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ISOPARAMETRIC FOLIATION AND YAU CONJECTURE ON THE FIRST EIGENVALUE

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Dedicated to Professor Banghe Li on his 70th birthday

Abstract

A well-known conjecture of Yau states that the first eigenvalue of every closed minimal hypersurface M^n in the unit sphere $S^{n+1}(1)$ is just its dimension n. The present paper shows that Yau conjecture is true for minimal isoparametric hypersurfaces. Moreover, the more fascinating result of this paper is that the first eigenvalues of the focal submanifolds are equal to their dimensions in the non-stable range.

1. Introduction

One of the most important operators acting on C^∞ functions on a Riemannian manifold is the Laplace-Beltrami operator. Over several decades, research on the spectrum of the Laplace-Beltrami operator has always been a core issue in the study of geometry. For instance, the geometry of closed minimal submanifolds in the unit sphere is closely related to the eigenvalue problem.

Let (M^n, g) be an n-dimensional compact connected Riemannian manifold without boundary and Δ be the Laplace-Beltrami operator acting on a C^∞ function f on M by $\Delta f = -\operatorname{div}(\nabla f)$, the negative of divergence of the gradient ∇f. It is well known that Δ is an elliptic operator and has a discrete spectrum

$$\{0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \cdots \leq \lambda_k(M), \cdots, \uparrow \infty\}$$

with each eigenvalue repeated a number of times equal to its multiplicity. As usual, we call $\lambda_1(M)$ the first eigenvalue of M. When M^n is a minimal hypersurface in the unit sphere $S^{n+1}(1)$, it follows from Takahashi's theorem that $\lambda_1(M)$ is not greater than n.

In this connection, S.T. Yau posed in 1982 the following conjecture:

Yau conjecture ([Yau]). *The first eigenvalue of every closed minimal hypersurface M^n in the unit sphere $S^{n+1}(1)$ is just n.*

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The most significant breakthrough to this problem was made by Choi and Wang ([**CW**]). They showed that the first eigenvalue of every (embedded) closed minimal hypersurface in $S^{n+1}(1)$ is not smaller than $\frac{n}{2}$. As is well known, the calculation of the spectrum of the Laplace-Beltrami operator, even of the first eigenvalue, is rather complicated and difficult. Up to now, Yau's conjecture is far from being solved.

In this paper, we consider a more restricted problem: the Yau conjecture for closed minimal isoparametric hypersurfaces M^n in $S^{n+1}(1)$. As one of the main results of this paper, we show

Theorem 1.1. *Let M^n be a closed minimal isoparametric hypersurface in $S^{n+1}(1)$. Then*

$$\lambda_1(M^n) = n.$$

Recall that a hypersurface M^n in the unit sphere $S^{n+1}(1)$ is called isoparametric if it has constant principal curvatures (*cf.* [**Car1**], [**Car2**], [**CR**]). Let ξ be a unit normal vector field along M^n in $S^{n+1}(1)$, g the number of distinct principal curvatures of M, $\cot\theta_\alpha$ ($\alpha = 1, \dots, g;\ 0 < \theta_1 < \cdots < \theta_g < \pi$) the principal curvatures with respect to ξ, and m_α the multiplicity of $\cot\theta_\alpha$. Using an elegant topological method, Münzner proved the remarkable result that the number g must be $1, 2, 3, 4$, or 6; $m_\alpha = m_{\alpha+2}$ (indices mod g); $\theta_\alpha = \theta_1 + \frac{\alpha-1}{g}\pi$ ($\alpha = 1, \dots, g$); and when g is odd, $m_1 = m_2$ (*cf.* [**Mün**]).

Attacking the Yau conjecture, Muto-Ohnita-Urakawa ([**MOU**]), Kotani ([**Kot**]), and Solomon ([**Sol1**, **Sol2**]) made a breakthrough for some of the minimal homogeneous (automatically isoparametric) hypersurfaces. More precisely, they verified Yau conjecture for all the homogeneous minimal hypersurfaces with $g = 1, 2, 3, 6$. However, when it came to the case $g = 4$, they were only able to deal with the cases $(m_1, m_2) = (2, 2)$ and $(1, k)$. As a matter of fact, by classification of the homogeneous hypersurfaces with four distinct principal curvatures, the pairs (m_1, m_2) are $(1, k)$, $(2, 2k-1)$, $(4, 4k-1)$, $(2, 2)$, $(4, 5)$, or $(6, 9)$. They explained in [**MOU**] that "*it seems to be difficult to compute their first eigenvalue because none of the homogeneous minimal hypersurfaces in the unit sphere except the great sphere and the generalized Clifford torus is symmetric or normal homogeneous.*"

Furthermore, another breakthrough made by Muto ([**Mut**]) showed that Yau's conjecture is also true for some families of nonhomogeneous minimal isoparametric hypersurfaces with four distinct principal curvatures. His remarkable result does not depend on the homogeneity of the isoparametric hypersurfaces. However, his conclusion covers only some isoparametric hypersurfaces with $\min(m_1, m_2) \leq 10$. Roughly speaking, the generic families of the isoparametric hypersurfaces in the unit sphere with four distinct principal curvatures have $\min(m_1, m_2) > 10$.

Based on all results mentioned above and the classification of isoparametric hypersurfaces in $S^{n+1}(1)$ (*cf.* [**CCJ**], [**Imm**], [**Chi**], [**DN**], and [**Miy**]), we show our Theorem 1.1 by establishing the following

Theorem 1.2. *Let M^n be a closed minimal isoparametric hypersurface in the unit sphere $S^{n+1}(1)$ with four distinct principal curvatures and $m_1, m_2 \geq 2$. Then*

$$\lambda_1(M^n) = n.$$

Remark 1.1. Cartan classified isoparametric hypersurfaces in the unit spheres with $g = 1, 2, 3$ to be homogeneous (*cf.* [**Car1**], [**Car2**]); Dorfmeister- Neher ([**DN**]) and Miyaoka ([**Miy**]) showed that they are homogeneous for $g = 6$. Thus the results of [**MOU**], [**Kot**] and [**Sol1**], [**Sol2**] complete the proof of Theorem 1.1 in cases $g = 1, 2, 3, 6$. Moreover, Takagi ([**Tak1**]) asserted that the isoparametric hypersurface with $g = 4$ and multiplicities $(1, k)$ must be homogeneous. By virtue of [**MOU**], Theorem 1.1 is true for the case $(1, k)$. Therefore, Theorem 1.2 completes in a direct way the proof of Theorem 1.1.

Remark 1.2. For isoparametric hypersurfaces with $g = 4$, Cecil-Chi-Jensen ([**CCJ**]), Immervoll ([**Imm**]), and Chi ([**Chi**]) proved a far reaching result that they are either homogeneous or of OT-FKM type except possibly for the case $(m_1, m_2) = (7, 8)$. Actually, Theorem 1.2 depends only on the values of (m_1, m_2), but not on the homogeneity. Besides, our method is also applicable to the case $g = 6$.

Remark 1.3. Chern conjectured that a closed, minimally immersed hypersurface in $S^{n+1}(1)$, whose second fundamental form has constant length, is isoparametric (*cf.* [**GT**]). If this conjecture is proven, we would have settled Yau conjecture for the minimal hypersurface whose second fundamental form has constant length, which gives us more confidence in Yau conjecture.

The more fascinating part of this paper is the determination of the first eigenvalues of the focal submanifolds in $S^{n+1}(1)$, which relies on the deeper geometric properties of the isoparametric foliation.

