J. DIFFERENTIAL GEOMETRY
92 (2012) 221-261

PROOF OF THE YANO-OBATA CONJECTURE FOR H-PROJECTIVE TRANSFORMATIONS

Vladimir S. Matveev & Stefan Rosemann

Abstract

We prove the classical Yano-Obata conjecture by showing that the connected component of the group of h-projective transformations of a closed, connected Riemannian Kähler manifold consists of isometries unless the manifold is the complex projective space with the standard Fubini-Study metric (up to a constant).

1. Introduction

1.1. Definitions and main result. Let (M, g, J) be a Riemannian Kähler manifold of real dimension $2n \geq 4$. We denote by ∇ the Levi-Civita connection of g. All objects we consider are assumed to be sufficiently smooth.

Definition 1. A regular curve $\gamma : I \to M$ is called *h-planar*, if there exist functions $\alpha, \beta : I \to \mathbb{R}$ such that the ODE

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = \alpha\dot\gamma(t) + \beta J(\dot\gamma(t)) \tag{1}$$

holds for all t, where $\dot\gamma = \frac{d}{dt}\gamma$.

In certain papers, h-planar curves are called complex geodesics. The reason is that if we view the action of J on the tangent space as the multiplication with the imaginary unit i, the property of a curve γ to be h-planar means that $\nabla_{\dot\gamma(t)}\dot\gamma(t)$ is proportional to $\dot\gamma(t)$ with a complex coefficient of the proportionality $\alpha(t) + i \cdot \beta(t)$. Recall that geodesics (in an arbitrary, not necessary arc length parameter t) of a metric can be defined as curves satisfying the equation $\nabla_{\dot\gamma(t)}\dot\gamma(t) = \alpha(t)\dot\gamma(t)$.

Example 1. Consider the complex projective space

$$\mathbb{C}P(n) = \{\text{1-dimensional complex subspaces of } \mathbb{C}^{n+1}\}$$

with the standard complex structure $J = J_{standard}$ and the standard Fubini-Study metric g_{FS}. Then, a regular curve γ is h-planar, if and only if it lies in a projective line.

Indeed, it is well known that every projective line L is a totally geodesic submanifold of real dimension two such that its tangent space is

Received 7/1/2011.

invariant with respect to J. Since L is totally geodesic, for every regular curve $\gamma : I \to L \subseteq \mathbb{C}P(n)$ we have $\nabla_{\dot\gamma(t)}\dot\gamma(t) \in T_{\gamma(t)}L$. Since L is two-dimensional, the vectors $\dot\gamma(t), J(\dot\gamma(t))$ form a basis in $T_{\gamma(t)}L$. Hence, $\nabla_{\dot\gamma(t)}\dot\gamma(t) = \alpha(t)\dot\gamma(t) + \beta(t)J(\dot\gamma(t))$ for certain $\alpha(t), \beta(t)$ as we claimed.

Conversely, given a regular curve σ in $\mathbb{C}P(n)$ that satisfies (1) for some functions α and β, we consider the projective line L such that $\sigma(0) \in L$ and $\dot\sigma(0) \in T_{\sigma(0)}L$. Solving the initial value problem $\gamma(0) = \sigma(0)$ and $\dot\gamma(0) = \dot\sigma(0)$ for the ODE (1) with these functions α and β on $(L, g_{FS|L}, J_{|L})$, we find a curve γ in L. Since L is totally geodesic, this curve satisfies (1) on $(\mathbb{C}P(n), g_{FS}, J)$. The uniqueness of a solution of an ODE implies that σ coincides with γ and, hence, is contained in L.

Definition 2. Let g and $\bar g$ be Riemannian metrics on M such that they are Kähler with respect to the same complex structure J. They are called *h-projectively equivalent*, if every h-planar curve of g is an h-planar curve of $\bar g$ and vice versa.

Remark 1. If two Kähler metrics g and $\bar g$ on (M, J) are *affinely equivalent* (i.e., if their Levi-Civita connections ∇ and $\bar\nabla$ coincide), then they are h-projectively equivalent. Indeed, the equation (1) for the first and for the second metric coincide if $\nabla = \bar\nabla$.

Definition 3. Let (M, g, J) be a Kähler manifold. A diffeomorphism $f : M \to M$ is called an *h-projective transformation*, if f is *holomorphic* (that is, if $f_*(J) = J$), and if f^*g is h-projectively equivalent to g. A vector field v is called h-projective if its local flow Φ^v_t consists of (local) h-projective transformations. Similarly, a diffeomorphism $f : M \to M$ is called an *affine transformation*, if it preserves the Levi-Civita connection of g. A vector field v is *affine*, if its local flow consists of (local) affine transformations. An h-projective transformation (resp. h-projective vector field) is called *essential*, if it is not an affine transformation (resp. affine vector field).

Clearly, the set of all h-projective transformations of (M, g, J) is a group. As it was shown in [**21**] and [**65**], it is a finite-dimensional Lie group (provided that $\dim(M) \geq 4$). We denote it by $\mathrm{HProj}(g, J)$. By Remark 1, holomorphic affine transformations and holomorphic isometries are h-projective transformations, $\mathrm{Iso}(g, J) \subseteq \mathrm{Aff}(g, J) \subseteq \mathrm{HProj}(g, J)$. Obviously, the same is true for the connected components of these groups containing the identity transformation: $\mathrm{Iso}_0(g, J) \subseteq \mathrm{Aff}_0(g, J) \subseteq \mathrm{HProj}_0(g, J)$.

Example 2 (Generalization of the Beltrami construction from [**7**, **34**]). Consider a non-degenerate complex linear transformation $A \in Gl_{n+1}(\mathbb{C})$ and the induced bi-holomorphic diffeomorphism $f_A : \mathbb{C}P(n) \to \mathbb{C}P(n)$. Since the mapping f_A sends projective lines to projective lines, it sends h-planar curves (of the Fubiny-Study metric g_{FS}) to h-planar curves; see Example 1. Then, the pullback $g_A := f_A^* g_{FS}$ is h-projectively equivalent to g_{FS} and f_A is an h-projective transformation. Note that the metric g_A coincides with g_{FS} (i.e., f_A is an isometry), if and only if A is proportional to a unitary matrix.

We see that for $(\mathbb{C}P(n), g_{FS}, J_{standard})$ we have $\mathrm{Iso}_0 \neq \mathrm{HProj}_0$. Our main result is

Theorem 1 (Yano-Obata conjecture)**.** *Let (M, g, J) be a closed, connected Riemannian Kähler manifold of real dimension $2n \geq 4$. Then, $Iso_0(g, J) = HProj_0(g, J)$ unless (M, g, J) is $(\mathbb{C}P(n), c \cdot g_{FS}, J_{standard})$ for some positive constant c.*

Remark 2. The above theorem is not true locally; one can construct counterexamples. We conjecture that Theorem 1 is also true if we replace closedness by completeness, but dealing with this case will require a lot of work. In particular, one will need to generalize the results of [**12**] to the complete metrics.

1.2. History and motivation. H-projective equivalence was introduced by Otsuki and Tashiro in [**48**, **58**]. They have shown that the classical projective equivalence is not interesting in the Kähler situation since only simple examples are possible, and have suggested h-projective equivalence as an interesting object of study instead. This suggestion appeared to be very fruitful and, between the 1960s and the 1970s, the theory of h-projectively equivalent metrics and h-projective transformations was one of the main research topics in Japanese and Soviet (mostly Odessa and Kazan) differential geometry schools. For a collection of results of these times, see for example the survey [**43**] with more than one hundred fifty references. Moreover, two classical books [**53**, **63**] contain chapters on h-projectivity.

New interest in h-projective equivalence is due to its connection with the so-called *hamiltonian 2-forms* defined and investigated in Apostolov et al [**3**, **4**, **5**, **6**]. Actually, a hamiltonian 2-form is essentially the same as an h-projectively equivalent metric $\bar{g}$: it is easy to see that the defining equation [**3**, equation (12)] of a hamiltonian 2-form is algebraically equivalent to (3), which is a reformulation of the condition "$\bar{g}$ is h-projectively equivalent to g" in the language of PDE, see also Remark 7. The motivation of Apostolov et al. to study hamiltonian 2-forms is different from that of Otsuki and Tashiro and is explained in [**3**, **4**]. Roughly speaking, they observed that many interesting problems in Kähler geometry lead to hamiltonian 2-forms and suggested studying them. The motivation is justified in [**5**, **6**], where they indeed constructed interesting and useful examples of Kähler manifolds. There is also a direct connection between h-projectively equivalent metrics and conformal Killing (or twistor) 2-forms studied in [**44**, **51**, **52**]; see Appendix A of [**3**] for details.

In private communications with the authors of [**3**, **4**, **5**, **6**] we were informed that they did not know that the object they considered was studied before under another name. Indeed, they re-derived certain facts that were well known in the theory of h-projectively equivalent metrics. On the other hand, the papers [**3**, **4**, **5**, **6**] contain several solutions of the problems studied in the framework of h-projectively equivalent metrics, for example the local [**3**] and global [**4**] description of h-projectively equivalent metrics—previously, only special cases of such descriptions were known (see for example [**24**]).

Additional interest in h-projectivity is due to the following connection between h-projectively equivalent metrics and integrable geodesic flows: it appears that the existence of $\bar{g}$ that is h-projectively equivalent to g allows one to construct quadratic and linear integrals for the geodesic flow of g. The existence of quadratic integrals has been proven by Topalov in [**60**]. Under certain nondegeneracy assumptions, the quadratic integrals of Topalov are as considered by Kiyohara in [**24**]; the existence of such integrals immediately implies the existence of Killing vector fields. In the general situation, the existence of the Killing vector fields follows from [**3**] and was also known to Topalov according to a private conversation. Altogether, in the most nondegenerate case studied by Kiyohara, we obtain n quadratic and n linear integrals on a $2n$-dimensional manifold; the integrals are in involution and are functionally independent so the geodesic flow of the metric is Liouville-integrable. In the present paper, we will actively use the existence of these integrals.

Note that the attribution of the Yano-Obata conjecture to Yano and Obata is in folklore—we did not find a paper of them where they state this conjecture explicitly. It is clear though that both Obata and Yano (and many other geometers) tried to prove this statement and did this under certain additional assumptions; see below. The conjectures of similar type were standard in the 1960s and the 1970s, in the time when Yano and Obata were active (and, unfortunately, it was also standard in that time not to publish conjectures or open questions). For example, another famous conjecture of that time states that an essential group of conformal transformations of a Riemannian manifold is possible if and only if the manifold is conformally equivalent to the standard sphere or to the Euclidean space; this conjecture is attributed to Lichnerowicz and Obata, though it seems that neither Lichnerowicz nor Obata published it as a conjecture or a question; it was solved in Alekseevskii [**2**], Ferrand [**13**], and Schoen [**50**]. One more example is the so-called projective Lichnerowicz-Obata conjecture stating that a complete Riemannian manifold, such that the connected component of the identity transformation of the projective group contains not only isometries, has constant positive sectional curvature. This conjecture was proven in [**32**, **33**, **35**, **40**]. Though it is also attributed to Lichnerowicz and

Obata in folklore, neither Lichnerowicz nor Obata published this conjecture (however, this particular conjecture was published as "a classical conjecture" in [**18**, **45**, **61**]). In view of these two examples, it would be natural to call the Yano-Obata conjecture the Lichnerowicz-Obata conjecture for h-projective transformations.

Special cases of Theorem 1 were known before. For example, under the additional assumption that the scalar curvature of g is constant, the conjecture was proven in [**20**, **64**]. The case when the Ricci tensor of g vanishes or is covariantly constant was proven earlier in [**21**, **22**, **23**]. Obata [**46**] and Tanno [**57**] proved this conjecture under the assumption that the h-projective vector field lies in the so-called k-nullity space of the curvature tensor. Many local results related to essential h-projective transformations are listed in the survey [**43**]. For example, in [**41**, **49**] it was shown that locally symmetric spaces of non-constant holomorphic sectional curvature do not admit h-projective transformations even locally.

A very important special case of Theorem 1 has been obtained in the recent paper [**12**]. There, the Yano-Obata conjecture was proven under the additional assumption that the degree of mobility (see Definition 4) is ≥ 3. In the present paper, we will actively use the results of [**12**]. Actually, we consider that both papers, [**12**] and the present one, are equally important for the proof of the Yano-Obata conjecture. The methods of [**12**] came from the theory of overdetermined PDE-systems of finite type and are very different from the methods used in this paper.

Acknowledgments. We thank N. Hitchin for noticing that h-projectively equivalent metrics and hamiltonian 2-forms are essentially the same object. This work benefited from discussions with D. Calderbank and C. Tønnesen-Friedman. We thank Deutsche Forschungsgemeinschaft (Research Training Group 1523—Quantum and Gravitational Fields) and FSU Jena for partial financial support.

This work was partially supported by GK 1523 of DFG.

2. The main equation of h-projective geometry and the scheme of the proof of Theorem 1

2.1. Main equation of h-projective geometry. Let g and $\bar g$ be two Riemannian (or pseudo-Riemannian) metrics on $M^{2n\geq 4}$ that are Kähler with respect to the same complex structure J. We consider the induced isomorphisms $g : TM \to T^*M$ and $\bar g^{-1} : T^*M \to TM$. Let us introduce the $(1,1)$-tensor $A(g, \bar g)$ by the formula

$$A(g,\bar g) = \left(\frac{\det \bar g}{\det g}\right)^{\frac{1}{2(n+1)}} \bar g^{-1} \circ g : TM \to TM \tag{2}$$

(in coordinates, the matrix of $\bar g^{-1} \circ g$ is the product of the inverse matrix of $\bar g$ and the matrix of g).

Obviously, $A(g,\bar g)$ is non-degenerate, complex (in the sense that $A\circ J = J\circ A$), and self-adjoint with respect to both metrics. Let ∇ be the Levi-Civita connection of g.

Theorem 2 ([**41**]). *The metric $\bar g$ is h-projectively equivalent to g, if and only if there exists a vector field Λ such that $A = A(g,\bar g)$ given by (2) satisfies*

$$(\nabla_X A)Y = g(Y,X)\Lambda + g(Y,\Lambda)X + g(Y,JX)\bar\Lambda + g(Y,\bar\Lambda)JX, \tag{3}$$

for all $x\in M$ and all $X,Y\in T_xM$, where $\bar\Lambda = J(\Lambda)$.

Remark 3. One may consider the equation (3) as a linear PDE-system on the unknown (A,Λ); the coefficients in this system depend on the metric g. Indeed, if the equation is fulfilled for X,Y being basis vectors, it is fulfilled for all vectors; see also (4) below.

One can also consider (3) as a linear PDE-system on the $(1,1)$-tensor A only, since the components of Λ can be obtained from the components of ∇A by linear algebraic manipulations. Indeed, fix X and calculate the trace of the $(1,1)$-tensors on the left- and right-hand sides of (3) acting on Y. The trace of the right-hand side equals $4g(\Lambda,X)$. Clearly, the trace of $\nabla_X A$ is $\mathrm{trace}(\nabla_X A) = X(\mathrm{trace}\,A)$. Then, $\Lambda = \mathrm{grad}\lambda$, where the function λ is equal to $\frac14\mathrm{trace}\,A$. In what follows, we prefer the last point of view and speak about a self-adjoint, complex solution A of (3), instead of explicitly mentioning the pair (A,Λ).

Remark 4. Let g and $\bar g$ be two h-projectively equivalent Kähler metrics and let $A(g,\bar g)$ be the corresponding solution of (3). It is easy to see that g and $\bar g$ are affinely equivalent, if and only if the corresponding vector field Λ vanishes identically on M.

Remark 5. The original and more standard form of the equation (3) uses index (tensor) notation and reads

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} - \bar\lambda_i J_{jk} - \bar\lambda_j J_{ik}. \tag{4}$$

Here a_{ij}, λ_i, and $\bar\lambda_i$ are related to A,Λ, and $\bar\Lambda$ by the formulas $a_{ij} = g_{ip}A^p_j$, $\lambda_i = g_{ip}\Lambda^p$, and $\bar\lambda_i = -g_{ip}\bar\Lambda^p$.

Remark 6. Note that formula (2) is invertible, if A is non-degenerate: the metric $\bar g$ can be reconstructed from g and A by

$$\bar g = (\det\,A)^{-\frac12} g\circ A^{-1} \tag{5}$$

(we understand g as the mapping $g : TM \to T^*M$; in coordinates, the matrix of $g\circ A^{-1}$ is the product of the matrices of g and A^{-1}).

Evidently, if A is g-self-adjoint and complex, then $\bar g$ given by (5) is symmetric and invariant with respect to the complex structure. It can be checked by direct calculations that if g is Kähler and if A is a non-degenerate g-self-adjoint and complex $(1,1)$-tensor satisfying (3), then $\bar{g}$ is also Kähler with respect to the same complex structure and is h-projectively equivalent to g.

