[redacted] www.sciencedirect.com

[redacted]IER

J. Differential Equations 260 (2016) 1993–2025

www.elsevier.com/locate/jde

On [redacted]fication of polynomial differential operators [redacted]n respect to the type of first integrals

Jinzhi Lei

[redacted] Center for Applied Mathematics, Tsinghua University, Beijing, 100084, PR China

Received 4 November 2010; revised 21 September 2012

Available online 21 October 2015

Abstrac[redacted]

This [redacted]lassification of polynomial differential operators $\mathscr{X} = X_1(x_1, x_2)\delta_1 + X_2(x_1, x_2)\delta_2$ ($\delta_i = \partial/$[redacted]sification is defined through an order derived from $\mathscr{X}$. Let $X = \mathscr{X} y$ be the associated different[redacted], the order is defined as the order of a differential ideal Λ that is an essential extension of $\{X\}$. T[redacted]t shows the order can only be four possible values: 0, 1, 2, 3, or ∞. Furthermore, when the order[redacted]ssential extension $\Lambda = \{X, A\}$, where A is a differential polynomial with coefficients obtained[redacted]ional solution of a partial differential equation given explicitly by coefficients of $\mathscr{X}$. When th[redacted]inite, the extension Λ is identical with $\{X\}$. In addition, if, and only if, the order is 0, 1, [redacted]iated polynomial differential equation has Liouvillian first integrals. Examples and connecti[redacted]billon–Vey sequences are also discussed.

MSC: 34[redacted]H05

Keywords[redacted]ifferential operator; Classification; Polynomial differential equation; Differential algebra; Liouv[redacted]ral; Godbillon–Vey sequence

E-ma[redacted]jzlei@tsinghua.edu.cn.

http://dx.doi.org/10.1016/j.jde.2015.09.050

1. Introduction

1.1. Background

This paper studies polynomial differential operators of form

$$\mathscr{X} = X_1(x_1, x_2)\delta_1 + X_2(x_1, x_2)\delta_2, \tag{1}$$

where $\delta_i = \partial/\partial x_i$ $(i = 1, 2)$, and $X_1(x_1, x_2)$, $X_2(x_1, x_2)$ are polynomials of x_1 and x_2. We further assume that $X_1 \not\equiv 0$ without loss of generality.[1] In this paper, we give a classification for all operators of form (1), according to which the solution of the first order partial differential equation

$$\mathscr{X}\omega = 0 \tag{2}$$

is discussed.

The operator (1) closely relates to the following polynomial differential equation

$$\frac{dx_1}{dt} = X_1(x_1, x_2), \quad \frac{dx_2}{dt} = X_2(x_1, x_2), \tag{3}$$

and any non-constant solutions of (2) give a first integral of (3). Therefore, our results also yield a classification of the polynomial systems (3) with respect to the function type of its first integral.

The current study was motivated by investigating integrating methods of a polynomial differential equation of form (3). First, we take a look at a simple situation of how we can integrate (3).

If the equation (3) has an integrating factor η that is a rational function, a first integral ω is then obtained by an integral of a rational function, and satisfies

$$\delta_1\omega - a = 0, \tag{4}$$

with $a = \eta X_1$ a rational function. Therefore, there is a non-constant function ω that satisfies both equations (2) and (4). In this case, the differential operator $\mathscr{D}_A$ defined by

$$\mathscr{D}_A\omega = \delta_1\omega - a \tag{5}$$

is compatible with $\mathscr{X}$. In other words, let two differential polynomials X and A be defined as

$$X = \mathscr{X}y, \quad A = \mathscr{D}_A y,$$

then the differential ideal $\{X, A\}$ is a *nontrivial extension* of the ideal $\{X\}$ (refer to detailed definitions below). On the other hand, for any such differential polynomial A, we can always solve $\delta_1\omega$ and $\delta_2\omega$ from the equation $\{\mathscr{X} = 0, \ \mathscr{D}_A\omega = 0\}$ and therefore lead to the integration of the equation (3). This simple situation suggests that to integrate the equation (3) for first integrals, we need to find a differential polynomial A so that $\{X, A\}$ is a nontrivial extension

