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Classification of non-degenerate projective varieties with non-zero prolongation and application to target rigidity

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Abstract The prolongation $\mathfrak{g}^{(k)}$ of a linear Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ plays an important role in the study of symmetries of G-structures. Cartan and Kobayashi-Nagano have given a complete classification of irreducible linear Lie algebras $\mathfrak{g} \subset \mathfrak{gl}(V)$ with non-zero prolongations.

If $\mathfrak{g}$ is the Lie algebra $\mathfrak{aut}(\hat{S})$ of infinitesimal linear automorphisms of a projective variety $S \subset \mathbb{P}V$, its prolongation $\mathfrak{g}^{(k)}$ is related to the symmetries of cone structures, an important example of which is the variety of minimal rational tangents in the study of uniruled projective manifolds. From this perspective, understanding the prolongation $\mathfrak{aut}(\hat{S})^{(k)}$ is useful in questions related to the automorphism groups of uniruled projective manifolds. Our main result is a complete classification of irreducible non-degenerate nonsingular variety $S \subset \mathbb{P}V$ with $\mathfrak{aut}(\hat{S})^{(k)} \neq 0$, which can be viewed as a generalization of the result of Cartan and Kobayashi-Nagano. As an application, we show that when S is linearly normal and $\mathrm{Sec}(S) \neq \mathbb{P}V$, the blow-up $\mathrm{Bl}_S(\mathbb{P}V)$ has the target rigidity property, i.e., any deformation of a surjective morphism $f: Y \to \mathrm{Bl}_S(\mathbb{P}V)$ comes from the automorphisms of $\mathrm{Bl}_S(\mathbb{P}V)$.

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1 Introduction

For a linear algebraic group $G \subset \mathrm{GL}(V)$, a G-structure on a complex manifold M with $\dim M = \dim V$ is a G-subbundle of the frame bundle on M. Many classical geometric structures in differential geometry are G-structures for various choices of G. For this reason, the (self)-equivalence problem for G-structures has been studied extensively. It turns out that the graded pieces (under a natural filtration) of the Lie algebra of infinitesimal symmetries of G-structure are contained in the prolongations $\mathfrak{g}^{(i)}, i \geq 1$, of the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ (cf. Definition 2.1 for a precise definition) and in fact, equal to the prolongations when the G-structure is flat (cf. Proposition 5.9). In other words, an essential information of the symmetries of G-structures is encoded in $\mathfrak{g}^{(i)}$. A fundamental result in the study of prolongations is the following result of E. Cartan, S. Kobayashi and T. Nagano.

Theorem 1.1 *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be an irreducible representation of a Lie algebra $\mathfrak{g}$.*

(1) *If $\mathfrak{g}^{(2)} \neq 0$, then $\mathfrak{g} = \mathfrak{gl}(V), \mathfrak{sl}(V), \mathfrak{sp}(V)$ or $\mathfrak{csp}(V)$ where* $\dim V$ *is even for the last two cases.*
(2) *If $\mathfrak{g}^{(2)} = 0$, but $\mathfrak{g}^{(1)} \neq 0$, then $\mathfrak{g} \subset \mathfrak{gl}(V)$ is isomorphic to the isotropy representation on the tangent space at a base point of an irreducible Hermitian symmetric space of compact type, different from the projective space.*

The proof of Theorem 1.1 as given in [17] is purely algebraic, and depends heavily on the theory of semi-simple Lie algebras and their representations. For that reason, there is little hope of generalizing it to non-reductive Lie algebras.

In [14], motivated by algebro-geometric questions, the prolongation of $\mathfrak{g} \subset \mathfrak{gl}(V)$ associated to a projective variety $S \subset \mathbb{P}V$ was studied. More precisely, for a projective subvariety $S \subsetneq \mathbb{P}(V)$, consider the Lie algebra $\mathfrak{aut}(\hat{S}) \subset \mathfrak{gl}(V)$ of infinitesimal linear automorphisms of the affine cone $\hat{S}$. Hwang and Mok [14] show that one can study prolongations $\mathfrak{aut}(\hat{S})^{(k)}$ using projective geometry of $S \subset \mathbb{P}V$ and the deformation theory of rational curves on S. Combining these two geometric tools, the following generalization of Theorem 1.1 (1) is proved in Theorem 1.1.2 of [14].

Theorem 1.2 *Let $S \subset \mathbb{P}V$ be an irreducible nonsingular non-degenerate projective variety. If $\mathfrak{aut}(\hat{S})^{(2)} \neq 0$, then $S = \mathbb{P}V$.*

It is easy to derive Theorem 1.1 (1) from Theorem 1.2. On the other hand, the latter is stronger than the former, because there is no a priori reason that $\mathfrak{aut}(\hat{S})$ is reductive in Theorem 1.2. For example, for the deformation rigidity studied in [14], it is essential to have this stronger result.

It is natural to ask the generalization of Theorem 1.1 (2) in the form of Theorem 1.2. Some partial results in this direction was obtained in [14] (e.g. Theorem 2.6 below). The goal of this paper is to give a complete answer to this question in the following form.

Main Theorem *Let $S \subsetneq \mathbb{P}V$ be an irreducible nonsingular non-degenerate variety such that $\mathfrak{aut}(\hat{S})^{(1)} \neq 0$. Then $S \subset \mathbb{P}V$ is projectively equivalent to one of the following*:

(A1) *the second Veronese embedding $v_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{1}{2}(n^2+3n)}$ for $n \geq 2$;*
(A2) *Segre embedding $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{ab+a+b}$ for $a, b \geq 2$;*
(A3) *a natural embedding*

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^k}(-1)^m \oplus \mathcal{O}_{\mathbb{P}^k}(-2)) \subset \mathbb{P}^{m(k+1)+\frac{1}{2}(k+2)(k+1)-1}$$

for $k \geq 2, m \geq 1$;
(B1) *odd-dimensional hyperquadrics $\mathbb{Q}^1, \mathbb{Q}^3, \ldots, \mathbb{Q}^{2\ell-1}, \ldots$;*
(B2) *even-dimensional hyperquadrics $\mathbb{Q}^2, \mathbb{Q}^4, \ldots, \mathbb{Q}^{2\ell}, \ldots$;*
(B3) *Segre embedding $\mathbb{P}^1 \times \mathbb{P}^m \subset \mathbb{P}^{2m+1}$ and Plücker embedding*

$$\mathrm{Gr}(2, \mathbb{C}^{m+3}) \subset \mathbb{P}^{\frac{1}{2}(m^2+5m+4)}$$

for $m \geq 3$;
(B4) *Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$, Plücker embedding $\mathrm{Gr}(2, \mathbb{C}^5) \subset \mathbb{P}^9$, spinor embedding $\mathbb{S}_5 \subset \mathbb{P}^{15}$ and the E_6-Severi embedding $\mathbb{OP}^2 \subset \mathbb{P}^{26}$;*
(B5) *general hyperplane sections of the first three in* (B4), *i.e.*, $(\mathbb{P}^1 \times \mathbb{P}^2) \cap H_0 \subset \mathbb{P}^4, \mathrm{Gr}(2, \mathbb{C}^5) \cap H_1 \subset \mathbb{P}^8, \mathbb{S}_5 \cap H_2 \subset \mathbb{P}^{14}$;

(B6) *general hyperplane section of the first in* (B3), *i.e.* $(\mathbb{P}^1 \times \mathbb{P}^m) \cap H \subset \mathbb{P}^{2m}$, *for* $m \geq 3$;

(C) *some biregular projections of* (A1), (A2), (A3) *and* $\mathrm{Gr}(2, \mathbb{C}^{m+3})$ *in* (B3).

The varieties in (A1)–(A3) and (C) satisfy $\mathrm{Sec}(S) \neq \mathbb{P}V$ while the first entries of (B1)–(B6) verify $\mathrm{Sec}(S) = \mathbb{P}V$. Note that the varieties in (B1)–(B5) are listed as sequences $S_0, S_1, S_2, \ldots$. The reason behind this way of listing the varieties will become clear in the course of the proof of Main Theorem. In fact, the variety S_i is the VMRT (cf. Definition 3.1) of the variety S_{i+1}, a crucial fact in the proof of Main Theorem. More detailed description of the varieties (A1)–(B6) and the explicit computation of the prolongation $\mathfrak{aut}(\hat{S})^{(1)}$ for each of them are given in Sect. 3. The set of biregular projections in (C) which have non-zero prolongations will be described completely in Sect. 4. One may have the impression that compared with the linearly normal cases of (A1)–(B6), the projections in (C) are mere technicalities. This is not the case. In fact, in the induction process of the proof of Main Theorem, it is crucial to understand the cases in (C). In other words, imposing the additional condition of linear normality on $S \subset \mathbb{P}V$ in Main Theorem would not make the proof any simpler, and it is essential to include varieties which are not necessarily linearly normal to carry out the proof of Main Theorem.

As we will explain in Sect. 5, $\mathfrak{aut}(\hat{S})^{(1)}$ is an essential part of the symmetries of cone structures, in particular, the structure coming from the varieties of minimal rational tangents, which is an important tool in the study of uniruled projective varieties. In this respect, Main Theorem will be useful in algebraic geometric questions involving automorphism groups of uniruled varieties. As an example, we will give a direct application of Main Theorem in Sect. 9, in the proof of the target rigidity for the blow-up of $\mathbb{P}V$ along S. More precisely, we shall show (cf. Theorem 9.6) that if $S \subset \mathbb{P}V$ is an irreducible nonsingular non-degenerate linearly normal variety such that $\mathrm{Sec}(S) \neq \mathbb{P}V$, then any deformation $f_t : Y \to \mathrm{Bl}_S(\mathbb{P}V)$ of a surjective morphism $f_0 : Y \to \mathrm{Bl}_S(\mathbb{P}V)$ comes from automorphisms of $\mathrm{Bl}_S(\mathbb{P}V)$.

Turning to the proof of Main Theorem, the main strategy is to carry out an induction on VMRT. In fact, by the partial result in [14] and the work of Ionescu-Russo [16], the question is quickly reduced to the case when $S \subset \mathbb{P}V$ has Picard number 1 and covered by lines. In this setting, we show in Proposition 6.7 and Theorem 6.15 that the VMRT of S at a general point, say, $S' \subset \mathbb{P}V'$, is again an irreducible nonsingular non-degenerate projective variety with $\mathfrak{aut}(\hat{S}')^{(1)} \neq 0$. By induction, we have a classification of $S' \subset \mathbb{P}V'$. From the information on $S' \subset \mathbb{P}V'$, we can recover $S \subset \mathbb{P}V$ by Cartan-Fubini type extension theorem as explained in Corollary 6.9. An essential ingredient in this induction process is the local flatness of the associated cone structure, or equivalently, G-structure. For that purpose, we develop some general theory for the latter differential-geometric machinery in Sect. 5.

The induction process enables us to prove Main Theorem, modulo the termination of the sequence of varieties in (B3)–(B6). Among these, the termination of (B3) and (B4) is an easy consequence of the condition on the secant varieties, via a result from [10]. The termination of (B5) and (B6) is more complicated and technically demanding. It will be proved in Sects. 7 and 8. Many of the geometric ideas in these two sections are borrowed from Sects. 6 and 8 of [14]. However, the main line of arguments and details of the proof are rather different from [14], except Propositions 7.6 and 8.6 whose proofs are essentially contained in those of Propositions 6.3.4 and 8.3.4 of [14], respectively.

2 Prolongation of a projective variety: basic properties

Definition 2.1 Let V be a complex vector space and $\mathfrak{g} \subset \mathrm{End}(V)$ a Lie subalgebra. The *kth prolongation* (denoted by $\mathfrak{g}^{(k)}$) of $\mathfrak{g}$ is the space of symmetric multi-linear homomorphisms $A : \mathrm{Sym}^{k+1} V \to V$ such that for any fixed $v_1, \ldots, v_k \in V$, the endomorphism $A_{v_1,\ldots,v_k} : V \to V$ defined by

$$v \in V \quad \mapsto \quad A_{v_1,\ldots,v_k,v} := A(v, v_1, \ldots, v_k) \in V$$

is in $\mathfrak{g}$. In other words, $\mathfrak{g}^{(k)} = \mathrm{Hom}(\mathrm{Sym}^{k+1} V, V) \cap \mathrm{Hom}(\mathrm{Sym}^k V, \mathfrak{g})$.

It is immediate from the definition that $\mathfrak{g}^{(0)} = \mathfrak{g}$ and if $\mathfrak{g}^{(k)} = 0$, then $\mathfrak{g}^{(k+1)} = 0$.

In this paper, we are interested in the case where $\mathfrak{g}$ arises from geometric situations. Let us first recall some basic definitions.

Definition 2.2 Let $S \subset \mathbb{P}V$ be an irreducible projective subvariety.

(i) S is said to be *non-degenerate* (resp. *linearly normal*) if the restriction map $H^0(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(1)) \to H^0(S, \mathcal{O}_S(1))$ is injective (resp. surjective).

(ii) S is said to be *covered by lines* if through each general point of S, there passes a line lying on S. S is *conic-connected* if through two general points of S, there passes an irreducible conic contained in S.

(iii) The *secant variety* $\mathrm{Sec}(S) \subset \mathbb{P}V$ of S is the closure of the union of lines through two points of S.

(iv) The projective automorphism group of $S \subset \mathbb{P}V$ is

$$\mathrm{Aut}(S) := \{g \in \mathrm{PGL}(V) | gS = S\}.$$

Its identity component will be denoted by $\mathrm{Aut}_0(S)$ and its Lie algebra will be denoted by $\mathfrak{aut}(S)$.

(v) Denote by $\hat{S} \subset V$ the affine cone of S and by $T_\alpha(\hat{S}) \subset V$ the tangent space at a smooth point $\alpha \in \hat{S}$. The Lie algebra of infinitesimal linear automorphisms of $\hat{S}$ is

$$\mathfrak{aut}(\hat{S}) := \{g \in \mathrm{End}(V) | g(\alpha) \in T_\alpha(\hat{S}) \text{ for any smooth point } \alpha \in \hat{S}\}.$$

Its prolongation $\mathfrak{aut}(\hat{S})^{(k)}$ will be called the kth *prolongation* of $S \subset \mathbb{P}V$. We will often call $\mathfrak{aut}(\hat{S})^{(1)}$ the prolongation of S.

We have the following vanishing result.

Theorem 2.3 ([14], Theorem 1.1.2) *Let $S \subsetneq \mathbb{P}V$ be an irreducible nonsingular non-degenerate subvariety. Then $\mathfrak{aut}(\hat{S})^{(k)} = 0$ for all $k \geq 2$.*

However, there are several examples of S with non-zero first prolongation $\mathfrak{aut}(\hat{S})^{(1)}$. In [14], some partial results on the structure of such varieties were obtained. Here we collect them with some immediate improvements.

Before stating the next theorem, it is convenient to introduce the notion of Euler vector field. The following is a well-known fact in Poincaré normal form theory of ordinary differential equations (e.g. [2], Sects. 3.3.2 and 4.1.2).

Lemma 2.4 *A germ of holomorphic vector fields at $(\mathbb{C}^n, 0)$ of the form*

$$\sum_{i=1}^{n} (z_i + h_i(z)) \frac{\partial}{\partial z_i}$$

with $h_i \in \mathbf{m}^2$ where $\mathbf{m} \subset \mathcal{O}_{\mathbb{C}^n,0}$ is the maximal ideal, can be expressed as $\sum_{i=1}^{n} w_i \frac{\partial}{\partial w_i}$ in a suitable holomorphic coordinate system w_i.

Definition 2.5 A germ of vector fields of the form in Lemma 2.4 is called an *Euler vector field.*

The following theorem is essentially proved in Theorem 1.1.3 of [14].

Theorem 2.6 *Let $S \subsetneq \mathbb{P}V$ be a nonsingular, non-degenerate and linearly normal projective subvariety. Then the following holds.*

(i) *If $\mathfrak{aut}(\hat{S})^{(1)} \neq 0$, then S is conic-connected.*
(ii) *For each non-zero $A \in \mathfrak{aut}(\hat{S})^{(1)}$, there exists a non-zero $\lambda_A \in V^*$ such that for each $\alpha \in \hat{S}$, $A_{\alpha,\alpha} = \lambda_A(\alpha)\alpha$. This defines an inclusion $\mathfrak{aut}(\hat{S})^{(1)} \subset V^*$.*
(iii) *In the notation of* (ii), *for any $\alpha \in \hat{S}$ and $\alpha' \in T_\alpha(\hat{S})$,*

$$\lambda_A(\alpha)\alpha' + \lambda_A(\alpha')\alpha = 2A_{\alpha,\alpha'}.$$

In particular, the endomorphism A_α *acts on the tangent space of* S

$$T_{[\alpha]}(S) = \mathrm{Hom}(\mathbb{C}\alpha, T_\alpha(\hat{S})/\mathbb{C}\alpha)$$

as the scalar multiplication by $\frac{1}{2}\lambda_A(\alpha)$.

(iv) *Suppose* $\mathfrak{aut}(\hat{S})^{(1)} \neq 0$. *Then for a general point* $s \in S$, *there exists an element* $E \in \mathfrak{aut}(\hat{S})$ *which generates a* $\mathbb{C}^\times$*-action on* S *with an isolated fixed point at* s *such that the isotropy action on* $T_s(S)$ *is the scalar multiplication. In particular, the germ of* E *at* s *is an Euler vector field.*

Proof By Theorem 1.1.3 (ii) [14], there exists a point $s_o \in S$ such that S is covered by conics passing through s_o. By Lemma 1 in [3], this implies that S is conic-connected, proving (i). Claim (ii) follows from Proposition 2.3.1 [14] while (iii) and (iv) follow from the proof of Theorem 1.1.3 (iii) in [14]. □

Let us recall the following from Lemma 2.3.3 in [14].

Lemma 2.7 *Let* $S \subsetneq \mathbb{P}V$ *be a nonsingular non-degenerate linearly normal projective subvariety which is not biregular to a projective space. Let* $A \in \mathfrak{aut}(\hat{S})^{(1)}$. *Suppose for some* $\alpha \in V$ *and a subspace* $H \subset V$ *of codimension* 1, *the endomorphism* A_α *satisfies* $A_{\alpha,\beta} = 0$ *for all* $\beta \in H \cap \hat{S}$. *Then* $A_\alpha = 0$.

Theorem 2.6 has the following consequences.

Proposition 2.8 *In the setting of Theorem* 2.6 (ii), *assume* S *is not biregular to a projective space. Choose a general point of the hyperplane section*

$$\alpha \in \hat{S} \cap (\lambda_A = 0).$$

Let $\bar{\alpha}$ *be the image of* α *under the natural projection* $\hat{S} \setminus \{0\} \to S$. *Then the vector field on* S *induced by* A_α *is not identically zero and vanishes at* $\bar{\alpha} \in S$ *to second order.*

Proof Suppose that for any general point $\alpha \in \hat{S} \cap (\lambda_A = 0)$, the vector field on S induced by A_α is identically zero on S, i.e., for each $\beta \in \hat{S}$, $A_{\alpha,\beta}$ is proportional to β. Then it is proportional to α by symmetry. We conclude that A_α is identically zero on $\hat{S}$, thus on V. By symmetry, for each $\gamma \in V$, A_γ vanishes on $\hat{S} \cap (\lambda_A = 0)$. Then by Lemma 2.7, A_γ is identically zero, a contradiction to $A \neq 0$. This shows that the vector field on S induced by A_α is not identically zero.

Now Theorem 2.6 (iii) says that this vector field on S vanishes to second order at $\alpha \in \hat{S}$ with $\lambda_A(\alpha) = 0$. □

Proposition 2.9 *Let $S \subset \mathbb{P}V$ be a nonsingular non-degenerate linearly normal subvariety. Then*

$$\dim \mathfrak{aut}(\hat{S})^{(1)} \neq 1.$$

Proof Let us write $\mathfrak{g} = \mathfrak{aut}(\hat{S}) \subset \mathfrak{gl}(V)$. Assuming that $\dim \mathfrak{g}^{(1)} = 1$, we will derive a contradiction. Let $G \subset \mathrm{GL}(V)$ be the connected component of the linear automorphism group of the cone $\hat{S} \subset V$ whose Lie algebra is $\mathfrak{g}$. Note that G contains the central subgroup $\mathbb{C}^\times \cdot \mathrm{Id}$. The natural G-action on $\mathrm{Hom}(\mathrm{Sym}^2 V, V)$ induces a G-action

$$\chi : G \to \mathrm{GL}(\mathfrak{g}^{(1)}) \cong \mathbb{C}^\times,$$

which is a character of G. Let $G' \subset G$ be the kernel of χ and $\mathfrak{g}' \subset \mathfrak{g}$ be its Lie algebra. Since $\mathfrak{g}^{(1)} \subset \mathrm{Hom}(\mathrm{Sym}^2 V, V)$, the central subgroup $\mathbb{C}^\times \cdot \mathrm{Id}$ acts non-trivially on $\mathfrak{g}^{(1)}$ and the normal subgroup $G' \subset G$ is complementary to $\mathbb{C}^\times \cdot \mathrm{Id}$. Thus we have a direct sum decomposition of the Lie algebra $\mathfrak{g} = \mathbb{C} \cdot \mathrm{Id} \oplus \mathfrak{g}'$. Let $\bar{G} \subset \mathrm{PGL}(V)$ be the image of G, under the projection $\mathrm{GL}(V) \to \mathrm{PGL}(V)$. Then $\bar{G}$ is the identity component of the projective automorphism group of S. The homomorphism $G' \to \bar{G}$ has finite kernel and the Lie algebra $\mathfrak{g}'$ is isomorphic to the Lie algebra of $\bar{G}$.

From $\mathfrak{g}^{(1)} \neq 0$ and Theorem 2.6 (iv), for each general point $x \in S$, we have a $\mathbb{C}^\times$-subgroup $G_x \subset G'$ which acts as the multiplication by $\mathbb{C}^\times$ on the tangent space $T_x(S)$. Let $T_x(\hat{S})$ be the affine tangent space at x. Since G_x has weight 1 on $T_x(S)$, it has exactly two distinct weights on $T_x(\hat{S})$. In fact, from $T_x(S) = \mathrm{Hom}(\hat{x}, T_x(\hat{S})/\hat{x})$, if it has weight k on $\hat{x}$, the other weight on $T_x(\hat{S})/\hat{x}$ must be $k+1$.

Pick a vector $\alpha \in \hat{x}$ and $\alpha' \in T_x(\hat{S})/\hat{x}$. For $A \in \mathfrak{g}^{(1)}$ and $g \in G'$, $g \cdot A = A$ implies that

$$A_{g\alpha, g\alpha'} = g \cdot A_{\alpha,\alpha'}.$$

On the other hand, by Theorem 2.6 (iii) we have

$$2A_{\alpha,\alpha'} = \lambda(\alpha)\alpha' + \lambda(\alpha')\alpha,$$

for some non-zero $\lambda \in V^*$. Thus for any $t \in G_x \cong \mathbb{C}^\times$,

$$2t \cdot A_{\alpha,\alpha'} = t^{k+1}\lambda(\alpha)\alpha' + t^k\lambda(\alpha')\alpha,$$

while

$$2A_{t\cdot\alpha,t\cdot\alpha'} = 2A_{t^k\alpha,t^{k+1}\alpha'} = 2t^{2k+1}A_{\alpha,\alpha'} = t^{2k+1}\lambda(\alpha)\alpha' + t^{2k+1}\lambda(\alpha')\alpha.$$

Thus either $\lambda(\alpha) = 0$ or $\lambda(\alpha') = 0$. Since the set of such α or α' spans the vector space V, we get $\lambda = 0$, a contradiction. □

3 Examples of linearly normal varieties with non-zero first prolongation

In this section, we will list examples of linearly normal $S \subset \mathbb{P}V$ with non-zero first prolongation. Before giving these examples, it is convenient to recall the notion of VMRT, because our examples arise as VMRT of some uniruled manifolds.

Definition 3.1 Let X be a uniruled projective manifold. An irreducible component $\mathcal{K}$ of the space $\mathrm{RatCurves}^n(X)$ of rational curves on X is called *a minimal rational component* if the subvariety $\mathcal{K}_x$ of $\mathcal{K}$ parameterizing curves passing through a general point $x \in X$ is non-empty and proper. Curves parameterized by $\mathcal{K}$ will be called *minimal rational curves*. Let $\rho : \mathcal{U} \to \mathcal{K}$ be the universal family and let $\mu : \mathcal{U} \to X$ be the evaluation map. The tangent map $\tau : \mathcal{U} \dashrightarrow \mathbb{P}T(X)$ is defined by $\tau(u) = [T_{\mu(u)}(\mu(\rho^{-1}\rho(u)))] \in \mathbb{P}T_{\mu(u)}(X)$. The closure $\mathcal{C} \subset \mathbb{P}T(X)$ of its image is the total space of *variety of minimal rational tangents*. The natural projection $\mathcal{C} \to X$ is a proper surjective morphism and a general fiber $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is called the *variety of minimal rational tangents* at the point $x \in X$. To simplify the terminology, we will use 'VMRT' for 'variety (or varieties) of minimal rational tangents'.

