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SIMPLICIAL VOLUME OF MODULI SPACES OF RIEMANN SURFACES

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Abstract

Motivated by results on the simplicial volume of locally symmetric spaces of finite volume, in this note, we observe that the simplicial volume of the moduli space $\mathcal{M}_{g,n}$ is equal to 0 if $g \geq 2$; $g = 1, n \geq 3$; or $g = 0, n \geq 6$; and the orbifold simplicial volume of $\mathcal{M}_{g,n}$ is positive if $g = 1, n = 0, 1$; $g = 0, n = 4$. We also observe that the simplicial volume of the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$ is equal to 0, and the simplicial volumes of the reductive Borel-Serre compactification of arithmetic locally symmetric spaces $\Gamma\backslash X$ and the Baily-Borel compactification of Hermitian arithmetic locally symmetric spaces $\Gamma\backslash X$ are also equal to 0 if the $\mathbb{Q}$-rank of $\Gamma\backslash X$ is at least 3 or if $\Gamma\backslash X$ is irreducible and of $\mathbb{Q}$-rank 2.

Contents

1. Introduction

The notion of simplicial volume of manifolds and algebraic varieties was first introduced by Gromov [**25**] in the early 1980s. One motivation was to give a lower bound of minimal volumes of complete Riemannian metrics with sectional curvature bounded between ± 1 on the manifolds.

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Another was to construct a new homotopy invariant for odd dimensional compact manifolds, for which the Euler characteristic is zero.

One early important application of the simplicial volume was a new proof by Gromov of the Mostow strong rigidity of hyperbolic manifolds of finite volume of dimension at least three [**3**]. This depends on a result of Thurston that the simplicial volume of a hyperbolic manifold of finite volume is positive and proportional to the hyperbolic volume by a universal constant [**25**, §0.3].

There are several results on vanishing and non-vanishing of simplicial volumes of special classes of manifolds. They are mainly concerned with the simplicial volume of hyperbolic manifolds and more generally locally symmetric spaces of finite volume. See §3 below for precise statements and references.

Motivated by the close analogy between locally symmetric spaces and moduli spaces, we study simplicial volumes of moduli spaces of Riemann surfaces. Let $\mathcal{M}_{g,n}$ be the moduli space of Riemann surfaces of genus g with n punctures. It is a noncompact complex orbifold and also a quasi-projective variety and hence has a well-defined fundamental class. For an orbifold M that admits finite smooth coverings, there is also a notion of orbifold simplicial volume, besides the usual simplicial volume. See Definition 2.12 below.

Theorem 1.1. *Both the simplicial volume and the orbifold simplicial volume of $\mathcal{M}_{g,n}$ are equal to 0 if $g \geq 2$; $g = 1, n \geq 3$; or $g = 0, n \geq 6$; and the orbifold simplicial volume of $\mathcal{M}_{g,n}$ is positive if $g = 1, n = 0, 1$; $g = 0, n = 4$.*

Combining with a result of Thurston [**25**, §0.3] that the simplicial volume of a complete negatively pinched manifold is positive, we obtain immediately the following corollary (see Corollary 6.9 below for more details).

Corollary 1.2. *When $g \geq 2$; $g = 1, n \geq 3$; or $g = 0, n \geq 6$, $\mathcal{M}_{g,n}$ does not admit a complete Riemannian metric of negatively pinched sectional curvature.*

One application of the vanishing of the simplicial volume of $\mathcal{M}_{1,0}$ is a new proof of the vanishing of the simplicial volume of $\mathbb{R}^2$ and hence of the simplicial volume of $\mathbb{R}^{2n}$ [**25**, p. 10] [**41**, §6.3.1]. See Corollaries 2.15 and 2.16 below.

For a noncompact space M, it is natural to compactify M for various applications and hence it also seems natural to consider the simplicial volume of those compactifications which admit a fundamental class.

When $\Gamma\backslash X$ is a noncompact arithmetic locally symmetric space, it admits several different compactifications: e.g., the Borel-Serre compactification $\overline{\Gamma\backslash X}^{BS}$, the reductive Borel-Serre compactification $\overline{\Gamma\backslash X}^{RBS}$. If

$\Gamma\backslash X$ is Hermitian, then it also admits the Baily-Borel compactification $\overline{\Gamma\backslash X}^{BB}$ (see [**9**]).

$\overline{\Gamma\backslash X}^{BS}$ is a real analytic manifold with corners when Γ is torsion-free and can be considered as a topological manifold with boundary. Therefore, it has the relative fundamental class and the relative simplicial volume.

An immediate corollary of Proposition 3.3 and a general result [**40**, Proposition 5.12] (see Proposition 2.7 below) is

Corollary 1.3. *The relative simplicial volume of* $\overline{\Gamma\backslash X}^{BS}$ *is equal to 0 if the* $\mathbb{Q}$*-rank of* $\Gamma\backslash X$ *is at least 3.*

The reductive Borel-Serre compactification $\overline{\Gamma\backslash X}^{RBS}$ is not a smooth manifold with corners or a complex variety. But it has a fundamental class and its simplicial volume is well-defined (see Proposition 2.4 below).

Proposition 1.4. *The simplicial volume of* $\overline{\Gamma\backslash X}^{RBS}$ *is equal to 0 when the* $\mathbb{Q}$*-rank of* $\Gamma\backslash X$ *is at least 3, or when* $\Gamma\backslash X$ *is irreducible, the rank of* X *is at least 2 and the* $\mathbb{Q}$*-rank of* $\Gamma\backslash X$ *is at least 1.*

The simplicial volume is a birational invariant of smooth projective varieties [**25**, p. 60]. A natural problem is to understand this invariant for some explicit or special complex manifolds and to understand its relation with algebro-geometric properties of the varieties. The paper [**53**] studied vanishing of the simplicial volume of certain complex varieties via an analogue of the Lefschetz theorem on hyperplane sections.

When $\Gamma\backslash X$ is Hermitian, the Baily-Borel compactification $\overline{\Gamma\backslash X}^{BB}$ is a normal projective variety and admits a fundamental class. A corollary of Proposition 1.4 is the following result.

Proposition 1.5. *The simplicial volume of* $\overline{\Gamma\backslash X}^{BB}$ *is equal to 0 when the* $\mathbb{Q}$*-rank of* $\Gamma\backslash X$ *is at least 3, or when* $\Gamma\backslash X$ *is irreducible and the* $\mathbb{Q}$*-rank of* $\Gamma\backslash X$ *is equal to 2.*

Using the above discussions, we can prove the following result.

Proposition 1.6. *If* M *is a compact Riemann surface, then the simplicial volume of* M *is nonzero if and only if* M *is of the general type, i.e., its Kodaira dimension is equal to* $\dim_{\mathbb{C}} M$.

This follows easily from the uniformization theorem for Riemann surfaces and the result of Thurston on simplicial volume of compact hyperbolic manifolds [**25**, §0.3]. More generally, we have

Proposition 1.7. *Let* X *be a Hermitian symmetric space, and* $\Gamma\backslash X$ *be a compact smooth quotient. Then the simplicial volume of* $\Gamma\backslash X$ *is positive if and only if the Kodaira dimension of* $\Gamma\backslash X$ *is equal to* $\dim \Gamma\backslash X$*, i.e.,* $\Gamma\backslash X$ *is also of general type.*

Its proof will be given after Proposition 2.9.

Remark 1.8. If a compact complex manifold M is a quotient of a homogeneous space, then it can also be shown that M is of the general type if and only if the simplicial volume is positive. Briefly, the Kodaira dimension has the additivity property for fibrations, and hence if fibers are not of general type, the whole fibration is not of general type. Given a fibration whose base and fiber and the total space are closed manifolds, and whose fiber is of positive dimension, the simplicial volume of the fibration is zero if the fibers have amenable fundamental groups. The above result follows from the Levi decomposition of Lie groups and results on simplicial volume of quotients of symmetric spaces in Proposition 1.7. See [**54**] for some discussion on homogeneous complex manifolds.