[1] If $X_1 \equiv 0$, one can exchange x_1 and x_2 (and also the derivations δ_1 and δ_2) in discussions below. Our classification is an intrinsic property of the operator $\mathscr{X}$, and is unchanged when we exchange x_1 and x_2, and even when we apply a linear transformation in (x_1, x_2). See our discussion following Theorem 24.

of the ideal $\{X\}$. Such a differential polynomial A, if exists, is not unique. However, the lowest order among these differential polynomials is uniquely determined by the differential operator $\mathscr{X}$ (called the order of $\mathscr{X}$, to be detailed below), and therefore provides an index for a classification.

The classification presented in this study is given through an order of the operator $\mathscr{X}$ defined below. This order is essential for understanding integrating methods of the polynomial differential equation (3) in different classes, and also the classification of un-integrable systems. Furthermore, for a given equation (3), the differential polynomial A in defining the nontrivial extension $\{X, A\}$ provides additional informations for the first integral, which are important for further investigations of the structure of integrating curves (or foliations) of the polynomial differential equation.

1.2. Preliminary definitions

Before stating the main results, we give some preliminary concepts from differential algebra. For details, refer to [1] and [2].

Let K be the field of all rational functions of (x_1, x_2) with complex number coefficients, δ_1, δ_2 are two *derivations* of K, and are commutable to each other. Then K together with the two derivations form a *differential field*, with $\mathbb{C}$ as the constant field. For a *differential indeterminate* y, there is a usual way to add y to the differential field K, by adding an infinite sequence of symbols

$$y, \delta_1 y, \delta_2 y, \delta_1\delta_2 y, \cdots, \delta_1^{i_1}\delta_2^{i_2} y, \cdots \tag{6}$$

to K [1]. This procedure results in a differential ring, denoted as $K\{y\}$. Each element in $K\{y\}$ is a polynomial of finite numbers of the symbols in (6), and therefore is a *differential polynomial* in y with coefficients in K.

We say an algebra ideal Λ in $K\{y\}$ to be a *differential ideal* if $a \in \Lambda$ implies $\delta_i a \in \Lambda$ $(i = 1, 2)$. Let Σ be any set of differential polynomials. The intersection of all differential ideals containing Σ is called the *differential ideal generated by* Σ, and is denoted as $\{\Sigma\}$. A differential polynomial A is in $\{\Sigma\}$ if, and only if, A is a linear combination of differential polynomials in Σ and of derivatives, of various orders, of such differential polynomials.

For the operator $\mathscr{X}$ given by (1), we have

$$X = \mathscr{X} y = X_1\delta_1 y + X_2\delta_2 y \in K\{y\}, \tag{7}$$

and $\{X\}$ denotes the differential ideal generated by X.

Definition 1. Let

$$w_1 = \delta_1^{i_1}\delta_2^{i_2} y, \quad w_2 = \delta_1^{j_1}\delta_2^{j_2} y,$$

be two derivatives of y, w_1 is **lower** than w_2 (or w_2 is **higher** than w_1) if either $i_1 < j_1$, or $i_1 = j_1$ and $i_2 < j_2$. Any element in K is lower than the indeterminate y.

The infinite sequence (6) can be organized from low to high (here $w_1 < w_2$ means that w_1 is lower than w_2) as

$$\begin{aligned} a < \ y\ &< \delta_2 y < \delta_2^2 y < \cdots \\ &< \delta_1 y < \delta_1\delta_2 y < \delta_1\delta_2^2 y < \cdots \\ &< \delta_1^2 y < \delta_1^2\delta_2 y < \delta_1^2\delta_2^2 y < \cdots \\ &< \cdots. \end{aligned} \tag{8}$$

Here a means any element in K.

Definition 2. Let A be a differential polynomial, if A contains y (or its derivatives) effectively, by the **leader** of A, we mean the highest of those derivatives of y which are involved in A.

Definition 3. Let A_1, A_2 be two differential polynomials, we say A_2 to be of higher **rank** than A_1, if either

(1) A_2 has higher leader than A_1; or
(2) A_1 and A_2 have the same leader, and the degree of A_2 (as a polynomial of the leader) in the leader exceeds that of A_1.

