

SPECIAL BIRATIONAL TRANSFORMATIONS OF TYPE (2, 1)

BAOHUA FU AND JUN-MUK HWANG

ABSTRACT. A birational transformation $\Phi : \mathbb{P}^n \dashrightarrow Z \subset \mathbb{P}^N$, where $Z \subset \mathbb{P}^N$ is a nonsingular variety of Picard number 1, is called a *special birational transformation of type (a, b)* if Φ is given by a linear system of degree a , its inverse Φ^{-1} is given by a linear system of degree b and the base locus $S \subset \mathbb{P}^n$ of Φ is irreducible and nonsingular. In this paper, we classify special birational transformations of type (2, 1). In addition to previous works [AS] and [R2] on this topic, our proof employs natural \mathbb{C}^* -actions on Z in a crucial way. These \mathbb{C}^* -actions also relate our result to the problem studied in [FH].

1. INTRODUCTION

Recall (e.g. Section 2 in [AS] or Definition 4.1 in [R2]) that a birational transformation

$$\Phi : \mathbb{P}^n \dashrightarrow Z \subset \mathbb{P}^N$$

where $Z \subset \mathbb{P}^N$ is a nonsingular projective variety of Picard number 1 is called a *special birational transformation of type (a, b)* if

- (1) the base locus $S \subset \mathbb{P}^n$ of Φ is irreducible and nonsingular;
- (2) the rational map Φ is given by a linear system belonging to $\mathcal{O}_{\mathbb{P}^n}(a)$; and
- (3) the inverse rational map Φ^{-1} is given by a linear system belonging to $\mathcal{O}_Z(b)$.

When $Z = \mathbb{P}^n$, this is a special Cremona transformation, a classical topic in projective algebraic geometry. It is a challenging problem to classify special birational transformations. Even for special Cremona transformations, a complete classification is still missing. Special Cremona transformations of type (2, 2) have been classified by Ein and Shepherd-Barron in [ES] by relating them to Severi varieties classified by Zak ([Z]). In [R2], special Cremona transformations of types (2, 3) and (2, 5) have been classified. Recently Alzati and Sierra ([AS]) have

Baohua Fu is supported by National Scientific Foundation of China (11225106 and 11321101).

extended [ES] to a classification of special birational transformations of type $(2, 2)$ for a wider class of Z .

In this paper, we will give a complete classification of special birational transformations of type $(2, 1)$. This classification can be described in terms of the classification of the base locus $S \subset \mathbb{P}^n$, which is contained in a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ for the type $(2, 1)$. Our main result is the following.

Theorem 1.1. *The base locus $S^d \subset \mathbb{P}^{n-1}$ of a special birational transformations of type $(2, 1)$ is projectively equivalent to one of the following:*

- (a) $\mathbb{Q}^d \subset \mathbb{P}^{d+1}$ for $d \geq 1$;
- (b) $\mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \mathbb{P}^{2d-1}$ for $d \geq 3$;
- (c) the 6-dimensional Grassmannian $\text{Gr}(2, 5) \subset \mathbb{P}^9$;
- (d) the 10-dimensional Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$;
- (e) a nonsingular codimension ≤ 2 linear section of $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$;
- (f) a nonsingular codimension ≤ 3 linear section of $\text{Gr}(2, 5) \subset \mathbb{P}^9$.

The description of the varieties in Theorem 1.1 as well as a more precise formulation of the result will be given in Section 2. Our proof of Theorem 1.1, to be given in Section 6, uses previous works on this topic in [AS] and [R2]. The main strategy is an inductive argument on VMRT (see Proposition 4.6) developed by Russo in [R2]. There are two new ingredients in our approach: the use of natural \mathbb{C}^* -actions on Z , which reveals topological relations between the base loci of Φ and Φ^{-1} , and a study of the intersection of entry loci on the base locus of Φ , which exhibits a delicate structure in the projective geometry of the base locus. The part on \mathbb{C}^* -action is presented in Section 3 and the part on the intersection of entry loci is presented in Section 5. The use of \mathbb{C}^* -actions on Z is motivated by our previous work [FH] on projective manifolds with nonzero prolongations. As a matter of fact, we will see in Section 7 that a prime Fano manifold Z is the target of a special birational transformation of type $(2, 1)$ if and only if it has nonzero prolongation. By this correspondence, we can use Theorem 1.1 to give a new proof (see Theorem 7.13) of the main classification result of [FH]. This new proof corrects an error in the classification in [FH], as explained in Remark 7.14.

We will work over complex numbers. For simplicity, a nonsingular irreducible variety will be called a manifold.

Acknowledgments: We are grateful to José Carlos Sierra for explaining to us the examples in Proposition 2.11 and Proposition 2.12 and

for a careful reading of the draft. We would like to thank Francesco Russo for pointing out a mistake in a previous version and for numerous suggestions. We are grateful to Qifeng Li for discussions on Section 5.

2. STATEMENT OF THE CLASSIFICATION OF SPECIAL QUADRATIC MANIFOLDS

Definition 2.1. Let U be a vector space and let $\sigma : \text{Sym}^2 U \rightarrow W$ be a surjective linear map to a vector space W . Let us denote by $W^* \subset \text{Sym}^2 U^*$ the annihilator of $\text{Ker}(\sigma)$, which is naturally dual to W .

- (1) Let $\psi^\circ : \mathbb{P}U \dashrightarrow \mathbb{P}W$ be the rational map defined by the linear system $W^* \subset \text{Sym}^2 U^* \simeq H^0(\mathbb{P}U, \mathcal{O}(2))$. The scheme-theoretic base locus of ψ° will be denoted by $B(\sigma) \subset \mathbb{P}U$ and the proper image of $\mathbb{P}U$ under ψ° will be denoted by $Y(\sigma) \subset \mathbb{P}W$.
- (2) Fix a 1-dimensional vector space T with a fixed identification $T = \mathbb{C}$. Define a rational map $\phi^\circ : \mathbb{P}(T \oplus U) \dashrightarrow \mathbb{P}(T \oplus U \oplus W)$ by

$$[t : u] \mapsto [t^2 : tu : \sigma(u, u)] \text{ for } t \in T, u \in U.$$

The proper image of $\mathbb{P}(T \oplus U)$ under ϕ° will be denoted by $Z(\sigma) \subset \mathbb{P}(T \oplus U \oplus W)$. The scheme-theoretic base locus of ϕ° coincides with $B(\sigma) \subset \mathbb{P}U = \mathbb{P}(0 \oplus U)$.

We will say that σ is a *special system of quadrics* if (a) the base locus subscheme $B(\sigma) \subset \mathbb{P}U$ is irreducible and nonsingular and (b) the image $Z(\sigma)$ of ϕ° is nonsingular. A projective manifold $S \subset \mathbb{P}U$ is called a *special quadratic manifold* if $S = B(\sigma)$ for a special system of quadrics $\sigma : \text{Sym}^2 U \rightarrow W$.

In this case, the rational map ψ° comes from a morphism

$$\psi : \text{Bl}_S(\mathbb{P}U) \rightarrow Y(\sigma) \subset \mathbb{P}W$$

where $\text{Bl}_S(\mathbb{P}U)$ is the blow-up of $\mathbb{P}U$ along S . We will denote by $F \subset \text{Bl}_S(\mathbb{P}U)$ its exceptional divisor. The rational map ϕ° comes from a morphism

$$\phi : \text{Bl}_S(\mathbb{P}(T \oplus U)) \rightarrow Z(\sigma) \subset \mathbb{P}(T \oplus U \oplus W)$$

where $\text{Bl}_S(\mathbb{P}(T \oplus U))$ is the blow-up of $\mathbb{P}(T \oplus U)$ along $S \subset \mathbb{P}U = \mathbb{P}(0 \oplus U)$. We will denote by $E \subset \text{Bl}_S(\mathbb{P}(T \oplus U))$ its exceptional divisor.

We have the commuting diagram

$$\begin{array}{ccccc} F \subset & \text{Bl}_S(\mathbb{P}U) & \xrightarrow{\psi} & Y(\sigma) \subset & \mathbb{P}W \\ \cap & \cap & & \cap & \cap \\ E \subset & \text{Bl}_S(\mathbb{P}(T \oplus U)) & \xrightarrow{\phi} & Z(\sigma) \subset & \mathbb{P}(T \oplus U \oplus W). \end{array}$$

Note that ϕ sends $U \cong \mathbb{P}(T \oplus U) \setminus \mathbb{P}U$ isomorphically to $Z(\sigma) \setminus \mathbb{P}(U \oplus W)$. Thus $Z(\sigma)$ is a rational variety and $\phi^\circ : \mathbb{P}(T \oplus U) \dashrightarrow Z(\sigma)$ is a birational map.

It follows immediately from the definition that for a special quadratic manifold $S \subset \mathbb{P}U$, the birational map $\phi^\circ : \mathbb{P}(T \oplus U) \dashrightarrow Z(\sigma)$ is a special birational transformation of type $(2, 1)$ in the sense of Section 1. Conversely we have

Proposition 2.2. *Let $\Phi : \mathbb{P}^n \dashrightarrow Z \subset \mathbb{P}^N$ be a special birational transformation of type $(2, 1)$ as defined in Section 1. Then the base locus S of Φ is contained in a hyperplane \mathbb{P}^{n-1} of \mathbb{P}^n , the subvariety $S \subset \mathbb{P}^{n-1}$ is a special quadratic manifold and Φ coincides with the birational map ϕ° for a special system of quadrics $\sigma : \text{Sym}^2 U \rightarrow W$.*

Proof. By Proposition 2.3 (a) [ES], the base locus $S \subset \mathbb{P}^n$ of Φ is contained in a hyperplane. Since S is defined by quadratic equations, $S \subset \mathbb{P}^{n-1}$ is also defined by quadratic equations. As S is the base locus of Φ , the rational map Φ is given by a linear subspace $W^* \subset H^0(\mathbb{P}^{n-1}, \mathcal{I}_S(2))$ and the full linear system $H^0(\mathbb{P}^n, \mathcal{I}_{\mathbb{P}^{n-1}}(2))$. Then the map ϕ° associated to W coincides with Φ . \square

Example 2.3. We list some homogeneous examples of special quadratic manifolds in the next table. The data in this table can be found from Theorem 3.8 in Chapter III of [Z]. Note that in all these examples, the dimension a of W is equal to $h^0(\mathbb{P}^{n-1}, \mathcal{I}_S(2))$, namely, we have $W^* = H^0(\mathbb{P}^{n-1}, \mathcal{I}_S(2))$.

S	\mathbb{Q}^d	$\mathbb{P}^1 \times \mathbb{P}^{d-1}$	$\text{Gr}(2, 5)$	\mathbb{S}_5
$S \subset \mathbb{P}^{n-1}$	$\mathcal{O}(1)$	Segre	Plücker	Spinor
Y	point	$\text{Gr}(2, d)$	\mathbb{P}^4	\mathbb{Q}^8
$Y \subset \mathbb{P}^{a-1}$	identity	Plücker	identity	$\mathcal{O}(1)$
Z	\mathbb{Q}^{d+2}	$\text{Gr}(2, d+2)$	\mathbb{S}_5	$\mathbb{O}\mathbb{P}^2$
$Z \subset \mathbb{P}^{n+a}$	$\mathcal{O}(1)$	Plücker	Spinor	Severi
$d = \dim S$	$d = n - 2$	d	6	10
$n = \dim Z$	$n = d + 2$	$2d$	10	16
$m = \dim Y$	0	$2(d - 2)$	4	8
a	1	$d(d - 1)$	5	10

Example 2.4. Here we give some examples of special birational transformations of type (2, 1) with $W^* \subsetneq H^0(\mathbb{P}^{n-1}, \mathcal{I}_S(2))$. Recall from Example 2.3 that the special quadratic manifold $S = \mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \mathbb{P}^{2d-1}$ with $d \geq 6$ is associated to the special system of quadrics

$$\sigma : \text{Sym}^2 U \rightarrow W = H^0(\mathbb{P}U, \mathcal{I}_S(2))^*.$$

The corresponding birational transformation is

$$\phi^\circ : \mathbb{P}(T \oplus U) \dashrightarrow Z := \text{Gr}(2, d+2) \subset \mathbb{P}(T \oplus U \oplus W)$$

with $Y = \text{Gr}(2, d) \subset \mathbb{P}W = \mathbb{P}(\wedge^2 \mathbb{C}^d)$. Note that $\text{Sec}(Z) \cap \mathbb{P}W = \text{Sec}(Y)$. Take any linear subspace $L \subset W$ such that $\mathbb{P}L \cap \text{Sec}(Y) = \emptyset$, then $\mathbb{P}L \cap \text{Sec}(Z) = \emptyset$. Let

$$p_L : \mathbb{P}(T \oplus U \oplus W) \dashrightarrow \mathbb{P}(T \oplus U \oplus W/L)$$

be the projection from $\mathbb{P}L$. Then p_L sends Z (resp. Y) isomorphically to a subvariety

$$Z_L \subset \mathbb{P}(T \oplus U \oplus W/L) \text{ (resp. } Y_L \subset \mathbb{P}(W/L)).$$

The map

$$\phi_L^\circ := p_L \circ \phi^\circ : \mathbb{P}(T \oplus U) \dashrightarrow Z_L$$

is a special birational transformation of type (2, 1) associated to the special system of quadrics

$$\sigma_L : \text{Sym}^2 U \rightarrow W/L.$$

with $Y(\sigma_L) = Y_L \subset \mathbb{P}(W/L)$ and $B(\sigma_L) \subset \mathbb{P}U$ being $\mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \mathbb{P}^{2d-1}$.

To discuss non-homogeneous examples, it is convenient to introduce the following notion.

Definition 2.5. Let $Z \subset \mathbb{P}V$ be a nondegenerate submanifold and let $W \subset V$ be a subspace such that $\mathbb{P}W \subset Z$. Denote by $(V/W)^* \subset V^*$ the set of linear functionals on V annihilating W such that $\mathbb{P}(V/W)^*$ parameterizes the set of hyperplanes in $\mathbb{P}V$ containing $\mathbb{P}W$. Then a general member of $\mathbb{P}(V/W)^*$ is called a $\mathbb{P}W$ -general hyperplane in $\mathbb{P}V$. More generally, a linear subspace of codimension- s in $\mathbb{P}V$ is $\mathbb{P}W$ -general if it is defined by a general member of $\text{Gr}(s, (V/W)^*)$, i.e., it is general among subspaces of codimension- s containing $\mathbb{P}W$.