Thus, the set of Kähler metrics that are h-projectively equivalent to g is essentially the same as the set of self-adjoint, complex (in the sense $J \circ A = A \circ J$) solutions of (3) (the only difference is the case when A is degenerate, but since adding const $\cdot$ Id to A does not change the property of A to be a solution, this difference is not important).

Remark 7. As we have already mentioned in Section 1.2, equation (3) is equivalent to the defining equation for a hamiltonian 2-form (see [**3**, equation (12)]). Indeed, for a complex and self-adjoint solution A of (3), the 2-form $\Phi(X, Y) := g(JAX, Y)$ is hamiltonian in the sense of [**3**].

By Remark 3, equation (3) is a system of linear PDEs on the $(1, 1)$-tensor A.

Definition 4. We denote by $\mathrm{Sol}(g)$ the linear space of complex, self-adjoint solutions of (3). The *degree of mobility* $D(g)$ of a Kähler metric g is the dimension of the space $\mathrm{Sol}(g)$.

Remark 8. We always have $1 \leq D(g) < \infty$. Indeed, since Id is always a solution of (3), we have $D(g) \geq 1$. We will not use the fact that $D(g) < \infty$; a proof of this statement can be found in [**12**] or in [**41**].

Let us now show that the degree of mobility is the same for h-projectively equivalent metrics: we construct an explicit isomorphism between $\mathrm{Sol}(g)$ and $\mathrm{Sol}(\bar{g})$.

Lemma 1. *Let g and $\bar{g}$ be two h-projectively equivalent Kähler metrics on (M, J). Then the solution spaces $Sol(g)$ and $Sol(\bar{g})$ are isomorphic. The isomorphism is given by*

$$A_1 \in Sol(g) \longmapsto A_1 \circ A(g, \bar{g})^{-1} \in Sol(\bar{g}),$$

where $A(g, \bar{g})$ is constructed by (2). In particular, $D(g)$ is equal to $D(\bar{g})$.

Proof. Let $A = A(g, \bar{g})$ be the solution of (3) constructed by formula (2). If $A_1 \in \mathrm{Sol}(g)$ is non-degenerate, then $g_1 = (\det A_1)^{-\frac{1}{2}} g \circ A_1^{-1}$ is h-projectively equivalent to g by Remark (6) and, hence, g_1 is h-projectively equivalent to $\bar{g}$. It follows that $A_2 = A(\bar{g}, g_1) \in \mathrm{Sol}(\bar{g})$. On the other hand, using formula (2) we can easily verify that $A_2 = A_1 \circ A^{-1}$. If A_1 is degenerate, we can choose a real number t such that $A_1 + t\mathrm{Id}$ is non-degenerate. As we have already shown, $(A_1 + t\mathrm{Id}) \circ A^{-1} = A_1 \circ A^{-1} + tA^{-1}$ is contained in $\mathrm{Sol}(\bar{g})$. Since $A^{-1} \in \mathrm{Sol}(\bar{g})$, the same is true for $A_1 \circ A^{-1}$. We obtain that the mapping $A_1 \longmapsto A_1 \circ A(g, \bar{g})^{-1}$ is a linear isomorphism between the spaces $\mathrm{Sol}(g)$ and $\mathrm{Sol}(\bar{g})$. q.e.d.

Lemma 2 (Folklore)**.** *Let (M, g, J) be a Kähler manifold and let v be an h-projective vector field. Then the $(1,1)$-tensor*

$$A_v := g^{-1} \circ \mathcal{L}_v g - \frac{\operatorname{trace}(g^{-1} \circ \mathcal{L}_v g)}{2(n+1)} \mathrm{Id} \tag{6}$$

(where $\mathcal{L}_v$ is the Lie derivative with respect to v) is contained in $Sol(g)$.

Proof. Since v is h-projective, $\bar{g}_t = (\Phi^v_t)^* g$ is h-projectively equivalent to g for every t. It follows that for every t the tensor $A_t = A(g, \bar{g}_t)$ is a solution of (3). Since (3) is linear, $A_v := \left(\frac{d}{dt} A_t\right)_{|t=0}$ is also a solution of (3) and it is clearly self-adjoint. Since the flow of v preserves the complex structure, A_v is complex. Using (2), we obtain that A_v is equal to

$$\begin{aligned}
&\frac{d}{dt}\left[\left(\frac{\det \bar{g}_t}{\det g}\right)^{\frac{1}{2(n+1)}} \bar{g}_t^{-1} \circ g\right]\Big|_{t=0} = \frac{1}{2(n+1)}\left(\frac{d}{dt}\frac{\det \bar{g}_t}{\det g}\Big|_{t=0}\right)\mathrm{Id} + \left(\frac{d}{dt}\bar{g}_t^{-1} \circ g\right)\Big|_{t=0} \\
&= \frac{1}{2(n+1)}\left(\frac{d}{dt}\frac{\det \bar{g}_t}{\det g}\right)\Big|_{t=0}\mathrm{Id} - \left(\bar{g}_t^{-1} \circ \left(\frac{d}{dt}\bar{g}_t\right) \circ \bar{g}_t^{-1} \circ g\right)\Big|_{t=0} \\
&= \frac{1}{2(n+1)}\left(\frac{d}{dt}\frac{\det \bar{g}_t}{\det g}\right)\Big|_{t=0}\mathrm{Id} + g^{-1} \circ \mathcal{L}_v g = -\frac{\operatorname{trace}(g^{-1} \circ \mathcal{L}_v g)}{2(n+1)} + g^{-1} \circ \mathcal{L}_v g.
\end{aligned}$$

Thus, $A_v \in Sol(g)$ as we claimed. q.e.d.

2.2. Scheme of the proof of Theorem 1. In the case when the degree of mobility $D(g)$ is ≥ 3, Theorem 1 is an immediate consequence of [**12**, Theorem 1]. Indeed, by [**12**, Theorem 1], if $D(g) \geq 3$ and the manifold is not $(\mathbb{C}P(n), c \cdot g_{FS}, J_{standard})$, every metric $\bar{g}$ that is h-projectively equivalent to g is actually affinely equivalent to g. By [**27**], the connected component of the identity transformation of the group of affine transformations on a closed manifold consists of isometries. This implies that $\mathrm{HProj}_0 = \mathrm{Iso}_0$.

If the degree of mobility is equal to one, every metric $\bar{g}$ that is h-projectively equivalent to g is proportional to g. Then, the group $\mathrm{HProj}_0(g, J)$ acts by homotheties. Since the manifold is closed, it acts by isometries. Again, we obtain $\mathrm{HProj}_0 = \mathrm{Iso}_0$.

Thus, in the proof of Theorem 1, we may (and will) assume that the degree of mobility of the metric g is equal to two.

The proof will be organized as follows. In Sections 3 and 4, we collect and prove basic facts that will be used in the proof of Theorem 1. Certain results of Sections 3 and 4 were known before; we will give precise references. The proofs in Sections 3 and 4 are based on different groups of methods and different ideas. In Section 3, we use the family of quadratic integrals for the geodesic flow of the metric g found by Topalov in [**60**]. With the help of these integrals, we prove that the eigenvalues of A behave quite regularly, in particular we show that they are globally ordered and that the multiplicity of every nonconstant eigenvalue is equal to two. The assumptions of this section are global (we assume that every two points can be connected by a geodesic).

In Section 4, we work locally with equation (3). We show that the vector fields Λ and $\bar{\Lambda}$ appearing in this equation are commuting holomorphic vector fields that are nonzero at almost every point. We also deduce from (3) certain equations on the eigenvectors and eigenvalues of A: in particular we show that the gradient of every eigenvalue is an eigenvector corresponding to this eigenvalue.

Beginning with Section 5, we require the assumption that the degree of mobility is equal to two. Moreover, we assume the existence of an h-projective vector field which is not an affine vector field. The main goal of Section 5 is to show that for every solution A of (3) with corresponding vector field Λ and for almost every point $y \in M$, there exists a neighborhood $U(y)$, a function $\mu : U(y) \to \mathbb{R}$, and a constant $B < 0$ (μ and B can a priori depend on the neighborhood) such that for all points $x \in U(y)$ and all $X, Y \in T_xM$ we have

$$\begin{aligned} (\nabla_X A)Y &= g(Y, X)\Lambda + g(Y, \Lambda)X + g(Y, JX)\bar{\Lambda} + g(Y, \bar{\Lambda})JX \\ \nabla_X \Lambda &= \mu X + BA(X) \\ \nabla_X \mu &= 2Bg(X, \Lambda). \end{aligned} \tag{7}$$

The equations (7) should be viewed as a PDE-system on (A, Λ, μ).

This is the longest and the most complicated part of the proof. First, in Section 5.1, we combine Lemma 2 with the assumption that the degree of mobility is two, to obtain the formulas (15, 20) that describe the evolution of A along the flow of the h-projective vector field. With the help of the results of Section 4, we deduce (in the proof of Lemma 8) an ODE for the eigenvalues of A along the trajectories of the h-projective vector field. This ODE can be solved; combining the solutions with the global ordering of the eigenvalues from Section 3, we obtain that A has at most three eigenvalues at every point; moreover, precisely one eigenvalue of A considered as a function on the manifold is not constant (unless our h-projective vector field is an affine vector field). As a consequence, in view of the results of Section 4, the vectors Λ and $\bar{\Lambda}$ are eigenvectors of A.

The equation (20) depends on two parameters. We prove that under the assumption that the manifold is closed, the parameters satisfy some algebraic equation (given in Lemma 15) so that in fact the equation (20) depends on one parameter only. In order to do it, we work with the distribution span $\{\Lambda, \bar{\Lambda}\}$ and show that its integral manifolds are totally geodesic. Equations (6, 20) contain enough information to calculate the restriction of the metric to this distribution; the metric depends on the same parameters as equation (20). We calculate the sectional curvature of this metric and see that it is unbounded (which cannot happen on a closed manifold), unless the parameters satisfy a certain algebraic equation.

In Section 5.3, we show that the algebraic equation mentioned above implies the local existence of B and μ such that (7) is fulfilled. This proves that the system (7) is satisfied in a neighborhood of almost every point of M, for certain B, μ that can a priori depend on the neighborhood.

We complete the proof of Theorem 1 in Section 6. First we recall that [**12**, Section 2.5] implies that the constant B is the same in all neighborhoods, implying that the system (7) is fulfilled on the whole manifold (for a certain globally defined constant B and a certain globally defined function μ).

Once we have shown that the system (7) holds globally, Theorem 1 is an immediate consequence of [**57**, Theorem 10.1].

2.3. Relation with projective equivalence. Two metrics g and $\bar{g}$ on the same manifold are *projectively equivalent*, if every geodesic of g, after an appropriate reparametrization, is a geodesic of $\bar{g}$. As we already mentioned above, the notion "h-projective equivalence" appeared as an attempt to adapt the notion "projective equivalence" to Kähler metrics. It is therefore not a surprise that certain methods from the theory of projectively equivalent metrics could be adapted to the h-projective situation. For example, the above mentioned papers [**1**, **20**, **64**] are actually h-projective analogs of the papers [**19**, **61**] (dealing with projective transformations); see also [**16**, **55**]. Moreover, [**58**, **65**] are h-projective analogs of [**23**, **56**], and many results listed in the survey [**43**] are h-projective analogs of those listed in [**42**].

The Yano-Obata conjecture is also an h-projective analog of the so-called projective Lichnerowicz-Obata conjecture mentioned above and recently proved in [**35**, **40**]; see also [**32**, **33**]. The general scheme of our proof of the Yano-Obata conjecture is similar to the scheme of the proof of the projective Lichnerowicz-Obata conjecture in [**40**]. More precisely, as in the projective case, the cases $D(g) = 2$ and $D(g) \geq 3$ were done using completely different groups of methods. As we mentioned above, the proof of the Yano-Obata conjecture for the metrics with degree of mobility ≥ 3 was done in [**12**]. This proof is based on other ideas than the corresponding part in the proof of the projective Lichnerowicz-Obata conjecture in [**37**, **40**].

Concerning the proof under the assumption that the degree of mobility is two, the first part of the proof (Sections 3, 5.1) is based on the same ideas as in the projective case. More precisely, the way to use integrals for the geodesic flow to show the regular behavior of the eigenvalues of A and their global ordering is very close to that of [**8**, **31**, **36**, **59**]. The way to obtain the equation (20) that describes the evolution of A along the orbits of the h-projective vector field is close to that used in [**9**] and is motivated by [**32**, **33**, **35**, **40**, **39**].

3. Quadratic integrals and the global ordering of the eigenvalues of solutions of (3)

3.1. Quadratic integrals for the geodesic flow of g. Let A be a self-adjoint, complex solution of (3). By [**60**] (see also the end of Appendix A of [**3**]), for every $t \in \mathbb{R}$, the function

(8) $$F_t : TM \to \mathbb{R}\,, \quad F_t(\zeta) := \sqrt{\det(A - t\mathrm{Id})}\, g((A - t\mathrm{Id})^{-1}\zeta, \zeta)$$

is an integral for the geodesic flow of g.

Remark 9. It is easy to prove (see formula (10) below) that the integrals are defined for all $t \in \mathbb{R}$ (i.e., even if $A - t\mathrm{Id}$ is degenerate). Actually, the family F_t is a polynomial of degree $n-1$ in t whose coefficients are certain functions on TM; these functions are automatically integrals.

Remark 10. The integrals are visually close to the integrals for the geodesic flows of projectively equivalent metrics constructed in [**28**].

Later it will be useful to consider the t-derivatives of the integrals defined above:

Lemma 3. *Let $\{F_t\}$ be the family of integrals given in* (8). *Then, for each integer $m \geq 0$ and for each number $t_0 \in \mathbb{R}$,*

(9) $$\left(\tfrac{d^m}{dt^m} F_t\right)_{|t=t_0}$$

is also an integral for the geodesic flow of g.

Proof. As we already mentioned above in Remark 9,

$$F_t(\zeta) = s_{n-1}(\zeta)t^{n-1} + \cdots + s_1(\zeta)t + s_0(\zeta)$$

for certain integrals $s_0, \ldots, s_{n-1} : TM \to \mathbb{R}$. Then, the t-derivatives (9) are also polynomials in t whose coefficients are integrals, i.e., the t-derivatives (9) are also integrals for every fixed t_0. q.e.d.

3.2. Global ordering of the eigenvalues of solutions of (3). During the whole subsection let A be an element of $\mathrm{Sol}(g)$; that is, A is a complex self-adjoint $(1,1)$-tensor such that it is a solution of (3). Since it is self-adjoint with respect to (a positively-definite metric) g, the eigenvalues of $A_{|x} := A_{|T_xM}$ are real.

Definition 5. We denote by $m(y)$ the number of different eigenvalues of A at the point y. Since $A \circ J = J \circ A$, each eigenvalue has even multiplicity ≥ 2. Hence, $m(y) \leq n$ for all $y \in M$. We say that $x \in M$ is a *typical point* for A if $m(x) = \max_{y \in M}\{m(y)\}$. The set of all typical points of A will be denoted by $M^0 \subseteq M$.

Let us denote by $\mu_1(x) \leq \cdots \leq \mu_n(x)$ the eigenvalues of A counted with half of their multiplicities. The functions $\mu_1, \ldots, \mu_n$ are real since A is self-adjoint and they are at least continuous. It follows that $M^0 \subseteq M$ is an open subset. The next theorem shows that M^0 is dense in M.

Theorem 3. *Let (M, g, J) be a Kähler manifold of real dimension $2n \geq 4$. Suppose every two points of M can be connected by a geodesic. Then, for every $A \in \mathrm{Sol}(g)$ and every $i = 1, \ldots, n-1$, the following statements hold:*

1) *$\mu_i(x) \leq \mu_{i+1}(y)$ for all $x, y \in M$.*
2) *If $\mu_i(x) < \mu_{i+1}(x)$ at least at one point, then the set of all points y such that $\mu_i(y) < \mu_{i+1}(y)$ is everywhere dense in M.*

Remark 11. If the Kähler manifold is compact, the global description of hamiltonian 2-forms [**4**, Theorem 5] implies the global ordering of the eigenvalues (the first part of Theorem 3), and this is sufficient for our further goals. However, we give an alternative proof which works under less general assumptions, and is based on other ideas.