A differential polynomial which effectively involves the intermediate y is of higher rank than one which does not. Two differential polynomials of which no difference in the rank as created above is said to be of the same rank.

The following fact is basic [2, p. 3]:

Proposition 4. *Any non-empty subset of differential polynomials $\Sigma \subseteq K\{y\}$ contains a differential polynomial which is not higher than any other differential polynomials in the subset Σ.*

For a differential polynomial $A \in K\{y\}$, we associate with A a *differential operator* $\mathscr{D}_A$ on analytic functions $\mathcal{A}(\Omega)$, where Ω is an open subset of $\mathbb{C}^2$, such that

$$\mathscr{D}_A u = A|_{y=u}, \quad \forall u \in \mathcal{A}(\Omega). \tag{9}$$

By $S(\mathscr{D}_A)$, we denote the singularity set of $\mathscr{D}_A$, which contains all singular points (*i.e.*, the poles) in the coefficients of the differential polynomial A. Because all coefficients of A are rational functions, the singularity set $\mathscr{D}_A$ is a closed subset in $\mathbb{C}^2$. Thus, for any $u \in \mathcal{A}(\Omega)$, $\mathscr{D}_A u$ is well defined in the open subset $\Omega\backslash S(\mathscr{D}_A)$.

Definition 5. A differential ideal Λ in $K\{y\}$ is an **extension** of $\{X\}$, or an **extension** of $\mathscr{X}$, if $\{X\} \subseteq \Lambda$. The extension Λ is a **nontrivial extension** if there exists an open subset $\Omega \subset \mathbb{C}^2$ and a non-constant function $\omega \in \mathcal{A}(\Omega)$, such that $\mathscr{D}_A\omega = 0$ in $\Omega\backslash S(\mathscr{D}_A)$ for any $A \in \Lambda$. Otherwise, Λ is a **trivial extension**.

It is obvious that $\{X\}$ itself is a nontrivial extension of $\mathscr{X}$, and $K\{y\}$ is a trivial extension of $\mathscr{X}$. Main results in this paper are to show that nontrivial extensions of $\mathscr{X}$ with the lowest order (to be defined below) are essential to provide the classification of $\mathscr{X}$.

Proposition 6. *Let Λ be an extension of $\{X\}$, then either $\Lambda = \{X\}$, or Λ contains a differential polynomial A of lower rank than X.*

Proof. We only need to show that if $\{X\} \subsetneq \Lambda$, then Λ contains a differential polynomial A of lower rank than X. Let $\Lambda_0 = \Lambda\backslash\{X\}$, then Λ_0 is a non-empty subset of Λ. Proposition 4 yields that there is a differential polynomial $A \in \Lambda_0$ that has the lowest rank.

We claim that A has lower rank than X. If otherwise, the leader of A, denoted as $\delta_1^i\delta_2^j y$, is not lower than $\delta_1 y$ (the leader of X), *i.e.*, $i \geq 1$. Hence, $G = \delta_1^{i-1}\delta_2^j X$ is an element contained in $\{X\}$ and has the same leader $\delta_1^i\delta_2^j y$ as in A, and the degree of $\delta_1^i\delta_2^j y$ in G is 1. Thus, there is a differential polynomial $Q \in K\{y\}$ so that

$$A_1 = A - QG$$

has lower rank than A. It is obvious $A_1 \in \Lambda_0$, but with lower rank than A, which is contradiction to our choice of A, and the proposition is proved. □

Let Λ be a nontrivial extension of $\{X\}$, and $A \in \Lambda$ with the lowest rank. Proposition 6 yields that the leader of A either has form $\delta_2^r y$ ($r \geq 0$), or equals $\delta_1 y$. This fact leads to the order of Λ given below.

Definition 7. Let Λ be a nontrivial extension of $\{X\}$, and $A \in \Lambda$ with the lowest rank. If the leader of A is $\delta_2^r y$ ($r \geq 0$), r is called the **order** of Λ, denoted by $\mathrm{ord}(\Lambda)$. If otherwise, $\Lambda = \{X\}$ and the order of Λ is infinity ($\mathrm{ord}(\Lambda) = \infty$).