Proposition 2.6. *Let $S = B(\sigma) \subset \mathbb{P}U$ be a special quadratic manifold defined by $\sigma : \text{Sym}^2 U \rightarrow W$. Assume that $\dim S \geq 2$, $\dim \mathbb{P}U > \dim Y(\sigma)$ and the intersection of $Z(\sigma)$ with a $\mathbb{P}W$ -general hyperplane of $\mathbb{P}(T \oplus U \oplus W)$ is nonsingular. Then for a general subspace $U' \subset U$ of codimension 1, the restriction $\sigma' : \text{Sym}^2 U' \rightarrow W$ of σ is a special system of quadrics such that*

- (i) the base locus scheme $B(\sigma') \subset \mathbb{P}U'$ coincides with the hyperplane section $S' = S \cap \mathbb{P}U' \subset \mathbb{P}U$ of $S = B(\sigma)$;
- (ii) $Y(\sigma') = Y(\sigma) \subset \mathbb{P}W$; and
- (iii) $Z(\sigma') = Z(\sigma) \cap \mathbb{P}(T \oplus U' \oplus W)$.

Proof. Firstly, we claim that for a general $U' \subset U$, the hyperplane section $Z(\sigma) \cap \mathbb{P}(T \oplus U' \oplus W)$ is nonsingular. This is a consequence of the assumption that a $\mathbb{P}W$ -general hyperplane section of $Z(\sigma) \subset \mathbb{P}(T \oplus U \oplus W)$ is nonsingular. To see this, associate to each vector $v \in U$ the linear automorphism g_v of $\mathbb{P}(T \oplus U \oplus W)$ defined by

$$g_v : [t : u : w] \mapsto [t : u + tv : w + 2\sigma(u, v) + t\sigma(v, v)].$$

For a general choice of $v \in U$ and $U' \subset U$, the automorphism g_v sends $\mathbb{P}(T \oplus U' \oplus W)$ to a $\mathbb{P}W$ -general hyperplane of $\mathbb{P}(T \oplus U \oplus W)$. The g_v -image of a general point $[1 : u : \sigma(u, u)] \in Z(\sigma)$ is

$$\begin{aligned} g_v([1 : u : \sigma(u, u)]) &= [1 : u + v : \sigma(u, u) + 2\sigma(u, v) + \sigma(v, v)] \\ &= [1 : u + v : \sigma(u + v, u + v)] \in Z(\sigma). \end{aligned}$$

Thus g_v preserves $Z(\sigma)$. Consequently, for a general choice of v , the automorphism g_v sends the hyperplane section $Z(\sigma) \cap \mathbb{P}(T \oplus U' \oplus W)$ to a $\mathbb{P}W$ -general hyperplane section of $Z(\sigma)$. This proves the claim.

Now we can choose a general subspace $U' \subset U$ of codimension 1 such that

- (1) $\dim \mathbb{P}U' \geq \dim Y(\sigma)$ and $\dim U - \dim U' < \dim S$;
- (2) the (scheme-theoretic) linear section $S' := S \cap \mathbb{P}U'$ is nonsingular; and
- (3) the restriction $\psi^o|_{\mathbb{P}U'}$ is dominant over $Y(\sigma)$.

Then $\sigma' : \text{Sym}^2 U' \rightarrow W$ is surjective, and $S' = B(\sigma') \subset \mathbb{P}U'$ is a special quadratic manifold with $Y(\sigma') = Y(\sigma)$, and $Z(\sigma') = Z(\sigma) \cap \mathbb{P}(T \oplus U' \oplus W)$. \square

To check the condition in Proposition 2.6, we need the following.

Proposition 2.7. *In the setting of Definition 2.5, let $Z \subset \mathbb{P}V$ be a nondegenerate submanifold containing $\mathbb{P}W$. Define $Z_W^* \subset \mathbb{P}V^*$ by*

$$Z_W^* = \{[H] \in \mathbb{P}(V/W)^* \mid H \cap Z \text{ is singular at a point of } \mathbb{P}W\}.$$

Assume that $\dim Z_W^ < \dim \mathbb{P}(V/W)^*$. Then for a $\mathbb{P}W$ -general hyperplane $[H] \in \mathbb{P}(V/W)^*$, the intersection $\mathcal{Z} := Z \cap H$ is a nonsingular subvariety containing $\mathbb{P}W$ and the submanifold $\mathcal{Z} \subset H$ satisfies $\dim \mathcal{Z}_W^* \leq \dim Z_W^*$.*

Proof. The intersection of Z with a $\mathbb{P}W$ -general H is nonsingular outside $\mathbb{P}W$ by Bertini. But we can choose $[H] \in \mathbb{P}(V/W)^*$ outside Z_W^*

by the dimension condition $\dim Z_W^* < \dim \mathbb{P}(V/W)^*$. Thus $\mathcal{Z} = Z \cap H$ is nonsingular and it contains $\mathbb{P}W$. It remains to check $\dim \mathcal{Z}_W^* \leq \dim Z_W^*$.

By definition,

$$\mathcal{Z}_W^* = \{[L] \in \mathbb{P}(H/W)^* \mid L \cap \mathcal{Z} \text{ is singular at a point of } \mathbb{P}W\}.$$

Given an element $[L]$ of \mathcal{Z}_W^* , then $L \cap \mathcal{Z}$ is singular at some point say $x \in \mathbb{P}W$. Take the hyperplane $\tilde{L} \subset V$ to be the linear span of L and $T_x Z$, then $L = \tilde{L} \cap H$ and $\tilde{L} \cap Z$ is singular at $x \in \mathbb{P}W$. This shows that the image of Z_W^* under the projection $\mathbb{P}(V/W)^* \setminus \{H\} \rightarrow \mathbb{P}(\hat{H}/W)^*$ contains \mathcal{Z}_W^* , where \hat{H} is the hyperplane in V corresponding to H . This implies that $\dim \mathcal{Z}_W^* \leq \dim Z_W^*$. \square

By applying Proposition 2.7 repeatedly, we have the following.

Corollary 2.8. *In Proposition 2.7, let s be a positive integer satisfying*

$$s < \dim Z \text{ and } \dim Z_W^* \leq \dim \mathbb{P}(V/W)^* - s.$$

Then a $\mathbb{P}W$ -general linear section of Z with codimension s is nonsingular.

Let us recall the following results from Proposition 2.19 and Remark 2.20 in Chapter III of [Z].

Proposition 2.9. *(i) Let $Y = \mathbb{P}W \subset Z \subset \mathbb{P}V$ be $\mathbb{P}^2 \subset \text{Gr}(2, 5) \subset \mathbb{P}^9$ from $S = \mathbb{P}^1 \times \mathbb{P}^2$ of Example 2.3. Then Z_W^* is isomorphic to a cone over S . In particular, $\dim \mathbb{P}(V/W)^* - \dim Z_W^* = 6 - 4 = 2$.*

(ii) Let $Y = \mathbb{P}W \subset Z \subset \mathbb{P}V$ be $\mathbb{P}^4 \subset \mathbb{S}_5 \subset \mathbb{P}^{15}$ from $S = \text{Gr}(2, 5)$ of Example 2.3. Then Z_W^ is isomorphic to a cone over S . In particular, $\dim \mathbb{P}(V/W)^* - \dim Z_W^* = 10 - 7 = 3$.*

Lemma 2.10. *Recall that a subvariety $S \subset \mathbb{P}^{n-1}$ is arithmetically Cohen-Macaulay if $H^i(\mathbb{P}^{n-1}, \mathcal{I}_S(l)) = 0$ for all l and for all $0 < i < \dim(S) + 1$. Let $S \subset \mathbb{P}^{n-1}$ be either the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ or the Plücker embedding $\text{Gr}(2, 5) \subset \mathbb{P}^9$. Let $S' \subset \mathbb{P}^{n-s-1}$ be any nonsingular linear section of $S \subset \mathbb{P}^{n-1}$ with codimension $s < \dim S$. Then $S' \subset \mathbb{P}^{n-s-1}$ is arithmetically Cohen-Macaulay and $H^0(\mathbb{P}^{n-1}, \mathcal{I}_S(2)) = H^0(\mathbb{P}^{n-s-1}, \mathcal{I}_{S'}(2))$.*

Proof. By [Z] (Chapter III, Theorem 1.2), $S \subset \mathbb{P}^{n-1}$ is arithmetically Cohen-Macaulay. On the other hand, for a nonsingular hyperplane section, we have the exact sequence

$$0 \rightarrow \mathcal{I}_S(-1) \rightarrow \mathcal{I}_S \rightarrow \mathcal{I}_{S \cap H \subset H} \rightarrow 0.$$

Using the associated long exact sequence, we deduce that $S \cap H$ is arithmetically Cohen-Macaulay. Repeating the same argument, we see that

any nonsingular linear section of S is arithmetically Cohen-Macaulay. The last claim then follows easily. \square

Now we can give some nonhomogeneous examples of special quadratic manifolds.

Proposition 2.11. *From Example 2.3, the Segre variety $S = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5 = \mathbb{P}U$ is a special quadratic manifold associated with*

$$\sigma : \text{Sym}^2 U \rightarrow W = H^0(\mathbb{P}^5, \mathcal{I}_S(2))^*$$

and $Z(\sigma)$ isomorphic to the Plücker embedding $\text{Gr}(2, 5) \subset \mathbb{P}^9$. A nonsingular linear section S' of S by a general subspace $U' \subset U$ of codimension $s \leq 2$ is a special quadratic manifold associated to a special system of quadrics

$$\sigma' : \text{Sym}^2 U' \rightarrow W = H^0(\mathbb{P}^5, \mathcal{I}_S(2))^* = H^0(\mathbb{P}^{5-s}, \mathcal{I}_{S'}(2))^*$$

and $Z(\sigma')$ is equal to a nonsingular linear section of $\text{Gr}(2, 5)$ with codimension s .

Proof. Applying Proposition 2.6 repeatedly in combination with Corollary 2.8 and Proposition 2.9, we see that a general linear section S' of $\mathbb{P}^1 \times \mathbb{P}^2$ with codimension $s \leq 2$ is a special quadratic manifold associated to a special system of quadrics $\sigma' : \text{Sym}^2 U' \rightarrow W = H^0(\mathbb{P}^5, \mathcal{I}_S(2))^*$ and $Z(\sigma')$ equal to a \mathbb{P}^2 -general linear section of $\text{Gr}(2, 5)$ with codimension s . It is well-known (e.g. Remark 3.3.2 in [IP]) that all nonsingular sections of S with codimension $s \leq 2$ are projectively equivalent and all nonsingular sections of $\text{Gr}(2, 5)$ with codimension $s \leq 2$ are projectively equivalent. Thus we may say that $Z(\sigma')$ is any nonsingular linear section of $\text{Gr}(2, 5)$ of codimension $s \leq 2$. Using Lemma 2.10, we see $W^* = H^0(\mathbb{P}^{5-s}, \mathcal{I}_{S'}(2))$. \square

Proposition 2.12. *From Example 2.3, the Grassmannian $S = \text{Gr}(2, 5) \subset \mathbb{P}^9 = \mathbb{P}U$ is a special quadratic manifold associated with*

$$\sigma : \text{Sym}^2 U \rightarrow W = H^0(\mathbb{P}^9, \mathcal{I}_S(2))^*$$

and $Z(\sigma)$ is isomorphic to the Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$. A nonsingular linear section S' of S by a general subspace $U' \subset U$ of codimension $s \leq 3$ is a special quadratic manifold associated to a special system of quadrics

$$\sigma' : \text{Sym}^2 U' \rightarrow W = H^0(\mathbb{P}^9, \mathcal{I}_S(2))^* = H^0(\mathbb{P}^{9-s}, \mathcal{I}_{S'}(2))^*$$

and $Z(\sigma')$ equals to a \mathbb{P}^4 -general linear section of $Z(\sigma) = \mathbb{S}_5$ with codimension $s \leq 3$.

Proof. As in the proof of Proposition 2.11, this follows from Proposition 2.6, Corollary 2.8, Proposition 2.9 and Lemma 2.10, modulo the fact (see Remark 3.3.2 in [IP]) that all nonsingular linear sections of $\text{Gr}(2, 5)$ of a fixed codimension $s \leq 3$ are projectively equivalent. \square

Remark 2.13. (1) Let $\mathbb{S}_5 = S \subset \mathbb{P}U$, $\dim U = 16$, be the 10-dimensional Spinor variety. It is well-known that the dual variety $S^* \subset \mathbb{P}U^*$ is isomorphic to $S \subset \mathbb{P}U$ and the automorphism group $\text{Aut}(S)$ acts transitively on S and $\mathbb{P}U \setminus S$. This implies that all nonsingular hyperplane sections of \mathbb{S}_5 are projectively equivalent.

(2) A general linear section of \mathbb{S}_5 with codimension 2 does *not* contain a linear \mathbb{P}^4 . One way to see this is using the fact that a general hyperplane section S_1 of $\mathbb{S}_5 \subset \mathbb{P}^{15}$ is isomorphic to a horospherical Fano manifold of Picard number 1, the case 4 in Theorem 1.7 of [P] (this fact follows from Mukai's classification [M]). From [P], the automorphism group $\text{Aut}(S_1)$ has two orbits, an open orbit and a closed orbit, say $Q \subset S_1$, which is isomorphic to the 6-dimensional hyperquadric \mathbb{Q}^6 . Let $\pi : \text{Bl}_Q(S_1) \rightarrow S_1$ be the blow-up of S_1 along Q and let $E \subset \text{Bl}_Q(S_1)$ be the exceptional divisor. Then by the proof of Lemma 1.17 [P], there exists a morphism $q : \text{Bl}_Q(S_1) \rightarrow \mathbb{Q}^5$ which is a \mathbb{P}^4 -bundle and E is homogeneous which is a \mathbb{P}^3 -bundle over \mathbb{Q}^5 under the map q . The fibers of q are mapped to linear \mathbb{P}^4 's contained in S_1 and any linear \mathbb{P}^4 in S_1 arises this way. This implies that the intersection of \mathbb{P}^4 in S_1 with Q is $\mathbb{P}^3 \subset \mathbb{P}^4$. Now take a general hyperplane section $H \subset S_1$, then $H \cap Q$ is a smooth \mathbb{Q}^5 . If H contains a \mathbb{P}^4 , then $H \cap Q$ will contain a \mathbb{P}^3 , a contradiction.

The following result from Corollary 3.21 in [AS] is a converse to Proposition 2.11 and Proposition 2.12.

Proposition 2.14. *A nonsingular linear section of the homogeneous special quadratic manifolds in Example 2.3 is a special quadratic manifold if and only if it is a hyperquadric or one provided by Proposition 2.11 and Proposition 2.12.*

Our aim is to show that Example 2.3, Example 2.4, Proposition 2.11 and Proposition 2.12 exhaust all special quadratic manifolds. More precisely, we have the following classification, which gives a complete classification of special birational transformations of type (2,1).