The following is well-known (cf. Proposition 1.5 in [6]).

Proposition 3.2 *Let $X \subset \mathbb{P}^N$ be a nonsingular projective variety covered by lines. A component of family of lines covering X is a minimal rational component and the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point $x \in X$ is nonsingular.*

The following is immediate.

Lemma 3.3 *In the setting of Proposition 3.2, let $X \cap H$ be a general hyperplane section. If $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is the VMRT of X at a general point $x \in X \cap H$ and* $\dim \mathcal{C}_x \geq 1$, *then the VMRT associated to a family of lines covering $X \cap H$ is*

$$\mathcal{C}_x \cap \mathbb{P}T_x(H) \subset \mathbb{P}T_x(X \cap H).$$

3.1 VMRT of an irreducible Hermitian symmetric space of compact type

An irreducible Hermitian symmetric space of compact type is a homogeneous space $M = G/P$ with a simple Lie group G and a maximal parabolic subgroup P such that the isotropy representation of P on $T_x(M)$ at a base point $x \in M$ is irreducible. The highest weight orbit of the isotropy action on $\mathbb{P}T_x(M)$ is exactly the VMRT at x.

The following table collects some well-known facts on irreducible Hermitian symmetric spaces of compact type (see e.g. [10], Sect. 6.2).

Type	I.H.S.S. M	VMRT S	$S \subset \mathbb{P}T_x(M)$	$\dim \mathbb{P}T_x(M)$	$\dim \mathrm{Sec}(S)$
I	$\mathrm{Gr}(a, a+b)$	$\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}$	Segre	$ab-1$	$2a+2b-5$
II	$\mathbb{S}_n$	$\mathrm{Gr}(2,n)$	Plücker	$\frac{1}{2}(n^2-n-2)$	$4n-11$
III	$\mathrm{Lag}(2n)$	$\mathbb{P}^{n-1}$	Veronese	$\frac{1}{2}(n^2+n-2)$	$2n-2$
IV	$\mathbb{Q}^n$	$\mathbb{Q}^{n-2}$	Hyperquadric	$n-1$	$n-1$
V	$\mathbb{OP}^2$	$\mathbb{S}_5$	Spinor	15	15
VI	$E_7/(E_6 \times U(1))$	$\mathbb{OP}^2$	Severi	26	25

Here $\mathrm{Gr}(a, a+b)$ is the Grassmannian of a-dimensional subspaces in an $(a+b)$-dimensional vector space. $\mathbb{S}_n$ is the spinor variety, i.e. the variety parameterizing n-dimensional isotropic linear subspaces in an orthogonal vector space of dimension $2n$. $\mathrm{Lag}(2n)$ is the Lagrangian Grassmannian, which parameterizes Lagrangian subspaces in a $2n$-dimensional symplectic vector space. $\mathbb{Q}^n$ denotes the n-dimensional hyperquadric. $\mathbb{OP}^2$ is the Cayley plane, which is of dimension 16 and homogeneous under the action of E_6.

In the following, we always assume that M is not a projective space. Let $o \in M = G/P$ be the point with the isotropy subgroup P and set $V = T_o(M)$. Let $S \subset \mathbb{P}V$ be the VMRT of M. There exists a depth 1 decomposition of the Lie algebra $\mathfrak{g}$ of G (cf. [14], Sect. (4.1)): if we denote by α_k the simple root corresponding to the maximal parabolic subgroup P, and Φ_i the set of roots whose coefficient in α_k equals to i, then Φ_i is not empty exactly for $i = 0, \pm 1$. Let $\mathfrak{g}_i = \bigoplus_{\alpha \in \Phi_i} \mathfrak{g}_\alpha$, then we get the decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ satisfying $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all i, j. There exist natural isomorphisms $\mathfrak{g}_{-1} \cong T_o(G/P) = V$, $\mathfrak{g}_1 \cong T_o^*(G/P) = V^*$ and $\mathfrak{g}_0 \cong \mathfrak{aut}(\hat{S})$. When G is of classical type, an explicit description of the gradation of $\mathfrak{g}$ can be found in Sect. 4.4 of [21].

We have a natural injective map:

$$\phi : \mathfrak{g}_1 \to \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_0), \quad \text{given by } \phi_X(Y) = [X, Y],\ \forall X \in \mathfrak{g}_1,\ Y \in \mathfrak{g}_{-1}.$$

By Theorem 5.2 of [21], the image $\mathrm{Im}(\phi)$ is exactly the prolongation $\mathfrak{g}_0^{(1)} = \mathfrak{aut}(\hat{S})^{(1)}$. This gives

Proposition 3.4 *Let $S \subsetneq \mathbb{P}V$ be the VMRT of an irreducible Hermitian symmetric space M of compact type. Then $\mathfrak{aut}(\hat{S})^{(1)} \cong \mathfrak{g}_1 \cong V^*$.*

As explained in Corollary 1.1.5 of [14], Theorem 2.6 implies the following result of [17].

Theorem 3.5 *Let $S \subseteq \mathbb{P}V$ be the highest weight variety of an irreducible representation. Then $\mathfrak{aut}(\hat{S})^{(1)} \neq 0$ if and only if S is the VMRT of an irreducible Hermitian symmetric space of compact type.*

3.2 VMRT of symplectic Grassmannians

Let Σ be an n-dimensional vector space endowed with a skew-symmetric 2-form ω of maximal rank. We denote by $\mathrm{Gr}_\omega(k, \Sigma)$ the variety of all k-dimensional isotropic subspaces of Σ. When n is even, this is the usual symplectic Grassmannian, which is homogeneous under the action of $\mathrm{Sp}(\Sigma)$. When n is odd, $\mathrm{Gr}_\omega(k, \Sigma)$ is the odd symplectic Grassmannian, which is not homogeneous and has two orbits under the action of its automorphism group (cf. [19], Proposition 1.12).

Let W and Q be vector spaces of dimensions $k \geq 2$ and m respectively. Let $v_2 : \mathbb{P}(W \oplus Q) \hookrightarrow \mathbb{P}(\mathrm{Sym}^2(W \oplus Q))$ be the second Veronese embedding. Let

$$U := (W \otimes Q) \oplus \mathrm{Sym}^2 W \subset \mathrm{Sym}^2(W \oplus Q).$$

Let $p_{\mathrm{Sym}^2 Q} : \mathbb{P}(\mathrm{Sym}^2(W \oplus Q)) \dashrightarrow \mathbb{P}U$ be the projection from $\mathbb{P}(\mathrm{Sym}^2 Q)$. We denote by Z the proper image of $v_2(\mathbb{P}(W \oplus Q))$ under the projection $p_{\mathrm{Sym}^2 Q}$. Then Z is isomorphic to the projective bundle $\mathbb{P}((Q \otimes \mathbf{t}) \oplus \mathbf{t}^{\otimes 2})$ over $\mathbb{P}W$, where $\mathbf{t}$ is the tautological line bundle over $\mathbb{P}W$. The embedding $Z \subset \mathbb{P}U$ is given by the complete linear system

$$H^0(\mathbb{P}W, (Q \otimes \mathbf{t}^*) \oplus (\mathbf{t}^*)^{\otimes 2}) = (W \otimes Q)^* \oplus \mathrm{Sym}^2 W^* = U^*.$$

The following lemma was proved in Proposition 3.2.1 [14] for the case of (even) symplectic Grassmannians. The proof there works also for odd symplectic Grassmannians.

Lemma 3.6 *The linearly normal embedding $Z \hookrightarrow \mathbb{P}U$ is isomorphic to the VMRT at a general point of the symplectic Grassmannian* $\mathrm{Gr}_\omega(k, \Sigma)$ (*with* $\dim \Sigma = m + 2k$).

We also have

Lemma 3.7 *If $k = 2$, then $Z \hookrightarrow \mathbb{P}U$ is the VMRT at a general point of a general hyperplane section of the Plücker embedding of* $\mathrm{Gr}(2, m + 4)$. *Equivalently, Z is a general hyperplane section of* $\mathbb{P}^1 \times \mathbb{P}^{m+1}$.

Proof Let Σ be a vector space of dimension $m + 4$, then we have the Plücker embedding $\mathrm{Gr}(2, \Sigma) \hookrightarrow \mathbb{P}(\bigwedge^2 \Sigma)$. Let ω be a general element of $\bigwedge^2 \Sigma^*$, i.e., a skew-symmetric 2-form on Σ with maximal rank. Let $H \subset \bigwedge^2 \Sigma$ be the kernel of $\omega \in \bigwedge^2 \Sigma^*$. Then we get $\mathrm{Gr}(2, \Sigma) \cap H = \mathrm{Gr}_\omega(2, \Sigma)$, the latter being the symplectic Grassmannian. The last statement is by Lemma 3.3. □

The following will be proved after Proposition 4.15.

Proposition 3.8 $\mathfrak{aut}(\hat{Z}) = (W^* \otimes Q) \rtimes (\mathfrak{gl}(W) \oplus \mathfrak{gl}(Q))$ *and* $\mathfrak{aut}(\hat{Z})^{(1)} \cong \mathrm{Sym}^2 W^*$.

3.3 Hyperplane section of $\mathbb{S}_5$

Let Q be a 7-dimensional orthogonal vector space and let W be the 8-dimensional spin representation of $\mathfrak{so}(Q) = \mathfrak{so}(7)$. There exists a Spin(7)-stable 9-dimensional Fano manifold Z of Picard number 1 with an embedding $Z \subset \mathbb{P}(W \oplus Q)$ which is isomorphic to a general hyperplane section of the 10-dimensional spinor variety (cf. Sect. 7 in [14] where it is denoted by $\mathcal{C}_o$). In fact, as explained in Sect. 7 of [14], $Z \subset \mathbb{P}(W \oplus Q)$ is isomorphic to the VMRT of a 15-dimensional F_4-homogeneous space. The variety Z is biregular to the horospherical Fano manifold of Picard number 1, the case 4 in Theorem 1.7 of [19]. The next proposition follows from Theorem 1.11 of [19].

Proposition 3.9 $\mathfrak{aut}(\hat{Z}) = \mathbb{C} \oplus W \rtimes (\mathfrak{so}(Q) \oplus \mathbb{C})$.

Here the central $\mathbb{C}$ corresponds to the scalar multiplication on $W \oplus Q$, while the second $\mathbb{C}$ acts with weight 1 on W and 0 on Q. The action of W on $W \oplus Q$ is annihilating W and given by $W \subset \mathrm{Hom}(Q, W)$ induced from the natural inclusion of W as an irreducible $\mathfrak{so}(7)$-factor of $\mathrm{Hom}(Q, W)$. The inclusion $\mathfrak{aut}(\hat{Z}) \subset \mathrm{End}(W \oplus Q)$ can be represented as follows:

$$\begin{pmatrix} \mathbb{C} \subset \mathrm{End}(W) & 0 \\ W \subset \mathrm{Hom}(Q, W) & \mathfrak{co}(Q) \subset \mathrm{End}(Q) \end{pmatrix}.$$

The following is from Proposition 7.2.3 of [14]. We give a more direct proof.

Proposition 3.10 $\mathfrak{aut}(\hat{Z})^{(1)} = Q^*$.

Proof By Theorem 2.6 (ii) and (iii), every $A \in \mathfrak{aut}(\hat{Z})^{(1)}$ is determined by an element $\lambda \in W^* \oplus Q^*$ such that

$$2A_{x,y} = \lambda(x)y + \lambda(y)x$$

for $x \in \hat{Z}$ and $y \in T_x(\hat{Z})$. As $A_x \in \mathfrak{aut}(\hat{Z})$, we can write

$$A_x = \begin{pmatrix} \mu_x & 0 \\ \phi_x & g_x \end{pmatrix}, \quad \mu_x \in \mathbb{C},\ \phi_x \in W \subset \mathrm{Hom}(Q, W),\ g_x \in \mathfrak{co}(Q).$$

If we write $x = (x_1, x_2)$ and $y = (y_1, y_2)$ with $x_1, y_1 \in W$ and $x_2, y_2 \in Q$, then we have

$$\begin{aligned}&2(\mu_x y_1 + \phi_x(y_2), g_x(y_2))\\ &\quad = 2A_x(y) = 2A_{x,y} = (\lambda(x)y_1, \lambda(x)y_2) + (\lambda(y)x_1, \lambda(y)x_2).\end{aligned}$$

As this holds for all $y \in T_x(\hat{Z})$, we have

$$\begin{aligned}\mu_x = \lambda(x)/2, \qquad \phi_x(y_2) = (\lambda(y)/2)x_1,\\ g_x(y_2) = (\lambda(x)/2)y_2 + (\lambda(y)/2)x_2.\end{aligned} \tag{3.1}$$

If we take $y_2 = 0$ in the previous equations, then $\lambda((y_1, 0)) = 0$ for all $y_1 \in W$, which implies $\lambda \in Q^*$. Conversely, for any $\lambda \in Q^*$, we can use formulae in (3.1) to construct A_x and one checks that $A \in \mathfrak{aut}(\hat{Z})^{(1)}$. □

3.4 Hyperplane section of Gr(2, 5)

Let Q be a 5-dimensional orthogonal vector space and let W be the 4-dimensional spin representation of $\mathfrak{so}(Q) = \mathfrak{so}(5)$. There exists a Spin(5)-stable 5-dimensional Fano manifold Z of Picard number 1 with an embedding $Z \subset \mathbb{P}(W \oplus Q)$. In fact, Z is a general hyperplane section of Gr(2, 5), which is isomorphic to a symplectic Grassmannian $\mathrm{Gr}_\omega(2, 5)$ by Lemma 3.7. This can be seen as follows. As $\mathfrak{sp}(4) \cong \mathfrak{so}(5)$, we can regard W as a 4-dimensional symplectic vector space. We have a decomposition of $\mathfrak{sp}(4)$-modules

$$\bigwedge^2 W \cong \mathbb{C} \oplus Q.$$

Equip $W \oplus \mathbb{C}$ with the skew symmetric form ω obtained from W with $\mathbb{C}$ as its null-space. A natural embedding $Z \subset \mathbb{P}(W \oplus Q)$ of the symplectic Grassmannian Z can be obtained by viewing Z as the hyperplane section of

$$\mathrm{Gr}(2, W \oplus \mathbb{C}) \subset \mathbb{P}\bigwedge^2 (W \oplus \mathbb{C}) \cong \mathbb{P}\left(\bigwedge^2 W \oplus W\right)$$

where the hyperplane is given by the kernel of ω in $\bigwedge^2 W = \mathbb{C} \oplus Q$. From the table in Sect. 3.1, we see that Z is the VMRT at a general point of a hyperplane section of $\mathbb{S}_5$. As Proposition 3.9, the next proposition follows from Theorem 1.11 Case 5 of [19].

Proposition 3.11 $\mathfrak{aut}(\hat{Z}) = \mathbb{C} \oplus W \rtimes (\mathfrak{so}(Q) \oplus \mathbb{C})$.

The next proposition can be proved in the same way as Proposition 3.10.

Proposition 3.12 $\mathfrak{aut}(\hat{Z})^{(1)} = Q^*$.

4 Prolongation and projection

Notation 4.1 Given a linear space $L \subset V$, denote by $p_L : \mathbb{P}V \dashrightarrow \mathbb{P}(V/L)$ the projection. When $L = \hat{x}$ for a point $x \in \mathbb{P}V$, we write $p_{\hat{x}}$ as p_x.

Let us recall the following two elementary facts.

Lemma 4.2 *Given an irreducible variety $S \subset \mathbb{P}V$ and a linear subspace $L \subset V$ with $S \not\subset \mathbb{P}L$, the proper image $p_L(S) \subset \mathbb{P}(V/L)$ is well-defined. When S is nonsingular, the restriction $p_L|_S$ is a morphism sending S biregularly to $p_L(S)$ if and only if $\mathbb{P}L \cap \mathrm{Sec}(S) = \emptyset$.*

Lemma 4.3 *Let $S \subset \mathbb{P}V$ be an irreducible closed subvariety and $L \subset V$ a linear subspace with $S \not\subset \mathbb{P}L$. Then $\mathrm{Sec}(p_L(S)) = p_L(\mathrm{Sec}(S))$.*

In this section, we study the prolongation of $p_L(S) \subset \mathbb{P}(V/L)$ for the examples $S \subset \mathbb{P}V$ listed in the previous section for suitable linear spaces L. It is convenient to introduce the following.

Definition 4.4 Let $S \subset \mathbb{P}V$ be an irreducible projective variety and let $L \subset V$ be a linear subspace. We define two Lie subalgebras of $\mathfrak{aut}(\hat{S})$ as follows.

$$\mathfrak{aut}(\hat{S}, L) := \{g \in \mathfrak{aut}(\hat{S}) | g(L) \subset L\} \subset \mathfrak{gl}(V),$$
$$\mathfrak{aut}(\hat{S}, L, 0) := \{g \in \mathfrak{aut}(\hat{S}) | g(L) = 0\} \subset \mathfrak{gl}(V).$$

Proposition 4.5 *Let $S \subset \mathbb{P}V$ be a non-degenerate irreducible subvariety. Let $L \subsetneq V$ be a linear subspace. Assume that*

(i) *the natural Lie algebra homomorphism $\mathfrak{aut}(\hat{S}, L) \to \mathfrak{aut}(\widehat{p_L(S)})$ is an isomorphism; and*
(ii) *for a general $\alpha \in \hat{S}$, $T_\alpha(\hat{S}) \cap L = 0$.*

Then we have an isomorphism of vector spaces

$$\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \cong \mathfrak{aut}(\hat{S}, L, 0)^{(1)} = \{A \in \mathfrak{aut}(\hat{S})^{(1)} | A_\alpha(L) = 0, \forall \alpha \in \hat{S}\}.$$

Proof For any element $A \in \mathfrak{aut}(\widehat{p_L(S)})^{(1)} \subset \mathrm{Hom}(V/L, \mathfrak{aut}(\widehat{p_L(S)}))$, we define an element $\tilde{A} \in \mathrm{Hom}(V, \mathfrak{aut}(\hat{S}, L))$ by composing A with the natural projection $V \to V/L$ and the isomorphism $\mathfrak{aut}(\widehat{p_L(S)}) \cong \mathfrak{aut}(\hat{S}, L)$ given by the condition (i).

For a general element $\alpha \in \hat{S}$, denote by $\bar{\alpha} \in \widehat{p_L(S)}$ its image in V/L. By the condition (ii), we have a natural identification $T_\alpha(\hat{S}) \cong T_{\bar{\alpha}}(\widehat{p_L(S)})$ for a general $\alpha \in \hat{S}$. Let $\beta \in \hat{S}$ be a general point. As $A_{\bar{\alpha}}(\bar{\beta}) = A_{\bar{\beta}}(\bar{\alpha}) \in$

$T_{\bar{\alpha}}(\widehat{p_L(S)}) \cap T_{\bar{\beta}}(\widehat{p_L(S)})$, we have $\tilde{A}_\alpha(\beta) \in (T_\alpha(\hat{S}) \cap T_\beta(\hat{S})) \oplus L$. On the other hand, as $\tilde{A}_\alpha \in \mathfrak{aut}(\hat{S})$, we have $\tilde{A}_\alpha(\beta) \in T_\beta(\hat{S})$ by Definition 2.2 (v). As $T_\beta(\hat{S}) \cap L = 0$, this implies that $\tilde{A}_\alpha(\beta) \in T_\alpha(\hat{S}) \cap T_\beta(\hat{S})$. In particular, we have $\mathrm{Im}(\tilde{A}_\alpha) \subset T_\alpha(\hat{S})$ and $\tilde{A}_\alpha(\beta) = \tilde{A}_\beta(\alpha)$ for all general $\alpha, \beta \in \hat{S}$ because $A_{\bar{\alpha}}(\bar{\beta}) = A_{\bar{\beta}}(\bar{\alpha})$. As S is non-degenerate, the equality $\tilde{A}_\alpha(\beta) = \tilde{A}_\beta(\alpha)$ holds for all $\alpha, \beta \in V$. Thus $\tilde{A} \in \mathrm{Hom}(\mathrm{Sym}^2 V, V)$.

As $\tilde{A}_\alpha \in \mathfrak{aut}(\hat{S}, L)$, we have $\tilde{A}_\alpha(L) \subset L$. This implies that $\tilde{A}_\alpha(L) \subset L \cap T_\alpha(\hat{S}) = 0$ for general $\alpha \in \hat{S}$ by the condition (ii). Consequently, $\tilde{A}_\alpha(L) = 0$ for a general $\alpha \in \hat{S}$, hence for any $\alpha \in V$ by the non-degeneracy of S. It follows that

$$\tilde{A} \in \mathrm{Hom}(V, \mathfrak{aut}(\hat{S}, L, 0)) \cap \mathrm{Hom}(\mathrm{Sym}^2 V, V) = \mathfrak{aut}(\hat{S}, L, 0)^{(1)}.$$

Conversely, for any $\tilde{A} \in \mathfrak{aut}(\hat{S}, L, 0)^{(1)}$, it is easy to see that it induces an element $A \in \mathfrak{aut}(\widehat{p_L(S)})^{(1)}$, proving the proposition. □

Proposition 4.6 *Let $S \subset \mathbb{P}V$ be a linearly normal nonsingular non-degenerate projective variety. If $L \subset V$ is a subspace with $\mathbb{P}L \cap \mathrm{Sec}(S) = \emptyset$, then it satisfies the two conditions in Proposition 4.5. In particular,*

$$\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \cong \mathfrak{aut}(\hat{S}, L, 0)^{(1)} = \{A \in \mathfrak{aut}(\hat{S})^{(1)} | A_\alpha(L) = 0, \forall \alpha \in \hat{S}\}.$$

Proof Let us identify $S \subset \mathbb{P}V$ with $S \subset \mathbb{P}H^0(S, \mathcal{O}(1))^*$ for the hyperplane line bundle $\mathcal{O}(1)$ on S. The condition (ii) is immediate from $\mathbb{P}T_\alpha(\hat{S}) \subset \mathrm{Sec}(S)$ for any $\alpha \in \hat{S}$. The condition (i) will follow from the next lemma. □

Lemma 4.7 *Let $S \subset \mathbb{P}V$ be a linearly normal nonsingular non-degenerate projective variety. Let $\mathbb{P}L_1, \mathbb{P}L_2 \subset \mathbb{P}V \setminus \mathrm{Sec}(S)$ be two linear subspaces and $p_{L_i} : S \to p_{L_i}(S) \subset \mathbb{P}(V/L_i)$ the projection from $\mathbb{P}L_i$. Suppose that there exists an isomorphism $\sigma : \mathbb{P}(V/L_1) \to \mathbb{P}(V/L_2)$ with $\sigma(p_{L_1}(S)) = p_{L_2}(S)$. Then there exists a unique isomorphism $\tilde{\sigma} : \mathbb{P}V \to \mathbb{P}V$ with $\tilde{\sigma}(S) = S$ and $\tilde{\sigma}(\mathbb{P}L_1) = \mathbb{P}L_2$.*

Proof The restriction $\sigma|_{p_{L_1}(S)} : p_{L_1}(S) \to p_{L_2}(S)$ is an automorphism $\bar{\sigma}$ of S with $\bar{\sigma}^*\mathcal{O}(1) \cong \mathcal{O}(1)$ such that sections of $\mathcal{O}(1)$ annihilated by $\mathbb{P}L_2 \subset \mathbb{P}H^0(S, \mathcal{O}(1))^*$ correspond to sections of $\mathcal{O}(1)$ annihilated by $\mathbb{P}L_1 \subset \mathbb{P}H^0(S, \mathcal{O}(1))^*$. Thus it induces a homomorphism $\tilde{\sigma} : \mathbb{P}H^0(S, \mathcal{O}(1))^* \to \mathbb{P}H^0(S, \mathcal{O}(1))^*$ with $\tilde{\sigma}(\mathbb{P}L_1) = \mathbb{P}L_2$. □

As an immediate consequence of Proposition 4.6, we have the following Corollary.