For a nontrivial extension Λ, a differential polynomial with the lowest rank can only take one of the following forms:

- a polynomial of y, with at least one coefficient that is non-constant ($\mathrm{ord}(\Lambda) = 0$); or
- a differential polynomial of y effectively involves derivatives ($1 \leq \mathrm{ord}(\Lambda) < \infty$); or
- the differential polynomial X, and therefore $\Lambda = \{X\}$ ($\mathrm{ord}(\Lambda) = \infty$).

Definition 8. An **essential extension** of $\mathscr{X}$ is a nontrivial extension of $\mathscr{X}$ that has order not higher than any other nontrivial extensions. By the **order** of $\mathscr{X}$, denoted as $\mathrm{ord}(\mathscr{X})$, we mean the order of an essential extension of $\mathscr{X}$.

We note that for a given differential operator $\mathscr{X}$, essential extensions of $\mathscr{X}$ may not be unique, but all essential extensions must have the same order. Thus, the order of $\mathscr{X}$ is well defined. Main results presented in the next subsection show that $\mathrm{ord}(\mathscr{X})$ provides a classification of polynomial differential operators.

1.3. Main results

Here we state the main results in this paper, with proofs given in the next section.

Theorem 9. *Let the polynomial differential operator $\mathscr{X}$ be given by (1), with coefficients $X_1, X_2 \in K$, then either $0 \leq \mathrm{ord}(\mathscr{X}) \leq 3$, or $\mathrm{ord}(\mathscr{X}) = \infty$. Furthermore, when $0 \leq \mathrm{ord}(\mathscr{X}) \leq 3$,* *we can always select an essential extension Λ of $\mathscr{X}$, such that $\Lambda = \{X, A\}$, with $A \in K\{y\}$ given below*

(1) if $\mathrm{ord}(\mathscr{X}) = 0$*, then*

$$A = y - a \quad (a \in K\backslash\mathbb{R}); \tag{10}$$

(2) if $\mathrm{ord}(\mathscr{X}) = 1$*, then*

$$A = (\delta_2 y)^n - a \quad (n \in \mathbb{N}, a \in K); \tag{11}$$

(3) if $\mathrm{ord}(\mathscr{X}) = 2$*, then*

$$A = \delta_2^2 y - a\delta_2 y \quad (a \in K); \tag{12}$$

(4) if $\mathrm{ord}(\mathscr{X}) = 3$*, then*

$$A = 2(\delta_2 y)(\delta_2^3 y) - 3(\delta_2^2 y)^2 - a(\delta_2 y)^2 \quad (a \in K). \tag{13}$$

From Theorem 9, when the order of a differential operator $\mathscr{X}$ is finite, an essential extension of $\mathscr{X}$ is given by $\Lambda = \{X, A\}$, with $A \in K\{y\}$ given by (10)–(13). Discussions in [2, Chapter 2] have proved that the system of equations

$$\mathscr{X} y = 0, \quad \mathscr{D}_A y = 0 \tag{14}$$

has solution in some extension field of K. It is easy to see that this solution gives a first integral of the polynomial differential equation (3). The following result for the classification of (3) is straightforward from Theorem 9.

Theorem 10. *Consider the polynomial differential equation (3), and let $\mathscr{X}$ be the corresponding differential operator given by (1); we have the following*

(1) if $\mathrm{ord}(\mathscr{X}) = 0$*, then (3) has a first integral $\omega \in K$;*
(2) if $\mathrm{ord}(\mathscr{X}) = 1$*, then (3) has a first integral ω, such that*

$$(\delta_2\omega)^n \in K$$

for some $n \in \mathbb{N}$;
(3) if $\mathrm{ord}(\mathscr{X}) = 2$*, then (3) has a first integral ω, such that*

$$\delta_2^2\omega/\delta_2\omega \in K;$$

(4) if $\mathrm{ord}(\mathscr{X}) = 3$*, then (3) has a first integral ω, such that*

$$\frac{2(\delta_2\omega)(\delta_2^3\omega) - 3(\delta_2^2\omega)^2}{(\delta_2\omega)^2} \in K;$$

(5) if $\mathrm{ord}(\mathscr{X}) = \infty$*, then any first integral of* (3) *does not satisfy any differential equation of form*

$$\mathscr{D}_A y = 0$$

with $A \in K\{y\}\backslash\{X\}$.