To state our Theorem 1.3 clearly, let us start with some preliminaries. A well-known result of Cartan states that isoparametric hypersurfaces come as a family of parallel hypersurfaces. To be more specific, given an isoparametric hypersurface M^n in $S^{n+1}(1)$ and a smooth field ξ of unit normals to M, for each $x \in M$ and $\theta \in \mathbb{R}$, we can define $\phi_\theta : M^n \to S^{n+1}(1)$ by

$$\phi_\theta(x) = \cos\theta\ x + \sin\theta\ \xi(x).$$

Clearly, $\phi_\theta(x)$ is the point at an oriented distance θ to M along the normal geodesic through x. If $\theta \neq \theta_\alpha$ for any $\alpha = 1, \ldots, g$, ϕ_θ is a parallel hypersurface to M at an oriented distance θ, which we will denote by M_θ henceforward. If $\theta = \theta_\alpha$ for some $\alpha = 1, \ldots, g$, it is easy to find that for any vector X in the principal distributions $E_\alpha(x) = \{X \in T_xM \mid A_\xi X = \cot\theta_\alpha X\}$, where A_ξ is the shape operator with respect to ξ, $(\phi_\theta)_* X = (\cos\theta - \sin\theta\cot\theta_\alpha)X = \frac{\sin(\theta_\alpha-\theta)}{\sin\theta_\alpha}X = 0$. In other words, if $\cot\theta = \cot\theta_\alpha$ is a principal curvature of M, ϕ_θ is not an immersion, but is actually a *focal submanifold* of codimension $m_\alpha + 1$ in $S^{n+1}(1)$.

Münzner asserted that regardless of the number of distinct principal curvatures of M, there are only two distinct focal submanifolds in a parallel family of isoparametric hypersurfaces, and every isoparametric hypersurface is a tube of constant radius over each focal submanifold. Denote by M_1 the focal submanifold in $S^{n+1}(1)$ at an oriented distance θ_1 along ξ from M with codimension $m_1 + 1$, and by M_2 the focal submanifold in $S^{n+1}(1)$ at an oriented distance $\frac{\pi}{g} - \theta_1$ along $-\xi$ from M with codimension $m_2 + 1$. In view of Cartan's identity, one sees that the focal submanifolds M_1 and M_2 are minimal in $S^{n+1}(1)$ (*cf.* [**CR**]).

Another main result of the present paper concerning the first eigenvalues of focal submanifolds in the non-stable range (*cf.* [**HH**]) is stated as follows.

Theorem 1.3. *Let M_1 be the focal submanifold of an isoparametric hypersurface with four distinct principal curvatures in the unit sphere $S^{n+1}(1)$ with codimension $m_1 + 1$. If $\dim M_1 \geq \frac{2}{3}n + 1$, then*

$$\lambda_1(M_1) = \dim M_1$$

with multiplicity $n+2$. A similar conclusion holds for M_2 under an analogous condition.

Recall the classification results of [**CCJ**] [**Chi**] which stated that except for the case $(m_1, m_2) = (7, 8)$, the isoparametric hypersurfaces in $S^{n+1}(1)$ with four distinct principal curvatures are either homogeneous with $(m_1, m_2) = (2, 2), (4, 5)$ or of OT-FKM type. Fortunately, as a simple application of Theorem 1.3, we obtain immediately that each focal submanifold with $g = 4$, $(m_1, m_2) = (4, 5)$ or $(7, 8)$ has its dimension as the first eigenvalue. Subsequently, we will look into the focal submanifolds of OT-FKM type and give their first eigenvalues.

We now recall the construction of the isoparametric hypersurfaces of OT-FKM type. For a symmetric Clifford system $\{P_0, \cdots, P_m\}$ on $\mathbb{R}^{2l}$—*i.e.*, P_i's are symmetric matrices satisfying $P_iP_j + P_jP_i = 2\delta_{ij}I_{2l}$—Ferus, Karcher, and Münzner ([**FKM**]) constructed a polynomial F on $\mathbb{R}^{2l}$:

$$\begin{aligned} F:\quad & \mathbb{R}^{2l} \to \mathbb{R} \\ F(x) &= |x|^4 - 2\sum_{i=0}^{m} \langle P_i x, x\rangle^2. \end{aligned} \tag{1}$$

It turns out that each level hypersurface of $f = F|_{S^{2l-1}}$, *i.e.*, the preimage of some regular value of f, has four distinct constant principal curvatures. Choosing $\xi = \frac{\nabla f}{|\nabla f|}$, we find $M_1 = f^{-1}(1)$, $M_2 = f^{-1}(-1)$, which have codimensions $m_1 + 1$ and $m_2 + 1$ in $S^{n+1}(1)$, respectively. The multiplicity pairs (m_1, m_2) of the OT-FKM type are $(m, l-m-1)$, provided $m > 0$ and $l-m-1 > 0$, where $l = k\delta(m)$ $(k = 1, 2, 3, \ldots)$ and $\delta(m)$ is the dimension of an irreducible module of the Clifford algebra C_{m-1}. In the following, we list the values of $\delta(m)$ corresponding to m:

m	1	2	3	4	5	6	7	8	$\cdots$ $m+8$
$\delta(m)$	1	2	4	4	8	8	8	8	$16\delta(m)$

First, we focus on the focal submanifold M_2. If $3\dim M_2 \geq 2n+3$, or equivalently, $m_1 \geq \frac{1}{2}(m_2+3)$, Theorem 1.3 gives $\lambda_1(M_2) = \dim M_2 = 2m_1 + m_2$. The assumption $3\dim M_2 \geq 2n+3$ is essential. For instance, Solomon ([**Sol3**]) constructed an eigenfunction on the focal submanifold M_2 of OT-FKM-type, which has $4m$ as an eigenvalue. It follows that $\lambda_1(M_2) \leq 4m$. Therefore, in the stable range $3\dim M_2 < 2(n+1) - 2$, *i.e.*, $m_1 < \frac{1}{2}m_2$, $\lambda_1(M_2) < 2m_1 + m_2 = \dim M_2$. Only three cases are left to estimate: $m_1 = \frac{1}{2}m_2$, $m_1 = \frac{1}{2}(m_2+1)$, and $m_1 = \frac{1}{2}(m_2+2)$, which are actually $(m_1, m_2) = (1,1)$, $(1,2)$, $(2,3)$, $(3,4)$, $(4,7)$, $(5,10)$, and $(8,15)$.

Next, we will be concerned with the focal submanifold M_1. Fortunately, the condition in Theorem 1.3 is almost always satisfied. Actually, the first eigenvalue of the focal submanifold M_1 of those OT-FKM type can be determined completely. By analyzing the conditions $m_1 \geq 1$, $m_2 \geq 1$, and $m_2 < \frac{1}{2}(m_1+3)$, we find that there are only five cases left, that is, $(m_1, m_2) = (1,1)$, $(2,1)$, $(4,3)$, $(5,2)$, and $(6,1)$. In view of [**FKM**], the families for multiplicities $(2,1)$, $(5,2)$, $(6,1)$, and one of the $(4,3)$-families are congruent to those with multiplicities $(1,2)$, $(2,5)$, $(1,6)$, and $(3,4)$, respectively, and the focal submanifolds interchange. For the case $(2,5)$, an effective estimate can be given by [**Sol3**], while for the cases $(1,2)$ and $(1,6)$, the following proposition determines the first eigenvalues.

Proposition 1.1. Let M_2 be the focal submanifold of OT-FKM type defined before with $(m_1, m_2) = (1, k)$. The following equality is valid:

$$\lambda_1(M_2) = \min\{4, 2+k\}.$$

As mentioned before, Takagi ([**Tak1**]) asserted that the isoparametric hypersurface with $g = 4$ and multiplicity $(1,k)$ must be homogeneous. Thus the corresponding focal submanifold of isoparametric hypersurface with four distinct principal curvatures and $\min\{m_1, m_2\} = 1$ has $\min\{4, 2+k\}$ as its first eigenvalue.