Proof. (1): Let $x \in M$ be an arbitrary point. At T_xM, we choose an orthonormal frame $\{U_i, JU_i\}_{i=1,\ldots,n}$ of eigenvectors (we assume $AU_i = \mu_i U_i$ and $g(U_i, U_i) = 1$ for all $i = 1, ..., n$). For $X \in T_xM$, we denote its components in the frame $\{U_i, JU_i\}_{i=1,\ldots,n}$ by $X_j := g(X, U_j)$ and $\bar{X}_j := g(X, JU_j)$. By direct calculations, we see that $F_t(X)$ given by (8) reads

$$
\begin{aligned}
F_t(X) = \quad & \sum_{i=1}^{n} \left[(X_i^2 + \bar{X}_i^2) \prod_{j=1; j\neq i}^{n} (\mu_j - \mu_i) \right] \\
= \quad & (\mu_2 - t) \cdot \cdots \cdot (\mu_n - t)(X_1^2 \\
& + \bar{X}_1^2) + \cdots + (\mu_1 - t) \cdot \cdots \cdot (\mu_{n-1} - t)(X_n^2 + \bar{X}_n^2).
\end{aligned} \tag{10}
$$

Obviously, $F_t(X)$ is a polynomial in t of degree $n-1$ whose leading coefficient is $(-1)^{n-1} g(X, X)$.

For every point $x \in M$ and every $X \in T_xM$ such that $X \neq 0$, let us consider the roots

$$t_1(x, X), \ldots, t_{n-1}(x, X) : T_xM \to \mathbb{R}$$

of the polynomial counted with their multiplicities. From the arguments below it will be clear that they are real. We assume that at every (x, X) we have $t_1(x, X) \leq \cdots \leq t_{n-1}(x, X)$. Since for every fixed t the polynomial F_t is an integral, the roots t_i are also integrals.

Let us show that for every $i = 1, \ldots, n-1$ the inequality

$$\mu_i(x) \leq t_i(x, X) \leq \mu_{i+1}(x) \tag{11}$$

holds.

We consider first the case when all eigenvalues are different from each other, i.e., $\mu_1(x) < \cdots < \mu_n(x)$, and all components X_i are different from zero. Substituting $t = \mu_i$ and $t = \mu_{i+1}$ into (10), we obtain

$$
\begin{aligned}
F_{\mu_i}(X) = & (\mu_1 - \mu_i) \cdot \cdots \cdot (\mu_{i-1} - \mu_i) \\
& \cdot (\mu_{i+1} - \mu_i) \cdot \cdots \cdot (\mu_n - \mu_i)(X_i^2 + \bar{X}_i^2),
\end{aligned}
$$

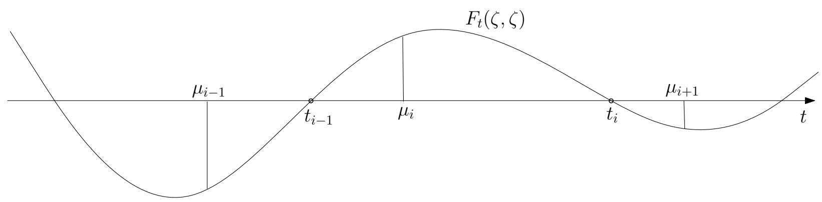

Figure 1. If $\mu_1 < \mu_2 < \cdots < \mu_n$ and all $X_i \neq 0$, the values of $F_t(X)$ have different signs at $t = \mu_i$ and $t = \mu_{i+1}$ implying the existence of a root t_i such that $\mu_i < t_i < \mu_{i+1}$.

$$\begin{aligned} F_{\mu_{i+1}}(X) =&(\mu_1 - \mu_{i+1}) \cdot \dots \cdot (\mu_i - \mu_{i+1}) \\ &\cdot (\mu_{i+2} - \mu_{i+1}) \cdot \dots \cdot (\mu_n - \mu_{i+1})(X_{i+1}^2 + \bar{X}_{i+1}^2). \end{aligned}$$

We see that $F_{\mu_i}(X)$ and $F_{\mu_{i+1}}(X)$ have different signs; see Figure 1. Then, every open interval (μ_i, μ_{i+1}) contains a root of the polynomial $F_t(X)$. Thus, all $n-1$ roots of the polynomial are real, and the inequality (11) holds as we claimed.

In the general case, since $F_t(X)$ depends continuously on the vector X and on the eigenvalues $\mu_1(x) \leq \cdots \leq \mu_n(x)$ of $A_{|x}$, its zeros also depend continuously on X and μ_i. It follows that for every x and for all $X \in T_xM$ we have that all zeros are real and that (11) holds.

Let us now show that for any two points x, y we have $\mu_i(x) \leq \mu_{i+1}(y)$.

We consider a geodesic $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Since F_t are integrals, we have $F_t(\dot\gamma(0)) = F_t(\dot\gamma(1))$ implying

$$t_i(\gamma(0), \dot\gamma(0)) = t_i(\gamma(1), \dot\gamma(1)). \tag{12}$$

Combining (11) and (12), we obtain

$$\mu_i(x) \overset{(11)}{\leq} t_i(x, \dot\gamma(0)) \overset{(12)}{=} t_i(y, \dot\gamma(1)) \overset{(11)}{\leq} \mu_{i+1}(y)$$

which proves the first part of Theorem 3.

(2): Assume $\mu_i(y) = \mu_{i+1}(y)$ for all points y in some nonempty open subset $U \subseteq M$. We need to prove that for every $x \in M$ we have $\mu_i(x) = \mu_{i+1}(x)$.

First let us show that $\mu := \mu_i = \mu_{i+1}$ is a constant on U. Indeed, suppose that $\mu_i(y_1) \leq \mu_i(y_2)$ for some points $y_1, y_2 \in U$. From the first part of Theorem 3 and from the assumption $\mu_i = \mu_{i+1}$ we obtain

$$\mu_i(y_1) \leq \mu_i(y_2) \leq \mu_{i+1}(y_1) = \mu_i(y_1),$$

implying $\mu_i(y_1) = \mu_i(y_2)$ for all $y_1, y_2 \in U$ as we claimed.

Now take an arbitrary point $x \in M$ and consider the set of all initial velocities of geodesics connecting x with points of U (we assume $\gamma(0) = x$ and $\gamma(1) \in U$); see figure 2. For every such geodesic γ we have

$$\mu = \mu_i(\gamma(1)) \leq t_i(\gamma(1), \dot\gamma(1)) \leq \mu_{i+1}(\gamma(1)) = \mu.$$

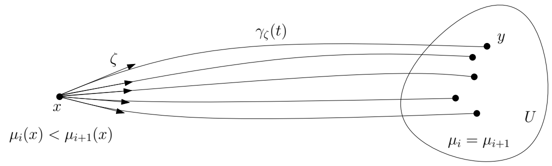

Figure 2. The initial velocity vectors X at x of the geodesics connecting the point x with points from U form a subset of nonzero measure and are contained in U_μ.

Thus, $t_i(\gamma(1), \dot\gamma(1)) = \mu$. Since the value $t_i(\gamma(t), \dot\gamma(t))$ is the same for all points of the geodesic, we obtain that $t_i(\gamma(0), \dot\gamma(0)) = \mu$. Then, the set

$$U_\mu := \{X \in T_xM : \ t_i(x, X) = \mu\}$$

has nonzero measure. Since U_μ is contained in the set

$$\{X \in T_xM : \ F_\mu(X) = 0\}$$

which is a quadric in T_xM, the latter must coincide with the whole T_xM. In view of formula (10), this implies that at least two eigenvalues of A at x should be equal to μ. Suppose the multiplicity of the eigenvalue μ is equal to $2k$. This implies that $\mu_{r+1}(x) = \cdots = \mu_{r+k}(x) = \mu$, $\mu_r(x) \neq \mu$, and $\mu_{r+k+1}(x) \neq \mu$. If $i \in \{r+1, \ldots, r+k-1\}$, we are done. We assume that $i \notin \{r+1, \ldots, r+k-1\}$ and find a contradiction.

In order to do it, we consider the function

$$\tilde F : \mathbb{R} \times TM \to \mathbb{R}\,, \quad \tilde F_t(\zeta) := F_t(\zeta)/(t-\mu)^{k-1}.$$

At the point x, each term of the sum (10) contains $(t-\mu)^{k-1}$, implying that $\tilde F_t(\zeta)$ is a polynomial in t (and is a quadratic function in ζ). Since for every fixed t_0 the function F_{t_0} is an integral, the function $\tilde F_{t_0}$ is also an integral. Let us show that for every geodesic γ with $\gamma(0) = x$ and $\gamma(1) \in U$ we have that $\left(\tilde F_t(\dot\gamma(0))\right)_{|t=\mu} = 0$. Indeed, we already have shown that $t_i(x, \dot\gamma(0)) = \mu$. By similar arguments, in view of inequality (11), we obtain $t_{r+1}(x, \dot\gamma(0)) = \cdots = t_{r+k-1}(x, \dot\gamma(0)) = \mu$. Then, $t = \mu$ is a root of multiplicity k of $F_t(\dot\gamma(0))$ and, therefore, a root of multiplicity $k-(k-1) = 1$ of $\tilde F_t(\dot\gamma(0)) = F_t(\dot\gamma(0))/(t-\mu)^{k-1}$. Finally, $\tilde F_\mu(\dot\gamma(0)) = 0$.

Now, in view of the formula (10), the set $\{\zeta \in T_xM : \ \tilde F_\mu(\zeta) = 0\}$ is a nontrivial (since $\mu_r \neq \mu \neq \mu_{r+k+1}$) quadric in T_xM, which contradicts the assumption that it contains a subset U_μ of nonzero measure. Finally, we have $i, i+1 \in \{r+1, \ldots, r+k\}$, implying $\mu_i(x) = \mu_{i+1}(x) = \mu$. q.e.d.

From Theorem 3, we immediately obtain the following two corollaries:

Corollary 1. *Let (M, g, J) be a complete, connected Riemannian Kähler manifold. Then, for every $A \in$ Sol(g), the set M^0 of typical points of A is open and dense in M.*

Corollary 2 ([**3**]). *Let (M, g, J) be a complete, connected Riemannian Kähler manifold and assume $A \in Sol(g)$. Then at almost every point the multiplicity of a non-constant eigenvalue ρ of A is equal to two.*

4. Basic properties of solutions of (3)

In this section, we collect some basic technical properties of solutions of (3). Most of the results are known in folklore; we will give precise references wherever possible.

4.1. The vector fields Λ and $\bar{\Lambda}$ are holomorphic.

Lemma 4 (Folklore; see equation (13) and the sentence below in [**41**], Proposition 3 of [**3**], and Corollary 3 of [**12**]). *Let (M, g, J) be a Kähler manifold of real dimension $2n \geq 4$ and let be $A \in Sol(g)$. Let Λ be the corresponding vector field defined by* (3). *Then $\bar{\Lambda}$ is a Killing vector field for the Kähler metric g, i.e.,*

$$g(\nabla_X \bar{\Lambda}, Y) + g(X, \nabla_Y \bar{\Lambda}) = 0$$

for all $X, Y \in TM$.

It is a well-known fact that if a Killing vector field K vanishes on some open nonempty subset U of the connected manifold M, then K vanishes on the whole M. From this, we conclude

Corollary 3. *Let (M, g, J) be a connected Kähler manifold of real dimension $2n \geq 4$ and let v be an h-projective vector field.*

1) *If v restricted to some open nonempty subset $U \subseteq M$ is a Killing vector field, then v is a Killing vector field on the whole M.*
2) *If v is not identically zero, the set of points $M_{v \neq 0} := \{x \in M : v(x) \neq 0\}$ is open and dense in M.*

Proof. (1) Suppose the restriction of v to an open subset U is a Killing vector field. Then $\bar{g}_t = (\Phi_t^v)^* g$ restricted to $U' \subset U$ is equal to $g_{|U'}$ for sufficiently small t. Hence, $A_{t|U'} = A(g, \bar{g}_t)_{|U'} = \mathrm{Id}$. The corresponding vector field $\Lambda_t = \frac{1}{4}\mathrm{grad}\,\mathrm{trace}\, A_t$ vanishes (on U'), implying $\bar{\Lambda}_t$ vanishes (on U') as well. Since $\bar{\Lambda}_t$ is a Killing vector field, $\bar{\Lambda}_t$ vanishes on the whole manifold, implying Λ_t is equal to zero on the whole M. Then, by (3), the $(1, 1)$-tensor $A_t - \mathrm{Id}$ is covariantly constant on the whole M. Since this tensor vanishes on U', it vanishes on the whole manifold. Finally, $A_t = \mathrm{Id}$ on M, implying that v is a Killing vector field on M. This proves part (1) of Corollary 3.

(2) Suppose v vanishes on some open subset $U \subseteq M$. To prove (2), we have to show that $v = 0$ everywhere on M. From part (1) we can conclude that v is a Killing vector field on M. Since v vanishes on the (open, nonempty) subset U, it vanishes on the whole M. q.e.d.

The next lemma is a standard result in Kähler geometry (we give a proof for self-containedness). Combined with Lemma 4, it shows that $\bar{\Lambda}$ is a holomorphic vector field.

Lemma 5. *Let (M, g, J) be a Kähler manifold. Let K be a vector field of the form $K = J\mathrm{grad}\, f$ for some function f. Then K is a Killing vector field for g, if and only if K is holomorphic.*

Proof. The equation $K = J\mathrm{grad}\, f$ just means that, with respect to the Kähler 2-form $\omega = g(., J.)$, K is a hamiltonian vector field corresponding to the hamiltonian function f. Then, K is automatically symplectic, i.e.,

$$0 = \mathcal{L}_K\omega = (\mathcal{L}_K g)(., J.) + g(., \mathcal{L}_K J.).$$

Since g and J are non-degenerate, it follows that $\mathcal{L}_K J = 0$ if and only if $\mathcal{L}_K g = 0$. q.e.d.

Corollary 4 ([**3**])**.** *Let (M, g, J) be a Kähler manifold of real dimension $2n \geq 4$. Then, for every $A \in Sol(g)$, the vector fields Λ and $\bar{\Lambda}$ from (3) are holomorphic and commuting, i.e.,*

$$\mathcal{L}_\Lambda J = \mathcal{L}_{\bar{\Lambda}} J = 0 \text{ and } [\Lambda, \bar{\Lambda}] = 0.$$

Proof. By Remark 3, Λ is the gradient of a function. Since $\bar{\Lambda} = J\Lambda$ is a Killing vector field, Lemma 5 implies that $\bar{\Lambda}$ is holomorphic. Since the multiplication with the complex structure sends holomorphic vector fields to holomorphic vector fields, Λ is holomorphic as well. By direct calculations, $[\Lambda, \bar{\Lambda}] = (\mathcal{L}_\Lambda J)\Lambda + J[\Lambda, \Lambda] = 0$. q.e.d.

4.2. Covariant derivatives of the eigenvectors of *A*. Let A be a complex, self-adjoint solution of (3). On M^0, the eigenspace distributions $E_A(\mu_i)$ are well-defined and differentiable. In general, they are not integrable (except for the trivial case when the metrics are affinely equivalent). The next proposition explains the behavior of these distributions; it is essentially equivalent to [**3**, Proposition 14 and equation (62)].

Proposition 1. *Let (M, g, J) be a Riemannian Kähler manifold and assume $A \in Sol(g)$. Let U be a smooth field of eigenvectors of A defined on some open subset of M^0. Let ρ be the corresponding eigenvalue. Then, for an arbitrary vector $X \in TM$, we have*

(13)
$$(A - \rho\mathrm{Id})\nabla_X U = X(\rho)U - g(U, X)\Lambda - g(U, \Lambda)X - g(U, JX)\bar{\Lambda} - g(U, \bar{\Lambda})JX.$$

Moreover, if V is an eigenvector of A corresponding to an eigenvalue $\tau \neq \rho$, then $V(\rho) = 0$ and, consequently, $\mathrm{grad}\, \rho \in E_A(\rho)$.

Proof. Using (3), we obtain

$$(\nabla_X A)U = g(U, X)\Lambda + g(U, \Lambda)X + g(U, JX)\bar{\Lambda} + g(U, \bar{\Lambda})JX$$

for arbitrary $X \in TM$. On the other hand, since $U \in E_A(\rho)$, we calculate

$$\nabla_X(AU) = \nabla_X(\rho U) = X(\rho)U + \rho\nabla_X U.$$

Inserting the last two equations in $\nabla_X(AU) = (\nabla_X A)U + A(\nabla_X U)$, we obtain (13).

Now let τ be another eigenvalue of A, such that $\rho \neq \tau$, and let $V \in E_A(\tau)$. Replacing V with X in (13) and using that $E_A(\rho) \perp E_A(\tau)$, we obtain

$$(A - \rho\mathrm{Id})\nabla_V U = V(\rho)U - g(U,\Lambda)V - g(U,\bar{\Lambda})JV.$$

Since the left-hand side of the equation above is orthogonal to $E_A(\rho)$, we immediately obtain $0 = V(\rho) = g(V, \operatorname{grad}\rho)$. Thus, $\operatorname{grad}\rho$ is orthogonal to all eigenvectors corresponding to eigenvalues different from ρ, implying it lies in $E_A(\rho)$ as we claimed. q.e.d.