In 1992, Singer has proved that the cases (1)–(3) in Theorem 10 (also refer to Theorem 24 below) are the only cases to have Liouvillian integrals, *i.e.*, the first integral obtained from rational functions using finite steps of exponentiation, integration, an algebraic functions [3] (also refer to [4]). In the cases (4)–(5), however, there is no first integral of (3) that can be obtained from rational functions in finite steps operation given above (refer to [3] or [4]). From the proof of Lemma 15 given below, when $\mathrm{ord}(\mathscr{X}) = 3$, a first integral of (3) can be expressed through finite step operations from rational functions and a solution of the partial differential equation of form (35). This result provides further classification of polynomial differential equations that are not Liouvillian integrable.

In the rest of this paper, we first give the proof of Theorem 9 in Section 2, then show examples for each type of equations in Section 3, and a discussion for connections between our results and Godbillon–Vey sequence in Section 4.

2. Proof of the main result

2.1. Outline of the proof

Hereinafter, we denote $\delta_2^i y$ by y_i ($y_0 = y$). For any essential extension Λ of $\mathscr{X}$, let $A \in \Lambda$ with the lowest rank. From the above definitions, if $\mathrm{ord}(\mathscr{X}) = r (< \infty)$, then A is a polynomial of $y_0, y_1, \cdots, y_r$, with coefficients in K. Write

$$A = \sum_{\mathbf{m}} a_{\mathbf{m}} y_0^{m_0} y_1^{m_1} \cdots y_r^{m_r}, \tag{15}$$

where $\mathbf{m} = (m_0, m_1, \cdots, m_r) \in \mathbb{Z}^{r+1}$, and $a_{\mathbf{m}} \in K$. To prove Theorem 9, we only need to determine all possible non-zero coefficients in A. Let

$$\mathcal{I}_A = \{\mathbf{m} \in \mathbb{Z}^{r+1} | a_{\mathbf{m}} \neq 0\}. \tag{16}$$

We only need to specify the finite set $\mathcal{I}_A$. The process is outlined below.

Let $\mathbf{m} = (m_0, m_1, \cdots, m_r) \in \mathbb{Z}^{r+1}$; we define an operator $\Delta_{i,j} : \mathbf{Z}^{r+1} \to \mathbf{Z}^{r+1}$ for $0 < i < j \le r$ such that $\Delta_{i,j}(\mathbf{m}) \in \mathbb{Z}^{r+1}$ is given by

$$\Delta_{i,j}(\mathbf{m}) = \mathbf{m} + \mathbf{e}_{j-i} - \mathbf{e}_j \tag{17}$$

where

$$\mathbf{e}_k = (\overset{\overset{0}{\downarrow}}{0}, \cdots, 0, \overset{\overset{k}{\downarrow}}{1}, 0, \cdots, 0).$$

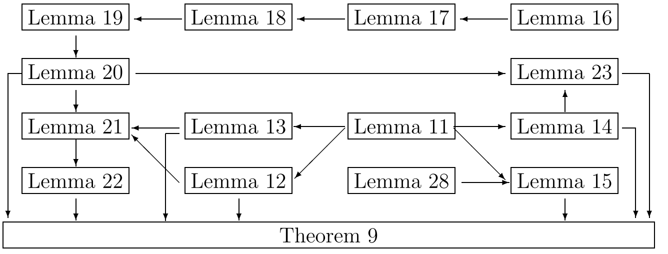

Fig. 1. Flow chart of the proof of Theorem 9.

