SURFACES WITH PARALLEL MEAN CURVATURE VECTOR IN COMPLEX SPACE FORMS

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Abstract

We consider surfaces with parallel mean curvature vector (pmc surfaces) in complex space forms and introduce a holomorphic differential on such surfaces. When the complex dimension of the ambient space is equal to two we find a second holomorphic differential and then determine those pmc surfaces on which both differentials vanish. We also provide a reduction of codimension theorem and prove a non-existence result for pmc 2-spheres in complex space forms.

1. Introduction

Sixty years ago, H. Hopf was the first to use a quadratic form in order to study surfaces immersed in a 3-dimensional Euclidean space. He proved, in 1951, that any such surface which is homeomorphic to a sphere and has constant mean curvature is actually isometric to a round sphere (see [13]). This result was extended by S.-S. Chern to surfaces immersed in 3-dimensional space forms (see [8]) and by U. Abresch and H. Rosenberg to surfaces in simply connected, homogeneous 3-dimensional Riemannian manifolds, whose group of isometries has dimension 4 (see [1, 2]). Very recently, H. Alencar, M. do Carmo and R. Tribuzy have made the next step by obtaining Hopf-type results in spaces with dimension higher than 3, namely in product spaces $M^n(c) \times \mathbb{R}$, where $M^n(c)$ is a simply connected $n$-dimensional space form with constant sectional curvature $c \neq 0$ (see [3, 4]). They have considered the case of surfaces with parallel mean curvature vector, as a natural generalization of those with constant mean curvature in a 3-dimensional ambient space. We also have to mention a recent paper of F. Torralbo and F. Urbano, which is devoted to the study of surfaces with parallel mean curvature vector in $S^2 \times S^2$ and $H^2 \times H^2$.

Minimal surfaces and surfaces with parallel mean curvature vector in complex space forms have been also a well studied subject in the
last two decades (see, for example, [5, 7, 9, 10, 12, 15, 16, 17, 18]). In all these papers the Kähler angle proved to play a decisive role in understanding the geometry of immersed surfaces in a complex space form, and, in several of them, important results were obtained when this angle was supposed to be constant (see [5, 16, 18]).

The main goal of our paper is to obtain some characterization results concerning surfaces with parallel mean curvature vector in complex space forms by using as a principal tool holomorphic quadratic forms defined on these surfaces. The paper is organized as follows. In Section 2 we introduce a quadratic form $Q$ on surfaces of an arbitrary complex space form and prove that its $(2,0)$-part is holomorphic when the mean curvature vector of the surface is parallel. In Section 3 we work in the complex space forms with complex dimension equal to 2 and find another quadratic form $Q'$ with holomorphic $(2,0)$-part. Then we determine surfaces with parallel mean curvature vector on which both $(2,0)$-part of $Q$ and $(2,0)$-part of $Q'$ vanish. As a by-product we reobtain a result in [12]. More precisely, we prove that a 2-sphere can be immersed as a surface with parallel mean curvature vector only in a flat complex space form and it is a round sphere in a hyperplane in $\mathbb{C}^2$. In Section 4 we deal with surfaces in $\mathbb{C}^n$ with parallel mean curvature vector, and we prove that the $(2,0)$-part of $Q$ vanishes on such a surface if and only if it is pseudo-umbilical. The main result of Section 5 is a reduction theorem, which states that a surface in a complex space form, with parallel mean curvature vector, either is totally real and pseudo-umbilical or it is not pseudo-umbilical and lies in a complex space form with complex dimension less or equal to 5. The last Section is devoted to the study of the 2-spheres with parallel mean curvature vector and constant Kähler angle. We prove that there are no non-pseudo-umbilical such spheres in a complex space form with constant holomorphic sectional curvature $\rho \neq 0$.

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2. A quadratic form

Let $\Sigma^2$ be an immersed surface in $N^n(\rho)$, where $N$ is a complex space form with complex dimension $n$, complex structure $(J, \langle , \rangle)$, and with constant holomorphic sectional curvature $\rho$; which is either $\mathbb{C}P^n(\rho)$, or $\mathbb{C}^n$, or $\mathbb{C}H^n(\rho)$, as $\rho > 0$, $\rho = 0$, and $\rho < 0$, respectively. Let us define a quadratic form $Q$ on $\Sigma^2$ by

$$Q(X,Y) = 8|H|^2\langle \sigma(X,Y), H \rangle + 3\rho\langle JX, H \rangle \langle JY, H \rangle,$$
where $\sigma$ is the second fundamental form of the surface and $H$ is its mean curvature vector field. Assume that $H$ is parallel in the normal bundle of $\Sigma^2$, i.e. $\nabla^\perp H = 0$, the normal connection $\nabla^\perp$ being defined by the equation of Weingarten

$$\nabla^N_X V = -A_X V + \nabla^\perp_X V,$$

for any vector field $X$ tangent to $\Sigma^2$ and any vector field $V$ normal to the surface, where $\nabla^N$ is the Levi-Civita connection on $N$ and $A$ is the shape operator.

We shall prove that the $(2, 0)$-part of $Q$ is holomorphic. In order to do that, let us first consider the isothermal coordinates $(u, v)$ on $\Sigma^2$. Then $ds^2 = \lambda^2(du^2 + dv^2)$ and define $z = u + iv$, $\bar{z} = u - iv$, $dz = \frac{1}{\sqrt{2}}(du + idv)$, $d\bar{z} = \frac{1}{\sqrt{2}}(du - idv)$ and

$$Z = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \bar{Z} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

We also have $\langle Z, \bar{Z} \rangle = \langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \rangle = \langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \rangle = \lambda^2$.

In the following we shall calculate

$$\bar{Z}(Q(Z, Z)) = \bar{Z}(8|H|^2\langle \sigma(Z, Z), H \rangle + 3\rho\langle JZ, H \rangle^2).$$

First, we get

$$\bar{Z}(\langle \sigma(Z, Z), H \rangle) = \langle \nabla^\perp_Z \sigma(Z, Z), H \rangle + \langle \sigma(Z, Z), \nabla^\perp_Z H \rangle$$

$$= \langle \nabla^\perp_Z \sigma(Z, Z), H \rangle + \langle \sigma(Z, Z), \nabla^\perp_Z H \rangle$$

$$= \langle (\nabla^\perp_Z \sigma)(Z, Z), H \rangle + \langle \sigma(Z, Z), \nabla^\perp_Z H \rangle,$$

where we have used that

$$(\nabla^\perp_Z \sigma)(Z, Z) = \nabla^\perp_Z \sigma(Z, Z) - 2\sigma(\nabla_Z Z, Z) = \nabla^\perp_Z \sigma(Z, Z)$$

since, from the definition of the connection $\nabla$ on the surface, we easily get $\nabla_Z Z = 0$.

