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SMOOTH YAMABE INVARIANT AND SURGERY

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Abstract

We prove a surgery formula for the smooth Yamabe invariant $\sigma(M)$ of a compact manifold M. Assume that N is obtained from M by surgery of codimension at least 3. We prove the existence of a positive constant Λ_n, depending only on the dimension n of M, such that

$$\sigma(N) \geq \min\{\sigma(M), \Lambda_n\}.$$

1. Introduction

1.1. Main result. The smooth Yamabe invariant, also called Schoen's σ-constant, of a compact manifold M is defined as

$$\sigma(M) := \sup \inf \int_M \mathrm{Scal}^g \, dv^g,$$

where the supremum runs over all conformal classes $[g_0]$ on M and the infimum runs over all metrics g of volume 1 in $[g_0]$. The integral $\mathcal{E}(g) := \int_M \mathrm{Scal}^g \, dv^g$ is the integral of the scalar curvature of g integrated with respect to the volume element of g and is known as the Einstein–Hilbert functional.

Let $n = \dim M$. We assume that N is obtained from M by surgery of codimension $n-k \geq 3$. That is for a given embedding $S^k \hookrightarrow M$, with trivial normal bundle, $0 \leq k \leq n-3$, we remove a tubular neighborhood $U_\epsilon(S^k)$ of this embedding. The resulting manifold has boundary $S^k \times S^{n-k-1}$. This boundary is glued together with the boundary of $B^{k+1} \times S^{n-k-1}$, and we thus obtain the closed smooth manifold

$$N := (M \setminus U_\epsilon(S^k)) \cup_{S^k \times S^{n-k-1}} (B^{k+1} \times S^{n-k-1}).$$

Our main result is the existence of a positive constant Λ_n depending only on n such that

$$\sigma(N) \geq \min\{\sigma(M), \Lambda_n\}.$$

This formula unifies and generalizes previous results obtained by Gromov and Lawson; Schoen and Yau; Kobayashi; and Petean and Yun. It also allows many conclusions by using bordism theory.

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In Section 1.2 we give a detailed description of the background of our result; a stronger version of the main result follows in Section 1.3, followed by a sketch of topological applications (Section 1.4). The construction of a generalization of surgery is recalled in Section 2. Then, in Section 3 the constant Λ_n is described and is proven to be positive. After the proof of some preliminary results on limit spaces in Section 4, we derive a key estimate in Section 5—namely, an estimate for the L^2-norm of solutions of a perturbed Yamabe equation on a special kind of sphere bundle, called WS-bundle. The last section contains the proof of the main theorem, Theorem 1.3.

1.2. Basic notions, the Yamabe problem, and some surgery formulas. We denote by $B^n(r)$ the open ball of radius r around 0 in $\mathbb{R}^n$ and we set $B^n := B^n(1)$. The unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1}. By ξ^n we denote the standard flat metric on $\mathbb{R}^n$ and by σ^{n-1} the standard metric of constant sectional curvature 1 on S^{n-1}. We denote the Riemannian manifold (S^{n-1}, σ^{n-1}) by $\mathbb{S}^{n-1}$. All manifolds in this article are manifolds without boundary unless stated differently.

Let (M, g) be a Riemannian manifold of dimension n. The Yamabe operator, or Conformal Laplacian, acting on smooth functions on M is defined by

$$L^g u = a\Delta^g u + \mathrm{Scal}^g u,$$

where $a := \frac{4(n-1)}{n-2}$ and where $\Delta^g = \mathrm{div}^g \mathrm{grad}^g$ is the non-negative Laplacian associated to the metric g. Let $p := \frac{2n}{n-2}$. Define the functional J^g acting on non-zero compactly supported smooth functions on M by

$$J^g(u) := \frac{\int_M u L^g u \, dv^g}{\left(\int_M u^p \, dv^g\right)^{\frac{2}{p}}}. \tag{1}$$

If g and $\tilde{g} = f^{\frac{4}{n-2}} g = f^{p-2} g$ are conformal metrics on M, then the corresponding Yamabe operators are related by

$$L^{\tilde{g}} u = f^{-\frac{n+2}{n-2}} L^g(fu) = f^{1-p} L^g(fu). \tag{2}$$

It follows that

$$J^{\tilde{g}}(u) = J^g(fu). \tag{3}$$

For a compact Riemannian manifold (M, g), the conformal Yamabe constant is defined by

$$\mu(M, g) := \inf J^g(u) \in \mathbb{R},$$

where the infimum is taken over all non-zero smooth functions u on M. The same value of $\mu(M, g)$ is obtained if one takes the infimum over positive smooth functions. From (3) it follows that the invariant μ depends only on the conformal class $[g]$ of g, and the notation $\mu(M,[g]) = \mu(M,g)$ is also used. For the standard sphere we have

$$\mu(\mathbb{S}^n) = n(n-1){\omega_n}^{2/n}, \tag{4}$$

where ω_n denotes the volume of $\mathbb{S}^n$. This value is a universal upper bound for μ.

Theorem 1.1 ([**10**, Lemma 3])**.** *The inequality*

$$\mu(M,g) \leq \mu(\mathbb{S}^n)$$

holds for any compact Riemannian manifold (M,g)*.*

For $u > 0$ the J^g-functional is related to the Einstein–Hilbert functional via

$$J^g(u) = \frac{\mathcal{E}(u^{4/(n-2)}g)}{\mathrm{Vol}(M, u^{4/(n-2)}g)^{\frac{n-2}{n}}}, \qquad \forall u \in C^\infty(M,\mathbb{R}^+),$$

and it follows that $\mu(M,g)$ has the alternative characterization

$$\mu(M,g) = \inf_{\tilde{g}\in[g]} \frac{\mathcal{E}(\tilde{g})}{\mathrm{Vol}(M,\tilde{g})^{\frac{n-2}{n}}}.$$

Critical points of the functional J^g are given by solutions of the Yamabe equation

$$L^g u = \mu |u|^{p-2} u$$

for some $\mu \in \mathbb{R}$. If the inequality in Theorem 1.1 is satisfied strictly—that is, if $\mu(M,g) < \mu(\mathbb{S}^n)$—then the infimum in the definition of $\mu(M,g)$ is attained.

Theorem 1.2 ([**56**, **10**])**.** *Let M be connected. If $\mu(M,g) < \mu(\mathbb{S}^n)$, then there exists a smooth positive function u with $J^g(u) = \mu$ and $\|u\|_{L^p} = 1$. This implies that u solves* (5) *with $\mu = \mu(M,g)$. The minimizer u is unique if $\mu \leq 0$.*

The inequality $\mu(M,g) < \mu(\mathbb{S}^n)$ was shown by Aubin [**10**] for non-conformally flat, compact manifolds of dimension at least 6. Later, Schoen [**48**] could apply the positive mass theorem to obtain this strict inequality for all compact manifolds not conformal to the standard sphere. We thus have a solution of

$$L^g u = \mu u^{p-1}, \qquad u > 0. \tag{5}$$

To explain the geometric meaning of these results we recall a few facts about the Yamabe problem; see, for example, [**38**] and [**53**, Chapter 5] for more details on this material. The name of Yamabe is associated to the problem, as Yamabe wrote the first article about this subject [**57**].

For a given compact Riemannian manifold (M,g), the Yamabe problem consists of finding a metric of constant scalar curvature in the conformal class of g. The above results yield a minimizer u for J^g. Equation (5) is equivalent to the fact that the scalar curvature of the metric

$u^{4/(n-2)}g$ is everywhere equal to μ. Thus, the above theorem, together with $\mu(M,g) < \mu(\mathbb{S}^n)$, resolves the Yamabe problem.

A conformal class $[g]$ on M contains a metric of positive scalar curvature if and only if $\mu(M,[g]) > 0$. If $M = M_1 \amalg M_2$ is a disjoint union of M_1 and M_2 and if g_i is the restriction of g to M_i, then an elementary argument where one rescales the components with different factors yields that

$$\mu(M,[g]) = \min\{\mu(M_1,[g_1]), \mu(M_2,[g_2])\}$$

if $\mu(M_1,[g_1]) \geq 0$ or $\mu(M_2,[g_2]) \geq 0$, and otherwise

$$\mu(M,[g]) = -\left(|\mu(M_1,[g_1])|^{n/2} + |\mu(M_2,[g_2])|^{n/2}\right)^{2/n}.$$

One now defines the smooth Yamabe invariant of an arbitrary compact manifold M of dimension at least 3 as

$$\sigma(M) := \sup \mu(M,[g]) \leq n(n-1)\omega_n^{2/n},$$

where the supremum is taken over all conformal classes $[g]$ on M.

The introduction of this invariant was originally motivated by Yamabe's attempt to find Einstein metrics on a given compact manifold; see [**49**, **35**]. Yamabe's idea in the early 1960s was to search for a conformal class $[g_{\sup}]$ that attains the supremum. The minimizer g_0 of $\mathcal{E}$ among all unit volume metrics in $[g_{\sup}]$ exists according to Theorem 1.2, and Yamabe hoped that the g_0 obtained with this minimax procedure would be a stationary point of $\mathcal{E}$ among all unit volume metrics (without fixed conformal class), which is equivalent to g_0 being an Einstein metric.

Yamabe's approach was very ambitious. If M is a simply connected compact 3-manifold, then an Einstein metric on M is necessarily a round metric on S^3, and hence the 3-dimensional Poincaré conjecture would follow. It turned out that his approach actually yields an Einstein metric in some special cases. For example, LeBrun [**36**] showed that if a compact 4-dimensional manifold M carries a Kähler–Einstein metric with non-positive scalar curvature, then the supremum is attained by the conformal class of this metric. Moreover, in any maximizing conformal class the minimizer is a Kähler–Einstein metric.

Compact quotients $M = \Gamma\backslash\mathbb{H}^3$ of 3-dimensional hyperbolic space $\mathbb{H}^3$ yield other examples on which Yamabe's approach yields an Einstein metric. On such quotients the supremum is attained by the hyperbolic metric on M. The proof of this statement uses Perelman's proof of the Geometrization conjecture, see [**9**] and [**30**, Section II.8]. In particular, $\sigma(\Gamma\backslash\mathbb{H}^3) = -6(v_\Gamma)^{2/3}$ where v_Γ is the volume of $\Gamma\backslash\mathbb{H}^3$ with respect to the hyperbolic metric.

On a general manifold, Yamabe's approach failed for various reasons. In dimension 3 and 4 obstructions against the existence of Einstein metrics are known today; see, for example, [**34**, **37**]. In many cases the supremum is not attained.

R. Schoen and O. Kobayashi started to study the smooth Yamabe invariant systematically in the late 1980s [**49**, **50**, **31**, **32**]. In particular, they determined $\sigma(S^{n-1}\times S^1)$ to be $\sigma(S^n)=n(n-1)\omega_n^{2/n}$. On $S^{n-1}\times S^1$ the supremum in the definition of σ is not attained. Because of Schoen's important results in these articles, the smooth Yamabe invariant is also often called Schoen's σ-constant.

The smooth Yamabe invariant determines the existence of positive scalar curvature metrics. Namely, it follows from above that the smooth Yamabe invariant $\sigma(M)$ is positive if and only if the manifold M admits a metric of positive scalar curvature. Thus the value of $\sigma(M)$ can be interpreted as a quantitative refinement of the property of admitting a positive scalar curvature metric.

In general calculating the σ-invariant is very difficult. LeBrun [**34**, Section 5], [**36**] showed that the σ-invariant of a complex algebraic surfaces is negative (resp. zero) if and only if it is of general type (resp. of Kodaira dimension 0 or 1), and the value of $\sigma(M)$ can be calculated explicitly in these cases. As already explained above, the σ-invariant can also be calculated for hyperbolic 3-manifolds; they are realized by the hyperbolic metrics.

There are many manifolds admitting a Ricci-flat metric, but no metric of positive scalar curvature—for example, tori, K3-surfaces and compact connected 8-dimensional manifolds admitting metrics with holonomy Spin(7). These conditions imply $\sigma(M)=0$, and the supremum is attained.

Conversely, Bourguignon showed that if $\sigma(M)=0$ and if the supremum is attained by a conformal class $[g_{\mathrm{sup}}]$, then $\mathcal{E}:[g_{\mathrm{sup}}]\to\mathbb{R}$ attains its minimum in a Ricci-flat metric $g_0\in[g_{\mathrm{sup}}]$. Thus Cheeger's splitting principle implies topological restrictions on M in this case. In particular, a compact quotient $\Gamma\backslash N$ of a non-abelian nilpotent Lie group N does not admit metrics of non-negative scalar curvature, but it admits a sequence of metrics g_i with $\mu(\Gamma\backslash N,g_i)\to 0$. Thus $\Gamma\backslash N$ is an example of a manifold for which $\sigma(\Gamma\backslash N)=0$, for which the supremum is not attained.

All the examples mentioned up to here have $\sigma(M)\leq 0$. Positive smooth Yamabe invariants are even harder to determine. The calculation of non-positive $\sigma(M)$ often relies on the formula

$$|\min\{\sigma(M),0\}|^{n/2}=\inf_g\int_M|\mathrm{Scal}^g|^{n/2}\,dv^g$$

where the infimum runs over all Riemannian metrics g on M; see [**35**, Proof of Proposition 2.1]. This formula does not distinguish between different positive values of $\sigma(M)$, and thus it cannot be used in the positive case.

It has been conjectured by Schoen [**50**, page 10, lines 6–11] that all finite quotients of round spheres satisfy $\sigma(S^n/\Gamma) = (\#\Gamma)^{-2/n} Y(\mathbb{S}^n)$, but this conjecture is only verified for $\mathbb{R}P^3$ [**13**]—namely, $\sigma(\mathbb{R}P^3) = 6(\omega_3/2)^{2/3}$. The smooth Yamabe invariant is also known for connected sums of copies of real projective space $\mathbb{R}P^3$ with copies of $S^2 \times S^1$ [**3**], for $\mathbb{C}P^2$ [**23**] and for connected sums of $\mathbb{C}P^2$ with several copies of $S^3 \times S^1$. With similar methods, it can also be determined for some related manifolds, but, for example, the value of $\sigma(S^2 \times S^2)$ is not known. To the knowledge of the authors there are no manifolds M of dimension $n \geq 5$ for which it has been shown that $0 < \sigma(M) < \sigma(S^n)$, but due to Schoen's conjecture finite quotients of spheres would be examples of such manifolds.

As explicit calculation of the Yamabe invariant is difficult, it is natural to use surgery theory to get estimates for more complicated examples. Several articles study the behavior of the smooth Yamabe invariant under surgery. In [**21**] and [**51**] it is proven that the existence of a positive scalar curvature metric is preserved under surgeries of codimension at least 3. In terms of the σ-invariant, this means that if N is obtained from a compact manifold M by surgery of codimension at least 3 and $\sigma(M) > 0$, then $\sigma(N) > 0$.

Later, Kobayashi proved in [**32**] that if N is obtained from M by 0-dimensional surgery, then $\sigma(N) \geq \sigma(M)$. A first consequence is an alternative deduction of $\sigma(S^{n-1} \times S^1) = \sigma(S^n)$ using the fact that $S^{n-1} \times S^1$ is obtained from S^n by 0-dimensional surgery. More generally, one sees that $\sigma(S^{n-1} \times S^1 \# \cdots \# S^{n-1} \times S^1) = \sigma(S^n)$ as this connected sum is obtained from S^n by 0-dimensional surgeries as well.

Note that it follows from what we said above that the smooth Yamabe invariant of disjoint unions $M = M_1 \amalg M_2$ satisfies

$$\sigma(M) = \min\{\sigma(M_1), \sigma(M_2)\}$$

if $\sigma(M_1) \geq 0$ or $\sigma(M_2) \geq 0$, and otherwise

$$\sigma(M) = -\left(|\sigma(M_1)|^{n/2} + |\sigma(M_2)|^{n/2}\right)^{2/n}.$$

Kobayashi's result then implies $\sigma(M_1 \# M_2) \geq \sigma(M_1 \amalg M_2)$, and thus yields a lower bound for $\sigma(M_1 \# M_2)$ in terms of $\sigma(M_1)$ and $\sigma(M_2)$.

A similar monotonicity formula for the σ-invariant was proved by Petean and Yun in [**45**]. They prove that $\sigma(N) \geq \min\{\sigma(M), 0\}$ if N is obtained from M by surgery of codimension at least 3. See also [**35**, Proposition 4.1] and [**1**] for other approaches to this result. Clearly, this surgery result is particularly interesting in the case $\sigma(M) \leq 0$, and it has several fruitful applications. In particular, any simply connected compact manifold of dimension at least 5 has $\sigma(M) \geq 0$; see [**44**]. This result has been generalized to manifolds with certain types of fundamental group in [**12**]. Further results in the same spirit for $n = 4$ can be found in [**55**].

1.3. Stronger version of the main result. In the present article we derive a surgery formula, which is stronger than the Gromov–Lawson resp. Schoen–Yau surgery formula, the Kobayashi surgery formula and the Petean-Yun surgery formula described above. Suppose that M_1 and M_2 are compact manifolds of dimension n and that W is a compact manifold of dimension k. Let embeddings $W \hookrightarrow M_1$ and $W \hookrightarrow M_2$ be given. We assume further that the normal bundles of these embeddings are trivial. Removing tubular neighborhoods of the images of W in M_1 and M_2, and gluing together these manifolds along their common boundary, we get a new compact manifold N, the connected sum of M_1 and M_2 along W. Strictly speaking, N also depends on the choice of trivialization of the normal bundle. See section 2 for more details.

Surgery is a special case of this construction: if $M_2 = S^n$, $W = S^k$, and if $S^k \hookrightarrow S^n$ is the standard embedding, then N is obtained from M_1 via k-dimensional surgery along $S^k \hookrightarrow M_1$.

Theorem 1.3. *Let M_1 and M_2 be compact manifolds of dimension n. If N is obtained as a connected sum of M_1 and M_2 along a k-dimensional submanifold where $k \leq n-3$, then*

$$\sigma(N) \geq \min\{\sigma(M_1 \amalg M_2), \Lambda_{n,k}\}$$

where $\Lambda_{n,k}$ is positive and only depends on n and k. Furthermore, $\Lambda_{n,0} = \sigma(S^n)$.

From Theorem 1.1 we know that $\sigma(M) \leq \sigma(S^n)$ and thus see that $\sigma(M \amalg S^n) = \sigma(M)$ for all compact M. Hence, we obtain for the special case of surgery the following corollary.

Corollary 1.4. *Let M be a compact manifold of dimension n. Assume that N is obtained from M via surgery along a k-dimensional sphere W, $k \leq n-3$. We then have*

$$\sigma(N) \geq \min\{\sigma(M), \Lambda_{n,k}\}.$$

The constants $\Lambda_{n,k}$ will be defined in Section 3. In Subsections 3.3 and 3.4 we prove that these constants are positive, and in Subsection 3.5 we prove that $\Lambda_{n,0} = \mu(\mathbb{S}^n)$. Explicit lower bounds for $\Lambda_{n,k}$ can be found for all $k \notin \{1, n-3\}$; however, these are not optimal (see Subsection 3.6). An explicit calculation of $\Lambda_{n,k}$ for $k > 0$ seems very difficult. The main problem consists in calculating the conformal Yamabe constant of certain Riemannian products, which in general is a hard problem. See [**2**, **5**] for recent progress on this problem.

1.4. Topological applications. The above surgery result can be combined with standard techniques of bordism theory. As these topological applications are not the main subject of this article, we will only sketch some typical conclusions as examples here.

The first corollary uses the fact that spin bordism groups and oriented bordism groups are finitely generated together with techniques developed for the proof of the h-cobordism theorem.

Corollary 1.5. *For any $n \geq 5$ there is a constant $C_n > 0$, depending only on n, such that*

$$\sigma(M) \in \{0\} \cup [C_n, \sigma(S^n)]$$

for any simply connected compact manifold M of dimension n.

We now sketch how interesting bordism invariants can be constructed using our main result. This construction will be explained here only for spin manifolds, but similar constructions can also be done for oriented, non-spin manifolds or for non-oriented manifolds.

Fix a finitely presented group Γ, and let $B\Gamma$ be the classifying space of Γ. We consider pairs (M, f) where M is a compact spin manifold and where $f : M \to B\Gamma$ is continuous. Two such pairs (M_1, f_1) and (M_2, f_2) are called spin bordant over $B\Gamma$ if there exists an $(n+1)$-dimensional spin manifold W with boundary $-M_1 \amalg M_2$ with a map $F : W \to B\Gamma$ such that the restriction of F to the boundary yields f_1 and f_2. It is implicitly required that the boundary carries the induced orientation and spin structure and $-M_1$ denotes M_1 with reversed orientation. Being spin bordant over $B\Gamma$ is an equivalence relation. The equivalence class of (M, f) under this equivalence relation is denoted by $[M, f]$, and the set of equivalence classes is called $\Omega_n^{\mathrm{Spin}}(B\Gamma)$. Disjoint union of manifolds defines a sum on $\Omega_n^{\mathrm{Spin}}(B\Gamma)$, which turns it into an abelian group.

We say that a pair (M, f) with $f : M \to B\Gamma$ is a *π_1-bijective representative of* $[M, f]$ if M is connected and if the induced map $f_* : \pi_1(M) \to \Gamma$ is a bijection. Any equivalence class in $\Omega_n^{\mathrm{Spin}}(B\Gamma)$ has a π_1-bijective representative.