At last, we would like to propose a problem on the first eigenvalue of the minimal submanifolds with dimensions in the non-stable range in $S^{n+1}(1)$, which could be regarded as an extension of Yau conjecture.

Problem: *Let M^d be a closed minimal submanifold in the unit sphere $S^{n+1}(1)$ with $d \geq \frac{2}{3}n + 1$. Is it true that*

$$\lambda_1(M^d) = d \ ?$$

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2. The first eigenvalue of the minimal isoparametric hypersurface

Let $\phi : M^n \to S^{n+1}(1)(\subset \mathbb{R}^{n+2})$ be a closed isoparametric hypersurface with g distinct principal curvatures in $S^{n+1}(1)$ and ξ be a smooth field of unit normals to M. Again, denote by E_α ($\alpha = 1, \ldots, g$) the principal distribution on M, *i.e.*, the eigenspace of the shape operator A_ξ corresponding to the eigenvalue $\cot\theta_\alpha$ ($0 < \theta_1 < \cdots < \theta_g < \pi$). The parallel hypersurface M_θ at an oriented distance θ from ϕ is defined by $\phi_\theta : M^n \to S^{n+1}(1)$ ($-\pi < \theta < \pi, \cot\theta \neq \cot\theta_\alpha$),

$$\phi_\theta(x) = \cos\theta\ x + \sin\theta\ \xi(x).$$

At first, let us prepare some formulae:

For $X \in E_\alpha$, it is easy to see

$$(\phi_\theta)_* X = \frac{\sin(\theta_\alpha - \theta)}{\sin\theta_\alpha} \widetilde{X}, \tag{2}$$

where $\widetilde{X} \mathbin{/\!/} X$ are vectors in $\mathbb{R}^{n+2}$.

Let H be the mean curvature of M^n in $S^{n+1}(1)$ with respect to ξ. Clearly,

$$\begin{aligned} nH &= \sum_{\alpha=1}^{g} m_\alpha \cot\theta_\alpha \\ &= \begin{cases} m_1 g \cot(g\theta_1) & \text{for } g \text{ odd} \\ \dfrac{m_1 g}{2} \cot\dfrac{g\theta_1}{2} - \dfrac{m_2 g}{2} \tan\dfrac{g\theta_1}{2} & \text{for } g \text{ even} \end{cases} \end{aligned} \tag{3}$$

In order to estimate the eigenvalues of M, we would recall a theorem that will play a crucial role in our work as Muto did in [**Mut**].

Theorem (Chavel and Feldman [CF], Ozawa [Oza]) *Let V be a closed, connected smooth Riemannian manifold and W a closed submanifold of V. For any sufficiently small $\varepsilon > 0$, set $W(\varepsilon) = \{x \in V : \ dist(x, W) < \varepsilon\}$. Let $\lambda_k^D(\varepsilon)$ ($k = 1, 2, \ldots$) be the k-th eigenvalue of* *the Laplace-Beltrami operator on $V - W(\varepsilon)$ under the Dirichlet boundary condition. If $\dim V \geq \dim W + 2$, then for any $k = 1, 2, \ldots$*

$$\lim_{\varepsilon \to 0} \lambda_k^D(\varepsilon) = \lambda_{k-1}(V). \tag{4}$$

We will apply this theorem to the case $V = S^{n+1}(1)$ and $W = M_1 \cup M_2$, the union of the focal submanifolds. By estimating the eigenvalue $\lambda_k(M^n)$ from below, we can prove Theorem 1.2.

Theorem 1.2. *Let M^n be a closed minimal isoparametric hypersurface in the unit sphere $S^{n+1}(1)$ with four distinct principal curvatures and $m_1, m_2 \geq 2$. Then*

$$\lambda_1(M^n) = n.$$

Proof. For sufficiently small $\varepsilon > 0$, set

$$M(\varepsilon) = \bigcup_{\theta \in [-\frac{\pi}{4} + \theta_1 + \varepsilon,\ \theta_1 - \varepsilon]} M_\theta.$$

Clearly, $M(\varepsilon)$ is a domain of $S^{n+1}(1)$ obtained by excluding ε-neighborhoods of M_1 and M_2 from $S^{n+1}(1)$. Alternatively, it can also be regarded as a tube around the minimal isoparametric hypersurface M. According to the theorem of Chavel, Feldman, and Ozawa,

$$\lim_{\varepsilon \to 0} \lambda_{k+1}^D(M(\varepsilon)) = \lambda_k(S^{n+1}(1)), \tag{5}$$

we need to estimate $\lambda_{k+1}^D(M(\varepsilon))$ from above in terms of $\lambda_k(M^n)$.

Let $\left\{ \widetilde{e}_{\alpha,i} \mid i = 1, \ldots, m_\alpha,\ \alpha = 1, \ldots, 4,\ \widetilde{e}_{\alpha,i} \in E_\alpha \right\}$ be a local orthonormal frame field on M. Then

$$\left\{ \frac{\partial}{\partial \theta},\ e_{\alpha,i} \mid e_{\alpha,i} = \frac{\sin \theta_\alpha}{\sin(\theta_\alpha - \theta)} (\phi_\theta)_* \widetilde{e}_{\alpha,i},\ i = 1, \ldots, m_\alpha, \right.$$
$$\left. \alpha = 1, \ldots, 4,\ \theta \in [-\frac{\pi}{4} + \theta_1 + \varepsilon,\ \theta_1 - \varepsilon] \right\}$$

is a local orthonormal frame field on $M(\varepsilon)$. From the formula (2), we derive immediately that the volume element of $M(\varepsilon)$ can be expressed in terms of the volume element of M:

$$dM(\varepsilon) = \frac{\sin^{m_1} 2(\theta_1 - \theta) \cos^{m_2} 2(\theta_1 - \theta)}{\sin^{m_1} 2\theta_1 \cos^{m_2} 2\theta_1} d\theta dM. \tag{6}$$

Following [**Mut**], let h be a nonnegative, increasing smooth function on $[0, \infty)$ satisfying $h = 1$ on $[2, \infty)$ and $h = 0$ on $[0, 1]$. For sufficiently small $\eta > 0$, let ψ_η be a nonnegative smooth function on $[\eta, \frac{\pi}{2} - \eta]$ satisfying

(*i*) $\psi_\eta(\eta) = \psi_\eta(\frac{\pi}{2} - \eta) = 0$,

(ii) ψ_η is symmetric with respect to $x=\frac{\pi}{4}$,
(iii) $\psi_\eta(x)=h(\frac{x}{\eta})$ on $[\eta,\frac{\pi}{4}]$.
Let f_k $(k=0,1,\dots)$ be the k-th eigenfunctions on M which are orthogonal to each other with respect to the square integral inner product on M and $L_{k+1}=Span\{f_0,f_1,\dots,f_k\}$.

For each fixed $\theta\in[-\frac{\pi}{4}+\theta_1+\varepsilon,\ \theta_1-\varepsilon]$, denote $\pi=\pi_\theta=\phi_\theta^{-1}:M_\theta\to M$. Then any $\varphi\in L_{k+1}$ on M can give rise to a function $\Phi_\varepsilon: M(\varepsilon)\to\mathbb{R}$ by

$$\Phi_\varepsilon(x)=\psi_{2\varepsilon}(2(\theta_1-\theta))(\varphi\circ\pi)(x),$$

where θ is characterized by $x\in M_\theta$, $\theta\in[-\frac{\pi}{4}+\theta_1+\varepsilon,\ \theta_1-\varepsilon]$. It is evident to see that Φ_ε is a smooth function on $M(\varepsilon)$ satisfying the Dirichlet boundary condition and square integrable.