Now, from the Codazzi equation, we obtain

$$\bar{Z}(\langle \sigma(Z, Z), H \rangle) = \langle (\nabla^\perp_Z \sigma)(Z, Z), H \rangle + \langle (R^N(Z, Z)Z)^\perp, H \rangle$$

$$+ \langle \sigma(Z, Z), \nabla^\perp_Z H \rangle$$

$$= \langle (\nabla^\perp_Z \sigma)(Z, Z), H \rangle + \langle R^N(Z, Z)Z, H \rangle$$

$$+ \langle \sigma(Z, Z), \nabla^\perp_Z H \rangle.$$

From the expression of the curvature tensor field of $N$

$$R^N(U, V)W = \frac{4}{\rho}(\langle V, W \rangle U - \langle U, W \rangle V + \langle JV, W \rangle JU - \langle JU, W \rangle JV + 2\langle JV, U \rangle JW),$$

it follows

$$\langle R^N(\bar{Z}, Z)Z, H \rangle = \frac{3\rho}{4}\langle \bar{Z}, JZ \rangle \langle H, JZ \rangle.$$  

We also have the following

**Lemma 2.1.**

$$\langle (\nabla^\perp_Z \sigma)(\bar{Z}, Z), H \rangle = \langle \bar{Z}, Z \rangle \langle \nabla^\perp_Z H, H \rangle.$$  

**Proof.** By using the definition of $(\nabla^\perp_Z \sigma)(\bar{Z}, Z)$ one obtains

$$(\nabla^\perp_Z \sigma)(\bar{Z}, Z) = \nabla^\perp_Z \sigma(\bar{Z}, Z) - \sigma(\nabla_Z \bar{Z}, Z) - \sigma(\bar{Z}, \nabla_Z Z)$$

$$= \nabla^\perp_Z \sigma(\bar{Z}, Z) - \sigma(\bar{Z}, \nabla_Z Z)$$

since $\nabla_Z \bar{Z} = 0$.

Next, let us consider the unit vector fields $e_1$ and $e_2$ corresponding to $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$, respectively, and $E = \frac{1}{\sqrt{2}}(e_1 - ie_2)$. Then we have $Z = \lambda E$ and

$$\sigma(\bar{Z}, Z) = \lambda^2 \frac{1}{2}\sigma(e_1 - ie_2, e_1 + ie_2) = \lambda^2 \frac{1}{2}(\sigma(e_1, e_1) + \sigma(e_2, e_2)) = \langle \bar{Z}, Z \rangle H.$$  

Since $\nabla_Z Z$ is tangent it follows that $\nabla_Z Z = a\bar{Z} + b\bar{Z}$ and then

$$0 = \langle \nabla_Z Z, Z \rangle = b\lambda^2,$$

where we have used the fact that $\langle Z, Z \rangle = 0$, and

$$a = \frac{1}{\lambda^2} \langle \nabla_Z Z, \bar{Z} \rangle.$$  

In conclusion

$$\langle (\nabla^\perp_Z \sigma)(\bar{Z}, Z), H \rangle = \langle \nabla^N_Z (\langle \bar{Z}, Z \rangle H), H \rangle - \langle \nabla_Z Z, \bar{Z} \rangle \langle H, H \rangle$$

$$= \langle \nabla_Z \bar{Z}, Z \rangle \langle H, H \rangle + \langle \nabla_Z Z, \bar{Z} \rangle \langle H, H \rangle$$

$$+ \langle \bar{Z}, Z \rangle \langle \nabla^\perp_Z H, H \rangle - \langle \nabla_Z Z, \bar{Z} \rangle \langle H, H \rangle$$

$$= \langle \bar{Z}, Z \rangle \langle \nabla^\perp_Z H, H \rangle.$$  

q.e.d.

**Lemma 2.2.**

$$\bar{Z}(\langle JZ, H \rangle^2) = 2\langle JZ, H \rangle \langle (JZ)^\perp, \nabla^\perp_Z H \rangle - 2|H|^2 \langle \bar{Z}, JZ \rangle \langle JZ, H \rangle$$
Proof. From the definitions of the Kähler structure and of the Levi-Civita connection we have
\[
\bar{Z}(\langle JZ, H \rangle^2) = 2\langle JZ, H \rangle \{\langle \nabla_{\frac{1}{2}} JZ, H \rangle - \langle (JZ)^\top, AH \bar{Z} \rangle \}
\]
\[
= 2\langle JZ, H \rangle \{\langle (JZ)^\perp, \nabla_{\frac{1}{2}} H \rangle - \langle \sigma((JZ)^\top, \bar{Z}), H \rangle \}
\]
\[
= 2\langle JZ, H \rangle \{\langle (JZ)^\perp, \nabla_{\frac{1}{2}} H \rangle - \langle (JZ)^\top, \bar{Z} \rangle |H|^2 \},
\]
where we have used \( \nabla_{\frac{1}{2}} Z = \sigma(\bar{Z}, Z) = \langle \bar{Z}, Z \rangle H \), as we have seen in the proof of the previous Lemma, and \( (JZ)^\top = \frac{1}{H}(JZ, \bar{Z})Z \), that can be easily checked. q.e.d.

Replacing (2.2), (2.3) and (2.4) into (2.1) we obtain that \( \bar{Z}(Q(Z, Z)) \) vanishes and then we come to the conclusion that

**Proposition 2.3.** If \( \Sigma^2 \) is an immersed surface in a complex space form \( N^a(\rho) \), with parallel mean curvature vector field, then the (2,0)-part of the quadratic form \( Q \), defined on \( \Sigma^2 \) by
\[
Q(X, Y) = 8|H|^2 \langle \sigma(X, Y), H \rangle + 3\rho\langle JX, H \rangle \langle JY, H \rangle,
\]
is holomorphic.

3. Quadratic forms and 2-spheres in 2-dimensional complex space forms

In this section we shall define a new quadratic form on a surface \( \Sigma^2 \) immersed in a complex space form \( N^2(\rho) \), with parallel mean curvature vector field \( H \neq 0 \), and prove that its (2,0)-part is holomorphic. Then, by using these two quadratic forms, we shall classify the 2-spheres with nonzero parallel mean curvature vector.

3.1. Another quadratic form. Let us consider an oriented orthonormal local frame \( \{\tilde{e}_1, \tilde{e}_2\} \) on the surface and denote by \( \theta \) the Kähler angle function defined by \( \langle Je_1, e_2 \rangle = \cos \theta \). The immersion \( x : \Sigma^2 \to N \) is said to be holomorphic if \( \cos \theta = 1 \), anti-holomorphic if \( \cos \theta = -1 \), and totally real if \( \cos \theta = 0 \). In the following we shall assume that \( x \) is neither holomorphic or anti-holomorphic.