Now we define

$$\Lambda_n := \min\{\Lambda_{n,1}, \dots \Lambda_{n,n-3}\} > 0,$$

$$\bar{\sigma}(M) := \min\{\sigma(M), \Lambda_n\}.$$

Proposition 1.6. *Let $n \geq 5$. Let (M_1, f_1) and (M_2, f_2) be compact spin manifolds with maps $f_i : M_i \to B\Gamma$. If (M_1, f_1) and (M_2, f_2) are spin bordant over $B\Gamma$ and if (M_2, f_2) is a π_1-bijective representative of its class, then*

$$\bar{\sigma}(M_1) \leq \bar{\sigma}(M_2).$$

We define $s_\Gamma : \Omega_n^{\mathrm{Spin}}(B\Gamma) \to \mathbb{R}$ by

$$s_\Gamma([M,f]) := \sup_{(M_1,f_1)\in[M,f]} \bar{\sigma}(M_1).$$

The proposition states $s_\Gamma([M,f]) = \bar{\sigma}(M)$ if (M,f) is a π_1-bijective representative of its class. The surgery formula further implies

$$s_\Gamma\Big([M_1,f_1]+[M_2,f_2]\Big) \geq \min\Big\{s_\Gamma([M_1,f_1]), s_\Gamma([M_2,f_2])\Big\}$$

if $s_\Gamma([M_1,f_1]) \geq 0$ or $s_\Gamma([M_2,f_2]) \geq 0$, and otherwise

$$s_\Gamma\Big([M_1,f_1]+[M_2,f_2]\Big) \geq -\Big(|s_\Gamma([M_1,f_1])|^{n/2} + |s_\Gamma([M_2,f_2])|^{n/2}\Big)^{2/n}.$$

We conclude, and obtain the following theorem.

Theorem 1.7. *Let $t \in \mathbb{R}$, $t \geq 0$, $n \in \mathbb{N}$, $n \geq 5$. Then the sets*

$$\Omega_n^{\mathrm{Spin}}(B\Gamma)^{>t} := \{[M,f] \in \Omega_n^{\mathrm{Spin}}(B\Gamma) \,|\, s_\Gamma([M,f]) > t\}$$

and

$$\Omega_n^{\mathrm{Spin}}(B\Gamma)^{\geq t} := \{[M,f] \in \Omega_n^{\mathrm{Spin}}(B\Gamma) \,|\, s_\Gamma([M,f]) \geq t\}$$

are subgroups of $\Omega_n^{\mathrm{Spin}}(B\Gamma)$.

The theorem admits—among other interesting conclusions—the following application. For a positive integer p we write $p\#M$ for the connected sum $M\#\cdots\#M$ where M appears p times. We already know that $\sigma(p\#M) \geq \sigma(M)$ if $\sigma(M) \geq 0$.

Corollary 1.8. *Suppose that M is a compact spin manifold of dimension at least 5 with $\sigma(M) \in (0,\Lambda_n)$. Let p and q be two relatively prime positive integers. If $\sigma(p\#M) > \sigma(M)$, then $\sigma(q\#M) = \sigma(M)$.*

If Schoen's conjecture about the σ-invariant of quotients of spheres holds true, then quotients of spheres by large fundamental groups yield examples of manifolds M with $\sigma(M) \in (0,\Lambda_n)$.

The determination of manifolds admitting positive scalar curvature metrics—that is, manifolds with $\sigma(M) > 0$—has led to interesting results and challenging problems in topology; see, for example, [**47**]. It would be interesting to develop similar topological tools for manifolds with $\sigma(M) > \epsilon$ for $\epsilon > 0$. As explained above, such manifolds form a subgroup on the bordism level.

The subgroups $\Omega_n^{\mathrm{Spin}}(B\Gamma)^{>t}$ also provide interesting algebraic structures. Any homomorphism $\Gamma_1 \to \Gamma_2$ provides a homomorphism

$$\Omega_n^{\mathrm{Spin}}(B\Gamma_1)^{>t} \to \Omega_n^{\mathrm{Spin}}(B\Gamma_2)^{>t}.$$

After introducing some factors and powers depending on the dimension, these subgroups carry an ideal-like structure. More precisely, it follows from [**5**] that for any numbers $t_3 > 0$ there is a sequence $t_n > 0$, $n \geq 3$, such that taking products of manifolds defines a $\mathbb{Z}$-bilinear map

$$\Omega^{\mathrm{Spin}}_{*\geq 3}(B\Gamma_1) \times \Omega^{\mathrm{Spin}}_{*\geq 3}(B\Gamma_2)^{>t_*} \to \Omega^{\mathrm{Spin}}_{*\geq 3}(B(\Gamma_1 \times \Gamma_2))^{>t_*}$$

where the index $* \geq 3$ indicates that we consider the spin bordism ring of manifolds whose dimension is at least 3. In particular, $\Omega^{\mathrm{Spin}}_{*\geq 3}(B\Gamma)^{>t_*}$ is a module over the ring $\Omega^{\mathrm{Spin}}_{*\geq 3} := \Omega^{\mathrm{Spin}}_{*\geq 3}(B\{1\})$ and $\Omega^{\mathrm{Spin}\,>t_*}_{*\geq 3}$ is an ideal in $\Omega^{\mathrm{Spin}}_{*\geq 3}$. Analogous structures exist for $\Omega^{\mathrm{Spin}}_n(B\Gamma)^{\geq t}$.

1.5. Comparison to other results. At the end of the section we want to mention some similar constructions in the literature. An analogous surgery formula holds if we replace the Conformal Laplacian by the Dirac operator; see [**4**] for details and applications. D. Joyce [**28**], followed by L. Mazzieri [**42**, **43**], considered a problem tightly related to our result: their goal is to construct a metric on a manifold obtained via a connected sum along a k-dimensional submanifold. For these metrics they construct a solution of the Yamabe equation on the new manifold that is close to solutions of the Yamabe equations on the original pieces. Such a construction was achieved by D. Joyce for $k = 0$ and by L. Mazzieri for $k \in \{1, \ldots, n-3\}$, provided that the embeddings defining the connected sum are isometric. In contrast to our article, their solutions on the new manifold are not necessarily minimizers of the volume-normalized Einstein–Hilbert functional. Similar constructions have also been developed by R. Mazzeo, D. Pollack, and K. Uhlenbeck [**41**] in order to glue together metrics of constant scalar curvature. Recently, J. Corvino, M. Eichmair, and P. Miao showed how to glue together metrics while preserving constant scalar curvature and volume; see [**16**]. Further, P. T. Chrusciel, J. Isenberg, and D. Pollack [**15**] found methods to glue together solutions of the vacuum Einstein constraint equations.

Other authors studied the equivariant analogues. In this setting one assumes that a compact Lie group G acts on the manifolds before and after surgery and that the surgery is compatible with this actions. Furthermore, all metrics are assumed to be G-invariant, and the Yamabe constant and Yamabe invariant are replaced by their equivariant analogues. The equivariant Yamabe problem is solved in many cases—in particular, on spin manifolds or in the case that all orbits have positive dimension; see [**27**, **39**, **40**]. An equivariant analogue of the Petean–Yun surgery formula was provided in [**54**]. B. Hanke [**24**] proved that the existence of G-invariant positive scalar curvature metrics is preserved under equivariant surgeries of the appropriate dimensions, which is the equivariant generalization of the result by Gromov and Lawson, respectively Schoen and Yau, cited above.

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2. The connected sum along a submanifold

In this section we are going to describe how two manifolds are joined along a common submanifold with trivialized normal bundle. Strictly speaking, this is a differential topological construction, but since we work with Riemannian manifolds we will make the construction adapted to the Riemannian metrics and use distance neighborhoods defined by the metrics etc.

Let (M_1, g_1) and (M_2, g_2) be complete Riemannian manifolds of dimension n. Let W be a compact manifold of dimension k, where $0 \leq k \leq n$. Let $\bar{w}_i : W \times \mathbb{R}^{n-k} \to TM_i$, $i = 1, 2$, be smooth embeddings. We assume that $\bar{w}_i$ restricted to $W \times \{0\}$ maps to the zero section of TM_i (which we identify with M_i) and thus gives an embedding $W \to M_i$. The image of this embedding is denoted by W_i'. Further, we assume that $\bar{w}_i$ restrict to linear isomorphisms $\{p\} \times \mathbb{R}^{n-k} \to N_{\bar{w}_i(p,0)} W_i'$ for all $p \in W_i$, where NW_i' denotes the normal bundle of W_i' defined using g_i.

By setting $w_i := \exp^{g_i} \circ \bar{w}_i$ we obtain the embeddings $w_i : W \times B^{n-k}(R_{\max}) \to M_i$ for some $R_{\max} > 0$ and $i = 1, 2$. We have $W_i' = w_i(W \times \{0\})$ and we define the disjoint union

$$(M, g) := (M_1 \amalg M_2, g_1 \amalg g_2),$$

and

$$W' := W_1' \amalg W_2'.$$

Let r_i be the function on M_i giving the distance to W_i'. Then $r_1 \circ w_1(p, x) = r_2 \circ w_2(p, x) = |x|$ for $p \in W$, $x \in B^{n-k}(R_{\max})$. Let r be the function on M defined by $r(x) := r_i(x)$ for $x \in M_i$, $i = 1, 2$. For $0 < \epsilon$ we set $U_i(\epsilon) := \{x \in M_i : r_i(x) < \epsilon\}$ and $U(\epsilon) := U_1(\epsilon) \cup U_2(\epsilon)$. For $0 < \epsilon < \theta$ we define

$$N_\epsilon := (M_1 \setminus U_1(\epsilon)) \cup (M_2 \setminus U_2(\epsilon))/\sim$$

and

$$U_\epsilon^N(\theta) := (U(\theta) \setminus U(\epsilon))/\sim$$

where $\sim$ indicates that we identify the point $x \in \partial U_1(\epsilon)$ in M_1 with the corresponding point $w_2 \circ w_1^{-1}(x) \in \partial U_2(\epsilon)$ in M_2. Hence,

$$N_\epsilon = (M \setminus U(\theta)) \cup U_\epsilon^N(\theta).$$

We say that N_ϵ is obtained from M_1, M_2 (and $\bar{w}_1$, $\bar{w}_2$) by a connected sum along W with parameter ϵ.

The diffeomorphism type of N_ϵ is independent of ϵ, and hence we will usually write $N = N_\epsilon$. However, in situations when dropping the index causes ambiguities, we will keep the notation N_ϵ. For example, the function $r : M \to [0,\infty)$ gives a continuous function $r_\epsilon : N_\epsilon \to [\epsilon,\infty)$ whose domain depends on ϵ. It is also going to be important to keep track of the subscript ϵ on $U_\epsilon^N(\theta)$ since crucial estimates on solutions of the Yamabe equation will be carried out on this set.

The surgery operation on a manifold is a special case of taking connected sum along a submanifold. Indeed, let M be a compact manifold of dimension n, and let $M_1 = M$, $M_2 = S^n$, $W = S^k$. Let $w_1 : S^k \times B^{n-k} \to M$ be an embedding defining a surgery, and let $w_2 : S^k \times B^{n-k} \to S^n$ be the standard embedding. Since $S^n \setminus w_2(S^k \times B^{n-k})$ is diffeomorphic to $B^{k+1} \times S^{n-k-1}$, we have in this situation that N is obtained from M by performing surgery on w_1; see [**33**, Section VI, 9].

3. The constants $\Lambda_{n,k}$

In Section 1.2 we defined the conformal Yamabe constant only for compact manifolds. There are several ways to generalize the conformal Yamabe constant to non-compact manifolds. In this section we define two such generalizations $\mu^{(0)}$ and $\mu^{(1)}$, and also introduce a related quantity called $\mu^{(2)}$. These invariants will be needed to define the constants $\Lambda_{n,k}$ and to prove their positivity on our model spaces $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$.

The definition of $\mu^{(2)}$ comes from a technical difficulty in the proof of Theorem 6.1 and is only relevant in the case $k = n-3 \geq 3$; see Remark 3.4.

3.1. The manifolds $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$. For $0 \leq k < n$ and $c \in \mathbb{R}$, we define the metric $\eta_c^{k+1} := e^{2ct}\xi^k + dt^2$ on $\mathbb{R}^k \times \mathbb{R}$ and write

$$\mathbb{H}_c^{k+1} := (\mathbb{R}^k \times \mathbb{R}, \eta_c^{k+1}).$$

This is a model of the simply connected complete manifold of constant curvature $-c^2$. We denote by

$$G_c := \eta_c^{k+1} + \sigma^{n-k-1}$$

the product metric on $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$. The scalar curvature of $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ is $\mathrm{Scal}^{G_c} = -k(k+1)c^2 + (n-k-1)(n-k-2)$.

Proposition 3.1. *$\mathbb{H}_1^{k+1} \times \mathbb{S}^{n-k-1}$ is conformal to $\mathbb{S}^n \setminus \mathbb{S}^k$.*

Proof. Let $\mathbb{S}^k$ be embedded in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ by setting the last $n-k$ coordinates to zero, and let $s := d(\cdot, \mathbb{S}^k)$ be the intrinsic distance to $\mathbb{S}^k$ in $\mathbb{S}^n$. Then the function $\sin s$ is smooth and positive on $S^n \setminus S^k$. The points of maximal distance $\pi/2$ to $\mathbb{S}^k$ lie on an $(n-k-1)$-sphere, denoted by $(\mathbb{S}^k)^\perp$. On $\mathbb{S}^n \setminus (\mathbb{S}^k \cup (\mathbb{S}^k)^\perp)$ the round metric is

$$\sigma^n = (\cos s)^2 \sigma^k + ds^2 + (\sin s)^2 \sigma^{n-k-1}.$$

Substitute $s \in (0, \pi/2)$ by $t \in (0, \infty)$ such that $\sinh t = \cot s$. Then $\cosh t = (\sin s)^{-1}$ and $\cosh t \, dt = -(\sin s)^{-2} \, ds$, so σ^n is conformal to

$$(\sin s)^{-2} \sigma^n = (\sinh t)^2 \sigma^k + dt^2 + \sigma^{n-k-1}.$$

Here we see that the first two terms give a metric

$$(\sinh t)^2 \sigma^k + dt^2$$

on $S^k \times (0, \infty)$. This is just the standard metric on $\mathbb{H}_1^{k+1} \setminus \{p_0\}$ where $t = d(\cdot, p_0)$, written in polar normal coordinates. In the case $k \geq 1$, it is evident that the conformal diffeomorphism $\mathbb{S}^n \setminus (\mathbb{S}^k \cup (\mathbb{S}^k)^\perp) \to (\mathbb{H}_1^{k+1} \setminus \{p_0\}) \times \mathbb{S}^{n-k-1}$ extends to a conformal diffeomorphism $\mathbb{S}^n \setminus \mathbb{S}^k \to \mathbb{H}_1^{k+1} \times \mathbb{S}^{n-k-1}$.

In the case $k = 0$, we equip s and t with a sign; that is, we let $s > 0$ and $t > 0$ on one of the components of $\mathbb{S}^n \setminus (\mathbb{S}^0 \cup (\mathbb{S}^0)^\perp)$, and $s < 0$ and $t < 0$ on the other component. The functions s and t are then smooth on $\mathbb{S}^n \setminus \mathbb{S}^0$ and take values $s \in (-\pi/2, \pi/2)$ and $t \in \mathbb{R}$. Then the argument is the same as above. q.e.d.

3.2. Definition of $\Lambda_{n,k}$. Let (N, h) be a Riemannian manifold of dimension n. For $i = 1, 2$ we let $\Omega^{(i)}(N, h)$ be the set of non-negative C^2 functions u that solve the Yamabe equation

$$L^h u = \mu u^{p-1} \tag{6}$$

for some $\mu = \mu(u) \in \mathbb{R}$ and satisfy

- $u \not\equiv 0$,
- $\|u\|_{L^p(N)} \leq 1$,
- $u \in L^\infty(N)$,

together with

- $u \in L^2(N)$, for $i = 1$,

or

- $\mu(u) \|u\|_{L^\infty(N)}^{p-2} \geq \frac{(n-k-2)^2(n-1)}{8(n-2)}$, for $i = 2$.

For $i = 1, 2$ we set

$$\mu^{(i)}(N, h) := \inf_{u \in \Omega^{(i)}(N,h)} \mu(u).$$

In particular, if $\Omega^{(i)}(N, h)$ is empty then $\mu^{(i)}(N, h) = \infty$.

Definition 3.2. For integers $n \geq 3$ and $0 \leq k \leq n-2$, let

$$\Lambda_{n,k}^{(i)} := \inf_{c \in [-1,1]} \mu^{(i)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$$

and

$$\Lambda_{n,k} := \min \left\{ \Lambda^{(1)}_{n,k}, \Lambda^{(2)}_{n,k} \right\}.$$

Note that the infimum could just as well be taken over $c \in [0,1]$ since $\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1}$ and $\mathbb{H}^{k+1}_{-c} \times \mathbb{S}^{n-k-1}$ are isometric. We are going to prove that these constants are positive.

Theorem 3.3. *For all $n \geq 3$ and $0 \leq k \leq n-3$, we have $\Lambda_{n,k} > 0$.*

The condition $k \leq n-3$ is important, as this implies that $\mathbb{S}^{n-k-1}$ has positive curvature.

To prove Theorem 3.3 we have to prove that $\Lambda^{(1)}_{n,k} > 0$ and that $\Lambda^{(2)}_{n,k} > 0$. This is the object of the following two subsections. In the final subsection we prove that $\Lambda_{n,0} = \mu(\mathbb{S}^n) = n(n-1)\omega_n^{2/n}$.

Remark 3.4. Suppose that either $k \leq n-4$ or $k = n-3 \leq 2$. One can then use methods similar to those used in Section 5 to show that any L^p-solution of (6) on the model spaces is also an L^2-solution; see [**6**]. An analogous argument also works in the case $(n,k) = (6,3)$, for model spaces with $c < 1$, and this allows similar conclusions; see [**7**]. This implies that $\Lambda^{(2)}_{n,k} \geq \Lambda^{(1)}_{n,k}$ if $k \leq n-4$ or $k = n-3 \leq 3$, and hence

$$\Lambda_{n,k} = \Lambda^{(1)}_{n,k}.$$

In the case $k = n-3 \geq 4$, there are L^p-solutions of (6) on $\mathbb{H}^{k+1}_1 \times \mathbb{S}^{n-k-1}$ that are not L^2-solutions.

3.3. Proof of $\Lambda^{(1)}_{n,k} > 0$. The proof proceeds in several steps. We first introduce a conformal Yamabe constant for non-compact manifolds and show that it gives a lower bound for $\mu^{(1)}$. We then conclude by studying this conformal invariant.

Let (N,h) be a Riemannian manifold that is not necessarily compact or complete. We define the conformal Yamabe constant $\mu^{(0)}$ of (N,h) following Schoen–Yau [**52**, Section 2]—see also [**29**]—as

$$\mu^{(0)}(N,h) := \inf J^h(u)$$

where J^h is defined in (1) and the infimum runs over the set of all non-zero compactly supported smooth functions u on N. If h and $\tilde{h}$ are conformal metrics on N, it follows from (3) that $\mu^{(0)}(N,h) = \mu^{(0)}(N,\tilde{h})$.

Lemma 3.5. *Let $0 \leq k \leq n-2$. Then*

$$\mu^{(1)}(\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1}) \geq \mu^{(0)}(\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1})$$

for all $c \in \mathbb{R}$.

Proof. Suppose that $u \in \Omega^{(1)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$ is a solution of (6) on $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ with $\mu = \mu(u) \in [\mu^{(1)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}), \mu^{(1)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) + \epsilon]$. Let χ_α be a cut-off function on $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ depending only on the distance r to a fixed point, such that $\chi_\alpha(r) = 1$ for $r \leq \alpha$, $\chi_\alpha(r) = 0$ for $r \geq \alpha + 2$, and $|d\chi_\alpha| \leq 1$. We are going to see that

$$\begin{aligned} \mu^{(0)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) &\leq \lim_{\alpha\to\infty} J^{G_c}(\chi_\alpha u) \\ &= \mu \|u\|^{p-2}_{L^p(\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1})} \\ &\leq \mu \\ &\leq \mu^{(1)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) + \epsilon. \end{aligned} \tag{7}$$

Integrating by parts and using equations (6) and (65), we get

$$\begin{aligned} \int_{\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1}} (\chi_\alpha u) L^{G_c}(\chi_\alpha u)\, dv^{G_c} &= \int_{\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1}} \chi_\alpha^2 u L^{G_c} u \, dv^{G_c} \\ &\quad + a \int_{\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1}} |d\chi_\alpha|^2 u^2 \, dv^{G_c} \\ &= \mu \int_{\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1}} \chi_\alpha^2 u^p \, dv^{G_c} \\ &\quad + a \int_{\mathrm{Supp}(d\chi_\alpha)} |d\chi_\alpha|^2 u^2 \, dv^{G_c}. \end{aligned}$$

Since $u \in L^2(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$ and $|d\chi_\alpha| \leq 1$, the last integral goes to zero as $\alpha \to \infty$ and we conclude that

$$\lim_{\alpha\to\infty} \int_{\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1}} (\chi_\alpha u) L^{G_c}(\chi_\alpha u)\, dv^{G_c} = \mu \|u\|^p_{L^p(\mathbb{H}_c^{k+1}\times\mathbb{S}^{n-k-1})}.$$

Going back to the definition of J^{G_c}, we easily get (7), and Lemma 3.5 follows. q.e.d.

Remark 3.6. It follows from [**22**, Theorem 13] and a straight-forward cut-off argument that

$$\mu^{(1)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) = \mu^{(0)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$$

if the space $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ has positive scalar curvature—that is, if we have $(n-k-1)(n-k-2) > c^2 k(k+1)$.

We define

$$\Lambda^{(0)}_{n,k} := \inf_{c\in[-1,1]} \mu^{(0)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}).$$

Then Lemma 3.5 tells us that $\Lambda^{(1)}_{n,k} \geq \Lambda^{(0)}_{n,k}$, so we are done if we prove that $\Lambda^{(0)}_{n,k} > 0$. To do this we need two lemmas.