Corollary 4.8 *Let $S \subset \mathbb{P}V$ be a nonsingular non-degenerate subvariety such that $\mathfrak{aut}(\hat{S})^{(1)} \neq 0$ and $\mathrm{Pic}(S) = \mathbb{Z}\mathcal{O}_S(1)$. Then for the linearly normal embedding $S \subset \mathbb{P}H^0(S, \mathcal{O}_S(1))^*$, we still have $\mathfrak{aut}(\hat{S})^{(1)} \neq 0$.*

By Propositions 4.5 and 4.6, studying the prolongation of $p_L(S)$ under a biregular projection of a linearly normal S is reduced to the study of $\mathfrak{aut}(\hat{S}, L, 0)^{(1)}$. Let us carry this out for the examples listed in Sect. 3.

For the VMRT of an irreducible Hermitian symmetric space of compact type, we have the following uniform description.

Proposition 4.9 *Let $S \subsetneq \mathbb{P}V$ be the VMRT of an irreducible Hermitian symmetric space M of compact type. Recall that from Sect. 3.1, we have a graded Lie algebra structure of $\mathfrak{g} := \mathfrak{aut}(M)$, $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{g}_{-1} \cong T_o(M) = V$ and $\mathfrak{aut}(\hat{S})^{(1)} \cong \mathfrak{g}_1$. For a subspace $L \subset \mathfrak{g}_{-1}$, we have*

$$\mathfrak{aut}(\hat{S}, L, 0)^{(1)} = \{X \in \mathfrak{g}_1 | [X, Z] = 0, \forall Z \in L\}.$$

Proof From the definition

$$\mathfrak{aut}(\hat{S}, L, 0)^{(1)} = \{A \in \mathfrak{aut}(\hat{S})^{(1)} | A_\alpha(L) = 0, \forall \alpha \in \hat{S}\}$$

and the isomorphism $\mathfrak{aut}(\hat{S})^{(1)} \cong \mathfrak{g}_1$ given in Sect. 3.1, we get

$$\begin{aligned}\mathfrak{aut}(\hat{S}, L, 0)^{(1)} &= \{X \in \mathfrak{g}_1 | [[X, Y], Z] = 0, \forall Y \in \mathfrak{g}_{-1}, \forall Z \in L\} \\ &= \{X \in \mathfrak{g}_1 | [[X, Z], Y] = 0, \forall Y \in \mathfrak{g}_{-1}, \forall Z \in L\} \\ &= \{X \in \mathfrak{g}_1 | [X, Z] = 0, \forall Z \in L\}.\end{aligned}$$

The last equality follows from the fact that if an element $u \in \mathfrak{g}_1 = \mathfrak{g}_0^{(1)}$ satisfies $[u, Y] = 0$ for all $Y \in \mathfrak{g}_{-1}$, then $u = 0$ (cf. [21], Lemma 3.2). □

The following four propositions give more explicit description of Proposition 4.9 when $\mathrm{Sec}(S) \neq \mathbb{P}V$. Here we will use the data in the table of Sect. 3.1 freely. Our main interest is when $\mathbb{P}L \cap \mathrm{Sec}(S) = \emptyset$. But for the classical types, we will treat also general L, because it will be needed later and requires little extra work.

Proposition 4.10 *Let A and B be vector spaces with $a := \dim A \geq b := \dim B \geq 3$. Let $V = \mathrm{Hom}(A, B)$ and let $\hat{S} \subset V$ be the set of elements of rank ≤ 1. For a subspace $L \subset \mathrm{Hom}(A, B)$, we define $\mathrm{Im}(L) \subset B$ as the linear span of $\{\mathrm{Im}(\phi) \subset B, \phi \in L\}$ and $\mathrm{Ker}(L) := \bigcap_{\phi \in L} \mathrm{Ker}(\phi)$. Then*

(i) *there is a canonical isomorphism of vector spaces*

$$\mathfrak{aut}(\hat{S}, L, 0)^{(1)} \cong \mathrm{Hom}(B/\mathrm{Im}(L), \mathrm{Ker}(L));$$

(ii) *for any* $\psi \in \mathrm{Hom}(B, A)$ *with* $\mathrm{Im}(L) \subset \mathrm{Ker}(\psi)$ *and* $\mathrm{Im}(\psi) \subset \mathrm{Ker}(L)$, L *is contained in*

$$L(\psi) := \{\phi \in \mathrm{Hom}(A, B) | \phi \circ \psi = 0, \psi \circ \phi = 0\}$$
$$\cong \mathrm{Hom}(A/\mathrm{Im}(\psi), \mathrm{Ker}(\psi));$$

(iii) *if* $\mathbb{P}L \cap \mathrm{Sec}(S) = \emptyset$ *and* $\mathfrak{aut}(\hat{S}, L, 0)^{(1)}$ *contains an element of* rank r *in* $\mathrm{Hom}(B/\mathrm{Im}(L), \mathrm{Ker}(L))$, *then*

$$\dim L \leq ab - 2(a+b) + 4 - r(a+b-r-4).$$

Proof From [21], p. 457, the grading on $\mathfrak{g}$ in Proposition 4.9 for $M = \mathrm{Gr}(a, a+b)$ can be identified with

$$\mathfrak{g}_{-1} = \mathrm{Hom}(A, B), \qquad \mathfrak{g}_0 = \mathrm{End}(A) \oplus \mathrm{End}(B), \qquad \mathfrak{g}_1 = \mathrm{Hom}(B, A).$$

The bracket $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \subset \mathfrak{g}_0$ is given by $[\phi, \psi] = \phi \circ \psi - \psi \circ \phi \in \mathfrak{g}_0$ for $\phi \in \mathfrak{g}_{-1}$ and $\psi \in \mathfrak{g}_1$. By Proposition 4.9, this gives

$$\begin{aligned}\mathfrak{aut}(\hat{S}, L, 0)^{(1)} &= \{\psi \in \mathrm{Hom}(B, A) | \psi \circ \phi = 0, \phi \circ \psi = 0, \forall \phi \in L\}\\ &= \{\psi \in \mathrm{Hom}(B, A) | \mathrm{Im}(\psi) \subset \mathrm{Ker}(\phi),\\ &\qquad \mathrm{Im}(\phi) \subset \mathrm{Ker}(\psi), \forall \phi \in L\}\\ &= \{\psi \in \mathrm{Hom}(B, A) | \mathrm{Im}(\psi) \subset \mathrm{Ker}(L), \mathrm{Im}(L) \subset \mathrm{Ker}(\psi)\}\\ &\cong \mathrm{Hom}(B/\mathrm{Im}(L), \mathrm{Ker}(L)),\end{aligned}$$

proving (i). For (ii), it is clear from above that L is contained in $L(\psi)$.

Now assume that $\mathbb{P}L \cap \mathrm{Sec}(S) = \emptyset$ and there is ψ in (ii) of rank r. Note that $\mathrm{Sec}(\hat{S}) \subset V = \mathrm{Hom}(A, B)$ consists of elements of rank ≤ 2 (e.g. [10], p. 188, Type I). Let $S_\psi \subset \mathbb{P}L(\psi) \cong \mathbb{P}(\mathrm{Hom}(A/\mathrm{Im}(\psi), \mathrm{Ker}(\psi)))$ be the set of elements of rank ≤ 1, then $\mathrm{Sec}(S_\psi)$ consists of elements of rank ≤ 2, which has dimension $2(a+b-2r)-5$. By the assumption, $\mathbb{P}L \subset \mathbb{P}L(\psi)$ is disjoint from $\mathrm{Sec}(S_\psi)$, which implies that $a-r \geq b-r \geq 3$ and

$$\begin{aligned}\dim L &\leq (a-r)(b-r) - (\dim \mathrm{Sec}(S_\psi)) - 1\\ &= ab - 2(a+b) + 4 - r(a+b-r-4).\end{aligned}$$

□

Proposition 4.11 *Let* W *be a vector space of dimension* $n \geq 6$. *For each* $\phi \in \bigwedge^2 W$, *denote by* $\phi^\sharp \in \mathrm{Hom}(W^*, W)$ *the corresponding element via the natural inclusion* $\bigwedge^2 W \subset W \otimes W = \mathrm{Hom}(W^*, W)$. *Let* $V = \bigwedge^2 W$ *and let* $\hat{S} \subset V$ *be the set of elements* ϕ *with* $\phi^\sharp$ *of* rank ≤ 2. *For a subspace* $L \subset V$, *define* $\mathrm{Im}(L) \subset W$ *as the linear span of* $\{\mathrm{Im}(\phi^\sharp) \subset W, \phi \in L\}$. *Then*

(i) *there is a canonical vector space isomorphism*

$$\mathfrak{aut}(\hat{S}, L, 0)^{(1)} \cong \bigwedge^2 (W/\mathrm{Im}(L))^*;$$

(ii) *for each* $\psi \in \bigwedge^2 (W/\mathrm{Im}(L))^* \subset \bigwedge^2 W^*$, *denoting by* $\psi^\sharp$ *the corresponding element in* $\mathrm{Hom}(W, W^*)$, L *is contained in*

$$L(\psi) := \left\{\phi \in \bigwedge^2 W \,|\, \mathrm{Im}(\phi^\sharp) \subset \mathrm{Ker}(\psi^\sharp)\right\} \cong \bigwedge^2 \mathrm{Ker}(\psi^\sharp);$$

(iii) *if* $\mathbb{P}L \cap \mathrm{Sec}(S) = \emptyset$ *and* $\mathfrak{aut}(\hat{S}, L, 0)^{(1)}$ *contains an element of* rank r *in* $\bigwedge^2 (W/\mathrm{Im}(L))^*$ (*i.e. the corresponding element in* $\mathrm{Hom}(W, W^*)$ *has* rank r), *then*

$$\dim L \le \frac{n(n-1)}{2} - 4n + 10 - \frac{r(2n - r - 9)}{2}.$$

Proof From [21], pp. 459–461, the grading on $\mathfrak{g}$ in Proposition 4.9 for $M = \mathbb{S}_n$ can be identified with

$$\mathfrak{g}_{-1} = \bigwedge^2 W, \qquad \mathfrak{g}_0 = \mathrm{End}(W), \qquad \mathfrak{g}_1 = \bigwedge^2 W^*.$$

For each $\psi \in \bigwedge^2 W^*$, denote by $\psi^\sharp \in \mathrm{Hom}(W, W^*)$, the corresponding element via the natural inclusion $\bigwedge^2 W^* \subset W^* \otimes W^* = \mathrm{Hom}(W, W^*)$.

For any $\phi \in \bigwedge^2 W$, $\psi \in \bigwedge^2 W^*$, the endomorphism $[\phi, \psi] \in \mathrm{End}(W)$ is given by

$$[\phi, \psi] = \phi^\sharp \circ \psi^\sharp.$$

Note that we have the following equivalences:

$$\begin{aligned}
[\phi, \psi] = 0 \quad &\Leftrightarrow \quad \mathrm{Im}(\psi^\sharp) \subset \mathrm{Ker}(\phi^\sharp) \quad \Leftrightarrow \quad \psi \in \bigwedge^2 \mathrm{Ker}(\phi^\sharp) \\
&\Leftrightarrow \quad \phi \in \bigwedge^2 \mathrm{Ker}(\psi^\sharp) \quad \Leftrightarrow \quad \mathrm{Im}(\phi^\sharp) \subset \mathrm{Ker}(\psi^\sharp).
\end{aligned}$$

By Proposition 4.9, this gives

$$\begin{aligned}
\mathfrak{aut}(\hat{S}, L, 0)^{(1)} &= \left\{\psi \in \bigwedge^2 W^* \,|\, \mathrm{Im}(\phi^\sharp) \subset \mathrm{Ker}(\psi^\sharp), \forall \phi \in L\right\} \\
&= \left\{\psi \in \bigwedge^2 W^* \,|\, \mathrm{Im}(L) \subset \mathrm{Ker}(\psi^\sharp)\right\} \\
&\cong \bigwedge^2 (W/\mathrm{Im}(L))^*,
\end{aligned}$$

proving (i). For (ii), it is clear from above that L is contained in $L(\psi)$.

Now assume that $\mathbb{P}L \cap \mathrm{Sec}(S) = \emptyset$ and there is ψ in (ii) of rank r. Note that $\mathrm{Sec}(\hat{S}) \subset V = \bigwedge^2 W$ consists of elements with rank ≤ 4 (e.g. [10], p. 188, Type II). Let $S_\psi \subset \mathbb{P}L(\psi)$ be the variety consisting of elements of rank ≤ 2, then we have $\dim \mathrm{Sec}(S_\psi) = 4n - 4r - 11$. By the hypothesis, $\mathbb{P}L \subset \mathbb{P}L(\psi)$ is disjoint from $\mathrm{Sec}(S_\psi)$, which implies that $n - r \geq 6$ and

$$\dim L \leq \frac{n(n-1)}{2} - 4n + 10 - \frac{r(2n-r-9)}{2}. \qquad \square$$

Proposition 4.12 *Let W be a vector space of dimension $n \geq 3$. For each $\phi \in \mathrm{Sym}^2 W$, denote by $\phi^\sharp \in \mathrm{Hom}(W^*, W)$ the corresponding element via the natural inclusion $\mathrm{Sym}^2 W \subset W \otimes W = \mathrm{Hom}(W^*, W)$. Let $V = \mathrm{Sym}^2 W$ and let $\hat{S} \subset V$ be the set of elements ϕ with $\phi^\sharp$ of rank ≤ 1. For a subspace $L \subset V$, define $\mathrm{Im}(L) \subset W$ as the linear span of $\{\mathrm{Im}(\phi^\sharp) \subset W, \phi \in L\}$. Then*

(i) *there is a canonical isomorphism of vector spaces*

$$\mathfrak{aut}(\hat{S}, L, 0)^{(1)} \cong \mathrm{Sym}^2(W/\mathrm{Im}(L))^*;$$

(ii) *for each $\psi \in \mathrm{Sym}^2(W/\mathrm{Im}(L))^* \subset \mathrm{Sym}^2 W^*$, denoting by $\psi^\sharp$ the corresponding element in $\mathrm{Hom}(W, W^*)$, L is contained in*

$$L(\psi) := \{\phi \in \mathrm{Sym}^2 W \,|\, \mathrm{Im}(\phi^\sharp) \subset \mathrm{Ker}(\psi^\sharp)\} \cong \mathrm{Sym}^2 \mathrm{Ker}(\psi^\sharp);$$

(iii) *if $\mathbb{P}L \cap \mathrm{Sec}(S) = \emptyset$ and $\mathfrak{aut}(\hat{S}, L, 0)^{(1)}$ contains an element of rank r in $\mathrm{Sym}^2(W/\mathrm{Im}(L))^*$ (i.e. the corresponding element in $\mathrm{Hom}(W, W^*)$ has rank r), then*

$$\dim L \leq \frac{n(n+1)}{2} - 2n + 1 - \frac{r(2n-r-3)}{2}.$$

Proof From [21], pp. 458–459, the grading on $\mathfrak{g}$ in Proposition 4.9 for $M = \mathrm{Lag}(2n)$ can be identified with

$$\mathfrak{g}_{-1} = \mathrm{Sym}^2 W, \qquad \mathfrak{g}_0 = \mathrm{End}(W), \qquad \mathfrak{g}_1 = \mathrm{Sym}^2 W^*.$$

For each $\psi \in \mathrm{Sym}^2 W^*$, denote by $\psi^\sharp \in \mathrm{Hom}(W, W^*)$, the corresponding element via the natural inclusion $\mathrm{Sym}^2 W^* \subset W^* \otimes W^* = \mathrm{Hom}(W, W^*)$. For any $\phi \in \mathrm{Sym}^2 W$, $\psi \in \mathrm{Sym}^2 W^*$, the endomorphism $[\phi, \psi] \in \mathrm{End}(W)$ is given by

$$[\phi, \psi] = \phi^\sharp \circ \psi^\sharp.$$

As in the proof of Proposition 4.11, $[\phi, \psi] = 0$ is equivalent to $\mathrm{Im}(\phi^\sharp) \subset \mathrm{Ker}(\psi^\sharp)$, which gives, by Proposition 4.9,

$$\begin{aligned}\mathfrak{aut}(\hat{S}, L, 0)^{(1)} &= \{\psi \in \mathrm{Sym}^2 W^* \mid \mathrm{Im}(\phi^\sharp) \subset \mathrm{Ker}(\psi^\sharp), \forall \phi \in L\} \\ &= \{\psi \in \mathrm{Sym}^2 W^* \mid \mathrm{Im}(L) \subset \mathrm{Ker}(\psi^\sharp)\} \\ &\cong \mathrm{Sym}^2(W/\mathrm{Im}(L))^*,\end{aligned}$$

proving (i). For (ii), it is clear from above that L is contained in

$$L(\psi) = \{\phi \in \mathrm{Sym}^2 W \mid [\phi, \psi] = 0\} = \{\phi \in \mathrm{Sym}^2 W \mid \mathrm{Im}(\phi^\sharp) \subset \mathrm{Ker}(\psi^\sharp)\}.$$

Now assume that $\mathbb{P}L \cap \mathrm{Sec}(S) = \emptyset$ and there is ψ in (ii) of rank r. Note that $\mathrm{Sec}(\hat{S}) \subset V = \mathrm{Sym}^2 W$ consists of elements with rank ≤ 2 (e.g. [10], p. 188, Type III). Let $S_\psi \subset \mathbb{P}L(\psi)$ be the variety consisting of elements of rank ≤ 1. Then we have $\dim \mathrm{Sec}(S_\psi) = 2(n - r - 1)$. By the hypothesis, $\mathbb{P}L \subset \mathbb{P}L(\psi)$ is disjoint from $\mathrm{Sec}(S_\psi)$, which implies that $n - r \geq 3$ and

$$\dim L \leq \frac{n(n+1)}{2} - 2n + 1 - \frac{r(2n - r - 3)}{2}. \qquad \square$$

Proposition 4.13 *Let $S \subset \mathbb{P}V$ be the minimal (Severi) embedding of the Cayley plane $\mathbb{OP}^2$ with $\dim V = 27$. By [22] (pp. 59–60), $\mathrm{Sec}(S) \subset \mathbb{P}V$ is a cubic hypersurface. For any 1-dimensional subspace $L \subset V$ with $\mathbb{P}L \not\in \mathrm{Sec}(S)$, $\mathfrak{aut}(\hat{S}, L, 0)^{(1)} = 0$.*

Proof It is known that $\mathfrak{aut}(\hat{S}, L)$ is a simple Lie algebra of type F_4 and the natural representation on V/L is the minimal irreducible representation of dimension 26 ([22], pp. 59–60). Let $S' \subset \mathbb{P}(V/L)$ be the highest weight variety of this F_4-representation, which is not biregular to the VMRT of an irreducible Hermitian symmetric space. From Theorem 3.5, we have $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} = \mathfrak{aut}(\hat{S}')^{(1)} = 0$. Thus by Proposition 4.6, $\mathfrak{aut}(\hat{S}, L, 0)^{(1)} = \mathfrak{aut}(\widehat{p_L(S)})^{(1)} = 0$. $\square$

At this point, we can give the postponed proof of Proposition 3.8. We start with examining a special case of Proposition 4.12.

Proposition 4.14 *Let W and Q be vector spaces of dimensions $k \geq 2$ and m respectively. Set $L := \mathrm{Sym}^2 Q \subset V := \mathrm{Sym}^2(W \oplus Q)$. Let*

$$S := v_2(\mathbb{P}(W \oplus Q)) \subset \mathbb{P}(\mathrm{Sym}^2(W \oplus Q))$$

be the second Veronese embedding of $\mathbb{P}(W \oplus Q)$. Then for a general point $\alpha \in \hat{S}$, the tangent space $T_\alpha(\hat{S})$ satisfies $T_\alpha(\hat{S}) \cap L = 0$. In particular, $\mathbb{P}L \not\subset \mathrm{Sec}(S)$.

Proof It suffices to exhibit a point $\alpha \in \hat{S}$ with $T_\alpha(\hat{S}) \cap L = 0$. Fix a non-zero $w \in W$ and let

$$\alpha := w^2 \in \mathrm{Sym}^2 W \subset \mathrm{Sym}^2(W \oplus Q).$$

Fix any $w' \in W, q \in Q$. The arc

$$\{w + t(w' + q) \in W \oplus Q | t \in \mathbb{C}\}$$

in $\hat{S}$ has its tangent vector

$$\left.\frac{d}{dt}\right|_{t=0} (w + t(w' + q))^2 = 2w(w' + q) \in \mathrm{Sym}^2 W \oplus (W \otimes Q).$$

Since such tangent vectors span $T_\alpha(\hat{S})$, $T_\alpha(\hat{S})$ intersects L at 0. □

Proposition 4.15 *In the setting of Proposition 4.14, let $Z = p_L(S)$ be the proper image of S under p_L. The natural Lie algebra homomorphism $\mathfrak{aut}(\hat{S}, L) \to \mathfrak{aut}(\hat{Z})$ is an isomorphism, inducing a Lie algebra isomorphism*

$$\mathfrak{aut}(\hat{Z}) \cong (W^* \otimes Q) \rtimes (\mathfrak{gl}(W) \oplus \mathfrak{gl}(Q)).$$

Proof It is clear that

$$\mathfrak{aut}(\hat{S}, L) \cong (W^* \otimes Q) \rtimes (\mathfrak{gl}(W) \oplus \mathfrak{gl}(Q)).$$

The Lie algebra homomorphism $\mathfrak{aut}(\hat{S}, L) \to \mathfrak{aut}(\widehat{p_L(S)})$ is clearly injective. Thus it suffices to show that $\dim \mathfrak{aut}(\hat{Z}) \le m^2 + km + k^2$, or equivalently, $\dim \mathfrak{aut}(Z) \le m^2 + km + k^2 - 1$. From Sect. 3.2, we have a natural projection $\psi : Z \to \mathbb{P}W$ realizing Z as the projectivization of the vector bundle $\mathcal{O}(-1)^m \oplus \mathcal{O}(-2)$ on $\mathbb{P}W \cong \mathbb{P}^{k-1}$. From the exact sequence

$$0 \to T^\psi \to T(Z) \to \psi^* T(\mathbb{P}W) \to 0,$$

where T^ψ denotes the relative tangent bundle and

$$\dim H^0(Z, T^\psi) = \dim H^0(\mathbb{P}^{k-1}, \mathrm{End}^0(\mathcal{O}(-1)^m \oplus \mathcal{O}(-2)) = m^2 + km,$$

where End^0 denotes the traceless endomorphisms, we have

$$\dim \mathfrak{aut}(Z) = \dim H^0(Z, T(Z)) \le m^2 + km + k^2 - 1. \qquad \square$$

Proof of Proposition 3.8 $\mathfrak{aut}(\hat{Z})$ is given by Proposition 4.15. From Propositions 4.14 and 4.15, we can apply Proposition 4.5 to S and L. Thus by Proposition 4.12,

$$\mathfrak{aut}(\hat{Z})^{(1)} = \mathfrak{aut}(\widehat{p_L(S)})^{(1)} \cong \mathfrak{aut}(\hat{S}, L, 0)^{(1)} \cong \mathrm{Sym}^2 W^*$$

because $\mathrm{Im}(L) = Q \subset (W \oplus Q)$. □

Now we turn to study the prolongation of the biregular projection of $Z \subset \mathbb{P}U$ with $U = (W \otimes Q) \oplus \mathrm{Sym}^2 W$ in Sect. 3.2. This can be reduced to Proposition 4.12 by the following.