Remark 1.9. In the above discussion of the simplicial volume of quotients of Hermitian symmetric spaces and complex homogeneous spaces, the assumption that the quotients are compact is important. Proposition 3.3 shows that if the $\mathbb{Q}$-rank of a Hermitian locally symmetric space is greater than or equal to 3, its simplicial volume vanishes.

Propositions 1.6 and 1.7 might suggest that if M is a compact complex manifold of general type, then its simplicial volume is positive. The next result shows that this is not true.

The moduli space $\mathcal{M}_{g,n}$ admits the Deligne-Mumford compactification $\overline{\mathcal{M}_{g,n}}^{DM}$ which is a compact complex orbifold and hence admits a fundamental class.

Proposition 1.10. *The simplicial volume of* $\overline{\mathcal{M}_{g,n}}^{DM}$ *is equal to 0 for all values of* g, n.

It is known that when $g \geq 24$, $\overline{\mathcal{M}_g}^{DM}$ is of general type (see [**19**] and references there).

Remark 1.11. Suitable finite smooth covers of $\overline{\mathcal{M}_{g,n}}^{DM}$ are simply connected and smooth, and hence their simplicial volumes vanish by Proposition 2.9, but they are also of general type (see [**8**]).

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After this paper was circulated and submitted, Enrico Leuzinger told me that he and Roman Sauer had also worked on and proved vanishing of the simplicial volume of quotients of the Teichmüller space $\mathcal{T}_{g,n}$ by finite index torsion-free subgroups of $\mathrm{Mod}_{g,n}$ under the same assumption on g, n as in Theorem 1.1, though it seems that nothing has been written up yet.

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2. Definitions and basic results on simplicial volumes

In this section, we recall definitions of several versions of simplicial volume and some of their basic properties.

Let N be a compact topological space. Assume that N is a compactification of an oriented manifold M. If N is a smooth manifold, then $M = N$. Denote $N - M$ by N_{sing} and call it the *singular locus*.

If N is a manifold with nonempty boundary whose interior is equal to M, then N_{sing} is equal to the boundary of N and denoted by ∂N.

If M is compact, let $H_i(M, \mathbb{Z})$ be the singular homology. If M is noncompact, let $H_i^{\text{lf}}(M, \mathbb{Z})$ be the locally finite homology group of M (see [**33**] for example).

If N is a compact oriented manifold of dimension n, then $H_n(N, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, and there is a canonical choice of generator of $H_n(N, \mathbb{Z})$ that corresponds to the orientation of N, i.e., restricts to a generator of $H_n(N, N - x, \mathbb{Z})$ for every point $x \in N$, which corresponds to the local orientation at x. It is called the *fundamental class* of N and denoted by $[N]$.

If N is a compact oriented manifold of dimension n with boundary ∂N, then $H_n(N, \partial N, \mathbb{Z})$ is also isomorphic to $\mathbb{Z}$, and a canonical generator determined by the orientation of the interior M is called the *relative fundamental class* of N and denoted by $[N, \partial N]$.

Proposition 2.1. *Assume that M is an oriented connected manifold of dimension n, and that N is a compactification of M such that there are arbitrarily small neighborhoods $N_{\text{sing},\varepsilon}$ of the singular locus N_{sing}, where ε is a small real parameter, such that*

1) *N_{sing} is of codimension at least 2,*
2) *$N - N_{\text{sing},\varepsilon}$ is a manifold with boundary,*
3) *$N_{\text{sing},\varepsilon}$ can be deformation retracted to N_{sing}. Furthermore, for any $\delta < \varepsilon$, $N_{\text{sing},\varepsilon}$ can be deformation retracted to $N_{\text{sing},\delta}$.*

Then $H_n(N, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, and the orientation of M determines a generator of $H_n(N, \mathbb{Z})$ such that it restricts to a generator of $H_n(N, N - x, \mathbb{Z})$ for every point $x \in N - N_{\text{sing}}$. It is called the fundamental class of N and denoted by $[N]$.

Proof. Since N_{sing} is of codimension at least 2, the long exact sequence shows that

$$H_n(N, \mathbb{Z}) \cong H_n(N, N_{\mathrm{sing}}, \mathbb{Z}).$$

Since $N - N_{\mathrm{sing},\varepsilon}$ is a manifold with boundary, $H_n(N - N_{\mathrm{sing},\varepsilon}, \partial(N - N_{\mathrm{sing},\varepsilon}), \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. Then the isomorphisms

$$H_n(N, N_{\mathrm{sing}}, \mathbb{Z}) \cong H_n(N, N_{\mathrm{sing},\varepsilon}, \mathbb{Z}) \cong H_n(N - N_{\mathrm{sing},\varepsilon}, \partial(N - N_{\mathrm{sing},\varepsilon}), \mathbb{Z})$$

imply that $H_n(N, N_{\mathrm{sing}}, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. In the last isomorphism, we have used the assumption that for $\delta < \varepsilon$, $N_{\mathrm{sing},\varepsilon}$ can be deformation retracted to $N_{\mathrm{sing},\delta}$ to show that $N_{\mathrm{sing},\varepsilon}$ can be excised since the closure of $N_{\mathrm{sing},\delta}$ is contained in $N_{\mathrm{sing},\varepsilon}$ and hence can be excised [**24**, Theorems 15.1 and 15.2, p. 82].

Proposition 2.2. *If V is an irreducible compact variety over $\mathbb{C}$ of complex dimension n, then $H_{2n}(V, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$ and has a canonical fundamental class $[V]$ determined by the orientation of the complex structure.*

For detailed constructions and definitions of $[V]$, see [**23**, Appendix B] (the book [**33**] also discusses the Borel-Moore homology group).

Remark 2.3. An intuitive way to understand the fundamental class $[V]$ is to start with a triangulation of V and endow each simplex with the canonical orientation from the complex structure. Then the sum of the top dimensional simplexes gives the fundamental class. The fact that the singular locus is of real codimension at least 2 implies that it is a cycle. One difficulty of this approach is that it is not easy to check that two different triangulations give the same homology class. Using the Whitney stratification of V, it can be seen that the conditions in Proposition 2.1 are satisfied, and hence a fundamental class can be constructed by Proposition 2.1 as well. Another way to understand the class $[V]$ is to take a resolution $\tilde{V}$ of V. Then the fundamental class of $\tilde{V}$ is pushed forward under the map $H_{2n}(\tilde{V}, \mathbb{Z}) \to H_{2n}(V, \mathbb{Z})$ to obtain the fundamental class $[V]$. To show that another resolution $\hat{V}$ of V gives the same homology class, we take a resolution of V that dominates both $\tilde{V}$ and $\hat{V}$, and note that the image of its fundamental class factors through the classes $[\tilde{V}]$ and $[\hat{V}]$.

Proposition 2.4. *Let $\Gamma\backslash X$ be a noncompact arithmetic locally symmetric space, and $\overline{\Gamma\backslash X}^{RBS}$ the reductive Borel-Serre compactification. Assume that Γ is torsion-free. Then the conditions in Proposition 2.1 are satisfied by $\overline{\Gamma\backslash X}^{RBS}$, and $\overline{\Gamma\backslash X}^{RBS}$ has a fundamental class.*

We will explain the idea of the proof of the above proposition in Proposition 5.3 after describing $\overline{\Gamma\backslash X}^{RBS}$.

Proposition 2.5. *If N is a compact connected complex orbifold of complex dimension n, then $H_{2n}(N, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. If M is a noncompact connected complex orbifold of dimension n, then $H_{2n}^{\mathrm{lf}}(M, \mathbb{Z})$ is also isomorphic to $\mathbb{Z}$.*

Since the singular locus of N is of codimension at least 2, the same proof as for Proposition 2.1 works.

Remark 2.6. Complex orbifolds are orientable. Any locally orientable orbifold M is a rational homology manifold in the sense that for every point $x \in M$, $H_n(M, M - x, \mathbb{Q}) \cong \mathbb{Q}$, where $n = \dim M$, and any orientable orbifold admits a rational fundamental class in $H_n(M, \mathbb{Q})$ (see [**1**]). Since the smooth locus of M is connected, we can normalize the rational fundamental class by requiring that for any smooth point x, the class restricts to the generator of $H_{2n}(M, M - x, \mathbb{Z})$ for the orientation. Then this class agrees with the class in Proposition 2.5.

