge e_{-1} and a generalized angle θ_0 adjacent to e_{-1} and e so that $l(e_{-1}) = +\infty$ and $\lim_{m\to\infty} \theta_0^{(m)} = 0$. As before, either there is a fundamental edge loop, contradicting the assumption that $z \in P_h(T)$, or the procedure terminates with an edge e_{-q} and a decorated ideal triangle Δ_{-q} adjacent to e_{-q} so that $l(e_{-q}) = +\infty$ and Δ_{-q} is of type IV. If the procedure stops at e_{-q} and Δ_{-q} of type IV, we get a fundamental edge path $(\Delta_{-q}, e_{-q}, \ldots, e_p, \Delta_{p+1})$, where $e_0 = e$, such that Δ_{-q} and Δ_p are of type IV with $l(e_{-q}) = +\infty$ and $l(e_p) = +\infty$, and $\lim_{m\to\infty} \theta_i^{(m)} = 0$, where θ_i is the generalized angle of Δ_i adjacent to e_{i-1} and e_i for $i = 1-q, \ldots, p$. Denote by a_{-q} the generalized angle of Δ_{-q} facing e_{-q}, and by a_p the generalized angle of Δ_{p+1} facing e_p. Let b_{-q} and c_{-q} be the generalized angles of Δ_{-q} adjacent to e_{-q}, and let b_p and c_p be the generalized angles of Δ_{p+1} adjacent to e_p. We find

$$\begin{aligned}
\sum_{i=-q}^{p} z(e_i) &= \lim_{m\to\infty} \sum_{i=-q}^{p} \Psi_h(l^{(m)})(e_i) \\
&= \lim_{m\to\infty} \left(\int_0^{\frac{b_{-q}^{(m)}+c_{-q}^{(m)}-a_{-q}^{(m)}}{2}} e^{ht^2} dt + \int_0^{\frac{b_p^{(m)}+c_p^{(m)}-a_p^{(m)}}{2}} e^{ht^2} dt \right. \\
&\quad \left. + \sum_{i=1-q}^{p} \left(\int_0^{\frac{\theta_i^{(m)}+\beta_i^{(m)}-\gamma_i^{(m)}}{2}} e^{ht^2} dt + \int_0^{\frac{\theta_i^{(m)}+\gamma_i^{(m)}-\beta_i^{(m)}}{2}} e^{ht^2} dt \right) \right) \\
&= \int_0^{-\infty} e^{ht^2} dt + \int_0^{-\infty} e^{ht^2} dt \\
&\quad + \sum_{i=1-q}^{p} \left(\int_0^{\frac{\bar\beta_i-\bar\gamma_i}{2}} e^{ht^2} dt + \int_0^{\frac{\bar\gamma_i-\bar\beta_i}{2}} e^{ht^2} dt \right) \\
&= -2 \int_0^{+\infty} e^{ht^2} dt,
\end{aligned}$$

which contradicts the assumption that $z \in P_h(T)$. q.e.d

5. Uniqueness of the energy function

Let Δ be a decorated ideal triangle with edge lengths l_1, l_2, l_3 with opposite generalized angles θ_1, θ_2, θ_3 and set $x_i = \frac{\theta_j+\theta_k-\theta_i}{2}$ for $i, j, k = 1, 2, 3$. The following theorem shows that Ψ_h is the unique possible deformation of Penner's simplicial coordinate by using the variational principle stated in Section 2.

Theorem 5.1. *Let μ and u be two non-constant smooth functions. Up to an overall scale, there is a unique closed 1-form $\omega = \sum_{i=1}^{3} \mu(x_i) du(l_i)$ which is given by*

$$w_h = \sum_{i=1}^{3} \int^{x_i} e^{ht^2} dt d \left(\int^{l_i} e^{-he^{-t}} dt \right)$$

for some $h \in \mathbb{R}$.

The proof of Theorem 5.1 makes use of the following lemma.