Proposition 4.16 *Let $S_1 \subset \mathbb{P}V_1$ be a non-degenerate subvariety. Let $L_1 \subsetneq V_1$ be a linear subspace. Let $V_2 := V_1/L_1$ and let $S_2 := p_{L_1}(S_1) \subset \mathbb{P}V_2$ be the proper image of S_1. Let $L_2 \subsetneq V_2$ be a linear subspace and let $L_3 \subset V_1$ be the subspace containing L_1 with $L_3/L_1 = L_2$. Suppose that (S_1, V_1, L_1) (resp. (S_2, V_2, L_2)) satisfies the two conditions in Proposition 4.5 with $S = S_1, V = V_1, L = L_1$ (resp. with $S = S_2$, $V = V_2$, $L = L_2$). Then (S_1, V_1, L_3) satisfies the two conditions in Proposition 4.5 with $S = S_1$, $V = V_1$, $L = L_3$. In particular, when the two conditions are satisfied by both (S_1, V_1, L_1) and (S_2, V_2, L_2), we have*

$$\mathfrak{aut}(\widehat{p_{L_2}(S_2)})^{(1)} \cong \mathfrak{aut}(\hat{S}_1, L_3, 0)^{(1)}.$$

Proof For the condition (i) in Proposition 4.5, it suffices to show that the homomorphism

$$\mathfrak{aut}(\hat{S}_1, L_3) \to \mathfrak{aut}(\widehat{p_{L_2}(S_2)})$$

is surjective. But under the isomorphism $\mathfrak{aut}(\hat{S}_2) \cong \mathfrak{aut}(\hat{S}_1, L_1)$, the subalgebra $\mathfrak{aut}(\hat{S}_2, L_2)$ is sent into

$$\mathfrak{aut}(\hat{S}_1, L_1 \subset L_3) := \{\sigma \in \mathfrak{aut}(\hat{S}_1, L_1), \sigma(L_3) \subset L_3\} \subset \mathfrak{aut}(\hat{S}_1, L_3)$$

from which the surjectivity is clear. Now for the condition (ii) in Proposition 4.5, if $v \in T_\alpha(\hat{S}_1) \cap L_3$ for a general $\alpha \in \hat{S}_1$, then its image $\bar{v} \in V_2$ satisfies $\bar{v} \in T_{\bar{\alpha}}(\hat{S}_2) \cap L_2$. Thus by condition (ii) for (S_2, V_2, L_2), we have $\bar{v} = 0$, i.e., $v \in L_1$. Then by condition (ii) for (S_1, V_1, L_1), we get $v = 0$. ☐

We have the following

Corollary 4.17 *In the notation of Proposition 4.14, let $Z := p_L(S) \subset \mathbb{P}U := \mathbb{P}(V/L)$ be the VMRT of a symplectic Grassmannian as explained in Sect. 3.2. For a subspace $L_2 \subset U$, let $L_3 \subset \mathrm{Sym}^2(W \oplus Q)$ be the inverse under the projection $\mathrm{Sym}^2(W \oplus Q) \to U$. If $\mathbb{P}L_2 \cap \mathrm{Sec}(Z) = \emptyset$, then*

$$\mathfrak{aut}(\widehat{p_{L_2}(Z)})^{(1)} \cong \mathfrak{aut}(\hat{S}, L_3, 0)^{(1)}.$$

Proof We just apply Proposition 4.16 with $S_1 = S$, $V_1 = V$, $L_1 = L$, together with Propositions 4.6, 4.14 and 4.15. ☐

We can make this more explicit as follows.

Proposition 4.18 *Let us assume the setting of Corollary 4.17. For* $\phi \in \mathrm{Sym}^2(W \oplus Q)$, *denote by* $\phi^\sharp \in \mathrm{Hom}(W^* \oplus Q^*, W \oplus Q)$ *the corresponding homomorphism. For* $L_2 \subset U$, *we denote by* $L_3 \subset \mathrm{Sym}^2(W \oplus Q)$ *the subspace satisfying* $L_3/(\mathrm{Sym}^2 Q) \cong L_2$ *and by* $\mathrm{Im}(L_2)$ *the linear span of* $\{\mathrm{Im}(\phi^\sharp) \subset W \oplus Q, \phi \in L_2\}$. *Define* $\mathrm{Im}_W(L_2) := p_Q(\mathrm{Im}(L_2)) \subset W$, *where* $p_Q : W \oplus Q \to W$ *is the projection to the first factor. Then*

(i) *there is a canonical vector space isomorphism*

$$\mathfrak{aut}(\hat{S}, L_3, 0)^{(1)} \cong \mathrm{Sym}^2(W/\mathrm{Im}_W(L_2))^*;$$

(ii) *for each* $\psi \in \mathrm{Sym}^2(W/\mathrm{Im}_W(L_2))^* \subset \mathrm{Sym}^2 W^*$, *denoting by* $\psi^\sharp$ *the corresponding element in* $\mathrm{Hom}(W, W^*)$ *and writing* $W' := \mathrm{Ker}(\psi^\sharp)$, L_2 *is contained in*

$$L'(\psi) := \{\phi \in U \,|\, \mathrm{Im}(\phi^\sharp) \subset \mathrm{Ker}(\psi^\sharp) \oplus Q\} \cong (W' \otimes Q) \oplus \mathrm{Sym}^2 W';$$

(iii) *if* $\mathbb{P}L_2 \cap \mathrm{Sec}(Z) = \emptyset$ *and* $\mathfrak{aut}(\hat{S}, L_3, 0)^{(1)}$ *contains an element of* $\mathrm{rank}\, r$ *in* $\mathrm{Sym}^2(W/\mathrm{Im}_W(L_2))^*$ (*i.e. the corresponding element in* $\mathrm{Hom}(W, W^*)$ *has* $\mathrm{rank}\, r$), *then*

$$\dim L_2 \leq mk + \frac{k(k+1)}{2} - 2m - 2k + 1 - \frac{r(2m+2k-r-3)}{2}.$$

The following lemma is immediate from Lemma 4.3 and the information on $\mathrm{Sec}(S)$ (S as in Proposition 4.14) from the table in Sect. 3.1.

Lemma 4.19 *In the notation of Proposition* 4.18, $\dim \mathrm{Sec}(Z) = 2m + 2k - 2$ *where* $m = \dim Q, k = \dim W$. *In particular,* $\mathrm{Sec}(Z) = \mathbb{P}(U)$ *if and only if* $k = 2$.

Proof of Proposition 4.18 From Proposition 4.12 (i), we have

$$\mathfrak{aut}(\hat{S}, L_3, 0)^{(1)} \cong \mathrm{Sym}^2((W \oplus Q)/\mathrm{Im}(L_3))^*.$$

From $L = \mathrm{Sym}^2 Q \subset L_3$ and $L_3/L = L_2$, we have $\mathrm{Im}(L_3) = Q \oplus \mathrm{Im}_W(L_2)$, proving (i).

Now for $\psi \in \mathrm{Sym}^2(W/\mathrm{Im}_W(L_2))^*$, denote by $\tilde{\psi}^\sharp \in \mathrm{Hom}(W \oplus Q, W^* \oplus Q^*)$ the element induced by $\psi^\sharp \in \mathrm{Hom}(W, W^*)$ via the composition

$$W \oplus Q \xrightarrow{p_Q} W \xrightarrow{\psi^\sharp} W^* \hookrightarrow W^* \oplus Q^*.$$

From Proposition 4.12 (ii), we see that L_3 is contained in

$$L(\psi) := \{\phi \in \mathrm{Sym}^2(W \oplus Q) \,|\, \mathrm{Im}(\phi^\sharp) \subset \mathrm{Ker}(\tilde{\psi}^\sharp)\}.$$

Clearly, $\mathrm{Sym}^2 Q \subset L(\psi)$ and the quotient $L(\psi)/(\mathrm{Sym}^2 Q)$ is naturally isomorphic to $L'(\psi)$, proving (ii).

For (iii), assume that $\mathbb{P}L_2 \cap \mathrm{Sec}(Z) = \emptyset$ and there is ψ in (ii) of rank r. Then $\dim W' = k - r$. Let $V' = \mathrm{Sym}^2(W' \oplus Q)$ and $S' \subset \mathbb{P}V'$ be the second Veronese embedding of $\mathbb{P}(W' \oplus Q)$. Then we have $\dim(\mathrm{Sec}(S')) = 2m + 2(k-r) - 2$. By the assumption, $\mathbb{P}L_2 \subset \mathbb{P}L'(\psi)$ is disjoint from

$$p_L(\mathrm{Sec}(S')) \subset \mathrm{Sec}(p_L(S)) = \mathrm{Sec}(Z),$$

which implies that $k - r \geq 3$ by Lemma 4.19 and also

$$\dim L_2 \leq m(k-r) + \frac{(k-r)(k+1-r)}{2} - 1 - (2m + 2(k-r) - 2). \quad \square$$

Let us derive an important consequence of our study of the prolongation of biregular projections of the examples in Sect. 3, Theorem 4.21 below, which is a key ingredient in the proof of Main Theorem.

Definition 4.20 A linear subspace $\emptyset \neq \mathbb{P}L \subset \mathbb{P}V \setminus \mathrm{Sec}(S)$ is called *maximal* if $\mathrm{Sec}(p_L(S)) = \mathbb{P}(V/L)$. In this case, $\dim L = \dim \mathbb{P}V - \dim \mathrm{Sec}(S)$ from Lemma 4.3.

Theorem 4.21 *Let $S \subset \mathbb{P}V$ be one of the linearly normal varieties listed in Main Theorem* (A1)–(B5) *with* $\mathrm{Sec}(S) \neq \mathbb{P}V$. *Let $\mathbb{P}L \subset \mathbb{P}V \setminus \mathrm{Sec}(S)$ be a linear space and p_L the projection along $\mathbb{P}L$. If $\mathbb{P}L$ contains a general point of $\mathbb{P}V$ or if $\mathbb{P}L$ is maximal, then* $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} = 0$.

Proof From $\mathrm{Sec}(S) \neq \mathbb{P}V$, it suffices to check those covered by Propositions 4.10, 4.11, 4.12, 4.13 and 4.18. In fact, the examples in Sects. 3.3 and 3.4 admit no biregular projections since $\mathrm{Sec}(S) = \mathbb{P}V$ by [22] (Chap. V, Corollary 1.13). There is nothing to check for the case of Proposition 4.13.

In Proposition 4.10, suppose L contains a general element ϕ of V. Then $\phi \in L$ is of maximal rank and $\mathrm{Im}(L) = B$, proving $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \cong \mathfrak{aut}(\hat{S}, L, 0)^{(1)} = 0$ from Proposition 4.10 (i). On the other hand, if L is maximal, $\dim L = ab - 2a - 2b + 4$. Thus from Proposition 4.10 (iii), the rank of any element of $\mathfrak{aut}(\hat{S}, L, 0)^{(1)}$ must be zero, i.e., $\mathfrak{aut}(\hat{S}, L, 0)^{(1)} = 0$.

In Proposition 4.11 (resp. Proposition 4.12), suppose L contains a general element ϕ of V. Then $\phi^\sharp$ is of maximal rank and $\mathrm{Im}(L) = W$, proving $\mathfrak{aut}(\widehat{p_L(S)})^{(1)} \cong \mathfrak{aut}(\hat{S}, L, 0)^{(1)} = 0$ from Proposition 4.11 (i) (resp. Proposition 4.12 (i)). On the other hand, if L is maximal, $\dim L = \frac{1}{2}(n^2 - n - 2) - 4n + 11$ (resp. $\frac{1}{2}(n^2 + n - 2) - (2n - 2)$). Thus from Proposition 4.11 (iii), (resp. Proposition 4.12 (iii)), the rank of any element of $\mathfrak{aut}(\hat{S}, L, 0)^{(1)}$ must be zero, i.e., $\mathfrak{aut}(\hat{S}, L, 0)^{(1)} = 0$.

In Proposition 4.18, suppose L_2 contains a general $\phi \in V_2$. Then $\phi^\sharp$ is of maximal rank and $\mathrm{Im}_W(L_2) = W$, proving $\mathfrak{aut}(\widehat{p_{L_2}(Z)})^{(1)} \cong \mathfrak{aut}(\hat{S}, L_3, 0)^{(1)} = 0$ from Proposition 4.18 (i). On the other hand, $\dim(\mathrm{Sec}(Z)) = 2m + 2k - 2$ by Lemma 4.3, which implies that if L_2 is maximal, $\dim L_2 = mk + \frac{k(k+1)}{2} - 1 - (2m + 2k - 2)$. Thus from Proposition 4.18 (iii), the rank of any element of $\mathfrak{aut}(\hat{S}, L_3, 0)^{(1)}$ must be zero, i.e., $\mathfrak{aut}(\widehat{p_{L_2}(Z)})^{(1)} = 0$. □

5 Cone structure and *G*-structure

This section collects some general facts on cone structures and G-structures. The main theme is to reveal the relationship between the existence of an Euler vector field, the local flatness of the cone structure and the prolongation of a linear Lie algebra.

Definition 5.1 A *cone structure* on a complex manifold M is a closed analytic subvariety $\mathcal{C} \subset \mathbb{P}T(M)$ such that the projection $\pi : \mathcal{C} \to M$ is proper, flat and surjective with connected fibers.

Lemma 5.2 *A cone structure $\mathcal{C} \subset \mathbb{P}T(M)$ on a complex manifold M induces a holomorphic equivalence relation on M: two points $x, y \in M$ are equivalent if the projective varieties $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ and $\mathcal{C}_y \subset \mathbb{P}T_y(M)$ are projectively equivalent. At a general point $x \in M$, there exists a neighborhood $x \in U \subset M$ such that the equivalence classes form a (regular) holomorphic foliation in U.*

Proof We can choose an open neighborhood U of x such that $T(U)$ is trivial bundle and the fibers of the family $\mathcal{C}|_U \subset U \times \mathbb{P}^{n-1}$, $n = \dim M$, are all reduced subvarieties of $\mathbb{P}^{n-1}$. We have the associated morphism to the Chow variety $\nu : U \to \mathrm{Chow}(\mathbb{P}^{n-1})$ which can be assumed to be a smooth morphism by shrinking U if necessary. Then the $GL(n)$-equivalence classes of points in $\mathrm{Chow}(\mathbb{P}^{n-1})$ induces a holomorphic foliation on U by shrinking U if necessary, from the generality of x. □

Definition 5.3 Given a cone structure $\mathcal{C} \subset \mathbb{P}T(M)$ on a complex manifold M, and a general point $x \in M$, the leaves of the foliation in Lemma 5.2 are *isotrivial leaves* of the cone structure in a neighborhood of x. The dimension of isotrivial leaves is denoted by $\delta(\mathcal{C})$. The cone structure is said to be *isotrivial*, if all general fibers of $\mathcal{C} \to M$ are projectively equivalent, i.e., $\delta(\mathcal{C}) = \dim M$. In this case, if we denote by Z the projective variety $\mathcal{C}_x \subset \mathbb{P}T_x M$ for a general point $x \in M$, then we call $\mathcal{C}$ a *Z-isotrivial cone structure*.

For an isotrivial cone structure, we can associate to it another geometric structure: the G-structure.

Definition 5.4 Given a complex manifold M, fix a vector space V with $\dim V = \dim M$. The frame bundle $\mathcal{F}(M)$ has the fiber at $x \in M$,

$$\mathcal{F}_x(M) := \mathrm{Isom}(V, T_x(M)).$$

For a closed connected subgroup $G \subset \mathrm{GL}(V)$, a *G-structure* on M is a G-subbundle $\mathcal{G} \subset \mathcal{F}(M)$. If G contains the scalar group $\mathbb{C}^\times \cdot \mathrm{Id} \subset \mathrm{GL}(V)$, we say that the G-structure is *of cone type*. An isotrivial cone structure $\mathcal{C} \subset \mathbb{P}T(M)$ induces a G-structure $\mathcal{G}$ of cone type on an open subset M^o of M where each fiber $\mathcal{C}_x \subset \mathbb{P}T_x(M), x \in M^o$ is projectively equivalent to $\mathcal{C}_z \subset \mathbb{P}T_z(M)$ for a base point $z \in M^o$. In fact, setting $V = T_z(M)$ and $G = \mathrm{Aut}_0(\hat{\mathcal{C}}_z)$, the fiber $\mathcal{G}_x \subset \mathcal{F}_x(M)$ is given by

$$\{\sigma \in \mathrm{Isom}(V, T_x(M)), \sigma(\hat{\mathcal{C}}_z) = \hat{\mathcal{C}}_x\}.$$

Definition 5.5 A G-structure $\mathcal{G} \subset \mathcal{F}(M)$ on M and a G-structure $\mathcal{G}' \subset \mathcal{F}(M')$ on M' are *equivalent* if there exists a biholomorphic map $\varphi : M \to M'$ such that the induced map $\varphi_* : \mathcal{F}(M) \to \mathcal{F}(M')$ sends $\mathcal{G}$ isomorphically to $\mathcal{G}'$. A cone structure $\mathcal{C} \subset \mathbb{P}T(M)$ on M and a cone structure $\mathcal{C}' \subset \mathbb{P}T(M')$ on M' are *equivalent* if there exists a biholomorphic map $\varphi : M \to M'$ such that the induced map $\varphi_* : \mathbb{P}T(M) \to \mathbb{P}T(M')$ sends $\mathcal{C}$ isomorphically to $\mathcal{C}'$.

Definition 5.6 On the vector space V as a complex manifold, we have a canonical trivialization $\mathcal{F}(V) = \mathrm{GL}(V) \times V$. For any subgroup $G \subset \mathrm{GL}(V)$, this induces the *flat G-structure* on V defined by

$$\mathcal{G} = G \times V \subset \mathrm{GL}(V) \times V = \mathcal{F}(V).$$

A G-structure $\mathcal{G} \subset \mathcal{F}(M)$ is *locally flat* if its restriction to some open subset is equivalent to the restriction of the flat G-structure to some open subset of V. An isotrivial cone structure is *locally flat* if its associated G-structure is locally flat.

Definition 5.7 Given a cone structure $\mathcal{C} \subset \mathbb{P}T(M)$ and a point $x \in M$, a germ of holomorphic vector field v at x is said to *preserve the cone structure* if the local 1-parameter family of biholomorphisms integrating v lifts to local biholomorphisms of $\mathbb{P}T(M)$ preserving $\mathcal{C}$. The flows of such a vector field must be tangent to the isotrivial leaves of $\mathcal{C}$. The set of all such germs form a Lie algebra, called the *Lie algebra of infinitesimal automorphisms of the cone structure* $\mathcal{C}$ at x, to be denoted by $\mathfrak{aut}(\mathcal{C}, x)$.

Definition 5.8 Let $\mathcal{C} \subset \mathbb{P}T(M)$ be a cone structure. For a non-negative integer ℓ, let $\mathfrak{aut}(\mathcal{C}, x)_\ell \subset \mathfrak{aut}(\mathcal{C}, x)$ be the subalgebra of vector fields vanishing at x to order $\geq \ell + 1$. This gives the structure of a filtered Lie algebra on $\mathfrak{aut}(\mathcal{C}, x)$, i.e., $[\mathfrak{aut}(\mathcal{C}, x)_\ell, \mathfrak{aut}(\mathcal{C}, x)_k] \subset \mathfrak{aut}(\mathcal{C}, x)_{\ell+k}$.

The following result is Proposition 1.2.1 [14].

Proposition 5.9 *For each $k \geq 0$, regard the quotient space $\mathfrak{aut}(\mathcal{C}, x)_k / \mathfrak{aut}(\mathcal{C}, x)_{k+1}$ as a subspace of $\mathrm{Hom}(\mathrm{Sym}^{k+1} T_x(M), T_x(M))$ by taking the leading terms of the Taylor expansion of the vector fields at x. Then*

$$\mathfrak{aut}(\mathcal{C}, x)_k / \mathfrak{aut}(\mathcal{C}, x)_{k+1} \subseteq \mathfrak{aut}(\hat{\mathcal{C}}_x)^{(k)}.$$

If the cone structure $\mathcal{C}$ is isotrivial and locally flat, then the equality in the above inclusion holds for all k.

Proposition 5.10 *Given a cone structure $\mathcal{C} \subset \mathbb{P}T(M)$ and a general point $x \in M$, denote by $\hat{\mathcal{C}}_x \subset T_x(M)$ the affine cone over the fiber $\mathcal{C}_x$ at x and let $\mathfrak{aut}(\hat{\mathcal{C}}_x) \subset \mathrm{End}(T_x(M))$ be the Lie algebra of infinitesimal automorphisms of the affine cone. Assume that $\mathfrak{aut}(\hat{\mathcal{C}}_x)^{(k+1)} = 0$ for some $k \geq 0$. Then*

$$\dim(\mathfrak{aut}(\mathcal{C}, x)) \leq \delta(\mathcal{C}) + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x) + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} + \cdots + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x)^{(k)}$$

and if the equality holds then there exists an Euler vector field (cf. Definition 2.5) in $\mathfrak{aut}(\mathcal{C}, x)_0$.

Proof The codimension of $\mathfrak{aut}(\mathcal{C}, x)_0$ in $\mathfrak{aut}(\mathcal{C}, x)$ is at most $\delta(\mathcal{C})$. That the dimension of $\mathfrak{aut}(\mathcal{C}, x)_0$ is bounded by

$$\dim \mathfrak{aut}(\hat{\mathcal{C}}_x) + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} + \cdots + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x)^{(k)}$$

follows from Proposition 5.9, which also shows that the equality holds only if each element of $\mathfrak{aut}(\hat{\mathcal{C}}_x) \subset \mathrm{End}(T_x(M))$ can be realized as the linear part of a vector field in $\mathfrak{aut}(\mathcal{C}, x)_0$. Thus if the equality holds, there exists an Euler vector field in $\mathfrak{aut}(\mathcal{C}, x)$ (cf. Lemma 2.4). □

The following is from [5] (also see Sect. 1 of [11]).

Theorem 5.11 *Given a G-structure $\mathcal{G} \subset \mathcal{F}(M)$, we can define vector-valued functions c^k, $k = 0, 1, 2, \ldots$ on $\mathcal{G}$ with the following properties.*

(1) *c^k is an $H^{k,2}(\mathfrak{g})$-valued function, well-defined if $c^{k-1} \equiv 0$. Here $H^{k,2}(\mathfrak{g})$ is the cohomology of the natural sequence*

$$\mathfrak{g}^{(k)} \otimes V^* \to \mathfrak{g}^{(k-1)} \otimes \bigwedge^2 V^* \to \mathfrak{g}^{(k-2)} \otimes \bigwedge^3 V^*$$

and by convention, $c^{-1} \equiv 0$, $\mathfrak{g}^{(-1)} = V$ and $\mathfrak{g}^{(-2)} = 0$.

(2) *Under the action of G on $\mathcal{G}$, the function c^k transforms like the G-module* $\mathfrak{g}^{(k-1)} \otimes \bigwedge^2 V^* \subset \mathrm{Hom}(\mathrm{Sym}^k V, V) \otimes \bigwedge^2 V^*$.
(3) *If $c^k \equiv 0$ for all non-negative integers k, then $\mathcal{G}$ is locally flat.*
(4) *c^k is an invariant of the G-structure, i.e., it is invariant under an automorphism of the G-structure.*

It has the following consequence.

Proposition 5.12 *Let $\mathcal{C} \subset \mathbb{P}T(M)$ be a cone structure. Assume that for a general point $x \in M$, there exists an Euler vector field in $\mathfrak{aut}(\mathcal{C}, x)_0$. Then the cone structure is isotrivial and locally flat.*

Proof In a neighborhood of a general point x, the isotrivial leaves of $\mathcal{C}$ form a regular foliation from Lemma 5.2. Given any vector field $v \in \mathfrak{aut}(\mathcal{C}, x)_0$, the flows of v must be tangent to the leaves of the foliation. But by Lemma 2.4, each flow of an Euler vector field v has limit x. Thus x is a singularity of the foliation, unless there is only one leaf. This shows that $\mathcal{C}$ is isotrivial.

To prove the local flatness, by Theorem 5.11, it suffices to show that the functions c^k on the associated G-structure of cone type are identically zero. By induction, assume that $c^{k-1} \equiv 0$ and c^k is well-defined. Pick a general point $x \in M$. The subgroup $\mathbb{C}^\times \cdot \mathrm{Id} \subset G$ acts on the fiber $\mathcal{G}_x$ and under this action, the characteristic function c^k is multiplied by $t^{-(k+1)} \in \mathbb{C}^\times$ by Theorem 5.11 (2). But by integrating the Euler vector field in $\mathfrak{aut}(\mathcal{C}, x)_0$, we get a 1-parameter family of local automorphisms of the G-structure which preserve the fiber $\mathcal{G}_x$ and act by $\mathbb{C}^\times \cdot \mathrm{Id}$-action on it. Since this is an automorphism of the G-structure, the functions c^k cannot change under this action by Theorem 5.11 (4), a contradiction unless c^k vanishes on $\mathcal{G}_x$. Since this is true for a general x, we get $c^k \equiv 0$. Thus by induction we have $c^k \equiv 0$ for all $k \geq 0$ and the local flatness of $\mathcal{G}$ from Theorem 5.11 (3). □

Corollary 5.13 *If the equality holds in Proposition 5.10, then the cone structure is locally flat.*

In fact, the converse is also true. The proof of the following can be found in Sect. 2.1 of [21].

Proposition 5.14 *Assume that $\mathfrak{g}^{(k+1)} = 0$ and that the G-structure $\mathcal{G}$ on a complex manifold M is locally flat. Then for any point $x \in M$, $\mathfrak{aut}(\mathcal{G}, x)$, the Lie algebra of germs of holomorphic vector fields at x preserving the G-structure, is isomorphic to the graded Lie algebra $V \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(k)}$ for a vector space V with $\dim V = \dim M$. It follows that when $\mathcal{C}$ is a locally* *flat cone structure on* M *with* $\mathfrak{aut}(\hat{\mathcal{C}}_x) = \mathfrak{g}$ *and* $\mathfrak{g}^{(k+1)} = 0$,

$$\mathfrak{aut}(\mathcal{C}, x) \cong V \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(k)}.$$

6 Proof of Main Theorem modulo Theorems 6.16 and 6.17

In this section, we prove Main Theorem modulo two technical results, Theorems 6.16 and 6.17, the proofs of which will be postponed to Sects. 7 and 8, respectively.