For any topological space with a fundamental class, we can define its simplicial volume. The fundamental class gives a natural class in the homology with $\mathbb{R}$-coefficients. In the definitions of simplicial volumes below, all fundamental cycles, i.e., cycles representing the fundamental class, have $\mathbb{R}$-coefficients.

For any chain of singular n-simplexes $c = \sum_\sigma a_\sigma \sigma$ with real coefficients on N, define its *ℓ_1-norm* by

$$||c||_1 = \sum_\sigma |a_\sigma|.$$

Assume that N is a connected oriented closed manifold (i.e., compact without boundary). Let $[N]$ also denote the fundamental class in the homology group with $\mathbb{R}$-*coefficients*, i.e., the image of the integral fundamental class $[N]$ under the natural map $H_n(N, \mathbb{Z}) \to H_n(N, \mathbb{R})$. Then the **simplicial volume** $||N||$ of N is defined by

$$||N|| = \inf\{||c||_1 \mid c \text{ is any cycle with } \mathbb{R}\text{-coefficients representing } [N]\}.$$

If N is a connected compact oriented manifold with nonempty boundary ∂N, using relative cycles with $\mathbb{R}$-coefficients representing the **relative fundamental class** $[N, \partial N]$, one can similarly define the relative simplicial volume $||N, \partial N||$.

If M is a noncompact connected oriented manifold, using the fundamental class $[M]$ in the locally finite homology group with $\mathbb{R}$-coefficients, the simplicial volume $||M||$ of M can be defined. Since cycles for the locally finite homology groups are usually not of compact support and may have infinite ℓ_1-norms, the simplicial volume $||M||$ may be equal to infinity. For example, $||\mathbb{R}^1|| = +\infty$.

If N is a compact topological space of dimension n such that $H_n(N, \mathbb{Z})$ is cyclic and $[N]$ is a fundamental class, then its simplicial volume $||N||$

can also be defined by ℓ_1-norms of all cycles with $\mathbb{R}$-coefficients representing the class $[N]$ in $H_n(N, \mathbb{R})$.

Proposition 2.7 ([**40**, Proposition 5.12])**.** *Let N be a compact manifold with nonempty boundary ∂N, and M the interior of N. Then*

$$||N, \partial N|| \leq ||M||.$$

Remark 2.8. Given a variety V, let V_{reg} be the subset consisting of smooth points, and $V_{sing} = V - V_{reg}$ its singular locus. Then the simplicial volumes $||V_{reg}||$ of the smooth locus and $||V||$ of the whole variety V should satisfy

$$||V|| \leq ||V_{reg}||.$$

The reason is that the complement $V - V_{sing,\varepsilon}$ is homeomorphic to V_{reg}, where $V_{sing,\varepsilon}$ is a small tubular neighborhood of V_{sing}, and a proof similar to the proof of [**40**, Proposition 5.12] should work.

Proposition 2.9. *Let N be a compact topological space of positive dimension and $[N]$ a fundamental class. If the fundamental group $\pi_1(N)$ is amenable, then the simplicial volume $||N|| = 0$.*

The reason is that the bounded cohomology groups of N vanish [**25**, §3.1].

As a corollary of Proposition 2.9, we now prove Proposition 1.7.

Proof of Proposition 1.7. Since the Ricci curvature of an irreducible symmetric space is negative if and only if it is of noncompact type, it follows that $\Gamma\backslash X$ is of general type if and only if X is of noncompact type. By Proposition 2.9, if X is compact type, flat, or products of them, then the simplicial volume of $\Gamma\backslash X$ is zero, and Proposition 3.2 below completes the proof.

Proposition 2.10. *Let N_1, N_2 be two compact topological spaces of the same dimension with fundamental classes $[N_1]$ and $[N_2]$ respectively. Let $f : N_1 \to N_2$ be a continuous map of degree d, i.e., $f_*([N_1]) = d[N_2]$. Then*

$$||N_1|| \geq d||N_2||.$$

A similar result holds for noncompact topological spaces.

Corollary 2.11. *If a compact topological space N with a fundamental class admits a self-map of degree greater than or equal to 2, then $||M|| = 0$. If a noncompact topological space N with a fundamental class admits a self-map of degree greater than or equal to 2, then $||M|| = 0$ or $||M|| = \infty$.*

Definition 2.12. For an orbifold M that is quotient of a smooth manifold Y, let d be the degree of the map $Y \to M$. Define the orbifold simplicial volume $||M||$ by

$$||M||_{orb} = \frac{||Y||}{d}.$$

In the above definition, if M admits several different finite coverings, then the Proportionality Theorem [**25**, p. 11] [**14**] [**20**] [**52**] implies that the orbifold simplicial volume of M is well-defined. We note that the simplicial volume of finite coverings of closed manifolds is multiplicative. This is similar to the definition of the Euler characteristic of virtually torsion-free groups in [**11**].

Remark 2.13. This definition of orbifold simplicial volume allows one to remove the restriction in [**43**, Theorem 1.7, and §1.6] where only neat arithmetic subgroups are considered, in view of the fact that most natural arithmetic subgroups are not torsion-free. The usual simplicial volume may not work.

The above two definitions of simplicial volumes of orbifolds satisfy the inequality

$$||M|| \leq ||M||_{orb},$$

but they may not agree. To prove the inequality, we note that for any finite smooth cover, $f : Y \to M$, $f_*([Y]) = d[M]$. Therefore, by Proposition 2.10,

$$||M|| \leq \frac{||Y||}{d}.$$

The following proposition gives an example of the strict inequality. A *natural problem* is to determine conditions on M under which $||M|| = ||M||_{orb}$.

Proposition 2.14. *Let* $M = \mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}^2$. *Then* $||M||_{orb} > 0$, *but* $||M|| = 0$.

Proof. For any finite smooth cover Y of M as an orbifold, $Y = \Gamma\backslash\mathbb{H}^2$ for some torsion-free subgroup of $\mathrm{SL}(2, \mathbb{Z})$ of finite index. By the result of Thurston for hyperbolic manifolds of finite volume [**25**, §0.3], $0 < ||Y|| < +\infty$, and hence the simplicial volume $||M||_{orb}$ is finite and positive. We note that $\mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}^2$ is homeomorphic to $\mathbb{R}^2$. One way to see this is to use the usual fundamental domain of $SL(2, \mathbb{Z})$ in $\mathbb{H}^2$ given by $\Omega = \{x + iy \in \mathbb{H}^2 \mid -\frac{1}{2} \leq x \leq \frac{1}{2}, x^2 + y^2 \geq 1\}$. When the boundary components of Ω are identified (i.e., the left is identified with the right), we obtain a space homeomorphic to $\mathbb{R}^2$. Another way is to use the j-invariant of elliptic curves to establish a homomorphism between the moduli space $\mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}^2$ of elliptic curves to the complex space $\mathbb{C}$, which is of course homeomorphic to $\mathbb{R}^2$.

We also note that the fundamental class $[\mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}^2]$ is mapped to the fundamental class of $\mathbb{R}^2$. Therefore,

$$||\mathbb{R}^2|| = ||M|| < +\infty.$$

Since $\mathbb{R}^2$ admits self-maps of degrees greater than 1, it follows from Proposition 2.11 that $||\mathbb{R}^2|| = 0$.

As a corollary, we recover a result on vanishing of $||\mathbb{R}^2||$ [**25**, p. 10] [**41**, §6.3.1].

Corollary 2.15. *The simplicial volume of* $\mathbb{R}^2$ *is equal to 0.*

Using the product formula for simplicial volume [**40**, Theorem C.7] [**25**], we obtain

Corollary 2.16. *For any* $n \geq 1$, $||\mathbb{R}^{2n}|| = 0$.

3. A summary of some known results on simplicial volume

Though the notion of simplicial volume is natural, it is not easy to compute it, or even to determine whether it vanishes or not. In this section, we summarize several known results, in particular those on the simplicial volume of locally symmetric spaces, which will motivate the results in this paper.