Next, we take \( e_3 = -\frac{H}{|H|} \) and let \( e_4 \) be the unique unit normal vector field orthogonal to \( e_3 \) compatible with the orientation of \( \Sigma^2 \) in \( N \). Since \( e_3 \) is parallel in the normal bundle so is \( e_4 \), and, as the Kähler angle is independent of the choice of the orthonormal frame on the surface (see, for example, \([9]\)), we have \( \langle Je_4, e_3 \rangle = \cos \theta \).
Now, we can consider the vector fields
\[ e_1 = \cot \theta e_3 - \frac{1}{\sin \theta} Je_4, \quad e_2 = \frac{1}{\sin \theta} Je_3 + \cot \theta e_4 \]
tangent to the surface and get an orthonormal frame field \( \{e_1, e_2, e_3, e_4\} \) adapted to \( \Sigma^2 \) in \( N \).
We define a quadratic form \( Q' \) on \( \Sigma^2 \) by
\[ Q'(X, Y) = 8i|H|\langle \sigma(X, Y), e_4 \rangle + 3\rho\langle JX, e_4 \rangle\langle JY, e_4 \rangle \]
and again consider the isothermal coordinates \((u, v)\) on \( \Sigma^2 \) and the tangent complex vector fields \( Z \) and \( \bar{Z} \). In the same way as in the case of \( Q \), using the Codazzi equation, the fact that \( H \) and \( e_4 \) are parallel and the expression of the curvature tensor field of \( N \), we get
\[ (3.1) \quad \bar{Z}(\langle \sigma(Z, Z), e_4 \rangle) = \frac{3\rho}{4} \langle \bar{Z}, JZ \rangle \langle JZ, e_4 \rangle. \]
On the other hand, we have
\[
\bar{Z}(\langle JZ, e_4 \rangle^2) = 2\langle JZ, e_4 \rangle \{ \langle \nabla^N_Z JZ, e_4 \rangle + \langle JZ, \nabla^N_Z e_4 \rangle \}
= 2\langle JZ, e_4 \rangle \{ \langle \bar{Z}, Z \rangle \langle JH, e_4 \rangle - \langle (JZ)^\top, A_{e_4} \bar{Z} \rangle \}
= -2|H|\langle JZ, e_4 \rangle \langle \bar{Z}, Z \rangle \langle Je_3, e_4 \rangle - 2\langle JZ, e_4 \rangle \langle \sigma((JZ)^\top, \bar{Z}), e_4 \rangle
= 2|H|\langle JZ, e_4 \rangle \langle \bar{Z}, Z \rangle \cos \theta - 2\langle JZ, e_4 \rangle \langle JZ, \bar{Z} \rangle \langle H, e_4 \rangle
= 2|H|\langle JZ, e_4 \rangle \langle \bar{Z}, Z \rangle \cos \theta,
\]
where we have used
\[ \nabla^N_Z Z = \sigma(\bar{Z}, Z) = \langle \bar{Z}, Z \rangle H, \quad (JZ)^\top = \frac{1}{\lambda^2} \langle JZ, \bar{Z} \rangle Z \]
and \( \langle Je_4, e_3 \rangle = \cos \theta \). We have
\[ \langle \bar{Z}, JZ \rangle = -i \langle \bar{Z}, Z \rangle \langle e_1, Je_2 \rangle = i \langle \bar{Z}, Z \rangle \cos \theta, \]
and, therefore, one obtains
\[ (3.2) \quad \bar{Z}(\langle JZ, e_4 \rangle^2) = -2i|H|\langle \bar{Z}, JZ \rangle \langle JZ, e_4 \rangle. \]
Hence, from (3.1) and (3.2), one obtains \( \bar{Z}(Q'(Z, Z)) = 0 \), which means that the \((2, 0)\)-part of the quadratic form \( Q' \) is holomorphic.
3.2. 2-spheres in 2-dimensional complex space forms. In order to classify the 2-spheres in 2-dimensional complex space forms, we shall need a result of T. Ogata in [16], which we will briefly recall in the following (see also [12] and [15]). Consider a surface $\Sigma^2$ isometrically immersed in a complex space form $N^2(\rho)$, with parallel mean curvature vector field $H \neq 0$. Using the frame field on $N^2(\rho)$ adapted to $\Sigma^2$, defined above, and considering isothermal coordinates $(u, v)$ on the surface, Ogata proved that there exist complex-valued functions $a$ and $c$ on $\Sigma^2$ such that $\theta$, $\lambda$, $a$ and $c$ satisfy

$$
\begin{aligned}
\frac{\partial a}{\partial z} &= \lambda(a + b), \\
\frac{\partial a}{\partial \lambda} &= -|\lambda|^2(\bar{a} - b) \cot \theta, \\
\frac{\partial a}{\partial \bar{z}} &= \lambda(2|a|^2 - 2ab + \frac{3\rho \sin^2 \theta}{8}) \cot \theta, \\
\frac{\partial c}{\partial z} &= 2\lambda(a - b)c \cot \theta, \\
|c|^2 &= |a|^2 + \frac{\rho (3\sin^2 \theta - 2)}{8}
\end{aligned}
$$

(3.3)

where $z = u + iv$ and $|H| = 2b$; and also the converse: if $\rho$ is a real constant, $b$ a positive constant, $\Sigma^2$ a 2-dimensional Riemannian manifold, and there exist some functions $\theta$, $a$ and $c$ on $\Sigma^2$ satisfying (3.3), then there is an isometric immersion of $\Sigma^2$ into $N^2(\rho)$ with parallel mean curvature vector field of length equal to $2b$ and with the Kähler angle $\theta$. The second fundamental form of $\Sigma^2$ in $N$ w.r.t. $\{e_1, e_2, e_3, e_4\}$ is given by

$$
\sigma^3 = \begin{pmatrix}
-2b - \Re(\bar{a} + c) & -\Im(\bar{a} + c) \\
-\Im(\bar{a} + c) & -2b + \Re(\bar{a} + c)
\end{pmatrix}
$$

and

$$
\sigma^4 = \begin{pmatrix}
\Im(\bar{a} - c) & -\Re(\bar{a} - c) \\
-\Re(\bar{a} - c) & -\Im(\bar{a} - c)
\end{pmatrix}
$$

and the Gaussian curvature of $\Sigma^2$ is

$$
K = 4b^2 - 4|c|^2 + \frac{\rho}{2}
$$

(see also [12]).