Lemma 5.2. *Let f and g be two non-constant smooth functions on $\mathbb{R}$. If $\frac{f(x_i)}{g(l_j)}$ is symmetric in $i, j = 1, 2$, then there are constants h, c_1 and c_2 so that*

$$f(t) = e^{ht^2+c_1} \quad \textit{and} \quad g(t) = e^{-he^{-t}+c_2}.$$

Proof. By taking $\frac{\partial}{\partial l_k}$ in the equality $\frac{f(x_i)}{g(l_j)} = \frac{f(x_j)}{g(l_i)}$, we have $\frac{f'(x_i)}{g(l_j)} \frac{\partial x_i}{\partial l_k} = \frac{f'(x_j)}{g(l_i)} \frac{\partial x_j}{\partial l_k}$ for $i, j, k = 1, 2, 3$. We deduce from (1) that $\frac{\partial x_i}{\partial l_j} = \frac{x_k}{2}$, so $\frac{f'(x_i)}{g(l_j)} \frac{x_j}{2} = \frac{f'(x_j)}{g(l_i)} \frac{x_i}{2}$. Thus, $\frac{f'(x_i)}{f'(x_j)} \frac{x_j}{x_i} = \frac{g(l_j)}{g(l_i)} = \frac{f(x_i)}{f(x_j)}$, which implies $\frac{f'(x_i)}{f(x_i)} \frac{1}{x_i} = \frac{f'(x_j)}{f(x_j)} \frac{1}{x_j}$ and $\frac{f'(t)}{f(t)} \frac{1}{t} = 2h_1$ for some $h_1 \in \mathbb{R}$. Solving this ordinary differential equation for f, we find

$$f(t) = e^{h_1 t^2 + c_1}$$

for some $c_1 \in \mathbb{R}$. By taking $\frac{\partial}{\partial x_k}$ in the equality $\frac{g(l_i)}{f(x_j)} = \frac{g(l_j)}{f(x_i)}$, we have $\frac{g'(l_i)}{f(x_j)} \frac{\partial l_i}{\partial x_k} = \frac{g'(l_j)}{f(x_i)} \frac{\partial l_j}{\partial x_k}$ for $i, j, k = 1, 2, 3$. From (1) again, we deduce that $\frac{\partial l_i}{\partial x_j} = -\frac{1}{\theta_k}$, so $-\frac{g'(l_i)}{f(x_j)} \frac{1}{\theta_j} = -\frac{g'(l_j)}{f(x_i)} \frac{1}{\theta_i}$. Thus, $\frac{g'(l_i)}{g'(l_j)} \frac{e^{l_i}}{e^{l_j}} = \frac{g'(l_i)}{g'(l_j)} \frac{\theta_i}{\theta_j} = \frac{f(x_j)}{f(x_i)} = \frac{g(l_i)}{g(l_j)}$, which implies $\frac{g'(l_i)}{g(l_i)} e^{l_i} = \frac{g'(l_j)}{g(l_j)} e^{l_j}$ and $\frac{g'(t)}{g(t)} e^t = h_2$ for some $h_2 \in \mathbb{R}$. Solving this ordinary differential equation for g, we find

$$g(t) = e^{-h_2 e^{-t} + c_2}$$

for some $c_1 \in \mathbb{R}$. From $f(t) = e^{h_1 t^2 + c_1}$ and the equality $\frac{f(x_i)}{g(l_j)} = \frac{f(x_j)}{g(l_i)}$, we conclude that $h_1 = h_2$. q.e.d

Proof of Theorem 5.1. The differential 1-form $\omega = \sum_{i=1}^{3} \mu(x_i) du(l_i)$ is closed if and only if $\frac{\partial \mu(x_i)}{\partial u(l_j)} = \frac{\mu'(x_i)}{u'(l_j)} \frac{\partial x_i}{\partial l_j}$ is symmetric in i and j. Since $\frac{\partial x_i}{\partial l_j} = \frac{\partial x_j}{\partial l_i} = \frac{x_k}{2}$, ω is closed if and only if $\frac{\mu'(x_i)}{u'(l_j)}$ is symmetric in i and j. By Lemma 5.2, if $\frac{\mu'(x_i)}{u'(l_j)}$ is symmetric in i and j, then $\mu'(x_i) = e^{hx_i^2 + c_1}$ and $u'(l_i) = e^{-he^{-l_i} + c_2}$ for some constants h, c_1 and c_2. q.e.d

6. Ψ_h and the Delaunay decomposition

We first review the construction of the Delaunay decomposition associated to a decorated hyperbolic metric following Bowditch-Epstein [4]. Suppose S is a punctured surface with a set of ideal vertices V, and let (d, r) be a decorated hyperbolic metric on S so that the horodisks associated to the ideal vertices do not intersect. Let B_v be the horodisks associated to the ideal vertex v, and let $B = \bigcup_{v \in V} B_v$. The *spine* $\Gamma_{d,r}$ of S is the set of points in S which have at least two distinct shortest geodesics to ∂B. The spine $\Gamma_{d,r}$ is shown [4] to be a graph whose edges are geodesic arcs on S.