Proposition 3.1. *If a compact manifold* M *is a quotient of a symmetric space of nonnegative sectional curvature, then the simplicial volume* $||M||$ *of* M *is equal to 0.*

The reason is that the fundamental group of such a manifold is a finite extension of an abelian group and hence amenable. It follows from Proposition 2.9 that $||M|| = 0$.

Proposition 3.2 ([**39**], [**15**])**.** *If* X *is a symmetric space of noncompact type and* $\Gamma\backslash X$ *is a compact smooth quotient by a discrete isometry group* Γ, *then the simplicial volume of* $\Gamma\backslash X$ *is positive.*

The paper [**15**] proved the nonvanishing of simplicial volume in Proposition 3.2 when $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$, and [**39**] proved all other cases.

The assumption of compactness of the quotient is crucial. For noncompact locally symmetric spaces, the answer can be quite different.

Proposition 3.3 ([**44**, Theorem 1.1])**.** *If* $\Gamma\backslash X$ *is an arithmetic locally symmetric space of* $\mathbb{Q}$*-rank at least 3, then the simplicial volume of* $\Gamma\backslash X$ *is equal to zero.*

We recall that an arithmetic locally symmetric space $\Gamma\backslash X$ is compact if and only if it is of $\mathbb{Q}$-rank 0. For example, the $\mathbb{Q}$-rank of $SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n)$ is equal to $n-1$.

The idea of the proof of Proposition 3.3 will be explained in §5 below. If the rank of X is equal to 1, then the sectional curvature of $\Gamma\backslash X$ is pinched between two negative numbers and hence the simplicial volume of $\Gamma\backslash X$ is positive [**25**, §0.3]. For some $\mathbb{Q}$-rank 1 arithmetic locally symmetric spaces including smooth Hilbert modular varieties, the simplicial volume is positive [**43**] [**37**]. The case of $\mathbb{Q}$-rank 2 locally symmetric spaces is still open (see Remark 5.4).

As mentioned in Remark 2.13 above, by introducing a notion of orbifold simplicial volume, the restriction to neat arithmetic subgroups in [**43**] (or Proposition 3.3 above) can be removed.

Besides these results, there are also other results on properties of simplicial volumes under operations such as taking products and fibrations [**28**] [**12**]. See [**40**] [**42**] for summaries and references.

The simplicial volume of a manifold is closely related to topological and geometric properties of the manifold [**1**] [**2**] [**4**] [**16**] [**17**] [**21**] [**29**] [**38**] [**47**] [**50**] [**51**] [**53**] [**55**] [**46**] [**22**].

4. Vanishing of simplicial volumes

In this section, we extract the main steps of the proof of vanishing of simplicial volume in Proposition 3.3 [**44**, Theorem 1.1] and formulate them as Propositions 4.2, 4.3, and 4.4. In the next sections, we explain how they can be applied to prove Proposition 3.3 and Propositions 1.4 and 1.5. This will show similarities between locally symmetric spaces and moduli spaces $\mathcal{M}_{g,n}$.

The proof of [**44**, Theorem 1.1] uses a vanishing criterion of Gromov [**25**, p. 58] [**44**, p. 221].

Definition 4.1. *Let N be a topological space. A subset $U \subset N$ is called amenable in N if for every point $x \in U$, the image of the map $\pi_1(U, x) \to \pi_1(N, x)$ is amenable. A sequence of subsets $U_i, i = 1, 2, \cdots$, is called amenable at infinity if there is an increasing exhausting sequence of compact subsets K_i of N, $i = 1, 2, \cdots$, with $U_i \subset N - K_i$, such that U_i is amenable in $N - K_i$ when $i \gg 1$.*

Proposition 4.2. *Let M be a manifold without boundary. Let $\{U_i\}_{i\in\mathbb{N}}$ be a locally finite covering of M by open relatively compact subsets such that each point of M is contained in at most n subsets of the covering, where $n = \dim M$. If each U_i is amenable in M and the sequence U_i is amenable at infinity, then the simplicial volume $||M|| = 0$.*

The construction of such a locally finite covering of locally symmetric spaces $\Gamma\backslash X$ is carried out in the following steps [**44**, Theorem 5.3, Corollary 5.4]:

Proposition 4.3. *Let M be a manifold and $\Gamma = \pi_1(M)$. Assume that Γ admits a finite classifying space $B\Gamma$ of dimension k. Then there is a locally finite covering of M by relatively compact, amenable, and open subsets U_i, $i = 1, \cdots$, such that every point of M is contained in at most $k + 2$ such subsets.*

Proposition 4.4. *Let M be the interior of a compact, n-dimensional manifold W with boundary ∂W. Assume that $B\pi_1(M)$ admits a finite* *model of dimension at most $n-2$ and for every point $x \in \partial W$, the inclusion induces an injection $\pi_1(\partial W, x) \to \pi_1(W, x)$. Then the covering U_i, $i = 1, 2, \cdots$, constructed in the previous proposition, is amenable at infinity, and every point in M meets at most n subsets U_i. Therefore,* $||M|| = 0$.

5. Simplicial volume of locally symmetric spaces and their compactifications

In this section, we recall the proof of Proposition 3.3 ([**44**, Theorem 1.1]) by showing that the conditions in Proposition 4.4 are satisfied by an arithmetic locally symmetric space $\Gamma\backslash X$ when its $\mathbb{Q}$-rank $r_{\mathbb{Q}}$ is at least 3. We will also prove Propositions 1.4 and 1.5 on simplicial volumes of compactifications of locally symmetric spaces.

First we note that since the invariant Riemannian metric of $\Gamma\backslash X$ has bounded sectional curvature and the volume of $\Gamma\backslash X$ is finite, it is clear from a lower bound of the minimal volume in terms of the simplicial volume that the simplicial volume of $\Gamma\backslash X$ is finite.

We explain the setup of Proposition 3.3 in more detail. Assume that $\mathbf{G} \subset \mathrm{GL}(n, \mathbb{C})$ is a linear algebraic group defined over $\mathbb{Q}$, and $G = \mathbf{G}(\mathbb{R})$ its real locus. By assumption, $\Gamma \subset G$ is a discrete subgroup commensurable with $\mathbf{G}(\mathbb{Q}) \cap \mathrm{GL}(n, \mathbb{Z})$.

Proof of Proposition 3.3. For a torsion-free arithmetic subgroup Γ, the locally symmetric space $\Gamma\backslash X$ is a model for $B\Gamma$. Since the $\mathbb{Q}$-rank is positive, it is not compact, and hence it is not a finite space (i.e., not a finite CW-complex). Since the Borel-Serre compactification $\overline{\Gamma\backslash X}^{BS}$ is a real analytic manifold with corners [**9**] and hence admits a finite triangulation, and the inclusion $\Gamma\backslash X \to \overline{\Gamma\backslash X}^{BS}$ is a homotopy equivalence, it follows that $\overline{\Gamma\backslash X}^{BS}$ is a finite model of $B\Gamma$.

It is also known that the cohomological dimension of Γ, cd Γ, is equal to $\dim \Gamma\backslash X - r_{\mathbb{Q}}$, where $r_{\mathbb{Q}}$ is the $\mathbb{Q}$-rank of $\Gamma\backslash X$. When $r_{\mathbb{Q}} \geq 3$, cd $\Gamma \leq \dim \Gamma\backslash X - 3$. When $r_{\mathbb{Q}} \geq 3$, it follows easily from the structure of symmetric spaces that $\dim \Gamma\backslash X - r_{\mathbb{Q}} \geq 3$. Then a general result on classifying spaces implies that Γ admits a finite $B\Gamma$-space of dimension equal to $\dim \Gamma\backslash X - r_{\mathbb{Q}}$. This shows that the first condition on existence of a finite model of $B\Gamma$ of dimension at most $\dim \Gamma\backslash X - 2$ in Proposition 4.4 is satisfied.

Since $\overline{\Gamma\backslash X}^{BS}$ is a topological manifold with boundary, it can be taken as W in Proposition 4.4. The remaining condition, that the inclusion induces an injection $\pi_1(\partial W, x) \to \pi_1(W, x)$, follows from [**7**, Proposition 2.3].