By the min-max principle, we have:

$$\lambda^D_{k+1}(M(\varepsilon))\le\sup_{\varphi\in L_{k+1}}\frac{\|\nabla\Phi_\varepsilon\|_2^2}{\|\Phi_\varepsilon\|_2^2}. \tag{7}$$

In the following, we will concentrate on the calculation of $\frac{\|\nabla\Phi_\varepsilon\|_2^2}{\|\Phi_\varepsilon\|_2^2}$. Observing that the normal geodesic starting from M is perpendicular to each parallel hypersurface M_θ, we obtain

$$\|\nabla\Phi_\varepsilon\|_2^2=\int_{M(\varepsilon)}4(\psi'_{2\varepsilon})^2\varphi(\pi)^2dM(\varepsilon)+\int_{M(\varepsilon)}\psi^2_{2\varepsilon}|\nabla\varphi(\pi)|^2dM(\varepsilon).$$

On the other hand, a simple calculation leads to

$$\begin{aligned}\|\Phi_\varepsilon\|_2^2 &= \int_{M(\varepsilon)}\psi^2_{2\varepsilon}(2(\theta_1-\theta))\varphi(\pi(x))^2dM(\varepsilon)\\ &= \int_M\int_{-\frac{\pi}{4}+\theta_1+\varepsilon}^{\theta_1-\varepsilon}\psi^2_{2\varepsilon}(2(\theta_1-\theta))\\ &\quad\frac{\sin^{m_1}2(\theta_1-\theta)\cos^{m_2}2(\theta_1-\theta)}{\sin^{m_1}2\theta_1\cos^{m_2}2\theta_1}\varphi(\pi(x))^2d\theta dM\\ &= \frac{\|\varphi\|_2^2}{2\sin^{m_1}2\theta_1\cos^{m_2}2\theta_1}\Big(\int_{2\varepsilon}^{\frac{\pi}{2}-2\varepsilon}\psi^2_{2\varepsilon}(x)\sin^{m_1}x\cos^{m_2}x\ dx\Big).\end{aligned}$$

For the sake of convenience, let us decompose

$$\frac{\|\nabla\Phi_\varepsilon\|_2^2}{\|\Phi_\varepsilon\|_2^2}=I(\varepsilon)+II(\varepsilon), \tag{8}$$

with

$$\begin{aligned}I(\varepsilon) &= \frac{\int_{M(\varepsilon)}4(\psi'_{2\varepsilon})^2\varphi(\pi)^2dM(\varepsilon)}{\int_{M(\varepsilon)}(\psi_{2\varepsilon})^2\varphi(\pi)^2\ dM(\varepsilon)}\\ &= \frac{4\int_{2\varepsilon}^{\frac{\pi}{2}-2\varepsilon}(\psi'_{2\varepsilon}(x))^2\sin^{m_1}x\cos^{m_2}x\ dx}{\int_{2\varepsilon}^{\frac{\pi}{2}-2\varepsilon}\psi^2_{2\varepsilon}(x)\sin^{m_1}x\cos^{m_2}x\ dx}\end{aligned} \tag{9}$$

and

$$II(\varepsilon) = \frac{\int_{M(\varepsilon)} \psi_{2\varepsilon}^2 |\nabla\varphi(\pi)|^2 dM(\varepsilon)}{\int_{M(\varepsilon)} \psi_{2\varepsilon}^2 \varphi(\pi)^2 dM(\varepsilon)}. \tag{10}$$

We shall take the first step by claiming that

$$\lim_{\varepsilon\to 0} I(\varepsilon) = 0. \tag{11}$$

In fact, for the smooth function h, we have a positive number C such that $|h'| \leq C$. It follows immediately that $|\psi_\eta'(x)| = |\frac{1}{\eta}h'(\frac{x}{\eta})| \leq \frac{1}{\eta}C$ for $x \in [\eta, \frac{\pi}{4}]$. Under the assumption $\min\{m_1, m_2\} \geq 2$, we deduce

$$\begin{aligned}
&\int_{2\varepsilon}^{\frac{\pi}{2}-2\varepsilon} (\psi_{2\varepsilon}'(x))^2 \sin^{m_1} x \cos^{m_2} x \; dx \\
\leq\; & \int_{2\varepsilon}^{4\varepsilon} (\psi_{2\varepsilon}'(x))^2 \sin^2 x \; dx + \int_{\frac{\pi}{2}-4\varepsilon}^{\frac{\pi}{2}-2\varepsilon} (\psi_{2\varepsilon}'(x))^2 \cos^2 x \; dx \\
\leq\; & \frac{C^2}{4}\int_{2\varepsilon}^{4\varepsilon} \frac{\sin^2 x}{\varepsilon^2} \; dx + \frac{C^2}{4}\int_{\frac{\pi}{2}-4\varepsilon}^{\frac{\pi}{2}-2\varepsilon} \frac{\cos^2 x}{\varepsilon^2} \; dx,
\end{aligned}$$

from which it follows that the numerator of $I(\varepsilon)$ in (9) approaches 0 as ε goes to 0. On the other hand, the denominator of $I(\varepsilon)$ approaches a non-zero number as ε goes to 0. Thus the claim (11) is established. q.e.d.

Next, we turn to the estimation of $II(\varepsilon)$.

Decompose $\nabla\varphi = Z_1 + Z_2 + Z_3 + Z_4 \in E_1 \oplus E_2 \oplus E_3 \oplus E_4$, and set $k_\alpha = \frac{\sin(\theta_\alpha - \theta)}{\sin\theta_\alpha}$ for $\alpha = 1, \ldots, 4$. Using the following identity,

$$\langle \nabla\varphi(\pi), X\rangle = \langle \nabla\varphi, \pi_* X\rangle, \quad for\ any\ X \in T_x M_\theta, \tag{12}$$

we have

$$\begin{cases} \quad |\nabla\varphi|^2 \;= |Z_1|^2 + |Z_2|^2 + |Z_3|^2 + |Z_4|^2 \\ |\nabla\varphi(\pi)|^2 = \frac{1}{k_1^2}|Z_1|^2 + \frac{1}{k_2^2}|Z_2|^2 + \frac{1}{k_3^2}|Z_3|^2 + \frac{1}{k_4^2}|Z_4|^2. \end{cases} \tag{13}$$

Moreover, for simplicity, for $\alpha = 1, \ldots, 4$, define

$$\begin{aligned}
K_\alpha \;&:=\; \int_{-\frac{\pi}{4}+\theta_1}^{\theta_1} \frac{\sin^{m_1} 2(\theta_1 - \theta)\cos^{m_2} 2(\theta_1 - \theta)}{k_\alpha^2} d\theta \qquad (14)\\
&=\; \sin^2\theta_\alpha \int_0^{\frac{\pi}{4}} \frac{\sin^{m_1} 2x \cos^{m_2} 2x}{\sin^2(\frac{\alpha-1}{4}\pi + x)} dx,
\end{aligned}$$

$$\begin{aligned}
G \;&:=\; \int_0^{\frac{\pi}{2}} \sin^{m_1} x \cos^{m_2} x \; dx \qquad (15)\\
&=\; 2\int_0^{\frac{\pi}{4}} \sin^{m_1} 2x \cos^{m_2} 2x \; dx.
\end{aligned}$$

Let $K = \max_\alpha\{K_\alpha\}$. Then combining with (8), (9), (10), (11), (13), (14), and (15), we arrive at

$$\lim_{\varepsilon\to 0}\frac{\|\nabla\Phi_\varepsilon\|_2^2}{\|\Phi_\varepsilon\|_2^2} = \frac{\sum_\alpha K_\alpha\|Z_\alpha\|_2^2}{\|\varphi\|_2^2\cdot\frac{1}{2}G} \leq \frac{2K}{G}\cdot\frac{\|\nabla\varphi\|_2^2}{\|\varphi\|_2^2}, \tag{16}$$