Assume now that the $(2,0)$-part of $Q$ and the $(2,0)$-part of $Q'$ vanish on the surface $\Sigma^2$. It follows, from the expression of the second fundamental form, that $\bar{c} + a \in \mathbb{R}$, $\bar{c} - a \in \mathbb{R}$ and

$$
32b(\bar{c} + a) - 3\rho \sin^2 \theta = 0, \quad 32b(\bar{c} - a) + 3\rho \sin^2 \theta = 0.
$$

Therefore $c = 0$ and $a = \frac{3\rho \sin^2 \theta}{32b}$ and, from the fifth equation of (3.3), it follows

$$
9\rho^2 \sin^4 \theta + 128b\rho^2(3\sin^2 \theta - 2) = 0.
$$

(3.4)

We have to split the study of this equation in two cases. First, if $\rho = 0$ then the above equation holds and $a = 0$. Next, if $\rho \neq 0$, we get that function $\theta$ is a constant. This, together with the first equation of (3.3), leads to $a = \frac{3\rho \sin^2 \theta}{32b} = -b$. By replacing in equation (3.4) we obtain...
\( \rho = -12b^2 \) and then \( \sin^2 \theta = \frac{8}{9} \). We note that in both cases the Gaussian curvature of \( \Sigma^2 \) is given by \( K = 4b^2 + \frac{\rho}{2} = \text{constant} \) (see \([12]\)). Thus, by using Theorem 1.1 in \([12]\), we have just proved that

**Theorem 3.1.** If the (2,0)-part of \( Q \) and the (2,0)-part of \( Q' \) vanish on a surface \( \Sigma^2 \) isometrically immersed in a complex space form \( N^2(\rho) \), with parallel mean curvature vector field of length \( 2b > 0 \), then either

1) \( N^2(\rho) = CH^2(-12b^2) \) and \( \Sigma^2 \) is the slant surface in \([6, \text{Theorem } 3(2)]\);

2) \( N^2(\rho) = C^2 \) and \( \Sigma^2 \) is a part of a round sphere in a hyperplane in \( C^2 \).

Since the Gaussian curvature \( K \) is nonnegative only in the second case of the Theorem, we have also reobtained the following result of S. Hirakawa in \([12]\).

**Corollary 3.2.** If \( S^2 \) is an isometrically immersed sphere in a 2-dimensional complex space form, with nonzero parallel mean curvature vector, then it is a round sphere in a hyperplane in \( C^2 \).

### 4. A remark on pmc 2-spheres in \( C^n \)

**Proposition 4.1.** Let \( \Sigma^2 \) be an isometrically immersed surface in \( C^n \), with nonzero parallel mean curvature vector. Then the (2,0)-part of the quadratic form \( Q \) vanishes on \( \Sigma^2 \) if and only if the surface is pseudo-umbilical, i.e. \( A_H = |H|^2 I \).

**Proof.** It can be easily seen that if \( \Sigma^2 \) is pseudo-umbilical then the (2,0)-part of \( Q \) vanishes and, therefore, we have to prove only the necessity.

From \( Q(Z, Z) = \frac{(Z, \bar{Z})^2}{2} Q(e_1 - ie_2, e_1 - ie_2) = 0 \) it follows

\[
\langle \sigma(e_1, e_1) - \sigma(e_2, e_2), H \rangle = 0
\]

and

\[
\langle \sigma(e_1, e_2), H \rangle = 0.
\]

But, since \( \langle \sigma(e_1, e_1) + \sigma(e_2, e_2), H \rangle = 2|H|^2 \), we obtain, for each \( i \in \{1, 2\} \),

\[
\langle A_H e_i, e_i \rangle = \langle \sigma(e_i, e_i), H \rangle = |H|^2.
\]

Therefore \( A_H = |H|^2 I \), i.e. \( \Sigma^2 \) is pseudo-umbilical. q.e.d.

S.-T. Yau proved, in \([21, \text{Theorem } 4]\), that if \( \Sigma^2 \) is a surface with parallel mean curvature vector \( H \) in a manifold \( N \) with constant sectional curvature, then either \( \Sigma^2 \) is a minimal surface of an umbilical hypersurface of \( N \) or \( \Sigma^2 \) lies in a 3-dimensional umbilical submanifold of \( N \) with constant mean curvature, as \( H \) is an umbilical direction or the second fundamental form of \( \Sigma^2 \) can be diagonalized simultaneously.
We note that, in the first case, the mean curvature vector field of $\Sigma^2$ in $\mathbb{C}^n$ is orthogonal to the hypersurface.

Applying this result, together with Proposition 4.1, to the 2-spheres in $\mathbb{C}^n$, and using the Gauss equation of a hypersurface in $\mathbb{C}^n$, we get

**Proposition 4.2.** If $S^2$ is an isometrically immersed sphere in $\mathbb{C}^n$, with nonzero parallel mean curvature vector field $H$, then it is a minimal surface of a hypersphere $S^{2n-1}(|H|) \subset \mathbb{C}^n$.

5. Reduction of codimension

Let $x : \Sigma^2 \to N^n(\rho)$, $n \geq 3$, $\rho \neq 0$, be an isometric immersion of a surface $\Sigma^2$ in a complex space form, with parallel mean curvature vector field $H \neq 0$.

**Lemma 5.1.** For any vector $V$ normal to $\Sigma^2$, which is also orthogonal to $JT\Sigma^2$ and to $JH$, we have $[A_H, A_V] = 0$, i.e. $A_H$ commutes with $A_V$.

**Proof.** The statement follows easily, from the Ricci equation

$$\langle R^\perp(X, Y)H, V \rangle = \langle [A_H, A_V]X, Y \rangle + \langle R^N(X, Y)H, V \rangle,$$

since

$$\langle R^N(X, Y)H, V \rangle = \frac{\rho}{4}\{\langle JY, H \rangle \langle JX, V \rangle - \langle JX, H \rangle \langle JY, V \rangle + 2\langle JY, X \rangle \langle JH, V \rangle \} = 0$$

and $R^\perp(X, Y)H = 0$. q.e.d.

**Remark 5.2.** If $n = 3$ and $H \perp JT\Sigma^2$ do not hold simultaneously, then there exists at least one normal vector $V$ as in Lemma 5.1. This can be proved by using the basis of the tangent space $TN$ along $\Sigma^2$ defined in [17], which construction we shall briefly explain in the following. Let us consider a local orthonormal frame $\{e_1, e_2\}$ of vector fields tangent to $\Sigma^2$. Since we have assumed that $H \neq 0$ it follows that $\Sigma^2$ is not holomorphic or antiholomorphic, which means that $\cos^2 \theta = 1$ only at isolated points, and we shall work in the open dense set of points where $\cos^2 \theta \neq 1$, where $\theta$ is the Kähler angle function. The next step is to define two normal vectors by

$$e_3 = -\cot \theta e_1 - \frac{1}{\sin \theta}Je_2 \quad \text{and} \quad e_4 = \frac{1}{\sin \theta}Je_1 - \cot \theta e_2.$$