Proposition 5.1. *When $r_{\mathbb{Q}} \geq 3$, the inclusion $\pi_1(\partial\overline{\Gamma\backslash X}^{BS}, x) \to \pi_1(\overline{\Gamma\backslash X}^{BS}, x)$ is an isomorphism.*

Before we explain the idea of the proof of Proposition 5.1 outlined in [**7**], we recall the construction of the Borel-Serre compactification $\overline{\Gamma\backslash X}^{BS}$. Fix a basepoint $x_0 \in X = G/K$ corresponding to a maximal compact subgroup K of G. For every proper $\mathbb{Q}$-parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, its real locus $P = \mathbf{P}(\mathbb{R})$ admits a rational Langlands decomposition

$$P = N_{\mathbf{P}} A_{\mathbf{P}} M_{\mathbf{P}} \cong N_{\mathbf{P}} \times A_{\mathbf{P}} \times M_{\mathbf{P}},$$

where $N_{\mathbf{P}}$ is the unipotent radical of P, $A_{\mathbf{P}}$ is the $\mathbb{Q}$-split component of P, and $M_{\mathbf{P}}$ is a reductive group. The product $A_{\mathbf{P}} M_{\mathbf{P}}$ is the Levi factor of P which is stable with respect to the Cartan involution of G associated with K. Define $X_{\mathbf{P}} = M_{\mathbf{P}}/(K \cap M_{\mathbf{P}})$. $X_{\mathbf{P}}$ is a symmetric space of nonpositive sectional curvature and is called the boundary symmetric space of the $\mathbb{Q}$-parabolic subgroup $\mathbf{P}$. The Langlands decomposition of P induces a horospherical decomposition of X:

$$X \cong N_{\mathbf{P}} \times A_{\mathbf{P}} \times X_{\mathbf{P}}.$$

When $X = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) \cong \mathbb{H}^2$ and P is the parabolic subgroup of upper triangular matrices, the associated horospherical decomposition of X is the x, y-coordinate decomposition of $\mathbb{H}^2$.

For every $\mathbf{P}$, define its Borel-Serre boundary component by

$$e(\mathbf{P}) = N_{\mathbf{P}} \times X_{\mathbf{P}}.$$

Attach these boundary components $e(\mathbf{P})$ to X at infinity by using the horospherical decomposition to obtain the **Borel-Serre partial compactification** $_{\mathbb{Q}}\overline{X}^{BS}$:

$$_{\mathbb{Q}}\overline{X}^{BS} = X \cup \coprod_{\mathbf{P}} e(\mathbf{P}).$$

It is known that $_{\mathbb{Q}}\overline{X}^{BS}$ is a real analytic manifold with corners and the Γ-action on X extends to a proper real analytic action on $_{\mathbb{Q}}\overline{X}^{BS}$ with a compact quotient $\Gamma\backslash_{\mathbb{Q}}\overline{X}^{BS}$. It is the Borel-Serre compactification $\overline{\Gamma\backslash X}^{BS}$ (see [**9**] for references).

When $X = \mathbb{H}^2$, $_{\mathbb{Q}}\overline{X}^{BS}$ is obtained by adding a copy $\mathbb{R}$ at every rational boundary point in $\partial\mathbb{H}^2 = \mathbb{R} \cup \{i\infty\}$, and the Borel-Serre compactification of $\Gamma\backslash\mathbb{H}^2$ is obtained by adding a circle to every end so that $\overline{\Gamma\backslash X}^{BS}$ is a compact manifold with boundary in this case. One way to visualize the Borel-Serre compactification of a higher rank locally symmetric space is to take products of $\Gamma\backslash\mathbb{H}^2$.

Proposition 5.2. *The boundary $\partial_{\mathbb{Q}}\overline{X}^{BS}$ is homotopy equivalent to a bouquet of infinitely many spheres of dimension $r_{\mathbb{Q}} - 1$. When $r_{\mathbb{Q}} \geq 3$, the boundary $\partial_{\mathbb{Q}}\overline{X}^{BS}$ is simply connected.*

Proof. For each $\mathbb{Q}$-parabolic subgroup $\mathbf{P}$, its boundary component $e(\mathbf{P})$ is contractible. For two proper parabolic subgroups $\mathbf{P}_1, \mathbf{P}_2$, the inclusion $\mathbf{P}_1 \subset \mathbf{P}_2$ holds if and only if $e(\mathbf{P}_1)$ is contained in the closure of $e(\mathbf{P}_2)$. This implies that $\partial_{\mathbb{Q}}\overline{X}^{BS}$ is homotopy equivalent to the $\mathbb{Q}$-Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ of $\mathbf{G}$, which is an infinite simplicial complex with one simplex $\sigma_{\mathbf{P}}$ for each $\mathbb{Q}$-parabolic subgroup $\mathbf{P}$ such that maximal $\mathbb{Q}$-parabolic subgroups $\mathbf{P}$ correspond to 0-dimensional simplexes, and $\sigma_{\mathbf{P}_1}$ is contained in $\sigma_{\mathbf{P}_2}$ as a face if and only if $\mathbf{P}_1$ is contained in $\mathbf{P}_2$. By the Solomon-Tits Theorem, $\Delta_{\mathbb{Q}}(\mathbf{G})$ is homotopy equivalent to a bouquet of infinitely many spheres of dimension $r_{\mathbb{Q}}-1$. This proves the proposition.

Proof of Proposition 5.1. Since ${}_{\mathbb{Q}}\overline{X}^{BS}$ is a manifold with corners whose interior is equal to X, it can be deformed into X and hence is contractible. Since the boundary $\partial_{\mathbb{Q}}\overline{X}^{BS}$ is simply connected and Γ also acts properly without fixed points, we conclude that the inclusion $\Gamma\backslash\partial_{\mathbb{Q}}\overline{X}^{BS} \to \Gamma\backslash{}_{\mathbb{Q}}\overline{X}^{BS}$ induces an isomorphism

$$\pi_1(\Gamma\backslash\partial_{\mathbb{Q}}\overline{X}^{BS}) \to \pi_1(\Gamma\backslash{}_{\mathbb{Q}}\overline{X}^{BS}),$$

i.e., $\pi_1(\partial\overline{\Gamma\backslash X}^{BS}) \to \pi_1(\overline{\Gamma\backslash X}^{BS})$ is an isomorphism.

Before we prove Proposition 1.4, we explain the reductive Borel-Serre compactification $\overline{\Gamma\backslash X}^{RBS}$. For each $\mathbb{Q}$-parabolic subgroup $\mathbf{P}$, define its boundary component by

$$\hat{e}(\mathbf{P}) = X_{\mathbf{P}}.$$

It is a quotient of the Borel-Serre boundary component $e(\mathbf{P})$ by collapsing the unipotent factor $N_{\mathbf{P}}$. By attaching these boundary components $\hat{e}(\mathbf{P})$ to X at infinity, we obtain the **reductive Borel-Serre partial compactification**

$${}_{\mathbb{Q}}\overline{X}^{RBS} = X \cup \coprod_{\mathbf{P}} \hat{e}(\mathbf{P}).$$

It can be shown that the Γ-action on X extends to a continuous action on ${}_{\mathbb{Q}}\overline{X}^{RBS}$ with a compact quotient, $\Gamma\backslash{}_{\mathbb{Q}}\overline{X}^{RBS}$, which is the reductive Borel-Serre compactification $\overline{\Gamma\backslash X}^{RBS}$. The extended Γ-action is not proper. From this definition, it is clear that the identity map on X extends to Γ-equivariant surjective map

$${}_{\mathbb{Q}}\overline{X}^{BS} \to {}_{\mathbb{Q}}\overline{X}^{RBS}.$$

It follows that the identity map on $\Gamma\backslash X$ extends to a continuous surjective map

$$\overline{\Gamma\backslash X}^{BS} \to \overline{\Gamma\backslash X}^{RBS}.$$

For every point in the boundary $\partial\overline{\Gamma\backslash X}^{RBS}$, its inverse image is a nilmanifold $\Gamma \cap N_{\mathbf{P}}\backslash N_{\mathbf{P}}$ for some parabolic subgroup. See [**9**] for more details.

For example, when $X = \mathbb{H}^2$, the compactification $\overline{\Gamma\backslash X}^{RBS}$ is obtained by adding one point to each end.