Ionescu and Russo's classification of conic-connected manifolds (cf. Definition 2.2 (ii)) in [16] will be essential in our proof. To recall their result, it is convenient to introduce the following definition.

Definition 6.1 A conic-connected manifold $X \subset \mathbb{P}^N$ is said to be *primitive*, if X is a Fano manifold with $\mathrm{Pic}(X)$ generated by $\mathcal{O}_X(1)$ and X is covered by lines (cf. Definition 2.2 (ii)).

Theorem 6.2 ([16], Theorem 2.2) *Let* $X \subset \mathbb{P}^N$ *be a conic-connected manifold of dimension* n. *Unless* X *is primitive, it is projectively equivalent to one of the following or their biregular projections*:

- (a1) *The second Veronese embedding of* $\mathbb{P}^n$.
- (a2) *The Segre embedding of* $\mathbb{P}^a \times \mathbb{P}^{n-a}$ *for* $1 \leq a \leq n-1$.
- (a3) *The VMRT of the symplectic Grassmannian* $\mathrm{Gr}_\omega(k, k+n+1)$ *for* $2 \leq k \leq n$.
- (a4) *A hyperplane section of the Segre embedding* $\mathbb{P}^a \times \mathbb{P}^{n+1-a}$ *with* $2 \leq a, n+1-a$.

We know the prolongations of the varieties (a1), (a2) and (a3) in Theorem 6.2 from Sect. 4. The prolongation of varieties (a4) in Theorem 6.2 turns out to be zero:

Proposition 6.3 *Let* $a \geq b \geq 2$ *be two integers. Let* $X = \mathbb{P}^a \times \mathbb{P}^b \hookrightarrow \mathbb{P}^{ab+a+b}$ *be the Segre embedding and let* $S = X \cap H$ *be a nonsingular hyperplane section, which is conic-connected. Then for the non-degenerate embedding* $S \subset H$, *we have* $\mathfrak{aut}(\hat{S})^{(1)} = 0$.

Proof The two projections $X \to \mathbb{P}^a$ and $X \to \mathbb{P}^b$ induce two fibrations: $\mathbb{P}^a \stackrel{\pi_1}{\longleftarrow} S \stackrel{\pi_2}{\longrightarrow} \mathbb{P}^b$, with fibers isomorphic to $\mathbb{P}^{b-1}$ and $\mathbb{P}^{a-1}$ respectively (cf. the proof of Theorem 2.2 in [16], Case II, subcase (b)). Let $F \subset T(S)$ be the distribution spanned by the tangent spaces of fibers of π_1 and π_2. Then F is a vector subbundle of rank $a+b-2$. Note that the projectivization $\mathbb{P}F \subset \mathbb{P}T(S)$ is a cone structure, which is invariant under $\mathrm{Aut}_0(S)$.

Supposing that $\mathfrak{aut}(\hat{S})^{(1)} \neq 0$, we will derive a contradiction. For a general point $s \in S$, $\mathrm{Aut}_0(S)$ contains a $\mathbb{C}^\times$ subgroup whose germ at s is an Euler vector field by Theorem 2.6 (iv). Thus any $\mathrm{Aut}_0(S)$-invariant cone structure on an open subset of S is locally flat by Proposition 5.12. This implies that the cone structure $\mathbb{P}F$ is locally flat, which is equivalent to the integrability of the distribution F. Thus the distribution F is a foliation with leaves of codimension 1. For a general point $x \in S$, let R_x be the set of points on S which can be connected by a chain of lines contained in the fiber of π_1 or π_2. Then R_x must agree with the leaf of F through x, and is a divisor on S. For $i = 1, 2$, let l_i be a line contained in the fiber of π_i such that l_1 meets l_2 at x. Then we have $0 = R_x \cdot l_1 = R_x \cdot l_2$ because the two lines are contained in a leaf. By Lefschetz hyperplane theorem, $H_2(S, \mathbb{C}) \cong \mathbb{C}^2$ is generated by the classes of l_1 and l_2. Thus R_x is a numerically trivial effective divisor, a contradiction. □

In the setting of Main Theorem, Theorem 6.2 has the following consequence.

Proposition 6.4 *Let $S \subset \mathbb{P}V$ be a nonsingular non-degenerate variety of dimension n such that $\mathfrak{aut}(\hat{S})^{(1)} \neq 0$. Unless $S \subset \mathbb{P}V$ is a primitive conic-connected manifold, it is projectively equivalent to one of the following or their biregular projections*:

- (a1) *The second Veronese embedding of $\mathbb{P}^n$.*
- (a2) *The Segre embedding of $\mathbb{P}^a \times \mathbb{P}^{n-a}$ for $1 \leq a \leq n-1$.*
- (a3) *The VMRT of the symplectic Grassmannian* $\mathrm{Gr}_\omega(k, k+n+1)$ *for* $2 \leq k \leq n$.

Proof By Corollary 4.8, we may assume that $S \subset \mathbb{P}^N$ is linearly normal. By Theorem 2.6 (i), S is conic-connected. Assuming that S is not primitive and applying Theorem 6.2, we see that S is projectively equivalent to one of (a1)–(a4) in Theorem 6.2 or their biregular projections. But by Proposition 6.3, varieties in (a4) have no prolongation. By Proposition 4.6, their biregular projections do not have prolongation, either. Thus we are left with (a1)–(a3). □

The following variation of Proposition 6.4 will be useful.

Proposition 6.5 *Let $S \subset \mathbb{P}V$ be a nonsingular non-degenerate variety such that* $\mathfrak{aut}(\hat{S})^{(1)} \neq 0$ *and* $\mathrm{Sec}(S) = \mathbb{P}V$. *Unless $S \subset \mathbb{P}V$ is a primitive conic-connected manifold, it is projectively equivalent to one of the following*:

- (i) *The second Veronese embedding $v_2(\mathbb{P}^1) \subset \mathbb{P}^2$, i.e. a plane conic.*
- (ii) *The Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^k$ with $k \geq 1$.*
- (iii) *The VMRT of a symplectic Grassmannian* $\mathrm{Gr}_\omega(2, V)$ *with* $\dim V \geq 5$.

Proof From the table in Sect. 3.1, Lemmas 4.3, and 4.19, a variety in (a1)–(a3) of Proposition 6.4, satisfying $\mathrm{Sec}(S) = \mathbb{P}V$ belongs to the above list. On the other hand, if $S \subset \mathbb{P}V$ is a biregular projection of (a1)–(a3) in Proposition 6.4 with $\mathrm{Sec}(S) = \mathbb{P}V$, then $\mathfrak{aut}(\hat{S})^{(1)} = 0$ by Theorem 4.21. □

From Proposition 6.4, the difficulty in proving Main Theorem lies in the study of primitive Fano manifolds which have nonzero prolongation. We will study at first the VMRT of such varieties.

Proposition 6.6 *Let X be a Fano manifold of Picard number* 1 *such that for a general point $x \in X$, there exists a holomorphic vector field v_x on X which is an Euler vector field at x* (*in the sense of Definition* 2.5) *and generates a $\mathbb{C}^\times$-action on X. Then for any choice of a minimal rational component on X, the associated VMRT at a general point $x \in X$ is irreducible.*

Proof Since v_x is Euler at x, it acts on $\mathbb{P}T_x(X)$ trivially. The $\mathbb{C}^\times$-action generated by v_x sends minimal rational curves through x to minimal rational curves through x fixing their tangent directions. On the other hand, the normal bundle (pull-back to the normalization) of a general minimal rational curve C is of the form $\mathcal{O}(1)^p \oplus \mathcal{O}^q$ for some non-negative integers p and q (cf. Sect. 1 in [6]). This implies that C does not have a non-trivial deformation fixing a point and the tangent direction at that point. Thus the $\mathbb{C}^\times$-action must send each minimal rational curve through x to itself, inducing a non-trivial $\mathbb{C}^\times$-action on each minimal rational curve through x. Denote by $\mathcal{N} \to \mathcal{C}$ the normalization of the total variety of minimal rational tangents $\mathcal{C}$ and $\mathcal{N} \to \mathcal{M} \to X$ the Stein factorization of $\mathcal{N} \to X$, where $f : \mathcal{M} \to X$ is a finite morphism. As $\mathcal{N}$ is irreducible, to prove Proposition 6.6, it suffices to show that f is birational. Suppose not and let $D \subset X$ be an irreducible component of the branch locus. Any $\mathbb{C}^\times$-action on X lifts to a $\mathbb{C}^\times$-action on $\mathcal{M}$ because it induces an action on the space of minimal rational curves. Thus any $\mathbb{C}^\times$-action on X preserves D. Let C be a general minimal rational curve through x. We can assume that C intersects D transversally. By Lemma 4.2 of [13], there exists a component C' of $f^{-1}(C)$ which is not birational to C. After normalizing C and C', the morphism $f|_{C'} : C' \to C$ has non-empty branch points at least at two points $z_1, z_2 \in D \cap C$. Since x is general, we have $x \neq z_1, z_2$. But v_x generates a $\mathbb{C}^\times$-action on C fixing $D \cap C$ and x. Since $z_1, z_2 \in D \cap C$, we have a non-trivial $\mathbb{C}^\times$-action on $\mathbb{P}^1$ with at least three fixed points, a contradiction. □

Proposition 6.7 *Let $X \subset \mathbb{P}^N$ be a primitive conic-connected manifold. Fix a minimal rational component consisting of lines covering X. Assume that $\mathfrak{aut}(\hat{X})^{(1)} \neq 0$. Then for a general point $x \in X$, the VMRT $\mathcal{C}_x$ is an irreducible nonsingular and non-degenerate projective variety satisfying* $\mathrm{Sec}(\mathcal{C}_x) = \mathbb{P}T_x(X)$.

Proof By Corollary 4.8, all conditions of the proposition remain unchanged if we replace $X \subset \mathbb{P}^N$ by the linearly normal embedding $X \subset \mathbb{P}H^0(X, \mathcal{O}_X(1))^*$. thus we may assume that $X \subset \mathbb{P}^N$ is linearly normal. We know that $\mathcal{C}_x$ is irreducible from Theorem 2.6 (iv) and Proposition 6.6. It is nonsingular from Proposition 3.2. It remains to prove that $\mathrm{Sec}(\mathcal{C}_x) = \mathbb{P}T_x(X)$, which implies the non-degeneracy. By the proof of Theorem 2.2 [16] (p. 155), if all conics joining two general points are irreducible, then X is isomorphic to the second Veronese embedding, a contradiction to the assumption that X is covered by lines. Thus two general points of X can be joined by a connected union of two lines. Then by Theorem 3.14 of [10], $\mathrm{Sec}(\mathcal{C}_x) = \mathbb{P}T_x(X)$. □

Remark that if the VMRT $\mathcal{C}_x \subsetneq \mathbb{P}T_xX$ at a general point x of a Fano manifold X of Picard number 1 is linear, then $\mathcal{C}_x$ cannot be irreducible (cf. the first paragraph in the proof of Proposition 6.10 below). Hence we can apply the main theorem in [13], which gives the following Cartan-Fubini type extension theorem.

Theorem 6.8 *Let X (resp. X') be a Fano manifold of Picard number 1 and let $\mathcal{C}$ (resp. $\mathcal{C}'$) be the VMRT associated to some minimal rational components. Assume that $\mathcal{C}_x \subsetneq \mathbb{P}T_xX$ is irreducible and nonsingular for a general point $x \in X$. Given any connected analytic open subsets $U \subset X$ and $U' \subset X'$ with a biholomorphic map $\phi : U \to U'$ such that the differential $\phi_* : \mathbb{P}T_x(X) \to \mathbb{P}T_{\phi(x)}(X')$ sends $\mathcal{C}_x$ to $\mathcal{C}'_{x'}$ isomorphically for all $x \in U$, we can extend ϕ to a biholomorphic map $\Phi : X \to X'$.*

An immediate consequence of Theorem 6.8 is the following which allows us to reconstruct some Fano manifolds of Picard number 1 from its VMRT.

Corollary 6.9 *Let X, X' be two Fano manifolds of Picard number 1 such that the VMRT at general points of both X and X' are projectively isomorphic to an irreducible nonsingular non-degenerate projective variety $Z \subsetneq \mathbb{P}V$. If the cone structures on open subsets of X and X' given by the VMRT are both locally flat, then X is biregular to X'.*

For the purpose of this article, one has to consider the situation of higher Picard numbers. The following proposition is a result for the general case.

Proposition 6.10 *Let X and X' be two nonsingular uniruled projective varieties. Let $\mathcal{C}$ (resp. $\mathcal{C}'$) be the VMRT associated to some minimal rational component $\mathcal{K}$ (resp. $\mathcal{K}'$) of X (resp. X'). Assume that*

(1) *for any effective divisor $D \subset X$, a member of $\mathcal{K}$ has positive intersection with D; and*
(2) *$\mathcal{C}_x \subsetneq \mathbb{P}T_xX$ is irreducible and nonsingular for a general point $x \in X$.*

Given any connected analytic open subsets $U \subset X$ and $U' \subset X'$ with a biholomorphic map $\phi : U \to U'$ such that the differential $\phi_* : \mathbb{P}T_x(X) \to \mathbb{P}T_{\phi(x)}(X')$ sends $\mathcal{C}_x$ to $\mathcal{C}'_{x'}$ isomorphically for all $x \in U$, we can extend ϕ to a generically finite rational map $\Phi : X \dashrightarrow X'$.

Proof If $\mathcal{C}_x \subsetneq \mathbb{P}T_x X$ is an irreducible linear subspace of dimension $k-1$, then by [1], there exists an open subset $W \subset X$ which has a $\mathbb{P}^k$-bundle structure $\phi : W \to T$ such that general members of $\mathcal{K}$ are lines in fibers of ϕ. As $\mathcal{C}_x \neq \mathbb{P}T_x X$, we have $\dim T \geq 1$. Take $D \subset X$ be the closure of the pre-image of a divisor on T, then the intersection of D with a general member of $\mathcal{K}$ is zero, a contradiction to assumption (1).

Since $\mathcal{C}_x$ is nonsingular and non-linear, hence its Gauss map is finite. The claim follows from Proposition 4.3 of [13], where it was stated under the assumption that both X and X' have Picard number 1. But the condition on the Picard number of X' was not used in the proof and the condition on the Picard number of X was used in the proof only to guarantee the assumption (1) above. □

Recall the following example from Example 1.7 in [9].

Example 6.11 Fix an irreducible nonsingular non-degenerate projective variety $Z \subset \mathbb{P}V$, $\dim V = n$. Regard $Z \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ as a submanifold in the hyperplane. Let Y be the blow-up of $\mathbb{P}^n$ along Z. Then lines on $\mathbb{P}^n$ intersecting Z give rise to minimal rational component on Y such that the cone structure $\mathcal{C}^Y$ given by the VMRT is locally flat. In fact, on the open set $Y^o \subset Y$ corresponding to $\mathbb{P}^n \setminus \mathbb{P}^{n-1}$, the VMRT induces the flat G-structure in Definition 5.6 with a suitable G.

We have the following corollary of Proposition 6.10.

Corollary 6.12 *Let X be a nonsingular uniruled projective variety with a minimal rational component satisfying the assumption* (1) *in Proposition 6.10. Assume that the cone structure $\mathcal{C}$ given by the VMRT is Z-isotrivial and locally flat for some nonsingular non-degenerate projective variety $Z \subset \mathbb{P}V$. Let Y and $\mathcal{C}^Y$ be as in Example 6.11. Then there exists a generically finite rational map $\Phi : X \dashrightarrow Y$ inducing the equivalence of the cone structures $\mathcal{C}$ and $\mathcal{C}^Y$ in some neighborhoods of a general point of X and its image in Y.*

Proposition 6.13 *In the setting of Corollary 6.12, assume furthermore that X has Picard number* 1. *Then X is an equivariant compactification of the affine space $\mathbb{C}^n$ where $n = \dim X$, i.e., there is an inclusion $\mathbb{C}^n \subset X$ such that the translation action of $\mathbb{C}^n$ extends to an action of the vector group on X.*

Proof Setting $\mathfrak{g} = \mathfrak{aut}(\hat{Z})$, we have $\mathfrak{g}^{(2)} = 0$ by Theorem 2.3 and for a point $y \in Y^o$ (cf. Example 6.11)

$$\mathfrak{aut}(\mathcal{C}^Y, y) \cong V \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)}$$

by Proposition 5.14. The abelian subalgebra V generates an algebraic action of the vector group $\mathbb{C}^n$ on Y, making Y an equivariant compactification of $\mathbb{C}^n$.

By Corollary 6.12, we get a generically finite rational map $\Phi : X \dashrightarrow Y$ which gives an equivalence of $\mathcal{C}$ at a general point $x \in X$ and $\mathcal{C}^Y$ at $\Phi(x)$, inducing an isomorphism

$$\Phi_* : \mathfrak{aut}(\mathcal{C}, x) \cong \mathfrak{aut}(\mathcal{C}^Y, \Phi(x)) \cong V \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)}.$$

The maximal abelian subalgebra $V \subset \mathfrak{aut}(\mathcal{C}, x)$ generates an *analytic* action of the vector group $\mathbb{C}^n$ on X by Theorem 6.8. Since V is a maximal abelian subalgebra, this analytic $\mathbb{C}^n$-action gives a commutative algebraic subgroup A of the automorphism group of X, acting with an open orbit on X. A priori, $A \cong \mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}$ for some $0 \le k \le n := \dim X$. By Ψ, the action of A descends to an algebraic action of the vector group $\mathbb{C}^n$ on Y, explained above. It follows that $k = n$ and A itself is the vector group acting algebraically on X. This shows that X is an equivariant compactification of $\mathbb{C}^n$. □

Now we continue the study of the VMRT of varieties with prolongation.

Proposition 6.14 *Let $S \subsetneq \mathbb{P}V$ be an n-dimensional non-degenerate primitive conic-connected manifold with $\mathfrak{aut}(\hat{S})^{(1)} \neq 0$. Then* (i) *the cone structure on a Zariski open subset of S defined by VMRT is locally flat, and* (ii) *S is an equivariant compactification of the affine space $\mathbb{C}^n$.*

Proof By Corollary 4.8, we may assume that $S \subset \mathbb{P}V$ is linearly normal. Then (i) follows from Theorem 2.6 and Proposition 5.12. By Proposition 6.7, (ii) follows from Proposition 6.13. □

The following theorem enables us to use induction to study prolongation, and is a crucial step in the proof of Main Theorem.

Theorem 6.15 *Let $S \subsetneq \mathbb{P}V$ be an irreducible non-degenerate primitive conic-connected manifold with $\mathfrak{aut}(\hat{S})^{(1)} \neq 0$. Then the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(S)$ at a general point $x \in S$ is an irreducible nonsingular non-degenerate variety satisfying $\mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} \neq 0$ and* $\mathrm{Sec}(\mathcal{C}_x) = \mathbb{P}T_x(S)$.

Proof All follow from Proposition 6.7, except $\mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} \neq 0$. By Corollary 4.8, we may assume that $S \subsetneq \mathbb{P}V$ is linearly normal.

By Proposition 6.14 (ii), we have an algebraic action of $\mathbb{C}^n$ on S with an open orbit. The complement of the open orbit is an irreducible hypersurface $H \subset S$ from $\mathrm{Pic}(S) \cong \mathbb{Z}$ (e.g. Proposition 1.2 (c) in [20]).

For each $A \in \mathfrak{aut}(\hat{S})^{(1)}$, we have an associated element $\lambda_A \in H^0(S, \mathcal{O}(1))$ in the sense of Theorem 2.6. As S is covered by lines and $S \subsetneq \mathbb{P}V$, S is not biregular to a projective space and we can apply Proposition 2.8 to get a point $x \in S \cap (\lambda_A = 0)$ such that $A_\alpha, \alpha \in \hat{x}$, induces a vector field v_x on S vanishing at x to second order.

Suppose that the divisor $S \cap (\lambda_A = 0)$ is different from the boundary divisor $H = S \setminus \mathbb{C}^n$. Then we can assume that x is in the open $\mathbb{C}^n$-orbit. In particular, the VMRT gives a cone structure $\mathcal{C}$ in a neighborhood of x. The global vector field v_x on S should preserve the cone structure $\mathcal{C}$. It follows that the germ of v_x at x belongs to $\mathfrak{aut}(\mathcal{C}, x)_1$, which implies that $\mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} \neq 0$ by Proposition 5.9. Since x lies in the $\mathbb{C}^n$-orbit, this x is a general point of S.

So we may assume that the hyperplane section $S \cap (\lambda_A = 0)$ is exactly the boundary divisor H for any non-zero $A \in \mathfrak{aut}(\hat{S})^{(1)}$. By Theorem 2.6 (ii), λ_A determines A. Hence $\dim \mathfrak{aut}(\hat{S})^{(1)} = 1$, a contradiction to Proposition 2.9. □

The following two theorems will be proved in the next two sections:

Theorem 6.16 *Let X be a Fano manifold with* $\mathrm{Pic}(X) = \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$. *Assume that X has minimal rational curves of degree* 1 *with respect to* $\mathcal{O}_X(1)$ *whose VMRT at a general point is isomorphic to the VMRT of a symplectic Grassmannian* $\mathrm{Gr}_\omega(2, m+4)$ *with* $m \geq 2$. *Then the cone structure given by the VMRT is not locally flat.*

Theorem 6.17 *Let X be a* 15*-dimensional Fano manifold with* $\mathrm{Pic}(X) = \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$. *Assume that X has minimal rational curves of degree* 1 *with respect to* $\mathcal{O}_X(1)$ *whose VMRT at a general point is isomorphic to a hyperplane section of the* 10*-dimensional spinor variety. Then the cone structure given by the VMRT is not locally flat.*

Conjecturally, the Fano manifold in Theorem 6.16 (resp. Theorem 6.17) is isomorphic to $\mathrm{Gr}_\omega(2, m+4)$ (resp. a general hyperplane section of the Cayley plane $\mathbb{OP}^2$). Theorems 6.16 and 6.17 are in contrast with the following two propositions. Note that $\mathrm{Gr}_\omega(2, 5)$ in Proposition 6.19 is the case $m = 1$ in the setting of Theorem 6.16.

Proposition 6.18 *Let* $\mathbb{S}_5 \subset \mathbb{P}^{15}$ *be the spinor embedding of the* 10*-dimensional spinor variety and let* $S \subset \mathbb{P}^{14}$ *be a general hyperplane section. The cone structure on S defined by the VMRT of lines covering S is locally flat at general points.*

Proof S is of Picard number 1 and covered by lines. Since $\mathfrak{aut}(\hat{S})^{(1)} \neq 0$ by Proposition 3.10, the cone structure on S is locally flat by Proposition 6.14 (i). □

Proposition 6.19 *Let* $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ *be the Plücker embedding of the Grassmannian and let* $S := \mathrm{Gr}_\omega(2,5) \subset \mathbb{P}^8$ *be a general hyperplane section of* $\mathrm{Gr}(2,5)$. *The cone structure on* S *defined by the VMRT of lines covering* S *is locally flat at general points.*

Proof This is by the same argument as in the proof of Proposition 6.18, replacing Proposition 3.10 by Proposition 3.12. □

We are now ready to prove Main Theorem.

Proof of Main Theorem Suppose that S is not primitive, then it belongs to (a1)–(a3) in Proposition 6.4 or their biregular projections. These varieties correspond to (A1)–(A3) and the first entries in (B1)–(B6) of Main Theorem, together with the biregular projections of (A1)–(A3). Here, note that the first entries in (B1)–(B6) do not have biregular projections (cf. Proposition 6.5).

Now suppose that S is primitive. By Theorem 6.15, the VMRT at a general point $x \in S$, $\mathcal{C}_x \subset \mathbb{P}T_x(S)$ is a nonsingular non-degenerate variety with $\mathfrak{aut}(\hat{\mathcal{C}}_x)^{(1)} \neq 0$ and $\mathrm{Sec}(\mathcal{C}_x) = \mathbb{P}T_x(S)$. Then we can apply Proposition 6.5 to $\mathcal{C}_x \subset \mathbb{P}T_x(S)$ to determine $\mathcal{C}_x$, unless $\mathcal{C}_x$ is again primitive. Repeating this, we end up with a positive integer ℓ and a sequence of irreducible nonsingular non-degenerate projective varieties

$$S_0 \subset \mathbb{P}V_0, \qquad S_1 \subset \mathbb{P}V_1, \ldots, \qquad S_i \subset \mathbb{P}V_i, \ldots, \qquad S_\ell \subset \mathbb{P}V_\ell,$$

such that

(a) $S_\ell := S$ and $V_\ell = V$,
(b) when x_i is a general point of S_i for $1 \leq i \leq \ell$, $S_{i-1} \subset \mathbb{P}V_{i-1}$ is isomorphic to $\mathcal{C}_{x_i} \subset \mathbb{P}T_{x_i}(S_i)$ and the cone structure given by this VMRT on S_i is locally flat;
(c) $\mathfrak{aut}(\hat{S}_i)^{(1)} \neq 0$ for each $0 \leq i \leq \ell$ and $\mathrm{Sec}(S_i) = \mathbb{P}V_i$ for each $0 \leq i \leq \ell - 1$;
(d) S_i is primitive for each $1 \leq i \leq \ell$ and S_0 is one of the varieties (i)–(iii) in Proposition 6.5.