Lemma 3.7. *Let $0 \leq k \leq n-2$. Then*

$$\mu^{(0)}(\mathbb{H}_1^{k+1} \times \mathbb{S}^{n-k-1}) = \mu(\mathbb{S}^n).$$

Proof. The inequality $\mu^{(0)}(\mathbb{H}_1^{k+1} \times \mathbb{S}^{n-k-1}) \leq \mu(\mathbb{S}^n)$ is completely analogous to [**10**, Lemma 3]. As we do not need this inequality later, we skip the proof. To prove the opposite inequality $\mu^{(0)}(\mathbb{H}_1^{k+1} \times \mathbb{S}^{n-k-1}) \geq \mu(\mathbb{S}^n)$, we use Proposition 3.1 and the conformal invariance of $\mu^{(0)}$, and we obtain

$$\mu^{(0)}(\mathbb{H}_1^{k+1} \times \mathbb{S}^{n-k-1}) = \mu^{(0)}(\mathbb{S}^n \setminus \mathbb{S}^k).$$

Clearly $\mu^{(0)}(\mathbb{S}^n \setminus \mathbb{S}^k) \geq \mu(\mathbb{S}^n)$ as the infimum defining the left-hand side runs over a smaller set of functions; see [**52**, Lemma 2.1]. q.e.d.

Lemma 3.8. *Let $0 \leq k \leq n-2$ and $0 < c_0 \leq c_1$. Then*

$$\mu^{(0)}(\mathbb{H}_{c_0}^{k+1} \times \mathbb{S}^{n-k-1}) \geq \left(\frac{c_0}{c_1}\right)^{\frac{2(n-k-1)}{n}} \mu^{(0)}(\mathbb{H}_{c_1}^{k+1} \times \mathbb{S}^{n-k-1}).$$

Proof. Let $c > 0$. Setting $s = ct + \ln c$, we see that

$$G_c = e^{2ct}\xi^k + dt^2 + \sigma^{n-k-1} = \frac{1}{c^2}\left(e^{2s}\xi^k + ds^2\right) + \sigma^{n-k-1}.$$

Hence G_c is conformal to the metric

$$\tilde{G}_c := e^{2s}\xi^k + ds^2 + c^2\sigma^{n-k-1},$$

and by the conformal invariance of $\mu^{(0)}$ we get that

$$\mu^{(0)}(\mathbb{H}_{c_i}^{k+1} \times \mathbb{S}^{n-k-1}) = \mu^{(0)}(\mathbb{R}^k \times \mathbb{R} \times S^{n-k-1}, \tilde{G}_{c_i})$$

for $i = 0, 1$. In these coordinates we easily compute that $\mathrm{Scal}^{\tilde{G}_{c_0}} \geq \mathrm{Scal}^{\tilde{G}_{c_1}}$, $|du|^2_{\tilde{G}_{c_0}} \geq |du|^2_{\tilde{G}_{c_1}}$, and $dv^{\tilde{G}_{c_0}} = \left(\frac{c_0}{c_1}\right)^{n-k-1} dv^{\tilde{G}_{c_1}}$. We conclude that

$$J^{\tilde{G}_{c_0}}(u) \geq \left(\frac{c_0}{c_1}\right)^{\frac{2(n-k-1)}{n}} J^{\tilde{G}_{c_1}}(u)$$

for all functions u on $\mathbb{R}^k \times \mathbb{R} \times S^{n-k-1}$, and Lemma 3.8 follows. q.e.d.

If we set $c_1 = 1$ and use Lemma 3.7 together with (4), we get the following result.

Corollary 3.9. *For $0 \leq k \leq n-2$ and $c_0 > 0$, we have*

$$\inf_{c \in [c_0, 1]} \mu^{(0)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) \geq n(n-1)\,\omega_n{}^{2/n} c_0{}^{4/n}.$$

Finally, we are ready to prove that $\Lambda_{n,k}^{(0)}$ is positive.

Theorem 3.10. *Let $0 \leq k \leq n-3$. Then $\Lambda_{n,k}^{(0)} > 0$.*

For this theorem the restriction $k \leq n-3$ is necessary. The proof needs the positive scalar curvature of $\mathbb{S}^{n-k-1}$, and it can be shown that the theorem no longer holds for $k = n-2$.

Proof. Choose $c_0 > 0$ small enough so that $\mathrm{Scal}^{G_{c_0}} > 0$. We then have $\mathrm{Scal}^{G_c} \geq \mathrm{Scal}^{G_{c_0}}$ for all $c \in [0, c_0]$. Hence,

$$\mu^{(0)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) \geq \inf \frac{\int_{\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}} \left(a|du|_{G_c}^2 + \mathrm{Scal}^{G_{c_0}} u^2\right) \, dv^{G_c}}{\|u\|^2_{L^p(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})}}$$

where the infimum is taken over all non-zero smooth functions u with compact support. By Hebey [**25**, Theorem 4.6, page 64], there exists a constant $A > 0$ such that for all $c \in [0, c_0]$ and all smooth non-zero functions u compactly supported in $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ we have

$$\|u\|^2_{L^p(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})} \leq A \int_{\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}} \left(|du|_{G_c}^2 + u^2\right) \, dv^{G_c}.$$

This implies that

$$\mu^{(0)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) \geq \frac{1}{A} \min\left\{a, \mathrm{Scal}^{G_{c_0}}\right\} > 0$$

for all $c \in [0, c_0]$, and together with Lemma 3.8 we obtain that

$$\inf_{c \in [0,1]} \mu^{(0)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) > 0.$$

Since $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ and $\mathbb{H}_{-c}^{k+1} \times \mathbb{S}^{n-k-1}$ are isometric, we have

$$\Lambda_{n,k}^{(0)} = \inf_{c \in [-1,1]} \mu^{(0)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}) > 0.$$

This ends the proof of Theorem 3.10. q.e.d.

As an immediate consequence, we obtain that $\Lambda_{n,k}^{(1)}$ is positive.

Corollary 3.11. *Let $0 \leq k \leq n-3$. Then $\Lambda_{n,k}^{(1)} > 0$.*

3.4. Proof of $\Lambda_{n,k}^{(2)} > 0$.

Theorem 3.12. *Let $0 \leq k \leq n-3$. Then $\Lambda_{n,k}^{(2)} > 0$.*

Proof. We prove this by contradiction. Assume that there exists a sequence (c_i) of $c_i \in [-1, 1]$ for which $\mu_i := \mu^{(2)}(\mathbb{H}_{c_i}^{k+1} \times \mathbb{S}^{n-k-1})$ tends to a limit $l \leq 0$ as $i \to \infty$. After removing the indices i for which μ_i is infinite we get for every i a positive solution $u_i \in \Omega^2(\mathbb{H}_{c_i}^{k+1} \times \mathbb{S}^{n-k-1})$ of the equation

$$L^{G_{c_i}} u_i = \mu_i u_i^{p-1}.$$

By definition of $\Omega^{(2)}(\mathbb{H}_{c_i}^{k+1} \times \mathbb{S}^{n-k-1})$, we have

$$\frac{(n-k-2)^2(n-1)}{8(n-2)} \leq \mu_i \|u_i\|_{L^\infty}^{p-2}, \tag{8}$$

which implies that $\mu_i > 0$. We conclude that $l := \lim_i \mu_i = 0$. We cannot assume that $\|u_i\|_{L^\infty}$ is attained, but we can choose points $x_i \in \mathbb{H}_{c_i}^{k+1} \times \mathbb{S}^{n-k-1}$ such that $u_i(x_i) \geq \frac{1}{2}\|u_i\|_{L^\infty}$. Moreover, we can compose the functions u_i with isometries so that all the x_i are the same point x. From (8) we get

$$\frac{1}{2}\left(\frac{(n-k-2)^2(n-1)}{8(n-2)\mu_i}\right)^{\frac{1}{p-2}} \leq u_i(x).$$

We define $m_i := u_i(x)$. Since $\lim_{i\to\infty}\mu_i = 0$, we have $\lim_{i\to\infty} m_i = \infty$. Restricting to a subsequence we can assume that $c := \lim_i c_i \in [-1,1]$ exists. Define $\tilde{g}_i := m_i^{\frac{4}{n-2}} G_{c_i}$. We apply Lemma 4.1 with $\alpha = 1/i$, $(V,\gamma_\alpha) = \mathbb{H}^{k+1}_{c_i}\times\mathbb{S}^{n-k-1}$, $(V,\gamma_0) = \mathbb{H}^{k+1}_{c}\times\mathbb{S}^{n-k-1}$, $q_\alpha = x_i = x$, and $b_\alpha = m_i^{\frac{2}{n-2}}$. For $r>0$ we obtain diffeomorphisms

$$\Theta_i : B^n(r) \to B^{G_{c_i}}(x, m_i^{-\frac{2}{n-2}} r)$$

such that the sequence $\Theta_i^*(\tilde{g}_i)$ tends to the flat metric ξ^n on $B^n(r)$. We let $\tilde{u}_i := m_i^{-1}u_i$. By (2) we then have

$$L^{\tilde{g}_i}\tilde{u}_i = \mu_i \tilde{u}_i^{p-1}$$

on $B^{G_{c_i}}(x_i, m_i^{-\frac{2}{n-2}} r)$ and

$$\begin{aligned}\int_{B^{G_{c_i}}(x_i, m_i^{-\frac{2}{n-2}} r)} \tilde{u}_i^p \, dv^{\tilde{g}_i} &= \int_{B^{G_{c_i}}(x_i, m_i^{-\frac{2}{n-2}} r)} u_i^p \, dv^{G_{c_i}} \\ &\leq \int_N u_i^p dv^{G_{c_i}} \\ &\leq 1.\end{aligned}$$

Here we used $dv^{\tilde{g}_i} = m_i^p \, dv^{G_{c_i}}$. The last inequality comes from the fact that any function in $\Omega^{(2)}(\mathbb{H}^{k+1}_{c_i}\times\mathbb{S}^{n-k-1})$ has L^p-norm smaller than 1. Since

$$\Theta_i : (B^n(r), \Theta_i^*(\tilde{g}_i)) \to (B^{G_{c_i}}(x, m_i^{-\frac{2}{n-2}} r), \tilde{g}_i)$$

is an isometry, we redefine $\tilde{u}_i$ as $\tilde{u}_i\circ\Theta_i$, which gives us solutions of

$$L^{\Theta_i^*(\tilde{g}_i)}\tilde{u}_i = \mu_i\tilde{u}_i^{p-1}$$

on $B^n(r)$ with $\int_{B^n(r)} \tilde{u}_i^p \, dv^{\Theta_i^*(\tilde{g}_i)} \leq 1$. Since $\|\tilde{u}_i\|_{L^\infty(B^n(r))} = \tilde{u}_i(0) = 1$, we can apply Lemma 4.2 with $V = \mathbb{R}^n$, $\alpha = 1/i$, $g_\alpha = \Theta_i^*(\tilde{g}_i)$, and $u_\alpha = \tilde{u}_i$ (we can apply this lemma since each compact set of $\mathbb{R}^n$ is contained in some ball $B^n(r)$). This shows that there exists a non-negative C^2 function u on $\mathbb{R}^n$ that does not vanish identically (since $u(0)=1$) and that satisfies

$$L^{\xi^n} u = a\Delta^{\xi^n} u = \bar{\mu} u^{p-1}$$

where $\bar{\mu} = 0$. By (12) we further have

$$\int_{B^n(r)} u^p \, dv^{\xi^n} = \lim_{i\to\infty}\int_{B^{G_{c_i}}(x, m_i^{-\frac{2}{n-2}} r)} u_i^p \, dv^{G_{c_i}} \leq 1$$

for any $r > 0$. In particular,

$$\int_{\mathbb{R}^n} u^p \, dv^{\xi^n} \leq 1.$$

Lemma 4.3 below then implies the contradiction $0 = \bar{\mu} \geq \mu(\mathbb{S}^n)$. This proves that $\Lambda^{(2)}_{n,k}$ is positive. q.e.d.

3.5. The constants $\Lambda_{n,0}$. Now we show that

$$\Lambda_{n,0} = \mu(\mathbb{S}^n) = n(n-1)\omega_n^{2/n}.$$

The corresponding model spaces $\mathbb{H}^1_c \times \mathbb{S}^{n-1}$ carry the standard product metric $dt^2 + \sigma^{n-1}$ of $\mathbb{R} \times \mathbb{S}^{n-1}$, independently of $c \in [-1, 1]$. Thus $\Lambda^{(i)}_{n,0} = \mu^{(i)}(\mathbb{R} \times \mathbb{S}^{n-1})$. Proposition 3.1 yields a conformal diffeomorphism from the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ to $\mathbb{S}^n \setminus \mathbb{S}^0$, the n-sphere with north and south poles removed.

Lemma 3.13.

$$\Lambda^{(i)}_{n,0} \leq \mu(\mathbb{S}^n) = n(n-1)\omega_n^{2/n}$$

for $i = 1, 2$.

Proof. We use the notation of Proposition 3.1 with $k = 0$. Then the standard metric on S^n is

$$\sigma^n = (\sin s)^2(dt^2 + \sigma^{n-1}) = (\cosh t)^{-2}(dt^2 + \sigma^{n-1}).$$

It follows that $(\omega_n)^{-2/n}(\cosh t)^{-2}(dt^2 + \sigma^{n-1})$ is a (non-complete) metric of volume 1 and scalar curvature $n(n-1)\omega^{2/n} = \mu(\mathbb{S}^n)$ on $\mathbb{H}^1_c \times \mathbb{S}^{n-1} = \mathbb{R} \times \mathbb{S}^{n-1}$. This is equivalent to saying that

$$u(t) := \omega_n^{-\frac{n-2}{2n}} (\cosh t)^{-\frac{n-2}{2}}$$

is a solution of (6) with $\mu = \mu(\mathbb{S}^n)$ and $\|u\|_{L^p} = 1$ on $\mathbb{H}^1_c \times \mathbb{S}^{n-1}$ equipped with the product metric. Clearly we have $u \in L^2$, and $\|u\|_{L^\infty} = \omega_n^{-\frac{n-2}{2n}} < \infty$. Thus $u \in \Omega^{(1)}(\mathbb{H}^1_c \times \mathbb{S}^{n-1})$. As a consequence, we obtain $\Lambda^{(1)}_{n,0} \leq n(n-1)\omega_n^{2/n}$.

Further, we have

$$\mu(\mathbb{S}^n)\|u\|^{p-2}_{L^\infty} = n(n-1) > \frac{(n-0-2)^2(n-1)}{8(n-2)},$$

and thus $u \in \Omega^{(2)}(\mathbb{H}^1_c \times \mathbb{S}^{n-1})$, which implies $\Lambda^{(2)}_{n,0} \leq n(n-1)\omega_n^{2/n}$. q.e.d.

Lemma 3.14. *Let $u \in C^2(\mathbb{R} \times \mathbb{S}^{n-1})$ be a solution of* (6) *on $\mathbb{R} \times \mathbb{S}^{n-1}$ with $\|u\|_{L^p} \leq 1$, $u \not\equiv 0$. Then $\mu \geq \mu(\mathbb{S}^n)$.*

Proof. As above $\sigma^n = (\sin s)^2(dt^2 + \sigma^{n-1})$. If u solves (6) with $h = dt^2 + \sigma^{n-1}$, then $\tilde{u} := (\sin s)^{-\frac{n-2}{2}} u$ solves

$$L^{\sigma^n} \tilde{u} = \mu \tilde{u}^{p-1}.$$

Further, $\tilde{u}^p \, dv^{\sigma^n} = u^p \, dv^h$; hence, $\nu := \|\tilde{u}\|_{L^p(S^n \setminus S^0, \sigma^n)} \leq 1$. For $\alpha > 0$ small, we choose a smooth cut-off function $\chi_\alpha : S^n \to [0,1]$ which is 1 on $S^n \setminus U_\alpha(S^0)$, with support disjoint from S^0, and with $|d\chi_\alpha|_{\sigma^n} \leq 2/\alpha$. Then using (65) in Appendix A.3, we see that

$$\int_{\mathbb{S}^n} (\chi_\alpha \tilde{u}) L^{\sigma^n} (\chi_\alpha \tilde{u}) \, dv^{\sigma^n} = \mu \int_{\mathbb{S}^n} u^p \chi_\alpha^2 \, dv^{\sigma^n} + a \int_{\mathbb{S}^n} |d\chi_\alpha|_{\sigma^n}^2 \tilde{u}^2 \, dv^{\sigma^n}.$$

The first summand tends to $\mu\nu^p$ as $\alpha \searrow 0$. By Hölder's inequality the second summand is bounded by

$$\frac{4a}{\alpha^2} \|\tilde{u}\|^2_{L^p(U_\alpha(S^0) \setminus S^0, \sigma^n)} \mathrm{Vol}(U_\alpha(S^0) \setminus S^0, \sigma^n)^{2/n} \leq C \|\tilde{u}\|^2_{L^p(U_\alpha(S^0) \setminus S^0, \sigma^n)} \to 0$$

as $\alpha \searrow 0$. Together with $\lim_{\alpha \searrow 0} \|\chi_\alpha \tilde{u}\|_{L^p(S^n \setminus S^0, \sigma^n)} = \nu$, we obtain

$$\mu(\mathbb{S}^n) \leq J^{\sigma^n}(\chi_\alpha \tilde{u}) \to \mu \nu^{p-2} \leq \mu$$

as $\alpha \searrow 0$. q.e.d.

This lemma obviously implies $\Lambda^{(i)}_{n,0} \geq \mu(\mathbb{S}^n)$ for $i = 1, 2$, and thus we have

$$\Lambda_{n,0} = \Lambda^{(1)}_{n,0} = \Lambda^{(2)}_{n,0} = \mu(\mathbb{S}^n).$$

3.6. The constants $\Lambda_{n,k}$ for $1 \leq k \leq n-3$. For $2 \leq k \leq n-4$, we have found an explicit positive lower bound on $\Lambda^{(0)}_{n,k}$, which will be published in [**5**]. Together with Remark 3.4 we obtain a lower bound for $\Lambda_{n,k}$; see also [**6**]. For $m := k+1 \in \{3, \dots, n-3\}$ we conclude

$$\Lambda_{n,m-1} \geq n \, a_n \left(\frac{Y_m}{m a_m} \right)^{\frac{m}{n}} \left(\frac{Y_{n-m}}{(n-m) a_{n-m}} \right)^{\frac{n-m}{n}}.$$

A lower bound in the case $k = 1$ and in the cases $(n,k) = (5,2)$ was established in [**7**]. These lower bounds are not optimal, but they are optimal up to a factor of at most 2.

We collected all known and conjectured values for $\Lambda_{n,k}$ for $n \leq 9$ in Figure 1. In the table, > 0 means that no explicit positive lower estimate has been worked out until now.

3.7. Speculation about $\Lambda_{n,k}$ for $k \geq 1$. We want to speculate about two relations that seem likely to us although we have no proof. Conformally, the model spaces $\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1}$ can be viewed as an interpolation between $\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1}$ (for $c = 0$) and the sphere $\mathbb{S}^n$ (for $c = 1$). Since the sphere has the largest possible value of the conformal Yamabe constant, we could hope that the function $c \mapsto \mu^{(0)}(\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1})$ is increasing for $c \in [0,1]$, or, in particular,

$$\mu^{(0)}(\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1}) \leq \mu^{(0)}(\mathbb{H}^{k+1}_c \times \mathbb{S}^{n-k-1})$$

n	k	Known $\Lambda_{n,k}$	Conjectured $\Lambda_{n,k}$	$\mu(\mathbb{S}^n)$
3	0	43.82323	43.82323	43.82323
4	0	61.56239	61.56239	61.56239
4	1	38.9	59.40481	61.56239
5	0	78.99686	78.99686	78.99686
5	1	56.6	78.18644	78.99686
5	2	45.1	75.39687	78.99686
6	0	96.29728	96.29728	96.29728
6	1	> 0	95.87367	96.29728
6	2	54.77904	94.71444	96.29728
6	3	49.98764	91.68339	96.29728
7	0	113.5272	113.5272	113.5272
7	1	> 0	113.2670	113.5272
7	2	74.50435	112.6214	113.5272
7	3	74.50435	111.2934	113.5272
7	4	> 0	108.1625	113.5272
8	0	130.7157	130.7157	130.7157
8	1	> 0	130.5398	130.7157
8	2	92.24278	130.1272	130.7157
8	3	95.76372	129.3551	130.7157
8	4	92.24278	127.9414	130.7157
8	5	> 0	124.7747	130.7157
9	0	147.8778	147.8778	147.8778
9	1	109.2993	147.7507	147.8778
9	2	109.4260	147.4615	147.8778
9	3	114.3250	146.9519	147.8778
9	4	114.3250	146.1089	147.8778
9	5	109.4260	144.6521	147.8778
9	6	> 0	141.4740	147.8778

Figure 1. Known and conjectured lower estimates for $\Lambda_{n,k}$.

for all $c \in [-1, 1]$. This would imply

$$\Lambda_{n,k} = \mu^{(0)}(\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1}).$$

To formulate the second potential relation, we define the following variant of $\mu^{(0)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1})$:

$$\mu^{(0)}_{\mathbb{H}_c^{k+1}}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k}) := \inf\{J^{G_c}(u) \,|\, u \in C_0^\infty(\mathbb{H}_c^{k+1})\}.$$

Here J^{G_c} is the functional of $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$, but we only evaluate it for functions that are constant along the sphere $\mathbb{S}^{n-k-1}$. We ask, similarly to the Question formulated in the Introduction in [**2**], whether

$$\mu^{(0)}_{\mathbb{H}_c^{k+1}}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k}) = \mu^{(0)}(\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k}).$$

It seems likely to us that the answer is yes, if and only if $|c| \leq 1$.