Now we prove the following result stated in Proposition 2.4.

Proposition 5.3. *Assume that* Γ *is torsion-free. Let* $\Gamma\backslash X$ *be a noncompact arithmetic locally symmetric space, and* $\overline{\Gamma\backslash X}^{RBS}$ *the reductive Borel-Serre compactification. The conditions in Proposition* 2.1 *are satisfied, and* $\overline{\Gamma\backslash X}^{RBS}$ *has a fundamental class.*

Proof. The singular locus of $\overline{\Gamma\backslash X}^{RBS}$ is contained in the boundary $\partial\overline{\Gamma\backslash X}^{RBS}$. A neighborhood of $\partial_{\mathbb{Q}}\overline{X}^{BS}$ can be deformation retracted to $\partial_{\mathbb{Q}}\overline{X}^{BS}$ by moving along the geodesic action of $\mathbb{Q}$-parabolic subgroups $\mathbf{P}$. The parameter ε in Proposition 2.1 is determined in terms of the split component of the horospherical decompositions of X. This deformation is Γ-equivariant and in the deformation, the $N_{\mathbf{P}}$-components are preserved (see [**48**]). Since we can collapse the nilpotent components during the deformation retraction, this implies that a small neighborhood of $\partial\overline{\Gamma\backslash X}^{RBS}$ can be deformation retracted to $\partial\overline{\Gamma\backslash X}^{RBS}$, and the other conditions in Proposition 2.1 are also satisfied.

Proof of Proposition 1.4. We note that the map $\overline{\Gamma\backslash X}^{BS} \to \overline{\Gamma\backslash X}^{RBS}$ is of degree one, and maps the relative fundamental cycle $[\overline{\Gamma\backslash X}^{BS}, \partial\overline{\Gamma\backslash X}^{BS}]$ to the fundamental class $[\overline{\Gamma\backslash X}^{RBS}]$.

When $r_{\mathbb{Q}} \geq 3$, by Proposition 1.3, the relative simplicial volume of $\overline{\Gamma\backslash X}^{BS}$ is equal to zero. It follows that the simplicial volume of $\overline{\Gamma\backslash X}^{RBS}$ is zero.

When $\Gamma\backslash X$ is irreducible, the rank of X is at least 2, and $r_{\mathbb{Q}} \geq 1$, by a result of [**35**], the fundamental group of $\overline{\Gamma\backslash X}^{RBS}$ is finite. By Proposition 2.9, the simplicial volume of $\overline{\Gamma\backslash X}^{RBS}$ is zero.

Proof of Proposition 1.5. When $\Gamma\backslash X$ is Hermitian, it admits the Baily-Borel compactification $\overline{\Gamma\backslash X}^{BB}$, which is a normal projective variety and hence admits a fundamental class.

For a finite index subgroup $\Gamma' \subset \Gamma$, the projection $\Gamma'\backslash X \to \Gamma\backslash X$ induces a holomorphic map $\overline{\Gamma'\backslash X}^{BB} \to \overline{\Gamma\backslash X}^{BB}$. Therefore, it suffices to prove vanishing of the simplicial volume of $\Gamma'\backslash X$ when Γ' is torsion-free. This can be proved as in the case of $\overline{\Gamma\backslash X}^{RBS}$ above or follows as a corollary of Proposition 1.4, since the fundamental class of $\overline{\Gamma\backslash X}^{RBS}$ is mapped to the fundamental class of $\overline{\Gamma\backslash X}^{BB}$.

Remark 5.4. One of the motivations to consider the simplicial volume of the reductive Borel-Serre compactification $\overline{\Gamma\backslash X}^{RBS}$ was an attempt to prove that the simplicial volume of arithmetic locally symmetric spaces $\Gamma\backslash X$ of $\mathbb{Q}$-rank 2 is zero and hence to extend the result [**44**, Theorem 1.1] (Proposition 3.3 above). It consists of two steps: (1) show that $||\Gamma\backslash X|| = ||\overline{\Gamma\backslash X}^{BS}, \partial\overline{\Gamma\backslash X}^{BS}||$, and (2) show that the dominating map $\overline{\Gamma\backslash X}^{BS} \to \overline{\Gamma\backslash X}^{RBS}$ which extends the identity on $\Gamma\backslash X$ induces an equality $||\overline{\Gamma\backslash X}^{BS}, \partial\overline{\Gamma\backslash X}^{BS}|| = ||\overline{\Gamma\backslash X}^{RBS}||$.

For Step (1), $\overline{\Gamma\backslash X}^{BS}$ is a topological manifold with boundary, and its interior is equal to $\Gamma\backslash X$. There are some results on when the simplicial volume of the interior of a compact manifold with nonempty boundary is equal to the relative simplicial volume (see [**40**, example 6.20] [**43**, Theorem 1.5]).

For Step (2), the fibers of the map $\overline{\Gamma\backslash X}^{BS} \to \overline{\Gamma\backslash X}^{RBS}$ are nilmanifolds. Since the fundamental group of nilmanifolds is amenable, this suggests some positive evidence.

In general, Step (2) is not correct. For example, consider a hyperbolic surface $\Sigma_{0,n}$ of genus 0 with n punctures, where $n \geq 3$, as a locally symmetric space $\Gamma\backslash X$ of finite volume. The Borel-Serre compactification $\overline{\Sigma_{0,n}}^{BS}$ is obtained by adding a circle to every end, and the reductive Borel-Serre compactification $\overline{\Sigma_{0,n}}^{RBS}$ is homeomorphic to the sphere S^2. By [**40**, example 6.20], $||\overline{\Sigma_{0,n}}^{BS}, \partial\overline{\Sigma_{0,n}}^{BS}|| = ||\Sigma_{0,n}||$. Since $||\Sigma_{0,n}|| > 0$ by [**25**, §0.3], $||\overline{\Sigma_{0,n}}^{BS}, \partial\overline{\Sigma_{0,n}}^{BS}|| > 0$. On the other hand, $||S^2|| = 0$, and hence $||\overline{\Sigma_{0,n}}^{RBS}|| = ||S^2|| = 0$. Therefore, $||\overline{\Sigma_{0,n}}^{BS}, \partial\overline{\Sigma_{0,n}}^{BS}|| \neq ||\overline{\Sigma_{0,n}}^{RBS}||$.

A natural question is whether Step (2) might hold in the higher rank case.

6. Simplicial volume of moduli spaces

Theorem 1.1 is proved by following the same procedure as in the proof of Proposition 3.3, i.e., the steps in §3. The difference is to replace the symmetric space X by the Teichmüller space $\mathcal{T}_{g,n}$, and the arithmetic subgroup Γ by the mapping class group $\mathrm{Mod}_{g,n}$. There is an analogue of the Borel-Serre partial compactification for $\mathcal{T}_{g,n}$, and the topology of its boundary is determined by an analogue of the Solomon-Tits Theorem for curve complexes.

Let $S_{g,n}$ be a Riemann surface of genus g with n punctures. The moduli space of marked complex structures on $S_{g,n}$ is the Teichmüller space $\mathcal{T}_{g,n}$. It is a complex manifold diffeomorphic to $\mathbb{R}^{2d}$, where $d = 3g-3+n$ if $2g-2+n > 0$ so that $S_{g,n}$ admits complete hyperbolic metrics of finite area, and $d = 0, 1$ otherwise. The mapping class group

$\mathrm{Mod}_{g,n} = \mathrm{Diff}^+(S_{g,n})/\mathrm{Diff}^0(S_{g,n})$ acts on $\mathcal{T}_{g,n}$ properly and holomorphically, and the quotient $\mathrm{Mod}_{g,n}\backslash\mathcal{T}_{g,n}$ is equal to the moduli space $\mathcal{M}_{g,n}$. This implies that $\mathcal{M}_{g,n}$ is a complex orbifold.