Therefore

$$\Delta_{i,j}^{-1}(\mathbf{m}) = \mathbf{m} - \mathbf{e}_{j-i} + \mathbf{e}_j. \tag{18}$$

For any $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^{r+1}$, we say $\mathbf{m} \succ \mathbf{n}$ if there exist $0 < i < j \leq r$, such that

$$\Delta_{i,j}(\mathbf{m}) = \mathbf{n}.$$

The proof below is done by showing that if $r = \mathrm{ord}(\mathscr{X}) < \infty$, then $\mathcal{I}_A$ can only be one of the following cases:

(1) $r = 0$, and $\mathcal{I}_A = \{(1), (0)\}$; or
(2) $r = 1$, and $\mathcal{I}_A = \{(0, n), (0, 0)\}$; or
(3) $r = 2$, and $\mathcal{I}_A = \{(0, 0, 1), (0, 1, 0)\}$, with

$$(0, 0, 1) \succ (0, 1, 0);$$

or
(4) $r = 3$, and $\mathcal{I}_A = \{(0, 1, 0, 1), (0, 0, 2, 0), (0, 2, 0, 0)\}$, with relations

$$\begin{array}{ccc} & (0,1,0,1) & \\ & \curlyvee & \\ \curlyvee & (0,1,1,0) & \prec (0,0,2,0) \\ & \curlyvee & \\ & (0,2,0,0) & \end{array}$$

Here $(0, 1, 1, 0)$ is an auxiliary index with $a_{(0,1,1,0)} = 0$.

The final proof is complete following fourteen preliminary lemmas as shown by the flow chart in Fig. 1.

2.2. *Preliminary notations*

Before proving Theorem 9, we introduce some notations as follows. Hereinafter, we note $X_1 \not\equiv 0$, and assume $X_1(x_1^0, x_2^0) \neq 0$.

Let

$$[\delta_2, \mathscr{X}] = \delta_2 \mathscr{X} - \mathscr{X} \delta_2 = (\delta_2 X_1)\delta_1 + (\delta_2 X_2)\delta_2,$$
$$b_0 = -X_1\,(\delta_2 \frac{X_2}{X_1}), \quad b_i = X_1\,(\delta_2 \frac{b_{i-1}}{X_1}) = -X_1\,(\delta_2^{i+1} \frac{X_2}{X_1}), \ \ i = 1, 2, \cdots$$

For $F \in K\{y\}$, and $\{X\}$ the differential ideal generated by $X = \mathscr{X} y$, we write

$$F \sim R \tag{19}$$

if $R \in K\{y\}$ such that $F - R \in \{X\}$.

Let $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^{r+1}$, the *degree* of $\mathbf{n}$ is higher than that of $\mathbf{m}$, denoted by $\mathbf{n} > \mathbf{m}$, if there exists $0 \le k \le r$ such that $n_k > m_k$ and

$$n_i = m_i, \ \ i = k+1, \cdots, r.$$

It is easy to verify that the relation $\succ$ implies $>$, and for any $\mathbf{m} \in \mathbb{Z}^{r+1}$ and $0 < i < j \le r$,

$$\Delta_{i,j}^{-1}(\mathbf{m}) \succ \mathbf{m} \succ \Delta_{i,j}(\mathbf{m}), \tag{20}$$

and

$$\Delta_{i,j}^{-1}(\mathbf{m}) > \mathbf{m} > \Delta_{i,j}(\mathbf{m}). \tag{21}$$

In the following discussion, by $\mathbf{m}^*$ we always denote the element in $\mathcal{I}_A$ with the highest degree, and always assume $A_{\mathbf{m}^*} = 1$ without loss of generality. This is possible as the coefficients in A are rational functions in K.

For any $\mathbf{m} \in \mathbb{Z}^{r+1}$, define

$$\mathcal{P}(\mathbf{m}) = \{\mathbf{p} \in \mathcal{I}_A \mid \mathbf{p} \succ \mathbf{m}\} \tag{22}$$

and $\#(\mathbf{m}) = |\mathcal{P}(\mathbf{m})|$, the number of elements in $\mathcal{P}(\mathbf{m})$.