Let $e_1^*, \ldots, e_N^*$ be the edges of $\Gamma_{d,r}$. By construction each interior point of an edge e_i^* has exactly two distinct shortest geodesics to ∂B. For each edge e_i^*, there are two horodisks B_1 and B_2 (possibly coincide) so that points in the interior of e_i^* have precisely two shortest geodesics to ∂B_1 and ∂B_2. Let e_i be the shortest geodesic from ∂B_1 to ∂B_2. It is known that e_i intersects e_i^* perpendicularly, and $\{e_1, \ldots, e_N\}$ are disjoint. The components of $S \setminus \{e_1, \ldots, e_N\}$ consists of decorated polygons (ideal polygons with horodisks associated to the ideal vertices) which are the 2-cells of the *Delaunay decomposition* $\Sigma_{d,r}$. The 1-cells of $\Sigma_{d,r}$ consist of the edges $\{e_1, \ldots, e_N\}$ and the arcs on ∂B which are the intersection of ∂B with the ideal polygons. For a generic decorated hyperbolic metric (d, r), each 2-cell of $\Sigma_{d,r}$ is a decorated ideal triangle, and $\Sigma_{d,r}$ is a decorated ideal triangulation of S.

Let D be a 2-cell of $\Sigma_{d,r}$. We call the hyperbolic circle on S tangent to all arcs of $D \cap \partial B$ the *inscribed circle* of D. By the construction of the Delaunay decomposition, for each 2-cell D of $\Sigma_{d,r}$, there is exactly one vertex v^* of the spine $\Gamma_{d,r}$ lying in the interior of D. Moreover, v^* is of equal distance to all arcs of $D \cap \partial B$, hence is the center of the inscribed circle of D. Thus, the center of the inscribed circle of each 2-cell D of the Delaunay decomposition is in the interior of D. We need the following proposition of Penner [17] whose proof is included here to the convenience of the readers.

Lemma 6.1. ([17]) *Suppose Δ is a decorated ideal triangle with edge lengths $l_i > 0$ and opposite generalized angles θ_i for $i = 1, 2, 3$. Then $x_i = \frac{\theta_j + \theta_k - \theta_i}{2} > 0$ for $i = 1, 2, 3$ if and only if the center of the inscribed circle of Δ is in the interior of Δ.*

Proof. For $i = 1, 2, 3$ let B_i be the horodisks associated to the ideal vertices of Δ, and let Z_i be the point of tangency of the inscribe circle of Δ and ∂B_i. Label the intersection of the horodisks and the edges of Δ by X_1, Y_1, X_2, Y_2, X_3 and Y_3 cyclically as in Figure 6(a). For two points A and B in the hyperbolic plane $\mathbb{H}^2$, let AB be the geodesic segment connecting A and B, and let $|AB|$ the length of AB. If the center v of the inscribed circle is in the interior of Δ, then $x_i = |X_i Z_{i+1}| > 0$ for $i = 1, 2, 3$. If v is on $X_i Y_i$, or v and Δ are on different sides of $X_i Y_i$ for some $i \in \{1, 2, 3\}$, then $x_i = -|X_i Z_{i+1}| \leqslant 0$. See Figure 6 (b). q.e.d

Proof of Theorem 1.3. Let (d, r) be a decorated hyperbolic metric so that the associated Delaunay decomposition $\Sigma_{d,r}$ is a decorated ideal triangulation of S. For each edge e of $\Sigma_{d,r}$, let Δ and Δ' be the decorated ideal triangles adjacent to e, and let θ_1 and θ_1' respectively be the