It is known that $\mathcal{M}_{g,n}$ is a quasi-projective variety and admits the Deligne-Mumford compactification $\overline{\mathcal{M}_{g,n}}^{DM}$, which is a projective variety and also a compact complex orbifold. The $\mathbb{C}$-dimension of $\mathcal{M}_{g,n}$ is given by

$$\dim_{\mathbb{C}} \mathcal{M}_{g,n} = \begin{cases} 3g-3+n & \text{if } 2g-2+n>0, \\ 1 & \text{if } g=1, n=0, \\ 0 & \text{if } n \leq 2. \end{cases} \tag{6.1}$$

A slightly stronger result than Theorem 1.1 holds.

Theorem 6.1. *For any finite index subgroup $\Gamma \subset \mathrm{Mod}_{g,n}$, the simplicial volume of $\Gamma\backslash\mathcal{T}_{g,n}$ is equal to 0 if $g \geq 2$; $g=1, n\geq 3$; or $g=0, n\geq 6$; and the orbifold simplicial volume of $\Gamma\backslash\mathcal{T}_{g,n}$ is positive if $g=1, n=0,1$; $g=0, n=4$.*

The only unsettled cases are $\mathcal{M}_{0,5}$ and $\mathcal{M}_{1,2}$.

To prove the vanishing of the simplicial volume of $\Gamma\backslash\mathcal{T}_{g,n}$, we need an analogue of the Borel-Serre compactification. Such a compactification was constructed in [**27**] (a different proof of a slightly weaker result was given in [**31**]). Instead we use the thick part of $\mathcal{T}_{g,n}$. For simplicity, we assume that $2g-2+n>0$, so that for each complex structure on $S_{g,n}$, there is a unique complete hyperbolic metric of finite area conformal to it. For each marked hyperbolic surface $\Sigma_{g,n} \in \mathcal{T}_{g,n}$, let $\ell(\Sigma_{g,n})$ be the length of the shortest geodesics in it. For any sufficiently small ε, define

$$\mathcal{T}_{g,n}(\varepsilon) = \{\Sigma_{g,n} \in \mathcal{T}_{g,n} \mid \ell(\Sigma_{g,n}) \geq \varepsilon\}.$$

Proposition 6.2. *The thick part $\mathcal{T}_{g,n}(\varepsilon)$ satisfies the following properties:*

1) *$\mathcal{T}_{g,n}(\varepsilon)$ is stable under $\mathrm{Mod}_{g,n}$, and the quotient $\mathrm{Mod}_{g,n}\backslash\mathcal{T}_{g,n}(\varepsilon)$ is compact.*
2) *$\mathcal{T}_{g,n}(\varepsilon)$ is a real analytic manifold with corners,*
3) *there is a Γ-equivariant deformation retract from $\mathcal{T}_{g,n}$ to $\mathcal{T}_{g,n}(\varepsilon)$, and hence $\mathcal{T}_{g,n}(\varepsilon)$ is contractible.*

See [**36**] for more details of the above proposition. The above proposition shows that $\mathcal{T}_{g,n}(\varepsilon)$ can be viewed as a partial compactification of $\mathcal{T}_{g,n}$, and the quotient $\Gamma\backslash\mathcal{T}_{g,n}(\varepsilon)$ can be viewed as a compactification of $\Gamma\backslash\mathcal{T}_{g,n}$.

By [**36**], we have

Proposition 6.3. *$\mathcal{T}_{g,n}(\varepsilon)$ is a cofinite model of the universal space for proper actions of Γ. If Γ is torsion-free, then $\Gamma\backslash\mathcal{T}_{g,n}(\varepsilon)$ is a finite model of $B\Gamma$.*

The topology of the boundary of $\mathcal{T}_{g,n}(\varepsilon)$ can be described in terms of the curve complex $\mathcal{C}(S_{g,n})$. Recall that a simple closed curve in $S_{g,n}$ is called essential if it does not bound a disk or a puncture. Vertices of $\mathcal{C}(S_{g,n})$ correspond to homotopy classes of essential simple closed curves in $S_{g,n}$, and $(k+1)$ vertices form the vertices of a k-simplex if and only if they admit disjoint representatives.

By [**26**] [**30**] [**34**], it is known that

Proposition 6.4. *$\mathcal{C}(S_{g,n})$ is homotopy equivalent to a bouquet of infinitely many spheres S^d, where*

$$d = \begin{cases} -\chi(S_g) = 2g-2 & \text{if } n = 0, \\ -\chi(S_{g,n}) - 1 = 2g-3+n & \text{if } g \geq 1 \text{ and } n > 0, \\ -\chi(S_{0,n}) - 2 = n-4 & \text{if } g = 0. \end{cases} \tag{6.2}$$

Proposition 6.5. *The boundary faces of $\mathcal{T}_{g,n}(\varepsilon)$ are contractible and parametrized by simplexes of $\mathcal{C}(S_{g,n})$. For each simplex $\sigma \in \mathcal{C}(S_{g,n})$, let f_σ be the corresponding boundary face. Then for any two simplexes σ_1, σ_2, the simplex σ_1 is a face of σ_2 if and only if f_{σ_1} contains f_{σ_2} as a face. Consequently, the boundary $\partial\mathcal{T}_{g,n}(\varepsilon)$ is homotopy equivalent to $\mathcal{C}(S_{g,n})$, and hence homotopy equivalent to a bouquet of spheres S^d, where d is given in the previous proposition.*

See [**34**, Proposition 2.1] for more detail. A corollary of the above proposition determines the virtual cohomological dimension of $\mathrm{Mod}_{g,n}$ [**26**] [**30**].

Corollary 6.6. *$\mathrm{Mod}_{g,n}$ is a virtual duality group of dimension d, where*

$$d = \begin{cases} 4g-5 = \dim \mathcal{T}_g + \chi(S_g) - 1 & \text{if } n = 0, \\ 4g-4+n = \dim \mathcal{T}_{g,n} + \chi(S_{g,n}) & \text{if } g \geq 1 \text{ and } n > 0, \\ n-3 = \dim \mathcal{T}_{0,n} + \chi(S_{0,n}) + 1 & \text{if } g = 0. \end{cases} \tag{6.3}$$

Hence for any torsion-free finite index subgroup Γ of $\mathrm{Mod}_{g,n}$, $cd\,\Gamma = d$.

Proof of Theorem 6.1 *and Theorem* 1.1. Under the assumption that $g \geq 2$; $g = 1, n \geq 3$; or $g = 0, n \geq 6$, the spheres in the homotopy type of $\mathcal{C}(S_{g,n})$ are of dimension at least 2, and hence $\mathcal{C}(S_{g,n})$ and $\partial\mathcal{T}_{g,n}(\varepsilon)$ are simply connected. Using the results recalled in this section, we see that the same proof of Proposition 3.3 in §4 goes through to give a proof of Theorem 6.1, i.e., the conditions of Proposition 4.4 are satisfied, and Theorem 6.1 can be proved for these cases.

When $g = 1, n = 0, 1$, $\mathcal{T}_{g,n}$ is equal to $\mathbb{H}^2$ and $\mathrm{Mod}_{g,n} = \mathrm{SL}(2,\mathbb{Z})$, and hence $\mathcal{M}_{g,n} = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$. By Proposition 2.14, $||\mathcal{M}_{g,n}||_{orb} > 0$. When $g = 0, n = 4$, $\mathcal{T}_{g,n}$ is equal to $\mathbb{H}^2$ and $\mathrm{Mod}_{g,n}$ is commensurable with $\mathrm{SL}(2,\mathbb{Z})$. It follows that $||\mathcal{M}_{g,n}||_{orb} > 0$. The same result holds for $\Gamma\backslash\mathcal{T}_{g,n}$. This completes the proof of Theorem 6.1.

Remark 6.7. In Theorem 1.1, when $g = 1, n = 0, 1$, $\mathcal{T}_{g,n}$ is equal to $\mathbb{H}^2$ and $\mathrm{Mod}_{g,n} = \mathrm{SL}(2,\mathbb{Z})$, and hence $\mathcal{M}_{g,n} = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}^2$. By Proposition 2.14, $||\mathcal{M}_{g,n}|| = 0$.