We define a function $C : \mathbb{Z}^{r+1} \to \mathbb{Z}$ by

$$C(\mathbf{m}) = \sum_{j=1}^{r} j m_j, \tag{23}$$

where $\mathbf{m} = (m_0, m_1, \cdots, m_r) \in \mathbb{Z}^{r+1}$. It is easy to verify that if $\mathbf{m} \succ \mathbf{p}$, then $C(\mathbf{m}) > C(\mathbf{p})$. In particular,

$$C(\mathbf{m}) - C(\Delta_{i,j}(\mathbf{m})) = i \quad (0 < i < j \le r). \tag{24}$$

2.3. Preliminary lemmas

Now, we start the proof process. First, Lemma 11 below is straightforward from the definition of nontrivial extension.

Lemma 11. *Let $A \in K\{y\}\backslash\{X\}$, the differential ideal $\Lambda = \{A, X\}$ is a nontrivial extension of $\mathscr{X}$ if, and only if, the equation*

$$\begin{cases} \mathscr{X} y &= 0 \\ \mathscr{D}_A y &= 0 \end{cases} \tag{25}$$

has a non-constant solution in $\mathcal{A}(\Omega)$, with Ω an open subset of $\mathbb{C}^2$.

The following result is a direct conclusion from Lemma 11:

Lemma 12. *If there exists $a \in K$, non-constant, such that $\mathscr{X} a = 0$, then let*

$$A = y - a,$$

the differential ideal $\Lambda = \{X, A\}$ is a nontrivial extension of $\mathscr{X}$.

Lemma 13. *If there exists $a \in K$, $a \neq 0$, such that*

$$\mathscr{X} a = n b_0 a, \tag{26}$$

where n is non-zero integer, let

$$A = (\delta_2 y)^{|n|} - a^{|n|/n}, \tag{27}$$

then $\Lambda = \{X, A\}$ is a nontrivial extension of $\mathscr{X}$.

Proof. From Lemma 11, we only need to show that there is a non-constant solution for the differential equation

$$\begin{cases} X_1\delta_1 y + X_2\delta_2 y &= 0 \\ (\delta_2 y)^{|n|} - a^{|n|/n} &= 0. \end{cases} \tag{28}$$

Let

$$u = a^{1/n}, \quad v = -\frac{X_2}{X_1} u,$$

and taking account of (26), direct calculations show that

$$\delta_1 u = \frac{1}{X_1}(b_0 u - X_2 \delta_2 u) = \delta_2 v.$$

Thus, the 1-form $v dx_1 + u dx_2$ is closed, and therefore the function of form

$$\omega = \int_{(x_1^0, x_2^0)}^{(x_1, x_2)} v dx_1 + u dx_2$$

is well defined and analytic on a neighborhood of some $(x_1^0, x_2^0) \in \mathbb{C}^2$. Further,

$$\delta_1 \omega = v, \quad \delta_2 \omega = u.$$

It is easy to verify that ω satisfies (28), and the lemma is proved. □

Lemma 14. *If there exists $a \in K$ that satisfies*

$$\mathscr{X} a = b_0 a + b_1, \tag{29}$$

let

$$A = \delta_2^2 y - a \delta_2 y, \tag{30}$$

then $\Lambda = \{X, A\}$ is a nontrivial extension of $\mathscr{X}$.

Proof. Let

$$b = -\frac{X_2}{X_1} a + \frac{b_0}{X_1}.$$

From (29), we have

$$\delta_1 a = -\frac{X_2}{X_1} \delta_2 a - (\delta_2 \frac{X_2}{X_1}) a + \delta_2 \frac{b_0}{X_1} = \delta_2 b.$$

Thus, the 1-form $b dx_1 + a dx_2$ is closed, and there exists a function μ that is analytic on a neighborhood of some $(x_1^0, x_2^0) \in \mathbb{C}^2$, such that

$$\delta_1 \mu = b, \quad \delta_2 \mu = a.$$

Let $u = \exp(\mu)$, then u is a non-zero function, and

$$\mathscr{X} u = u(X_1 \delta_1 \mu + X_2 \delta_2 \mu) = u(X_1 b + X_2 a) = b_0 u.$$