Theorem 2.15. *A special quadratic manifold $S^d \subset \mathbb{P}^{n-1}$ is projectively equivalent to one of the following:*

- (a) $\mathbb{Q}^d \subset \mathbb{P}^{d+1}$ for $d \geq 1$;
- (b) $\mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \mathbb{P}^{2d-1}$ for $d \geq 3$;

- (c) the 6-dimensional Grassmannian $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$;
- (d) the 10-dimensional Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$;
- (e) a nonsingular codimension ≤ 2 linear section of $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$;
- (f) a nonsingular codimension ≤ 3 linear section of $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$.

The birational map ψ° is given by $H^0(\mathbb{P}^{n-1}, \mathcal{I}_S(2))$ except for (b) with $d \geq 6$, where ψ° can be given by some proper subspaces of $H^0(\mathbb{P}^{n-1}, \mathcal{I}_S(2))$ as in Example 2.4.

The proof of Theorem 2.15 will be given in Section 6.

3. CLASSIFICATION OF $Y(\sigma)$

In this section, we classify varieties that can occur as the ψ -image $Y(\sigma)$ of a special quadratic manifold $S \subset \mathbb{P}U$. To simplify the notation, we will write Y for $Y(\sigma)$ and Z for $Z(\sigma)$. We will use the notation of Definition 2.1.

Proposition 3.1. *Let $S \subset \mathbb{P}U$ be a special quadratic manifold. Then Y is nonsingular.*

Proof. Consider the \mathbb{C}^* -action on $\mathbb{P}(T \oplus U)$ given by $\lambda \cdot [t : u] = [t : \lambda u]$ for all $\lambda \in \mathbb{C}^*$. The fixed locus of this \mathbb{C}^* -action has two components: the isolated point $\mathbb{P}T$ and the hyperplane $\mathbb{P}U$. As S is contained in the fixed locus, this \mathbb{C}^* -action lifts to $\mathrm{Bl}_S(\mathbb{P}(T \oplus U))$. Since the morphism $\phi : \mathrm{Bl}_S(\mathbb{P}(T \oplus U)) \rightarrow Z$ has connected fibers by Zariski main theorem, the \mathbb{C}^* -action on $\mathrm{Bl}_S(\mathbb{P}(T \oplus U))$ descends to a \mathbb{C}^* -action on Z such that ϕ is \mathbb{C}^* -equivariant. This implies that the proper image of $\mathbb{P}U$ under ϕ° , which is Y , is contained in the fixed locus of the \mathbb{C}^* -action on Z . On the other hand, the \mathbb{C}^* -orbit of a general point of Z has a limit point in Y , hence Y is an irreducible component of the fixed locus of the \mathbb{C}^* -action on Z . This shows that Y is nonsingular because Z is nonsingular. \square

Definition 3.2. For a projective submanifold $M \subset \mathbb{P}^N$ and two distinct points $x \neq y \in M$, denote by $\ell_{x,y} \subset \mathbb{P}^N$ the line joining x and y . Such a line is called a *secant line* of M . The *secant variety* of M is

$$\mathrm{Sec}(M) = \text{the closure of } \cup_{x \neq y \in M} \ell_{x,y}.$$

The *secant defect* of M is defined by $\delta_M = 2 \dim M + 1 - \dim \mathrm{Sec}(M)$.

Proposition 3.3. *Let $S \subset \mathbb{P}U$ be a special quadratic manifold. Let $\pi : \mathrm{Bl}_S(\mathbb{P}U) \rightarrow \mathbb{P}U$ be the blow-up morphism. Then $\mathrm{Sec}(S) = \mathbb{P}U$ and the morphism $\psi : \mathrm{Bl}_S(\mathbb{P}U) \rightarrow Y$ has the following properties:*

- (1) ψ contracts the proper transform $\pi^{-1}[\ell]$ of a secant line ℓ of S not contained in S to a point in Y ;

- (2) *the fibers of the exceptional divisor $\pi|_F : F \rightarrow S$ are sent isomorphically to linear subspaces in Y ; and*
- (3) $\psi(F) = Y$.

Proof. The equality $\text{Sec}(S) = \mathbb{P}U$ is from Proposition 2.3(a) of [ES].

Recall that $\psi^o : \mathbb{P}U \dashrightarrow Y$ is induced by the projection $\sigma : \text{Sym}^2 U \rightarrow W$ composed with the second Veronese embedding $v_2 : \mathbb{P}U \rightarrow \mathbb{P}\text{Sym}^2 U$. Let $\ell \subset \mathbb{P}U$ be a secant line of S not contained in S . Let $L \subset \text{Sym}^2 U$ be the 3-dimensional subspace spanned by $v_2(\ell)$. Since $v_2(\ell)$ intersects $\mathbb{P}\text{Ker}(\sigma)$ at two distinct points, we have $\dim L \cap \text{Ker}(\sigma) = 2$. Thus $\sigma(L)$ is 1-dimensional, i.e., ψ contracts ℓ , proving (1).

Note that ψ can be viewed as the restriction of the natural projection $p : \text{Bl}_{\mathbb{P}\text{Ker}(\sigma)}(\mathbb{P}\text{Sym}^2 U) \rightarrow \mathbb{P}W$ to $\text{Bl}_S(v_2(\mathbb{P}U))$. As the normal bundle of $\mathbb{P}\text{Ker}(\sigma)$ in $\mathbb{P}\text{Sym}^2 U$ is $\mathcal{O}(1)^s$ for some s , the exceptional divisor of the blow-up is a product of two projective spaces, which can only be contracted by the projection to the other factor. Hence the projection p sends a fiber of the blow-up isomorphically to a linear subspace in $\mathbb{P}W$. This implies (2).

Finally, (3) is a consequence of $\text{Sec}(S) = \mathbb{P}U$ and (1). \square

Let us recall the following basic terminology.

Definition 3.4. Let $M \subset \mathbb{P}^N$ be a projective submanifold.

- (1) M is *conic-connected* if there exists an irreducible conic curve through two general points of M .
- (2) M is a *prime Fano manifold* if $\text{Pic}(M)$ is generated by $\mathcal{O}_M(1)$ and M is covered by lines.

Proposition 3.5. *Let $S \subset \mathbb{P}U$ be a special quadratic manifold. Then Y is a conic-connected prime Fano manifold.*

Proof. Recall that $\psi^o : \mathbb{P}U \dashrightarrow Y$ is induced by the projection $\sigma : \text{Sym}^2 U \rightarrow W$ composed with the second Veronese embedding $v_2 : \mathbb{P}U \rightarrow \mathbb{P}\text{Sym}^2 U$. Since the Veronese variety $v_2(\mathbb{P}U) \subset \mathbb{P}(\text{Sym}^2 U)$ is conic-connected, the image Y must be conic-connected.

Since $\text{Bl}_S(\mathbb{P}U)$ has Picard number 2 and $\psi : \text{Bl}_S(\mathbb{P}U) \rightarrow Y$ contracts some curves by Proposition 3.3 (1), we see that Y has Picard number 1. By Proposition 3.3 (2), lines cover Y . Thus Y is a prime Fano manifold. \square

Proposition 3.6. *The secant defect $\delta = \delta_S$ of a special quadratic manifold $S \subset \mathbb{P}U$ satisfies $\delta = 2 \dim S + 2 - \dim U$ and*

$$\dim Y = 2(\dim S - \delta) = 2(\dim U - \dim S - 2).$$

Proof. The equality $\delta = 2 \dim S + 2 - \dim U$ is a consequence of $\text{Sec}(S) = \mathbb{P}U$ in Proposition 3.3 and the equality $\dim Y = 2(\dim S - \delta)$ is from Corollary 2.3 of [AS]. Finally, $\dim S - \delta = \dim U - \dim S - 2$ from $\delta = 2 \dim S + 2 - \dim U$. \square

Proposition 3.7. *Let $S \subset \mathbb{P}U$ be a special quadratic manifold with secant defect δ . Then there is a natural identification $\text{Bl}_S(\mathbb{P}(T \oplus U)) = \text{Bl}_Y(Z)$ such that $\phi : \text{Bl}_S(\mathbb{P}(T \oplus U)) \rightarrow Z$ coincides with the blow-up of Z along Y with the exceptional divisor $\text{Bl}_S(\mathbb{P}U)$. In particular, the morphism $\psi : \text{Bl}_S(\mathbb{P}U) \rightarrow Y$ is a $\mathbb{P}^{\delta+1}$ -bundle.*

Proof. Since both Y and Z are nonsingular, Proposition 3.3 and Theorem 1.1 of [ES] imply that $\text{Bl}_S(\mathbb{P}(T \oplus U))$ is also the blow-up of Z along Y . Thus ψ is a \mathbb{P}^k -bundle, where $k = \dim Z - \dim Y - 1$. By Proposition 3.6, we have $k = \delta + 1$. \square

The following two propositions are direct consequences of Proposition 3.7.

Proposition 3.8. *Let $S \subset \mathbb{P}U$ be a special quadratic manifold with secant defect δ . A general fiber of the $\mathbb{P}^{\delta+1}$ -bundle $\psi : \text{Bl}_S(\mathbb{P}U) \rightarrow Y$ in Proposition 3.7 is sent to a linear subspace in $\mathbb{P}U$ by the blow-up morphism $\text{Bl}_S(\mathbb{P}U) \rightarrow \mathbb{P}U$.*

Proof. Let $z \in \text{Sec}(S) = \mathbb{P}U$ be a general point. Let $C_z(S) \subset \mathbb{P}U$ be the union of secant lines of S passing through z . By Proposition 2.3 (b) of [ES], the cone $C_z(S) \subset \mathbb{P}U$ is a linear subspace of dimension $\delta + 1$. But by Proposition 3.3 (1), the cone $C_z(S)$ is contained in the image of a fiber of ψ . Thus $C_z(S)$ must coincide with the image of a fiber of ψ . \square

Proposition 3.9. *For a special quadratic manifold S , set $n = \dim U$ and $d = \dim S$. Then the Euler numbers of S and Y are related by*

$$\chi(S) = \frac{(\delta + 2)\chi(Y) - n}{n - d - 2}.$$

Proof. Since the exceptional divisor F of the blow-up $\text{Bl}_S(\mathbb{P}^{n-1}) \rightarrow \mathbb{P}^{n-1}$ is a \mathbb{P}^{n-d-2} -bundle over S , we have

$$\chi(\mathbb{P}^{n-1}) - \chi(S) = \chi(\text{Bl}_S(\mathbb{P}^{n-1})) - \chi(F) = \chi(\text{Bl}_S(\mathbb{P}^{n-1})) - (n-d-1)\chi(S),$$

which gives $\chi(\text{Bl}_S(\mathbb{P}^{n-1})) = n + (n-d-2)\chi(S)$. On the other hand, the map $\text{Bl}_S(\mathbb{P}^{n-1}) \rightarrow Y$ is a $\mathbb{P}^{\delta+1}$ -bundle from Proposition 3.7, which implies $\chi(\text{Bl}_S(\mathbb{P}^{n-1})) = (\delta + 2)\chi(Y)$. Combining the two gives the desired equality. \square

We are ready to have a classification of Y .

Theorem 3.10. *Let $S \subset \mathbb{P}U$ be a special quadric manifold and let $c = \dim U - \dim S - 1$ be its codimension. Recall that $\dim Y = 2(c-1)$ from Proposition 3.6. Then $Y \subset \mathbb{P}W$ is isomorphic to one of the following:*

- (Y1) $\mathbb{P}^{2(c-1)} \cong Y = \mathbb{P}W$;
- (Y2) a nonsingular quadric hypersurface $\mathbb{Q}^{2(c-1)} \cong Y \subset \mathbb{P}W \cong \mathbb{P}^{2c-1}$;
or
- (Y3) a biregular projection of the Plücker embedding $\text{Gr}(2, \mathbb{C}^{c+1}) \subset \mathbb{P}(\wedge^2 \mathbb{C}^{c+1})$.

Proof. By Proposition 3.3, the (nondegenerate) projective submanifold $Y \subset \mathbb{P}W$ of dimension $2(c-1)$ is covered by linear subspaces of dimension $c-1$. Thus we can use Eichi Sato's classification, Main Theorem in [S] (see also [NO] Corollary 5.3, which completes the classification), of nonsingular projective varieties of dimension $\leq 2(c-1)$ covered by linear subspaces of dimension $(c-1)$. Since Y has Picard number 1 from Proposition 3.5, Sato's classification shows that $Y \subset \mathbb{P}W$ must be one of the three listed varieties. \square

We have the following topological consequence.

Corollary 3.11. *Let $S \subset \mathbb{P}U$ be a special quadratic manifold with $d = \dim S$ and $n = \dim U$. Then all odd Betti numbers of S vanish. In particular, the Euler number of S satisfies $\chi(S) \geq d + 1$.*

Proof. As the morphism $\text{Bl}_S(\mathbb{P}^{n-1}) \rightarrow Y$ is a $\mathbb{P}^{\delta+1}$ -bundle by Proposition 3.7 and odd Betti numbers of Y vanish from Theorem 3.10, the odd Betti numbers of $\text{Bl}_S(\mathbb{P}^{n-1})$ vanish. This implies that odd Betti numbers of S vanish by the formulae for blow-ups in Chapter 4, Section 6, p605 of [GH]. \square

In the case of (Y2) of Theorem 3.10, we have the following topological consequence.

Proposition 3.12. *For a special quadratic manifold $S^d \subset \mathbb{P}^{n-1}$ with codimension c and $n \geq 3$, assume that $Y = \mathbb{Q}^{2(c-1)}$. Then either $\delta \geq \frac{d}{2}$ or $\delta = \frac{d}{2} - 1$.*

Proof. Putting $\chi(\mathbb{Q}^{2(c-1)}) = 2c$ in Proposition 3.9, we obtain

$$(3.1) \quad \chi(S) = 2\delta + \frac{n}{n-d-2}.$$

Set $k := \frac{n}{n-d-2}$. The above equation implies that k is a natural number, hence $2 \leq k \leq n$.

Assume that $\delta < \frac{d}{2}$, i.e. $2d + 2 - n < \frac{d}{2}$, which is equivalent to $3d + 5 \leq 2n$, implying $d \leq \frac{2n - 5}{3}$ and

$$n = k(n - d - 2) \geq k\left(n - \frac{2n - 5}{3} - 2\right) = \frac{k(n - 1)}{3}.$$

Let us check case-by-case the possible values of k .