Therefore, putting (5), (7), and (16) together, we see that

$$\lambda_k(S^{n+1}(1)) = \lim_{\varepsilon\to 0}\lambda^D_{k+1}(M(\varepsilon)) \leq \lim_{\varepsilon\to 0}\sup_{\varphi\in L_{k+1}}\frac{\|\nabla\Phi_\varepsilon\|_2^2}{\|\Phi_\varepsilon\|_2^2} \leq \lambda_k(M^n)\frac{2K}{G}. \tag{17}$$

Comparing the leftmost side with the rightmost side of (17), it is sufficient to complete the proof of Theorem 1.2, if we can verify the inequality

$$K < \frac{n+2}{n}G. \tag{18}$$

Since then, $\lambda_{n+3}(S^{n+1}(1)) = 2(n+2) < \lambda_{n+3}(M^n)\cdot\frac{2(n+2)}{n}$, which implies immediately that $\lambda_{n+3}(M^n) > n$. Recall that n is an eigenvalue of M^n with multiplicity at least $n+2$. Therefore, the first eigenvalue of M^n must be n with multiplicity $n+2$.

We are now in a position to verify the inequality (18), which is equivalent to

$$K_\alpha < \frac{n+2}{n}G, \quad for\ each\ \alpha = 1,2,3,4. \tag{19}$$

First, we observe that the certifications for K_2 and K_3 are similar; so are those for K_1 and K_4. Thus we just need to give two verifications.

(i) Given $0 < x < \frac{\pi}{4}$, since $0 < \theta_1 < \frac{\pi}{4}$, it follows straightforwardly that

$$K_2 < 2\sin^2\theta_2\int_0^{\frac{\pi}{4}}\sin^{m_1}2x\cos^{m_2}2x\,dx < 2\int_0^{\frac{\pi}{4}}\sin^{m_1}2x\cos^{m_2}2x\,dx = G.$$

Similarly, we have

$$K_3 < 2\sin^2\theta_3\int_0^{\frac{\pi}{4}}\sin^{m_1}2x\cos^{m_2}2x\,dx < 2\int_0^{\frac{\pi}{4}}\sin^{m_1}2x\cos^{m_2}2x\,dx = G.$$

(ii) Express K_1 and G in terms of the beta function $B(x,y) = 2\int_0^{\frac{\pi}{2}}\sin^x\theta\cos^y\theta\ d\theta$:

$$\begin{aligned} K_1 &= \sin^2\theta_1\int_0^{\frac{\pi}{4}}\frac{\sin^{m_1}2x\cos^{m_2}2x}{\sin^2 x}dx \\ &= \frac{1}{2}\sin^2\theta_1\Big[B(\frac{m_1-1}{2},\frac{m_2+1}{2}) + B(\frac{m_1-1}{2},\frac{m_2+2}{2})\Big], \end{aligned} \tag{20}$$

$$G = \int_0^{\frac{\pi}{2}}\sin^{m_1}x\cos^{m_2}x\ dx = \frac{1}{2}B(\frac{m_1+1}{2},\frac{m_2+1}{2}). \tag{21}$$

Using the properties of the beta function and the gamma function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad \textit{and} \quad \Gamma(x+1) = x\Gamma(x) \quad \textit{for any } x > 0,$$

it follows from (20) and (21) that

$$\frac{K_1}{G} = \sin^2\theta_1 \cdot \frac{m_1+m_2}{m_1-1} \cdot \left(1 + \frac{\Gamma(\frac{m_2+2}{2})\Gamma(\frac{m_1+m_2}{2})}{\Gamma(\frac{m_2+1}{2})\Gamma(\frac{m_1+m_2+1}{2})}\right).$$

Define

$$S(m_1,m_2) := \frac{\Gamma(\frac{m_2+2}{2})\Gamma(\frac{m_1+m_2}{2})}{\Gamma(\frac{m_2+1}{2})\Gamma(\frac{m_1+m_2+1}{2})}$$

and

$$A(m_1,m_2) := \frac{n+2}{n}\frac{1}{\sin^2\theta_1}\frac{m_1-1}{m_1+m_2}.$$

Then it is clear that

$$K_1 < \frac{n+2}{n}G \iff 1 + S(m_1,m_2) < A(m_1,m_2). \tag{22}$$

We conclude this section by establishing two inequalities $S(m_1,m_2) < 1$ and $A(m_1,m_2) \geq 2$.

Lemma 2.1. The multiplicities m_1, m_2 of the principal curvatures of isoparametric hypersurfaces with four distinct principal curvatures with $m_1, m_2 \geq 2$ satisfy

$$S(m_1,m_2) < 1.$$

Proof. Recall a well-known result that when $g = 4$, m_1 and m_2 cannot both be even except for $(2,2)$ (*cf.* [**Mün**], [**Abr**], [**Tan**]). It suffices to estimate $S(m_1,m_2)$ in the following three cases.
Case 1: When $(m_1,m_2) = (2,2)$,

$$S(2,2) := \frac{\Gamma(2)\Gamma(2)}{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})} = \frac{8}{3\pi} < 1.$$

Case 2: When $m_1 = 2p+1$, it is obvious that

$$S(m_1,m_2) = \frac{\frac{m_2+1}{2}\cdot(\frac{m_2+1}{2}+1)\cdots(\frac{m_2+1}{2}+p-1)}{\frac{m_2+2}{2}\cdot(\frac{m_2+2}{2}+1)\cdots(\frac{m_2+2}{2}+p-1)} < 1.$$

Case 3: When $m_1 = 2p$, $m_2 = 2q+1$, for simplicity, we define

$$T(p,q) := S(m_1,m_2) = \frac{(2q+1)!!(2p+2q-1)!!\cdot\pi}{q!(p+q)!\cdot 2^{p+2q+1}}.$$

It is straightforward to see that $T(p,q)$ is strictly decreasing with p for a fixed q, and strictly increasing with q for a fixed p. It follows that

$$T(p,q) < T(p-1,q) < \cdots < T(1,q) < T(1,q+1) < \cdots < T(1,\infty).$$

Using the Stirling Formula:

$$\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}(\frac{n}{e})^n}=1,$$

we obtain that

$$\begin{aligned}T(1,\infty) &= \lim_{q\to\infty}\frac{[(2q+1)!]^2\pi}{(q!)^3(q+1)!2^{4q+2}}\\ &= \lim_{q\to\infty}\frac{(2q+1)^{3q+\frac{3}{2}}}{(2q)^{3q+\frac{3}{2}}}\cdot\frac{(2q+1)^{q+\frac{3}{2}}}{(2q+2)^{q+\frac{3}{2}}}\cdot\frac{1}{e}\\ &= e^{\frac{3}{2}}\cdot\frac{1}{e^{\frac{1}{2}}}\cdot\frac{1}{e}\\ &= 1.\end{aligned}$$

This completes the proof of Lemma 2.1. q.e.d.

Lemma 2.1 reduces the proof of (19) for K_1 to proving that $A(m_1,m_2)\geq 2$.