To prove Proposition 1.10, we only need to use Proposition 2.9 and the following result [**45**] [**8**, Proposition 1.2]:

Proposition 6.8. $\overline{\mathcal{M}_{g,n}}^{DM}$ *is simply connected.*

More generally, for some finite index subgroups Γ of $\mathrm{Mod}_{g,n}$, a corresponding compactification $\overline{\Gamma\backslash\mathcal{T}_{g,n}}^{DM}$ is also simply connected. Hence the simplicial volume of $\overline{\Gamma\backslash\mathcal{T}_{g,n}}^{DM}$ is also equal to 0. See [**8**, §3] for definition and detailed discussion of such groups.

As mentioned in the introduction, a corollary of the vanishing of the simplicial volume of $\mathcal{M}_{g,n}$ in Theorem 6.1 gives an obstruction to existence of certain metrics on $\mathcal{M}_{g,n}$.

Corollary 6.9. *If $g \geq 2$; $g = 1, n \geq 3$; or $g = 0, n \geq 6$, then $\mathcal{M}_{g,n}$ and any finite cover of it do not admit any complete finite volume Riemannian metric of negatively pinched sectional curvature.*

This result (in fact, a stronger result) was proved in [**33**, Theorem 1.2]. See also [**10**, p. 12], [**10**, Theorem 1.3], and [**10**, question 6.1] for related results and questions on existence of complete nonpositively curved metrics on $\mathcal{M}_{g,n}$.

7. Open problems

A natural problem is to settle the remaining cases of Theorem 1.1, i.e., when $g = 1, n = 2$, and $g = 0, n = 5$. It is not clear whether $||\mathcal{M}_{g,n}||_{orb} = 0$ in these two cases. They are probably analogues of locally symmetric spaces of $\mathbb{Q}$-rank 2 and will require different techniques to solve them.

It was shown in [**44**] that unlike the usual simplicial volume, the Lipschitz simplicial volume of every locally symmetric space of finite volume is positive. A natural question is

Question 7.1. Is the Lipschitz orbifold simplicial volume of $\mathcal{M}_{g,n}$ positive for all choices of g, n?

A guess is that the answer should be yes. It will probably require techniques different from the case of locally symmetric spaces. One important ingredient in [**44**] is that for every noncompact locally symmetric space, its universal cover also admits a compact quotient, i.e., a compact locally symmetric space, which has positive simplicial volume. It is known that the Teichmüller space $\mathcal{T}_{g,n}$ does not admit cocompact actions of discrete groups which preserve natural geometric structures of $\mathcal{T}_{g,n}$ [**18**].

If a discrete group Γ admits a model of classifying space $B\Gamma$ by a manifold M, i.e., $\pi_1(M) = \Gamma$, and $\pi_i(M) = \{1\}$ for $i \geq 2$, then we can define a notion of *simplicial volume* of the group Γ.

If M is a compact manifold, then the dimension of M is equal to the cohomological dimension $cd\ \Gamma$ of Γ. If M' is another compact manifold model of $B\Gamma$, then M, M' are homotopy equivalent and have the same simplicial volume, $||M|| = ||M'||$. In this case, the simplicial volume $||M||$ is an invariant of Γ, and we define the **simplicial volume** $||\Gamma||$ of Γ by

$$||\Gamma|| = ||M||.$$

It is known that the existence of a compact manifold model M of $B\Gamma$ imposes strong conditions on Γ. For example, it implies that Γ is a Poincaré duality group of dimension n.

When Γ is not a Poincaré duality group, then we have to consider manifold models of $B\Gamma$ which are not compact. In this case, we note that if M is a model of $B\Gamma$, then for every $\mathbb{R}^n$, the product $M \times \mathbb{R}^n$ is also a model of $B\Gamma$. We also note that for every manifold model M of $B\Gamma$, $\dim M$ is bounded from below by the cohomological dimension $cd\ \Gamma$.

For any discrete group Γ of finite cohomological dimension, there exist manifold models of $B\Gamma$. Briefly, there exists a finite dimensional contractible complex $E\Gamma$ on which Γ acts properly and fixed point freely, and $\Gamma\backslash E\Gamma$ gives a model of $B\Gamma$. By embedding $E\Gamma$ into some $\mathbb{R}^n$ and taking a tubular neighborhood [**6**, p. 226, p. 220], we obtain a proper and fixed point free action of Γ on a contractible manifold W, and $\Gamma\backslash W$ is a manifold model of $B\Gamma$.

For a group Γ, let n_Γ be the smallest dimension of such manifold models. By the previous graph, $n_\Gamma \geq cd\ \Gamma$.

Define the **simplicial volume** $||\Gamma||$ by

$$||\Gamma|| = \inf\{||M|| \mid M \text{ is a manifold of dimension equal to } n_\Gamma\}.$$

It is tempting to conjecture that for any two manifold models M, M' of $B\Gamma$ of dimension equal to n_Γ, $||M|| = ||M'||$. A natural problem is to relate the simplicial volume $||\Gamma||$ to properties of Γ. For example, if Γ is amenable and does act properly and freely on some contractible manifold, then $||\Gamma|| = 0$. As the discussion below shows, the situation is quite unclear if Γ is not amenable.

By a result of [**5**], for a torsion free lattice Γ of a semisimple Lie group G, $n_\Gamma = \dim X$, where $X = G/K$ is the symmetric space of noncompact type associated with G. When Γ is not a uniform lattice, then $cd\ \Gamma < \dim X$, and hence $n_\Gamma > cd\ \Gamma$ in this case.

In terms of the notations just introduced, the results on locally symmetric spaces mentioned above can be formulated as follows.

Proposition 7.2. *Let Γ be a torsion-free arithmetic subgroup of a linear algebraic semisimple Lie group G defined over $\mathbb{Q}$. If Γ is uniform, i.e., the $\mathbb{Q}$-rank of G is equal to 0, then $||\Gamma|| > 0$; if the $\mathbb{Q}$-rank of G is greater than or equal to 3, then $||\Gamma|| = 0$.*

Note that the arithmetic subgroup Γ in the proposition is not amenable in both cases.

It is natural to ask whether the following analogue of [**5**] (see [**6**, p. 234]) holds for $\mathrm{Mod}_{g,n}$.

Question 7.3. If $\mathrm{Mod}_{g,n}$ acts properly on a contractible manifold W, is it true that $\dim W \geq \dim \mathcal{T}_{g,n} = 6g - 6 + 2n$?

If the answer to the above question is true, then Theorem 1.1 implies that for any finite index torsion-free subgroup Γ of the mapping class group $\mathrm{Mod}_{g,n}$, the simplicial volume $||\Gamma|| = 0$ when $g \geq 2$; $g = 1, n \geq 3$; or $g = 0, n \geq 6$.

The outer automorphism group $Out(F_n)$ of the free group F_n shares many properties with arithmetic subgroups of Lie groups and mapping class groups $\mathrm{Mod}_{g,n}$, and a lot of work on $Out(F_n)$ has been inspired by them. Then the following two problems seem natural.

Problem 7.4. Determine the smallest dimension of contractible manifolds W on which $Out(F_n)$ acts properly.

It is known that the outer space X_n of marked metric graphs with fundamental group isomorphic to F_n is contractible and $Out(F_n)$ acts properly on it, but X_n is not a manifold and $Out(F_n)\backslash X_n$ does not have a fundamental class.

It is tempting to conjecture that the smallest dimension of such manifolds W in the above problem is equal to $4n - 6$, based on the computation on [**6**, p. 233]. It is also an interesting problem to find natural contractible manifolds on which $Out(F_n)$ acts properly. (Since the virtual cohomological dimension of $Out(F_n)$ is finite, finite index torsion-free subgroup Γ of $Out(F_n)$ acts properly and fixed point freely on a contractible manifold. It does not seem to be obvious whether $Out(F_n)$ admits a *proper* action on contractible manifolds, though it admits many nontrivial actions on contractible manifolds via homomorphisms to arithmetic groups.)

Problem 7.5. Show that when $n \geq 3$, every finite index torsion-free subgroup $\Gamma \subset Out(F_n)$ has vanishing simplicial volume, i.e., $||\Gamma|| = 0$.

This problem depends on the assumption that $Out(F_n)$ does act properly on contractible manifolds. For any such contractible manifold W, one can also ask whether the simplicial volume $||\Gamma\backslash W|| = 0$.

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