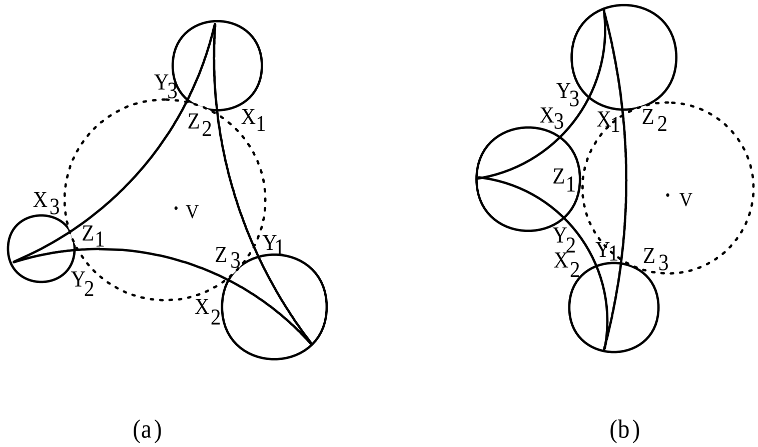

Figure 6. The inscribed circle.

generalized angles of Δ and Δ' facing e, and θ_2, θ_3, θ'_2 and θ'_3 be the generalized angles adjacent to e. Let $x(e) = \frac{\theta_2+\theta_3-\theta_1}{2}$ and $x'(e) = \frac{\theta'_2+\theta'_3-\theta'_1}{2}$. From Lemma 6.1 and the fact that the center of the inscribed circle of each 2-cell of the Delaunay decomposition is in the interior of the 2-cell, we conclude that $x(e)$ and $x'(e)$ are positive, and

$$\Psi_h(d,r)(e) = \int_0^{x(e)} e^{ht^2}dt + \int_0^{x'(e)} e^{ht^2}dt > 0.$$

On the other hand, if T is an ideal triangulation of S such that $\Psi_h(d,r)(e) \leqslant 0$ for some edge e, then at least one of $x(e)$ and $x'(e)$, say $x(e)$, is less than or equal to zero. By Lemma 6.1, the center of the inscribed circle of Δ is not in the interior of Δ. Since the center of the inscribed circle of each 2-cell of the Delaunay decomposition has to be in the interior of the 2-cell, T cannot be the Delaunay decomposition $\Sigma_{d,r}$ of S. q.e.d

7. Further questions

Suppose Δ is a decorated ideal triangle with edge lengths l_1, l_2 and l_3 and opposite generalized angles θ_1, θ_2 and θ_3. For each $h \neq -1$, the differential 1-form $\omega_h = \sum_{i=1}^{3} \theta_i^{h+1} de^{-(h+1)l_i}$ is closed in $\mathbb{R}^3$. However, the primitive $F_h(u) = \int_0^u \omega_h$ is not strictly concave on $\mathbb{R}^3$. Let (S,T) be an ideally triangulated punctured surface. For each $h \neq -1$, we define a map $\Phi_h\colon\ T_c(S) \times \mathbb{R}^V_{>0} \to \mathbb{R}^E$ by

$$\Phi_h(d,r)(e) = \theta^{h+1} + \theta'^{h+1},$$

where θ and θ' are the generalized angles facing e. To the best of the author's knowledge, there is no counterexample to the following

Conjecture 7.1. *The map* $\Phi_h\colon T_c(S) \times \mathbb{R}^V_{>0} \to \mathbb{R}^E$ *is a smooth embedding, and the image of* Φ_h *is a convex polytope.*

The motivation of this conjecture is as follows. Penner's simplical coordinate Ψ and its deformation Ψ_h are in some sense analogues to Colin de Vedière's invariant [5] for circle packings in a different setting, and the quantities Φ_h are the corresponding analogues to Rivin's invariant [18] for the polyhedra surfaces in this setting, see also [1] and [11].

By Corollary 1.4, for each $h \geqslant 0$, there is a homeomorphism

$$\Pi_h\colon T_c(S) \times \mathbb{R}^V_{>0} \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

equivariant under the mapping class group action. If $h \neq h'$, then $\Pi_{h'}^{-1}\Pi_h$ is a self-homeomorphism of the decorated Teichmüller space equivariant under the mapping class group action. These self-homeomorphisms deserve a further study. We do not know yet if these self-homeomorphisms are smooth on the decorated Teichmüller space. As suggested by the referee of this article, it also seems natural to ask if these self-homeomorphisms have bounded distortion.

The Weil-Pertersson Kähler form on the Teichmüller space was computed in the length coordinates in [16]. How to express the Weil-Petersson symplectic form on the decorated Teichmüller space in terms of the simplicial coordinate Ψ and in terms of the Ψ_h coordinate, and how to relate the Ψ_h coordinate to the quantum Teichmüller space are interesting problems ([2], [3], [14] and [17]).

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