Thus $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis in $\text{span}\{e_1, e_2, Je_1, Je_2\}$. Moreover, we can set

$$\bar{e}_1 = \cos \left(\frac{\theta}{2}\right)e_1 + \sin \left(\frac{\theta}{2}\right)e_3, \quad \bar{e}_2 = \cos \left(\frac{\theta}{2}\right)e_2 + \sin \left(\frac{\theta}{2}\right)e_4$$
\[ \bar{e}_3 = \sin \left( \frac{\theta}{2} \right) e_1 - \cos \left( \frac{\theta}{2} \right) e_3, \quad \bar{e}_4 = -\sin \left( \frac{\theta}{2} \right) e_2 + \cos \left( \frac{\theta}{2} \right) e_4 \]

and obtain a \( J \)-canonical basis of \( \text{span}\{e_1, e_2, Je_1, Je_2\} \), i.e. \( \mathcal{J} \bar{e}_{2i-1} = \bar{e}_{2i} \). Finally, let us consider a \( J \)-basis of \( TN \) along \( \Sigma^2 \), of the form \( \{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4, \bar{e}_5, \bar{e}_6 = J\bar{e}_5, \ldots, \bar{e}_{2n-1}, \bar{e}_{2n} = J\bar{e}_{2n-1}\} \). Now, three situations can occur:

1) \( H \in (JT\Sigma^2)^\perp \), and then \( e_5 \perp JT\Sigma^2 \) and \( e_5 \perp JH \), where we have denoted by \( (JT\Sigma^2)^\perp = \{(JX)^\perp : X \text{ tangent to } \Sigma^2\} \);

2) \( \bar{e}_5 \perp JT\Sigma^2 \), and then, if we choose \( \bar{e}_1 = H \) and \( \bar{e}_6 = JH \), we have \( \bar{e}_7 \perp JT\Sigma^2 \) and \( \bar{e}_7 \perp JH \) (obviously, this case can occur only if \( n > 3 \));

3) \( H \notin (JT\Sigma^2)^\perp \) and \( H \) is not orthogonal to \( JT\Sigma^2 \). In this case we may consider the vector \( \bar{u} \), the projection of \( H \) on the complementary space of \( (JT\Sigma^2)^\perp \) in \( TN \) (along \( \Sigma^2 \)) and set \( \bar{e}_5 = \frac{\bar{u}}{|\bar{u}|} \). It follows that \( \bar{e}_5 \perp JT\Sigma^2 \) and \( \bar{e}_5 \perp JH \).

If \( n = 3 \) and \( H \perp JT\Sigma^2 \) it is easy to see that

\[ \langle R^N(X, Y)H, e_3 \rangle = \langle R^N(X, Y)H, e_4 \rangle = 0 \]

for any vector fields \( X \) and \( Y \) tangent to \( \Sigma^2 \), and then that \( A_H \) commutes with \( A_{e_3} \) and \( A_{e_4} \).

Conclusively, we get the following

**Corollary 5.3.** Either \( H \) is an umbilical direction or there exists a basis that diagonalizes simultaneously \( A_H \) and \( A_V \), for all normal vectors satisfying \( V \perp JH \), if \( n = 3 \) and \( H \perp JT\Sigma^2 \), or the conditions in Lemma 5.1, otherwise.

**Lemma 5.4.** Assume that \( H \) is nowhere an umbilical direction. Then there exists a parallel subbundle of the normal bundle which contains the image of the second fundamental form \( \sigma \) and has dimension less or equal to 8.

**Proof.** We consider the following subbundle \( L \) of the normal bundle

\[ L = \text{span}\{\text{Im } \sigma \cup (J\text{ Im } \sigma)^\perp \cup (JT\Sigma^2)^\perp\}, \]

and we will show that \( L \) is parallel.

First, we shall prove that, if \( V \) is orthogonal to \( L \), then \( \nabla^N_{\bar{e}_i}V \) is orthogonal to \( JT\Sigma^2 \) and to \( JH \), where \( \{e_1, e_2\} \) is an orthonormal frame w.r.t. which we have \( \langle \sigma(e_1, e_2), V \rangle = \langle \sigma(e_1, e_2), H \rangle = 0 \). Indeed, we get

\begin{align*}
\langle (JH)^\perp, \nabla^N_{\bar{e}_i}V \rangle &= \langle (JH)^\perp, \nabla^N_{\bar{e}_i}V \rangle = -\langle \nabla^N_{\bar{e}_i}(JH)^\perp, V \rangle \\
&= -\langle \nabla^N_{\bar{e}_i}JH, V \rangle + \langle \nabla^N_{\bar{e}_i}(JH)^\top, V \rangle \\
&= \langle JA_{H}e_i, V \rangle + \langle \sigma(e_i, (JH)^\top), V \rangle \\
&= 0
\end{align*}
and
\[ \langle (Je_j)^\perp, \nabla_{e_i}^\perp V \rangle = -\langle \nabla_{e_i}^N (Je_j)^\perp, V \rangle \]
\[ = -\langle \nabla_{e_i}^N Je_j, V \rangle + \langle \nabla_{e_i}^N (Je_j)^\top, V \rangle \]
\[ = -\langle J\nabla_{e_i} e_j, V \rangle - \langle J\sigma(e_i, e_j), V \rangle \]
\[ + \langle \sigma(e_i, (Je_j)^\top), V \rangle \]
\[ = 0. \]

Next, we shall prove that if a normal subbundle \( S \) is orthogonal to \( L \), then so is \( \nabla^\perp S \), i.e.
\[ \langle \sigma(e_i, e_j), \nabla_{e_k}^\perp V \rangle = 0, \quad \langle J\sigma(e_i, e_j), \nabla_{e_k}^\perp V \rangle = 0 \quad \text{and} \quad \langle Je_i, \nabla_{e_k}^\perp V \rangle = 0 \]
for any \( V \in S \) and \( i, j, k \in \{ 1, 2 \} \). Since we have just proved the last property, it remains only to verify the first two of them.

We denote \( A_{ijk} = \langle \nabla_{e_k}^\perp \sigma(e_i, e_j), V \rangle \) and, since \( \sigma \) is symmetric, we have \( A_{ijk} = A_{jik} \). We also obtain \( A_{ijk} = -\langle \sigma(e_i, e_j), \nabla_{e_k}^\perp V \rangle \), since \( V \) is orthogonal to \( L \). We get
\[ \langle (\nabla_{e_k}^\perp \sigma)(e_i, e_j), V \rangle = \langle \nabla_{e_k}^\perp \sigma(e_i, e_j), V \rangle - \langle \sigma(\nabla_{e_k} e_i, e_j), V \rangle \]
\[ - \langle \sigma(e_i, \nabla_{e_k} e_j), V \rangle \]
\[ = \langle \nabla_{e_k}^\perp \sigma(e_i, e_j), V \rangle, \]
and, from the Codazzi equation,
\[ \langle (\nabla_{e_k}^\perp \sigma)(e_i, e_j), V \rangle = \langle \langle (\nabla_{e_k}^\perp \sigma)(e_k, e_j) \rangle + (R^N(e_k, e_i)e_j)^\perp, V \rangle \]
\[ = \langle \langle (\nabla_{e_j} \sigma)(e_k, e_i) \rangle + (R^N(e_k, e_j)e_i)^\perp, V \rangle \]
\[ = \langle (\nabla_{e_i}^\perp \sigma)(e_k, e_j), V \rangle = \langle (\nabla_{e_j}^\perp \sigma)(e_k, e_i), V \rangle. \]

We have just proved that \( A_{ijk} = A_{kji} = A_{ikj} \).