An affirmative answer for $|c| \leq 1$ would imply, using a reflection argument, that we can restrict not only to functions that are constant along the sphere, but even to radial functions. Here a radial function is defined as a function of the form $u(x, y) = u(d^{\mathbb{H}_c^{k+1}}(x))$ where $d^{\mathbb{H}_c^{k+1}}(x)$ is the distance from x to a fixed point in $\mathbb{H}_c^{k+1}$. The constants $\Lambda_{n,k}$ could then be calculated numerically. For example, we would obtain

$$\Lambda_{4,1} = \mu^{(0)}(\mathbb{R}^2 \times \mathbb{S}^2) = 59.40481\ldots$$

and thus $\sigma(S^2 \times S^2) \geq 59.40481\ldots$, which should be compared to $\mu(\mathbb{S}^4) = 61.56239\ldots$ and $\mu(\mathbb{S}^2 \times \mathbb{S}^2) = 16\pi = 50.26548\ldots$.

Using the handle reduction techniques of the proof of the h-cobordism theorem, together with information about the spin bordism groups in low dimensions, we would be able to conclude the following lower bounds on $\sigma(M)$ for simply connected spin manifolds of dimension n (and with vanishing index in the case $n = 8$):

n	$\sigma(M) >$
5	75.3968
6	91.683
7	108.162
8	124.774

If $n = 5, 6, 7$, we use that M is spin bordant to a sphere; for $n = 8$, we have that M is spin bordant to a number of copies of $\mathbb{H}P^2$. For the standard metric we have $\mu(\mathbb{H}P^2) = 144.959\ldots$. In all four cases we would have $\sigma(M)/\sigma(\mathbb{S}^n) > 0.95$. Similar conclusions can be drawn for non-spin manifolds.

These inequalities would imply for example that $\sigma(\mathbb{C}P^3)$ is not attained by the Fubini–Study metric, as $\mu(\mathbb{C}P^3) = 82.9864\ldots$ for this conformal class.

4. Limit spaces and limit solutions

In the proofs of the main theorems, we will construct limit solutions of the Yamabe equation on certain limit spaces. For this we need the following two lemmas.

Lemma 4.1. *Let V be an n-dimensional manifold. Let (q_α) be a sequence of points in V that converges to a point q as $\alpha \searrow 0$. Let (γ_α) be a sequence of metrics defined on a neighborhood O of q that converges to a metric γ_0 in the $C^2(O)$-topology. Finally, let (b_α) be a sequence of* *positive real numbers such that* $\lim_{\alpha \searrow 0} b_\alpha = \infty$. *Then for* $r > 0$ *there exists for* α *small enough a diffeomorphism*

$$\Theta_\alpha : B^n(r) \to B^{\gamma_\alpha}(q_\alpha, b_\alpha^{-1} r)$$

with $\Theta_\alpha(0) = q_\alpha$ *such that the metric* $\Theta_\alpha^*(b_\alpha^2 \gamma_\alpha)$ *tends to the flat metric* ξ^n *in* $C^2(B^n(r))$.

Proof. Denote by $\exp_{q_\alpha}^{\gamma_\alpha} : U_\alpha \to O_\alpha$ the exponential map at the point q_α defined with respect to the metric γ_α. Here O_α is a neighborhood of q_α in V and U_α is a neighborhood of the origin in $\mathbb{R}^n$. We set

$$\Theta_\alpha : B^n(r) \ni x \mapsto \exp_{q_\alpha}^{\gamma_\alpha}(b_\alpha^{-1} x) \in B^{\gamma_\alpha}(q_\alpha, b_\alpha^{-1} r).$$

It is easily checked that Θ_α is the desired diffeomorphism. q.e.d.

Lemma 4.2. *Let* V *be an* n*-dimensional manifold. Let* (g_α) *be a sequence of metrics that converges to a metric* g *in* C^2 *on all compact sets* $K \subset V$ *as* $\alpha \searrow 0$. *Assume that* (U_α) *is an increasing sequence of subdomains of* V *such that* $\bigcup_\alpha U_\alpha = V$. *Let* $u_\alpha \in C^2(U_\alpha)$ *be a sequence of positive functions such that* $\|u_\alpha\|_{L^\infty(U_\alpha)}$ *is bounded independently of* α. *We assume*

$$L^{g_\alpha} u_\alpha = \mu_\alpha u_\alpha^{p-1} \tag{9}$$

where the μ_α *are numbers tending to* $\bar{\mu}$. *Then there exists a non-negative function* $u \in C^2(V)$ *satisfying*

$$L^g u = \bar{\mu} u^{p-1} \tag{10}$$

on V *and a subsequence of* u_α *that tends to* u *in* C^1 *on each open set* $\Omega \subset V$ *with compact closure. In particular,*

$$\|u\|_{L^\infty(K)} = \lim_{\alpha \searrow 0} \|u_\alpha\|_{L^\infty(K)} \tag{11}$$

and

$$\int_K u^r \, dv^g = \lim_{\alpha \searrow 0} \int_K u_\alpha^r \, dv^{g_\alpha} \tag{12}$$

for any compact set K *and any* $r \geq 1$.

Proof. Let K be a compact subset of V, and let Ω be an open set with smooth boundary and compact closure in V such that $K \subset \Omega$. From equation (9) and the boundedness of $\|u_\alpha\|_\infty$ we see with standard results on elliptic regularity (see, for example, [**18**]) that (u_α) is bounded in the Sobolev space $H^{2,2n}(\Omega, g)$; that is, all derivatives of $u_\alpha|_\Omega$ up to second order are bounded in $L^{2n}(\Omega)$). As this Sobolev space embeds compactly into $C^1(\Omega)$, a subsequence of (u_α) converges in $C^1(\Omega)$ to a function $u^\Omega \in C^1(\Omega)$, $u^\Omega \geq 0$, depending on Ω. Let $\varphi \in C^\infty(\Omega)$ be compactly supported in Ω. Multiplying equation (9) by φ and integrating over Ω, we obtain that u^Ω satisfies equation (10) weakly on Ω. By standard regularity results $u^\Omega \in C^2(\Omega)$ and satisfies equation (10).

As a next step, we choose an increasing sequence of compact sets K_m satisfying $\bigcup_m K_m = V$. Using the above arguments and taking successive subsequences, it follows that (u_α) converges to functions $u_m \in C^2(K_m)$ that solve equation (10) and satisfy $u_m \geq 0$ and $u_m|_{K_{m-1}} = u_{m-1}$. We define u on V by $u = u_m$ on K_m. By taking a diagonal subsequence of (u_α), we get that (u_α) tends to u in C^1 on any compact set $K \subset V$. This ends the proof of Lemma 4.2. q.e.d.

The next Lemma is useful when the sequence of metrics in Lemma 4.2 converges to the flat metric ξ^n on $\mathbb{R}^n$.

Lemma 4.3. *Let ξ^n be the standard flat metric on $\mathbb{R}^n$, and assume that $u \in C^2(\mathbb{R}^n)$, $u \geq 0$, $u \not\equiv 0$ satisfies*

$$L^{\xi^n} u = \mu u^{p-1} \tag{13}$$

for some $\mu \in \mathbb{R}$. Assume in addition that $u \in L^p(\mathbb{R}^n)$ and that

$$\|u\|_{L^p(\mathbb{R}^n)} \leq 1.$$

Then $\mu \geq \mu(\mathbb{S}^n)$.

Proof. The map $\varphi : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{R}^n \setminus \{0\}$, $\varphi(t, x) = e^t x$, is a conformal diffeomorphism with

$$dt^2 + \sigma^{n-1} = e^{-2t} \varphi^* \xi^n.$$

Thus if u is a solution of (13), then $\hat{u} := e^{(n-2)t/2} u \circ \varphi$ is a solution of $L^{dt^2 + \sigma^{n-1}} \hat{u} = \mu \hat{u}^{p-1}$ and $\|\hat{u}\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} = \|u\|_{L^p(\mathbb{R}^n)} \leq 1$. The result now follows from Lemma 3.14. q.e.d.

5. L^2-estimates on WS-bundles

Manifolds with a certain structure of a double bundle will appear in the proofs of our main results. In this section we derive L^2-estimates for solutions to a perturbed Yamabe equation on a WS-bundle.

5.1. Definition and statement of the result. Let $n \geq 1$ and $0 \leq k \leq n - 3$ be integers. Let W be a closed manifold of dimension k, and let I be an interval. By a *WS-bundle* we will mean the product $P := I \times W \times S^{n-k-1}$ equipped with a metric of the form

$$g_{\mathrm{WS}} = dt^2 + e^{2\varphi(t)} h_t + \sigma^{n-k-1} \tag{14}$$

where h_t is a smooth family of metrics on W depending on $t \in I$ and φ is a function on I. The condition $k \leq n-3$ implies that the sphere S^{n-k-1} carries positive scalar curvature, which is an essential ingredient in the proof of Theorem 5.2. Let $\pi : P \to I$ be the projection onto the first factor, and let $F_t := \pi^{-1}(t) = \{t\} \times W \times S^{n-k-1}$. The metric induced on F_t is $g_t := e^{2\varphi(t)} h_t + \sigma^{n-k-1}$. Let H_t be the mean curvature of F_t in P; that is, $H_t \partial_t$ is the mean curvature vector of F_t. We always use the sign convention for the mean curvature vector for which it points in the direction of decreasing volume of F_t. The mean curvature is given by the formula

$$H_t = -\frac{k}{n-1}\varphi'(t) - e(h_t) \tag{15}$$

with $e(h_t) := \frac{1}{2(n-1)}\mathrm{tr}_{h_t}(\partial_t h_t)$. Clearly, $e(h_t) = 0$ if $t \mapsto h_t$ is constant. The derivative of the volume element dv^{g_t} of F_t is

$$\partial_t dv^{g_t} = -(n-1)H_t dv^{g_t}.$$

It is straightforward to check that the scalar curvatures of g_{WS} and h_t are related by (see Appendix A.2 for details)

$$\begin{aligned} \mathrm{Scal}^{g_{\mathrm{WS}}} &= e^{-2\varphi(t)}\mathrm{Scal}^{h_t} + (n-k-1)(n-k-2) \\ &\quad - k(k+1)\varphi'(t)^2 - 2k\varphi''(t) - (k+1)\varphi'(t)\mathrm{tr}(h_t^{-1}\partial_t h_t) \\ &\quad + \frac{3}{4}\mathrm{tr}((h_t^{-1}\partial_t h_t)^2) - \frac{1}{4}(\mathrm{tr}(h_t^{-1}\partial_t h_t))^2 - \mathrm{tr}({h_t}^{-1}\partial_t^2 h_t). \end{aligned} \tag{16}$$

Definition 5.1. We say that condition (A_t) holds if the following assumptions are true:

$$\left.\begin{array}{ll} 1. & t \mapsto h_t \text{ is constant,} \\ 2. & e^{-2\varphi(t)} \inf_{x\in W} \mathrm{Scal}^{h_t}(x) \geq -\frac{n-k-2}{32}a, \\ 3. & |\varphi'(t)| \leq 1, \\ 4. & 0 \leq -2k\varphi''(t) \leq \frac{1}{2}(n-1)(n-k-2)^2. \end{array}\right. \tag{A_t}$$

Similarly, we say that condition (B_t) holds if the following assumptions are true:

$$\left.\begin{array}{ll} 1. & t \mapsto \varphi(t) \text{ is constant,} \\ 2. & \inf_{x\in F_t} \mathrm{Scal}^{g_{\mathrm{WS}}}(x) \geq \frac{1}{2}\mathrm{Scal}^{\sigma^{n-k-1}} = \frac{1}{2}(n-k-1)(n-k-2), \\ 3. & \frac{(n-1)^2}{2}e(h_t)^2 + \frac{n-1}{2}\partial_t e(h_t) \geq -\frac{3}{64}(n-k-2). \end{array}\right. \tag{B_t}$$

Let P be WS-bundle equipped with a metric G that is close to g_{WS} in a sense to be made precise later. Let $\alpha, \beta \in \mathbb{R}$ be such that $[\alpha, \beta] \subset I$. Our goal is to derive an estimate for the distribution of L^2-norm of a positive solution to the Yamabe equation

$$L^G u = \mu u^{p-1}.$$

If we write this equation in terms of the metric g_{WS}, we get a perturbed version of the Yamabe equation for g_{WS}. We assume that we have a smooth positive solution u of the equation

$$L^{g_{\mathrm{WS}}}u = a\Delta^{g_{\mathrm{WS}}}u + \mathrm{Scal}^{g_{\mathrm{WS}}}u = \mu u^{p-1} + d^*A(du) + Xu + \epsilon\partial_t u - su \tag{17}$$

where $s, \epsilon \in C^\infty(P)$, $A \in \mathrm{End}(T^*P)$, and $X \in \Gamma(TP)$ are perturbation terms coming from the difference between G and g_{WS}. We assume that the endomorphism A is symmetric and that X and A are vertical; that is, $dt(X) = 0$ and $A(dt) = 0$.

Theorem 5.2. *Assume that P carries a metric g_{WS} of the form* (14)*. Let $\alpha, \beta \in \mathbb{R}$ be such that $[\alpha, \beta] \subset I$. Assume further that for each $t \in I$ either condition (A_t) or condition (B_t) is true. We also assume that u is a positive solution of* (17) *satisfying*

$$\mu \|u\|_{L^\infty(P)}^{p-2} \leq \frac{(n-k-2)^2(n-1)}{8(n-2)}. \tag{18}$$

Then there exists $c_0 > 0$ independent of α, β, and φ, such that if

$$\|A\|_{L^\infty(P)}, \|X\|_{L^\infty(P)}, \|s\|_{L^\infty(P)}, \|\epsilon\|_{L^\infty(P)}, \|e(h_t)\|_{L^\infty(P)} \leq c_0$$

then

$$\int_{\pi^{-1}((\alpha+\gamma, \beta-\gamma))} u^2 \, dv^{g_{\mathrm{WS}}} \leq \frac{4\|u\|_{L^\infty}^2}{n-k-2} \left(\mathrm{Vol}^{g_\alpha}(F_\alpha) + \mathrm{Vol}^{g_\beta}(F_\beta) \right),$$

where $\gamma := \frac{\sqrt{32}}{n-k-2}$.

Note that this theorem only gives information when $\beta - \alpha > 2\gamma$.

5.2. Proof of Theorem 5.2.

For the proof of Theorem 5.2, we need the following lemma.

Lemma 5.3. *Let T and γ be positive numbers, and assume that $w : [-T-\gamma, T+\gamma] \to \mathbb{R}$ is a smooth positive function satisfying*

$$w''(t) \geq \frac{w(t)}{\gamma^2}. \tag{19}$$

Then

$$\int_{-T}^{T} w(t)^m \, dt \leq \frac{\gamma}{m} \Big((w(T+\gamma))^m + (w(-T-\gamma))^m \Big) \tag{20}$$

for all $m \geq 1$.

Proof. Assume that $w|_{[-T-\gamma, T+\gamma]}$ attains its minimum at t_0. Since $w'' \geq w/\gamma^2 > 0$, we have $w'(t) > 0$ for $t \in (t_0, T+\gamma)$, and $w'(t) < 0$ for $t \in (-T-\gamma, t_0)$. We first study the case when $t_0 \in (-T, T)$. We define $W(t) := w(t) + \gamma w'(t)$. As w and w' are increasing on $(t_0, T+\gamma)$, we get

$$\begin{aligned} W(T) &= w(T) + \int_T^{T+\gamma} w'(T) \, dt \\ &\leq w(T) + \int_T^{T+\gamma} w'(t) \, dt \\ &= w(T+\gamma). \end{aligned} \tag{21}$$

From (19) we see that $W'(t) \geq W(t)/\gamma$, or $\partial_t \ln W(t) \geq 1/\gamma$. Integrating this relation between $t \in (t_0, T)$ and T, we get

$$W(t) \leq e^{-\frac{T-t}{\gamma}} W(T).$$

Using that $w \leq W$ on (t_0, T) together with (21), we obtain

$$w(t) \leq W(t) \leq e^{-\frac{T-t}{\gamma}} w(T+\gamma),$$

and hence

$$w(t)^m \leq e^{-m\frac{T-t}{\gamma}} (w(T+\gamma))^m$$

for all $t \in [t_0, T]$ and $m \geq 1$. Integrating this relation over $t \in [t_0, T]$, we get

$$\int_{t_0}^{T} w(t)^m \, dt \leq \frac{\gamma(1 - e^{-m\frac{T-t_0}{\gamma}})}{m} (w(T+\gamma))^m \leq \frac{\gamma}{m} (w(T+\gamma))^m. \tag{22}$$

Similarly, we conclude that

$$\int_{-T}^{t_0} w(t)^m \, dt \leq \frac{\gamma}{m} (w(-T-\gamma))^m. \tag{23}$$

This proves relation (20) in this case. In the case that $t_0 \leq -T$, relation (22) remains valid. Using

$$\int_{-T}^{T} w(t)^m \, dt \leq \int_{t_0}^{T} w(t)^m \, dt$$

and

$$(w(T+\gamma))^m \leq (w(T+\gamma))^m + (w(-T-\gamma))^m,$$

we obtain relation (20). We proceed in a similar way using (23) in case $t_0 \geq T$. This ends the proof of Lemma 5.3. q.e.d.

Proof of Theorem 5.2. The Laplacian $\Delta^{g_{\mathrm{WS}}}$ on P is related to the Laplacian Δ^{g_t} on F_t through the formula

$$\Delta^{g_{\mathrm{WS}}} = \Delta^{g_t} - \partial_t^2 + (n-1) H_t \partial_t,$$

so

$$\begin{aligned}
\int_{F_t} u \Delta^{g_{\mathrm{WS}}} u \, dv^{g_t} &= \int_{F_t} \left(u \Delta^{g_t} u - u(\partial_t^2 u) + (n-1) H_t u (\partial_t u) \right) \, dv^{g_t} \\
&= \int_{F_t} \left(|d_{\mathrm{vert}} u|^2 - u(\partial_t^2 u) + (n-1) H_t u (\partial_t u) \right) \, dv^{g_t}.
\end{aligned}$$

Together with (17) we get

$$\begin{aligned}
a \int_{F_t} u \partial_t^2 u \, dv^{g_t} = \int_{F_t} \Big(& a|d_{\mathrm{vert}} u|^2 + a(n-1) H_t u \partial_t u \\
& - \langle d_{\mathrm{vert}} u, A(d_{\mathrm{vert}} u) \rangle - uXu - \epsilon u \partial_t u \\
& + (\mathrm{Scal}^{g_{\mathrm{WS}}} + s) u^2 - \mu u^p \Big) \, dv^{g_t}.
\end{aligned}$$

In the following we denote by $\delta(c_0)$ a positive constant that goes to 0 if c_0 tends to 0 and whose convergence depends only on n, μ, and h. We set $S_t := \inf_{F_t} \mathrm{Scal}^{g_{\mathrm{WS}}}$. If we use the inequality $2\int |ab| \leq \int (a^2 + b^2)$ to simplify the terms involving X and ϵ, we obtain

$$a \int_{F_t} u \partial_t^2 u \, dv^{g_t} \geq \int_{F_t} \Big((a - \delta(c_0)) |d_{\mathrm{vert}} u|^2 + a(n-1) H_t u \partial_t u \\ - \delta(c_0)(\partial_t u)^2 + (S_t - \delta(c_0)) u^2 - \mu u^p \Big) \, dv^{g_t}.$$

If c_0 is small enough so that $a - \delta(c_0) > 0$, we conclude that

$$\begin{aligned} & a \int_{F_t} \Big(u \partial_t^2 u - (n-1) H_t u (\partial_t u) \Big) \, dv^{g_t} \\ & \quad \geq (S_t - \delta(c_0)) w(t)^2 - \int_{F_t} \Big(\delta(c_0)(\partial_t u)^2 + \mu u^p \Big) \, dv^{g_t}, \end{aligned} \tag{24}$$

where we define

$$w(t) := \|u\|_{L^2(F_t)} = \left(\int_{F_t} u^2 \, dv^{g_t} \right)^{1/2}.$$

Differentiating this, we get

$$\begin{aligned} 2w'(t) w(t) &= \partial_t \int_{F_t} u^2 \, dv^{g_t} \\ &= \int_{F_t} \Big(2u(\partial_t u) - (n-1) H_t u^2 \Big) \, dv^{g_t}. \end{aligned} \tag{25}$$

We now assume that (A_t) holds. Then (15) tells us that

$$H_t = -\frac{k}{n-1} \varphi'(t),$$

so (25) becomes

$$w'(t) w(t) = \int_{F_t} u(\partial_t u) \, dv^{g_t} + \frac{k}{2} \varphi'(t) w(t)^2. \tag{26}$$

We differentiate this and obtain

$$\begin{aligned} w'(t)^2 + w''(t) w(t) &= \int_{F_t} (\partial_t u)^2 \, dv^{g_t} \\ &\quad + \int_{F_t} \Big(u \partial_t^2 u - (n-1) H_t u \partial_t u \Big) \, dv^{g_t} \\ &\quad + \frac{k}{2} \varphi''(t) w(t)^2 + k \varphi'(t) w'(t) w(t). \end{aligned}$$

From (24) we get

$$
\begin{aligned}
w'(t)^2 + w''(t)w(t) \geq & \left(1 - \frac{\delta(c_0)}{a}\right) \int_{F_t} (\partial_t u)^2 \, dv^{g_t} \\
& + \left(\frac{1}{a}\left(S_t - \delta(c_0)\right) + \frac{k}{2}\varphi''(t)\right) w(t)^2 \\
& - \frac{1}{a}\int_{F_t} \mu u^p \, dv^{g_t} + k\varphi'(t)w'(t)w(t).
\end{aligned}
\tag{27}
$$

We now use Cauchy–Schwarz and (26) to get

$$
\begin{aligned}
w(t)^2 \int_{F_t} (\partial_t u)^2 \, dv^{g_t} &\geq \left(\int_{F_t} u(\partial_t u) \, dv^{g_t}\right)^2 \\
&= \left(w'(t)w(t) - \frac{k}{2}\varphi'(t)w(t)^2\right)^2,
\end{aligned}
$$

and thus

$$
\int_{F_t} (\partial_t u)^2 \, dv^{g_t} \geq \left(w'(t) - \frac{k}{2}\varphi'(t)w(t)\right)^2. \tag{28}
$$