Since M^n is the minimal isoparametric hypersurface in $S^{n+1}(1)$, from Formula (3), we derive that $\sin^2\theta_1=\frac{1}{2}(1-\frac{\sqrt{m_2}}{\sqrt{m_1+m_2}})$. On the other hand, in our case $g=4$, we have $n=\frac{g}{2}(m_1+m_2)=2(m_1+m_2)$; thus

$$A(m_1,m_2)=\frac{m_1-1}{m_1+m_2}\cdot\frac{m_1+m_2+1}{m_1+m_2}\cdot\frac{2}{1-\frac{\sqrt{m_2}}{\sqrt{m_1+m_2}}}.$$

A simple calculation shows

$$A(m_1,m_2)\geq 2\Longleftrightarrow m_2(m_1+m_2)^3\geq(m_2^2+m_1m_2+m_2+1)^2.$$

It is not difficult to see that the following three inequalities guarantee the right hand of the equivalence above.

$$\begin{cases} 3m_1\geq 2m_1+2\\ 3{m_1}^2\geq m_1^2+2m_1+3\\ m_1^3\geq 2m_1+3\end{cases}$$

Fortunately, the last three inequalities are satisfied simultaneously if $m_1\geq 2$. Thus $1+S(m_1,m_2)<2\leq A(m_1,m_2)$ under the assumption $\min\{m_1,m_2\}\geq 2$; equivalently, the inequality $K_1<\frac{n+2}{n}G$ we required holds true.

Similarly, $K_4<\frac{n+2}{n}G$.

The proof of Theorem 1.2 is now complete. □

3. The first eigenvalue of the focal submanifolds

At the beginning of this section, we should investigate the multiplicity of the dimension $n-m_i$ as an eigenvalue of the focal submanifold M_i $(i=1,2)$ of an isoparametric hypersurface with g distinct principal curvatures. For this purpose, we first prepare the following lemma.

Lemma 3.1. Both M_1 and M_2 are fully embedded in $S^{n+1}(1)$ if $g \geq 3$; namely, they cannot be embedded into a hypersphere.

Proof. We are mainly concerned with the proof for M_1; the other case is verbatim with obvious changes on index ranges.

Suppose M_1 is not fully embedded in $S^{n+1}(1)$; then we can find a point $q \in S^{n+1}(1)$ such that $\langle x, q\rangle = 0$ for any $x \in M_1$. For any $p \in S^{n+1}(1)$, define the spherical distance function $L_p : M_1 \to \mathbb{R}$ by:

$$L_p(x) = \cos^{-1}\langle p, x\rangle.$$

Since L_p is a Morse function on M_1 when $p \in S^{n+1}(1) - (M_1 \cup M_2)$ (*cf.* [**CR**], p. 285), we need only to deal with the two remaining cases:

(1) $p \in M_1$. Since the function $\langle x, p\rangle$ can achieve 1 at $x = p$, the point q cannot lie in M_1.

(2) $p \in M_2$. If L_p is a constant, then from each point $x \in M_1$, there exists one normal geodesic (normal to M_1 at x, normal to M_2 at p, geodesic in $S^{n+1}(1)$), which connects x and p. Thus we can define a smooth map f from the unit normal space of M_2 at p to M_1 by:

$$\begin{aligned} f : S(T_p^{\perp} M_2) &\longrightarrow M_1 \\ \xi &\longmapsto x \end{aligned}$$

where x is the first intersection point of M_1 and the normal geodesic starting from p along the initial direction ξ after ξ passes through the isoparametric hypersurface M. Under our assumption, f would be surjective. According to Sard's theorem, this implies an inequality $m_2 \geq \frac{g}{2}(m_1 + m_2) - m_1$. Obviously, this inequality holds true only when $g \leq 2$.

This completes the proof of Lemma 3.1. q.e.d.

Remark 3.1. The assumption $g \geq 3$ in Lemma 3.1 is essential. For instance, for $g = 2$, both the focal submanifolds of the isoparametric hypersurface (generalized Clifford torus) are not full, but are actually totally geodesic.

As a direct result of Lemma 3.1, the dimension $n - m_1$ (*resp.* $n - m_2$) of M_1 is an eigenvalue of M_1 (resp. M_2) with multiplicity at least $n + 2$.

Now, we are ready to prove Theorem 1.3.

Theorem 1.3. *Let M_1 be the focal submanifold of an isoparametric hypersurface with four distinct principal curvatures in the unit sphere $S^{n+1}(1)$ with codimension $m_1 + 1$. If $\dim M_1 \geq \frac{2}{3}n + 1$, then*

$$\lambda_1(M_1) = \dim M_1$$

with multiplicity $n + 2$. A similar conclusion holds for M_2 under an analogous condition.

Proof. For sufficiently small $\varepsilon > 0$, set

$$M_1(\varepsilon) := S^{n+1}(1) - B_\varepsilon(M_2) = \bigcup_{\theta\in[0,\frac{\pi}{4}-\varepsilon]} M_\theta$$

where $B_\varepsilon(M_2) = \{x \in S^{n+1}(1) \mid dist(x, M_2) < \varepsilon\}$, and M_θ is the isoparametric hypersurface with an oriented distance θ from M_1. Notice that the notation M_θ here is different from that we used before.

Given $\theta \in (0, \frac{\pi}{4}-\varepsilon]$, let $\{e_{\alpha,i} \mid i = 1, \dots, m_\alpha,\ \alpha = 1, \dots, 4,\ e_{\alpha,i} \in E_\alpha\}$ be a local orthonormal frame field on M_θ and ξ be the unit normal field of M_θ toward M_1. After a parallel translation from any point $x \in M_\theta$ to a point $p = \phi_\theta(x) \in M_1$ (where $\phi_\theta : M_\theta \to M_1$ is the focal map, whose meaning is a little different from that in the last section), ξ is still a unit normal vector at p, which we also denote by ξ; $e_{1,i}$ $(i = 1, \dots, m_1)$ become normal vectors on M_1, while the others are still tangent vectors on M_1, which we will denote by $\{\widetilde{e}_{1,i}, \widetilde{e}_{2,i}, \widetilde{e}_{3,i}, \widetilde{e}_{4,i}\}$ determined by x.

We can decompose any $X \in T_xM_\theta$ as $X = X_1 + X_2 + X_3 + X_4 \in E_1 \oplus E_2 \oplus E_3 \oplus E_4$. Identify the principal distribution $E_\alpha(x)$ ($\alpha = 2, 3, 4$, $x \in M_\theta$) with its parallel translation at $p = \phi_\theta(x) \in M_1$. The shape operator A_ξ at p is given in terms of its eigenvectors $\widetilde{X}_\alpha$ (the parallel translation of $X_\alpha, \alpha = 2, 3, 4$) by (*cf.* [**Mün**]):

$$\begin{aligned} A_\xi \widetilde{X}_2 &= \cot(\theta_2 - \theta_1)\widetilde{X}_2 = \widetilde{X}_2, \\ A_\xi \widetilde{X}_3 &= \cot(\theta_3 - \theta_1)\widetilde{X}_3 = 0, \\ A_\xi \widetilde{X}_4 &= \cot(\theta_4 - \theta_1)\widetilde{X}_4 = -\widetilde{X}_4. \end{aligned} \tag{23}$$

Namely, $\widetilde{X}_2, \widetilde{X}_3, \widetilde{X}_4$ belong to the eigenspaces $E(1), E(0), E(-1)$ of A_ξ, respectively.

On the other hand, for a fixed θ, define $\rho = \phi_\theta : M_\theta \to M_1$. For any point $p \in M_1$, at a point $x \in \rho^{-1}(p)$, we have a distribution $E_1 \oplus E_2 \oplus E_3 \oplus E_4$. Among them, the first one is projected to be 0 under ρ_*; for the others, we have

$$\begin{aligned} \rho_* e_{\alpha,i} &= \frac{\sin(\theta_\alpha - \theta)}{\sin\theta_\alpha}\widetilde{e}_{\alpha,i} = \frac{\sin\frac{\alpha-1}{4}\pi}{\sin(\frac{\alpha-1}{4}\pi + \theta)}\widetilde{e}_{\alpha,i} \\ &:= \widetilde{k}_{\alpha-1}\widetilde{e}_{\alpha,i}, \qquad i = 1, \dots, m_\alpha,\ \alpha = 2, 3, 4. \end{aligned}$$

Denote by $\{\theta_{\alpha,i} \mid \alpha = 1, 2, 3, 4, i = 1, \dots, m_\alpha\}$ the dual frame of $e_{\alpha,i}$. We then conclude that (up to a sign)

$$dM_\theta = \prod_{j=1}^{m_\alpha}\prod_{\alpha=2}^{4}\theta_{\alpha,j} \wedge \prod_{i=1}^{m_1}\theta_{1,i} = \frac{1}{\widetilde{k}_1^{m_2}\widetilde{k}_2^{m_1}\widetilde{k}_3^{m_2}}\rho^*(dM_1) \wedge \prod_{i=1}^{m_1}\theta_{1,i}. \tag{24}$$

Notice that here the submanifold M_1 may be non-orientable, but the notation dM_1 still makes sense locally, up to a sign.