5. Kähler manifolds of degree of mobility $D(g) = 2$ admitting essential h-projective vector fields

For closed manifolds, the condition $\mathrm{HProj}_0 \neq \mathrm{Iso}_0$ is equivalent to the existence of an essential (i.e., not affine) h-projective vector field. The goal of this section is to prove the following

Theorem 4. *Let (M, g, J) be a closed, connected Riemannian Kähler manifold of real dimension $2n \geq 4$ and of degree of mobility $D(g) = 2$ admitting an essential h-projective vector field. Let $A \in Sol(g)$ with corresponding vector field Λ.*

Then, for almost every point $y \in M$, there exists a neighborhood $U(y)$, a constant $B < 0$, and a smooth function $\mu : U(y) \to \mathbb{R}$ such that the system

$$\begin{aligned} (\nabla_X A)Y &= g(Y,X)\Lambda + g(Y,\Lambda)X + g(Y,JX)\bar{\Lambda} + g(Y,\bar{\Lambda})JX \\ \nabla_X\Lambda &= \mu X + BA(X) \\ \nabla_X\mu &= 2Bg(X,\Lambda) \end{aligned} \tag{14}$$

is satisfied for all x in $U(y)$ and all $X, Y \in T_xM$.

One should understand (14) as a system of PDEs on the components of (A, Λ, μ). Actually, in the system (14), the first equation is the equation (3) and is fulfilled by the definition of $\mathrm{Sol}(g)$, so our goal is to prove the local existence of B and μ such that the second and the third equations of (14) are fulfilled.

Remark 12. If $D(g) \geq 3$, the conclusion of this theorem is still true if we allow all, i.e., not necessary negative, values of B. In this case we even do not need the "closedness" assumption (i.e., the statement is local) and the existence of an h-projective vector field see [**12**]. Theorem

4 essentially requires the existence of an h-projective vector field and is not true locally.

5.1. The tensor A has at most two constant and precisely one non-constant eigenvalue. First let us prove

Lemma 6. *Let (M, g, J) be a Kähler manifold of real dimension $2n \geq 4$ and of degree of mobility $D(g) = 2$. Suppose $f : M \to M$ is an h-projective transformation for g and let A be an element of Sol(g). Then f maps the set M^0 of typical points of A onto M^0.*

Proof. Let x be a point of M^0. Since the characteristic polynomial of $(f^*A)_{|x}$ is the same as for $A_{|f(x)}$, we have to show that the number of different eigenvalues of $(f^*A)_{|x}$ and $A_{|x}$ coincide. If A is proportional to the identity on TM, the assertion follows immediately. Let us therefore assume that $\{A, \mathrm{Id}\}$ is a basis for $\mathrm{Sol}(g)$. We can find neighborhoods U_x and $f(U_x)$ of x and $f(x)$ respectively, such that A is non-degenerate in these neighborhoods (otherwise we add $t \cdot \mathrm{Id}$ to A with a sufficiently large $t \in \mathbb{R}_+$). By (5), $\bar{g} = (\det A)^{-\frac{1}{2}} g \circ A^{-1}, g, f^*g$, and $f^*\bar{g}$ are h-projectively equivalent to each other in U_x. By direct calculation, we see that $f^*A = f^*A(g, \bar{g}) = A(f^*g, f^*\bar{g})$. Hence, f^*A is contained in $\mathrm{Sol}(f^*g)$. First suppose that $A(g, f^*g)$ is proportional to the identity. We obtain that

$$f^*A = \alpha A + \beta \mathrm{Id}$$

for some constants α, β. Since $\alpha \neq 0$ (if A is non-proportional to Id, the same holds for f^*A), the number of different eigenvalues of $(f^*A)_{|x}$ is the same as for $A_{|x}$. It follows that $f(x) \in M^0$. Now suppose that $A(g, f^*g)$ is non-proportional to Id. Then the numbers of different eigenvalues for $A_{|x}$ and $A(g, f^*g)_{|x}$ coincide. By Lemma 1, $D(f^*g) = 2$ and $\{A(g, f^*g)^{-1}, \mathrm{Id}\}$ is a basis for $\mathrm{Sol}(f^*g)$. We obtain that

$$f^*A = \gamma A(g, f^*g)^{-1} + \delta \mathrm{Id}$$

for some constants $\gamma \neq 0$ and δ. It follows that the numbers of different eigenvalues of $(f^*A)_{|x}$ and $A(g, f^*g)^{-1}_{|x}$ coincide. Thus, the number of different eigenvalues of $(f^*A)_{|x}$ is equal to the number of different eigenvalues of $A_{|x}$. Again we have that $f(x) \in M^0$ as we claimed. q.e.d.

Convention. *In what follows, (M, g, J) is a closed, connected Riemannian Kähler manifold of real dimension $2n \geq 4$ and of degree of mobility $D(g) = 2$. We assume that v is an h-projective vector field which is not affine. We chose a real number t_0 such that the pullback $\bar{g} := (\Phi^v_{t_0})^* g$ is not affinely equivalent to g. Let $A = A(g, \bar{g})$ be the corresponding element in Sol(g) constructed by formula* (2).

Lemma 7. *The tensor A and the h-projective vector field v satisfy*

$$\mathcal{L}_v A = c_2 A^2 + c_1 A + c_0 \mathrm{Id} \tag{15}$$

for some constants $c_2 \neq 0, c_1, c_0$.

Proof. Note that the vector field v is also h-projective with respect to the metric $\bar g$ and the degrees of mobility of the metrics g and $\bar g$ are both equal to two (see Lemma 1). Since $A = A(g, \bar g)$ is not proportional to the identity and $A(\bar g, g) = A(g, \bar g)^{-1} \in \mathrm{Sol}(\bar g)$, we obtain that $\{A, \mathrm{Id}\}$ and $\{A^{-1}, \mathrm{Id}\}$ form bases for $\mathrm{Sol}(g)$ and $\mathrm{Sol}(\bar g)$ respectively. It follows from Lemma 2 that

$$\begin{aligned} g^{-1} \circ \mathcal{L}_v g - \tfrac{\mathrm{trace}(g^{-1} \circ \mathcal{L}_v g)}{2(n+1)} \mathrm{Id} &= \beta_1 A + \beta_2 \mathrm{Id}, \\ \bar g^{-1} \circ \mathcal{L}_v \bar g - \tfrac{\mathrm{trace}(\bar g^{-1} \circ \mathcal{L}_v \bar g)}{2(n+1)} \mathrm{Id} &= \beta_3 A^{-1} + \beta_4 \mathrm{Id} \end{aligned} \tag{16}$$

for some constants $\beta_1, \beta_2, \beta_3$, and β_4. Taking the trace on both sides of the above equations, we see that they are equivalent to

$$\begin{aligned} g^{-1} \circ \mathcal{L}_v g &= \beta_1 A + \left(\tfrac{1}{2}\beta_1 \,\mathrm{trace}\, A + (n+1)\beta_2\right) \mathrm{Id}, \\ \bar g^{-1} \circ \mathcal{L}_v \bar g &= \beta_3 A^{-1} + \left(\tfrac{1}{2}\beta_3 \,\mathrm{trace}\, A^{-1} + (n+1)\beta_4\right) \mathrm{Id}. \end{aligned} \tag{17}$$

By (5), $\bar g$ can be written as $\bar g = (\det\, A)^{-\frac{1}{2}} g \circ A^{-1}$. Then,

$$\begin{aligned} \bar g^{-1} \circ \mathcal{L}_v \bar g &\overset{(5)}{=} (\det\, A)^{\frac{1}{2}} A \circ g^{-1} \circ \mathcal{L}_v((\det\, A)^{-\frac{1}{2}} g \circ A^{-1}) \\ &= -\frac{1}{2} (\det\, A)^{-1} (\mathcal{L}_v \det\, A) \mathrm{Id} + A \circ (g^{-1} \circ \mathcal{L}_v g) \circ A^{-1} - (\mathcal{L}_v A) \circ A^{-1}. \end{aligned}$$

We insert the second equation of (17) in the left-hand side, the first equation of (17) in the right-hand side, and multiply with A from the right to obtain

$$\begin{aligned} \beta_3 \mathrm{Id} + \left(\frac{1}{2}\beta_3 \,\mathrm{trace}\, A^{-1} + (n+1)\beta_4\right) A &= -\frac{1}{2}(\det\, A)^{-1}(\mathcal{L}_v \det\, A) A \\ &\quad + \beta_1 A^2 + \left(\frac{1}{2}\beta_1 \,\mathrm{trace}\, A + (n+1)\beta_2\right) A - \mathcal{L}_v A. \end{aligned}$$

Rearranging the terms in the last equation, we obtain

$$\mathcal{L}_v A = c_2 A^2 + c_1 A + c_0 \mathrm{Id} \tag{18}$$

for constants $c_2 = \beta_1$, $c_0 = -\beta_3$, and a certain function c_1.

Remark 13. Our way to obtain the equation (18) is based on an idea of Fubini from [**14**] used in the theory of projective vector fields.

Our next goal is to show that $c_2 = \beta_1 \neq 0$. If $\beta_1 = 0$, the first equation of (17) reads

$$\mathcal{L}_v g = (n+1)\beta g$$

and, hence v is an infinitesimal homothety for g. This contradicts the assumption that v is essential and we obtain that $c_2 = \beta_1 \neq 0$.

Now let us show that the function c_1 is a constant. Since A is non-degenerate, c_1 is a smooth function, so it is sufficient to show that its differential vanishes at every point of M^0. We will work in a neighborhood of a point of M^0. Let $U \in E_A(\rho)$ be an eigenvector of A with corresponding eigenvalue ρ. Using the Leibniz rule for the Lie derivative and the condition that $U \in E_A(\rho)$, we obtain the equations

$$\mathcal{L}_v(AU) = \mathcal{L}_v(\rho U) = v(\rho)U + \rho[v,U] \text{ and } \mathcal{L}_v(AU) = (\mathcal{L}_v A)U + A([v,U]).$$

Combining both equations and inserting $\mathcal{L}_v A$ from (18), we obtain

$$(v(\rho) - c_2\rho^2 - c_1\rho - c_0)U = (A - \rho\mathrm{Id})[v,U].$$

In a basis of eigenvectors $\{U_i, JU_i\}$ of A from the proof of Theorem 3, we see that the right-hand side does not contain any component from $E_A(\rho)$ (i.e., the right-hand side is a linear combination of eigenvectors corresponding to other eigenvalues). Then,

$$c_1 = v(ln(\rho)) - c_2\rho - \frac{c_0}{\rho} \text{ and } (A - \rho\mathrm{Id})[v,U] = 0. \tag{19}$$

These equations are true for all eigenvalues ρ of A and corresponding eigenvectors U. Note that $\rho \neq 0$ since A is non-degenerate. By construction, the metric $\bar{g}$ (such that $A = A(g,\bar{g})$) is not affinely equivalent to g; in particular, A has more than one eigenvalue. Let $W \in E_A(\mu)$ and $\rho \neq \mu$. Applying W to the first equation in (19) and using Proposition 1, we obtain

$$W(c_1) = [W,v](ln(\rho)).$$

The second equation of (19) shows that $[v,W] = 0$ modulo $E_A(\mu)$. Hence,

$$W(c_1) = 0.$$

We obtain that $U(c_1) = 0$ for all eigenvectors U of A. Then, $dc_1 \equiv 0$ on M^0. Since M^0 is dense in M, we obtain that $dc_1 \equiv 0$ on the whole M, implying c_1 is a constant. This completes the proof of Lemma 7. q.e.d.

Convention. *Since $c_2 \neq 0$, we can replace v by the h-projective vector field $\frac{1}{c_2}v$. For simplicity, we denote the new vector field again by v; this implies that* (15) *is now satisfied for $c_2 = 1$: instead of* (15) *we have*

$$\mathcal{L}_v A = A^2 + c_1 A + c_0\mathrm{Id} \tag{20}$$

for some constants c_1, c_0.

Remark 14. Note that the constant β_1 in the proof of Lemma 7 is equal to c_2. With the convention above, the first equation in (16) now reads

$$A_v = g^{-1} \circ \mathcal{L}_v g - \frac{\mathrm{trace}(g^{-1} \circ \mathcal{L}_v g)}{2(n+1)}\mathrm{Id} = A + \beta\mathrm{Id} \tag{21}$$

for some $\beta \in \mathbb{R}$.

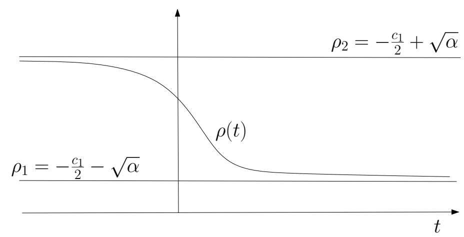

Figure 3. The behavior of the restriction of the eigenvalues to the integral curve of v: at most two eigenvalues, ρ_1 and ρ_2, are constant; they are roots of the quadratic polynomial $X^2 + c_1 X + c_0$. Precisely one eigenvalue, ρ, is not constant along the integral curve and is given by (23).

Remark 15. In the proof of Lemma 7, we had to do some additional work to show that c_1 is indeed a constant. This problem does not appear if we use the h-projectively invariant formulation of (3). We introduce this approach in Appendix A where we also give an alternative proof of Lemma 7.

Lemma 8. *The tensor A has precisely one non-constant eigenvalue ρ of multiplicity 2 and at least one and at most two constant eigenvalues. (We denote the constant eigenvalues by $\rho_1 < \rho_2$ and their multiplicities by $2k_1$ and $2k_2 = 2n - 2k_1 - 2$ respectively; we allow k_1 to be equal to 0 and $n-1$; if $k_1 = 0$, A has only one constant eigenvalue ρ_2 and if $k_1 = n-1$, then A has only one constant eigenvalue ρ_1.) Moreover, the eigenvalues satisfy the equations*

$$\begin{aligned} 0 &= \rho_1^2 + c_1\rho_1 + c_0 = \rho_2^2 + c_1\rho_2 + c_0 \\ v(\rho) &= \rho^2 + c_1\rho + c_0 \end{aligned} \tag{22}$$

for the constants c_1, c_0 from (20). For every point $x \in M^0$ such that $d\rho_{|x} \neq 0$ and $v(x) \neq 0$, the evolution of the non-constant eigenvalue ρ along the flow line $\Phi_t^v(x)$ is given by

$$\rho(t) = -\frac{c_1}{2} - \sqrt{\alpha}\,\tanh(\sqrt{\alpha}(t+d)), \tag{23}$$

where $\alpha = \frac{1}{4}c_1^2 - c_0$ is necessarily a positive real number.

Proof. We proceed as in the proof of Lemma 7. Applying the equation (20) to an eigenvector U of A, corresponding to the eigenvalue ρ, yields

$$(\rho^2 + c_1\rho + c_0 - v(\rho))U = -(A - \rho\mathrm{Id})[v, U].$$

Since the right-hand side does not contain any components lying in $E_A(\rho)$, we obtain that

$$(A-\rho\mathrm{Id})[v,U]=0 \text{ and } v(\rho)=\rho^2+c_1\rho+c_0 \tag{24}$$

for all eigenvalues ρ of A and all eigenvectors $U\in E_A(\rho)$.

In particular, each constant eigenvalue is a solution of the equation $\rho^2+c_1\rho+c_0=0$. This implies that there are at most two different constant eigenvalues ρ_1 and ρ_2 for A as we claimed.

On the other hand, let ρ be a non-constant eigenvalue of A (there is always a non-constant eigenvalue since otherwise, the vector field Λ vanishes identically on M and, therefore, the metrics g and $\bar g$ (such that $A=A(g,\bar g)$) are affinely equivalent; see Remark 4), and let $x\in M^0$ be a point such that $d\rho_{|x}\neq 0$ and $v(x)\neq 0$. The second equation in (24) shows that the restriction of ρ to the flow line $\Phi^v_t(x)$ of v (i.e., $\rho(t):=\rho(\Phi^v_t(x))$) satisfies the ordinary differential equation

$$\dot\rho=\rho^2+c_1\rho+c_0, \quad \text{where } \dot\rho \text{ stays for } \tfrac{d}{dt}\rho. \tag{25}$$

This ODE can be solved explicitly; the solution (depending on the parameters c_1, c_0) is given by the following list. We put $\alpha=\frac{c_1^2}{4}-c_0$.

- For $\alpha<0$, the non-constant solutions of (25) are of the form
$$-\frac{c_1}{2}-\sqrt{-\alpha}\tan(\sqrt{-\alpha}(-t+d)).$$
- For $\alpha>0$, the non-constant solutions of (25) take the form
$$-\frac{c_1}{2}-\sqrt{\alpha}\tanh(\sqrt{\alpha}(t+d)) \text{ or } -\frac{c_1}{2}-\sqrt{\alpha}\coth(\sqrt{\alpha}(t+d)).$$
- For $\alpha=0$, the non-constant solutions of (25) are given by
$$-\frac{c_1}{2}-\frac{1}{t+d}.$$

Since the degree of mobility is equal to two, we can apply Lemma 6 to obtain that the flow Φ^v_t maps preserves M^0. It follows that $\rho(t)$ satisfies (25) for all $t\in\mathbb{R}$. However, the only solution of (25) which does not reach infinity in finite time is

$$-\frac{c_1}{2}-\sqrt{\alpha}\tanh(\sqrt{\alpha}(t+d)),$$

where $\alpha=\frac{c_1^2}{4}-c_0$ is necessarily a positive real number.