From assumption (18) it follows that

$$
\frac{\mu}{a}\int_{F_t} u^p \, dv^{g_t} \leq \frac{(n-k-2)^2}{32} w(t)^2. \tag{29}
$$

Inserting (28) and (29) into (27), we obtain

$$
\begin{aligned}
w'(t)^2 + w''(t)w(t) \geq & \left(1 - \frac{\delta(c_0)}{a}\right) \left(w'(t) - \frac{k}{2}\varphi'(t)w(t)\right)^2 \\
& + \left(\frac{1}{a}\left(S_t - \delta(c_0)\right) + \frac{k}{2}\varphi''(t)\right) w(t)^2 \\
& - \frac{(n-k-2)^2}{32} w(t)^2 + k\varphi'(t)w'(t)w(t),
\end{aligned}
$$

or, after some rearranging,

$$
\begin{aligned}
w''(t)w(t) \geq & -\frac{\delta(c_0)}{a}\left(w'(t) - \frac{k}{2}\varphi'(t)w(t)\right)^2 \\
& + \left(\frac{1}{a}\left(S_t - \delta(c_0)\right) + \frac{k}{2}\varphi''(t) + \frac{k^2}{4}\varphi'(t)^2 - \frac{(n-k-2)^2}{32}\right) w(t)^2.
\end{aligned}
\tag{30}
$$

Next, we estimate the coefficient of $w(t)^2$ in the last line of (30). We denote this coefficient by D. Using (16) and assumption 1 of (A_t), which tells us that $e(h_t) = 0$, we get

$$\begin{aligned}
D &= \frac{1}{a}\big(e^{-2\varphi(t)} \inf_{x\in W} \mathrm{Scal}^{h_t}(x) - k(k+1)\varphi'(t)^2 - 2k\varphi''(t) + (n-k-1)(n-k-2)\big)\\
&\quad - \frac{\delta(c_0)}{a} + \frac{k}{2}\varphi''(t) + \frac{k^2}{4}\varphi'(t)^2 - \frac{(n-k-2)^2}{32}\\
&= \frac{1}{a} e^{-2\varphi(t)} \inf_{x\in W} \mathrm{Scal}^{h_t}(x) + \frac{1}{a}\left((n-k-1)(n-k-2) - \delta(c_0)\right) + \frac{k}{2(n-1)}\varphi''(t)\\
&\quad - \frac{k}{4(n-1)}(n-k-2)\varphi'(t)^2 - \frac{(n-k-2)^2}{32}.
\end{aligned}$$

From assumptions 2 and 3 of (A_t), we obtain

$$\begin{aligned}
D &\geq -\frac{n-k-2}{32} + \frac{1}{a}\left((n-k-1)(n-k-2) - \delta(c_0)\right) + \frac{k}{2(n-1)}\varphi''(t)\\
&\quad - \frac{k}{4(n-1)}(n-k-2) - \frac{(n-k-2)^2}{32}\\
&= \frac{1}{4(n-1)}\left((n-1)(n-k-2)^2 + 2k\varphi''(t)\right)\\
&\quad - \frac{n-k-2}{32} - \frac{(n-k-2)^2}{32} - \frac{\delta(c_0)}{a}.
\end{aligned}$$

Using assumption 4 of (A_t) and $n-k-2 \geq 1$, we further obtain

$$\begin{aligned}
D &\geq \frac{1}{4(n-1)}\left(\frac{1}{2}(n-1)(n-k-2)^2\right)\\
&\quad - \frac{(n-k-2)^2}{32} - \frac{(n-k-2)^2}{32} - \frac{\delta(c_0)}{a}\\
&= \frac{(n-k-2)^2}{16} - \frac{\delta(c_0)}{a}.
\end{aligned}$$

Inserting this in (30), we get

$$\begin{aligned}
w''(t)w(t) &\geq -\frac{\delta(c_0)}{a}\left(w'(t) - \frac{k}{2}\varphi'(t)w(t)\right)^2\\
&\quad + \left(\frac{(n-k-2)^2}{16} - \frac{\delta(c_0)}{a}\right) w(t)^2\\
&\geq -\frac{2\delta(c_0)}{a} w'(t)^2\\
&\quad + \left(-\frac{2\delta(c_0)}{a}\frac{k^2}{4}\varphi'(t)^2 + \frac{(n-k-2)^2}{16} - \frac{\delta(c_0)}{a}\right) w(t)^2,
\end{aligned}$$

where we also used the elementary inequality $(a-b)^2 \leq 2a^2+2b^2$. Again using assumption 3 of (A_t), we conclude

$$
\begin{aligned}
w''(t)w(t) \geq & -\frac{2\delta(c_0)}{a}w'(t)^2 \\
& + \left(\frac{(n-k-2)^2}{16} - \frac{\delta(c_0)}{a}\left(1+\frac{k^2}{2}\right)\right) w(t)^2.
\end{aligned}
\tag{31}
$$

Fix a small positive number $\hat{\delta}$. Choose c_0 small so that $\delta(c_0)$ is also small. Then (31) tells us that

$$
w''(t)w(t) \geq \frac{(n-k-2)^2}{32} w(t)^2 - \hat{\delta} w'(t)^2. \tag{32}
$$

Define $v(t) := w(t)^{1+\hat{\delta}}$. This function satisfies

$$
\begin{aligned}
v''(t) &= (1+\hat{\delta})w''(t)w(t)^{\hat{\delta}} + \hat{\delta}(1+\hat{\delta})w'(t)^2 w(t)^{\hat{\delta}-1} \\
&\geq (1+\hat{\delta})\frac{(n-k-2)^2}{32} w(t)^{1+\hat{\delta}} \\
&\geq \frac{(n-k-2)^2}{32} v(t).
\end{aligned}
$$

Next, we assume that (B_t) holds. Then (15) becomes

$$
H_t = -e(h_t),
$$

and from (25) we get

$$
w'(t)w(t) = \int_{F_t} \left(u(\partial_t u) + \frac{n-1}{2} e(h_t) u^2\right) dv^{g_t}. \tag{33}
$$

Differentiating this, we get

$$
\begin{aligned}
w'(t)^2 + w''(t)w(t) = & \int_{F_t} \Bigg((\partial_t u)^2 + (n-1)e(h_t)u\partial_t u \\
& \qquad + \left(\frac{(n-1)^2}{2} e(h_t)^2 + \frac{n-1}{2}\partial_t e(h_t)\right) u^2\Bigg) dv^{g_t} \\
& + \int_{F_t} \left(u\partial_t^2 u - (n-1)H_t u \partial_t u\right) dv^{g_t}.
\end{aligned}
$$

Next, we use (24) followed by assumptions 2 and 3 of (B_t) to obtain

$$\begin{aligned}
w'(t)^2 + w''(t)w(t) \geq & \int_{F_t} \Bigg((\partial_t u)^2 + (n-1)e(h_t)u\partial_t u \\
& + \left(\frac{(n-1)^2}{2} e(h_t)^2 + \frac{n-1}{2}\partial_t e(h_t) \right) u^2 \\
& - \frac{\delta(c_0)}{a}(\partial_t u)^2 - \frac{\mu}{a}u^p \Bigg) \, dv^{g_t} \\
& + \frac{1}{a}(S_t - \delta(c_0))w(t)^2 \\
\geq & \int_{F_t} \left(\left(1 - \frac{\delta(c_0)}{a}\right)(\partial_t u)^2 + (n-1)e(h_t)u\partial_t u - \frac{\mu}{a}u^p \right) dv^{g_t} \\
& + \left(\frac{1}{2a}(n-k-1)(n-k-2) - \frac{3}{64}(n-k-2) - \frac{\delta(c_0)}{a} \right) w(t)^2.
\end{aligned}$$

From (29) we further get, using $k \leq n-3$ in the last step,

(34)

$$\begin{aligned}
w'(t)^2 + w''(t)w(t) \geq & \int_{F_t} \left(\left(1 - \frac{\delta(c_0)}{a}\right)(\partial_t u)^2 + (n-1)e(h_t)u\partial_t u \right) dv^{g_t} \\
& + \Bigg(\frac{1}{2a}(n-k-1)(n-k-2) - \frac{3}{64}(n-k-2) \\
& - \frac{1}{32}(n-k-2)^2 - \frac{\delta(c_0)}{a} \Bigg) w(t)^2 \\
\geq & \int_{F_t} \left(\left(1 - \frac{\delta(c_0)}{a}\right)(\partial_t u)^2 + (n-1)e(h_t)u\partial_t u \right) dv^{g_t} \\
& + \left(\frac{1}{32}(n-k-2)(n-k-3/2) - \frac{\delta(c_0)}{a} \right) w(t)^2 \\
\geq & \int_{F_t} \left(\left(1 - \frac{\delta(c_0)}{a}\right)(\partial_t u)^2 + (n-1)e(h_t)u\partial_t u \right) dv^{g_t} \\
& + \left(\frac{1}{32}(n-k-2)^2 + \frac{1}{64} - \frac{\delta(c_0)}{a} \right) w(t)^2.
\end{aligned}$$

We set $E_t := \sup_{F_t} |e(h_t)|$ and use (33) to compute

$$\begin{aligned}
w(t)^2 \int_{F_t} (\partial_t u)^2 \, dv^{g_t} &\geq \left(\int_{F_t} u(\partial_t u) \, dv^{g_t} \right)^2 \\
&= \left(w'(t)w(t) - \frac{n-1}{2} \int_{F_t} e(h_t) u^2 \, dv^{g_t} \right)^2 \\
&= \big(w'(t)w(t) \big)^2 + \left(\frac{n-1}{2} \int_{F_t} e(h_t) u^2 \, dv^{g_t} \right)^2 \\
&\quad - (n-1) w'(t) w(t) \int_{F_t} e(h_t) u^2 \, dv^{g_t} \\
&\geq w'(t)^2 w(t)^2 - \left(\frac{n-1}{2} \right)^2 E_t^2 w(t)^4 \\
&\quad - (n-1) |w'(t)| w(t) \int_{F_t} |e(h_t)| u^2 \, dv^{g_t} \\
&\geq w'(t)^2 w(t)^2 - \left(\frac{n-1}{2} \right)^2 E_t^2 w(t)^4 \\
&\quad - (n-1) E_t |w'(t)| w(t)^3.
\end{aligned}$$

Next, we divide by $w(t)^2$ and obtain

(35)

$$\begin{aligned}
\int_{F_t} (\partial_t u)^2 \, dv^{g_t} &\geq w'(t)^2 - \left(\frac{n-1}{2} \right)^2 E_t^2 w(t)^2 - (n-1) E_t |w'(t)| w(t) \\
&\geq w'(t)^2 - \left(\frac{n-1}{2} \right)^2 E_t^2 w(t)^2 - \frac{n-1}{2} E_t \left(w'(t)^2 + w(t)^2 \right) \\
&= \left(1 - \frac{n-1}{2} E_t \right) w'(t)^2 - \left(\frac{n-1}{2} E_t + \left(\frac{n-1}{2} \right)^2 E_t^2 \right) w(t)^2.
\end{aligned}$$

Also,

$$\begin{aligned}
\left| \int_{F_t} e(h_t) u \partial_t u \, dv^{g_t} \right| &\leq \int_{F_t} |e(h_t) u \partial_t u| \, dv^{g_t} \\
&\leq E_t \int_{F_t} |u \partial_t u| \, dv^{g_t} \\
&\leq \frac{1}{2} E_t \int_{F_t} \left(u^2 + (\partial_t u)^2 \right) \, dv^{g_t},
\end{aligned}$$

so

$$\int_{F_t} (n-1) e(h_t) u \partial_t u \, dv^{g_t} \geq -\frac{n-1}{2} E_t \int_{F_t} \left(u^2 + (\partial_t u)^2 \right) \, dv^{g_t}. \tag{36}$$

Fix a small number $\hat{\delta} > 0$. We insert (35) and (36) in (34) and choose c_0 small enough so that $\delta(c_0)$ and E_t are small. Then we get that $w(t)$

satisfies the same inequality (32) as we obtained under the assumption (A_t). We have showed that in both cases (A_t) and (B_t) the function $v(t) = w(t)^{1+\hat{\delta}}$ satisfies

$$v''(t) \geq v(t)/\gamma^2$$

since $\frac{32}{(n-k-2)^2} = \gamma^2$.

Now we apply Lemma 5.3 to the function $\tilde{v}(t) := v(t + \frac{\beta+\alpha}{2})$ with $T = \frac{\beta-\alpha}{2} - \gamma$ and $m = \frac{2}{1+\hat{\delta}}$. From this we obtain

$$\frac{\gamma}{m}\left((\tilde{v}(T+\gamma))^m + (\tilde{v}(-T-\gamma))^m\right) \geq \int_{-T}^{T} \tilde{v}^m \, dt. \tag{37}$$

With $s = t + \frac{\beta+\alpha}{2}$ we further have

$$\int_{-T}^{T} \tilde{v}^m \, dt = \int_{\alpha+\gamma}^{\beta-\gamma} (w(s))^{(1+\hat{\delta})m} \, dt = \int_{\alpha+\gamma}^{\beta-\gamma} w^2 \, ds.$$

From the definition of w we obtain

$$\int_{-T}^{T} \tilde{v}^m \, dt = \int_{\pi^{-1}((\alpha+\gamma,\beta-\gamma))} u^2 \, dv^{g_{\mathrm{WS}}}.$$

In addition, we have

$$\begin{aligned}
\left((\tilde{v}(T+\gamma))^m + (\tilde{v}(-T-\gamma))^m\right) &= \int_{F_\alpha} u^2 \, dv^{g_\alpha} + \int_{F_\beta} u^2 \, dv^{g_\beta} \\
&\leq \|u\|^2_{L^\infty(P)} \left(\mathrm{Vol}^{g_\alpha}(F_\alpha) + \mathrm{Vol}^{g_\beta}(F_\beta)\right).
\end{aligned}$$

Choosing $\hat{\delta}$ small, we may assume $m \geq \sqrt{2}$. This together with (37) and $\gamma = \frac{\sqrt{32}}{n-k-2}$ gives us

$$\int_{\pi^{-1}((\alpha+\gamma,\beta-\gamma))} u^2 \, dv^{g_{\mathrm{WS}}} \leq \frac{4\|u\|^2_{L^\infty}}{n-k-2} \left(\mathrm{Vol}^{g_\alpha}(F_\alpha) + \mathrm{Vol}^{g_\beta}(F_\beta)\right).$$

This proves Theorem 5.2. q.e.d.

6. Proof of Theorem 1.3

6.1. Stronger version of Theorem 1.3. In this section we prove the following Theorem 6.1. By taking the supremum over all conformal classes, Theorem 6.1 implies Theorem 1.3.

Theorem 6.1. *Suppose that (M_1, g_1) and (M_2, g_2) are compact Riemannian manifolds of dimension n. Let N be obtained from M_1, M_2, by a connected sum along W as described in Section 2. Then there is a* *family of metrics g_θ, $\theta \in (0, \theta_0)$ on N satisfying*

$$\begin{aligned}\min\{\mu(M_1 \amalg M_2, g_1 \amalg g_2), \Lambda_{n,k}\} &\leq \liminf_{\theta \searrow 0} \mu(N, g_\theta) \\ &\leq \limsup_{\theta \searrow 0} \mu(N, g_\theta) \\ &\leq \mu(M_1 \amalg M_2, g_1 \amalg g_2).\end{aligned}$$

In the following we define suitable metrics g_θ, and then we show that they satisfy these inequalities.

6.2. Definition of the metrics g_θ. We continue to use the notation of Section 2. In the following, C denotes a constant that might change its value between lines. Recall that $(M, g) = (M_1 \amalg M_2, g_1 \amalg g_2)$. For $i = 1, 2$ we define the metric h_i as the restriction of g_i to $W_i' = w_i(W \times \{0\})$, and we set $h := h_1 \amalg h_2$ on $W' = W_1' \amalg W_2'$. As already explained, the normal exponential map of $W' \subset M$ defines a diffeomorphism

$$w_i : W \times B^{n-k}(R_{\max}) \to U_i(R_{\max}), \quad i = 1, 2,$$

which decomposes $U(R_{\max}) = U_1(R_{\max}) \amalg U_2(R_{\max})$ as a product $W' \times B^{n-k}(R_{\max})$.

In general, the Riemannian metric g does not have a corresponding product structure, and we introduce an error term T measuring the difference from the product metric. If r denotes the distance function to W', then the metric g can be written as

$$g = h + \xi^{n-k} + T = h + dr^2 + r^2\sigma^{n-k-1} + T \tag{38}$$

on $U(R_{\max}) \setminus W' \cong W' \times (0, R_{\max}) \times S^{n-k-1}$. Here T is a symmetric $(2,0)$-tensor vanishing on W' (in the sense of sections of $(T^*M \otimes T^*M)|_{W'}$). We also define the product metric

$$g' := h + \xi^{n-k} = h + dr^2 + r^2\sigma^{n-k-1}, \tag{39}$$

on $U(R_{\max}) \setminus W'$. Thus $g = g' + T$. Since T vanishes on W', we have for sufficiently small r

$$|T(X, Y)| \leq Cr|X|_{g'}|Y|_{g'} \tag{40}$$

for any $X, Y \in T_xM$ where $x \in U(R_{\max})$. Since T is smooth, we have for sufficiently small r

$$|(\nabla_U T)(X, Y)| \leq C|X|_{g'}|Y|_{g'}|U|_{g'},$$

and

$$|(\nabla^2_{U,V})T(X, Y)| \leq C|X|_{g'}|Y|_{g'}|U|_{g'}|V|_{g'},$$

for $X, Y, U, V \in T_xM$. We define $T_i := T|_{M_i}$ for $i = 1, 2$.

For a fixed $R_0 \in (0, R_{\max})$, $R_0 < 1$, and sufficiently small in the sense of equation (40) and the following equations, we choose a smooth

HIERARCHY OF PARAMETERS

$$R_{\max} > R_0 > \theta > \delta_0 > \epsilon > 0$$

We choose parameters in the order $R_{\max}, R_0, \theta, \delta_0, A_\theta$. We then set $\epsilon := e^{-A_\theta}\delta_0$.

This implies $|t| = A_\theta \Leftrightarrow r_i = \delta_0$.

Figure 2. Hierarchy of parameters.

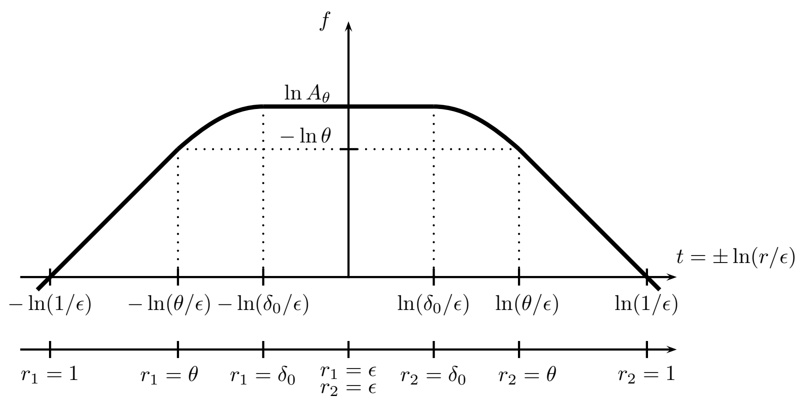

Figure 3. The function f.

positive function $F : M \setminus W' \to \mathbb{R}$ such that

$$F(x) = \begin{cases} 1, & \text{if } x \in M_i \setminus U_i(R_{\max}), \\ r_i(x)^{-1}, & \text{if } x \in U_i(R_0) \setminus W'. \end{cases}$$

Next we choose small numbers $\theta, \delta_0 \in (0, R_0)$ with $\theta > \delta_0 > 0$. Here "small" means that for a given small number θ we choose a number $\delta_0 = \delta_0(\theta) \in (0, \theta)$ such that all arguments that need δ_0 to be small will hold; see Figure 2. For any $\theta > 0$ and sufficiently small δ_0 there is $A_\theta \in (\theta^{-1}, (\delta_0)^{-1})$ and a family of smooth functions $f = f_{\theta,\delta_0} : U(R_{\max}) \to \mathbb{R}$ depending only on the coordinate r such that

$$f(x) = \begin{cases} -\ln r(x), & \text{if } x \in U(R_{\max}) \setminus U(\theta); \\ \ln A_\theta, & \text{if } x \in U(\delta_0), \end{cases}$$

and such that

(41)
$$\left| r\frac{df}{dr} \right| = \left| \frac{df}{d(\ln r)} \right| \leq 1, \quad \text{and} \quad \left\| r\frac{d}{dr}\left(r\frac{df}{dr} \right) \right\|_{L^\infty} = \left\| \frac{d^2 f}{d^2(\ln r)} \right\|_{L^\infty} \to 0$$

as $\theta \searrow 0$. See Figure 3.

We set $\epsilon = e^{-A_\theta}\delta_0$. We can and will assume that $\epsilon < 1$.