- (1) When $4 \leq k \leq n$. Then $3n \geq 4(n - 1)$ gives $n = 4, d = 1, \delta = 0$ and $\chi(S) = 4$. But there is no smooth curve with $\chi(S) = 4$, a contradiction.
- (2) When $k = 3$. Then $2n = 3d + 6$ and $\delta = \frac{d}{2} - 1$.
- (3) When $k = 2$. We have $n = 2(n - d - 2)$, which yields $n = 2d + 4$, a contradiction to $\text{Sec}(S) = \mathbb{P}^{n-1}$ of Proposition 3.3.

We conclude that $\delta = \frac{d}{2} - 1$ is the only possibility. \square

4. REVIEW OF RESULTS ON QEL-MANIFOLDS

For the proof of Theorem 2.15, we will use the notion of QEL-manifolds introduced by Russo in [R2]. In this section, we review some results on QEL-manifolds from [IR2] and [R2] that are needed for our purpose.

Definition 4.1. Let $S \subset \mathbb{P}V$ be a nondegenerate submanifold with secant defect δ .

- (1) For a point $z \in \text{Sec}(S) \setminus S$, the *entry locus* of S with respect to z is the subvariety $\Sigma_z(S) \subset S$ defined by

$$\Sigma_z(S) = \text{the closure of } \{x \in S \mid z \in \ell_{x,y} \text{ for some } y \in S, y \neq x\}.$$

The cone over $\Sigma_z(S)$ with the vertex at z is denoted by $C_z(S)$. If z is general, then $\Sigma_z(S) = S \cap C_z(S)$.

- (2) S is said to be a *quadratic entry locus manifold* (QEL-manifold in abbreviation) if for a general $z \in \text{Sec}(S)$, the cone $C_z(S) \subset \mathbb{P}V$ is a linear subspace of dimension $\delta + 1$ and the entry locus $\Sigma_z(S) \subset C_z(S)$ is a nonsingular quadratic hypersurface in the linear subspace.

This notion is relevant to us by the following.

Proposition 4.2. *A special quadratic manifold $S \subset \mathbb{P}U$ is a QEL-manifold and is linearly normal.*

Proof. Proposition 2.3 (b) of [ES] says that a special quadratic manifold S is a QEL-manifold. Proposition 1.3 of [R2] says that a QEL-manifold $S \subset \mathbb{P}U$ is linearly normal if $\text{Sec}(S) = \mathbb{P}U$. But the latter condition holds for a special quadratic manifold by Proposition 3.3. \square

In what way is Proposition 4.2 useful to us? The biggest advantage of considering QEL-manifolds is that one can use an inductive argument via varieties of minimal rational tangents. Let us recall the definition.

Definition 4.3. Let $M \subset \mathbb{P}^N$ be a prime Fano manifold. The VMRT at a point x is the subvariety $\mathcal{C}_x \subset \mathbb{P}T_x M$ consisting of tangent directions to lines on M through x . When x is a general point of M , the VMRT at x is nonsingular.

Examples 4.4. An irreducible Hermitian symmetric space of compact type is a homogeneous space $M = G/P$ with a simple Lie group G and a maximal parabolic subgroup P such that the isotropy representation of P on $T_x(M)$ at a base point $x \in M$ is irreducible. The highest weight orbit of the isotropy action on $\mathbb{P}T_x(M)$ is exactly the VMRT at x . The following table (e.g. Section 3.1 [FH]) collects basic information on these varieties.

Type	I.H.S.S. M	VMRT S	$S \subset \mathbb{P}T_x(M)$
I	$\mathrm{Gr}(a, a+b)$	$\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}$	Segre
II	\mathbb{S}_n	$\mathrm{Gr}(2, n)$	Plücker
III	$\mathrm{Lag}(2n)$	\mathbb{P}^{n-1}	Veronese
IV	\mathbb{Q}^n	\mathbb{Q}^{n-2}	Hyperquadric
V	$\mathbb{O}\mathbb{P}^2$	\mathbb{S}_5	Spinor
VI	$E_7/(E_6 \times U(1))$	$\mathbb{O}\mathbb{P}^2$	Severi

Examples 4.5. Let Σ be an n -dimensional vector space endowed with a skew-symmetric 2-form ω of maximal rank. The symplectic Grassmannian $M = \mathrm{Gr}_\omega(k, \Sigma)$ is the variety of all k -dimensional isotropic subspaces of Σ , which is not homogeneous if n is odd. Let W and Q be vector spaces of dimensions $k \geq 2$ and m respectively. Let \mathbf{t} be the tautological line bundle over $\mathbb{P}W$. The VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ of $\mathrm{Gr}_\omega(k, \Sigma)$ at a general point is isomorphic to the projective bundle $\mathbb{P}((Q \otimes \mathbf{t}) \oplus \mathbf{t}^{\otimes 2})$ over $\mathbb{P}W$ with the projective embedding given by the complete linear system

$$H^0(\mathbb{P}W, (Q \otimes \mathbf{t}^*) \oplus (\mathbf{t}^*)^{\otimes 2}) = (W \otimes Q)^* \oplus \mathrm{Sym}^2 W^*.$$

An inductive argument in studying QEL-manifolds is provided by the following result from Theorem 2.1, Theorem 2.3 and Theorem 2.8 in [R2].

Proposition 4.6 ([R2]). *Let $S^d \subset \mathbb{P}U$ be a d -dimensional QEL-manifold with secant defect δ .*

- (1) *If $\delta \geq 1$ and S is a prime Fano manifold, then $d + \delta$ is even and $K_S^{-1} = \mathcal{O}(\frac{d+\delta}{2})$.*

- (2) If $\delta \geq 3$, then $d - \delta$ is divisible by $2^{\lfloor \frac{\delta-1}{2} \rfloor}$, the QEL-manifold S is a prime Fano manifold and the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(S)$ at a general point is itself a QEL-manifold with $\dim \mathcal{C}_x = \frac{d+\delta-4}{2}$ and $\delta_{\mathcal{C}_x} = \delta - 2$. In fact, points of \mathcal{C}_x corresponding to the lines through x contained in an entry locus of S through x form an entry locus of \mathcal{C}_x .

Since in many cases the VMRT \mathcal{C}_x at a general $x \in S$ gives a good deal of information for S , if a QEL-manifold S has $\delta \geq 3$, Proposition 4.6 enables us to reduce a problem on S to a lower-dimensional QEL-manifold \mathcal{C}_x . Notice that such an inductive argument is not available for special quadratic manifolds. This is why Proposition 4.2 is useful to us. The inductive argument for QEL-manifolds works perfectly when $\delta_S \geq \frac{1}{2} \dim S$ and we have the following two classification results due to Russo.

Proposition 4.7 (Corollary 3.1 in [R2]). *Let $S^d \subset \mathbb{P}^N$ be a d -dimensional QEL-manifold with $\frac{d}{2} < \delta < d$. Then $S \subset \mathbb{P}^N$ is isomorphic to one of the followings.*

- (A1) the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$;
- (A2) the Plücker embedding $\text{Gr}(2, 5) \subset \mathbb{P}^9$;
- (A3) the Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$;
- (A4) a general hyperplane section of the Plücker embedding $\text{Gr}(2, 5) \subset \mathbb{P}^9$;
- (A5) a general hyperplane section of the Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$.

Proposition 4.8 (Corollary 3.2 in [R2]). *Let $S^d \subset \mathbb{P}^N$ be a d -dimensional QEL-manifold with $\frac{d}{2} = \delta$. Then $d = 2, 4, 8$ or 16 and $S \subset \mathbb{P}^N$ is isomorphic to one of the followings.*

- (B1) a general hyperplane section of the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$;
- (B2) the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$;
- (B3) the Segre variety $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$;
- (B4) the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$;
- (B5) a general codimension-2 linear section of the Plücker embedding $\text{Gr}(2, 5) \subset \mathbb{P}^9$;
- (B6) a general codimension-2 linear section of the Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$;
- (B7) the Plücker embedding $\text{Gr}(2, 6) \subset \mathbb{P}^{14}$;
- (B8) the E_6 -Severi variety $\mathbb{O}\mathbb{P}^2 \subset \mathbb{P}^{26}$.

Remark 4.9. Corollary 3.2 of [R2] lists one hypothetical case of $d = 16$ in addition to (B8). That case does not exist in view of Main Theorem of Section 2 in [Mo]. Corollary 3.2 of [R2] is stated for “LQEL-manifolds”, a weaker notion than QEL-manifolds, with $\frac{d}{2} = \delta$. Because

of this, the list there includes some biregular projections of QEL-manifolds. Since biregular projections of QEL-manifolds can not be QEL-manifolds by Proposition 1.3 of [R2], we have the above list.

Note that the inductive argument using Proposition 4.6 cannot be continued if $\delta \leq 2$. This is a major difficulty in the study of QEL-manifolds. In our case, however, thanks to the restriction coming from Theorem 3.10, such low defect cases can be handled by a number of explicit classification results on QEL-manifolds in the following two extreme cases.

Proposition 4.10. *Let $S^d \subset \mathbb{P}^N$ be a QEL-manifold with $\text{Sec}(S) = \mathbb{P}^N$ and $\delta \geq 2$. If S is a prime Fano manifold with $K_S^{-1} = \mathcal{O}(d-2)$, then it is projectively equivalent to one of the following:*

- (M1) *the 10-dimensional Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$;*
- (M2) *a nonsingular linear section of $\mathbb{S}_5 \subset \mathbb{P}^{15}$ of codimension ≤ 4 .*

Proof. The assumption $\text{Sec}(S) = \mathbb{P}^N$ implies that $S \subset \mathbb{P}^N$ is linearly normal by Proposition 1.3 of [R2]. By Proposition 4.6, we have $d + \delta = 2(d-2)$, hence $\delta = d-4$ and $N = 2d+1-\delta = d+5$. By our assumption $\delta \geq 2$, we obtain $d \geq 6$.

We will use Mukai's classification in [M] of linearly normal prime Fano manifolds $S^d \subset \mathbb{P}^N$ with $K_S^{-1} = \mathcal{O}(d-2)$. Mukai's classification is in terms of the genus $g := \frac{1}{2}\text{deg}(S) + 1$. If $g \leq 5$, then such S is a complete intersection by Remark 2 in [M]. Since a QEL-manifold that is a complete intersection must be a hyperquadric by Proposition 3.4 of [IR2], we have only $g \geq 6$. The classification for $g \geq 6$ is listed in Example 5.2.2 and Theorem 5.2.3 of [IP]. In the list, the only possibilities of $S^d \subset \mathbb{P}^{d+5}$ with $d \geq 6$ occur in Example 5.2.2 (vi) and these are exactly (M1) and (M2). \square

For the next classification result, we need the notion of rational normal scrolls.

Definition 4.11. For integers $a_1, \dots, a_n \geq 1$, the *rational normal scroll* $S(a_1, \dots, a_n)$ is the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n))$ over \mathbb{P}^1 , embedded into \mathbb{P}^N with $N = \sum_i (a_i + 1) - 1$ such that the fiber of the projective bundle is mapped to linear subspaces in \mathbb{P}^N .

The following is proved in Theorem 3 of [R1].

Proposition 4.12. *Let $S \subset \mathbb{P}^{2d+1}$ be a QEL-manifold with $\delta = 0$.*

- (1) *When $\dim S = 1$, the curve $S \subset \mathbb{P}^3$ is isomorphic to the twisted cubic.*

- (2) When $\dim S = 2$, there are three possibilities for the surface $S \subset \mathbb{P}^5$:
- (a) a Del Pezzo surface of degree 5 in \mathbb{P}^5 ;
 - (b) the rational normal scroll $S(2, 2)$;
 - (c) the rational normal scroll $S(1, 3)$.

5. INTERSECTION OF ENTRY LOCI ON QEL-MANIFOLDS

In this section, we study the intersection of entry loci on QEL-manifolds satisfying certain additional conditions. Although our main purpose is to use this in the proof of Theorem 2.15, our results here have independent interest in projective geometry of QEL-manifolds. The following notation will be used.

Notation 5.1. For any subset $A \subset \mathbb{P}^{n-1}$, its linear span will be denoted by $\langle A \rangle \subset \mathbb{P}^{n-1}$. For a point $s \in \mathbb{P}^{n-1}$, we will write $\langle A, s \rangle$ in place of $\langle A \cup \{s\} \rangle$. For a projective variety $S \subset \mathbb{P}^{n-1}$ covered by lines and a point $s \in S$, we denote by $\text{Locus}(s) \subset S$ the locus of lines on S passing through s . Thus for a general point $x \in S$,

$$\dim \text{Locus}(x) = \dim \mathcal{C}_x + 1.$$

If $s \in S$ is a nonsingular point, then $\mathbf{T}_s S \subset \mathbb{P}^{n-1}$ denotes the projective tangent space of S at s . When $S \subset \mathbb{P}^{n-1}$ is a QEL-manifold with $\delta > 0$, for a point $u \in \text{Sec}(S)$, we recall from Definition 4.1 that $\Sigma_u(S)$ is the entry locus of S and $C_u(S) = \langle \Sigma_u(S) \rangle$.

Condition 5.2. We consider the following condition on S :

For a general $u \in \text{Sec}(S)$, we have $\Sigma_t(S) = \Sigma_u(S)$ for all $t \in \langle \Sigma_u \rangle \setminus \Sigma_u$.

Condition 5.2 is relevant to us by the following fact from Remark 2.4 [AS].

Lemma 5.3. *A special quadratic manifold satisfies Condition 5.2.*

Proposition 5.4. *Let $S \subsetneq \mathbb{P}^{n-1}$ be a QEL-manifold with $\delta > 0$ satisfying Condition 5.2. Fix a general entry locus $\Sigma = \Sigma_z(S)$ by choosing a general point $z \in \text{Sec}(S)$ and fix a general point $x \in \Sigma$. There exists a Zariski open subset $O \subset S$ with $O \cap \Sigma = \emptyset$ such that for any $s \in O$, we have the following.*

- (1) *Either $\text{Locus}(s) \cap \Sigma = \emptyset$ for all $s \in O$ or $\text{Locus}(s) \cap \Sigma \not\subseteq \mathbf{T}_x \Sigma \cap \Sigma$ for all $s \in O$.*
- (2) *There exists a unique entry locus of S passing through x and s , to be denoted by $\Sigma_s \in S$, such that $\Sigma_s \subset \langle \Sigma_s \rangle$ is a smooth hyperquadric.*
- (3) *$\langle \mathbf{T}_x S, s \rangle \cap S = (\mathbf{T}_x S \cap S) \cup \Sigma_s$.*

- (4) The line $\ell_{s,x}$ is not contained in S .
- (5) For any $t \in \langle \Sigma_s \rangle \setminus \Sigma_s$, we have $C_t(S) = \langle \Sigma_s \rangle$ and $\Sigma_t(S) = \Sigma_s$.
- (6) Define $P_s := \Sigma_s \cap \Sigma$. Then P_s coincides with $\langle \Sigma_s \rangle \cap \langle \Sigma \rangle$ and is a linear subspace of \mathbb{P}^{n-1} . In particular, we have $P_s \subset \mathbf{T}_x \Sigma \cap \Sigma$.
- (7) Let $p_\Sigma : S \setminus \Sigma \rightarrow \mathbb{P}^{n-\delta-3}$ be the projection from $\langle \Sigma \rangle$. Then p_Σ is a smooth morphism at s .