We obtain that the non-constant eigenvalues of A satisfy (23); in particular, their images contain the open interval $(-\frac{c_1}{2}-\sqrt{\alpha},-\frac{c_1}{2}+\sqrt{\alpha})$. Suppose that there are two different non-constant eigenvalues $\rho=-\frac{c_1}{2}-\sqrt{\alpha}\tanh(\sqrt{\alpha}(t+d))$ and $\tilde\rho=-\frac{c_1}{2}-\sqrt{\alpha}\tanh(\sqrt{\alpha}(t+\tilde d))$ of A. Then we can find points $x_0,x_1,x_2\in M$ such that $\rho(x_0)<\tilde\rho(x_1)<\rho(x_2)$. This contradicts the global ordering of the eigenvalues of A; see Theorem 3(1). It follows that A has precisely one non-constant eigenvalue ρ. This eigenvalue restricted to flow lines of v satisfies (23). By Corollary 2, the multiplicity of ρ is equal to two. We obtain that there must be at least one constant eigenvalue of A. Finally, Lemma 8 is proven. q.e.d.

Corollary 5. *In the notation above, all eigenvalues ρ_1, ρ, ρ_2 are smooth functions on the manifold.*

Proof. The eigenvalues ρ_1, ρ_2 are constant and are therefore smooth. The non-constant eigenvalue ρ is equal to $\frac{1}{2}\mathrm{trace}\, A - k_1\rho_1 - (n-1-k_1)\rho_2$ and is therefore also smooth. q.e.d.

Lemma 9. *Let ρ be the only non-constant eigenvalue of A. On $M_{d\rho\neq 0} := \{x \in M : d\rho_{|x} \neq 0\}$, the vector fields Λ and $\bar{\Lambda}$ are eigenvectors of A corresponding to the eigenvalue ρ, i.e., $E_A(\rho) = \mathrm{span}\{\Lambda, \bar{\Lambda}\}$.*

Moreover, $M_{d\rho\neq 0}$ is open and dense in M and $\Lambda(\rho) \neq 0$ on $M_{d\rho\neq 0}$.

Remark 16. Note that the second part of the assertion above is still true even locally and even if there are more than just one non-constant eigenvalue. The proof is based on the existence of a family of Killing vector fields (one for each non-constant eigenvalue) and is given in [**3**, Proposition 14].

Proof. First of all, since ρ is the only non-constant eigenvalue of A and ρ has multiplicity equal to two (see Corollary 2), we obtain $\Lambda = \frac{1}{4}\mathrm{grad}\,\mathrm{trace}\, A = \frac{1}{2}\mathrm{grad}\,\rho$.

By Proposition 1, Λ is an eigenvector of A corresponding to the eigenvalue ρ. Since the eigenspaces of A are invariant with respect to the complex structure J, we immediately obtain $E_A(\rho) = \mathrm{span}\{\Lambda, \bar{\Lambda}\}$. Moreover, since $\mathrm{grad}\,\rho$ is proportional to Λ, we have $\bar{\Lambda}(\rho) = 0$ and $\Lambda(\rho) \neq 0$ at every point of $M_{d\rho\neq 0}$.

Obviously, $M_{d\rho\neq 0}$ is an open subset of M. As we explained above, $d\rho_{|x} = 0$, if and only if $\Lambda(x) = \bar{\Lambda}(x) = 0$. Then $M \setminus M_{d\rho\neq 0}$ coincides with the set of zeros of the non-trivial Killing vector field $\bar{\Lambda}$. We obtain that $M_{d\rho\neq 0}$ is dense in M. q.e.d.

Let us now consider the critical points of the non-constant eigenvalue ρ:

Lemma 10. *At every x such that $d\rho_{|x} = 0$, ρ takes its maximum or minimum values $\rho = -\frac{c_1}{2} \pm \sqrt{\alpha}$, where $\alpha = \frac{c_1^2}{4} - c_0$ and c_1, c_0 are the constants from* (20)*. Moreover, $v \neq 0$ on $M_{d\rho\neq 0}$.*

Proof. Since the subsets $M_{v\neq 0}$ and $M_{d\rho\neq 0}$ are both open and dense in M (see Corollary 3 and Lemma 9), we obtain that $M^1 = M_{v\neq 0} \cap M_{d\rho\neq 0}$ is open and dense in M as well. Equation (23) shows that $-\frac{c_1}{2} - \sqrt{\alpha} < \rho(x) < -\frac{c_1}{2} + \sqrt{\alpha}$ for all $x \in M^1$. Since M^1 is dense, we obtain

$$-\frac{c_1}{2} - \sqrt{\alpha} \leq \rho(x) \leq -\frac{c_1}{2} + \sqrt{\alpha}$$

for all $x \in M$. Now suppose that $d\rho_{|x} = 0$ for some $x \in M$. It follows from (22) that $\rho(x)$ satisfies $0 = d\rho_{|x}(v) = \rho(x)^2 + c_1\rho(x) + c_0$, hence,

$\rho(x)$ is equal to the maximum or minimum value of ρ. Now suppose $v(x) = 0$. By (22), ρ takes its maximum or minimum value at x. It follows that $d\rho_{|x} = 0$. q.e.d.

5.2. Metric components on integral manifolds of $\operatorname{span}\{\Lambda, \bar{\Lambda}\}$**.** By Lemma 8, A has precisely one non-constant eigenvalue ρ and at most two constant eigenvalues ρ_1 and ρ_2. The goal of this section is to calculate the components of the restriction of the metric g to the integral manifolds of the eigenspace distribution $E_A(\rho) = \operatorname{span}\{\Lambda, \bar{\Lambda}\}$. In order to do it, we split the tangent bundle on $M_{d\rho\neq 0}$ into the direct product of two distributions:

$$D_1 := \operatorname{span}\{\Lambda\} \text{ and } D_2 := D_1^{\perp} = \operatorname{span}\{\bar{\Lambda}\} \oplus E_A(\rho_1) \oplus E_A(\rho_2).$$

First let us show

Lemma 11. *The distributions D_1, D_2, and $E_A(\rho)$ are integrable on $M_{d\rho\neq 0}$. Moreover, integral manifolds of D_1 and $E_A(\rho)$ are totally geodesic.*

Proof. Since Λ is a gradient, the distribution D_2 is integrable. On the other hand, Corollary 4 immediately implies that $E_A(\rho)$ is integrable. The distribution D_1 is one-dimensional and is therefore integrable. In order to show that the integral manifolds of D_1 and $E_A(\rho)$ are totally geodesic, we consider the (quadratic in velocities) integrals $I_0, I_1, I_2 : TM \to \mathbb{R}$ given by

$$I_0(\zeta) = g(\bar{\Lambda}, \zeta)^2, I_1(\zeta) = \left(\frac{d^{k_1-1}}{dt^{k_1-1}} F_t(\zeta)\right)|_{t=\rho_1}, \text{ and}$$

$$I_2(\zeta) = \left(\frac{d^{k_2-1}}{dt^{k_2-1}} F_t(\zeta)\right)|_{t=\rho_2}, \tag{26}$$

where $2k_1, 2k_2$ are the multiplicities of the constant eigenvalues ρ_1, ρ_2 of A. Recall from Lemma 3 and Lemma 4 that these functions are indeed integrals.

If $s : TM \to \mathbb{R}$ is a quadratic polynomial in the velocities, we define the *nullity* of s by

$$\operatorname{null} s := \{\zeta \in TM : s(\zeta) = 0\}.$$

In the orthonormal frame of eigenvectors of A from the proof of Theorem 3, the integrals F_t are given by (10), and it is easy to see that

$$\operatorname{null} I_1 = E_A(\rho) \oplus E_A(\rho_2),\ \operatorname{null} I_2 = E_A(\rho) \oplus E_A(\rho_1),\ \text{and}$$
$$\operatorname{null} I_0 = \operatorname{span}\{\Lambda\} \oplus E_A(\rho_1) \oplus E_A(\rho_2).$$

It follows that $D_1 = \operatorname{null} I_0 \cap \operatorname{null} I_1 \cap \operatorname{null} I_2$ and $E_A(\rho) = \operatorname{null} I_1 \cap \operatorname{null} I_2$. Since the functions are integrals, if $\dot\gamma(0) \in \operatorname{null} I_i$, then $\dot\gamma(t) \in \operatorname{null} I_i$ for all t. Then every geodesic γ such that $\dot\gamma(0) \in D_1$ (resp. $E_A(\rho)$) remains tangent to D_1 (resp. $E_A(\rho)$). Thus, the integral manifolds of D_1 and $E_A(\rho)$ are totally geodesic. q.e.d.

Let us introduce local coordinates $x^1, x^2, \dots, x^{2n}$ in a neighborhood of a point of $M_{d\rho\neq 0}$ such that (for all constants $C_1, \dots, C_{2n}$) the equation $x^1 = C_1$ defines an integral manifold of D_2 and the system $\{x^i = C_i\}_{i=2,\dots,2n}$ defines an integral manifold of D_1. In these coordinates, the metric g has the block-diagonal form

$$g = g_{11} dx^1 \otimes dx^1 + \sum_{i,j=2}^{2n} \tilde{g}_{ij} dx^i \otimes dx^j.$$

In what follows we call such coordinates *adapted to the decomposition* $TM_{|M_{d\rho\neq 0}} = D_1 \oplus D_2$. Let us show that the h-projective vector field v splits into two independent components with respect to this decomposition:

Lemma 12. *In the coordinates $x^1, x^2, \dots, x^{2n}$ adapted to the decomposition $TM_{|M_{d\rho\neq 0}} = D_1 \oplus D_2$, the h-projective vector field v is given by*

$$v = \underbrace{v^1(x^1)\partial_1}_{=:v_1\in D_1} + \underbrace{v^2(x^2,\dots,x^{2n})\partial_2 + \dots + v^{2n}(x^2,\dots,x^{2n})\partial_{2n}}_{=:v_2\in D_2} \tag{27}$$

Proof. Since $\bar{\Lambda}$ is an eigenvector of A corresponding to the non-constant eigenvalue ρ, the first equation in (24) implies that

$$[v, \bar{\Lambda}] = f\bar{\Lambda} + h\Lambda$$

for some functions f, h. If we apply $d\rho$ to both sides of the equation above, we obtain $\bar{\Lambda}(v(\rho)) = \bar{\Lambda}(\rho^2 + c_1\rho + c_0) = 0$ on the left-hand side and $h\Lambda(\rho)$ on the right-hand side. Since $\Lambda(\rho) \neq 0$ on $M_{d\rho\neq 0}$, we necessarily have $h = 0$. By definition v is holomorphic and since $\bar{\Lambda} = J\Lambda$, we see that the equations

$$[v, \bar{\Lambda}] = f\bar{\Lambda} \text{ and } [v, \Lambda] = f\Lambda \tag{28}$$

are satisfied.

For an eigenvector U of A, corresponding to some constant eigenvalue μ, the first equation in (24) shows that

$$[v, U] \in E_A(\mu). \tag{29}$$

For each index $i \geq 2$, ∂_i is contained in D_2. On the other hand, ∂_1 is always proportional to Λ. We obtain

$$\partial_i \sim \bar{\Lambda} \bmod E_A(\rho_1) \oplus E_A(\rho_2) \text{ and } \partial_1 \sim \Lambda.$$

Using (28) and (29), we see that

$$[v, \partial_i] \in D_2 \text{ for all } i \geq 2 \text{ and } [v, \partial_1] \in D_1.$$

This means that $\partial_i v^1 = 0$ and $\partial_1 v^i = 0$ for all $i \geq 2$. Hence,

$$v = (v^1(x^1), v^2(x^2, \dots, x^{2n}), \dots, v^{2n}(x^2, \dots, x^{2n}))$$

as we claimed. q.e.d.

Let us write $v = v_1 + v_2$ with respect to the decomposition $TM_{|M_{d\rho\neq 0}} = D_1 \oplus D_2$ (as in (27)). The vector fields v_1 and v_2 are well-defined and smooth on $M_{d\rho\neq 0}$. By Lemma 12, we have $[v_1, v_2] = 0$.

Lemma 13. *The non-constant eigenvalue ρ satisfies the equation $v_1(\rho) = \rho^2 + c_1\rho + c_0$ and the evolution of ρ along the flow-lines of v_1 is given by* (23)*. Moreover, v_1 is a non-vanishing complete vector field on $M_{d\rho\neq 0}$.*

Proof. Since by Proposition 1 and Lemma 9 we have $d\rho(V) = 0$ for all $V \in D_2$, we have $v_2(\rho) = 0$ and, hence, $v_1(\rho) = v(\rho) = \rho^2 + c_1\rho + c_0$. Using Lemma 12, we obtain that the restriction of ρ on the flow line $\Phi_t^{v_1}(x)$ coincides with the restriction of ρ on $\Phi_t^{v}(x)$ for all $x \in M_{d\rho\neq 0}$. Therefore the evolution of ρ along flow lines of v_1 is again given by (23).

Let us assume that $v_1(x) = 0$ for some point $x \in M_{d\rho\neq 0}$. We obtain that $0 = \rho(x)^2 + c_1\rho(x) + c_0$, which implies that $\rho(x)$ is a maximum or minimum value of ρ (see Lemma 10). It follows that $d\rho_{|x} = 0$, contradicting our assumptions.

Finally, let us show that v_1 is complete. Take a maximal integral curve $\sigma : (a, b) \to M_{d\rho\neq 0}$ of v_1 and assume $b < \infty$. Since M is closed, there exists a sequence $\{b_n\} \subset (a, b)$, converging to b such that $\lim_{n\to\infty}\sigma(b_n) = y$ for some $y \in M$. Then, $y \in M \setminus M_{d\rho\neq 0}$, since otherwise the maximal interval (a, b) of σ can be extended beyond b. Then, $d\rho_{|y} = 0$, and Lemma 10 implies that $\rho(y)$ is equal to the minimum value $-\frac{c_1}{2} - \sqrt{\alpha}$. We obtain that $\lim_{n\to\infty}\rho(\sigma(b_n)) = -\frac{c_1}{2} - \sqrt{\alpha}$. On the other hand, formula (23) shows that this value cannot be obtained in finite time $b < \infty$. This gives us a contradiction, implying v_1 is a complete vector field on $M_{d\rho\neq 0}$. q.e.d.

Let us now calculate the restriction of the metric g to the integral manifolds of the distribution $E_A(\rho) = \mathrm{span}\{v_1, \bar{\Lambda}\}$.

Lemma 14. *In a neighborhood of each point of $M_{d\rho\neq 0}$, it is possible to choose the coordinates $t = x^1, x^2, \ldots, x^{2n}$ adapted to the decomposition $TM_{|M_{d\rho\neq 0}} = D_1 \oplus D_2$ in such a way that $v_1 = \partial_1$, $\bar{\Lambda} = \partial_2$, and*

$$g = \left(\begin{array}{c|c|ccc} h & 0 & 0 & \ldots & 0 \\ \hline 0 & g(\Lambda,\Lambda) & * & \ldots & * \\ \hline 0 & * & * & \ldots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & \ldots & * \end{array}\right). \tag{30}$$

The functions $h = g(v_1, v_1), g(\Lambda, \Lambda)$, and ρ depend on the first coordinate t only and are given explicitly by the formulas

$$h(t) = D\frac{e^{(C-c_1)t}}{\cosh^2(\sqrt{\alpha}(t+d))},$$

$$g(\Lambda, \Lambda) = \frac{\dot{\rho}^2}{4h} \text{ (where } \dot{\rho} = \tfrac{d\rho}{dt}\text{), and} \tag{31}$$

$$\rho(t) = -\frac{c_1}{2} - \sqrt{\alpha}\tanh(\sqrt{\alpha}(t+d)).$$

The constants $\alpha > 0$ and C in (31) are defined as $\alpha = \frac{c_1^2}{4} - c_0$ and $C = -\frac{n-1}{2}c_1 - (2k_1 + 1 - n)\sqrt{\alpha} + (n+1)\beta$, where $D > 0, d, \beta, c_1, c_0 \in \mathbb{R}$, and $2k_1$ is the multiplicity of the constant eigenvalue ρ_1. The constants c_1, c_0 are the same as in (20). Moreover, c_1, c_0, and β are global constants, i.e., they are the same for each coordinate system of the above type.