Let h be the same function as in Section 2. For sufficiently small $\eta > 0$, define $\widetilde{\psi}_\eta$ to be a nonnegative smooth function on $[0, \frac{\pi}{2} - \eta]$ by

$$\widetilde{\psi}_\eta(x) := \begin{cases} 1, & x \in [0, \frac{\pi}{4}] \\ h(\frac{\frac{\pi}{2}-x}{\eta}), & x \in [\frac{\pi}{4}, \frac{\pi}{2} - \eta] \end{cases}$$

Let f_k $(k = 0, 1, \ldots)$ be the k-th eigenfunctions on M_1 which are orthogonal to each other with respect to the square integral inner product on M_1 and $L_{k+1} = Span\{f_0, f_1, \ldots, f_k\}$. Then any $\varphi \in L_{k+1}$ on M_1 can give rise to a function $\widetilde{\Phi}_\varepsilon : M_1(\varepsilon) \to \mathbb{R}$ by:

$$\widetilde{\Phi}_\varepsilon(x) = \widetilde{\psi}_{2\varepsilon}(2\theta)(\varphi \circ \rho)(x).$$

Evidently, similarly as in the last section, $\widetilde{\Phi}_\varepsilon$ is a smooth function on $M_1(\varepsilon)$ satisfying the Dirichlet boundary condition and square integrable on $M_1(\varepsilon)$.

As in Section 2, the calculation of $\|\nabla\widetilde{\Phi}_\varepsilon\|_2^2$ is closely related to $|\nabla\varphi(\rho)|^2$. According to the decomposition (23), in the tangent space of M_1 at p, we can decompose $\nabla\varphi$ as $\nabla\varphi = Z_1 + Z_2 + Z_3 \in E(1) \oplus E(0) \oplus E(-1)$. Thus we have

$$\begin{cases} |\nabla\varphi|_p^2 = |Z_1|^2 + |Z_2|^2 + |Z_3|^2 \\ |\nabla\varphi(\rho)|_x^2 = \widetilde{k}_1^2|Z_1|^2 + \widetilde{k}_2^2|Z_2|^2 + \widetilde{k}_3^2|Z_3|^2 \end{cases} \tag{25}$$

In the following, we will investigate the change of $|\nabla\varphi(\rho)|^2$ along with the point x in the fiber sphere at p. For this purpose, we recall

Lemma (see, for example, [CCJ]) *Let M^n be an isoparametric hypersurface in the unit sphere $S^{n+1}(1)$. Then the curvature distributions are completely integrable. Their integral submanifolds corresponding to $\cot\theta_j$ are totally geodesic in M^n and have constant sectional curvature $1 + \cot^2\theta_j$.*

Denote by $S^{m_1}(\frac{1}{\sqrt{1+\cot^2\theta}}) \subset M_\theta$ the fiber sphere at p. Clearly, for any pair of antipodal points x, $x' \in \rho^{-1}(p) = S^{m_1}(\frac{1}{\sqrt{1+\cot^2\theta}})$, we have $\xi(x') = -\xi(x)$ by the parallel translations from x and x' to p, respectively. Denote by $E'(1), E'(0), E'(-1)$ the eigenspaces of $A_{\xi(x')}$ at p. Then we can also decompose $\nabla\varphi$ as $\nabla\varphi = Z_3 + Z_2 + Z_1 \in E'(1) \oplus E'(0) \oplus E'(-1)$ with respect to x'. In other words,

$$\begin{cases} |\nabla\varphi(\rho)|_x^2 = \widetilde{k}_1^2|Z_1|^2 + \widetilde{k}_2^2|Z_2|^2 + \widetilde{k}_3^2|Z_3|^2 \\ |\nabla\varphi(\rho)|_{x'}^2 = \widetilde{k}_3^2|Z_1|^2 + \widetilde{k}_2^2|Z_2|^2 + \widetilde{k}_1^2|Z_3|^2. \end{cases}$$

Thus, at the pair of two antipodal points x and x', we have

$$\frac{1}{2}\Big(|\nabla\varphi(\rho)|_x^2 + |\nabla\varphi(\rho)|_{x'}^2\Big) = \frac{\widetilde{k}_1^2 + \widetilde{k}_3^2}{2}\Big(|Z_1|^2 + |Z_3|^2\Big) + \widetilde{k}_2^2|Z_2|^2.$$

Set $\widetilde{K} := \max\{\frac{\widetilde{k}_1^2+\widetilde{k}_3^2}{2}, \widetilde{k}_2^2\}$ for $\theta \in (0, \frac{\pi}{4}-\varepsilon]$. It is clear to see $\widetilde{K} = \frac{1}{\cos^2 2\theta}$ by the definition of $\widetilde{k}_{\alpha-1}$. Since the assumption $3\dim M_1 \geq 2n+3$ implies $m_2 \geq 2$, this guarantees that $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{\frac{\pi}{4}-2\varepsilon}^{\frac{\pi}{4}-\varepsilon}\cos^{m_2}2\theta d\theta = 0$. Then a similar discussion to that in Section 2 leads to

$$\lim_{\varepsilon\to 0}\int_{M_1(\varepsilon)}(\widetilde{\psi}'_{2\varepsilon}(2\theta))^2\varphi(\rho)^2 dM_1(\varepsilon) = 0.$$

Hence

$$\begin{aligned}
\lim_{\varepsilon\to 0}\|\nabla\widetilde{\Phi}_\varepsilon\|_2^2 &= \lim_{\varepsilon\to 0}\int_{M_1(\varepsilon)}(\widetilde{\psi}_{2\varepsilon}(2\theta))^2|\nabla(\varphi\circ\rho)|^2 dM_1(\varepsilon)\\
&= \int_0^{\frac{\pi}{4}}\Big(\int_{M_\theta}\frac{|\nabla(\varphi\circ\rho)|^2}{\widetilde{k}_1^{m_2}\widetilde{k}_2^{m_1}\widetilde{k}_3^{m_2}}\rho^*(dM_1)dS^{m_1}(\frac{1}{\sqrt{1+\cot^2\theta}})\Big)d\theta \qquad (26)\\
&\leq \int_0^{\frac{\pi}{4}}\Big(\int_{M_\theta}|\nabla\varphi|^2\rho^*(dM_1)dS^{m_1}(\frac{1}{\sqrt{1+\cot^2\theta}})\Big)\cdot\frac{\widetilde{K}}{\widetilde{k}_1^{m_2}\widetilde{k}_2^{m_1}\widetilde{k}_3^{m_2}}d\theta\\
&= \int_0^{\frac{\pi}{4}}\Big(\int_{M_1}|\nabla\varphi|^2 dM_1\Big)\cdot\mathrm{Vol}(S^{m_1}(\frac{1}{\sqrt{1+\cot^2\theta}}))\cdot\frac{\widetilde{K}}{\widetilde{k}_1^{m_2}\widetilde{k}_2^{m_1}\widetilde{k}_3^{m_2}}d\theta\\
&= \|\nabla\varphi\|_2^2\cdot\frac{C_{m_1}}{2^{m_1+1}}\int_0^{\frac{\pi}{2}}\sin^{m_1}\theta\cos^{m_2-2}\theta\ d\theta\\
&= \|\nabla\varphi\|_2^2\cdot\frac{C_{m_1}}{2^{m_1+2}}\cdot B(\frac{m_1+1}{2},\frac{m_2-1}{2}),
\end{aligned}$$