We will use the following lemma.

Lemma 5.5 ([IR2], Lemma 1.2 and Theorem 2.3). *Let $S^d \subset \mathbb{P}^{n-1}$ be a QEL-manifold with $\delta > 0$. Then*

- (i) *Through two general points of S , there passes a unique entry locus, which is a smooth hyperquadric in its linear span.*
- (ii) *Let $x \in S$ be a general point and let $p_x : S \dashrightarrow \mathbb{P}^{n-d-2}$ be the tangential projection from $\mathbf{T}_x S$. Then the closure of the fiber of p_x through a general point $s \in S$ is the entry locus passing through s and x , hence it is a smooth hyperquadric of dimension δ .*

Proof of Proposition 5.4. For each of the conditions (1)-(7), we will show that there exists a Zariski open subset of S every element of which satisfies the condition. Then O can be taken as the intersection of these open subsets.

For (1), assume that $\text{Locus}(s) \cap \Sigma$ is non-empty for a general $s \in S$. Then

$$\bigcup_{y \in \Sigma \setminus \mathbf{T}_x \Sigma} \text{Locus}(y)$$

covers an open subset of S and for any s inside this open subset,

$$\text{Locus}(s) \cap \Sigma \not\subset \mathbf{T}_x \Sigma \cap \Sigma.$$

(2) follows from Lemma 5.5 (i) while (3) follows from Lemma 5.5 (ii).

Note that we have $\text{Locus}(x) \neq S$ as $S \neq \mathbb{P}^{n-1}$. Thus any point $s \in S \setminus \text{Locus}(x)$ satisfies the condition (4).

(5) is clear from Condition 5.2.

For (6), we take the open subset in (5). If $\langle \Sigma_s \rangle \cap \langle \Sigma \rangle$ contains a point not on S , say z , then $\langle \Sigma_s \rangle = C_z = \langle \Sigma \rangle$ by (5), which gives a contradiction. Thus $\langle \Sigma_s \rangle \cap \langle \Sigma \rangle \subset S$, hence $\Sigma_s \cap \Sigma = \langle \Sigma_s \rangle \cap \langle \Sigma \rangle$ is a linear subspace. The inclusion $P_s \subset \mathbf{T}_x \Sigma \cap \Sigma$ follows from the fact that $P_s \subset \Sigma$ is linear and $x \in P_s$.

(7) is obviously true for a general point s . \square

Proposition 5.6. *In Proposition 5.4, there exists a linear subspace $W_s \subset \mathbb{P}^{n-1}$ for each $s \in O$ such that*

$s \in W_s$, $\dim W_s = \dim P_s$, $\langle P_s \cap W_s, s \rangle = W_s$ and $\langle P_s, s \rangle \cap \Sigma_s = P_s \cup W_s$.

In particular, $\langle P_s, s \rangle \not\subseteq \Sigma_s$ for each $s \in O$.

Proof. As the line $\ell_{s,x}$ is not contained in S by Proposition 5.4 (4), we see that $\langle P_s, s \rangle \not\subseteq \Sigma_s$. As $\langle P_s, s \rangle$ is a linear subspace of $\langle \Sigma_s \rangle$ and Σ_s is a quadric hypersurface in $\langle \Sigma_s \rangle$, the intersection $\langle P_s, s \rangle \cap \Sigma_s$ is a hypersurface of degree ≤ 2 in $\langle P_s, s \rangle$, which contains already the linear subspace P_s . As $s \notin P_s$, there exists a linear subspace W_s with $\dim W_s = \dim P_s$ satisfying

$$s \in W_s \text{ and } \langle P_s, s \rangle \cap \Sigma_s = P_s \cup W_s.$$

Moreover $P_s \cap W_s$ must be a hyperplane in W_s disjoint from s . Thus $\langle P_s \cap W_s, s \rangle = W_s$. \square

Proposition 5.7. *In Proposition 5.4, assume furthermore that $S \subset \mathbb{P}^{n-1}$ is defined by quadratic equations. Then for any $s \in O \subset S$, there exists a linear subspace $F_s \subset S$ such that*

$$(5.1) \quad \langle \Sigma, s \rangle \cap S = \Sigma \cup F_s.$$

Moreover, we have $F_s = \langle \text{Locus}(s) \cap \Sigma, s \rangle$ and $\dim F_s = \dim(\text{Locus}(s) \cap \Sigma) + 1$. In particular, $\dim F_s = 0$ if and only if $\text{Locus}(s) \cap \Sigma = \emptyset$.

Proof. Let F_s be the closure of the fiber of p_Σ through s , which is nonsingular at s by Proposition 5.4 (7). We claim that F_s is a cone over $\text{Locus}(s) \cap \Sigma$ with vertex s . Then F_s must be linear as it is nonsingular at s and Proposition 5.7 follows.

To prove the claim, pick any $s' \in (\langle \Sigma, s \rangle \cap S) \setminus \Sigma$ with $s \neq s'$. It suffices to show that $\ell_{s,s'} \subset S$, which implies

$$s' \in \langle \text{Locus}(s) \cap \Sigma, s \rangle \subset \langle \Sigma, s \rangle \cap S,$$

proving the claim.

As $\langle s, \Sigma \rangle = \langle s', \Sigma \rangle$, the line $\ell_{s,s'}$ intersects $\langle \Sigma \rangle$ at a single point, say $t \in \langle \Sigma \rangle$. Suppose that $t \notin \Sigma$. Then $\Sigma_t(S) = \Sigma$ by Condition 5.2. As $\ell_{s,s'}$ passes through t , we have $s, s' \in \Sigma_t(S) = \Sigma$, a contradiction. Thus $t \in \Sigma$, which implies that the line $\ell_{s,s'}$ has 3 intersection points with S . From the assumption that S is defined by quadratic equations, we have $\ell_{s,s'} \subset S$, proving the claim. \square

Proposition 5.8. *Assume that $S \subset \mathbb{P}^{n-1}$ is a QEL-manifold with $\delta > 0$ that is defined by quadrics and satisfies Condition 5.2. Let O be as in Proposition 5.4. Then for any $s \in O$, we have*

$$(5.2) \quad \langle \mathbf{T}_x \Sigma, s \rangle \cap S = (\mathbf{T}_x \Sigma \cap S) \cup W_s$$

where W_s is as in Proposition 5.6.

It is convenient to recall the following straightforward lemma.

Lemma 5.9. *Let $L_1, L_2 \subset \mathbb{P}^{n-1}$ be two linear subspaces. Then*

- (i) $\langle L_1 \cap L_2, s \rangle = \langle L_1, s \rangle \cap L_2$ if $s \in L_2$; and
- (ii) $\langle L_1, s \rangle \cap L_2 = L_1$ if $L_1 \subset L_2$ and $s \notin L_2$.

Proof of Proposition 5.8. To start with, we claim the following relation

$$(5.3) \quad \langle P_s, s \rangle \cap \Sigma_s \subset \langle \mathbf{T}_x \Sigma, s \rangle \cap \Sigma_s \subsetneq \langle \Sigma, s \rangle \cap \langle \Sigma_s \rangle = \langle P_s, s \rangle.$$

The first inclusion is from $P_s \subset \mathbf{T}_x \Sigma$ in Proposition 5.4 (6). The second inclusion is from $\mathbf{T}_x \Sigma \subset \langle \Sigma, s \rangle$. The last equality follows from

$$\langle P_s, s \rangle = \langle \langle \Sigma \rangle \cap \langle \Sigma_s \rangle, s \rangle = \langle \Sigma, s \rangle \cap \langle \Sigma_s \rangle$$

which is a consequence of Proposition 5.4 (6) and Lemma 5.9 (i). Finally, the inequality in the middle is by $\langle P_s, s \rangle \not\subset \Sigma_s$ from Proposition 5.6.

In (5.3), the proper subvariety $\langle \mathbf{T}_x \Sigma, s \rangle \cap \Sigma_s$ of $\langle P_s, s \rangle$ is a subvariety of degree ≤ 2 . Since $\langle P_s, s \rangle$ is a linear subspace of $\langle \Sigma_s \rangle$ from Proposition 5.4 (6) and $\Sigma_s \subset \langle \Sigma_s \rangle$ is a quadric hypersurface from Proposition 5.4 (2), the intersection $\langle P_s, s \rangle \cap \Sigma_s$ must be a hypersurface of degree 2 in $\langle P_s, s \rangle$. It follows that

$$(5.4) \quad \langle P_s, s \rangle \cap \Sigma_s = \langle \mathbf{T}_x \Sigma, s \rangle \cap \Sigma_s.$$

From $\mathbf{T}_x \Sigma \subset \mathbf{T}_x S$, we have the tautological relation

$$(5.5) \quad \langle \mathbf{T}_x \Sigma, s \rangle \cap S = \langle \mathbf{T}_x \Sigma, s \rangle \cap (\langle \mathbf{T}_x S, s \rangle \cap S)$$

Combining (5.5) with $\langle \mathbf{T}_x S, s \rangle \cap S = \Sigma_s \cup (\mathbf{T}_x S \cap S)$ from Proposition 5.4 (3), we obtain

$$(5.6) \quad \langle \mathbf{T}_x \Sigma, s \rangle \cap S = (\langle \mathbf{T}_x \Sigma, s \rangle \cap \Sigma_s) \cup (\langle \mathbf{T}_x \Sigma, s \rangle \cap \mathbf{T}_x S \cap S).$$

For the first term on the righthand side of (5.6), we have

$$(5.7) \quad \langle \mathbf{T}_x \Sigma, s \rangle \cap \Sigma_s = \langle P_s, s \rangle \cap \Sigma_s = P_s \cup W_s$$

where the first equality is from (5.4) and the second equality is from Proposition 5.6. For the second term on the right hand side of (5.6), we have

$$(5.8) \quad \langle \mathbf{T}_x \Sigma, s \rangle \cap \mathbf{T}_x S \cap S = \mathbf{T}_x \Sigma \cap S$$

because $\langle \mathbf{T}_x \Sigma, s \rangle \cap \mathbf{T}_x S = \mathbf{T}_x \Sigma$ by Lemma 5.9 (ii). Putting (5.7) and (5.8) into (5.6), we obtain

$$\langle \mathbf{T}_x \Sigma, s \rangle \cap S = P_s \cup W_s \cup (\mathbf{T}_x \Sigma \cap S).$$

Recalling $P_s \subset \mathbf{T}_x \Sigma \cap S$ from Proposition 5.4 (6), we have (5.2). \square

Theorem 5.10. *In Proposition 5.8, assume that S is prime Fano and that $\text{Locus}(s) \cap \Sigma \neq \emptyset$ for $s \in O$. Then*

- (i) $\dim F_s = \frac{3\delta-d}{2}$;

- (ii) W_s is a codimension-1 linear subspace of F_s ; and
- (iii) $\Sigma \cap \Sigma_s \simeq \mathbb{P}^{\frac{3\delta-d-2}{2}}$.

Proof. Consider the incidence variety

$$I = \text{the closure of } \{(t, y) \in \Sigma \times O \mid L_{ty} \subset S\}$$

and the two projections $p_1 : I \rightarrow \Sigma$ and $p_2 : I \rightarrow S$. As S is a prime Fano QEL-manifold, Proposition 4.6 gives $\dim \mathcal{C}_t = \frac{d+\delta}{2} - 2$ for a general $t \in S$, which implies

$$\dim \text{Locus}(t) = \dim \mathcal{C}_t + 1 = \frac{d+\delta}{2} - 1.$$

Since we have chosen our Σ generally, the dimension of a general fiber of p_1 is equal to $\dim \text{Locus}(t)$ for general $t \in S$. Thus

$$\dim I - \dim \Sigma = \frac{d+\delta}{2} - 1 \text{ and } \dim I = \delta + \frac{d+\delta}{2} - 1.$$

The assumption $\text{Locus}(s) \cap \Sigma \neq \emptyset$ for $s \in O$ implies that p_2 is surjective by Proposition 5.4 (1). Thus the dimension of a general fiber of p_2 is

$$\dim I - \dim S = \delta + \frac{d+\delta}{2} - 1 - d = \frac{3\delta-d}{2} - 1.$$

Since the fiber of p_2 over a general $s \in O$ is $\text{Locus}(s) \cap \Sigma$ and $F_s = \langle \text{Locus}(s) \cap \Sigma, s \rangle$ from Proposition 5.7, we have $\dim F_s = \frac{3\delta-d}{2}$, proving (i).

Comparing equations (5.1) and (5.2), we see that $W_s \subset F_s$ is a linear subspace of codimension ≤ 1 . Suppose that $W_s = F_s$. Note that $P_s \cap W_s \subset \text{Locus}(s) \cap \Sigma$ from $P_s \subset \Sigma$ and $W_s \subset \text{Locus}(s)$. As $F_s = \langle s, \text{Locus}(s) \cap \Sigma \rangle$ from Proposition 5.7 and $W_s = \langle s, P_s \cap W_s \rangle$ from Proposition 5.6, the hypothesis $F_s = W_s$ implies $\text{Locus}(s) \cap \Sigma = P_s \cap W_s$. Combining this with Proposition 5.4 (6), we have

$$\text{Locus}(s) \cap \Sigma \subset P_s \subset \Sigma \cap \mathbf{T}_x \Sigma$$

which contradicts Proposition 5.4 (1). Thus W_s is of codimension one in F_s , proving (ii).

Finally, the equality $\dim P_s = \dim W_s$ in Proposition 5.6 implies

$$\dim(\Sigma \cap \Sigma_s) = \dim W_s = \dim F_s - 1 = \frac{3\delta-d-2}{2},$$

proving (iii). □

We close this section with one explicit example of the intersection of entry loci. It is a simple consequence of the following elementary lemma, whose proof will be skipped.