Proof. In a neighborhood of an arbitrary point of $M_{d\rho\neq 0}$, let us introduce a chart $x^1, x^2, \dots, x^{2n}$, adapted to the decomposition $TM_{|M_{d\rho\neq 0}} = D_1 \oplus D_2$. By Lemma 12 and Lemma 13, we can choose these coordinates such that the flow line parameter t of v_1 coincides with x^1 (i.e., such that the first component of v in the coordinate system equals $\frac{\partial}{\partial x^1}$). By (28), we have $[v, \bar{\Lambda}] \in D_2$. Moreover, $[v_2, \bar{\Lambda}] \in D_2$ since D_2 is integrable. It follows that $[v_1, \bar{\Lambda}] \in D_2$. On the other hand, since $v_1 = f\Lambda$ for some function f and $[\Lambda, \bar{\Lambda}] = 0$, we obtain that $[v_1, \bar{\Lambda}] = -\bar{\Lambda}(f)\Lambda \in D_1$, implying

$$[v_1, \bar{\Lambda}] = 0.$$

It follows that we can choose the second coordinate x^2 in such a way that $\bar{\Lambda} = \partial_2$.

Next let us show that $h = g_{11}$ depends on the first coordinate of the adapted chart only. For this, let I be an integral of second order for the geodesic flow of g such that I is block-diagonal with respect to the adapted coordinates $t, x^2, \dots, x^{2n}$. For the moment we adopt the convention that Latin indices run from 2 to $2n$ such that I, considered as a polynomial on T^*M, can be written as $I = I^{11}p_1^2 + I^{ij}p_ip_j$. We calculate the Poisson bracket $0 = \{H, I\}$ to obtain the equations

$$0 = I^{ik}\partial_k g^{11} - g^{ik}\partial_k I^{11} \text{ for all } i = 2, \dots, 2n. \tag{32}$$

Inserting integrals I of special type, we can impose restrictions on the metric. Obviously the integrals I_0, I_1, I_2 defined in (26) are block-diagonal. On the other hand, in the proof of Lemma 11 it was shown that they satisfy $\operatorname{null} I_1 = E_A(\rho) \oplus E_A(\rho_2)$, $\operatorname{null} I_2 = E_A(\rho) \oplus E_A(\rho_1)$, and $\operatorname{null} I_0 = \operatorname{span}\{\Lambda\} \oplus E_A(\rho_1) \oplus E_A(\rho_2)$. It follows that the integral $F = I_0 + I_1 + I_2$ is block-diagonal and that its nullity is equal to D_1. Then F can be written as $F^{ij}p_ip_j$ and the matrix $(F^{ij})_{i,j\geq 2}$ is invertible at each point where the coordinates are defined. Replacing the integral

I in (32) with F yields

$$\partial_i g^{11} = 0$$

for all $2 \leq i \leq 2n$; hence, the metric component $g_{11} = (g^{11})^{-1}$ depends only on t.

Now let us show the explicit dependence of the functions h, ρ, and $g(\Lambda, \Lambda)$ on the parameter t. We already know that $h = g_{11}$ and ρ depend only on t (for ρ this follows from Proposition 1 and Lemma 9), and by Lemma 13, the dependence of ρ on the first coordinate t is given by (23).

Recall that $\lambda = \frac{1}{4}\mathrm{trace}\, A = \frac{1}{2}\rho + \mathrm{const}$. It follows that $d\lambda = \frac{1}{2}\dot\rho\, dt$ and, hence, $\Lambda = \mathrm{grad}\, \lambda = \frac{\dot\rho}{2h}\partial_1$. We obtain

$$g(\Lambda, \Lambda) = \frac{\dot\rho^2}{4h}.$$

What is left is to clarify the dependence of the function h on the parameter t. Note that in the coordinates $t, x^2, \ldots, x^{2n}$, the h-projective vector field v is given by $v = \partial_1 + v_2$. Let us denote by $\dot h$ and $\dot\rho$ the derivatives of h and ρ with respect to the coordinate t and denote the restriction of g to the distribution D_2 by $\tilde g$. Then we calculate

$$\mathcal{L}_v g = \mathcal{L}_{v_1} g + \mathcal{L}_{v_2} g = \dot h\, dt \otimes dt + \mathcal{L}_{v_1}\tilde g + \mathcal{L}_{v_2}\tilde g, \tag{33}$$

where we used that $v_2(h) = 0$ and $\mathcal{L}_{v_2} dt = 0$, which follows from $[v_1, v_2] = 0$ and $[v_2, \partial_i] \in D_2$ for all $i \geq 2$. Note that $\mathcal{L}_{v_1}\tilde g$ and $\mathcal{L}_{v_2}\tilde g$ do not contain any expressions involving $dt \otimes dx^i$, $dx^i \otimes dt$, or $dt \otimes dt$. On the other hand, we already know that A_v given in formula (6) satisfies (21). After multiplication with g from the left, (21) can be written as

$$\mathcal{L}_v g - \frac{\mathrm{trace}(g^{-1} \circ \mathcal{L}_v g)}{2(n+1)} g = a + \beta g$$

for $a = g \circ A$ and some constant β. Calculating the trace on both sides yields

$$\mathcal{L}_v g = a + (\beta + \frac{1}{2}\mathrm{trace}(A + \beta \mathrm{Id}))g = a + ((n+1)\beta + \rho + k_1\rho_1 + k_2\rho_2)g.$$

Now we can insert (33) into the left-hand side to obtain

$$\dot h\, dt \otimes dt + \mathcal{L}_{v_1}\tilde g + \mathcal{L}_{v_2}\tilde g = a + ((n+1)\beta + \rho + k_1\rho_1 + k_2\rho_2)g. \tag{34}$$

Since (34) is in block-diagonal form, it splits up into two separate equations. The first equation which belongs to the matrix entry on the upper left reads

$$\dot h = (2\rho + C)h, \text{ where we defined } C = k_1\rho_1 + k_2\rho_2 + (n+1)\beta.$$

Integration of this differential equation yields

$$h(t) = De^{Ct + 2\int \rho dt} = De^{(C - c_1)t - 2\ln(\cosh(\sqrt{\alpha}(t+d)))}$$

for $\alpha = \frac{c_1^2}{4} - c_0 > 0$ and some constants d and $D > 0$. If we insert the formulas $\rho_1 = -\frac{c_1}{2} - \sqrt{\alpha}$ and $\rho_2 = -\frac{c_1}{2} + \sqrt{\alpha}$ for the constant eigenvalues into the definition of the constant C, we obtain

$$C = -\frac{n-1}{2} c_1 - (2k_1 + 1 - n)\sqrt{\alpha} + (n+1)\beta.$$

Finally, Lemma 14 is proven. q.e.d.

The formulas (31) in Lemma 14 show that the restriction

$$g_{|E_A(\rho)} = \begin{pmatrix} h & 0 \\ 0 & g(\Lambda, \Lambda) \end{pmatrix} \tag{35}$$

of the metric to the integral manifolds of the distribution $E_A(\rho) = \operatorname{span}\{v_1, \bar{\Lambda}\}$ (the coordinates are as in Lemma 14, i.e., $\partial_1 = v_1$ and $\partial_2 = \bar{\Lambda}$) depends on the global constants c_1, c_0, k_1, and β. The constants D and d are not interesting; they can depend a priori on the particular choice of the coordinate neighborhood. Note that c_1 and c_0 are subject to the condition $\alpha = c_1^2/4 - c_0 > 0$. Now our goal is to show that we can impose further constraints on the constants such that the only metric which is left is the metric of positive constant holomorphic sectional curvature. So far, we did not really use that the manifold is closed; indeed, most of the statements listed above still would be true if this condition is omitted. However, as the next lemma shows, the condition that M is closed imposes strong restrictions on the constants from Lemma 14:

Lemma 15. *The constants from the formulas* (31) *of Lemma* 14 *satisfy $C = c_1$. In particular, the function $h = g(v_1, v_1)$ has the form*

$$h(t) = \frac{D}{\cosh^2(\sqrt{\alpha}(t+d))}. \tag{36}$$

Proof. First we will show that certain integral curves of v_1 always have finite length. Let $x_{\max}$ and $x_{\min}$ be points where ρ takes its maximum and minimum value respectively and let $\gamma : [0,1] \to M$ be a geodesic joining the points $\gamma(0) = x_{\max}$ and $\gamma(1) = x_{\min}$. Consider the integrals $I_0, I_1, I_2 : TM \to \mathbb{R}$ given by (26). Since the Killing vector field $\bar{\Lambda}$ vanishes at $x_{\max}$, we obtain that $0 = I_0(\dot{\gamma}(0)) = I_0(\dot{\gamma}(t))$ for all $t \in [0,1]$. By Lemma 13, $\rho(x_{\max})$ is equal to the constant eigenvalue $\rho_2 = -\frac{c_1}{2} + \sqrt{\alpha}$. It follows that $I_2(\zeta) = 0$ for all $\zeta \in T_{x_{\max}}M$; in particular, $I_2(\dot{\gamma}(0)) = 0$. This implies that $I_2(\dot{\gamma}(t)) = 0$ for all $t \in [0,1]$. Similarly, considering the point $x_{\min}$, we obtain $I_1(\dot{\gamma}(t)) = 0$ for all $t \in [0,1]$. In the proof of Lemma 11, we already remarked that the distribution D_1 is equal to the intersection of the nullities of I_0, I_1, and I_2. It follows that $\dot{\gamma}(t)$ is contained in D_1 for all $0 < t < 1$. This implies that $\gamma_{|(0,1)}$ is a reparametrized integral curve

$\sigma : \mathbb{R} \to M$ of the complete vector field v_1. In particular, the length

$$l_g(\sigma) = \int_{-\infty}^{+\infty} \sqrt{g(\dot\sigma(t), \dot\sigma(t))}dt = \int_{-\infty}^{+\infty} \sqrt{g(v_1, v_1)(\sigma(t))}dt$$
$$= \int_{-\infty}^{+\infty} \sqrt{h(t)}dt \tag{37}$$

of the curve σ is equal to the length $l_g(\gamma_{|[0,1]})$ of the geodesic γ. We obtain that $l_g(\sigma)$ is finite. By (37), a necessary condition for $l_g(\sigma)$ to be finite is that $\sqrt{h(t)} \to 0$ when $t \to \infty$. Note that $h(t)$ is given by the first equation in (31) (for some constants D, d that can depend on the particular integral curve σ). From formula (31), we obtain that $\sqrt{h(t)}$ for $t \to \infty$ is asymptotically equal to

$$\sqrt{h(t)} \sim e^{\left(\frac{C-c_1}{2\sqrt{\alpha}}-1\right)t}.$$

The finiteness of $l_g(\sigma)$ now implies the condition

$$-\frac{C-c_1}{2\sqrt{\alpha}} + 1 > 0 \tag{38}$$

on the global constants given in (31). Let us find further conditions on the constants. Since M is assumed to be closed, the sectional curvature

$$K_{E_A(\rho)} = \frac{g(v_1, R(v_1, \bar\Lambda)\bar\Lambda)}{g(v_1, v_1)g(\Lambda, \Lambda)} = \frac{R_{1212}}{h\, g(\Lambda, \Lambda)}$$

of $E_A(\rho)$ has to be bounded on M. Since the integral manifolds of $E_A(\rho)$ are totally geodesic (by Lemma 11), the sectional curvature $K_{E_A(\rho)}$ is equal to the curvature of the two-dimensional metric (35). After a straight-forward calculation using the formulas (31) for h and $g(\Lambda, \Lambda)$, we obtain

$$\begin{aligned} K_{E_A(\rho)}(t) = &\frac{1}{4D}\Bigg[\underbrace{(-4c_0 - C^2 + 2Cc_1)}_{=:\gamma_1} e^{-(C-c_1)t} \\ &\underbrace{-(C-c_1)^2}_{=:\gamma_2} \cosh(2\sqrt{\alpha}(t+d))e^{-(C-c_1)t} \\ &\underbrace{-2(C-c_1)\sqrt{\alpha}}_{=:\gamma_3} \sinh(2\sqrt{\alpha}(t+d))e^{-(C-c_1)t}\Bigg] \\ =: &\frac{1}{4D}(\gamma_1 f_1(t) + \gamma_2 f_2(t) + \gamma_3 f_3(t)). \end{aligned} \tag{39}$$

Similar to the first part of the proof, we can consider the asymptotic behavior $t \to \infty$ of the functions $f_2(t), f_3(t)$ appearing as coefficients of the constants γ_2, γ_3 in formula (39). We substitute $s = 2\sqrt{\alpha}(t+d)$ and obtain

$$f_2(s) \sim \cosh(s) e^{-\frac{C-c_1}{2\sqrt{\alpha}}s} \underset{t\gg 0}{\sim} e^{\left(-\frac{C-c_1}{2\sqrt{\alpha}}+1\right)t} \underset{t\to\infty}{\overset{(38)}{\longrightarrow}} \infty,$$
$$f_3(s) \sim \sinh(s) e^{-\frac{C-c_1}{2\sqrt{\alpha}}s} \underset{t\gg 0}{\sim} e^{\left(-\frac{C-c_1}{2\sqrt{\alpha}}+1\right)t} \underset{t\to\infty}{\overset{(38)}{\longrightarrow}} \infty.$$

As we have already mentioned, the sectional curvatures of a closed manifold are bounded and, hence, $K_{E_A(\rho)}(t)$ must be finite when t approaches the limit $t \to \infty$. Using the formulas for the asymptotic behavior of $f_2(t)$ and $f_3(t)$ given above, this condition imposes the restriction $\gamma_2 = -\gamma_3$ on the constants in (39). Similarly, considering the asymptotic behavior for $t \to -\infty$, we obtain $\gamma_2 = \gamma_3$. Note that the dominating part in $\sinh(2\sqrt{\alpha}(t+d))$ now comes with the minus sign. It follows that $\gamma_2 = \gamma_3 = 0$; hence,

$$C - c_1 = 0 \tag{40}$$

as we claimed. Inserting (40) into the first formula of (31), the metric component $g_{11} = h$ takes the form (36). Lemma 15 is proven. q.e.d.

Remark 17. If we insert $\gamma_2 = \gamma_3 = 0$ and $C = c_1$ in the formula (39) for the sectional curvature of $E_A(\rho)$, we obtain that $K_{E_A(\rho)} = \frac{\alpha}{D}$ is constant and positive as we claimed.

5.3. Proof of Theorem 4. Our goal is to prove Theorem 4: we need to show the local existence of a function μ and a constant B such that the system (14) is satisfied.

Lemma 16. *At every point $x \in M$, the tensor A and the covariant differential $\nabla\Lambda$ are simultaneously diagonalizable in an orthogonal basis. More precisely, let $U \in E_A(\rho_1)$ and $W \in E_A(\rho_2)$ be eigenvectors of A corresponding to the constant eigenvalues. Then we obtain*

$$\begin{aligned}
\nabla_\Lambda \Lambda &= (\dot{\phi} + \phi\psi)\Lambda, \\
\nabla_{\bar{\Lambda}} \Lambda &= (\dot{\phi} + \phi\psi)\bar{\Lambda}, \\
\nabla_U \Lambda &= \tfrac{g(\Lambda,\Lambda)}{\rho-\rho_1} U, \\
\nabla_W \Lambda &= \tfrac{g(\Lambda,\Lambda)}{\rho-\rho_2} W.
\end{aligned} \tag{41}$$

The functions ϕ and ψ are given by the formulas

$$\phi = \frac{1}{2}\frac{\dot{\rho}}{h} \text{ and } \psi = \frac{1}{2}\frac{\dot{h}}{h}. \tag{42}$$

Proof. Since the distribution D_1 has totally geodesic integral manifolds (see Lemma 11), $\nabla_{v_1} v_1$ is proportional to v_1. Let us define two functions ϕ and ψ by setting

$$\Lambda =: \phi v_1 \text{ and } \nabla_{v_1} v_1 =: \psi v_1. \tag{43}$$

It follows immediately that $g(\Lambda, \Lambda) = \phi^2 h$. On the other hand, $\dot{h} = 2g(\nabla_{v_1} v_1, v_1) = 2\psi h$. Using the equations (31) in Lemma 14, we obtain

$$\phi = \frac{1}{2}\frac{\dot{\rho}}{h} \text{ and } \psi = \frac{1}{2}\frac{\dot{h}}{h}. \tag{44}$$

Note that the function ϕ has to be negative since ρ decreases along the flow-lines of v_1 while it increases along the flow-lines of $\Lambda = \frac{1}{2}\operatorname{grad}\rho$. By direct calculation, we obtain

$$\nabla_\Lambda \Lambda = \phi \nabla_{v_1}(\phi v_1) = \phi\dot{\phi} v_1 + \phi^2 \nabla_{v_1} v_1 = (\phi\dot{\phi} + \phi^2\psi) v_1 = (\dot{\phi} + \phi\psi)\Lambda.$$

From the above equation, the relation $\bar{\Lambda} = J\Lambda$, and the fact that Λ is a holomorphic vector field, we immediately obtain

$$\nabla_{\bar{\Lambda}} \Lambda = J \nabla_\Lambda \Lambda = (\dot{\phi} + \phi\psi)\bar{\Lambda}$$

and, hence, the first two equations in (41) are proven.