Let N be obtained from M by a connected sum along W with parameter ϵ, as described in Section 2. In particular, $U_\epsilon^N(s) = (U(s) \setminus U(\epsilon)) /\sim$ for all $s \in [\epsilon, R_{\max}]$. On the set $U_\epsilon^N(R_{\max}) = (U(R_{\max}) \setminus U(\epsilon)) /\sim$, we define the variable t by

$$t := \begin{cases} -\ln r_1 + \ln \epsilon, & \text{on } U_1(R_{\max}) \setminus U_1(\epsilon), \\ \ln r_2 - \ln \epsilon, & \text{on } U_2(R_{\max}) \setminus U_2(\epsilon). \end{cases}$$

Note that $t \leq 0$ on $U_1(R_{\max}) \setminus U_1(\epsilon)$ and $t \geq 0$ on $U_2(R_{\max}) \setminus U_2(\epsilon)$, with $t = 0$ precisely on the common boundary $\partial U_1(\epsilon)$ identified with $\partial U_2(\epsilon)$ in N. It follows that

$$r_i = e^{|t| + \ln \epsilon} = \epsilon e^{|t|}.$$

We can arrange that $t : U_\epsilon^N(R_{\max}) \to \mathbb{R}$ is smooth. Expressed in the variable t, we have

$$F(x) = \epsilon^{-1} e^{-|t|}$$

for $x \in U_\epsilon^N(R_0)$, or, in other words, if $|t| + \ln \epsilon \leq \ln R_0$. Then equation (38) tells us that

$$F^2 g = \epsilon^{-2} e^{-2|t|}(h + T) + dt^2 + \sigma^{n-k-1}$$

on $U_\epsilon^N(R_0)$. If we view f as a function of t, then

$$f(t) = \begin{cases} -|t| - \ln \epsilon, & \text{if } \ln \theta - \ln \epsilon \leq |t| \leq \ln R_{\max} - \ln \epsilon, \\ \ln A_\theta, & \text{if } |t| \leq \ln \delta_0 - \ln \epsilon, \end{cases}$$

and $|df/dt| \leq 1$, $\|d^2 f/dt^2\|_{L^\infty} \to 0$ as θ tends to 0. We choose a cut-off function $\chi : \mathbb{R} \to [0,1]$ such that $\chi = 0$ on $(-\infty, -1]$, $|d\chi| \leq 1$, and $\chi = 1$ on $[1, \infty)$. With these choices we define

$$g_\theta := \begin{cases} F^2 g_i, & \text{on } M_i \setminus U_i(\theta), \\ e^{2f(t)}(h_i + T_i) + dt^2 + \sigma^{n-k-1}, & \text{on } U_i(\theta) \setminus U_i(\delta_0), \\ \left.\begin{array}{l} A_\theta^2 \chi(t/A_\theta)(h_2 + T_2) \\ \quad + A_\theta^2 (1 - \chi(t/A_\theta))(h_1 + T_1) \\ \quad + dt^2 + \sigma^{n-k-1}, \end{array}\right\} & \text{on } U_\epsilon^N(\delta_0). \end{cases}$$

On $U_\epsilon^N(R_0)$ we write g_θ as

$$g_\theta = e^{2f(t)} \tilde{h}_t + dt^2 + \sigma^{n-k-1} + \widetilde{T}_t,$$

where the metric $\tilde{h}_t$ is defined by

$$\tilde{h}_t := \chi(t/A_\theta) h_2 + (1 - \chi(t/A_\theta)) h_1,$$

for $t \in \mathbb{R}$, and where the error term $\widetilde{T}_t$ is equal to

$$\widetilde{T}_t := e^{2f(t)} \left(\chi(t/A_\theta) T_2 + (1 - \chi(t/A_\theta)) T_1 \right).$$

See also Figure 4. On $U_\epsilon^N(R_0)$ we also define the metric without error term

$$g'_\theta := g_\theta - \widetilde{T}_t = e^{2f(t)} \tilde{h}_t + dt^2 + \sigma^{n-k-1}. \tag{42}$$

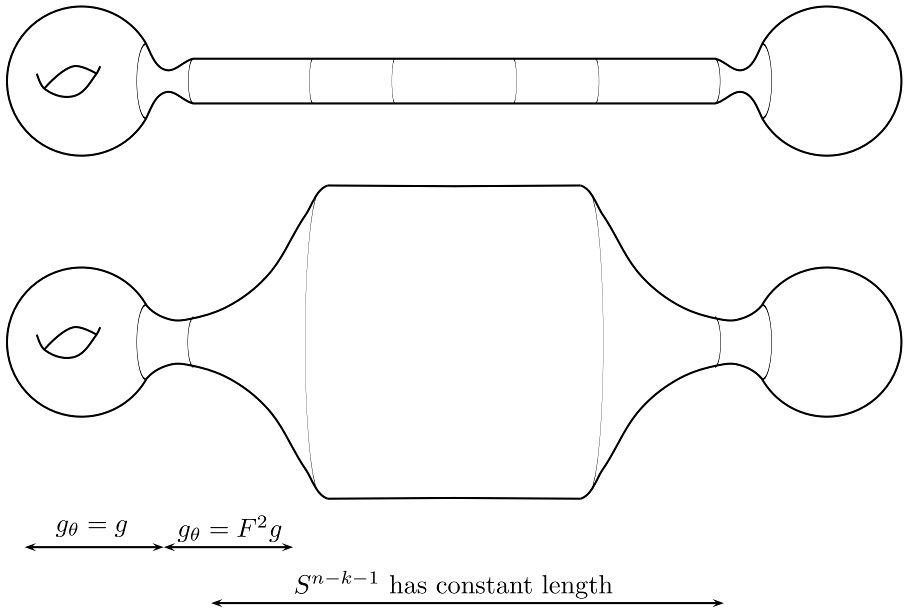

Figure 4. The metrics g_θ. The horizontal direction in both drawings corresponds to the t-variable. The vertical direction in the upper drawing corresponds to the projection to S^{n-k-1}, and in the lower drawing it corresponds to the projection to S^k. In the lower drawing, the curved parts close to the middle part are not drawn realistically. Their curvature tends to 0 for $\theta \to 0$, and the middle becomes huge in this limit, and thus it would be too large for our picture.

An upper bound for the error term $\tilde{T}_t$ will be needed in the following. We claim that

$$|X|_{g'} \leq Ce^{-f(t)}|X|_{g'_\theta} \tag{43}$$

for $X \in T_xN$, where g' is the metric defined by (39). To prove the claim, we decompose X in a radial part, a part parallel to W', and a part parallel to S^{n-k-1}. This decomposition is orthogonal with respect to both g' and g'_θ. For $X = \frac{\partial}{\partial t} = \pm\epsilon e^{|t|}\frac{\partial}{\partial r}$, we have that $1 = |X|_{g'_\theta}$ and $|X|_{g'} = \epsilon e^{|t|} \leq e^{-f(t)}$ since $f(t) \leq -|t| - \ln \epsilon$. The argument is similar if X is parallel to S^{n-k-1}. If X is tangent to W', then $|X|_g = |X|_h \leq C|X|_{\tilde{h}_t} \leq Ce^{-f(t)}|X|_{g'_\theta}$, and the claim follows.

The relations (40) and (43) imply

$$\begin{aligned}|\widetilde{T}_t(X,Y)| &\le Ce^{2f(t)}|T(X,Y)|\\ &\le Ce^{2f(t)}r|X|_{g'}|Y|_{g'}\\ &\le Cr|X|_{g'_\theta}|Y|_{g'_\theta}\end{aligned}$$

for all X, Y. In other words, this means

$$|\widetilde{T}_t|_{g'_\theta} \le Cr = C\epsilon e^{|t|} \le Ce^{-f(t)}. \tag{44}$$

Further, one can calculate that

$$|\nabla\widetilde{T}_t|_{g'_\theta} \le Ce^{-f(t)} \tag{45}$$

and

$$|\nabla^2\widetilde{T}_t|_{g'_\theta} \le Ce^{-f(t)}. \tag{46}$$

Here ∇ denotes the Levi-Civita-connection with respect to g'_θ. In particular, we see with Corollary A.2

$$|\mathrm{Scal}^{g_\theta} - \mathrm{Scal}^{g'_\theta}| \le Ce^{-f(t)}. \tag{47}$$

6.3. Geometric description of the new metrics. In this subsection we collect some facts about the geometry of F^2g and g'_θ introduced in the previous subsection. Most of the results are not needed for the proof of our result, but are useful to understand the underlying geometric concept of the argument. We will thus skip most of the proofs in this subsection.

The first proposition explains the special role of $\mathbb{H}^{k+1} \times \mathbb{S}^{n-k-1}$.

Proposition 6.2. *Let x_i be a sequence of points in $M \setminus W$, converging to W. Then the Riemann tensor of F^2g at x_i converges to the Riemann tensor of $\mathbb{H}^{k+1} \times \mathbb{S}^{n-k-1}$. The covariant derivative of the Riemann tensor of F^2g converges to zero. For any fixed $R > 0$, these convergences are uniform on balls (with respect to the metric F^2g) of radius R. In particular, for any fixed $R > 0$ the balls $(B^{F^2g}(x_i, R), x_i, F^2g)$ converge to a ball of radius R in $\mathbb{H}^{k+1} \times \mathbb{S}^{n-k-1}$ in the $C^{2,\alpha}$-topology of Riemannian manifolds with base point.*

The $C^{2,\alpha}$-topology of Riemannian manifolds with base point has its origins in Cheeger's finiteness theorem [**14**] and in the work of Gromov [**19**, **20**]. The article by Petersen [**46**, Pages 167–202] is a good introduction to the subject.

In the limit $r \searrow 0$ (or equivalently $t \to \infty$), the W-component of the metric F^2g grows exponentially. The motivation for introducing the function f into the definition of g_θ is to slow down this exponential growth: the diameter of the W-component with respect to g_θ is then bounded by $A_\theta \mathrm{diam}(W, g)$, where $\mathrm{diam}(W, g)$ is the diameter of W with respect to g. This slowing down has to be done carefully in order to get nice limit spaces. The properties claimed for f imply the following result.

Proposition 6.3. *Let θ_i be a sequence of positive numbers tending to zero, and let $x_i \in U_\epsilon^N(R_{max})$ be a sequence of points such that the limit $c := \lim(\frac{\partial}{\partial t} f)(t(x_i))$ exists. Then the Riemann tensor of g_{θ_i} at x_i converges to the Riemann tensor of $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$. The covariant derivative of the Riemann tensor of g_{θ_i} converges to zero. For any fixed $R > 0$, these convergences are uniform on balls (with respect to the metric g_{θ_i}) of radius R. In particular, for any fixed $R > 0$ the balls $(B^{g_{\theta_i}}(x_i, R), x_i, g_{\theta_i})$ converge to a ball of radius R in $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ in the $C^{2,\alpha}$-topology of Riemannian manifolds with base point.*

From this proposition it follows that the balls $(B^{F^2 g}(x_i, R), x_i, F^2 g)$ converge to a ball of radius R in $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ in the $C^{2,\alpha}$-topology of Riemannian manifolds with base point. Thus, we get an explanation why the spaces $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ appear as limit spaces.

The sectional curvature of $\mathbb{H}_c^{k+1}$ is $-c^2$. Hence the sectional curvatures of the product $\mathbb{H}_c^{k+1} \times \mathbb{S}^{n-k-1}$ are in the interval $[-c^2, 1]$. Using this fact, we can prove the following proposition.

Proposition 6.4. *The scalar curvatures of g_θ and g'_θ are bounded by a constant independent of θ.*

Proof. The metric g'_θ is the metric of a WS-bundle. Hence (16) is valid. We calculate $\partial_t \tilde{h}_t = (1/A_\theta)\chi'(t/A_\theta)(h_2 - h_1)$ and $\partial_t^2 \tilde{h}_t = (1/A_\theta)^2 \chi''(t/A_\theta)(h_2 - h_1)$. This implies $|\mathrm{tr}^{\tilde{h}_t} \partial_t \tilde{h}_t| \leq C/A_\theta$, $|\mathrm{tr}(\tilde{h}_t^{-1} \partial_t \tilde{h}_t)^2| \leq C/A_\theta^2$, and $|\mathrm{tr}^{\tilde{h}_t} \partial_t^2 \tilde{h}_t| \leq C/A_\theta^2$. From (16) it follows that $\mathrm{Scal}^{g'_\theta}$ is bounded. Equation (47) then implies that Scal^{g_θ} is bounded. q.e.d.

The geometry close to the gluing of $M_1 \setminus U_1(\epsilon)$ with $M_2 \setminus U_2(\epsilon)$ is described by the following simple proposition.

Proposition 6.5. *Let H be the metric on $W \times (-1, 1)$ given by $(\chi(t)h_2 + (1 - \chi(t))h_1) + dt^2$. Then $(U_\epsilon^N(\delta_0), g'_\theta)$ is isometric to $(W \times (-1, 1) \times S^{n-k-1}, A_\theta^2 H + \sigma^{n-k-1})$.*

6.4. Proof of Theorem 6.1. The metrics g_θ are defined for small $\theta > 0$ as described above. In order to prove Theorem 6.1, it is sufficient to prove

$$\min\{\mu(M, g), \Lambda_{n,k}\} \leq \lim_{i \to \infty} \mu(N, g_{\theta_i}) \leq \mu(M, g)$$

for any sequence $\theta_i \searrow 0$ as $i \to \infty$ for which $\lim_{i \to \infty} \mu(N, g_{\theta_i})$ exists. Recall that $(M, g) = (M_1 \amalg M_2, g_1 \amalg g_2)$.

The upper bound on $\lim_{i \to \infty} \mu(N, g_{\theta_i})$ is easy to prove. The proof of the lower bound is more complicated, and our arguments for this part are inspired by the compactness-concentration principle in analysis;

see, for example, [**17**]. In the case of a concentration, we will use blow-up analysis in order to construct a non-trivial solution to the Yamabe equation on some limit space. Here we follow and generalize a similar construction of a blow-up limit in lecture notes by Schoen; see [**53**, Chapter V.2].

For each metric g_θ we have a solution of the Yamabe equation (5). We take a sequence of θ tending to 0. Following the compactness-concentration principle, this sequence of solutions can concentrate in points or converge to a non-trivial solution or do both at the same time. The concentration in points can be used to construct a non-trivial solution on a sphere by blowing up the metrics.

In our situation we may have concentration in a fixed point (subcase I.1) or in a wandering point (subcase I.2), and we may have convergence to a non-trivial solution on the original manifold (subcase II.1.2) or in the attached part (subcases II.1.1 and II.2). In each of these cases we obtain a different lower bound for $\lim_{i\to\infty} \mu(N, g_{\theta_i})$: In the subcases I.1 and I.2, the lower bound is $\mu(\mathbb{S}^n)$, in subcase II.1.2 it is $\mu(M, g)$, and in the subcases II.1.1 and II.2 we obtain $\Lambda^{(1)}_{n,k}$ and $\Lambda^{(2)}_{n,k}$ as lower bounds. Together these cases give the lower bound of Theorem 6.1.

The cases here are not exclusive. For example, it is possible that the solutions may both concentrate in a point and converge to a non-trivial solution on the original manifold.

In our arguments we will often pass to subsequences. To avoid complicated notation, we write $\theta \searrow 0$ for a sequence $(\theta_i)_{i\in\mathbb{N}}$ converging to zero, and we will pass successively to subsequences without changing notation. Similarly, $\lim_{\theta\searrow 0} h(\theta)$ should be read as $\lim_{i\to\infty} h(\theta_i)$.

We set $\mu := \mu(M, g)$ and $\mu_\theta := \mu(N, g_\theta)$. From Theorem 1.1 we have

$$\mu, \mu_\theta \leq \mu(\mathbb{S}^n). \tag{48}$$

After passing to a subsequence, the limit

$$\bar{\mu} := \lim_{\theta\searrow 0} \mu_\theta \in [-\infty, \mu(\mathbb{S}^n)]$$

exists. Let $J := J^g$ and $J_\theta := J^{g_\theta}$ be defined as in (1).

We start with the easier part of the argument—namely, with

$$\bar{\mu} \leq \mu. \tag{49}$$

For this let $\alpha > 0$ be a small number. We choose a smooth cut-off function χ_α on M such that $\chi_\alpha = 1$ on $M \setminus U(2\alpha)$, $|d\chi_\alpha| \leq 2/\alpha$, and $\chi_\alpha = 0$ on $U(\alpha)$. Let u be a smooth non-zero function such that $J(u) \leq \mu + \delta$ where δ is a small positive number. On the support of χ_α the metrics g and g_θ are conformal since $g_\theta = F^2 g$, and hence by (3) we have

$$\mu_\theta \leq J_\theta\left(\chi_\alpha F^{-\frac{n-2}{2}} u\right) = J(\chi_\alpha u)$$

for $\theta < \alpha$. It is straightforward to compute that $\lim_{\alpha \searrow 0} J(\chi_\alpha u) = J(u) \le \mu + \delta$. From this Relation (49) follows.

Now we turn to the more difficult part of the proof—namely, the inequality

$$\bar{\mu} \ge \min\{\mu, \Lambda_{n,k}\}. \tag{50}$$

In the case $\bar{\mu} = \mu(\mathbb{S}^n)$ this inequality follows trivially from (48). Hence we assume $\bar{\mu} < \mu(\mathbb{S}^n)$ in the following, which implies $\mu_\theta < \mu(\mathbb{S}^n)$ if θ is sufficiently small. From Theorem 1.2 we know that there exist positive functions $u_\theta \in C^2(M)$ such that

$$L^{g_\theta} u_\theta = \mu_\theta u_\theta^{p-1} \tag{51}$$

and

$$\int_N u_\theta^p \, dv^{g_\theta} = 1.$$

We begin by proving a lemma that yields a bound of the L^2-norm of u_θ in terms of the L^∞-norm. This result is non-trivial since $\mathrm{Vol}(N, g_\theta) \to \infty$ as $\theta \searrow 0$.

Lemma 6.6. *Assume that there exists $b > 0$ such that*

$$\mu_\theta \sup_{U_\epsilon^N(b)} u_\theta^{p-2} \le \frac{(n-k-2)^2(n-1)}{8(n-2)}$$

for θ small enough. Then there exist constants $c_1, c_2 > 0$ independent of θ such that

$$\int_N u_\theta^2 \, dv^{g_\theta} \le c_1 \|u_\theta\|^2_{L^\infty(N)} + c_2$$

for all sufficiently small θ. In particular, if $\|u_\theta\|_{L^\infty(N)}$ is bounded, so is $\|u_\theta\|_{L^2(N)}$.

Proof. Let $\tilde{r} \in (0, b)$ be fixed, and set $P = U(\tilde{r})$. Then P is a WS-bundle where, with the notation of Section 5, $I = (\alpha, \beta)$ with $\alpha = -\ln \tilde{r} + \ln \epsilon$ and $\beta = \ln \tilde{r} - \ln \epsilon$. On P we have two natural metrics: g_θ and $g_{\mathrm{WS}} = g'_\theta = g_\theta - \widetilde{T}_t$. The metric g_{WS} has exactly the form (14) with $\varphi = f$ and $h_t = \tilde{h}_t$. Let θ be small enough, and let $t \in (-\ln \tilde{r} + \ln \epsilon, -\ln \delta_0 + \ln \epsilon) \cup (\ln \delta_0 - \ln \epsilon, \ln \tilde{r} - \ln \epsilon)$. Then assumption (A_t) of Theorem 5.2 is true. Now, again if θ is small enough, we have for all $t \in (-\ln \delta_0 + \ln \epsilon, \ln \delta_0 - \ln \epsilon)$ the relation $\mathrm{Scal}^{g_{\mathrm{WS}}} = \mathrm{Scal}^{\sigma^{n-k-1}} + O(1/A_\theta)$. The error term $e(\tilde{h}_t)$ from (B_t) in this case satisfies

$$2(n-1)|e(\tilde{h}_t)| \le \left| \mathrm{tr}^{\tilde{h}_t} \partial_t \tilde{h}_t \right| = \left| \mathrm{tr}^{\tilde{h}_t} \left(\chi'(t/A_\theta) \frac{h_2 - h_1}{A_\theta} \right) \right| \le \frac{C}{A_\theta}$$

and

$$2(n-1)|\partial_t e(\tilde{h}_t)| = \left| \mathrm{tr}\left(\tilde{h}_t^{-1} (\partial_t \tilde{h}_t) \tilde{h}_t^{-1} (\partial_t \tilde{h}_t) \right) \right| + \left| \mathrm{tr}^{\tilde{h}_t} \partial_t^2 \tilde{h}_t \right| \le \frac{C}{A_\theta^2}.$$

Because of $1/A_\theta \leq \theta$ condition (B_t) is true. Equation (51) is written in the metric g_θ. Using the expression of the Laplacian in local coordinates,

$$\Delta^{g_\theta} u = -\sum_{i,j} (\det g_\theta)^{-1/2} \partial_i \left(g_\theta^{ij} (\det g_\theta)^{1/2} \partial_j u \right),$$

one can check that if we write equation (51) in the metric g_{WS}, we obtain an equation of the form (17) with $\mu = \mu_\theta$. Together with (44), (45), and (47), one verifies that the error terms satisfy

$$|A(x)|_{g_{\mathrm{WS}}}, |X(x)|_{g_{\mathrm{WS}}}, |s(x)|_{g_{\mathrm{WS}}}, |\epsilon(x)|_{g_{\mathrm{WS}}} \leq Ce^{-f(t)},$$

where $|\cdot|_{g_{\mathrm{WS}}}$ denotes the pointwise norm at a point in $U_\epsilon^N(R_0)$, and where C is a constant independent of θ. In particular, for any $c_0 > 0$, we obtain

$$|A(x)|_{g_{\mathrm{WS}}}, |X(x)|_{g_{\mathrm{WS}}}, |s(x)|_{g_{\mathrm{WS}}}, |e(\tilde{h}_t)(x)|_{g_{\mathrm{WS}}}, |\epsilon(x)|_{g_{\mathrm{WS}}} \leq c_0$$

on $U_\epsilon^N(\theta)$ for small θ. These estimates allow us to apply Theorem 5.2. By the assumptions of Lemma 6.6, if $\tilde{r} \in (0, b)$ is small enough, assumption (18) of Theorem 5.2 is true. Thus, all hypotheses of Theorem 5.2 hold for $\alpha := -\ln \tilde{r} + \ln \epsilon$, $\beta := \ln \tilde{r} - \ln \epsilon$, and hence

$$\int_{P'} u_\theta^2 \, dv^{g_{\mathrm{WS}}} \leq \frac{4\|u_\theta\|_{L^\infty}^2}{n-k-2} \left(\mathrm{Vol}^{g_\alpha}(F_\alpha) + \mathrm{Vol}^{g_\beta}(F_\beta) \right).$$

where $P' := U_\epsilon^N(\tilde{r}e^{-\gamma})$. Now observe that

$$C := \frac{4}{n-k-2} \left(\mathrm{Vol}^{g_\alpha}(F_\alpha) + \mathrm{Vol}^{g_\beta}(F_\beta) \right)$$

does not depend on θ (since F_α and F_β correspond to the hypersurface $r = \tilde{r}$). This implies that

$$\int_{P'} u_\theta^2 \, dv^{g_{\mathrm{WS}}} \leq C \|u_\theta\|_{L^\infty(N)}^2$$

where $C > 0$ is independent of θ. Since if $\tilde{r}$ is small enough, we clearly have

$$dv^{g_\theta} \leq 2 dv^{g_{\mathrm{WS}}},$$

and we obtain that

$$\int_{P'} u_\theta^2 \, dv^{g_\theta} \leq c_1 \|u_\theta\|_{L^\infty(N)}^2$$

where $c_1 := 2C > 0$ is independent of θ. Now observe that $\mathrm{Vol}^{g_\theta}(N \setminus P')$ is bounded by a constant independent of θ. Using the Hölder inequality, we obtain

$$\begin{aligned} \int_N u_\theta^2 \, dv^{g_\theta} &= \int_{P'} u_\theta^2 \, dv^{g_\theta} + \int_{N \setminus P'} u_\theta^2 \, dv^{g_\theta} \\ &\leq c_1 \|u_\theta\|_{L^\infty(N)}^2 + \mathrm{Vol}^{g_\theta}(N \setminus P')^{\frac{2}{n}} \left(\int_{N \setminus P'} u_\theta^p \, dv \right)^{\frac{n-2}{n}}. \end{aligned}$$

Since $\|u_\theta\|_{L^p(N)} = 1$, this proves Lemma 6.6 with c_1 as defined above and with $c_2 := \mathrm{Vol}^{g_\theta}(N \setminus P')^{\frac{2}{n}}$. For small θ, the metric $g_\theta|_{N\setminus P'}$ is independent of θ, and thus c_2 does not depend on θ. q.e.d.