Lemma 5.11. *Let V be a vector space of dimension $2m \geq 6$, equipped with a non-degenerate quadratic form $Q : V \times V \rightarrow \mathbb{C}$.*

- (1) *The two components $\mathbb{S}^+(V; Q)$ and $\mathbb{S}^-(V; Q)$ of the space of m -dimensional Q -isotropic subspaces of V admit projective embeddings, realizing them as prime Fano manifolds biregular to each other.*
- (2) *Let $W \subset V$ be a Q -isotropic subspace of dimension $\leq m - 3$ and let*

$$W^\perp := \{v \in V, Q(v, w) = 0 \text{ for all } w \in W\}.$$

The quotient space W^\perp/W with $\dim W^\perp/W \geq 6$ is equipped with a non-degenerate quadratic form Q^W induced by Q and we have natural inclusions

$$\mathbb{S}^+(W^\perp/W; Q^W) \subset \mathbb{S}^+(V; Q) \text{ and } \mathbb{S}^-(W^\perp/W; Q^W) \subset \mathbb{S}^-(V; Q)$$

which induce isomorphisms between their Picard groups.

- (3) *When $W = \mathbb{C}v$ for some $0 \neq v \in V$, let us write $\mathbb{S}^+(W^\perp/W; Q^W)$ as \mathbb{S}^v . For two independent vectors $v, v' \in V$, if $Q(v, v') \neq 0$, then $\mathbb{S}^v \cap \mathbb{S}^{v'} = \emptyset$, while if $Q(v, v') = 0$, then*

$$\mathbb{S}^v \cap \mathbb{S}^{v'} = \mathbb{S}^+(W^\perp/W; Q^W) \text{ for } W = \mathbb{C}v + \mathbb{C}v'.$$

We denote by \mathbb{S}_m the Spinor variety $\mathbb{S}^+(V; Q)$ of Lemma 5.11. We have the following consequence.

Proposition 5.12. *The intersection of any two distinct entry loci of $\mathbb{S}_5 \subset \mathbb{P}^{15}$ is either empty or \mathbb{P}^3 .*

Proof. It is well-known that the embedding of \mathbb{S}_m as a prime Fano manifold in Lemma 5.11 becomes an isomorphism $\mathbb{S}_3 \cong \mathbb{P}^3$ when $m = 3$ and an isomorphism $\mathbb{S}_4 \cong \mathbb{Q}^6 \subset \mathbb{P}^7$ when $m = 4$. Applying Lemma 5.11 with $m = 5$, we have a natural inclusion $\mathbb{S}_4 \cong \mathbb{S}^v \subset \mathbb{S}_5$ for any isotropic vector $v \in V$. From the isomorphism $\mathbb{S}_4 \simeq \mathbb{Q}^6$, it is an entry locus of \mathbb{S}_5 . As these \mathbb{S}^v cover \mathbb{S}_5 , every entry locus of \mathbb{S}_5 is of the form \mathbb{S}^v for some isotropic vector v . Take any two distinct entry loci \mathbb{S}^v and $\mathbb{S}^{v'}$. If their intersection is non-empty, then $Q(v, v') = 0$ and $\mathbb{S}^v \cap \mathbb{S}^{v'} \simeq \mathbb{S}_3 \simeq \mathbb{P}^3$. \square

Recall (e.g. Proposition 1.3 (ii) in [IR1]) that the entry loci of a linear section with codimension $\leq \delta$ of a QEL-manifold S with $\delta \geq 1$ come from the corresponding linear section of entry loci of S . Thus Proposition 5.12 implies the following.

Corollary 5.13. *The intersection of any two distinct entry loci of a nonsingular linear section \mathbb{S}_5 with codimension $t \leq 2$ is either empty or isomorphic to \mathbb{P}^k with $k \geq 3 - t$.*

6. PROOF OF THEOREM 2.15

In this section, we will prove Theorem 2.15 by showing that a pair $(S \subset \mathbb{P}U, Y \subset \mathbb{P}W)$ determined by a special system of quadrics must be one of the examples in Example 2.3, Example 2.4, Proposition 2.11 or Proposition 2.12. We will divide our argument into a number of subsections. In the first two subsections, we will classify a certain type of special quadratic manifolds. In the next three subsections, we carry out the classification of the pair (S, Y) .

6.1. Classification when $\delta \geq \frac{1}{2} \dim S$. As an easy application of Proposition 4.7 and Proposition 4.8, we have the following.

Proposition 6.1. *A d -dimensional special quadratic manifold $S^d \subset \mathbb{P}U$ with secant defect $\delta \geq \frac{d}{2}$ is projectively equivalent to one of the following. For our later use, we specify one possibility of $Y = Y(\sigma)$ in each case in the list.*

- (i) the smooth hyperquadric $\mathbb{Q}^d \subset \mathbb{P}^{d+1}$ with Y a point;
- (ii) the Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ with $Y = \mathbb{P}^2$;
- (iii) the Plücker embedding of $\text{Gr}(2, 5) \subset \mathbb{P}^9$ with $Y = \mathbb{P}^4$;
- (iv) the 10-dimensional Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$ with $Y = \mathbb{Q}^8$;
- (v) a general hyperplane section of (iii) with $Y = \mathbb{P}^4$;
- (vi) a general hyperplane section of (ii) with $Y = \mathbb{P}^2$;
- (vii) a general codimension 2 linear section of (iii) with $Y = \mathbb{P}^4$;
- (viii) the Segre fourfold $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ with $Y = \text{Gr}(2, 4) \simeq \mathbb{Q}^4$.

Proof. If $\delta = d$, the fact $\text{Sec}(S) = \mathbb{P}U$ and $\delta = 2d + 1 - \dim \text{Sec}(S)$ implies that S is a hyperquadric. This gives case (i).

As S is a QEL-manifold by Proposition 4.2, if $d > \delta \geq d/2$, we can apply Proposition 4.7 and Proposition 4.8. In Proposition 4.7, (A5) is excluded by Proposition 2.14. Thus only (A1)-(A4) are possible, which gives (ii)-(v).

In Proposition 4.8, the four Severi varieties (B2), (B4), (B7) and (B8) do not satisfy $\text{Sec}(S) = \mathbb{P}U$ of Proposition 3.3. Also (B6) is excluded by Proposition 2.14. Thus only (B1), (B5) and (B3) are possible, which gives (vi)-(viii).

The example of $Y(\sigma)$ in each case follows from the table in Example 2.3, Proposition 2.11 and Proposition 2.12. \square

6.2. Classification when $\delta \geq 1$ and S is not a prime Fano manifold. Special quadratic manifolds with $\delta \geq 1$ that are not prime Fano manifolds can be classified as follows.

Proposition 6.2. *Let $S \subset \mathbb{P}U$ be a special quadratic manifold with $\delta \geq 1$. If S is not a prime Fano manifold, then $S \subset \mathbb{P}U$ is projectively equivalent to one of the following:*

- (c1) *a smooth conic in \mathbb{P}^2 ;*
- (c2) *a general hyperplane section of the Segre 3-fold $\mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^5$;*
- (c3) *the Segre variety $\mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \mathbb{P}^{2d-1}$.*

The proof is a direct consequence of the following result of Ionescu and Russo.

Theorem 6.3 (Theorem 2.2 in [IR1]). *Let $S \subset \mathbb{P}U$ be a nondegenerate, linearly normal, conic-connected manifold of dimension d . Then either S is a prime Fano manifold or it is projectively equivalent to one of the following:*

- (C1) *the second Veronese embedding of \mathbb{P}^d .*
- (C2) *the Segre embedding of $\mathbb{P}^a \times \mathbb{P}^{d-a}$ for $1 \leq a \leq d-1$.*
- (C3) *the VMRT of the symplectic Grassmannian $\text{Gr}_\omega(k, k+d+1)$ for $2 \leq k \leq d$.*
- (C4) *a hyperplane section of the Segre embedding $\mathbb{P}^a \times \mathbb{P}^{d+1-a}$ with $2 \leq a, d+1-a$.*

Proof of Proposition 6.2. By Proposition 4.2, we know that $S \subset \mathbb{P}U$ is linearly normal. By Theorem 2.1 of [R2], the condition $\delta \geq 1$ implies that S is conic-connected. Thus if S is not a prime Fano manifold, it must be one of (C1)-(C4) in Theorem 6.3.

Recall that we have $\text{Sec}(S) = \mathbb{P}U$ by Proposition 3.3. It is well known (e.g. the table in p.466 of [FH]) that only (c1) in (C1) and (c2) in (C2) satisfy $\text{Sec}(S) = \mathbb{P}U$ and no cases in (C4) satisfy $\text{Sec}(S) = \mathbb{P}U$. In (C3), only a general hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^d \subset \mathbb{P}^{2d+1}$ satisfies $\text{Sec}(S) = \mathbb{P}U$ by Lemma 4.19 [FH]. By Proposition 2.14, only (c3) can occur. \square

6.3. Classification when $Y = \mathbb{P}^{2(c-1)}$. We can classify special quadratic manifolds in the case of (Y1) of Theorem 3.10:

Proposition 6.4. *Assume that $Y = \mathbb{P}^{2(c-1)}$, then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to one of the following:*

- (i) $\mathbb{Q}^d \subset \mathbb{P}^{d+1}$;
- (ii) $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$;
- (iii) $\text{Gr}(2, 5) \subset \mathbb{P}^9$;
- (iv) *a general hyperplane section of (ii);*
- (v) *a general codimension ≤ 2 linear section of (iii);*
- (vi) *a general codimension-3 linear section of (iii);*
- (vii) *a general codimension-2 linear section of (ii).*

All of the above cases do occur with $Y = \mathbb{P}^{2(c-1)}$ from Example 2.3, Proposition 2.11 and Proposition 2.12.

Proof. As $Y = \mathbb{P}^{2(c-1)}$, we have $\chi(Y) = 2c - 1$. Proposition 3.9 yields $\chi(S) = 2\delta + 2$. By Corollary 3.11, we obtain $2\delta + 2 \geq d + 1$, i.e., $\delta \geq (d-1)/2$. If $\delta \geq d/2$, then Proposition 6.1 gives (i)-(v). It remains to handle the case $\delta = (d-1)/2$ and $\chi(S) = d+1$. The latter condition combined with Corollary 3.11 implies that all even Betti numbers of S must be 1. We will argue case-by-case, depending on the values of δ .

- (1) If $\delta \geq 3$, then $d - \delta = \delta + 1$ is divisible by $2^{\lfloor \frac{\delta-1}{2} \rfloor}$ by Proposition 4.6. This implies that either $(\delta, d, n) = (3, 7, 13)$ or $(\delta, d, n) = (7, 15, 25)$.
 - (1a) When $(\delta, d, n) = (3, 7, 13)$, Proposition 4.6 shows that S is a prime Fano manifold with $K_S^{-1} = \mathcal{O}(\frac{d+\delta}{2}) = \mathcal{O}(5)$. By Proposition 4.10, the only possibility is (M2), a linear section of \mathbb{S}_5 . But such a case can not occur by Proposition 2.14.
 - (1b) When $(\delta, d, n) = (7, 15, 25)$, we see that S is a prime Fano manifold and its VMRT $\mathcal{C}_x \subset \mathbb{P}^{14}$ at a general point $x \in S$ is a QEL-manifold of dimension $(d+\delta)/2 - 2 = 9$ and secant defect $\delta - 2 = 5$ from Proposition 4.6. It follows that the subvariety $\text{Locus}(x) \subset S$ has dimension 10 and has nonempty intersection with a general entry locus $\Sigma \subset S$ of dimension 7, because all even Betti numbers of S are 1. Since S is defined by quadrics and satisfies Condition 5.2 in Section 5 by Lemma 5.3, we can apply Theorem 5.10 to conclude that two general entry loci Σ_1 and Σ_2 of S through a general point $x \in S$ intersect along \mathbb{P}^2 . By Proposition 4.6, the lines on Σ_1 and Σ_2 through x give two entry loci on $\mathcal{C}_x \subset \mathbb{P}^{14}$. The intersection of these entry loci of \mathcal{C}_x must be \mathbb{P}^1 corresponding to the lines through x in $\Sigma_1 \cap \Sigma_2 \cong \mathbb{P}^2$. On the other hand, applying Proposition 4.6 to \mathcal{C}_x , we see that \mathcal{C}_x is a prime Fano manifold with $K_{\mathcal{C}_x}^{-1} = \mathcal{O}(7)$. Proposition 4.10 shows that $\mathcal{C}_x \subset \mathbb{P}^{14}$ is projectively equivalent to a codimension 1 linear section of $\mathbb{S}_5 \subset \mathbb{P}^{15}$. Thus the intersection of any two distinct entry loci of \mathcal{C}_x must be either empty or \mathbb{P}^k with $k \geq 2$, from Corollary 5.13, a contradiction.
- (2) If $\delta = 2$, then $d = 5$ and $n = 10$. As $d + \delta$ is odd, S is not a prime Fano manifold by Proposition 4.6. Applying Proposition 6.2, we see that $S \subset \mathbb{P}^9$ is the Segre 5-fold $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$, a contradiction to $\chi(S) = d + 1$.

- (3) If $\delta = 1$, then $d = 3$ and $n = 7$. Since none of the varieties in the list in Proposition 6.2 have $(d, n) = (3, 7)$, we see that S is a prime Fano manifold. Thus $K_S^{-1} = \mathcal{O}(2)$ by Proposition 4.6 (1). From the classification of prime Fano threefolds with $K_S^{-1} = \mathcal{O}(2)$ (e.g. 12.1 in [IP]), S is a general codimension-3 linear section of $\text{Gr}(2, 5) \subset \mathbb{P}^9$, which gives case (vi).
- (4) If $\delta = 0$, then $d = 1$ and $n = 4$. Thus S is a twisted cubic by Proposition 4.12, which is (vii).

□

6.4. Classification when $Y = \mathbb{Q}^{2(c-1)}$. We can classify special quadratic manifolds in the case of (Y2) of Theorem 3.10:

Proposition 6.5. *For a special quadratic manifold $S \subset \mathbb{P}^{n-1}$ of codimension c , assume that $Y = \mathbb{Q}^{2(c-1)}$. Then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to one of the following*

- (i) the Segre 4-fold $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$;
- (ii) the 10-dimensional Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$.

Both cases do occur with $Y = \mathbb{Q}^{2(c-1)}$ from Example 2.3.

Proof. If $\delta \geq \frac{d}{2}$, then Proposition 6.1 gives (i) and (ii).

To show that $\delta < \frac{d}{2}$ cannot occur, we may assume $d = 2\delta + 2$ and $n = 2d + 2 - \delta = 3\delta + 6$ by Proposition 3.12 and derive a contradiction. Under this assumption, putting $\chi(\mathbb{Q}^{2(c-1)}) = 2c$ in Proposition 3.9, we have $\chi(S) = d + 1$, which combined with Corollary 3.11 implies that all even Betti numbers of S must be 1. We will argue case-by-case depending on the value of δ .