Now let $U \in E_A(\rho_1)$ be an eigenvector of A corresponding to the constant eigenvalue ρ_1. Using Proposition 1, we obtain

$$\nabla_U \bar{\Lambda} = -\frac{g(\Lambda, \Lambda)}{\rho_1 - \rho} JU + f\bar{\Lambda} + \tilde{f}\Lambda \text{ and } \nabla_{\bar{\Lambda}} U = 0 \bmod E_A(\rho_1) \tag{45}$$

for some functions f and $\tilde{f}$. The lie bracket of U and $\bar{\Lambda}$ is given by

$$[U, \bar{\Lambda}] = f\bar{\Lambda} + \tilde{f}\Lambda \bmod E_A(\rho_1).$$

Applying $d\rho$ to both sides of the above equation yields $\tilde{f}\Lambda(\rho) = 0$. Since $\Lambda(\rho) \neq 0$ on $M_{d\rho \neq 0}$, it follows that $\tilde{f} = 0$. On the other hand, the first equation in (45) shows that

$$\frac{1}{2} U(g(\Lambda, \Lambda)) = g(\nabla_U \bar{\Lambda}, \bar{\Lambda}) = fg(\Lambda, \Lambda).$$

Since $dg(\Lambda, \Lambda)$ is zero when restricted to the distribution D_2 (as can be seen by using the coordinates given in Lemma 14), the left-hand side of the above equation vanishes and, hence, $f = 0$. Inserting $f = \tilde{f} = 0$ into the first equation of (45), we obtain the third equation in (41). If we replace ρ_1 and U by ρ_2 and $W \in E_A(\rho_2)$, the same arguments can be applied to obtain the last equation in (41). Lemma 16 is proven. q.e.d.

Let (M, g, J) be a closed, connected Riemannian Kähler manifold of real dimension $2n \geq 4$ and of degree of mobility $D(g) = 2$. Let v be an essential h-projective vector field and t_0 a real number, such that $\bar{g} = (\Phi^v_{t_0})^* g$ is not affinely equivalent to g. Let us denote by $A = A(g, \bar{g})$ the corresponding solution of (3).

We want to show that any point of $M_{d\rho \neq 0}$ has a small neighborhood such that on this neighborhood there exist a function μ and a constant $B < 0$ such that the covariant differential $\nabla\Lambda$ satisfies the second equation

$$\nabla\Lambda = \mu \mathrm{Id} + BA \tag{46}$$

in (14). By Lemma 16, at every point of $M_{d\rho\neq 0}$, each eigenvector of A is an eigenvector of $\nabla\Lambda$. Since A has (at most) three different eigenvalues, (46) is equivalent to an inhomogeneous linear system of three equations on the two unknown real numbers μ and B. Using the formulas (41) from Lemma 16, we see that for $x \in M_{d\rho\neq 0}$, $\nabla\Lambda$ satisfies (46) for some numbers μ and B, if and only if the inhomogeneous linear system of equations

$$\begin{aligned} \mu + \rho B &= \dot{\phi} + \phi\psi, \\ \mu + \rho_1 B &= \tfrac{g(\Lambda,\Lambda)}{\rho-\rho_1}, \\ \mu + \rho_2 B &= \tfrac{g(\Lambda,\Lambda)}{\rho-\rho_2} \end{aligned} \tag{47}$$

is satisfied. Now, according to Lemma 14 and Lemma 15, in a neighborhood of a point of $M_{d\rho\neq 0}$, the functions ρ, $g(\Lambda,\Lambda)$, h, ϕ, and ψ are given explicitly by (31), (36), and (42). Let us insert these functions and the formulas $-\frac{c_1}{2} \pm \sqrt{\alpha}$ for the constant eigenvalues $\rho_1 < \rho_2$ (see Lemma 8) in (47). After a straight-forward calculation, we obtain that (47) is satisfied for

$$\mu = -\frac{\alpha(\frac{c_1}{2} - \sqrt{\alpha}\tanh(\sqrt{\alpha}(t+d)))}{4D} = B(c_1+\rho) \text{ and } B = -\frac{\alpha}{4D}. \tag{48}$$

We also see that the constant B is negative (as we claimed in Section 2.2).

Using the equation $\lambda = \frac{1}{4}\mathrm{trace}\, A = \frac{1}{2}\rho + \mathrm{const}$, we obtain that μ given by (48) satisfies $d\mu = Bd\rho = 2Bd\lambda$. Since Λ is the gradient of λ, this is easily seen to be equivalent to the third equation in the system (14).

We have shown that in a neighborhood of almost every point of M, there exists a smooth function μ and a constant $B < 0$, such that the system (14) is satisfied for the triple (A, Λ, μ).

If $\tilde{A}$ is another element in $\mathrm{Sol}(g)$ with corresponding vector field $\tilde{\Lambda}$, then $\tilde{A} = aA + b\mathrm{Id}$ for some $a, b \in \mathbb{R}$, implying $\tilde{\Lambda} = a\Lambda$. By direct calculations we see that for an appropriate local function $\tilde{\mu}$ the triple $(\tilde{A}, \tilde{\Lambda}, \tilde{\mu})$ satisfies the system (14) for the same constant $\tilde{B} = B$. Finally, Theorem 4 is proven.

6. Final step in the proof of Theorem 1

As we explained in Section 2.2, it is sufficient to prove Theorem 1 under the additional assumption that the degree of mobility is equal to two. By Theorem 4, for every $A \in \mathrm{Sol}(g)$ with corresponding vector field $\Lambda = \frac{1}{4}\mathrm{grad}\,\mathrm{trace}\, A$, we find an open neighborhood $U(x)$ for almost every point $x \in M$, such that there exists a local function $\mu : U(x) \to \mathbb{R}$ and a negative constant B such that the triple (A, Λ, μ) satisfies the system (14).

Now, in [**12**, §2.5] it was shown that under these assumptions the constant B is the same for all such neighborhoods, implying that the system (14) is satisfied on the whole M (for a certain smooth function $\mu : M \to \mathbb{R}$). Note that in view of the third equation of (14), μ is not a constant (if A is chosen to be non-proportional to the identity on TM).

By direct calculation (differentiating μ covariantly and replacing the derivatives using the system (14)), we obtain

$$(\nabla\nabla\mu)(Y,Z) = \nabla_Y(\nabla_Z\mu) - \nabla_{\nabla_Y Z}\mu \overset{\text{eq. 3 of (14)}}{=} 2Bg(Z, \nabla_Y\Lambda) \overset{\text{eq. 2 of (14)}}{=} 2B(\mu g(Y,Z) + Bg(AY,Z)).$$

Then,

$$(\nabla\nabla\nabla\mu)(X,Y,Z) = 2B((\nabla_X\mu)g(Y,Z) + Bg((\nabla_X A)Y,Z))$$
$$\overset{\text{eq. 1 of (14)}}{=} B(2(\nabla_X\mu)g(Y,Z) + 2Bg(Z,\Lambda)g(X,Y) + 2Bg(Y,\Lambda)g(X,Z) + 2Bg(Z,\bar\Lambda)g(JX,Y) + 2Bg(Y,\bar\Lambda)g(JX,Z)).$$

Inserting the third equation of (14), we obtain that μ satisfies the equation

$$(\nabla\nabla\nabla\mu)(X,Y,Z) = B[2(\nabla_X\mu)g(Y,Z) + (\nabla_Z\mu)g(X,Y) + (\nabla_Y\mu)g(X,Z) - (\nabla_{JZ}\mu)g(JX,Y) - (\nabla_{JY}\mu)g(JX,Z)] \tag{49}$$

for all $X, Y, Z \in TM$.

Now by [**57**, Theorem 10.1], the existence of a non-constant solution of (49) with $B < 0$ on a closed, connected Riemannian Kähler manifold implies that the manifold has constant holomorphic sectional curvature equal to $-4B$. On the other hand, since every isometry of $(\mathbb{C}P(n), g_{FS}, J_{standard})$ is induced by a unitary matrix of $\mathbb{C}^{n+1}$ and, hence, always has a fixed point, $(\mathbb{C}P(n), g_{FS}, J_{standard})$ has no isometric quotients. Consequently, $(M, -4Bg, J)$ is $(\mathbb{C}P(n), g_{FS}, J_{standard})$ and Theorem 1 is proven.

Appendix A. H-projectively invariant formulation of the main equation (3)

A.1. H-projective structure. Let (M, J) be a complex manifold of real dimension $2n \geq 4$. Note that the defining equation (1) for h-planar curves only involves the connection—it does not depend on the metric.

Definition 6. Two symmetric complex (i.e., $DJ = \bar{D}J = 0$) affine connections D and $\bar{D}$ are called *h-projectively equivalent* if each h-planar curve with respect to D is h-planar with respect to $\bar{D}$ and vice versa.

It is a classical result (see for example [**48**, **58**]) that two symmetric complex affine connections D and $\bar{D}$ are h-projectively equivalent if and only if for a certain 1-form Φ we have

$$\bar{D}_X Y - D_X Y = \Phi(Y)X + \Phi(X)Y - \Phi(JY)JX - \Phi(JX)JY \tag{50}$$

satisfied for all vector fields X, Y.

REMARK 18. If the symmetric affine connections D and $\bar{D}$ are related by (50) and $DJ = 0$, then $\bar{D}J = 0$ as well.

Definition 7. An *h-projective structure* on (M, J) is an equivalence class $[D]$ of h-projectively equivalent symmetric complex affine connections.

A.2. H-projectively invariant version of (3). Let (M, J) be a complex manifold of real dimension $2n \geq 4$. Denote by $\wedge^{2n} := \wedge^{2n} T^*M$ the bundle of $2n$-forms on M. Note that it is a trivial line bundle since the complex manifold (M, J) is always orientable. The bundle $(\wedge^{2n})^{\frac{w}{2(n+1)}}$ of $2n$-forms of "h-projective weight" w can be constructed by the requirement that it has the transition functions of $\wedge^{2n}$ (which can be chosen to have positive values) to the power $\frac{w}{2(n+1)}$. Let us consider the bundle $S^2_J TM$ of symmetric Hermitian (with respect to J) $(2, 0)$-tensors and define its "weighted" version

$$S^2_J TM(w) := S^2_J TM \otimes (\wedge^{2n})^{\frac{w}{2(n+1)}}.$$

For each choice of local coordinates $x^1, \ldots, x^{2n}$, the local section $dx^1 \wedge \cdots \wedge dx^{2n}$ of $\wedge^{2n}$ gives us a trivialization for $(\wedge^{2n})^{\frac{w}{2(n+1)}}$. Then, we can think of a section σ in $S^2_J TM(w)$ as a symmetric Hermitian $2n \times 2n-$matrix with components $\sigma^{ij} = \sigma^{ij}(x^1, \ldots, x^{2n})$. If we make an orientation-preserving change of coordinates $x^1, \ldots, x^{2n} \longmapsto \tilde{x}^1, \ldots, \tilde{x}^{2n}$, the components σ^{ij} transform according to the rule

$$\tilde{\sigma}^{ij} = \left(\det\left(\frac{\partial\, \tilde{x}^k}{\partial\, x^l}\right)\right)^{-\frac{w}{2(n+1)}} \frac{\partial\, \tilde{x}^i}{\partial\, x^\alpha} \frac{\partial\, \tilde{x}^j}{\partial\, x^\beta} \sigma^{\alpha\beta}. \tag{51}$$

The covariant derivative of elements $\sigma \in \Gamma(S^2_J TM(w))$ with respect to an affine connection D is given by

$$D_k \sigma^{ij} = \underbrace{\partial_k \sigma^{ij} + \Gamma^i_{kl}\sigma^{lj} + \Gamma^j_{kl}\sigma^{il}}_{\text{usual covariant derivative for 2-tensors}} \quad \underbrace{-\frac{w}{2(n+1)}\Gamma^l_{kl}\sigma^{ij}}_{\substack{\text{addition corresponding to} \\ 2n\text{-forms of weight } w}}, \tag{52}$$

where Γ^i_{jk} are the Christoffel symbols of D.

Theorem 5. *Let σ be an element of $\Gamma(S^2_J TM(2))$. Consider the equation*

$$D_k \sigma^{ij} - \frac{1}{2n}(\delta^i_k D_l \sigma^{lj} + \delta^j_k D_l \sigma^{li} + J^i_k J^j_m D_l \sigma^{lm} + J^j_k J^i_m D_l \sigma^{lm}) = 0. \tag{53}$$

Then, the following holds:

1) *Equation* (53) *is h-projectively invariant, i.e., independent of the connection* $D \in [D]$.
2) *Equation* (53) *has a non-degenerate solution* σ *(where non-degeneracy means that the matrix of components* (σ^{ij}) *is invertible everywhere), if and only if there is a connection* $\nabla \in [D]$*, such that* ∇ *is the Levi-Civita connection of some Kähler metric.*

Remark 19. We do not pretend that Theorem 5 is new; it was known to D. Calderbank (private communication). The statement is analogous to the projective case treated in [**11**].

Proof. (1) The condition (50) for the h-projective equivalence of the connections D and $\bar{D}$ can be rewritten locally as

$$\bar{\Gamma}^i_{jk} - \Gamma^i_{jk} = \delta^i_j \Phi_k + \delta^i_k \Phi_j - J^i_j J^l_k \Phi_l - J^i_k J^l_j \Phi_l, \tag{54}$$

where Γ^i_{jk} and $\bar{\Gamma}^i_{jk}$ are the Christoffel symbols of D and $\bar{D}$ respectively. Combining (54) and (52), we can calculate the difference between the connections D and $\bar{D}$ when they are acting on $\sigma \in \Gamma(S^2_J TM(2))$. We obtain

$$\bar{D}_k \sigma^{ij} = D_k \sigma^{ij} + \delta^i_k \Phi_l \sigma^{lj} + \delta^j_k \Phi_l \sigma^{il} + J^i_k J^j_m \Phi_l \sigma^{lm} + J^j_k J^i_m \Phi_l \sigma^{lm}, \tag{55}$$

and in particular,

$$\bar{D}_l \sigma^{lj} = D_l \sigma^{lj} + 2n \Phi_l \sigma^{lj}. \tag{56}$$

Replacing D with $\bar{D}$ in (53) and inserting the transformation laws (55) and (56), we obtain that (53) remains unchanged if D is replaced by $\bar{D} \in [D]$.

(2) In one direction, (2) is trivial. Suppose that g is a Kähler metric that is h-projectively equivalent to D. Let us denote by $g^{-1} \in \Gamma(S^2_J TM)$ the dual of g (i.e., $g \circ g^{-1} = Id$). We consider the non-degenerate element

$$\sigma = g^{-1} \otimes (\mathrm{vol}_g)^{\frac{1}{n+1}} \in \Gamma(S^2_J TM(2)).$$

Evidently, $\nabla \sigma = 0$, where ∇ is the Levi-Civita connection of g. By the first part of Theorem 5, D can be replaced with ∇ in (53) and we obtain that σ is a solution of (53).

Let us prove (2) in the opposite direction. Let $\sigma \in \Gamma(S^2_J TM(2))$ be a non-degenerate solution of (53). Using the transformation law (51), it is easy to see that

$$g^{ij} = \sigma^{ij} |\det(\sigma^{ij})|^{\frac{1}{2}}$$

defines the components of a symmetric Hermitian (with respect to J) $(2,0)$-tensor. Thus the corresponding dual $(0,2)$-tensor g is a Hermitian metric. Note that σ and g are related by

$$\sigma = g^{-1} \otimes (\mathrm{vol}_g)^{\frac{1}{n+1}}.$$

It remains to show that the Levi-Civita connection of g is contained in $[D]$. We consider a connection $\bar{D} \in [D]$ related to D by (54) such that

$$\Phi_i = -\tfrac{1}{2n}\sigma_{im}D_l\sigma^{lm}, \tag{57}$$

where the components σ_{ij} are defined by $\sigma^{i\alpha}\sigma_{\alpha j} = \delta^i_j$. Substituting (57) into (56) shows that

$$\bar{D}_l\sigma^{lj} = 0. \tag{58}$$

Replacing D with $\bar{D}$ in (53) and substituting (58), we obtain $\bar{D}_k\sigma^{ij} = 0$. Thus, $\bar{D}$ is the Levi-Civita connection of g. By Remark 18, $\bar{D}$ satisfies $\bar{D}J = 0$, which implies that g is indeed a Kähler metric. q.e.d.

Definition 8. Let $[D]$ be an h-projective structure on the complex manifold (M, J). We denote by $\mathrm{Sol}([D]) \subseteq \Gamma(S^2_J TM(2))$ the linear space of solutions of (53).