Corollary 6.7.

$$\liminf_{\theta \searrow 0} \|u_\theta\|_{L^\infty(N)} > 0.$$

Proof. We set $m_\theta := \|u_\theta\|_{L^\infty(N)}$. In order to prove the corollary by contradiction, we assume $\lim_{\theta\searrow 0} m_\theta = 0$. Then since $\mu_\theta \leq \mu(\mathbb{S}^n)$ the assumption of Lemma 6.6 is satisfied for all $\theta > 0$ sufficiently small, and for all $b > 0$ for which $U_\epsilon^N(b)$ is defined. We get the contradiction

$$1 = \int_N u_\theta^p dv^{g_\theta} \leq m_\theta^{p-2} \int_N u_\theta^2 dv^g \leq m_\theta^{p-2}(c_1 m_\theta^2 + c_2) \to 0$$

as $\theta \searrow 0$. q.e.d.

Corollary 6.8.

$$\bar{\mu} = \lim_{\theta\searrow 0} \mu_\theta > -\infty.$$

Proof. Choose x_θ as above. We then have $\Delta^{g_\theta} u_\theta(x_\theta) \geq 0$, which together with (51) gives us

$$\mathrm{Scal}^{g_\theta}(x_\theta)\|u_\theta\|_{L^\infty(N)} \leq \mu_\theta \|u_\theta\|^{p-1}_{L^\infty(N)}.$$

Proposition 6.4 and the previous corollary then imply that μ_θ is bounded from below. q.e.d.

In addition, by Theorem 1.1, μ_θ is bounded from above by $\mu(\mathbb{S}^n)$. It follows that $\bar{\mu} \in \mathbb{R}$. The rest of the proof is divided into cases.

Case I. $\limsup_{\theta\searrow 0} \|u_\theta\|_{L^\infty(N)} = \infty$.

As before, we set $m_\theta := \|u_\theta\|_{L^\infty(N)}$ and choose $x_\theta \in N$ with $u_\theta(x_\theta) = m_\theta$. After again taking a subsequence, we can assume that $\lim_{\theta\searrow 0} m_\theta = \infty$. We consider two subcases.

Subcase I.1. There exists $b > 0$ such that $x_\theta \in N \setminus U_\epsilon^N(b)$ for an infinite number of θ.

We recall that $N_\epsilon \setminus U_\epsilon^N(b) = M_1 \amalg M_2 \setminus U(b)$. By taking a subsequence, we can assume that there exists $\bar{x} \in M_1 \amalg M_2 \setminus U(b)$ such that $\lim_{\theta\searrow 0} x_\theta = \bar{x}$. We let $\tilde{g}_\theta := m_\theta^{\frac{4}{n-2}} g_\theta$. In a neighborhood U of $\bar{x}$ the metric $g_\theta = F^2 g$ does not depend on θ. We apply Lemma 4.1 with $O = U$, $\alpha = \theta$, $q_\alpha = x_\theta$, $q = \bar{x}$, $\gamma_\alpha = g_\theta = F^2 g$, and $b_\alpha = m_\theta^{\frac{2}{n-2}}$. Let $r > 0$. For θ small enough Lemma 4.1 gives us a diffeomorphism

$$\Theta_\theta : B^n(r) \to B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-2}} r)$$

such that the sequence of metrics $(\Theta_\theta^*(\tilde g_\theta))$ tends to the flat metric ξ^n in $C^2(B^n(r))$. We let $\tilde u_\theta := m_\theta^{-1} u_\theta$. By (2) we then have

$$L^{\tilde g_\theta} \tilde u_\theta = \mu_\theta \tilde u_\theta^{p-1}$$

on $B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-2}} r)$, and, using the fact that $dv^{\tilde g_\theta} = m_\theta^p \, dv^{g_\theta}$, we have

$$\begin{aligned}\int_{B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-2}} r)} \tilde u_\theta^p \, dv^{\tilde g_\theta} &= \int_{B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-2}} r)} u_\theta^p \, dv^{g_\theta} \\ &\leq \int_N u_\theta^p dv^{g_\theta} \\ &= 1.\end{aligned}$$

Since

$$\Theta_\theta : (B^n(r), \Theta_\theta^*(\tilde g_\theta)) \to (B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-2}} r), \tilde g_\theta)$$

is an isometry, we can consider $\tilde u_\theta$ as a solution of

$$L^{\Theta_\theta^*(\tilde g_\theta)} \tilde u_\theta = \mu_\theta \tilde u_\theta^{p-1}$$

on $B^n(r)$ with $\int_{B^n(r)} \tilde u_\theta^p \, dv^{\Theta_\theta^*(\tilde g_\theta)} \leq 1$. Since $\|\tilde u_\theta\|_{L^\infty(B^n(r))} = |\tilde u_\theta(0)| = 1$, we can apply Lemma 4.2 with $V = \mathbb{R}^n$, $\alpha = \theta$, $g_\alpha = \Theta_\theta^*(\tilde g_\theta)$, and $u_\alpha = \tilde u_\theta$ (we can apply this lemma since each compact set of $\mathbb{R}^n$ is contained in some ball $B^n(r)$). This shows that there exists a non-negative function $u \not\equiv 0$ (since $u(0) = 1$) of class C^2 on $(\mathbb{R}^n, \xi^n)$ that satisfies

$$L^{\xi^n} u = a\Delta^{\xi^n} u = \bar\mu u^{p-1}.$$

By (12) we further have

$$\int_{B^n(r)} u^p \, dv^{\xi^n} = \lim_{\theta \searrow 0} \int_{B^{g_\theta}(x_\theta, m_\theta^{-\frac{2}{n-2}} r)} u_\theta^p \, dv^{g_\theta} \leq 1$$

for any $r > 0$. In particular,

$$\int_{\mathbb{R}^n} u^p \, dv^{\xi^n} \leq 1.$$

From Lemma 4.3, we get that $\bar\mu \geq \mu(\mathbb{S}^n) \geq \min\{\mu, \Lambda_{n,k}\}$. We have proved inequality (50) in this subcase.

Subcase I.2. For all $b > 0$ it holds that $x_\theta \in U_\epsilon^N(b)$ for θ sufficiently small.

The subset $U_\epsilon^N(b)$ is diffeomorphic to $W \times I \times S^{n-k-1}$ where I is an interval. We identify

$$x_\theta = (y_\theta, t_\theta, z_\theta)$$

where $y_\theta \in W$, $t_\theta \in (-\ln R_0 + \ln \epsilon, -\ln \epsilon + \ln R_0)$, and $z_\theta \in S^{n-k-1}$. By taking a subsequence we can assume that y_θ, $\frac{t_\theta}{A_\theta}$, and z_θ converge, respectively, to $y \in W$, $T \in [-\infty, +\infty]$, and $z \in S^{n-k-1}$. First, we apply Lemma 4.1 with $V = W$, $\alpha = \theta$, $q_\alpha = y_\theta$, $q = y$, $\gamma_\alpha = \tilde h_{t_\theta}$, $\gamma_0 = \tilde h_T$

(we define $\tilde{h}_{-\infty} = h_1$ and $\tilde{h}_{+\infty} = h_2$), and $b_\alpha = m_\theta^{\frac{2}{n-2}} e^{f(t_\theta)}$. The lemma provides diffeomorphisms

$$\Theta_\theta^y : B^k(r) \to B^{\tilde{h}_{t_\theta}}(y_\theta, m_\theta^{-\frac{2}{n-2}} e^{-f(t_\theta)} r)$$

for $r > 0$ such that $(\Theta_\theta^y)^*(m_\theta^{\frac{4}{n-2}} e^{2f(t_\theta)} \tilde{h}_{t_\theta})$ tends to the flat metric ξ^k on $B^k(r)$ as $\theta \searrow 0$. Second, we apply Lemma 4.1 with $V = S^{n-k-1}$, $\alpha = \theta$, $q_\alpha = z_\theta$, $\gamma_\alpha = \gamma_0 = \sigma^{n-k-1}$, and $b_\alpha = m_\theta^{\frac{2}{n-2}}$. For $r' > 0$ we get diffeomorphisms

$$\Theta_\theta^z : B^{n-k-1}(r') \to B^{\sigma^{n-k-1}}(z_\theta, m_\theta^{-\frac{2}{n-2}} r')$$

such that $(\Theta_\theta^z)^*(m_\theta^{\frac{4}{n-2}} \sigma^{n-k-1})$ converges to ξ^{n-k-1} on $B^{n-k-1}(r')$ as $\theta \searrow 0$. For $r, r', r'' > 0$ we define

$$\begin{aligned} U_\theta(r, r', r'') := B^{\tilde{h}_{t_\theta}}(y_\theta, m_\theta^{-\frac{2}{n-2}} e^{-f(t_\theta)} r) \times [t_\theta - m_\theta^{-\frac{2}{n-2}} r'', t_\theta + m_\theta^{-\frac{2}{n-2}} r''] \\ \times B^{\sigma^{n-k-1}}(z_\theta, m_\theta^{-\frac{2}{n-2}} r'), \end{aligned}$$

and

$$\Theta_\theta : B^k(r) \times [-r'', r''] \times B^{n-k-1}(r') \to U_\theta(r, r', r'')$$

by

$$\Theta_\theta(y, s, z) := (\Theta_\theta^y(y), t(s), \Theta_\theta^z(z)),$$

where $t(s) := t_\theta + m_\theta^{\frac{2}{n-2}} s$. By construction Θ_θ is a diffeomorphism, and we see that

$$\begin{aligned} \Theta_\theta^*(m_\theta^{\frac{4}{n-2}} g_\theta) = (\Theta_\theta^y)^*(m_\theta^{\frac{4}{n-2}} e^{2f(t)} \tilde{h}_t) + ds^2 \\ + (\Theta_\theta^z)^*(m_\theta^{\frac{4}{n-2}} \sigma^{n-k-1}) + \Theta_\theta^*(m_\theta^{\frac{4}{n-2}} \widetilde{T}_t). \end{aligned} \tag{52}$$

Next, we study the first term on the right-hand side of (52). Note that it is here evaluated at t, while we have information above when evaluated at t_θ. By construction of $f(t)$, one can verify that

$$\lim_{\theta \searrow 0} \left\| \frac{e^{f(t_\theta)}}{e^{f(t)}} - 1 \right\|_{C^2([t_\theta - m_\theta^{-\frac{2}{n-2}} r'', t_\theta + m_\theta^{-\frac{2}{n-2}} r''])} = 0$$

since $\frac{df}{dt}$ and $\frac{d^2 f}{dt^2}$ are uniformly bounded. Moreover, it is clear that

$$\lim_{\theta \searrow 0} \left\| \tilde{h}_t - \tilde{h}_{t_\theta} \right\|_{C^2(B^{\tilde{h}_{t_\theta}}(y_\theta, m_\theta^{-\frac{2}{n-2}} e^{-f(t_\theta)} r))} = 0$$

uniformly in $t \in [t_\theta - m_\theta^{-\frac{2}{n-2}} r'', t_\theta + m_\theta^{-\frac{2}{n-2}} r'']$. As a consequence,

$$\lim_{\theta \searrow 0} \left\| (\Theta_\theta^y)^* \left(m_\theta^{\frac{4}{n-2}} \left(e^{2f(t)} \tilde{h}_t - e^{2f(t_\theta)} \tilde{h}_{t_\theta} \right) \right) \right\|_{C^2(B^k(r))} = 0$$

uniformly for $t \in [t_\theta - m_\theta^{-\frac{2}{n-2}} r'', t_\theta + m_\theta^{-\frac{2}{n-2}} r'']$. This implies that the sequence $(\Theta_\theta^y)^*(m_\theta^{\frac{4}{n-2}} e^{2f(t)} \tilde{h}_t)$ tends to the flat metric ξ^k in $C^2(B^k(r))$ uniformly in t as $\theta \searrow 0$. Further, we also know that the sequence $(\Theta_\theta^z)^*(m_\theta^{\frac{4}{n-2}} \sigma^{n-k-1})$ tends to ξ^{n-k-1} in $C^2(B^{n-k-1}(r'))$ as $\theta \searrow 0$. Recalling from (42) that $g'_\theta = g_\theta - \widetilde{T}_t$, we have proved that $\Theta_\theta^*(m_\theta^{\frac{4}{n-2}} g'_\theta)$ tends to the flat metric in $C^2(B^k(r) \times [-r'', r''] \times B^{n-k-1}(r'))$. Finally, we are going to show that the last term of (52) tends to zero in C^2. It follows from (44) that

$$\lim_{\theta \searrow 0} \left\| \Theta_\theta^*(m_\theta^{\frac{4}{n-2}} \widetilde{T}_t) \right\|_{C^2(B^k(r) \times [-r'', r''] \times B^{n-k-1}(r'))} = 0. \tag{53}$$

Indeed, (44) tells us that

$$\begin{aligned}
\left| \Theta_\theta^*(m_\theta^{\frac{4}{n-2}} \widetilde{T}_t)(X, Y) \right| &= m_\theta^{\frac{4}{n-2}} \left| \widetilde{T}_t(\Theta_{\theta*}(X), \Theta_{\theta*}(Y)) \right| \\
&\le C r m_\theta^{\frac{4}{n-2}} |\Theta_{\theta*}(X)|_{g'_\theta} |\Theta_{\theta*}(Y)|_{g'_\theta} \\
&\le C r |X|_{\Theta_\theta^*(m_\theta^{\frac{4}{n-2}} g'_\theta)} |X|_{\Theta_\theta^*(m_\theta^{\frac{4}{n-2}} g'_\theta)},
\end{aligned}$$

and since $\Theta_\theta^*(m_\theta^{\frac{4}{n-2}} g'_\theta)$ tends to the flat metric, we get (53). Doing the same with $\nabla \widetilde{T}_t$ and $\nabla^2 \widetilde{T}_t$ using (45) and (46), we obtain that

$$\lim_{\theta \searrow 0} \Theta_\theta^*(m_\theta^{\frac{4}{n-2}} \widetilde{T}_t) = 0 \tag{54}$$

in $C^2(B^k(r) \times [-r'', r''] \times B^{n-k-1}(r'))$. Returning to (52), we see that the sequence $\Theta_\theta^*(m_\theta^{\frac{4}{n-2}} g_\theta)$ tends to $\xi^n = \xi^k + ds^2 + \xi^{n-k-1}$ on $B^k(r) \times [-r'', r''] \times B^{n-k-1}(r')$. We proceed as in Subcase I.1 to show that $\bar{\mu} \ge \mu(\mathbb{S}^n) \ge \min\{\mu, \Lambda_{n,k}\}$, which proves relation (50) in the present subcase. This ends the proof of Theorem 6.1 in Case I.

Case II. There exists a constant C_1 such that $\|u_\theta\|_{L^\infty(N)} \le C_1$ for all θ.

As in Case I, we consider two subcases.

Subcase II.1. There exists $b > 0$ such that

$$\liminf_{\theta \searrow 0} \left(\mu_\theta \sup_{U_\epsilon^N(b)} u_\theta^{p-2} \right) < \frac{(n-k-2)^2(n-1)}{8(n-2)}.$$

By restricting to a subsequence, we can assume that

$$\mu_\theta \sup_{U_\epsilon^N(b)} u_\theta^{p-2} < \frac{(n-k-2)^2(n-1)}{8(n-2)}$$

for all θ. Lemma 6.6 tells us that there is a constant $A_0 > 0$ such that

$$\|u_\theta\|_{L^2(N,g_\theta)} \le A_0. \tag{55}$$

We split the treatment of Subcase II.1. into two subsubcases.

Subsubcase II.1.1. $\limsup_{b\searrow 0}\limsup_{\theta\searrow 0}\sup_{U_\epsilon^N(b)} u_\theta > 0$.

We set $D_0 := \frac{1}{2}\limsup_{b\searrow 0}\limsup_{\theta\searrow 0}\sup_{U_\epsilon^N(b)} u_\theta > 0$. Then there are sequences (b_i) and (θ_i) of positive numbers converging to 0 such that

$$\sup_{U_\epsilon^N(b_i)} u_{\theta_i} \ge D_0$$

for all i. For brevity of notation we write θ for θ_i and b_θ for b_i. Let $x'_\theta \in \overline{U_\epsilon^N(b_\theta)}$ be such that

$$u_\theta(x'_\theta) \ge D_0. \tag{56}$$

As in Subcase I.2 above we write $x'_\theta = (y_\theta, t_\theta, z_\theta)$ where $y_\theta \in W$, $t_\theta \in (-\ln R_0 + \ln\epsilon, -\ln\epsilon + \ln R_0)$, and $z_\theta \in S^{n-k-1}$. By restricting to a subsequence we can assume that y_θ, $\frac{t_\theta}{A_\theta}$, and z_θ converge, respectively, to $y \in W$, $T \in [-\infty, +\infty]$, and $z \in S^{n-k-1}$. We apply Lemma 4.1 with $V = W$, $\alpha = \theta$, $q_\alpha = y_\theta$, $q = y$, $\gamma_\alpha = \tilde{h}_{t_\theta}$, $\gamma_0 = \tilde{h}_T$, and $b_\alpha = e^{f(t_\theta)}$, and we conclude that there is a diffeomorphism

$$\Theta_\theta^y : B^k(r) \to B^{\tilde{h}_{t_\theta}}(y_\theta, e^{-f(t_\theta)}r)$$

for $r > 0$ such that $(\Theta_\theta^y)^*(e^{2f(t_\theta)}\tilde{h}_{t_\theta})$ converges to the flat metric ξ^k on $B^k(r)$. For $r, r' > 0$ we set

$$U_\theta(r,r') := B^{\tilde{h}_{t_\theta}}(y_\theta, e^{-f(t_\theta)}r) \times [t_\theta - r', t_\theta + r'] \times S^{n-k-1}$$

and we define

$$\Theta_\theta : B^k(r) \times [-r', r'] \times S^{n-k-1} \to U_\theta(r,r')$$

by

$$\Theta_\theta(y,s,z) := (\Theta_\theta^y(y), t(s), z),$$

where $t(s) := t_\theta + s$. By construction, Θ_θ is a diffeomorphism, and we see that

$$\Theta_\theta^*(g_\theta) = \frac{e^{2f(t)}}{e^{2f(t_\theta)}}(\Theta_\theta^y)^*(e^{2f(t_\theta)}\tilde{h}_t) + ds^2 + \sigma^{n-k-1} + \Theta_\theta^*(\widetilde{T}_t). \tag{57}$$

We will now find the limit of $\Theta_\theta^*(g_\theta)$ in the C^2 topology. We define $c := \lim_{\theta\searrow 0} f'(t_\theta)$, which can be assumed to exist without loss of generality.

Lemma 6.9. *For fixed $r, r' > 0$ the sequence of metrics $\Theta_\theta^*(g_\theta)$ tends to $G_c = \eta_c^{k+1} + \sigma^{n-k-1} = e^{2cs}\xi^k + ds^2 + \sigma^{n-k-1}$ in the topological space $C^2(B^k(r) \times [-r', r'] \times S^{n-k-1})$.*

As this lemma coincides with [**4**, Lemma 4.1], we only sketch the proof.