Next, since \( \nabla_{e_k}^\perp V \) is orthogonal to \( JT\Sigma^2 \) and to \( JH \), it follows that the frame field \( \{ e_1, e_2 \} \) diagonalizes \( A_{\nabla_{e_k}^\perp V} \) and we get
\[ A_{ijk} = -\langle \sigma(e_i, e_j), \nabla_{e_k}^\perp V \rangle = -\langle e_i, A_{\nabla_{e_k}^\perp V} e_j \rangle = 0 \]
for any \( i \neq j \). Hence, we have obtained that \( A_{ijk} = 0 \) if two indices are different from each other.

Finally, we only have to prove that \( A_{iii} = 0 \). Indeed, we have
\[ A_{iii} = -\langle \sigma(e_i, e_j), \nabla_{e_i}^\perp V \rangle = -\langle 2H, \nabla_{e_i}^\perp V \rangle + \langle \sigma(e_j, e_j), \nabla_{e_i}^\perp V \rangle \]
\[ = \langle 2\nabla_{e_i} H, V \rangle - A_{jii} = 0. \]
It is easy to see that if $V$ is orthogonal to $L$, then $JV$ is normal and orthogonal to $L$. It follows that

$$\langle (J\sigma(e_i, e_j))^\perp, \nabla^\perp_{e_k} V \rangle = -\langle \nabla^N_{e_k} (J\sigma(e_i, e_j))^\perp, V \rangle$$

$$= -\langle \nabla^N_{e_k} J\sigma(e_i, e_j), V \rangle$$

$$+ \langle \nabla^N_{e_k} (J\sigma(e_i, e_j))^\top, V \rangle$$

$$= \langle JA\sigma(e_i, e_j)e_k, V \rangle - \langle J\nabla^1_{e_k} \sigma(e_i, e_j), V \rangle$$

$$+ \langle \sigma(e_k, (J\sigma(e_i, e_j))^\top), V \rangle$$

$$= \langle \nabla^1_{e_k} \sigma(e_i, e_j), JV \rangle = 0.$$

Thus, we come to the conclusion that the subbundle $L$ is parallel.

q.e.d.

When $H$ is umbilical we can use the quadratic form $Q$ to prove the following

**Lemma 5.5.** Let $\Sigma^2$ be an immersed surface in a complex space form $N^n(\rho)$, $\rho \neq 0$, with nonzero parallel mean curvature vector $H$. If $H$ is an umbilical direction everywhere, then $\Sigma^2$ is a totally real pseudo-umbilical surface of $N$.

**Proof.** Since $H$ is umbilical it follows that $\langle \sigma(Z, Z), H \rangle = 0$, which implies that $\Sigma^2$ is pseudo-umbilical and that $Q(Z, Z) = 3\rho (JZ, H)^2$.

Next, as the $(2, 0)$-part of $Q$ is holomorphic, we have $\bar{Z}(Q(Z, Z)) = 0$, and further

$$0 = \bar{Z}(\langle JZ, H \rangle^2) = -2|H|^2 \langle JZ, H \rangle \langle JZ, \bar{Z} \rangle,$$

as we have seen in a previous section. Hence, $\langle JZ, \bar{Z} \rangle = 0$ or $\langle JZ, H \rangle = 0$. Assume that the set of zeroes of $\langle JZ, \bar{Z} \rangle = 0$ is not the entire $\Sigma^2$. Then, by analyticity, it is a closed set without interior points and its complement is an open dense set in $\Sigma^2$. In this last set we have $\langle JZ, H \rangle = 0$ and then, since $H$ is parallel and $\Sigma^2$ is pseudo-umbilical,

$$0 = \bar{Z}(\langle JZ, H \rangle) = \langle J\nabla^N_{\bar{Z}} Z, H \rangle + \langle JZ, \nabla^N_{\bar{Z}} H \rangle$$

$$= -\langle \bar{Z}, Z \rangle \langle JH, H \rangle - \langle JZ, A_H \bar{Z} \rangle$$

$$= -|H|^2 \langle JZ, \bar{Z} \rangle,$$

which means that $\Sigma^2$ is also totally real.

q.e.d.

**Remark 5.6.** Some kind of a converse result was obtained by B.-Y. Chen and K. Ogiue since they proved in [7] that if a unit normal vector
field to a 2-sphere, immersed in a complex space form as a totally real surface, is parallel and isoperimetric, then it is umbilical.

**Remark 5.7.** In [19] N. Sato proved that, if $M$ is a pseudo-umbilical submanifold of a complex projective space $\mathbb{C}P^n(\rho)$, with nonzero parallel mean curvature vector field, then it is a totally real submanifold. Moreover, the mean curvature vector field $H$ is orthogonal to $JTM$. Therefore, if $M$ is a surface, it follows that the $(2,0)$-part of $Q$ vanishes on $M$.

**Remark 5.8.** In order to show that only the two situations exposed in Lemma 5.4 and Lemma 5.5 can occur, we shall use an argument similar to that in Remark 5 in [4]. Thus, since the map $p \in \Sigma^2 \to (AH - \mu I)(p)$, where $\mu$ is a constant, is analytic, it follows that if $H$ is an umbilical direction, then this either holds on $\Sigma^2$ or only for a closed set without interior points. In this second case $H$ is not an umbilical direction in an open dense set, and then Lemma 5.4 holds on this set. By continuity it holds on $\Sigma^2$.

By using Lemma 5.4 and Lemma 5.5 we can state

**Proposition 5.9.** Either $H$ is everywhere an umbilical direction, and $\Sigma^2$ is a totally real pseudo-umbilical surface of $N$, or $H$ is nowhere an umbilical direction, and there exists a subbundle of the normal bundle that is parallel, contains the image of the second fundamental form and its dimension is less or equal to 8.

Now, from Proposition 5.9 and a result of J. H. Eschenburg and R. Tribuzy [11, Theorem 2], it follows

**Theorem 5.10.** Let $\Sigma^2$ be an isometrically immersed surface in a complex space form $N^n(\rho)$, $n \geq 3$, $\rho \neq 0$, with nonzero parallel mean curvature vector. Then, one of the following holds:

1) $\Sigma^2$ is a totally real pseudo-umbilical surface of $N^n(\rho)$, or
2) $\Sigma^2$ is not pseudo-umbilical and it lies in a complex space form $N^r(\rho)$, where $r \leq 5$.

**Remark 5.11.** The case when $\rho = 0$ is solved by Theorem 4 in [21].

**Remark 5.12.** We have seen (Remark 5.6) that if $\Sigma^2$ is a totally real 2-sphere then it is pseudo-umbilical and therefore the second case of the previous Theorem cannot occur for such surfaces.