A.3. An alternative proof of Lemma 7. Let (M, g, J) be a Kähler manifold and let ∇ be the Levi-Civita connection of g. We assume that the degree of mobility (see Definition 4) is equal to two. Clearly, we have that $\mathrm{Sol}([\nabla])$ is 2-dimensional. Suppose that the Kähler metric $\bar{g}$ is non-proportional and h-projectively equivalent to g and consider the corresponding elements

$$\sigma = g^{-1} \otimes (\mathrm{vol}_g)^{\frac{1}{n+1}} \text{ and } \bar{\sigma} = \bar{g}^{-1} \otimes (\mathrm{vol}_{\bar{g}})^{\frac{1}{n+1}} \tag{59}$$

of $\mathrm{Sol}([\nabla])$. Since g and $\bar{g}$ are non-proportional, σ and $\bar{\sigma}$ form a basis for $\mathrm{Sol}([\nabla])$.

Now let v be an h-projective vector field for (M, g, J) (see Definition 3). Thus, the Lie derivative $\mathcal{L}_v$ maps solutions of (53) to solutions of (53) and, hence, restricts to an endomorphism of the 2-dimensional vector space $\mathrm{Sol}([\nabla])$. With respect to the basis $\sigma, \bar{\sigma}$ of $\mathrm{Sol}([\nabla])$, the endomorphism $\mathcal{L}_v$ is given by

$$\begin{aligned} \mathcal{L}_v\sigma &= \kappa_{11}\sigma + \kappa_{12}\bar{\sigma}, \\ \mathcal{L}_v\bar{\sigma} &= \kappa_{21}\sigma + \kappa_{22}\bar{\sigma} \end{aligned} \tag{60}$$

for some real numbers $\kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22}$.

Consider the $(1,1)$-tensor $A := \bar{\sigma}\sigma^{-1}$. Combining (59) with (2), we see that A coincides with $A(g, \bar{g})$. We calculate

$$\mathcal{L}_v A = (\mathcal{L}_v\bar{\sigma})\sigma^{-1} + \bar{\sigma}(\mathcal{L}_v\sigma^{-1}) = (\mathcal{L}_v\bar{\sigma})\sigma^{-1} - \bar{\sigma}\sigma^{-1}(\mathcal{L}_v\sigma)\sigma^{-1}.$$

Substituting (60), we obtain

$$\begin{aligned} \mathcal{L}_v A &= (\kappa_{21}\sigma + \kappa_{22}\bar{\sigma})\sigma^{-1} - \bar{\sigma}\sigma^{-1}(\kappa_{11}\sigma + \kappa_{12}\bar{\sigma})\sigma^{-1} \\ &= \kappa_{21}Id + (\kappa_{22} - \kappa_{11})A - \kappa_{12}A^2, \end{aligned}$$

hence,

$$\mathcal{L}_v A = c_2A^2 + c_1A + c_0Id$$

for some constants c_2, c_1, c_0. This is the assertion of Lemma 7.

References

[1] H. Akbar-Zadeh, *Transformations holomorphiquement projectives des variétés hermitiennes et kählériennes*, J. Math. Pures Appl. (9) **67**, no. 3 (1988), 237–261, MR 0964172, Zbl 0611.53059.

[2] D.V. Alekseevsky, *Groups of conformal transformations of Riemannian spaces*, Mat. Sb. (N.S.) **89(131)** (1972), 280–296, MR 0334077, Zbl 0263.53029.

[3] V. Apostolov, D. Calderbank & P. Gauduchon, *Hamiltonian 2-forms in Kähler geometry, I. General theory*, J. Differential Geom. **73** (2006), no. 3, 359–412, MR 2228318, Zbl 1101.53041.

[4] V. Apostolov, D. Calderbank, P. Gauduchon & C. Tønnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry, II. Global classification*, J. Differential Geom. **68** (2004), no. 2, 277–345, MR 2144249, Zbl 1079.32012.

[5] V. Apostolov, D. Calderbank, P. Gauduchon & C. Tønnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry, III. Extremal metrics and stability*, Invent. Math. **173** (2008), no. 3, 547–601, MR 2425136, Zbl 1145.53055.

[6] V. Apostolov, D. Calderbank, P. Gauduchon & C. Tønnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry, IV. Weakly Bochner-flat Kähler manifolds*, Comm. Anal. Geom. **16** (2008), no. 1, 91–126, MR 2411469, Zbl 1145.53054.

[7] E. Beltrami, *Risoluzione del problema: riportare i punti di una superficie sopra un piano in modo che le linee geodetische vengano rappresentante da linee rette*, Ann. di Mat., **1** (1865), no. 7, 185–204.

[8] A.V. Bolsinov & V.S. Matveev, *Geometrical interpretation of Benenti's systems*, J. of Geometry and Physics **44** (2003), no. 4, 489–506, MR 1943174, Zbl 1010.37035.

[9] A.V. Bolsinov & V.S. Matveev, *Splitting and gluing lemmas for geodesically equivalent pseudo-Riemannian metrics*, Trans. Amer. Math. Soc. **363** (2011), no. 8, 4081–4107, MR 2792981, Zbl 1220.53020.

[10] U. Dini, *Sopra un problema che si presenta nella teoria generale delle rappresentazioni geografiche di una superficie su un'altra*, Ann. Mat., ser. 2, **3** (1869), 269–293.

[11] M. Eastwood & V.S. Matveev, *Metric Connections in Projective Differential Geometry*, appeared in *Symmetries and overdetermined systems of partial differential equations*, 339–350, IMA Vol. Math. Appl., 144, Springer, New York, 2008, MR 2384718, Zbl 1144.53027.

[12] A. Fedorova, V. Kiosak, V. Matveev & S. Rosemann, *The only Kähler manifold with degree of mobility at least 3 is* $(CP(n), g_{Fubini-Study})$, Proc. London Math. Soc. (2012), doi: 10.1112/plms/pdr053.

[13] J. Ferrand, *Action du groupe conforme sur une variete riemannienne*, C. R. Acad. Sci. Paris Ser. I Math. **318** (1994), no. 4, 347–350, MR 1267613, Zbl 0831.53024.

[14] G. Fubini, *Sui gruppi trasformazioni geodetiche*, Mem. Acc. Torino **53**, 261–313 (1903), JFM 34.0658.03.

[15] S. Fujimura, *Indefinite Kähler metrics of constant holomorphic sectional curvature*, J. Math. Kyoto Univ. **30** (1990), no. 3, 493–516, MR 1075300, Zbl 0718.53018.

[16] I. Hasegawa, *H-projective-recurrent Kählerian manifolds and Bochner-recurrent Kählerian manifolds*, Hokkaido Math. J. **3** (1974), 271–278, MR 0362113, Zbl 0292.53025.

[17] I. Hasegawa & S. Fujimura, *On holomorphically projective transformations of Kaehlerian manifolds*, Math. Japon. **42** (1995), no. 1, 99–104, MR 1344635, Zbl 0846.53012.

[18] I. Hasegawa & K. Yamauchi, *Infinitesimal projective transformations on tangent bundles with lift connections*, Sci. Math. Jpn. **57** (2003), no. 3, 469–483, MR 1975964, Zbl 1050.53026.

[19] H. Hiramatu, *Riemannian manifolds admitting a projective vector field*, Kodai Math. J. **3** (1980), no. 3, 397–406, MR 0604484, Zbl 0454.53029.

[20] H. Hiramatu, *Integral inequalities in Kählerian manifolds and their applications*, Period. Math. Hungar. **12** (1981), no. 1, 37–47, MR 0607627, Zbl 0427.53032.

[21] S. Ishihara, *Holomorphically projective changes and their groups in an almost complex manifold*, Tohoku Math. J. (2) **9** (1957), 273–297, MR 0102096, Zbl 0090.38604.

[22] S. Ishihara & S. Tachibana, *On infinitesimal holomorphically projective transformations in Kählerian manifolds*, Tohoku Math. J. (2) **12** (1960), 77–101, MR 0120599, Zbl 0093.35404.

[23] S. Ishihara & S. Tachibana, *A note on holomorphic projective transformations of a Kaehlerian space with parallel Ricci tensor*, Tohoku Math. J. (2) **13** (1961), 193–200, MR 0139128, Zbl 0113.15402.

[24] K. Kiyohara, *Two classes of Riemannian manifolds whose geodesic flows are integrable*, Mem. Amer. Math. Soc. **130** (1997), no. 619, pp. viii+143, MR 1396959, Zbl 0904.53007.

[25] K. Kiyohara & P.J. Topalov, *On Liouville integrability of h-projectively equivalent Kähler metrics*, Proc. Amer. Math. Soc. **139** (2011), no. 1, 231–242, MR 2729086, Zbl pre05863815.

[26] T. Levi-Civita, *Sulle trasformazioni delle equazioni dinamiche*, Ann. di math. **24** (1896), 255–300.

[27] A. Lichnerowicz, *Geometry of groups of transformations*, Translated from the French and edited by Michael Cole, Noordhoff International Publishing, Leyden, 1977. pp. xiv+234, MR 0438364, Zbl 0348.53001.

[28] V.S. Matveev & P.J. Topalov, *Trajectory equivalence and corresponding integrals*, Regular and Chaotic Dynamics **3** (1998), no. 2, 30–45, MR 1693470, Zbl 0928.37003.

[29] V.S. Matveev, *Low-dimensional manifolds admitting metrics with the same geodesics*, Contemporary Mathematics **308** (2002), 229–243, MR 1955639, Zbl 1076.53051.

[30] V.S. Matveev, *Three-dimensional manifolds having metrics with the same geodesics*, Topology **42** (2003), no. 6, 1371–1395, MR 1981360, Zbl 1035.53117.

[31] V.S. Matveev, *Hyperbolic manifolds are geodesically rigid*, Invent. Math. **151** (2003), no. 3, 579–609, MR 1961339, Zbl 1039.53046.

[32] V.S. Matveev, *Die Vermutung von Obata für Dimension 2*, Arch. Math. **82** (2004), no. 3, 273–281, MR 2053631, Zbl 1076.53052.

[33] V.S. Matveev, *Solodovnikov's theorem in dimension two*, Dokl. Akad. Nauk **396** (2004), no. 1, 25–27, MR 2115905, Zbl pre05168081.

[34] V.S. Matveev, *Closed manifolds admitting metrics with the same geodesics*, SPT 2004-Symmetry and perturbation theory, 198–208, World Sci. Publ., Hackensack, NJ, 2005. MR 2331222, Zbl 1136.53315.

[35] V.S. Matveev, *Lichnerowicz-Obata conjecture in dimension two*, Comm. Math. Helv. **80** (2005), no. 3, 541–570, MR 2165202, Zbl 1113.53025.

[36] V.S. Matveev, *The eigenvalues of the Sinyukov mapping for geodesically equivalent metrics are globally ordered*, Mathematical Notes, **77**(2005) no. 3-4, 380-390, MR 2157900, Zbl 1114.53039.

[37] V.S. Matveev, *On degree of mobility for complete metrics*, Adv. Stud. Pure Math. **43** (2006), 221–250, MR 2325140, Zbl 1131.53013.

[38] V.S. Matveev, *Geometric explanation of the Beltrami theorem*, Int. J. Geom. Methods Mod. Phys. **3** (2006), no. 3, 623–629, MR 2232875, Zbl 1095.53023.

[39] V.S. Matveev, *Two-dimensional metrics admitting precisely one projective vector field*, appendix "Dini theorem for pseudoriemannian metrics" by A.V. Bolsinov, V.S. Matveev, and G. Pucacco, Math. Ann. **352**(2012), no. 4, 865–909, Zbl 1237.53013.

[40] V.S. Matveev, *Proof of the projective Lichnerowicz-Obata conjecture*, J. Diff. Geom. **75** (2007), no. 3, 459–502, MR 2301453, Zbl 1115.53029.

[41] J. Mikes, V.V. Domashev, *On The Theory Of Holomorphically Projective Mappings Of Kaehlerian Spaces*, Math. Zametki **23** (1978), no. 2, 297–303, MR 0492674, Zbl 0379.53035.

[42] J. Mikes, *Geodesic Mappings Of Affine-Connected And Riemannian Spaces*, Journal of Math. Sciences **78** (1996), no. 3, 311–333, MR 1384327, Zbl 0866.53028.

[43] J. Mikes, *Holomorphically projective mappings and their generalizations*, J. Math. Sci. (New York) **89** (1998), no. 3, 1334–1353, MR 1619720, Zbl 0983.53013.

[44] A. Moroianu & U. Semmelmann, *Twistor forms on Kähler manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2 (2003), no. 4, 823–845, MR 2040645, Zbl 1121.53050.

[45] T. Nagano & T. Ochiai, *On compact Riemannian manifolds admitting essential projective transformations*, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **33** (1986), 233–246, MR 0866391, Zbl 0645.53022.

[46] M. Obata, *Riemannian manifolds admitting a solution of a certain system of differential equations*, Proc. U.S.-Japan Seminar in Differential Geometry (Kyoto, 1965) pp. 101–114, MR 0216430, Zbl 0144.20903.

[47] M. Obata, *The conjectures on conformal transformations of Riemannian manifolds*, J. Diff. Geometry **6** (1971), 247–258, MR 0303464, Zbl 0236.53042.

[48] T. Otsuki & Y. Tashiro, *On curves in Kaehlerian spaces*, Math. Journal of Okayama University **4** (1954), 57–78, MR 0066024, Zbl 0057.14101.

[49] T. Sakaguchi, *On the holomorphically projective correspondence between Kählerian spaces preserving complex structure*, Hokkaido Mathematical Journal **3** (1974), 203–212, MR 0370411, Zbl 0305.53024.

[50] R. Schoen, *On the conformal and CR automorphism groups*, Geom. Funct. Anal. **5** (1995), no. 2, 464–481, MR 1334876, Zbl 0835.53015.

[51] U. Semmelmann, *Conformal Killing forms on Riemannian manifolds*, Habilitationschrift, Universität München (2002), arXiv:math.DG/0206117.

[52] U. Semmelmann, *Conformal Killing forms on Riemannian manifolds*, Math. Z. **245** (2003), no. 3, 503–527, MR 2021568, Zbl 1061.53033.

[53] N.S. Sinjukov, *Geodesic mappings of Riemannian spaces*, (in Russian) "Nauka," Moscow, 1979, MR 0552022, Zbl 0637.53020.

[54] N.S. Sinyukov & E.N. Sinyukova, *Holomorphically projective mappings of special Kählerian spaces*, Mat. Zametki, **36** (1984), no. 3, 417–423, MR 0767221, Zbl 0577.53046.

[55] H. Takeda, & Y. Watanabe, *On Riemannian spaces with parallel Weyl's projective curvature tensor*, Kyungpook Math. J. **12** (1972), 37–41, MR 0303479, Zbl 0245.53021.

[56] N. Tanaka, *Projective connections and projective transformations*, Nagoya Math. J. **12** (1957), 1–24, MR 0105154, Zbl 0081.38404.

[57] S. Tanno, *Some Differential Equations On Riemannian Manifolds*, J. Math. Soc. Japan **30** (1978), no. 3, 509–531, MR 0500721, Zbl 0387.53015.

[58] Y. Tashiro, *On A Holomorphically Projective Correspondence In An Almost Complex Space*, Math. Journal of Okayama University **6** (1957), 147–152, MR 0087181, Zbl 0077.35501.

[59] P.J. Topalov & V.S. Matveev, *Geodesic equivalence via integrability*, Geometriae Dedicata **96** (2003), 91–115, MR 1956835, Zbl 1017.37029.

[60] P.J. Topalov, *Geodesic compatibility and integrability of geodesic flows*, Journal of Mathematical Physics **44** (2003), no. 2, 913–929, MR 1953103, Zbl 1061.37042.

[61] K. Yamauchi, *On infinitesimal projective transformations*, Hokkaido Math. J. **3**(1974), 262–270, MR 0358628, Zbl 0299.53028.

[62] K. Yano, *On harmonic and Killing vector fields*, Ann. of Math. (2) **55** (1952), 38–45, MR 0046122, Zbl 0046.15603.

[63] K. Yano, *Differential geometry on complex and almost complex spaces*, International Series of Monographs in Pure and Applied Mathematics, **49** Pergamon Press, Macmillan, New York (1965) pp. xii+326, MR 0187181, Zbl 0127.12405.

[64] K. Yano & H. Hiramatu, *Isometry Of Kaehlerian Manifolds To Complex Projective Spaces*, J. Math. Soc. Japan **33** (1981), no. 1, 67–78, MR 0597481, Zbl 0466.53034.

[65] Y. Yoshimatsu, *H-projective connections and H-projective transformations*, Osaka J. Math. **15** (1978), no. 2, 435–459, MR 0500679, Zbl 0411.53028.

Institute of Mathematics
FSU Jena
07737 Jena, Germany
E-mail address: vladimir.matveev@uni-jena.de

Institute of Mathematics
FSU Jena
07737 Jena, Germany
E-mail address: stefan.rosemann@uni-jena.de