- (1) Assume $\delta \geq 3$. Proposition 4.6 shows that the VMRT $\mathcal{C}_x \subset \mathbb{P}^{d-1}$ at a general point $x \in S$ is a QEL-manifold of dimension $\frac{1}{2}(d + \delta - 4) = \frac{1}{2}(3\delta - 2)$ and secant defect $\delta - 2$. Since $d - \delta = \delta + 2$ is divisible by $2^{\lfloor \frac{\delta-1}{2} \rfloor}$ from Proposition 4.6, we have two possibilities: $(\delta, d, n) = (4, 10, 18)$ or $(\delta, d, n) = (6, 14, 24)$.
 - (1a) When $(\delta, d, n) = (4, 10, 18)$, the VMRT $\mathcal{C}_x \subset \mathbb{P}^9$ is a QEL manifold of dimension 5 and secant defect 2. By Proposition 4.6, this implies that $\mathcal{C}_x \subset \mathbb{P}^9$ is not a prime Fano manifold. Thus Proposition 6.2 shows that $\mathcal{C}_x \subset \mathbb{P}^9$ is projectively equivalent to the Segre 5-fold $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$. By Main Theorem of Section 2 in [Mo], S is isomorphic to $\text{Gr}(2, 7)$. But $S \subset \mathbb{P}^{17}$ is linearly normal by Proposition 4.2, which is not possible for $S = \text{Gr}(2, 7)$.
 - (1b) When $(\delta, d, n) = (6, 14, 24)$, we see that S is a prime Fano manifold and its VMRT $\mathcal{C}_x \subset \mathbb{P}^{13}$ at a general point $x \in S$

is a QEL-manifold of dimension 8 and secant defect 4 from Proposition 4.6. It follows that the subvariety $\text{Locus}(x) \subset S$ has dimension 9 and has nonempty intersection with a general entry locus $\Sigma \subset S$ of dimension 6, because all even Betti numbers of S are 1. Since S is defined by quadrics and satisfies Condition 5.2 in Section 5 by Lemma 5.3, we can apply Theorem 5.10 to conclude that two general entry loci Σ_1 and Σ_2 of S through a general point $x \in S$ intersect along \mathbb{P}^1 . By Proposition 4.6, the lines on Σ_1 and Σ_2 through s give two entry loci on $\mathcal{C}_x \subset \mathbb{P}^{13}$. The intersection of these entry loci of \mathcal{C}_x must be one point corresponding to the unique line through x in $\Sigma_1 \cap \Sigma_2$. On the other hand, applying Proposition 4.6 to \mathcal{C}_x , we see that \mathcal{C}_x is a prime Fano manifold with $K_{\mathcal{C}_x}^{-1} = \mathcal{O}(6)$. Proposition 4.10 shows that $\mathcal{C}_x \subset \mathbb{P}^{13}$ is projectively equivalent to a codimension 2 linear section of $\mathbb{S}_{10} \subset \mathbb{P}^{15}$. Thus the intersection of any two distinct entry loci of \mathcal{C}_x must be either empty or \mathbb{P}^k with $k \geq 1$ from Corollary 5.13, a contradiction.

- (2) If $\delta = 2$, then $d = 6$ and $n = 12$.
 - (2a) If S is not a prime Fano manifold, then Proposition 6.2 shows that S is the Segre 6-fold $\mathbb{P}^1 \times \mathbb{P}^5 \subset \mathbb{P}^{11}$. This contradicts $\chi(S) = d + 1$.
 - (2b) If S is a prime Fano manifold, then Proposition 4.10 is applicable. The only possibility is $s = 4$ of (M2), i.e., a codimension-4 linear section of the 10-dimensional Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$. But this can not be a special quadratic manifold by Proposition 2.14.
- (3) If $\delta = 1$, then $d = 4$ and $n = 9$. As $d + \delta$ is odd, Proposition 4.6 shows that S is not a prime Fano manifold. This is not possible by Proposition 6.2.
- (4) If $\delta = 0$, then $d = 2$ and Proposition 4.12 is applicable. But the surfaces in Proposition 4.12 do not have Picard number 1, a contradiction.

□

6.5. Classification when $Y = \text{Gr}(2, c + 1)$. To handle the case (Y3) of Theorem 3.10, we use the following result.

Proposition 6.6 (Theorem 2.17 ($b = 2$) [AS]). *Let V be a vector space with $\dim V = 2k + 1 \geq 7$. Let $\mu : \text{Sym}^2 V \rightarrow W$ be a system of quadrics such that the base locus subscheme $B(\mu) \subset \mathbb{P}V$ is irreducible, nonsingular and nondegenerate. Let $\nu : \mathbb{P}V \dashrightarrow \mathbb{P}W$ be the associated*

rational map. Suppose that ν is generically injective and its proper image $\text{Im}(\nu)$ is biregular to $\text{Gr}(2, k+2)$. Then $B(\mu) \subset \mathbb{P}V$ is one of the rational normal scrolls. In particular, $B(\mu)$ has Picard number 2 and is covered by lines.

We can classify special quadratic manifolds in the case of (Y3) of Theorem 3.10:

Proposition 6.7. *Let $S \subset \mathbb{P}^{n-1}$ be a special quadratic manifold of codimension $c \geq 4$ with $Y \subset \mathbb{P}W$ being an isomorphic projection of the Plücker variety $\text{Gr}(2, c+1) \subset \mathbb{P}(\wedge^2 \mathbb{C}^{c+1})$. Then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to $\mathbb{P}^1 \times \mathbb{P}^c \subset \mathbb{P}^{2c+1}$.*

Proof. Let $\sigma : \text{Sym}^2 U \rightarrow W, \dim U = n$, be the special system of quadrics such that our S is the base locus $B(\sigma) \subset \mathbb{P}U$. Recall that a general fiber of $\psi : \text{Bl}_S(\mathbb{P}U) \rightarrow Y(\sigma) \cong \text{Gr}(2, c+1)$ is sent to a linear space of dimension $\delta + 1$ in $\mathbb{P}U$ by Proposition 3.8. Take a general subspace $V \subset U$ with

$$\dim U - \dim V = \delta + 1, \quad \dim V = 2c - 1 \geq 7$$

and denote by $S' \subset \mathbb{P}V$ the linear section $S \cap \mathbb{P}V$. Then the restriction

$$\psi|_{\text{Bl}_{S'}(\mathbb{P}V)} : \text{Bl}_{S'}(\mathbb{P}V) \rightarrow Y(\sigma)$$

is birational. The birational map $\mathbb{P}V \dashrightarrow Y(\sigma)$ is given by a system of quadrics $\mu : \text{Sym}^2 V \rightarrow W$ induced by the restriction of σ . Since the base locus $B(\mu)$ is S' , which is irreducible, nonsingular and nondegenerate in $\mathbb{P}V$, we can apply Proposition 6.6 to see that S' is a rational normal scroll. We will argue case-by-case as follows.

- (1) When $\delta = 0$. Putting $n = 2d + 2$ and $\chi(\text{Gr}(2, c+1)) = \frac{c(c+1)}{2}$ in Proposition 3.9, we have $\chi(S) = d + 1$. This implies that S has Picard number 1.
 - (1a) If $c \geq 5$, then $\dim S' = c - 2 \geq 3$. Thus S' and S have the same Picard number by Lefschetz. Since S' has Picard number 2, this gives a contradiction.
 - (1b) If $c = 4$, then $d = 3$. We conclude that $S \subset \mathbb{P}^7$ is a Fano threefold of Picard number 1 with $\chi(S) = 4$. Furthermore, a general hyperplane section S' of S is a rational normal scroll. In particular, S is covered by lines, which implies that $K_S^{-1} = \mathcal{O}(k), k \geq 2$. From the classification of Fano threefolds (e.g. 12.1 of [IP]), there are three possibilities for such Fano threefolds: $\mathbb{P}^3, \mathbb{Q}^3$ and a codimension-3 linear section of $\text{Gr}(2, 5) \subset \mathbb{P}^9$. But none of these can have $\delta = 0$.
- (2) When $\delta \geq 1$.

- (2a) Assume that S is a prime Fano manifold and $c \geq 5$. Then $\dim S' = c - 2 \geq 3$, which implies that S' and S have the same Picard number by Lefschetz. Since S' has Picard number 2, this gives a contradiction.
- (2b) Assume that S is a prime Fano manifold and $c = 4$. Then $n = d + 5$ and $d - \delta = 3$ is odd. This is a contradiction to Proposition 4.6.
- (2c) Assume that S is not a prime Fano manifold. Proposition 6.2 shows that $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^c \subset \mathbb{P}^{2c+1}$.

□

7. QUADRATICALLY SYMMETRIC VARIETIES AND PROLONGATIONS

In this section, we give an intrinsic characterization of $Z(\sigma)$ and relate it to the problem studied in [FH]. Throughout this section, $X \subset \mathbb{P}V$ denotes an n -dimensional irreducible nondegenerate projective subvariety.

Definition 7.1. Let $X \subset \mathbb{P}V$ be an n -dimensional irreducible nondegenerate projective subvariety. We say that X is a *quadratically symmetric variety* if there exists a Zariski open subset $X^\circ \subset \text{Sm}(X)$ in the nonsingular locus of X such that for each $P \in X^\circ$, there exists a linear representation of the \mathbb{C}^* -group denoted by $E_P : \mathbb{C}^* \times V \rightarrow V$ on V satisfying the following conditions.

- (i) Denoting by $\widehat{P} \subset V$ the tautological line over P and by $T_P(\widehat{X}) \subset V$ the affine tangent space, the \mathbb{C}^* -action E_P has weight 0 on \widehat{P} , weight 1 on $T_P(\widehat{X})/\widehat{P}$ and weight 2 on $V/T_P(\widehat{X})$. In other words, there exist subspaces $T'_P \subset T_P(\widehat{X})$ and $N'_P \subset V$ with

$$V = \widehat{P} \oplus T'_P \oplus N'_P \text{ and } T_P(\widehat{X}) = \widehat{P} \oplus T'_P$$

such that $\lambda \in \mathbb{C}^*$ acts by

$$E_P^\lambda(t \in \widehat{P}, u \in T'_P, w \in N'_P) = (t, \lambda u, \lambda^2 w).$$

- (2) The induced \mathbb{C}^* -action on $\mathbb{P}V$ preserves X .

Notation 7.2. For a projective manifold $M \subset \mathbb{P}^N$, the second fundamental form at a point $x \in M$ is the non-empty linear system of quadrics on the tangent space $T_x M$, denoted by $|\text{II}_{x,M}| \subset \mathbb{P}(\text{Sym}^2 T_x M)^*$, which comes from the derivative of the Gauss map (see Section 3.2 of [IL] or p.602 of [R2] for details).

Proposition 7.3. *Let $X \subset \mathbb{P}V$, $\dim V = N + 1$, be a quadratically symmetric variety of dimension n . For a fixed $P \in X^\circ$ and the representation E_P of Definition 7.1, let (x_0, x_1, \dots, x_N) be linear coordinates on V such that the dual action of $\lambda \in \mathbb{C}^*$, denoted by the same symbol E_P^λ for simplicity, is given by $E_P^\lambda(x_0) = x_0$ and*

$$E_P^\lambda(x_i) = \begin{cases} \lambda x_i & \text{if } 1 \leq i \leq n \\ \lambda^2 x_i & \text{if } n+1 \leq i \leq N. \end{cases}$$

Let $z_i = \frac{x_i}{x_0}$, $1 \leq i \leq N$, be the inhomogeneous coordinates on $\mathbb{P}V$ centered at P . View the germ of $X \subset \mathbb{P}V$ at P as the graph of the germ of a holomorphic map

$$z_k = F^k(z_1, \dots, z_n), n+1 \leq k \leq N.$$

Then $F^k(z_1, \dots, z_n)$ are homogeneous quadratic polynomials and the system of quadrics on $T_P X$ defined by $\{F^k, n+1 \leq k \leq N\}$ under the identification of $T_P^* X = \mathbb{C}z_1 + \dots + \mathbb{C}z_n$ is isomorphic to the second fundamental form $|\Pi_{P,X}|$ of X at P .

Proof. It is well-known (e.g. p. 108 of [IL]) that the quadratic terms in the Taylor expansion of F^k 's give the second fundamental form $|\Pi_{P,X}|$. So it suffices to show that F^k 's are quadratic. The \mathbb{C}^* -action E_P induces a \mathbb{C}^* -action on the affine space $\mathbb{P}V \setminus (x_0 = 0) = \{(z_1, \dots, z_N)\}$ by

$$E_P^\lambda(z_i) = \begin{cases} \lambda z_i & \text{if } 1 \leq i \leq n \\ \lambda^2 z_i & \text{if } n+1 \leq i \leq N. \end{cases}$$

Since this action preserves the germ of X at P , the equations

$$z_k = F^k(z_1, \dots, z_n), n+1 \leq k \leq N$$

must remain unchanged under the \mathbb{C}^* -action. Thus the Taylor expansion of F^k can have only quadratic terms. \square

Proposition 7.4. *Let X_1 and X_2 be two quadratically symmetric varieties of the same dimension in $\mathbb{P}V$. Assume that there are points $P_1 \in X_1^\circ$ and $P_2 \in X_2^\circ$ such that the second fundamental forms Π_{P_1, X_1} and Π_{P_2, X_2} are isomorphic as systems of quadrics. Then X_1 and X_2 are projectively equivalent.*

Proof. We can choose the inhomogeneous coordinates in Proposition 7.3 such that the germ of $X_i \subset \mathbb{P}V$ at P_i for $i = 1, 2$ is defined by equations $z_k = F_i^k(z_1, \dots, z_n)$ for some quadratic polynomials F_i^k , $n+1 \leq k, i = 1, 2$. Since Π_{P_1, X_1} and Π_{P_2, X_2} are isomorphic, we can assume that $F_1^k = F_2^k$ by linear coordinate changes. It follows that X_1 and X_2 are projectively equivalent. \square

The following is a generalization of $Z(\sigma)$ in Definition 2.1.

Notation 7.5. Let U, W be two vector spaces and let $\sigma : \text{Sym}^2 U \rightarrow W$ be a surjective homomorphism. Fix a 1-dimensional vector space T with a fixed identification $T = \mathbb{C}$. Define a rational map $\phi^\circ : \mathbb{P}(T \oplus U) \dashrightarrow \mathbb{P}(T \oplus U \oplus W)$ by

$$[t : u] \mapsto [t^2 : tu : \sigma(u, u)] \text{ for } t \in T, u \in U.$$

The proper image of $\mathbb{P}(T \oplus U)$ under ϕ° will be denoted by $Z = Z(\sigma) \subset \mathbb{P}(T \oplus U \oplus W)$. Note that ϕ sends $U \cong \mathbb{P}(T \oplus U) \setminus \mathbb{P}U$ isomorphically to $Z \setminus \mathbb{P}(U \oplus W)$. Thus Z is a rational variety and $\phi^\circ : \mathbb{P}(T \oplus U) \dashrightarrow Z$ is a birational map.