We claim that the sequence of varieties $S_0, \ldots, S_\ell$ must be biregular to one of the following sequences of varieties.

(b1) $\mathbb{P}^1, \mathbb{Q}^3, \mathbb{Q}^5, \ldots, \mathbb{Q}^{2\ell-1}, \mathbb{Q}^{2\ell+1}$.
(b2) $\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q}^4, \ldots, \mathbb{Q}^{2\ell}, \mathbb{Q}^{2\ell+2}$.
(b3) $\ell = 1$ with $S_0 \cong \mathbb{P}^1 \times \mathbb{P}^k$ and $S_1 \cong \mathrm{Gr}(2, k+3)$ for some $k \geq 3$.

(b4) $\ell = 1, 2$ or 3, with $S_0 \cong \mathbb{P}^1 \times \mathbb{P}^2$, $S_1 \cong \mathrm{Gr}(2,5)$, $S_2 \cong \mathbb{S}_5$ and $S_3 \cong \mathbb{OP}^2$.
(b5) $\ell = 1$ or 2, with $S_0 \cong (\mathbb{P}^1 \times \mathbb{P}^2) \cap H_0$, $S_1 \cong \mathrm{Gr}(2,5) \cap H_1 (\cong \mathrm{Gr}_\omega(2,5))$ and $S_2 \cong \mathbb{S}_5 \cap H_2$, where H_0, H_1, H_2 are general hyperplanes.

Once the claim is proved, then by the property (c) of $\{S_i\}$ and Theorem 4.21, the embedding $S_i \subset \mathbb{P}V_i$ for $0 \le i \le \ell - 1$, is determined by the biregular type of S_i and is linearly normal, while $S_\ell \subset \mathbb{P}V_\ell$ is determined up to biregular projections. It is easy to check that among S_ℓ in (bi), $1 \le i \le 5$, only (b3) with $\ell = 1$ and (b4) with $\ell = 3$ admit biregular projections. In fact, $\mathrm{Sec}(S_\ell) = \mathbb{P}V_\ell$ holds for (b1), (b2) trivially, and also for (b5), as we have seen in the proof of Theorem 4.21. For (b4), we have $\mathrm{Sec}(\mathbb{OP}^2) \neq \mathbb{P}^{26}$, but its biregular projection has zero prolongation, by Proposition 4.13. Thus only S_1 in (b3) has non-trivial biregular projections with non-zero prolongation. In conclusion, the list in (bi), $1 \le i \le 5$, gives rise to the projective varieties in (Bi) and (C) of Main Theorem, completing the proof of Main Theorem.

To prove the claim let us recall that $S_0 \subset \mathbb{P}V_0$ must be one of the following from Proposition 6.5.

(i) The second Veronese embedding of $\mathbb{P}^1 \subset \mathbb{P}^2$.
(ii) The Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^k$ with $k \ge 1$.
(iii) The VMRT of a symplectic Grassmannian $\mathrm{Gr}_\omega(2, V)$ with $\dim V \ge 5$.

In Case (i), by a successive application of Corollary 6.9 combined with the property (b) of the sequence $\{S_i\}$, we obtain that S_i is isomorphic to an odd-dimensional hyperquadric, getting (b1).

In Case (ii), S_1 is biregular to $\mathrm{Gr}(2, k+3)$ by Corollary 6.9. If $\ell = 1$, we end up with the sequence (b3). If $\ell \ge 2$, by the property (c) of S_1 combined with Theorems 3.5 and 4.21, $S_1 \subset \mathbb{P}V_1$ must be the Plücker embedding of $\mathrm{Gr}(2, k+3)$ with $k = 1$ or 2. If $k = 1$, the Plücker embedding of $\mathrm{Gr}(2, k+3)$ is the natural embedding of the 4-dimensional hyperquadric $\mathbb{Q}^4 \subset \mathbb{P}^5$. By a successive application of Corollary 6.9, we get that S_i is an even-dimensional hyperquadric, yielding the sequence (b2). Now assume $k = 2$, then by Corollary 6.9, we get that S_2 is biregular to $\mathbb{S}_5$, i.e. the 10-dimensional spinor variety. If $\ell = 2$, we stop here, ending up with the case of $\ell = 2$ in the sequence (b4). On the other hand, if $\ell \ge 3$, the embedding $S_2 \subset \mathbb{P}V_2$ must be the spinor embedding by Theorems 3.5 and 4.21. Then S_3 is biregular to the Cayley plane $\mathbb{OP}^2$ by Corollary 6.9, giving $\ell = 3$ in (b4). It remains to show that $\ell \le 3$. If $\ell \ge 4$, then by (c) the embedding $S_3 \subset \mathbb{P}V_3$ must be the projection along a general point of the minimal embedding $\mathbb{OP}^2 \subset \mathbb{P}^{26}$, which has no prolongation by Proposition 4.13, a contradiction.

In Case (iii), if $\dim V \ge 6$, then Theorem 6.16 contradicts the property (b) of S_1. Thus this case, corresponding to (B6) in Main Theorem, does not give rise to a sequence with $\ell \ge 1$. Now we consider the case $\dim V = 5$. We want to show that the sequence must be (b5). First, S_0 is isomorphic to a hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^2$ under the Segre embedding, which is the VMRT of

$\mathrm{Gr}_\omega(2,5)$ at a general point. Then By Corollary 6.9 and Proposition 6.19, this implies that S_1 is biregular to $\mathrm{Gr}_\omega(2,5)$ and we are done if $\ell = 1$. If $\ell \geq 2$, then from the property (c) of S_1, the embedding $S_1 \subset \mathbb{P}V_1$ must be the hyperplane section of $\mathrm{Gr}(2,5)$ under the Plücker embedding. By Corollary 6.9 and Proposition 6.18, this implies that S_2 is biregular to a hyperplane section of the 10-dimensional spinor variety and we are done if $\ell = 2$. So it remains to show that $\ell \leq 2$. Suppose $\ell \geq 3$, then from the property (c) of S_2 and Theorem 4.21, the embedding $S_2 \subset \mathbb{P}V_2$ must be the hyperplane section of the spinor embedding $\mathbb{S}_5 \subset \mathbb{P}^{15}$ and the cone structure on S_3 given by S_2 cannot be locally flat by Theorem 6.17. This contradicts the property (b) of S_3, completing the proof of the claim. □

7 Proof of Theorem 6.16

This section is devoted to the proof of Theorem 6.16.

To start with, let us recall some facts about Grassmannians. Let W be a complex vector space of dimension 2 and let Q be a complex vector space of dimension $m \geq 2$. Let $\mathrm{Gr}(2, W^* \oplus Q)$ be the Grassmannian of 2-dimensional subspaces in $W^* \oplus Q$. There exists a canonical embedding

$$W \otimes Q = \mathrm{Hom}(W^*, Q) \subset \mathrm{Gr}(2, W^* \oplus Q)$$

by associating to an element of $\mathrm{Hom}(W^*, Q)$ the plane in $W^* \oplus Q$ given by its graph. The next proposition is elementary.

Proposition 7.1 *Consider a $\mathbb{C}^\times$-action with weight* 0 *on W and weight* 1 *on Q. This induces a $\mathbb{C}^\times$-action on $\mathrm{Gr}(2, W^* \oplus Q)$ whose fixed point set consists of the following three components*:

(i) *the isolated point $[W^*]$ corresponding to the plane $W^* \subset (W^* \oplus Q)$*;
(ii) *the subvariety $\mathrm{Gr}(2, Q) \subset \mathrm{Gr}(2, W^* \oplus Q)$ consisting of planes of $W^* \oplus Q$ contained in Q*;
(iii) *the subvariety $\mathbb{P}W^* \times \mathbb{P}Q$ consisting of planes which can be written as the direct sum of a line in W^* and a line in Q.*

Moreover under this $\mathbb{C}^\times$-action, the orbit $\mathbb{C}^\times \cdot z$ of any point

$$z \in \mathrm{Gr}(2, W^* \oplus Q) \setminus ((W \otimes Q) \cup (\mathbb{P}W^* \times \mathbb{P}Q))$$

has a limit point in $\mathrm{Gr}(2, Q)$.

Next, we need to look at the geometry of a certain Grassmannian bundle on a Lagrangian Grassmannian. Let Σ be a symplectic vector space of dimension 4 and denote by $\mathrm{Sp}(\Sigma)$ (resp. $\mathfrak{sp}(\Sigma)$) the Lie group (resp. algebra) of symplectic automorphisms (resp. endomorphisms) of Σ. Let Lag(Σ) be the Lagrangian Grassmannian, i.e., the space of Lagrangian subspaces in Σ. This is homogeneous under Sp(Σ) and is biregular to the 3-dimensional hyperquadric $\mathbb{Q}^3$. Let $\mathcal{W}$ be the universal quotient bundle on Lag(Σ), i.e., the rank 2 vector bundle satisfying $\mathrm{Sym}^2\,\mathcal{W} = T(\mathrm{Lag}(\Sigma))$. Its dual bundle is the tautological bundle $\mathcal{W}^* \subset \Sigma \times \mathrm{Lag}(\Sigma)$ whose fiber over the point $[W^*] \in \mathrm{Lag}(\Sigma)$ corresponding to a Lagrangian subspace $W^* \subset \Sigma$ is W^* itself. Fix a vector space Q of dimension $m \geq 2$ and denote by $\mathcal{Q}$ the trivial vector bundle on Lag(Σ) with a fiber Q.

Proposition 7.2 *Let* $\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q})$ *be the Grassmannian bundle of* 2*-planes in the vector bundle* $\mathcal{W}^* \oplus \mathcal{Q}$*. Then the Lie algebra* $\mathfrak{g}$ *of the automorphism group of the projective variety* $\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q})$ *is isomorphic to* $(\Sigma^* \otimes Q) \rtimes (\mathfrak{sp}(\Sigma) \oplus \mathfrak{gl}(Q))$*. The vector bundle* $\mathcal{W} \otimes \mathcal{Q}$ *has a natural embedding into* $\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q})$ *whose complement*

$$D := \mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q}) \setminus (\mathcal{W} \otimes \mathcal{Q})$$

is a hypersurface consisting of 2*-planes in* $\mathcal{W}^* \oplus \mathcal{Q}$ *which have positive-dimensional intersection with* $\mathcal{Q}$*.*

Proof The group $\Sigma^* \otimes Q = \mathrm{Hom}(\Sigma, Q)$ acts on the vector space $\Sigma \oplus Q$ by the following rule: $f \cdot (x, y) = (x, y + f(x))$ for any $x \in \Sigma$, $y \in Q$ and $f \in \mathrm{Hom}(\Sigma, Q)$. This action preserves $\mathcal{W}^* \oplus \mathcal{Q} \subset (\Sigma \oplus Q) \times \mathrm{Lag}(\Sigma)$, inducing an action of $\Sigma^* \otimes Q$ on $\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q})$. From this, we can see there is a natural inclusion

$$(\Sigma^* \otimes Q) \rtimes (\mathfrak{sp}(\Sigma) \oplus \mathfrak{gl}(Q)) \subset \mathfrak{g}.$$

To show that this is an isomorphism, it suffices to compare their dimensions. Let

$$\psi : \mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q}) \to \mathrm{Lag}(\Sigma)$$

be the natural projection. We have an exact sequence

$$0 \to T^{\psi} \to T(\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q})) \to \psi^* T(\mathrm{Lag}(\Sigma)) \to 0, \tag{7.1}$$

where T^{ψ} denotes the relative tangent bundle. We have $\psi_* T^{\psi} = \mathrm{End}^0(\mathcal{W}^* \oplus \mathcal{Q})$, the bundle of traceless endomorphisms, and $R^i \psi_* T^{\psi} = 0$ for $i \geq 1$. Write

$$\psi_* T^{\psi} = \mathcal{F} \oplus (\mathcal{W}^* \otimes \mathcal{Q}^*) \oplus (\mathcal{W} \otimes \mathcal{Q}),$$

where $\mathcal{F}$ is given by the exact sequence

$$0 \to \mathcal{O} \to \mathrm{End}(\mathcal{W}^*) \oplus \mathrm{End}(\mathcal{Q}) \to \mathcal{F} \to 0. \tag{7.2}$$

Here the map $\mathcal{O} \to \mathrm{End}(\mathcal{W}^*) \oplus \mathrm{End}(\mathcal{Q})$ is given by $s \mapsto s\,\mathrm{Id}_{\mathcal{W}^*} \oplus s\,\mathrm{Id}_{\mathcal{Q}}$. It is well-known that

$$\begin{aligned}
&H^0(\mathrm{Lag}(\Sigma), \mathcal{W}) = \Sigma^*, \qquad H^0(\mathrm{Lag}(\Sigma), \mathcal{W}^*) = 0,\\
&H^0(\mathrm{Lag}(\Sigma), \mathrm{End}(\mathcal{W}^*)) = \mathbb{C},\\
&H^1(\mathrm{Lag}(\Sigma), \mathcal{W}) = H^1(\mathrm{Lag}(\Sigma), \mathcal{W}^*) = H^1(\mathrm{Lag}(\Sigma), \mathrm{End}(\mathcal{W}^*)) = 0,\\
&H^1(\mathrm{Lag}(\Sigma), \mathcal{O}) = H^2(\mathrm{Lag}(\Sigma), \mathcal{O}) = 0.
\end{aligned}$$

Thus by the long-exact sequence associated to (7.2), we have

$$H^0(\mathrm{Lag}(\Sigma), \mathcal{F}) = \mathfrak{gl}(Q), \qquad H^1(\mathrm{Lag}(\Sigma), \mathcal{F}) = 0$$

and consequently,

$$H^1(\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q}), T^{\psi}) = 0,$$

$$H^0(\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q}), T^{\psi}) = H^0(\mathrm{Lag}(\Sigma), \psi_* T^{\psi}) = (\Sigma^* \otimes Q) \oplus \mathfrak{gl}(Q).$$

Since $H^0(\mathrm{Lag}(\Sigma), T(\mathrm{Lag}(\Sigma)) = \mathfrak{sp}(\Sigma)$, the long-exact sequence associated to (7.1) shows that

$$\begin{aligned}
\dim \mathfrak{g} &= \dim H^0(\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q}), T(\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q}))\\
&= \dim((\Sigma^* \otimes Q) \rtimes (\mathfrak{sp}(\Sigma) \oplus \mathfrak{gl}(Q))).
\end{aligned}$$

Now the vector bundle $\mathcal{W} \otimes \mathcal{Q} = \mathrm{Hom}(\mathcal{W}^*, \mathcal{Q})$ can be regarded as a subset of $\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q})$ by associating to a homomorphism to its graph. The statement on the complement D is immediate. □

Proposition 7.3 *Let G be the simply connected group with Lie algebra $\mathfrak{g}$ of Proposition 7.2. The open subset $\mathcal{W} \otimes \mathcal{Q} \subset \mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q})$ is homogeneous under the action of G and has a natural isotrivial cone structure $\mathcal{C}$ invariant under the G-action such that each fiber $\mathcal{C}_x \subset \mathbb{P}T_x(\mathcal{W} \otimes \mathcal{Q})$ is isomorphic to $Z \subset \mathbb{P}((W \otimes Q) \oplus \mathrm{Sym}^2 W)$, the VMRT of the symplectic Grassmannian $\mathrm{Gr}_\omega(2, m+4)$ in the notation of Sect. 3.2. This cone structure $\mathcal{C}$ is locally flat and $\mathfrak{aut}(\mathcal{C}, x) \cong \mathfrak{g}$ for each point $x \in \mathcal{W} \otimes \mathcal{Q}$.*

Proof Note that the hypersurface D in Proposition 7.2 is invariant under the action of G, hence $\mathcal{W} \otimes \mathcal{Q}$ is also G-invariant. The base $\mathrm{Lag}(\Sigma)$ is homogeneous under the action of $\mathrm{Sp}(\Sigma)$. Let $W^* \subset \Sigma$ be a Lagrangian subspace with quotient $W = \Sigma/W^*$. The subgroup $\mathrm{Hom}(\Sigma, Q) \subset G$ acts on the fiber $W \otimes Q = \mathrm{Hom}(W^*, Q)$ of $\mathcal{W} \otimes \mathcal{Q}$ over $[W^*] \in \mathrm{Lag}(\Sigma)$ by translation via the restriction to $W^* \subset \Sigma$ of the action of $\mathrm{Hom}(\Sigma, Q)$ on $\Sigma \oplus Q$ described at the beginning of the proof of Proposition 7.2. This action is transitive on the fiber with isotropy subgroup

$$\{\kappa \in \mathrm{Hom}(\Sigma, Q) \mid \kappa(W^*) = 0\} \cong \mathrm{Hom}(\Sigma/W^*, Q).$$

This shows that $\mathcal{W} \otimes \mathcal{Q}$ is G-homogeneous.

Regard $\mathrm{Lag}(\Sigma)$ as a submanifold of $\mathcal{W} \otimes \mathcal{Q} \subset \mathrm{Gr}(\mathcal{W}^* \oplus \mathcal{Q})$ via the zero section of the vector bundle. The Lie algebra of the isotropy subgroup in $\mathrm{Sp}(\Sigma)$ of $[W^*] \in \mathrm{Lag}(\Sigma)$ is a parabolic subalgebra $\mathfrak{p}_{[W^*]} \subset \mathfrak{sp}(\Sigma)$. At the point $[W^*] \in \mathrm{Lag}(\Sigma) \subset \mathcal{W} \otimes \mathcal{Q}$, the isotropy subgroup $G_{[W^*]}$ has Lie algebra

$$\mathfrak{g}_{[W^*]} := (W^* \otimes Q) \rtimes (\mathfrak{p}_{[W^*]} \oplus \mathfrak{gl}(Q)) \ \subset \ (\Sigma^* \otimes Q) \rtimes (\mathfrak{sp}(\Sigma) \oplus \mathfrak{gl}(Q)) = \mathfrak{g}.$$

The tangent space at the point $[W^*] \in \mathrm{Lag}(\Sigma) \subset \mathcal{W} \otimes \mathcal{Q}$

$$T_{[W^*]}(\mathcal{W} \otimes \mathcal{Q}) = (W \otimes Q) \oplus T_{[W^*]}(\mathrm{Lag}(\Sigma)) = (W \otimes Q) \oplus \mathrm{Sym}^2 W$$

contains the affine cone $\hat{Z}$ in a natural way. The isotropy representation of $\mathfrak{g}_{[W^*]}$ on $T_{[W^*]}(\mathcal{W} \otimes \mathcal{Q}) = (W \otimes Q) \oplus \mathrm{Sym}^2 W$ satisfies

(1) $(W^* \otimes Q)$-component of $\mathfrak{g}_{[W^*]}$ acts trivially;
(2) $\mathfrak{p}_{[W^*]}$ acts naturally as $\mathfrak{gl}(W)$ on $W \otimes Q$ and on $\mathrm{Sym}^2 W$;
(3) $\mathfrak{gl}(Q)$ acts naturally on $W \otimes Q$ and trivially on $\mathrm{Sym}^2 W$.

Thus $\hat{Z}$ is preserved under the isotropy representation of the isotropy subgroup $G_{[W^*]}$ and the G-action defines a natural isotrivial cone structure $\mathcal{C}$ on the open set $\mathcal{W} \otimes \mathcal{Q}$ whose fiber is isomorphic to Z. As $\mathfrak{aut}(\hat{Z})^{(2)} = 0$ by Theorem 2.3, we have the following inequalities from Proposition 5.10 where $x = [W^*]$:

$$\dim \mathfrak{g} \leq \dim \mathfrak{aut}(\mathcal{C}, x) \leq \dim(\mathfrak{aut}(\hat{Z})^{(1)} \oplus \mathfrak{aut}(\hat{Z}) \oplus (W \otimes Q) \oplus \mathrm{Sym}^2 W).$$

From Proposition 3.8,

$$\dim(\mathfrak{aut}(\hat{Z})^{(1)} \oplus \mathfrak{aut}(\hat{Z}) \oplus (W \otimes Q) \oplus \mathrm{Sym}^2 W) = m^2 + 4m + 10 = \dim \mathfrak{g}$$

implying $\mathfrak{g} \cong \mathfrak{aut}(\mathcal{C}, x)$. Now Corollary 5.13 gives the local flatness of the cone structure $\mathcal{C}$. □

Now to prove Theorem 6.16, we will make the following assumption and derive a contradiction.

(*Assumption*) Let X be a Fano manifold with $\mathrm{Pic}(X) = \mathbb{Z} \cdot L$ for an ample line bundle L. Assume that X has minimal rational curves of degree 1 with respect to L whose VMRT at a general point is isomorphic to $Z \subset \mathbb{P}((W \otimes Q) \oplus \mathrm{Sym}^2 W)$ and the cone structure is locally flat.

Proposition 7.4 *Under* (Assumption), *the group G in Proposition* 7.2 *acts on X with an open orbit X_o such that the complement $X \setminus X_o$ has codimension ≥ 2. There exists a G-biregular morphism $\chi : \mathcal{W} \otimes \mathcal{Q} \to X_o$, sending the Z-isotrivial cone structure of Proposition* 7.3 *to the VMRT-structure on X_o, inducing a fibration $\rho : X_o \to \mathrm{Lag}(\Sigma)$.*

Proof Since the isotrivial cone structure on X is locally flat, it is locally equivalent to the cone structure $\mathcal{C}$ of Proposition 7.3. By Theorem 6.8, we have $\mathfrak{aut}(X) = \mathfrak{aut}(\mathcal{C}, x) = \mathfrak{g}$ for $x \in \mathcal{W} \otimes \mathcal{Q}$, which implies that the group G acts on X with an open orbit X_o. As $\mathcal{W} \otimes \mathcal{Q}$ is simply connected, we have a G-equivariant unramified covering morphism $\chi : \mathcal{W} \otimes \mathcal{Q} \to X_o$. The image of the zero-section $\mathrm{Lag}(\Sigma) \subset (\mathcal{W} \otimes \mathcal{Q})$ is a positive-dimensional subvariety in X_o. Thus the complement $X \setminus X_o$ must be of codimension ≥ 2 because X has Picard number 1. In particular, X_o is simply connected and the morphism $\chi : \mathcal{W} \otimes \mathcal{Q} \to X_o$ is biregular. It certainly preserves the cone structure. The fibration $\mathcal{W} \otimes \mathcal{Q} \to \mathrm{Lag}(\Sigma)$ induces a fibration $\rho : X_o \to \mathrm{Lag}(\Sigma)$. □

The following lemma is elementary. See Proposition 4.4 of [13] for a proof.

Lemma 7.5 *Let Y_1 be a Fano manifold of Picard number one. Let Y_2 be a compact complex manifold. Assume there exist subsets $E_i \subset Y_i, i = 1, 2$, of codimension ≥ 2 and a biholomorphic morphism $\varphi : Y_2 \setminus E_2 \to Y_1 \setminus E_1$. Then φ can be extended to a biholomorphic morphism $\tilde{\varphi} : Y_2 \to Y_1$.*

The proof of the next proposition is essentially contained in the proof of Proposition 6.3.3 of [14]. We recall the proof for the reader's convenience.