where $\mathrm{Vol}(S^{m_1}(\frac{1}{\sqrt{1+\cot^2\theta}})) = C_{m_1}\cdot\sin^{m_1}\theta$; C_{m_1} is the volume of $S^{m_1}(1)$. Besides, with a simple calculation, we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}\|\widetilde{\Phi}_\varepsilon\|_2^2 &= \int_0^{\frac{\pi}{4}}\frac{1}{\widetilde{k}_1^{m_2}\widetilde{k}_2^{m_1}\widetilde{k}_3^{m_2}}\int_{M_1}\int_{S^{m_1}(\frac{1}{\sqrt{1+\cot^2\theta}})}\varphi(\rho)^2 dS^{m_1}dM_1 d\theta\\
&= \|\varphi\|_2^2\cdot\int_0^{\frac{\pi}{4}}\frac{1}{\widetilde{k}_1^{m_2}\widetilde{k}_2^{m_1}\widetilde{k}_3^{m_2}}\mathrm{Vol}(S^{m_1})\ d\theta \qquad (27)\\
&= \|\varphi\|_2^2\cdot\frac{C_{m_1}}{2^{m_1+2}}\cdot B(\frac{m_1+1}{2},\frac{m_2+1}{2}).
\end{aligned}$$

Consequently, combining (26) and (27), we arrive at

$$\lim_{\varepsilon\to 0}\frac{\|\nabla\widetilde{\Phi}_\varepsilon\|_2^2}{\|\widetilde{\Phi}_\varepsilon\|_2^2} \leq \frac{\|\nabla\varphi\|_2^2}{\|\varphi\|_2^2}\cdot\frac{B(\frac{m_1+1}{2},\frac{m_2-1}{2})}{B(\frac{m_1+1}{2},\frac{m_2+1}{2})} = \frac{\|\nabla\varphi\|_2^2}{\|\varphi\|_2^2}\cdot\frac{m_1+m_2}{m_2-1}.$$

A similar argument as in Section 2 leads us to

$$\lambda_k(S^{n+1}(1)) \leq \lambda_k(M_1)\frac{m_1+m_2}{m_2-1}. \qquad (28)$$

This inequality connects the eigenvalues of $S^{n+1}(1)$ and that of the focal submanifold M_1 in a concise manner. It contains rich information. Now we take $k = n+3$. The inequality (28) becomes

$$\frac{2(n+2)(m_2-1)}{m_1+m_2} \leq \lambda_{n+3}(M_1).$$

Based on this inequality, in order to complete the proof of Theorem 1.3, we just need to establish the following inequality:

$$\dim M_1 = m_1 + 2m_2 < \frac{2(n+2)(m_2-1)}{m_1+m_2}. \tag{29}$$

Due to the relation $n = 2(m_1+m_2)$, we get a sufficient condition on the positive integers m_1, m_2 which is almost optimal for the inequality (29) to hold:

$$m_2 \geq \frac{1}{2}(m_1+3).$$

At last, combining with Lemma 3.1, we can conclude that

$$\lambda_1(M_1) = \dim M_1 = m_1 + 2m_2, \text{ with multiplicity } n+2,$$
$$\text{provided } \ m_2 \geq \frac{1}{2}(m_1+3)$$

as we required. q.e.d.

Remark 3.2. When $g = 1$, the focal submanifolds are just two points. When $g = 2$, as is well known, the isoparametric hypersurface in $S^{n+1}(1)$ is isometric to the generalized Clifford torus $S^p(\sqrt{\frac{p}{n}}) \times S^q(\sqrt{\frac{q}{n}})$ $(p+q=n)$. The focal submanifolds are isometric to $S^p(1)$ and $S^q(1)$. Clearly, their first eigenvalues are their dimensions. When $g = 3$, E. Cartan asserted that $m_1 = m_2 = 1, 2, 4$ *or* 8. The focal submanifolds in the unit sphere $S^4(1)$, $S^7(1)$, $S^{13}(1)$, and $S^{25}(1)$ are the Veronese embedding of $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$, and $\mathbb{O}P^2$, respectively. The induced metric of this $\mathbb{R}P^2$ minimally embedded in $S^4(1)$ differs from the standard metric of constant Gaussian curvature $K = 1$ by a constant factor such that $K = \frac{1}{3}$; thus $\lambda_1(\mathbb{R}P^2) = 2$. As for $\mathbb{C}P^2$, $\mathbb{H}P^2$, and $\mathbb{O}P^2$, these are minimally embedded in the unit spheres $S^7(1)$, $S^{13}(1)$, and $S^{25}(1)$, respectively, while the induced metric differs from the symmetric space metric by a constant factor such that $\frac{1}{3} \leq \mathrm{Sec} \leq \frac{4}{3}$. By [**Str**] and [**Mas**], the first eigenvalues of the focal submanifolds $\mathbb{C}P^2$, $\mathbb{H}P^2$, and $\mathbb{O}P^2$ are equal to their dimensions, respectively.

Therefore, for $g = 2, 3$,

$$\lambda_1(M_i) = \dim M_i, \quad i = 1, 2.$$

□

We conclude this paper with a proof of Proposition 1.1.

Proposition 1.1. *Let M_2 be the focal submanifold of OT-FKM type defined before with $(m_1, m_2) = (1, k)$. The following equality is valid:*

$$\lambda_1(M_2) = \min\{4, 2+k\}.$$

Proof. When $m_1 = 1$, $m_2 = k$, the OT-FKM-type polynomial can be written as

$$F : \mathbb{R}^{2k+4} \longrightarrow \mathbb{R}$$
$$F(x) = |x|^4 - 2(\langle P_0 x, x\rangle^2 + \langle P_1 x, x\rangle^2).$$

By orthogonal transformations, we can always choose P_0 and P_1 to be

$$P_0 = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right), \quad P_1 = \left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right). \tag{30}$$

Writing any point $x \in S^{2k+3}(1)$ as $x = (z, w) \in \mathbb{R}^{k+2} \times \mathbb{R}^{k+2}$, the focal submanifold $M_2 = f^{-1}(-1)$ ($f = F|_{S^{2k+3}(1)}$) can be characterized as

$$M_2^{k+2} = \{(z, w) \in S^{2k+3}(1) \mid z \parallel w\}.$$

Define a map

$$\begin{array}{rcl} \Psi : S^1(1) \times S^{k+1}(1) & \longrightarrow & M_2^{k+2} \subset \mathbb{R}^{2k+4} \\ e^{i\theta}, x = (x_1, \ldots, x_{k+2}) & \mapsto & (e^{i\theta}x_1, \ldots, e^{i\theta}x_{k+2}). \end{array}$$

It satisfies $\Psi(\theta + \pi, -x) = \Psi(\theta, x)$. In this way, we can identify M_2 isometrically with the metric induced from $S^{2k+3}(1)$ as

$$M_2^{k+2} \cong S^1(1) \times S^{k+1}(1) \Big/ (\theta, x) \sim (\theta + \pi, -x).$$

The eigenfunctions of M_2 are those products of eigenfunctions from $S^1(1)$ and $S^{k+1}(1)$ which take the same values at (θ, x) and $(\theta+\pi, -x)$. Hence $\lambda_1(M_2^{k+2}) = \min\{4, k+2\}$, as we claimed. q.e.d.

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