Proof. The intermediate value theorem tells us that

$$\big|f(t) - f(t_\theta) - f'(t_\theta)(t - t_\theta)\big| \le \frac{r'^2}{2} \max_{s\in[t_\theta - r', t_\theta + r']} \big|f''(s)\big|$$

for all $t \in [t_\theta - r', t_\theta + r']$. Because of (41) we also have $\|f''\|_{L^\infty} \to 0$ for $\theta \searrow 0$, and hence

$$\lim_{\theta\searrow 0} \big\|f(t) - f(t_\theta) - f'(t_\theta)(t - t_\theta)\big\|_{C^0([t_\theta - r', t_\theta + r'])} = 0$$

for r' fixed. Further, we have

$$\begin{aligned}
\left|\frac{d}{dt}\Big(f(t) - f(t_\theta) - f'(t_\theta)(t - t_\theta)\Big)\right| &= \big|f'(t) - f'(t_\theta)\big| \\
&= \left|\int_{t_\theta}^{t} f''(s)\, ds\right| \\
&\le r' \max_{s\in[t_\theta - r', t_\theta + r']} \big|f''(s)\big| \\
&\to 0
\end{aligned}$$

as $\theta \searrow 0$, and finally

$$\left|\frac{d^2}{dt^2}\big(f(t) - f(t_\theta) - f'(t_\theta)(t - t_\theta)\big)\right| = \big|f''(t)\big| \to 0$$

as $\theta \searrow 0$. Together with $c = \lim_{\theta\searrow 0} f'(t_\theta)$, we have shown that

$$\lim_{\theta\searrow 0} \|f(t) - f(t_\theta) - c(t - t_\theta)\|_{C^2([t_\theta - r', t_\theta + r'])} = 0.$$

Hence

$$\lim_{\theta\searrow 0} \left\|e^{f(t)-f(t_\theta)} - e^{c(t-t_\theta)}\right\|_{C^2([t_\theta - r', t_\theta + r'])} = 0.$$

We now write $e^{2f(t)}\tilde{h}_t = e^{2f(t)}(\tilde{h}_t - \tilde{h}_{t_\theta}) + \frac{e^{2f(t)}}{e^{2f(t_\theta)}} e^{2f(t_\theta)}\tilde{h}_{t_\theta}$. Using the fact that

$$\lim_{\theta\searrow 0} \left\|\tilde{h}_t - \tilde{h}_{t_\theta}\right\|_{C^2(B^{\tilde{h}_{t_\theta}}(y_\theta, e^{-f(t_\theta)}r))} = 0$$

holds uniformly for $t \in [t_\theta - r', t_\theta + r']$, we obtain that the sequence $\frac{e^{2f(t)}}{e^{2f(t_\theta)}}(\Theta_\theta^y)^*(e^{2f(t_\theta)}\tilde{h}_t)$ tends to $e^{2cs}\xi^k$ in the $C^2(B^k(r))$-topology where, as before, $s = t - t_\theta \in [-r', r']$. Finally, proceeding exactly as we did to get relation (54), we have that

$$\lim_{\theta\searrow 0} \Theta_\theta^*(\widetilde{T}_t) = 0$$

in $C^2(B^k(r) \times [-r', r'] \times S^{n-k-1})$. Now Lemma 6.9 follows from (57). q.e.d.

We continue with the proof of Subsubcase II.1.1. As in Subcases I.1 and I.2 we apply Lemma 4.2 with $(V, g) = (\mathbb{R}^{k+1} \times S^{n-k-1}, G_c)$, $\alpha = \theta$, and $g_\alpha = \Theta_\theta^*(g_\theta)$ (we can apply this lemma since any compact subset of $\mathbb{R}^{k+1} \times S^{n-k-1}$ is contained in some $B^k(r) \times [-r', r'] \times S^{n-k-1}$). We obtain a C^2 function $u \geq 0$ that is a solution of

$$L^{G_c} u = \bar{\mu} u^{p-1}$$

on $\mathbb{R}^{k+1} \times S^{n-k-1}$. From (12) it follows that

$$\int_{\mathbb{R}^{k+1} \times S^{n-k-1}} u^p \, dv^{G_c} \leq 1.$$

From (11) it follows that $u \in L^\infty(\mathbb{R}^{k+1} \times S^{n-k-1})$. With (56), we see that $u(0) \geq D_0$ and thus, $u \not\equiv 0$. By (55) we also get that $u \in L^2(\mathbb{R}^{k+1} \times S^{n-k-1})$. By the definition of $\Lambda^{(1)}_{n,k}$ we have that $\bar{\mu} \geq \Lambda^{(1)}_{n,k} \geq \Lambda_{n,k}$. This ends the proof of Theorem 6.1 in this subsubcase.

Subsubcase II.1.2. $\lim_{b \searrow 0} \limsup_{\theta \searrow 0} \sup_{U^N_\epsilon(b)} u_\theta = 0$.

The proof in this subsubcase proceeds in several steps.

Step 1. *We prove* $\lim_{b \searrow 0} \limsup_{\theta \searrow 0} \int_{U^N_\epsilon(b)} u_\theta^p \, dv^{g_\theta} = 0$.

Let $b > 0$. Using (55), we have

$$\int_{U^N_\epsilon(b)} u_\theta^p \, dv^{g_\theta} \leq A_0 \sup_{U^N_\epsilon(b)} u_\theta^{p-2}$$

where the constant A_0 is independent of b and θ. Step 1 follows.

Step 2. *We show* $\liminf_{b \searrow 0} \liminf_{\theta \searrow 0} \int_{U^N_\epsilon(2b) \setminus U^N_\epsilon(b)} u_\theta^2 \, dv^{g_\theta} = 0$.

Let

$$d_0 := \liminf_{b \searrow 0} \liminf_{\theta \searrow 0} \int_{U^N_\epsilon(2b) \setminus U^N_\epsilon(b)} u_\theta^2 \, dv^{g_\theta}.$$

We prove this step by contradiction and assume that $d_0 > 0$. Then there exists $\delta > 0$ such that for all $b \in (0, \delta]$,

$$\liminf_{\theta \searrow 0} \int_{U^N_\epsilon(2b) \setminus U^N_\epsilon(b)} u_\theta^2 \, dv^{g_\theta} \geq \frac{d_0}{2}.$$

For $m \in \mathbb{N}$ we set $V_m := U(2^{-m}\delta) \setminus U(2^{-(m+1)}\delta)$. In particular, we have

$$\liminf_{\theta \searrow 0} \int_{V_m} u_\theta^2 \, dv^{g_\theta} \geq \frac{d_0}{2}$$

for all m. Let $N_0 \in \mathbb{N}$. For $m \neq m'$ the sets V_m and $V_{m'}$ are disjoint. Hence we can write

$$\int_N u_\theta^2 \, dv^{g_\theta} \geq \int_{\bigcup_{m=0}^{N_0} V_m} u_\theta^2 \, dv^{g_\theta} \geq \sum_{m=0}^{N_0} \int_{V_m} u_\theta^2 \, dv^{g_\theta}$$

for θ small enough. This leads to

$$\begin{aligned}\liminf_{\theta\searrow 0}\int_N u_\theta^2\,dv^{g_\theta} &\geq \liminf_{\theta\searrow 0}\sum_{m=0}^{N_0}\int_{V_m} u_\theta^2\,dv^{g_\theta}\\ &\geq \sum_{m=0}^{N_0}\liminf_{\theta\searrow 0}\int_{V_m} u_\theta^2\,dv^{g_\theta}\\ &\geq (N_0+1)\frac{d_0}{2}.\end{aligned}$$

Since N_0 is arbitrary, this contradicts that (u_θ) is bounded in $L^2(N)$ and proves Step 2.

Step 3. *Conclusion.*

Let $d_0>0$. By Steps 1 and 2 we can find $b>0$ such that after passing to a subsequence, we have for all θ close to 0

$$\int_{N\setminus U_\epsilon^N(2b)} u_\theta^p\,dv^{g_\theta}\geq 1-d_0 \tag{58}$$

and

$$\int_{U_\epsilon^N(2b)\setminus U_\epsilon^N(b)} u_\theta^2\,dv^{g_\theta}\leq d_0. \tag{59}$$

Let $\chi\in C^\infty(M)$, $0\leq\chi\leq 1$, be a cut-off function equal to 0 on $U_\epsilon^N(b)$ and equal to 1 on $N\setminus U_\epsilon^N(2b)$. Since the set $U_\epsilon^N(2b)\setminus U_\epsilon^N(b)$ corresponds to $t\in[t_0-\ln 2,t_0]\cup[t_1,t_1+\ln 2]$ with $t_0=-\ln b+\ln\epsilon$ and $t_1=\ln b-\ln\epsilon$, we can assume that

$$|d\chi|_{g_\theta}\leq 2\ln 2. \tag{60}$$

We will use the function χu_θ to estimate μ. This function is supported in $N\setminus U_\epsilon^N(b)$. If θ is smaller than b, then $(N\setminus U_\epsilon^N(b),g_\theta)$ is isometric to $(M\setminus U^M(b),F^2g)$. In other words, $(N\setminus U_\epsilon^N(b),g_\theta)$ is conformally equivalent to $(M\setminus U^M(b),g)$. Relation (3) implies that

$$\mu\leq J_\theta(\chi u_\theta)=\frac{\int_N(a|d(\chi u_\theta)|_{g_\theta}^2+\mathrm{Scal}^{g_\theta}(\chi u_\theta)^2)\,dv^{g_\theta}}{\left(\int_N(\chi u_\theta)^p\,dv^{g_\theta}\right)^{\frac{n-2}{n}}}. \tag{61}$$

We multiply equation (51) by $\chi^2 u_\theta$ and integrate over N. We can rewrite the result using the following form of (65),

$$\int_N|d(\chi u_\theta)|_{g_\theta}^2\,dv^{g_\theta}=\int_N\chi^2u_\theta\Delta^{g_\theta}u_\theta\,dv^{g_\theta}+\int_N|d\chi|_{g_\theta}^2u_\theta^2\,dv^{g_\theta},$$

to obtain

$$\begin{aligned}\int_N \big(a|d(\chi u_\theta)|^2_{g_\theta} + \mathrm{Scal}^{g_\theta}(\chi u_\theta)^2\big)\, dv^{g_\theta} \\ &= \mu_\theta \int_N u_\theta^p \chi^2\, dv^{g_\theta} + a \int_N |d\chi|^2_{g_\theta} u_\theta^2\, dv^{g_\theta} \\ &\le \mu_\theta \int_N u_\theta^p\, dv^{g_\theta} + |\mu_\theta| \int_{U_\epsilon^N(2b)} u_\theta^p\, dv^{g_\theta} \\ &\quad + a \int_N |d\chi|^2_{g_\theta} u_\theta^2\, dv^{g_\theta}.\end{aligned}$$

Using (59) and (60), we have

$$\int_N |d\chi|^2_{g_\theta} u_\theta^2\, dv^{g_\theta} = \int_{U_\epsilon^N(2b)\setminus U_\epsilon^N(b)} |d\chi|^2_{g_\theta} u_\theta^2\, dv^{g_\theta} \le 4(\ln 2)^2 d_0.$$

Relation (58) implies $\int_{U_\epsilon^N(2b)} u_\theta^p\, dv^{g_\theta} \le d_0$. Together with $\int_N u_\theta^p\, dv^{g_\theta} = 1$, we have

$$\int_N (a|d(\chi u_\theta)|^2_{g_\theta} + \mathrm{Scal}^{g_\theta}(\chi u_\theta)^2)\, dv^{g_\theta} \le \mu_\theta + |\mu_\theta| d_0 + 4(\ln 2)^2 a d_0. \tag{62}$$

In addition, by Relation (58),

$$\int_N (\chi u_\theta)^p\, dv^{g_\theta} \ge 1 - d_0. \tag{63}$$

Plugging (62) and (63) in (61), we get

$$\mu \le \frac{\mu_\theta + |\mu_\theta| d_0 + 4(\ln 2)^2 a d_0}{(1-d_0)^{\frac{n-2}{n}}}$$

for small θ. By taking the limit $\theta \searrow 0$, we can replace μ_θ by $\bar\mu$ in this inequality. Since d_0 can be chosen arbitrarily small, we finally obtain $\mu \le \bar\mu$. This proves Theorem 6.1 in Subcase II.1.

Subcase II.2. For all $b > 0$, we have

$$\liminf_{\theta \searrow 0} \left(\mu_\theta \sup_{U_\epsilon^N(b)} u_\theta^{p-2} \right) \ge \frac{(n-k-2)^2(n-1)}{8(n-2)}.$$

Hence, we can construct a subsequence of θ and a sequence (b_θ) of positive numbers converging to 0 with

$$\liminf_{\theta \searrow 0} \left(\mu_\theta \sup_{U_\epsilon^N(b_\theta)} u_\theta^{p-2} \right) \ge \frac{(n-k-2)^2(n-1)}{8(n-2)}.$$

Choose a point $x''_\theta \in \overline{U_\epsilon^N(b_\theta)}$ such that $u_\theta(x''_\theta) = \sup_{U_\epsilon^N(b_\theta)} u_\theta$. Since $\mu_\theta \le \mu(\mathbb{S}^n)$, we have

$$u_\theta(x''_\theta) \ge D_1 := \left(\frac{(n-k-2)^2(n-1)}{8\mu(\mathbb{S}^n)(n-2)} \right)^{\frac{1}{p-2}}.$$

With similar arguments as in Subcase II.1.1 (just replace x'_θ by x''_θ and D_0 by D_1), we get the existence of a C^2 function $u \geq 0$ that is a solution of

$$L^{G_c} u = \bar{\mu} u^{p-1}$$

on $\mathbb{H}_c^{k+1} \times S^{n-k-1}$. As in Subsubcase II.1.1, $u \not\equiv 0$, $u \in L^\infty(\mathbb{H}_c^{k+1} \times S^{n-k-1})$, and

$$\int_{\mathbb{R}^{k+1} \times S^{n-k-1}} u^p \, dv^{G_c} \leq 1.$$

Moreover, the assumption of Subcase II.2 implies that

$$\bar{\mu} u^{p-2}(0) = \lim_{\theta \searrow 0} \mu_\theta u_\theta^{p-2}(x''_\theta) \geq \frac{(n-k-2)^2(n-1)}{8(n-2)}.$$

By the definition of $\Lambda^{(2)}_{n,k}$, we have that $\bar{\mu} \geq \Lambda^{(2)}_{n,k} \geq \Lambda_{n,k}$.

Appendix A. Some details

A.1. Scalar curvature. In this section U denotes an open subset of a manifold and $q \in U$ a fixed point.

Proposition A.1. *Let g be a Riemannian metric on U and T a symmetric 2-tensor such that $\tilde{g} := g + T$ is also a Riemannian metric. Then the scalar curvature $\mathrm{Scal}^{\tilde{g}}(q)$ of $\tilde{g}$ in $q \in U$ is a smooth function of the Riemann tensor $R^g(q)$ of g at q, $T(q)$, $\nabla^g T(q)$, and $(\nabla^g)^2 T(q)$. Moreover, the operator $T \mapsto \mathrm{Scal}^{g+T}(q)$ is a quasilinear partial differential operator of second order.*

Proof. The proof is straightforward; we will just give a sketch using notation from [**8**], which coincides with that of [**26**]. We denote the components of the curvature tensors of g and $\tilde{g}$ by

$$R_{ijkl} = g(R^g(\partial_k, \partial_l)\partial_j, \partial_i), \qquad \tilde{R}_{ijkl} = \tilde{g}(R^{\tilde{g}}(\partial_k, \partial_l)\partial_j, \partial_i).$$

We work in normal coordinates for the metric g centered in q. Indices of partial derivatives in coordinates are added and separated with a comma "," and covariant ones with respect to g separated with a semi-colon ";". In particular $T = T_{ij} dx^i \, dx^j$,

$$T_{kl;i} = (\nabla_i T)(\partial_k, \partial_l) = \partial_i T_{kl} - T_{ml}\Gamma^m_{ik} - T_{km}\Gamma^m_{il}.$$

At the point q, we have $\tilde{g}_{kl,i} = T_{kl;i}$. As explained in [**8**, Formula (13)], we have

$$\nabla_\alpha \Gamma^k_{ij} = \partial_\alpha \Gamma^k_{ij} = -\frac{1}{3}\left(R_{ik\alpha j} + R_{i\alpha kj}\right)$$

at the point q. Hence in that point,

$$\begin{aligned} T_{kl;rs} &= (\nabla^2_{rs} T)(\partial_k, \partial_l) \\ &= \partial_r \partial_s T_{kl} + \frac{1}{3} T_{ml}(R_{smrk} + R_{srmk}) + \frac{1}{3} T_{mk}(R_{smrl} + R_{srml}). \end{aligned}$$

In order to calculate the scalar curvature $\mathrm{Scal}^{\tilde g}(q)$ of $\tilde g$ in q, we use the curvature formula as in [**26**] and contract twice. We obtain

$$\mathrm{Scal}^{\tilde g}(q) = \tilde g^{ik}\tilde g^{jm}(\tilde g_{km,ij} - \tilde g_{ki,mj}) + P(\tilde g^{rm}, \tilde g_{ij,k}) \tag{64}$$

where P is a polynomial expression in $\tilde g^{-1}$ and $\partial\tilde g$ that is cubic in $\tilde g^{-1} = \tilde g^{rm}$ and quadratic in $\tilde g_{ij,k}$. Note that formula (64) holds for an arbitrary metric in arbitrary coordinates. The polynomial P vanishes for $T = 0$ in normal coordinates for g. q.e.d.

Corollary A.2. *Let $\mathcal{R} \subset T_q^*M \otimes T_q^*M \otimes T_q^*M \otimes T_qM$ be a bounded set of curvature tensors. Then there is an $\epsilon > 0$ and $C \in \mathbb{R}$ such that for all metrics g on U with $R^g|_q \in \mathcal{R}$ we have the following: if*

$$\max_{i\in\{0,1,2\}} \left|(\nabla^g)^i T(q)\right| < \epsilon,$$

then

$$|\mathrm{Scal}^{g+T}(q) - \mathrm{Scal}^g(q)| \le C\left(\left|(\nabla^g)^2 T(q)\right| + |\nabla^g T(q)|^2 + |T(q)|\right).$$

A.2. Details for equation (16)**.** We compute the scalar curvature of the metric $dt^2 + e^{2\varphi(t)}h_t$ on $I \times W$. This is a generalized cylinder metric as studied in [**11**]. In the following computations we use the notation from [**11**], so $g_t = e^{2\varphi(t)}h_t$ and we have

$$\dot g_t = 2\varphi'(t)e^{2\varphi(t)}h_t + e^{2\varphi(t)}\partial_t h_t$$

and

$$\ddot g_t = (2\varphi''(t) + 4\varphi'(t)^2)e^{2\varphi(t)}h_t + 4\varphi'(t)e^{2\varphi(t)}\partial_t h_t + e^{2\varphi(t)}\partial_t^2 h_t.$$

This implies that the shape operator S of the hypersurfaces defined by having constant value t is given by

$$S = -\varphi'\mathrm{Id} - \frac{1}{2}h_t^{-1}\partial_t h_t,$$

so

$$\mathrm{tr}(S^2) = k\varphi'(t)^2 + \varphi'(t)\mathrm{tr}(h_t^{-1}\partial_t h_t) + \frac{1}{4}\mathrm{tr}((h_t^{-1}\partial_t h_t)^2)$$

and

$$(\mathrm{tr}S)^2 = k^2\varphi'(t)^2 + k\varphi'(t)\mathrm{tr}(h_t^{-1}\partial_t h_t) + \frac{1}{4}(\mathrm{tr}(h_t^{-1}\partial_t h_t))^2.$$

Further,

$$\begin{aligned}\mathrm{tr}^{g_t}\ddot g_t &= (2\varphi''(t) + 4\varphi'(t)^2)k + 4\varphi'(t)\mathrm{tr}^{h_t}(\partial_t h_t) + \mathrm{tr}^{h_t}(\partial_t^2 h_t)\\ &= (2\varphi''(t) + 4\varphi'(t)^2)k + 4\varphi'(t)\mathrm{tr}(h_t^{-1}\partial_t h_t) + \mathrm{tr}(h_t^{-1}\partial_t^2 h_t).\end{aligned}$$

From [**11**, Proposition 4.1, (21)] we have

$$
\begin{aligned}
\mathrm{Scal}^{e^{2\varphi(t)}h_t+dt^2} &= \mathrm{Scal}^{e^{2\varphi(t)}h_t} + 3\mathrm{tr}(S^2) - (\mathrm{tr}S)^2 - \mathrm{tr}^{g_t}\ddot{g}_t \\
&= e^{-2\varphi(t)}\mathrm{Scal}^{h_t} - k(k+1)\varphi'(t)^2 \\
&\quad - (k+1)\varphi'(t)\mathrm{tr}(h_t^{-1}\partial_t h_t) - 2k\varphi''(t) + \frac{3}{4}\mathrm{tr}((h_t^{-1}\partial_t h_t)^2) \\
&\quad - \frac{1}{4}(\mathrm{tr}(h_t^{-1}\partial_t h_t))^2 - \mathrm{tr}(h_t^{-1}\partial_t^2 h_t).
\end{aligned}
$$

When we add the scalar curvature of σ^{n-k-1} we get Formula (16) for the scalar curvature of $g_{\mathrm{WS}} = dt^2 + e^{2\varphi(t)}h_t + \sigma^{n-k-1}$.

A.3. A cut-off formula. Here we state a formula used several times in the article. Assume that u and χ are smooth functions on a Riemannian manifold (N, h), and that χ has compact support. Then

(65)

$$
\begin{aligned}
\int_N |d(\chi u)|^2 \, dv^h &= \int_N \left(u^2|d\chi|^2 + \langle u d\chi, \chi du\rangle + \langle \chi du, d(\chi u)\rangle\right) \, dv^h \\
&= \int_N \left(u^2|d\chi|^2 + \chi u\langle d\chi, du\rangle + \langle du, \chi d(\chi u)\rangle\right) \, dv^h \\
&= \int_N \left(u^2|d\chi|^2 + \chi u\langle d\chi, du\rangle + \langle du, d(\chi^2 u) - \chi u d\chi\rangle\right) \, dv^h \\
&= \int_N \left(u^2|d\chi|^2 + \langle du, d(\chi^2 u)\rangle\right) \, dv^h \\
&= \int_N \left(u^2|d\chi|^2 + \chi^2 u\Delta^h u\right) dv^h.
\end{aligned}
$$

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