6. 2-spheres with constant Kähler angle in complex space forms

This section is devoted to the study of immersed surfaces $\Sigma^2$ in a complex space form $N^n(\rho)$, $n \geq 3$, $\rho \neq 0$, with nonzero non-umbilical parallel mean curvature vector $H$ and constant Kähler angle, on which
the $(2,0)$-part of $Q$ vanishes. We shall compute the Laplacian of the function $|A_H|^2$ for such a surface and show that there are no 2-spheres with these properties.

Let \( \{e_1, e_2\} \) be an orthonormal frame on \( \Sigma^2 \) such that \( H \perp Je_1 \). The fact that the $(2,0)$-part of the quadratic form $Q$ vanishes can be written as

\[
\begin{align*}
8|H|^2\langle \sigma(e_1, e_1) - \sigma(e_2, e_2), H \rangle &= -3\rho(\langle Je_1, H \rangle^2 - \langle Je_2, H \rangle^2) \\
8|H|^2\langle \sigma(e_1, e_2), H \rangle &= 3\rho\langle Je_1, H \rangle\langle Je_2, H \rangle,
\end{align*}
\]

and, from the second equation, we see that $\langle \sigma(e_1, e_2), H \rangle = 0$. It follows that the frame $\{e_1, e_2\}$ diagonalizes simultaneously $A_H$ and $A_V$, for all normal vectors $V$ as in Corollary 5.3, since we are in the second case of Theorem 5.10.

Next, since $\Sigma^2$ is not holomorphic or anti-holomorphic, we have that $\cos \theta \neq \pm 1$ on an open dense set and then we can consider again the normal vectors

\[
e_3 = -\cot \theta e_1 - \frac{1}{\sin \theta} Je_2 \quad \text{and} \quad e_4 = \frac{1}{\sin \theta} Je_1 - \cot \theta e_2
\]

and get an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ in $\text{span}\{e_1, e_2, Je_1, Je_2\}$, where $\theta$ is the Kähler angle on $\Sigma^2$.

It is easy to see that if $H \perp JT\Sigma^2$ it results that the surface is pseudo-umbilical, which is a contradiction.

On the other hand, if we assume that $H \in \text{span}\{e_3, e_4\}$ it follows $H = \pm |H|e_3$, since $Je_1 \perp H$, and then $e_3$ is parallel. Also, since all normal vectors but $e_4$ verify conditions in Corollary 5.3 we have $\sigma(e_1, e_2) \parallel e_4$. By using these facts and the expression of $e_3$ we obtain that $\sigma(e_i, e_j) \in \text{span}\{e_3, e_4\}$ for $i, j \in \{1, 2\}$, and then $\dim L = 2$, where $L$ is the subbundle in Lemma 5.4. Therefore, again by the meaning of Theorem 2 in [11], we get that $\Sigma^2$ lies in a complex space form $N^2(\rho)$, which case was studied earlier in this paper.

In the following, we shall assume that $H \not\in \text{span}\{e_3, e_4\}$, and, as we also know that $H$ is not orthogonal to $JT\Sigma^2$, one obtains that the mean curvature vector can be written as

\[
H = |H|(\cos \beta e_3 + \sin \beta e_5)
\]

where $\beta$ is a real-valued function defined locally on $\Sigma^2$ and $e_5$ is a unit normal vector field such that $e_5 \perp JT\Sigma^2$. We consider the orthonormal frame field $\{e_1, e_2, e_3, e_4, e_5, e_6 = Je_5, \ldots, e_{2n-1}, e_{2n} = Je_{2n-1}\}$ on $N$ and its dual frame $\{\theta_i\}_{i=1}^{2n}$. These are well defined at the points of $\Sigma^2$ where $\sin(2\beta) \neq 0$, which, due to our assumptions, form an open dense set in $\Sigma^2$. The structure equations of the surface are

\[
d\phi = -i\theta_{12} \wedge \phi \quad \text{and} \quad d\theta_{12} = -\frac{i}{2} K \phi \wedge \bar{\phi},
\]
where \( \phi = \theta_1 + i \theta_2 \), the real 1-form \( \theta_{12} \) is the connection form of the Riemannian metric on \( \Sigma^2 \) and \( K \) is the Gaussian curvature.

A result of T. Ogata in [17], together with \( H \perp e_i \) for any \( i \geq 4, i \neq 5 \), implies that, w.r.t. the above orthonormal frame, the components of the second fundamental form are given by

\[
\sigma^3 = \begin{pmatrix}
|H| \cos \beta - \Im(\bar{a} + c) & -\Im(\bar{a} + c) \\
-\Im(\bar{a} + c) & |H| \cos \beta + \Re(\bar{a} + c)
\end{pmatrix}
\]

\[
\sigma^4 = \begin{pmatrix}
\Im(\bar{a} - c) & -\Re(\bar{a} - c) \\
-\Re(\bar{a} - c) & \Im(\bar{a} - c)
\end{pmatrix}
\]

\[
\sigma^5 = \begin{pmatrix}
|H| \sin \beta - \Re(\bar{a}_3 + c_3) & -\Im(\bar{a}_3 + c_3) \\
-\Im(\bar{a}_3 + c_3) & |H| \sin \beta + \Re(\bar{a}_3 + c_3)
\end{pmatrix}
\]

\[
\sigma^6 = \begin{pmatrix}
\Im(\bar{a}_3 - c_3) & -\Re(\bar{a}_3 - c_3) \\
-\Re(\bar{a}_3 - c_3) & \Im(\bar{a}_3 - c_3)
\end{pmatrix}
\]

and

\[
\sigma^{2\alpha-1} = \begin{pmatrix}
-\Re(\bar{a}_\alpha + c_\alpha) & -\Im(\bar{a}_\alpha + c_\alpha) \\
-\Im(\bar{a}_\alpha + c_\alpha) & \Re(\bar{a}_\alpha + c_\alpha)
\end{pmatrix}
\]

\[
\sigma^{2\alpha} = \begin{pmatrix}
\Im(\bar{a}_\alpha - c_\alpha) & -\Re(\bar{a}_\alpha - c_\alpha) \\
-\Re(\bar{a}_\alpha - c_\alpha) & -\Im(\bar{a}_\alpha - c_\alpha)
\end{pmatrix}
\]

where \( a, c, a_\alpha, c_\alpha \), with \( \alpha \in \{3, \ldots, n\} \), are complex-valued functions defined locally on the surface \( \Sigma^2 \). We note that, since \( \sigma(e_1, e_2) \perp H \) and \( \sigma(e_1, e_2) \perp e_5 \), it follows \( \sigma(e_1, e_2) \perp e_3 \). Moreover, since \( \sigma(e_1, e_2) \perp e_i \) for any \( i \in \{1, \ldots, 2n\} \setminus \{4, 6\} \), we have \( \bar{a} + c \in \mathbb{R}, \bar{a}_3 + c_3 \in \mathbb{R} \) and \( a_\alpha = c_\alpha \) for any \( \alpha \geq 4 \).