Proposition 7.6 *Let $\rho : X_o \to \mathrm{Lag}(\Sigma)$ be as in Proposition* 7.4. *Given a point $[W^*] \in \mathrm{Lag}(\Sigma)$, the closure in X of the fiber $\rho^{-1}([W^*])$ is a projective submanifold biregular to the Grassmannian $\mathrm{Gr}(2, W^* \oplus Q)$ such that $\rho^{-1}([W^*]) \subset \mathrm{Gr}(2, W^* \oplus Q)$ is isomorphic to $W \otimes Q \subset \mathrm{Gr}(2, W^* \oplus Q)$. Consequently, the biregular morphism*

$$\chi : \mathcal{W} \otimes \mathcal{Q} = \mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q}) \setminus D \to X_o$$

in Proposition 7.4 *can be extended to a morphism $\tilde{\chi} : \mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q}) \to X$.*

Proof As in Proposition 7.3, regard $\mathrm{Lag}(\Sigma)$ as a submanifold of

$$\mathcal{W} \otimes \mathcal{Q} \subset \mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q}).$$

From the description of the isotropy subgroup $G_{[W^*]}$ in the proof of Proposition 7.3, we see that $G_{[W^*]}$ contains a subgroup isomorphic to $\mathrm{GL}(W) \times \mathrm{GL}(Q)$. Choose $\mathbb{C}^\times \subset \mathrm{GL}(W) \times \mathrm{GL}(Q)$ with weight 1 on W and weight -1

on Q. Then it has weight 0 on $W \otimes Q$ and weight 2 on $\mathrm{Sym}^2 W$. It follows that this $\mathbb{C}^\times$ action on $\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q})$ fixes the point $[W^*]$ and the isotropy action on

$$T_{[W^*]}(\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q})) = (W \otimes Q) \oplus \mathrm{Sym}^2 W$$

fixes exactly $W \otimes Q$. Thus the fixed point set of this $\mathbb{C}^\times$-action on $\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q})$ has the fiber $\mathrm{Gr}(2, W^* \oplus Q)$ as a connected component. Consequently, the corresponding $\mathbb{C}^\times$-action on X has the closure $S \subset X$ of the fiber $\rho^{-1}([W^*])$ as a connected component of its fixed point set. Since the fixed point set of a $\mathbb{C}^\times$-action on the projective manifold X is nonsingular, the closure S is a projective submanifold.

To show that this submanifold S is biregular to the Grassmannian, we need to show that the birational map $\delta : S \dashrightarrow \mathrm{Gr}(2, W^* \oplus Q)$ induced by $\chi^{-1} : X_o \to \mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q})$ is biholomorphic. This is essentially Lemma 6.3.2 in [14]. Let us recall the argument.

For $Z \subset \mathbb{P}((W \otimes Q) \oplus \mathrm{Sym}^2 W)$, let $Z' = Z \cap \mathbb{P}(W \otimes Q)$, which is equivalent to a Segre embedding of $\mathbb{P}W \times \mathbb{P}Q$. The Z-isotrivial cone structure on $\mathcal{W} \otimes \mathcal{Q} \subset \mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q})$ induces a Z'-isotrivial cone structure on $\mathrm{Gr}(2, W^* \oplus Q)$. This cone structure is exactly the VMRT of lines on the Grassmannian. The Z-isotrivial cone structure on X_o also induces a Z'-isotrivial cone structure on $\rho^{-1}([W^*])$. This cone structure is the VMRT of S given by the minimal rational curves of X lying on S. The map δ induces an isomorphism of these Z'-isotrivial cone structures. Thus δ sends minimal rational curves of X lying on S to lines in the Grassmannian $\mathrm{Gr}(2, W^* \otimes Q)$.

Let $\mathcal{H} \subset S$ be the union of hypersurfaces where δ is ramified (note that δ is always well-defined in codimension 1). Suppose $\mathcal{H} \neq \emptyset$. Let A be the proper image of $\mathcal{H}$ under δ. Then A is a subset of codimension ≥ 2 in $\mathrm{Gr}(2, W^* \oplus Q)$. Choose a family of minimal rational curves $\{\ell_s \mid s \in \Delta\}$ on $\mathrm{Gr}(2, W^* \oplus Q)$ such that ℓ_0 intersects A but is not contained in A; all ℓ_s with $s \neq 0$ are disjoint from A and are the strict images of a family of minimal rational curves $C_s, s \neq 0$, on S. Then the limit C_0 is an irreducible curve because C_0 has degree 1 with respect to the line bundle L on X. This implies that the proper image of C_0 must be ℓ_0 and C_0 intersects $\mathcal{H}$. But $C_s, s \neq 0$ is disjoint from $\mathcal{H}$. Thus we have a family of irreducible curves $C_s, s \in \Delta$, on the projective manifold S and a hypersurface $\mathcal{H} \subset S$ such that $C_0 \cdot \mathcal{H} \neq 0$ but $C_s \cdot \mathcal{H} = 0$ for $s \neq 0$, a contradiction. We conclude that $\mathcal{H} = \emptyset$.

Since $\mathcal{H} = \emptyset$, we see that δ is unramified outside a subset $E \subset S$ of codimension ≥ 2. The image $\delta(S \setminus E) \subset \mathrm{Gr}(2, W^* \oplus Q)$ is not an affine subset, because $S \setminus E$ contains projective curves (general minimal rational curves of S). But $\delta(S \setminus E)$ contains the open subset $W \otimes Q \subset \mathrm{Gr}(2, W^* \oplus Q)$ and its complement $\mathrm{Gr}(2, W^* \oplus Q) \setminus (W \otimes Q)$ is an irreducible hypersurface. Thus the complement $\mathrm{Gr}(2, W^* \oplus Q) \setminus \delta(S \setminus E)$ is of codimension ≥ 2. By Lemma 7.5, δ extends to a biregular morphism $S \to \mathrm{Gr}(2, W^* \oplus Q)$. □

Proposition 7.7 *In the setting of Proposition 7.6, let* $\mathrm{Gr}(2,\mathcal{Q}) \subset \mathrm{Gr}(2,\mathcal{W}^*\oplus\mathcal{Q})$ *be the trivial fiber subbundle whose fiber over* $[W^*] \in \mathrm{Lag}(\Sigma)$ *corresponds to* $\mathrm{Gr}(2,Q) \subset \mathrm{Gr}(2,W^*\oplus Q)$ *of Proposition 7.1. Then the restriction* $\tilde{\chi}|_{\mathrm{Gr}(2,\mathcal{Q})}$ *agrees with the projection*

$$\mathrm{Gr}(2,\mathcal{Q}) = \mathrm{Gr}(2,Q) \times \mathrm{Lag}(\Sigma) \to \mathrm{Gr}(2,Q).$$

Proof The center of $\mathrm{GL}(Q) \subset G$, which is isomorphic to $\mathbb{C}^\times$, acts on $\mathrm{Gr}(2,\mathcal{W}^*\oplus\mathcal{Q})$ such that on each fiber it induces the $\mathbb{C}^\times$-action of Proposition 7.1. From Proposition 7.1, $\mathrm{Gr}(2,\mathcal{Q})$ is a component of the fixed point set of this action such that all general orbits in the divisor D have limit points in $\mathrm{Gr}(2,\mathcal{Q})$.

The morphism $\tilde{\chi} : \mathrm{Gr}(2,\mathcal{W}^*\oplus\mathcal{Q}) \to X$ defined in Proposition 7.6 sends the divisor D to $X \setminus X_o$, a subset of codimension ≥ 2 in X from Proposition 7.4. Let $A \subset D$ be a general fiber of the contraction $\tilde{\chi}|_D : D \to X \setminus X_0$. The limit of A under the $\mathbb{C}^\times$-action contains a positive-dimensional subvariety A' in $\mathrm{Gr}(2,\mathcal{Q})$. By the $\mathbb{C}^\times$-equivariance, A' must be contracted by $\tilde{\chi}$, too. But the action of $\mathrm{GL}(Q) \subset G$ is transitive on $\mathrm{Gr}(2,\mathcal{Q}_{[W^*]}) = \mathrm{Gr}(2,Q)$ for each $[W^*] \in \mathrm{Lag}(\Sigma)$. Thus $\tilde{\chi}(\mathrm{Gr}(2,\mathcal{Q}))$ has dimension strictly less than that of $\mathrm{Gr}(2,\mathcal{Q})$, i.e., $\mathrm{Gr}(2,\mathcal{Q})$ is contracted by $\tilde{\chi}$. By the definition of $\tilde{\chi}$ in Proposition 7.6, the line bundle $\tilde{\chi}^*L$ is ample on the $\mathrm{Gr}(2,Q)$-factor of

$$\mathrm{Gr}(2,\mathcal{Q}) = \mathrm{Gr}(2,Q) \times \mathrm{Lag}(\Sigma).$$

Thus the fibers of $\tilde{\chi}|_{\mathrm{Gr}(2,\mathcal{Q})}$ must be contained in the $\mathrm{Lag}(\Sigma)$-factor. Since $\mathrm{Lag}(\Sigma)$ has Picard number 1, $\tilde{\chi}$ must contract $\mathrm{Lag}(\Sigma)$ to one point. □

Proposition 7.8 *Pick a subspace* $Q' \subset Q$ *of dimension* 2, *defining a fiber subbundle* $\mathrm{Gr}(2,\mathcal{W}^*\oplus\mathcal{Q}')$ *of* $\mathrm{Gr}(2,\mathcal{W}^*\oplus\mathcal{Q})$. *Then the image of* $\mathrm{Gr}(2,\mathcal{W}^*\oplus\mathcal{Q}')$ *under* $\tilde{\chi}$ *is a* 7*-dimensional projective submanifold* $X' \subset X$ *such that the restriction of* $\tilde{\chi}$ *to* $\mathrm{Gr}(2,\mathcal{W}^*\oplus\mathcal{Q}')$

$$\mu : \mathrm{Gr}(2,\mathcal{W}^*\oplus\mathcal{Q}') \to X'$$

sends each fiber of $\psi : \mathrm{Gr}(2,\mathcal{W}^*\oplus\mathcal{Q}') \to \mathrm{Lag}(\Sigma)$ *isomorphically to a projective submanifold of* X' *and contracts the submanifold*

$$[Q'] \times \mathrm{Lag}(\Sigma) = \mathrm{Gr}(2,\mathcal{Q}') \subset \mathrm{Gr}(2,\mathcal{W}^*\oplus\mathcal{Q}')$$

to one point in X'.

Proof From Propositions 7.6 and 7.7, all are obvious except the smoothness of the image X'. To see this, fix a decomposition $Q = Q' \oplus Q''$ and choose a copy of $\mathbb{C}^\times \subset \mathrm{GL}(Q)$ which acts with weight 0 on Q' and weight 1 on Q''.

The induced $\mathbb{C}^\times$-action on $\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q})$ has $\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q}')$ as a component of its fixed point set. Since the morphism $\tilde{\chi}$ is equivariant under this $\mathbb{C}^\times$-action on $\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q})$ and the corresponding $\mathbb{C}^\times$-action on X, the image X' is a component of the fixed point set of this $\mathbb{C}^\times$-action. Thus X' is nonsingular. □

End of the proof of Theorem 6.16 Let $\iota \in X'$ be the image $\mu(\mathrm{Gr}(2, \mathcal{Q}'))$ in Proposition 7.8. The group $\mathrm{Sp}(\Sigma)$, with Lie algebra $\mathfrak{sp}(4) = \mathfrak{so}(5)$ acts on $\mathrm{Gr}(2, \mathcal{W}^* \oplus \mathcal{Q}')$ preserving $\mathrm{Gr}(2, \mathcal{Q}')$. Thus it acts on X' with ι fixed, inducing the isotropy representation of $\mathfrak{so}(5)$ on $T_\iota(X')$. This representation is non-trivial as a non-trivial action of a reductive group gives a non-trivial isotropy action on the tangent space of a fixed point. As non-trivial irreducible representations of $\mathfrak{so}(5)$ of dimension ≤ 7 can be either of dimension 4 (the spin representation) or 5 (the standard representation), the fixed point set of this $\mathfrak{so}(5)$-action has a component $E \subset X'$ with $\dim E = 3$ or 2 through ι.

For any $[W^*] \in \mathrm{Lag}(\Sigma)$, the isotropy subgroup in $\mathrm{Sp}(\Sigma)$ contains the subgroup $\mathrm{GL}(W^*)$, which acts in a natural way on $\mathrm{Gr}(2, W^* \oplus Q')$. The fixed point set of this $\mathrm{GL}(W^*)$-action consists of two isolated points: $[W^*]$ and $[Q']$. As $\mathrm{Gr}(2, W^* \oplus Q')$ is mapped isomorphically and equivariantly to a projective submanifold of X', the germ of E at ι intersects this image submanifold only at the point ι. As this is true for all $[W^*] \in \mathrm{Lag}(\Sigma)$ and the union of all such images is X', we deduce that $E = \iota$, a contradiction to the dimension of E. □

8 Proof of Theorem 6.17

This section is devoted to the proof of Theorem 6.17. The argument is overall parallel to that of Sect. 7, replacing Grassmannians by hyperquadrics. In fact, Proposition 8.i is a direct analogue of Proposition 7.i, etc.

To start with, let us recall some facts about hyperquadrics. By an orthogonal vector space we mean a vector space U equipped with a non-degenerate quadratic form β. Given an orthogonal vector space, the hyperquadric $\mathbb{Q}(U) \subset \mathbb{P}U$ is the set of null-vectors, i.e., its affine cone is

$$\widehat{\mathbb{Q}(U)} := \{u \in U \mid \beta(u, u) = 0\}.$$

Lemma 8.1 *Let S be an orthogonal vector space with a quadratic form α. Define a 2-dimensional orthogonal space $(\mathbb{C} \oplus \mathbb{C}, \gamma)$ by the multiplication $\gamma(s, t) = st \in \mathbb{C}$. The direct sum $(S \oplus (\mathbb{C} \oplus \mathbb{C}), \alpha \oplus \gamma)$ is an orthogonal space. Consider the hyperquadric $\mathbb{Q}(S \oplus (\mathbb{C} \oplus \mathbb{C}))$ of this orthogonal space. There is a natural embedding of S into $\mathbb{Q}(S \oplus (\mathbb{C} \oplus \mathbb{C}))$ as a Zariski open subset whose complement D is an irreducible divisor defined by $t = 0$. The divisor*

D has a unique singular point, to be denoted by Γ. The $\mathbb{C}^\times$-action on S given by the scalar multiplication extends to a $\mathbb{C}^\times$-action on $\mathbb{Q}(S \oplus (\mathbb{C} \oplus \mathbb{C}))$ such that a general orbit in D has Γ as a limit point.

Proof Choose coordinates $z_1, \dots, z_n$ on S with respect to which the quadratic form α is given by $z_1^2 + \cdots + z_n^2$. In terms of the homogeneous coordinates $[z_1, \dots, z_n, s, t]$ on $\mathbb{P}(S \oplus (\mathbb{C} \oplus \mathbb{C}))$, the hyperquadric $\mathbb{Q}(S \oplus (\mathbb{C} \oplus \mathbb{C}))$ is defined by

$$z_1^2 + \cdots + z_n^2 + st = 0.$$

The open embedding of S is given by

$$t = 1, \qquad s = -z_1^2 - \cdots - z_n^2.$$

Its complement is the divisor D defined by $t = 0$ and D has a unique singular point

$$\Gamma := (z_1 = \cdots = z_n = t = 0).$$

The $\mathbb{C}^\times$-action of scalar multiplication on S is given in these coordinates as the action of $\lambda \in \mathbb{C}^\times$ by

$$(z_1, \dots, z_n, s, t) \mapsto (\lambda z_1, \dots, \lambda z_n, \lambda^2 s, t).$$

This certainly induces a $\mathbb{C}^\times$-action on $\mathbb{Q}(S \oplus (\mathbb{C} \oplus \mathbb{C}))$ preserving D. For any point $(z_1, \dots, z_n, s, 0) \in D$ with $s \neq 0$, the orbit

$$\{[\lambda z_1, \dots, \lambda z_n, \lambda^2 s, 0)], \lambda \in \mathbb{C}^\times\} = \{[\lambda^{-1} z_1, \dots, \lambda^{-1} z_n, s, 0], \lambda \in \mathbb{C}^\times\}$$

has Γ as a limit point as λ^{-1} approaches 0. □

Next we need to look at the geometry of a certain hyperquadric bundle over a 7-dimensional hyperquadric. Fix a 9-dimensional orthogonal vector space U. The hyperquadric $\mathbb{Q}(U)$ is a 7-dimensional projective manifold homogeneous under $\mathrm{SO}(U)$. The semi-simple part of the isotropy group at a point of $\mathbb{Q}(U)$ has Lie algebra $\mathfrak{so}(7)$. The 8-dimensional spin representation W of $\mathfrak{so}(7)$ induces a homogeneous vector bundle $\mathcal{S}^*$ of rank 8 on $\mathbb{Q}(U)$, called the *dual spinor bundle* and its dual is called the *spinor bundle* $\mathcal{S}$. See [18] for details.

Proposition 8.2

(i) *Denoting by L the ample generator of* $\mathrm{Pic}(\mathbb{Q}(U))$, *we have $\mathcal{S}^* \cong \mathcal{S} \otimes L$ and*

$$H^0(\mathbb{Q}(U), \mathcal{S}^* \otimes L^{-1}) = H^1(\mathbb{Q}(U), \mathcal{S}^* \otimes L^{-1}) = H^1(\mathbb{Q}(U), \mathcal{S}^*) = 0.$$

(ii) *For all* $i \geq 0$,

$$H^i\left(\mathbb{Q}(U), \bigwedge^2 \mathcal{S}^* \otimes L^{-1}\right) = 0.$$

(iii) *The global sections of* $\mathcal{S}^*$ *generate the vector bundle* $\mathcal{S}^*$ *and* $H^0(\mathbb{Q}(U), \mathcal{S}^*)$ *is the* 16*-dimensional spin representation of* $\mathfrak{so}(U) = \mathfrak{so}(9)$.

Proof Claim (i) and (iii) follow from Theorems 2.3 and 2.8 of [18]. We now prove claim (ii). Denoting by Q the standard 7-dimensional representation of $\mathfrak{so}(7)$, we have $\bigwedge^2 W \cong Q \oplus \bigwedge^2 Q$ as representations of $\mathfrak{so}(7)$. Let P be the isotropy group of a point on $\mathbb{Q}(U)$. As the center of P acts trivially on W, we see that, as P-representations, the highest weight of Q (resp. $\bigwedge^2 Q$) is λ_2 (resp. λ_3), where λ_i is the ith fundamental weight of the simple Lie algebra of type B_4. Note that the line bundle L^{-1} is induced by the representation of highest weight $-\lambda_1$. This gives that the bundle $\bigwedge^2 \mathcal{S}^* \otimes L^{-1}$ is a direct sum of two equivariant vector bundles with highest weights $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$. Let δ be the sum of all fundamental weights, then we see $\delta + \lambda_2 - \lambda_1$ and $\delta + \lambda_3 - \lambda_1$ contain no λ_1, i.e. these sums are singular weights. This implies $H^i(\mathbb{Q}(U), \bigwedge^2 \mathcal{S}^* \otimes L^{-1}) = 0$ for all $i \geq 0$ by Borel-Weil-Bott's theorem. □

Note that the spin representation of $\mathfrak{so}(7)$ carries an invariant non-degenerate quadratic form (e.g. [4], Exercise 20.38). Thus there exists a fiberwise non-degenerate quadratic form on $\mathcal{S}^*$ with values in a line bundle M on $\mathbb{Q}(U)$, i.e., $\mathrm{Sym}^2 \mathcal{S}^* \to M$ inducing an isomorphism $(\mathcal{S} \otimes M) \cong \mathcal{S}^*$. From Proposition 8.2, we have

$$\mathcal{S} \otimes M \cong \mathcal{S} \otimes L,$$

implying $M = L$. Consequently, we have a fiberwise non-degenerate quadratic form

$$\alpha : \mathrm{Sym}^2(\mathcal{S}^*) \to L.$$

On the other hand the natural multiplication $L \otimes \mathcal{O} \to L$, where $\mathcal{O} = \mathcal{O}_{\mathbb{Q}(U)}$, induces a fiberwise non-degenerate quadratic form

$$\gamma : \mathrm{Sym}^2(L \oplus \mathcal{O}) \to L.$$

Thus the vector bundle $\mathcal{S}^* \oplus (L \oplus \mathcal{O})$ of rank 10 is equipped with the fiberwise non-degenerate quadratic form

$$\alpha \oplus \gamma : \mathrm{Sym}^2(\mathcal{S}^* \oplus (L \oplus \mathcal{O})) \to L.$$

The associated hyperquadric bundle

$$\psi : \mathbb{Q}(\mathcal{S}^* \oplus (L \oplus \mathcal{O})) \to \mathbb{Q}(U)$$

is a fiber bundle on $\mathbb{Q}(U)$ whose fiber is an 8-dimensional hyperquadric. We will denote this projective manifold $\mathbb{Q}(\mathcal{S}^* \oplus (L \oplus \mathcal{O}))$ by Y.

Proposition 8.3 *Let $\psi : Y \to \mathbb{Q}(U)$ be the hyperquadric bundle of the orthogonal vector bundle $\mathcal{S}^* \oplus (L \oplus \mathcal{O})$. Let $\Xi = H^0(\mathbb{Q}(U), \mathcal{S}^*)$ be the 16-dimensional spin representation of $\mathfrak{so}(U)$. Then the Lie algebra $\mathfrak{g}$ of the automorphism group of the projective variety Y is isomorphic to $\Xi \rtimes (\mathfrak{so}(U) \oplus \mathbb{C})$, where $\mathbb{C}$ corresponds to the scalar multiplication on the vector bundle $\mathcal{S}^*$. The vector bundle $\mathcal{S}^*$ has a natural embedding into $Y = \mathbb{Q}(\mathcal{S}^* \oplus (L \oplus \mathcal{O}))$. Its complement D is an irreducible divisor and the singular locus of D is a section Γ of ψ.*

Proof By Proposition 8.2 (iii), we have a surjective map $\Xi \otimes \mathcal{O} \to \mathcal{S}^*$, which gives for any $x \in \mathbb{Q}(U)$ a surjective map $\zeta_x : \Xi \to \mathcal{S}^*_x$. The vector group Ξ acts on $\mathcal{S}^* \oplus L \oplus \mathcal{O}$ by the following rule: for any $(v, s, t) \in \mathcal{S}^*_x \oplus L_x \oplus \mathcal{O}_x$ and any $f \in \Xi$,

$$f \cdot (v, s, t) = (v + t\zeta_x(f), s - 2\alpha(v, \zeta_x(f)) - t\alpha(\zeta_x(f), \zeta_x(f)), t).$$

One checks easily that this action preserves the quadratic form on $\mathcal{S}^* \oplus L \oplus \mathcal{O}$. This induces an action of Ξ on Y. From this, we see that there is a natural inclusion

$$\Xi \rtimes (\mathfrak{so}(U) \oplus \mathbb{C}) \subset \mathfrak{g}.$$

To show that this is an isomorphism, it suffices to compare their dimensions.

Let $\psi : Y \to \mathbb{Q}(U)$ be the natural projection. We have an exact sequence

$$0 \to T^\psi \to T(Y) \to \psi^* T(\mathbb{Q}(U)) \to 0, \tag{8.1}$$

where T^ψ denotes the relative tangent bundle. Recall that for an orthogonal vector space $\mathbb{C}^m$, there is a natural identification

$$H^0(\mathbb{Q}(\mathbb{C}^m), T(\mathbb{Q}(\mathbb{C}^m))) = \mathfrak{so}(\mathbb{C}^m) = \wedge^2 \mathbb{C}^m.$$

Translating it into relative setting, we get

$$\begin{aligned} \psi_* T^\psi &= \bigwedge^2 (\mathcal{S}^* \oplus (L \oplus \mathcal{O})) \otimes L^{-1} \\ &= \left(\left(\bigwedge^2 \mathcal{S}^*\right) \otimes L^{-1}\right) \oplus \mathcal{O} \oplus \mathcal{S}^* \oplus (\mathcal{S}^* \otimes L^{-1}). \end{aligned}$$

By $R^i \psi_* T^\psi = 0$ for $i \geq 1$ and Proposition 8.2, we have $H^1(Y, T^\psi) = 0$ and

$$H^0(Y, T^\psi) = H^0(\mathbb{Q}(U), \psi_* T^\psi) = H^0(\mathbb{Q}(U), \mathcal{O} \oplus \mathcal{S}^*) = \mathbb{C} \oplus \Xi.$$

Since $H^0(\mathbb{Q}(U), T(\mathbb{Q}(U)) = \mathfrak{so}(U)$, the long exact sequence associated to (8.1) shows that

$$\dim \mathfrak{g} = \dim H^0(Y, T(Y)) = \dim(\Xi \rtimes (\mathfrak{so}(U) \oplus \mathbb{C})).$$

Now the rest of Proposition 8.3 is a globalization of Lemma 8.1. The hyperquadric bundle $\psi : Y \to \mathbb{Q}(U)$ has a natural section $\Gamma \subset Y$ over $\mathbb{Q}(U)$ determined by

$$\Gamma := \mathbb{P}\mathcal{O} \subset \mathbb{Q}(\mathcal{S}^* \oplus (L \oplus \mathcal{O})) = Y$$

because the $\mathcal{O}$-factor of $\mathcal{S}^* \oplus (L \oplus \mathcal{O})$ is a null-vector with respect to the quadratic form $\alpha \oplus \gamma$. Given a point $v \in \mathcal{S}^*$, let $v' \in L$ be the unique vector defined by

$$\alpha(v, v) + \gamma(v', 1) = 0$$

where 1 denotes the section of $\mathcal{O}$ determined by the constant function 1 on $\mathbb{Q}(U)$. Then we have a canonical embedding of $\mathcal{S}^*$ into the hyperquadric bundle $Y = \mathbb{Q}(\mathcal{S}^* \oplus (L \oplus \mathcal{O}))$ as a Zariski open subset by

$$v \in \mathcal{S} \mapsto (v, (v', 1)).$$

Its complement is an irreducible divisor D determined by the zero section of $\mathcal{O}$ and Γ is the singular locus of D, which can be seen immediately from Lemma 8.1. □

Proposition 8.4 *Let G be the simply connected group with Lie algebra $\mathfrak{g}$ of Proposition 8.3. The open subset $\mathcal{S}^* \subset Y$ described in Proposition 8.3 is G-homogeneous and has a natural isotrivial cone structure $\mathcal{C}$ invariant under the G-action such that each fiber $\mathcal{C}_x \subset \mathbb{P}T_x(\mathcal{S}^*)$ is isomorphic to $Z \subset \mathbb{P}(W \oplus Q)$ in the notation of Sect. 3.3. This cone structure $\mathcal{C}$ is locally flat and $\mathfrak{aut}(\mathcal{C}, x) \cong \mathfrak{g}$ for each $x \in \mathcal{S}^*$.*

Proof It is easy to see that the open subset $\mathcal{S}^*$ is G-invariant. The base $\mathbb{Q}(U)$ is homogeneous under the action of $\mathrm{SO}(U)$. From the proof of Proposition 8.3, the vector group Ξ acts on the fiber $\mathcal{S}^*_x$ by translation of images of ζ_x, thus this action is transitive on the fibers of $\mathcal{S}^* \to \mathbb{Q}(U)$. This shows that $\mathcal{S}^*$ is G-homogeneous.