Proposition 7.6. *In Notation 7.5, the variety $Z(\sigma)$ is a quadratically symmetric variety.*

Proof. Consider the \mathbb{C}^* -action on $\mathbb{P}(T \oplus U \oplus W)$ given by $\lambda \cdot [t : u : w] = [t : \lambda u : \lambda^2 w]$ for all $\lambda \in \mathbb{C}^*$. Then it preserves $Z(\sigma)$. This gives E_P in Definition 7.1 when $P = \mathbb{P}T \in Z(\sigma)$.

As in the proof of Proposition 2.6, we can associate to each vector $v \in U$ the linear automorphism g_v of $\mathbb{P}(T \oplus U \oplus W)$ defined by

$$g_v : [t : u : w] \mapsto [t : u + tv : w + 2\sigma(u, v) + t\sigma(v, v)].$$

Then g_v preserves $Z(\sigma)$ as we saw in the proof of Proposition 2.6. In particular, the linear automorphism group of $Z(\sigma) \subset \mathbb{P}(T \oplus U \oplus W)$ acts transitively on $Z(\sigma) \setminus \mathbb{P}(U \oplus W)$. Thus E_P exists for any $P \in Z(\sigma) \setminus \mathbb{P}(U \oplus W)$. \square

Proposition 7.7. *Any quadratically symmetric variety $X \subset \mathbb{P}V$ is projectively equivalent to $Z(\sigma)$, where $\sigma : \text{Sym}^2 T_P X \rightarrow (T_P \mathbb{P}V)/T_P X$ is the dual of the second fundamental form of X at a point $P \in X^\circ$.*

Proof. By Section 3 of [L], the second fundamental form of $Z(\sigma)$ at a point in $Z(\sigma) \setminus \mathbb{P}(U \oplus W)$ is isomorphic to σ . Hence X is projectively equivalent to $Z(\sigma)$ by Proposition 7.4 and Proposition 7.6. \square

Now we can give an intrinsic characterization of the projective manifold $Z(\sigma)$ associated to a special system of quadrics σ as follows.

Theorem 7.8. *Let $X \subset \mathbb{P}V$ be a prime Fano manifold. Then $X \subset \mathbb{P}V$ is a quadratically symmetric variety if and only if $X = Z(\sigma)$ for a special system of quadrics σ . In particular, Theorem 2.15 gives a classification of quadratically symmetric prime Fano manifolds.*

Proof. From Theorem 2.15 and Proposition 7.6, the manifold $Z(\sigma)$ associated to a special system of quadrics σ is a quadratically symmetric prime Fano manifold.

Conversely, a quadratically symmetric prime Fano manifold is of the form $Z(\sigma)$ by Proposition 7.7 where σ is its second fundamental form. By Proposition 2.15 of [AS], the locus $\sigma(u, u) = 0$ is exactly the VMRT at a general point of X . Since $Z(\sigma)$ is a prime Fano manifold, its VMRT at a general point is smooth and irreducible by Proposition 6.6 of [FH]. This implies that $X \subset \mathbb{P}V$ is $Z(\sigma)$ associated to a special system of quadratics σ . \square

Theorem 7.8 enables us to relate special quadratic manifolds to the problem studied in [FH]. Let us recall the basic definitions.

Definition 7.9. Let V be a complex vector space and $\mathfrak{g} \subset \text{End}(V)$ a Lie subalgebra. The *first prolongation* (denoted by $\mathfrak{g}^{(1)}$) of \mathfrak{g} is the space of bilinear homomorphisms $A : \text{Sym}^2 V \rightarrow V$ such that for any fixed $w \in V$, the endomorphism $A_w : V \rightarrow V$ defined by

$$v \in V \mapsto A_{w,v} := A(w, v) \in V$$

is in \mathfrak{g} . In other words, $\mathfrak{g}^{(1)} = \text{Hom}(\text{Sym}^2 V, V) \cap \text{Hom}(V, \mathfrak{g})$.

Definition 7.10. Let $X \subset \mathbb{P}V$ be an irreducible subvariety. Denote by $\widehat{X} \subset V$ the affine cone of X and by $T_\alpha(\widehat{X}) \subset V$ the tangent space at a smooth point $\alpha \in \widehat{X}$. The Lie algebra of infinitesimal linear automorphisms of \widehat{X} is

$$\mathbf{aut}(\widehat{X}) := \{g \in \text{End}(V) \mid g(\alpha) \in T_\alpha(\widehat{X}) \text{ for any smooth point } \alpha \in \widehat{X}\}.$$

We will call $\mathbf{aut}(\widehat{X})^{(1)}$ the *prolongation* of $X \subset \mathbb{P}V$. We say that X has *nonzero prolongation* if $\mathbf{aut}(\widehat{X})^{(1)} \neq 0$.

Proposition 7.11. *Let $X \subset \mathbb{P}V$ be a quadratically symmetric variety. Then $\mathbf{aut}(\widehat{X})^{(1)} \neq 0$.*

Proof. By Proposition 7.7, we may assume that $X = Z(\sigma) \subset \mathbb{P}V$, with $V = T \oplus U \oplus W$ associated to a surjective homomorphism $\sigma : \text{Sym}^2 U \rightarrow W$. Define $A : \text{Sym}^2 V \rightarrow V$ by

$$A((t_1, u_1, w_1), (t_2, u_2, w_2)) = \left(t_1 t_2, \frac{t_1 u_2 + t_2 u_1}{2}, \sigma(u_1, u_2) \right).$$

We claim that $A \in \mathbf{aut}(\widehat{X})^{(1)}$ which proves the proposition.

To check the claim, we have to show that $A(v, \alpha) \in T_\alpha \widehat{X}$ for any $v \in V$ and a general smooth point α of \widehat{X} . It suffices to consider α in the open subset $\{(t, u, w) \mid t \neq 0, tw = \sigma(u, u)\}$ of \widehat{X} . Fix

$$\alpha = (t, u, w = \frac{1}{t}\sigma(u, u)) \in V, t \neq 0.$$

Then $T_\alpha \widehat{X}$ is the subspace of V consisting of (t', u', w') satisfying

$$(t + \epsilon t')(w + \epsilon w') = \sigma(u + \epsilon u', u + \epsilon u') \text{ modulo } \epsilon^2.$$

Thus

$$\begin{aligned} T_\alpha \widehat{X} &= \{(t', u', w') \in V \mid tw' + t'w - 2\sigma(u, u') = 0\} \\ &= \{(t', u', w') \mid tw' + \sigma(u, \frac{t'}{t}u - 2u') = 0\}. \end{aligned}$$

For any $v = (t_0, u_0, w_0)$, we have

$$A(v, \alpha) = A((t_0, u_0, w_0), (t, u, w)) = (t_0t, \frac{t_0u + tu_0}{2}, \sigma(u_0, u)).$$

Writing the right hand side as (t', u', w') , we have

$$tw' + \sigma(u, \frac{t'}{t}u - 2u') = t\sigma(u_0, u) + \sigma(u, \frac{t_0t}{t}u - 2\frac{t_0u + tu_0}{2}) = 0.$$

This proves the claim. \square

The following is a partial converse of Proposition 7.11.

Proposition 7.12. *Let $X \subset \mathbb{P}V$ be a nonsingular nondegenerate linearly normal projective variety. If X has nonzero prolongation, then X is a quadratically symmetric variety.*

Proof. This is contained in the proof of Theorem 1.1.3 [HM05]. In fact, it is shown in p. 606 of [HM05] that for a nonzero element $A : \text{Sym}^2 V \rightarrow V$ of the prolongation of X and a general point $\alpha \in \widehat{X}$, the endomorphism A_α of V satisfies $A_\alpha^3 = 0$ and the semisimple part of the endomorphism A_α generates a \mathbb{C}^* -action with weight 0 on α and weight 1 on $T_\alpha X$. The orbits of this \mathbb{C}^* -action on X have degree ≤ 2 from $A_\alpha^3 = 0$. Since X is nondegenerate in $\mathbb{P}V$, this implies that the weight on the complement of $T_\alpha \widehat{X}$ is 2. Thus X must be quadratically symmetric. \square

Using the above results, we can derive from Theorem 2.15 the following classification result.

Theorem 7.13. *Let $X \subsetneq \mathbb{P}V$ be an irreducible nonsingular nondegenerate variety such that $\mathbf{aut}(\widehat{X})^{(1)} \neq 0$. Then $X \subset \mathbb{P}V$ is projectively equivalent to one in the following list.*

- (1) *The VMRT of an irreducible Hermitian symmetric space of compact type of rank ≥ 2 from Example 4.4.*
- (2) *The VMRT of a symplectic Grassmannian from Example 4.5.*
- (3) *A nonsingular linear section of $\text{Gr}(2, 5) \subset \mathbb{P}^9$ of codimension ≤ 2 .*

- (4) A \mathbb{P}^4 -general linear section of $\mathbb{S}_5 \subset \mathbb{P}^{10}$ of codimension ≤ 3 .
- (5) Biregular projections of (1) and (2) with nonzero prolongation, which are completely described in Section 4 of [FH].

Proof. The assumption $\mathbf{aut}(\widehat{X})^{(1)} \neq 0$ implies that X is conic-connected by Theorem 1.1.3 [HM05].

If X is not a prime Fano manifold and X is linearly normal, then $X \subset \mathbb{P}V$ can be classified by Theorem 6.3. This is done in Proposition 6.4 of [FH]: they are exactly (1) and (2), excepting the prime Fano manifolds belonging to (1). Note that prime Fano manifolds belonging to (1) are exactly those appearing as $Z(\sigma)$ in Example 2.3.

If X is a prime Fano manifold and $X \subset \mathbb{P}V$ is linearly normal, then by Proposition 7.7 and Proposition 7.12, we see that $X \subset \mathbb{P}V$ is projectively equivalent to $Z(\sigma)$, where σ is the second fundamental form of X at a general point. Thus we have (3), (4) and prime Fano manifolds in (1), from Theorem 2.15 and the data on (1) and (2) in Section 3 of [FH].

Finally, if X is not linearly normal, it must be a biregular projection of the linearly normal one by Corollary 4.8 [FH]. (3) and (4) do not have biregular projections. So we have (5). \square

Remark 7.14. Main Theorem in [FH] asserted the classification result in Theorem 7.13, but there was a flaw: the list there missed the case of (3) with codimension 2 and the cases of (4) with codimension 2 and 3. This omission is caused by Proposition 2.9 of [FH], where it was claimed that $\dim \mathbf{aut}(\widehat{X})^{(1)} \neq 1$ for any nonsingular nondegenerate linearly normal variety $X \subset \mathbb{P}V$. This proposition is incorrect: the codimension-2 case of (3) and the codimension-3 case of (4) satisfy $\dim \mathbf{aut}(\widehat{X})^{(1)} = 1$. The error in the proof of Proposition 2.9 occurred at the very last line: the statement that the set of α and α' (in the notation therein) spans the vector space V is wrong.

REFERENCES

- [AS] A. Alzati and J. C. Sierra: Quadro-quadric special birational transformations of projective spaces, *Int. Math. Res. Not.* (2013), doi:10.1093/imrn/rnt173
- [ES] L. Ein and N. Shepherd-Barron: Some special Cremona transformations, *Amer. J. Math.*, **111** (1989), 783-800
- [FH] B. Fu and J.-M. Hwang: Classification of non-degenerate projective varieties with non-zero prolongation and application to target rigidity. *Invent. math.* **189** (2012) 457-513
- [GH] P. Griffiths and J. Harris: *Principles of Algebraic Geometry*, Wiley, New York, 1978
- [Ha] R. Hartshorne: *Algebraic Geometry*, Springer, New York, 1977

- [HM05] J.-M. Hwang and N. Mok: Prolongations of infinitesimal linear automorphisms of projective varieties and rigidity of rational homogeneous spaces of Picard number 1 under Kähler deformation. *Invent. Math.* **160** (2005) 591-645
- [IR1] P. Ionescu and F. Russo: Conic-connected manifolds, *J. Reine Angew. Math.* **644** (2010), 145-157
- [IR2] P. Ionescu and F. Russo: Varieties with quadratic entry locus, II, *Compos. Math.* **144** (2008), 949-962.
- [IP] V. A. Iskovskikh and Yu. G. Prokhorov: Fano varieties. *Algebraic geometry, V*, Encyclopaedia Math. Sci., 47, Springer, Berlin, 1999.
- [IL] T. A. Ivey and J. M. Landsberg: *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems*, Graduate Studies in Mathematics, vol. 61 (2003)
- [L] J. M. Landsberg: On second fundamental forms of projective varieties, *Invent. Math.* **117** (1994), 303-315
- [Mo] N. Mok: Recognizing certain rational homogeneous manifolds of Picard number 1 from their varieties of minimal rational tangents. *Third International Congress of Chinese Mathematicians. Part 1, 2*, 41–61, AMS/IP Stud. Adv. Math., 42, pt.1, 2, Amer. Math. Soc., Providence, RI, 2008
- [M] S. Mukai: Biregular classification of Fano 3-folds and Fano manifolds of coindex 3, *Proc. Natl. Acad. Sci. USA* **86** (1989), 3000-3002
- [NO] C. Novelli and G. Occhetta: Projective manifolds containing a large linear subspace with nef normal bundle. *Michigan Math. J.* **60** (2011), no. 2, 441-462.
- [P] B. Pasquier: On some smooth projective two-orbit varieties with Picard number 1. *Math. Ann.* **344** (2009), no. 4, 963-987.
- [R1] F. Russo: On a theorem of Severi, *Math. Ann.* **316** (2000), 1-17
- [R2] F. Russo: Varieties with quadratic entry locus, I, *Math. Ann.* **344** (2009), 597-617
- [S] E. Sato: Projective manifolds swept out by large-dimensional linear spaces. *Tôhoku Math. J.* **49** (1997), no. 3, 299–321
- [Z] F. L. Zak: *Tangents and secants of algebraic varieties*. Translations of Mathematical Monographs, 127. American Mathematical Society, Providence, RI, 1993

Baohua Fu

Institute of Mathematics, AMSS, Chinese Academy of Sciences,
55 ZhongGuanCun East Road, Beijing, 100190, China
bhfu@math.ac.cn

Jun-Muk Hwang

Korea Institute for Advanced Study, Hoegiro 85,
Seoul, 130-722, Korea
jmhwang@kias.re.kr