In the same paper [17], amongst others, the author computed the differential of the Kähler angle function \( \theta \) for a minimal surface. In the same way, this time for our surface, we get

\[
d\theta = \left( a - \frac{|H|}{2} \cos \beta \right) \phi + \left( \bar{a} - \frac{|H|}{2} \cos \beta \right) \bar{\phi}.
\]

The next step is to determine the connection form \( \theta_{12} \) and the differential of the function \( \beta \), by using the property of \( H \) being parallel. We have

\[
\nabla^\perp_{e_i} H = (-\sin \beta e_3 + \cos \beta e_5) d\beta(e_i) + \cos \beta \nabla^\perp_{e_i} e_3 + \sin \beta \nabla^\perp_{e_i} e_5 = 0
\]

(6.2)
for \( i \in \{1, 2\} \), and then
\[
\cos \beta \langle \nabla_{e_i}^N e_3, e_4 \rangle + \sin \beta \langle \nabla_{e_i}^N e_3, e_4 \rangle = 0, \quad i \in \{1, 2\}
\]
from where, by using the expressions of \( e_3 \) in the first term, of \( e_4 \) in the second one and of the second fundamental form of \( \Sigma^2 \), we get
\[
\theta_{12}(e_1) = \cot \theta \Im(i - c) - \frac{\tan \beta}{\sin \theta} \Im(i - c)
\]
\[
\theta_{12}(e_2) = -|H| \frac{\cot \theta}{\cos \beta} - 2 \cot \theta \Re a + \tan \beta \left( \tan \left( \frac{\theta}{2} \right) \Re a - \cot \left( \frac{\theta}{2} \right) \Re c \right)
\]
and finally
\[
\theta_{12} = f_1 + \bar{f}_1 \phi, \quad f_1 = \frac{i}{2} \left( |H| \frac{\cot \theta}{\cos \beta} + 2 \cot \theta a - \frac{\tan \beta}{\sin \theta} (a_3 - c_3) + \cot \theta \tan \beta (a_3 + c_3) \right).
\]
Now, from equation \((6.2)\), we also obtain
\[
d\beta(e_i) + \langle \nabla_{e_i}^N e_3, e_5 \rangle = 0, \quad i \in \{1, 2\}
\]
and then, replacing \( e_3 \) by its expression and also using the expression of the second fundamental form, we get
\[
d\beta(e_1) = |H| \cot \theta \sin \beta + \tan \left( \frac{\theta}{2} \right) \Re a - \cot \left( \frac{\theta}{2} \right) \Re c
\]
and
\[
d\beta(e_2) = \frac{1}{\sin \theta} \Im(i - c).
\]
Hence the differential of \( \beta \) is given by
\[
d\beta = f_2 \phi + \bar{f}_2 \bar{\phi}, \quad f_2 = \frac{1}{2} \left( |H| \cot \theta \sin \beta + \frac{1}{\sin \theta} (a_3 - c_3) - \cot \theta (a_3 + c_3) \right).
\]
We note that if the Kähler angle \( \theta \) is constant, then \( a = \bar{a} = \frac{|H|}{2} \cos \beta \), and, from \((6.3)\), it results
\[
f_1 = \frac{i}{2} \left( |H| \cot \theta \left( \cos \beta + \frac{1}{\cos \beta} \right) - \frac{\tan \beta}{\sin \theta} (a_3 - c_3) + \cot \theta \tan \beta (a_3 + c_3) \right).
\]
Let us now return to the first equation of \((6.1)\), which can be rewritten as
\[
\mu_1 - \mu_2 = \frac{3}{8} \rho \sin^2 \theta \cos^2 \beta,
\]
where \( A_H e_i = \mu_i e_i \). Since \( \mu_1 + \mu_2 = 2|H|^2 \), we have \( \mu_1 = |H|^2 + \frac{3}{16} \rho \sin^2 \theta \cos^2 \beta \) and \( \mu_2 = |H|^2 - \frac{3}{16} \rho \sin^2 \theta \cos^2 \beta \). Thus
\[
|A_H|^2 = \mu_1^2 + \mu_2^2 = 2|H|^4 + \frac{9}{128} \rho^2 \sin^4 \theta \cos^4 \beta.
\]
In the following, we shall assume that the Kähler angle of the surface \( \Sigma^2 \) is constant and then the Laplacian of \( |A_H|^2 \) is given by
\[
\Delta |A_H|^2 = \frac{9}{128} \rho^2 \sin^4 \theta \Delta (\cos^4 \beta).
\]
In order to compute the Laplacian of \(\cos^4 \beta\) we need the following formula, obtained by using (6.4) and (6.5),
\[
d(\cos^4 \beta) = -4 \sin \beta \cos^3 \beta d\beta = -4 \sin \beta \cos^3 \beta (f_2 \phi + \bar{f}_2 \bar{\phi})
\]
\[
= -4 \cos^4 \beta \left\{ (i f_1 + |H| \cot \frac{\theta}{\cos \beta}) \phi + (-i \bar{f}_1 + |H| \cot \frac{\theta}{\cos \beta}) \bar{\phi} \right\}.
\]
We also have
\[
d^c(\cos^4 \beta) = \frac{i}{2} (\Delta(\cos^4 \beta)) \phi \wedge \bar{\phi}
\]
and
\[
d^c(\cos^4 \beta) = -4i \cos^4 \beta \left\{ (-i \bar{f}_1 + |H| \cot \frac{\theta}{\cos \beta}) \bar{\phi} - (i f_1 + |H| \cot \frac{\theta}{\cos \beta}) \phi \right\}.
\]
After a straightforward computation, we get
\[
\Delta(\cos^4 \beta) = 4 \cos^4 \beta \left( K + 4 |f_1|^2 + 12 \left| i f_1 + |H| \cot \frac{\theta}{\cos \beta} \right|^2 \right)
\]
and then
\[
\Delta |A_H|^2 = \frac{9}{32} \rho^2 \sin^4 \theta \cos^4 \beta \left( K + 4 |f_1|^2 + 12 \left| i f_1 + |H| \cot \frac{\theta}{\cos \beta} \right|^2 \right).
\]
Assume now that \(\Sigma^2\) is complete and has nonnegative Gaussian curvature. It follows, from a result of A. Huber in [14], that \(\Sigma^2\) is parabolic. Then, from the above formula, we get that \(|A_H|^2\) is a subharmonic function, and, since \(|A_H|^2\) is bounded (due to (6.6)), it results \(K = 0\), which, together with the Gauss-Bonnet Theorem, leads to the following non-existence result.

**Theorem 6.1.** There are no 2-spheres with nonzero non-umbilical parallel mean curvature vector and constant Kähler angle in a non-flat complex space form.

**References**