For a point $z \in \mathbb{Q}(U)$, the Lie algebra of the isotropy subgroup in $\mathrm{SO}(U)$ of z is a parabolic subalgebra $\mathfrak{p}_z \subset \mathfrak{so}(U)$. It is known that the reductive part of $\mathfrak{p}_z$ is isomorphic to $\mathfrak{co}(7)$. Regard $\mathbb{Q}(U)$ as a submanifold of $\mathcal{S}^* \subset Y$ via the zero section of the vector bundle. Let $\Xi_z := \mathrm{Ker}(\zeta_z) \subset \Xi$, which corresponds to the sections of $\mathcal{S}^*$ vanishing at z. At the point $z \in \mathbb{Q}(U)$ the isotropy subgroup G_z has Lie algebra

$$\mathfrak{g}_z := \Xi_z \rtimes (\mathfrak{p}_z \oplus \mathbb{C}) \subset \Xi \rtimes (\mathfrak{so}(U) \oplus \mathbb{C}) = \mathfrak{g}.$$

The tangent space

$$T_z(\mathcal{S}^*) = \mathcal{S}^*_z \oplus T_z(\mathbb{Q}(U)) \cong W \oplus Q$$

contains the affine cone $\hat{Z}$ in a natural way. The isotropy representation of $\mathfrak{g}_z = \Xi_z \rtimes (\mathfrak{p}_z \oplus \mathbb{C})$ on $T_z(\mathcal{S}^*) = W \oplus Q$ satisfies

(1) Ξ_z-component of $\mathfrak{g}_z$ acts trivially.
(2) $\mathfrak{p}_z$-factor acts as $\mathfrak{co}(7)$ in a natural way on W and on Q.
(3) The $\mathbb{C}$-factor has weight 1 on W and weight 0 on Q.

From Proposition 3.9, $\hat{Z}$ is preserved under the isotropy representation of the isotropy subgroup G_z. Thus the G-action defines a natural Z-isotrivial cone structure on the open set $\mathcal{S}^*$.

As $\mathfrak{aut}(\hat{Z})^{(2)} = 0$ by Theorem 2.3, we have the following inequalities from Proposition 5.10

$$\dim \mathfrak{g} \leq \dim \mathfrak{aut}(\mathcal{C}, x) \leq \dim(\mathfrak{aut}(\hat{Z})^{(1)} \oplus \mathfrak{aut}(\hat{Z}) \oplus W \oplus Q).$$

From Propositions 3.9 and 3.10,

$$\dim(\mathfrak{aut}(\hat{Z})^{(1)} \oplus \mathfrak{aut}(\hat{Z}) \oplus (W \otimes Q) \oplus \mathrm{Sym}^2 W) = 53 = \dim \mathfrak{g}$$

implying $\mathfrak{g} \cong \mathfrak{aut}(\mathcal{C}, z)$. Then Corollary 5.13 shows that the Z-isotrivial cone structure on $\mathcal{S}^* \subset Y$ is locally flat. □

Now to prove Theorem 6.17, we will make the following assumption and derive a contradiction.

(*Assumption*) Let X be a 15-dimensional Fano manifold with $\mathrm{Pic}(X) = \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$. Assume that X has minimal rational curves of degree 1 with respect to $\mathcal{O}_X(1)$ whose VMRT at a general point is isomorphic to $Z \subset \mathbb{P}(W \oplus Q)$ and the cone structure is locally flat.

Proposition 8.5 *Under* (Assumption), *the group G in Proposition 8.4 acts on X with an open orbit X_o such that the complement $X \setminus X_o$ has codimension ≥ 2. There exists a G-biregular morphism $\chi : \mathcal{S}^* \to X_o$, sending the Z-isotrivial cone structure of Proposition 8.4 to the Z-isotrivial VMRT cone structure on X. This induces a fibration $\rho : X_o \to \mathbb{Q}(U)$.*

Proof Since the Z-isotrivial cone structure on X is locally flat, it is locally isomorphic to the cone structure $\mathcal{C}$ of Proposition 8.4. By Theorem 6.8, we have $\mathfrak{aut}(X) = \mathfrak{aut}(\mathcal{C}, x) = \mathfrak{g}$ for $x \in \mathcal{S}^*$ general, which implies that the group G acts on X with an open orbit X_o. As $\mathcal{S}^*$ is simply connected, we have a G-equivariant unramified covering morphism $\chi : \mathcal{S}^* \to X_o$. The image of the zero-section $\mathbb{Q}(U) \subset \mathcal{S}^*$ is a positive-dimensional subvariety in X_o.

Thus the complement $X \setminus X_o$ must be of codimension ≥ 2 because X has Picard number 1. In particular, X_o is simply connected and the morphism $\chi : \mathcal{S}^* \to X_o$ is biregular. It certainly preserves the cone structure. The fibration $\psi : \mathcal{S}^* \to \mathbb{Q}(U)$ induces a fibration $\rho : X_o \to \mathbb{Q}(U)$. □

The proof of the next proposition is essentially the same as that of Proposition 8.3.4 of [14]. We recall the proof for the reader's convenience.

Proposition 8.6 *Let $\rho : X_o \to \mathbb{Q}(U)$ be as in Proposition 8.5. Given a point $z \in \mathbb{Q}(U)$, the closure in X of the fiber $\rho^{-1}(z)$ is a projective submanifold biregular to the hyperquadric $\psi^{-1}(z)$ such that $\rho^{-1}(z)$ corresponds to $\mathcal{S}^*_z \subset \psi^{-1}(z)$. Consequently, the biregular morphism*

$$\chi : \mathcal{S}^* = Y \setminus D \;\to\; X_o$$

in Proposition 8.5 can be extended to a morphism $\tilde{\chi} : Y \to X$.

Proof As in the proof of Proposition 8.4, regard $\mathbb{Q}(U)$ as a submanifold of $\mathcal{S}^* \subset Y$. From the description of the isotropy subgroup G_z in the proof of Proposition 8.4, we see that $\mathfrak{g}_z$ contains a subalgebra isomorphic to $\mathfrak{co}(Q) \subset \mathfrak{p}_z$, whose center has weight 1 on both W and Q. Also, there is a $\mathbb{C}$-factor in $\mathfrak{g}_z$ with weight 1 on W and 0 on Q. This implies that there exists a subgroup $\mathbb{C}^\times \subset G$ which acts with weight 1 on Q and weight 0 on W. It follows that this $\mathbb{C}^\times$ action on Y fixes the point z and the isotropy action on

$$T_x(Y) = W \oplus Q$$

fixes exactly W. Thus the fixed point set of this $\mathbb{C}^\times$-action on Y has the fiber $Y_z := \psi^{-1}(z)$ as a connected component. Consequently, the corresponding $\mathbb{C}^\times$-action on X has the closure $S_z \subset X$ of the fiber $\rho^{-1}(z)$ as a connected component of its fixed point set. Since the fixed point set of a $\mathbb{C}^\times$-action on the projective manifold X is nonsingular, the closure S_z is a projective submanifold.

To show that this submanifold S_z is biregular to the hyperquadric $\psi^{-1}(z)$, we need to show that the birational map $\delta : S_z \dashrightarrow \psi^{-1}(z)$ induced by $\chi^{-1} : X_o \to Y$ is biholomorphic.

Recall that $Z' = Z \cap \mathbb{P}W$ is a 6-dimensional hyperquadric $\mathbb{Q}(W)$ determined by the orthogonal structure on the 8-dimensional spin representation W. The Z-isotrivial cone structure on $\mathcal{S}^* \subset Y$ induces a Z'-isotrivial cone structure on the fiber $\mathcal{S}^*_z$. This cone structure is exactly the VMRT of lines on the hyperquadric. The Z-isotrivial cone structure on X_o also induces a Z'-isotrivial cone structure on $\rho^{-1}(z)$. This cone structure is the VMRT of S_z given by the minimal rational curves of X lying on S_z. The map δ induces an isomorphism of these Z'-isotrivial cone structures. Thus δ sends minimal rational curves of X lying on S_z to lines in the hyperquadric $\psi^{-1}(z)$. Then the same argument as in the proof of Proposition 7.6, shows that δ extends to a biregular morphism $S_z \to \psi^{-1}(z)$. □

Proposition 8.7 *In the setting of Proposition 8.6, let* $\Gamma \subset Y$ *be the section of* ψ *given by the singular locus of the divisor* D. *Then* $\tilde{\chi}(\Gamma)$ *is one point.*

Proof We can choose a subgroup $\mathbb{C}^\times \subset G$ which corresponds to the scalar multiplication of the vector bundle $\mathcal{S}^*$, corresponding to the $\mathbb{C}^\times$-action of Lemma 8.1 on each fiber of ψ. From Lemma 8.1, Γ is a component of the fixed point set of this action such that all general orbits in the divisor D have limit points in Γ.

The morphism $\tilde{\chi} : Y \to X$ defined in Proposition 8.6 sends the divisor D to $X \setminus X_o$, a subset of codimension ≥ 2 in X from Proposition 8.5. Let $A \subset D$ be a general fiber of the contraction $\tilde{\chi}|_D : D \to X \setminus X_0$. The limit of A under the $\mathbb{C}^\times$-action contains a positive-dimensional subvariety A' in Γ. By the $\mathbb{C}^\times$-equivariance, A' must be contracted by $\tilde{\chi}$, too. But Γ is an orbit of the action of a subgroup of G with Lie algebra $\mathfrak{so}(U) \subset \mathfrak{g}$. Thus Γ is contracted by $\tilde{\chi}$. Since $\Gamma \cong \mathbb{Q}(U)$ is of Picard number 1, $\tilde{\chi}$ must contract Γ to one point. □

End of the proof of Theorem 6.17 Let $\iota \in X$ be the image $\tilde{\chi}(\Gamma)$ in Proposition 8.7. The group Spin(9) $\subset G$ acts on Y preserving Γ. Thus it acts on X with ι fixed, inducing the isotropy representation of $\mathfrak{so}(9)$ on $T_\iota(X)$. This representation is non-trivial, because a non-trivial action of a reductive group gives a non-trivial isotropy action on the tangent space of a fixed point. Since an irreducible representation of $\mathfrak{so}(9)$ with dimension ≤ 15 must be the 9-dimensional standard representation, $T_\iota(X)$ decomposes as a $\mathfrak{so}(9)$-module into the sum of the orthogonal space U and a complementary subspace of dimension 6 where $\mathfrak{so}(9)$ acts trivially. This implies that the fixed point set of the Spin(9)-action on X has a component E of dimension 6 through ι.

For any $z \in \mathbb{Q}(U)$, the stabilizer of Spin(9) contains the subgroup Spin(7), which acts in a natural way on the hyperquadric $Y_z = \psi^{-1}(z)$. This action when restricted to $\mathcal{S}_z^*$ is the spin representation, which has no fixed point in $\mathcal{S}_z^*$ except the zero point z. This action on $D_z := D \cap Y_z$ has only one isolated fixed point, which is its singular point $\Gamma \cap Y_z$. As Y_z is mapped isomorphically and equivariantly to its image in X, the germ of E at ι intersects this image only at the point ι. As this holds for all z and the union of such images covers X, we deduce that $E = \iota$, a contradiction to $\dim E = 6$. □

9 Application to target rigidity

In this section, we will give an application of Main Theorem and Theorem 4.21 in the study of the following notion.

Definition 9.1 A projective variety X is said to have the *target rigidity property* if for any surjective morphism $f : Y \to X$, and any deformation

$$\{f_t : Y \to X, |t| < 1, f_0 = f\},$$

there exist automorphisms $\sigma_t : X \to X$ such that $f_t = \sigma_t \circ f$.

All projective varieties which are not uniruled have the target rigidity property (modulo étale factorizations) by [15]. All known examples of Fano manifolds of Picard number 1, except projective space, have the target rigidity property (cf. [8]). Some examples of nonsingular uniruled projective varieties of higher Picard number have been studied (e.g. [7]).

For nonsingular uniruled projective varieties, the target rigidity property is related to the following.

Definition 9.2 Let X be a nonsingular uniruled projective variety and let $\mathcal{K}$ be a minimal rational component. Let $\mathcal{C} \subset \mathbb{P}T(X)$ be the cone structure associated to the VMRT of $\mathcal{K}$. X is said to have the *Liouville property with respect to* $\mathcal{K}$ if for a general $x \in X$, every local vector field in $\mathfrak{aut}(\mathcal{C}, x)$ can be extended to a global holomorphic vector field on a Zariski open subset of X.

Example 9.3 In Example 6.11, suppose that $\mathfrak{aut}(\hat{Z})^{(1)} = 0$. Then for a general point $y \in Y$, we have $\mathfrak{aut}(\mathcal{C}^Y, y) \cong V \oplus \mathfrak{aut}(\hat{Z})$ by Proposition 5.14. It is obvious that the subgroup of the affine group with Lie algebra $V \oplus \mathfrak{aut}(\hat{Z})$ acts on Y. Thus $\mathfrak{aut}(\mathcal{C}^Y, y)$ can be extended to a global holomorphic vector fields on Y and Y has the Liouville property.

Proposition 9.4 *Let X, X' be two nonsingular uniruled projective varieties with minimal components $\mathcal{K}, \mathcal{K}'$, respectively. Assume that X' has the Liouville property with respect to $\mathcal{K}'$ and there exists a generically finite rational map $\Phi : X \dashrightarrow X'$ sending the cone structure given by the VMRT of $\mathcal{K}$ to that of $\mathcal{K}'$ at general points. Then X has the Liouville property with respect to $\mathcal{K}$.*

Proof Φ induces an isomorphism $\mathfrak{aut}(\mathcal{C}, x) \cong \mathfrak{aut}(\mathcal{C}', \Phi(x))$ when $\mathcal{C}, \mathcal{C}'$ denote the cone structures and $x \in X$ is a general point. By the Liouville property for X', elements of $\mathfrak{aut}(\mathcal{C}', \Phi(x))$ can be extended to a global holomorphic vector field on a Zariski open subset of X'. By pulling it back to X via Φ, we get a global holomorphic vector field on a Zariski open subset of X. $\square$

It turns out that the target rigidity follows from the Liouville property:

Proposition 9.5 *Let X be a nonsingular uniruled projective variety which has the Liouville property with respect to a minimal component $\mathcal{K}$. Then X has the target rigidity property.*

Proof By the Stein factorization, it is easy to see that it suffices to check the condition in Definition 9.1 for generically finite surjective morphisms $\{f_t : Y \to X, |t| < 1\}$ (cf. [15], Sect. 2.2 for details). Let $\tau \in H^0(Y, f^*T(X))$ be the Kodaira-Spencer class of the deformation f_t at 0. As f_t is generically finite, we can regard τ as a multi-valued holomorphic vector field on X. It suffices to show that τ is univalent on X, namely, the germ of τ at a general point of X can be extended by analytic continuation to a global vector field on a Zariski open subset of X. In fact, this implies that $\tau \in f^*H^0(X, T(X))$ and the integration of τ generates the required automorphisms of X.

Let $\mathcal{C}$ be the cone structure given by the VMRT of X, for which the Liouville property holds. Take an analytic open subset near a general point $U \subset Y$ such that $f_t|_U : U \to f_t(U)$ is biholomorphic for $|t| < \epsilon$, then $\tau|_U$ can be regarded as a vector field on $f(U)$. By Proposition 3 in [12], there are countably many subvarieties $\mathcal{D}_i \subset \mathbb{P}T(Y), i = 1, 2, \ldots,$ (called varieties of distinguished tangents in [12]) such that for any generically finite morphism $h : Y \to X$ and the dominant rational map $dh : \mathbb{P}T(Y) \dashrightarrow \mathbb{P}T(X)$ defined by the differential of h, the proper inverse image $dh^{-1}(\mathcal{C}|_{h(U)})$ coincides with some $\mathcal{D}_i$. As the family $df_t^{-1}(\mathcal{C}), |t| < \epsilon$ is uncountable, we have $(df_t^{-1}(\mathcal{C}))|_U = (df_0^{-1}(\mathcal{C}))|_U$ for all t small. This implies that $\tau|_{f(U)}$ preserves $\mathcal{C}$, i.e. its germ at $x \in f_0(U)$ is an element of $\mathfrak{aut}(\mathcal{C}, x)$. By the Liouville property, this local vector field comes from a global vector field on a Zariski open subset of X. □

We will apply this to the following setting.

Theorem 9.6 *Let $S \subset \mathbb{P}V$ be a linearly normal nonsingular non-degenerate projective variety such that* $\mathrm{Sec}(S) \neq \mathbb{P}V$. *On the blow-up* $\mathrm{Bl}_S(\mathbb{P}V)$, *the proper transforms of lines on $\mathbb{P}V$ intersecting S determine a minimal rational component $\mathcal{K}$. Then* $\mathrm{Bl}_S(\mathbb{P}V)$ *has the Liouville property with respect to $\mathcal{K}$. In particular, it has the target rigidity property.*

To prove Theorem 9.6, we note the following two properties of $\mathcal{K}$. The proof of the first one is immediate.

Proposition 9.7 *In the setting of Theorem 9.6, if $x \in \mathrm{Bl}_S(\mathbb{P}V)$ is a general point corresponding to a point (using the same symbol by abuse of notation) $x \in \mathbb{P}V \setminus \mathrm{Sec}(S)$, the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at x is isomorphic to the subvariety in $\mathbb{P}T_x(\mathbb{P}V)$ consisting of tangents to lines joining x to S. Denoting by $p_x : S \to \mathbb{P}(V/\hat{x})$ the biregular projection (cf. Notation 4.1 and Lemma 4.2), the VMRT at x is isomorphic to $p_x(S) \subset \mathbb{P}(V/\hat{x})$.*

Proposition 9.8 *In the setting of Theorem 9.6, for any effective divisor D on* $\mathrm{Bl}_S(\mathbb{P}V)$, *a member of $\mathcal{K}$ has positive intersection with D.*

Proof If D is the exceptional divisor of the blow-up, this is obvious. We may assume that the image $\bar{D} \subset \mathbb{P}V$ of D is a hypersurface. It suffices to show that a general line intersecting S must contain a point of $\bar{D}$ outside S. Suppose otherwise. Pick a general point $x \in S$. Since a general line through x intersects $\bar{D}$ only at x, $p_x(\bar{D})$ is a divisor in $\mathbb{P}(V/\hat{x})$, i.e., $\bar{D}$ is a cone with vertex at x. But this should be true for all general $x \in S$, a contradiction to the non-degeneracy of S. □

To prove Theorem 9.6, let us recall the following consequence of Main Theorem and Theorem 4.21.

Proposition 9.9 *Let $S \subset \mathbb{P}V$ be a nonsingular non-degenerate projective variety with* $\mathrm{Sec}(S) \neq \mathbb{P}V$. *Then* $\mathfrak{aut}(\widehat{p_x(S)})^{(1)} = 0$ *for a general point $x \in \mathbb{P}V$.*

Proof Suppose that $\mathfrak{aut}(\widehat{p_x(S)})^{(1)} \neq 0$. From Main Theorem, $p_x(S) \subset \mathbb{P}(V/\hat{x})$ must be a biregular projection of the linearly normal embedding $S \subset \mathbb{P}W$, $W = H^0(S, \mathcal{O}(1))^*$, of the varieties in (A1), (A2), (A3) or (B3) in Main Theorem. Since x is general, it is a biregular projection from a subspace $L \subset W$ passing through a general point. This contradicts Theorem 4.21. □

We are ready to prove Theorem 9.6.

Proof of Theorem 9.6 Let $\mathcal{C}$ be the cone structure defined by the VMRT on general points of $X := \mathrm{Bl}_S(\mathbb{P}V)$. We can regard it as a cone structure on the open subset $M := \mathbb{P}V \setminus \mathrm{Sec}(S)$ as described in Proposition 9.7.

By Proposition 9.9, we have $\mathfrak{aut}(\widehat{p_x(S)})^{(1)} = 0$ for a general point $x \in \mathbb{P}V$. From Proposition 5.10, this implies that for a general point $x \in M$,

$$\dim(\mathfrak{aut}(\mathcal{C}, x)) \leq \delta(\mathcal{C}) + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x).$$

By Lemma 4.7, for the cone structure $\mathcal{C}$, the isotrivial leaf through a point x is exactly the orbit of x under the projective automorphism group $\mathrm{Aut}(S)$, which implies that

$$\delta(\mathcal{C}) = \dim \mathrm{Aut}(S) \cdot x = \dim \mathrm{Aut}(S) - \dim \mathrm{Aut}(S, x),$$

where $\mathrm{Aut}(S, x) \subset \mathrm{Aut}(S)$ is the isotropy subgroup at x. Since $\dim \mathfrak{aut}(\hat{\mathcal{C}}_x) = \dim \mathrm{Aut}(S, x) + 1$ by Lemma 4.7, we have $\dim(\mathfrak{aut}(\mathcal{C}, x)) \leq \dim \mathrm{Aut}(S) + 1$. But we have a natural inclusion $\mathfrak{aut}(S) \subset \mathfrak{aut}(\mathcal{C}, x)$ because the action of $\mathrm{Aut}(S)$ preserves the cone structure $\mathcal{C}$ on M. Thus

$$\dim \mathrm{Aut}(S) \leq \dim(\mathfrak{aut}(\mathcal{C}, x)) \leq \dim \mathrm{Aut}(S) + 1.$$

If $\dim \mathrm{Aut}(S) = \dim(\mathfrak{aut}(\mathcal{C}, x))$, then $\mathfrak{aut}(S) \cong \mathfrak{aut}(\mathcal{C}, x)$ and elements of $\mathfrak{aut}(\mathcal{C}, x)$ extend to global vector fields on X by the action of $\mathrm{Aut}(S)$ on X.

Thus we may assume that $\dim(\mathfrak{aut}(\mathcal{C}, x)) = \dim \mathrm{Aut}(S) + 1$. The assumption means that the equality holds in

$$\dim(\mathfrak{aut}(\mathcal{C}, x)) \leq \delta(\mathcal{C}) + \dim \mathfrak{aut}(\hat{\mathcal{C}}_x).$$

Then by Corollary 5.13, the cone structure is Z-isotrivial and locally flat, for some $Z \subset \mathbb{P}V$. By Proposition 9.8, we can apply Corollary 6.12 to get a rational map $\Phi : X \dashrightarrow Y$ inducing $\Phi_* : \mathfrak{aut}(\mathcal{C}, x) \cong \mathfrak{aut}(\mathcal{C}^Y, \Phi(x))$ for a general point $x \in X$. Since

$$\mathfrak{aut}(\hat{Z})^{(1)} = \mathfrak{aut}(\widehat{p_x(S)})^{(1)} = 0,$$

Y has the Liouville property from Example 9.3. Thus we are done by Proposition 9.4. □

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