Thus, following the proof of Lemma 13, let

$$v = -\frac{X_2}{X_1} u,$$

then $vdx_1 + udx_2$ is a closed 1-form, and the function

$$\omega = \int_{(x_1^0, x_2^0)}^{(x_1, x_2)} vdx_1 + udx_2$$

is well defined in a neighborhood of (x_1^0, x_2^0), non-constant, and satisfies

$$X_1\delta_1\omega + X_2\delta_2\omega = 0, \quad \delta_2\omega - u = 0.$$

Therefore,

$$X_1\delta_1\omega + X_2\delta_2\omega = 0, \quad \delta_2^2\omega - a\delta_2 u = 0.$$

Thus, the non-constant function ω satisfies equations

$$\begin{cases} X_1\delta_1 y + X_2\delta_2 y = 0 \\ \delta_2^2 y - a\delta_2 y = 0 \end{cases} \tag{31}$$

and hence the lemma is concluded from Lemma 11. $\square$

Lemma 15. *If there exists $a \in K$ that satisfies*

$$\mathscr{X} a = 2b_0 a + b_2, \tag{32}$$

let

$$A = 2(\delta_2 y)(\delta_2^3 y) - 3(\delta_2^2 y)^2 - a(\delta_2 y)^2, \tag{33}$$

then $\Lambda = \{X, A\}$ is a nontrivial extension of $\mathscr{X}$.

Proof. We only need to show that there is a function ω that is analytic on an open subset of $\mathbb{C}^2$, non-constant, and satisfies

$$\begin{cases} X_1\delta_1\omega + X_2\delta_2\omega = 0 \\ 2(\delta_2\omega)(\delta_2^3\omega) - 3(\delta_2^2\omega)^2 - a(\delta_2\omega)^2 = 0. \end{cases} \tag{34}$$

We divide the proof into two steps. First, define

$$\begin{aligned} f(x_1, x_2, u) &= -\delta_2^2\frac{X_2}{X_1} - \frac{X_2}{X_1}a - (\delta_2\frac{X_2}{X_1})u - \frac{1}{2}(\frac{X_2}{X_1})u^2, \\ g(x_1, x_2, u) &= a + \frac{1}{2}u^2. \end{aligned}$$

Then f and g are analytic at some point $(x_1^0, x_2^0, u^0) \in \mathbb{C}^3$. Now, we prove that there is a function $u(x_1, x_2)$ that is analytic on a neighborhood of (x_1^0, x_2^0), and $u(x_1^0, x_2^0) = u^0$, such that

$$\begin{cases} \delta_1 u &= f(x_1, x_2, u), \\ \delta_2 u &= g(x_1, x_2, u) \end{cases} \tag{35}$$

is satisfied in a neighborhood of (x_1^0, x_2^0). To this end, we need to show

$$(\frac{\partial}{\partial x_2} + g(x_1, x_2, u)\frac{\partial}{\partial u}) f(x_1, x_2, u) = (\frac{\partial}{\partial x_1} + f(x_1, x_2, u)\frac{\partial}{\partial u}) g(x_1, x_2, u) \tag{36}$$

and apply Lemma 28 in Appendix A.

From (32), we have

$$\delta_1 a = -\delta_2^3 \frac{X_2}{X_1} - \delta_2(\frac{X_2}{X_1} a) - (\delta_2 \frac{X_2}{X_1}) a.$$

Thus, from (35), we have

$$\begin{aligned}
&(\frac{\partial}{\partial x_2} + g(x_1, x_2, u)\frac{\partial}{\partial u}) f(x_1, x_2, u) \\
&= -\delta_2^3 \frac{X_2}{X_1} - \delta_2(\frac{X_2}{X_1} a) - (\delta_2^2 \frac{X_2}{X_1}) u - \frac{1}{2}(\delta_2 \frac{X_2}{X_1}) u^2 \\
&\quad - g(x_1, x_2, u)(\delta_2 \frac{X_2}{X_1} + \frac{X_2}{X_1} u) \\
&= -\delta_2^3 \frac{X_2}{X_1} - \delta_2(\frac{X_2}{X_1} a) - (\delta_2 \frac{X_2}{X_1}) a - (\delta_2^2 \frac{X_2}{X_1}) u - (\frac{X_2}{X_1} a) u \\
&\quad - (\delta_2 \frac{X_2}{X_1}) u^2 - \frac{1}{2}(\frac{X_2}{X_1}) u^3, \\
&= -\delta_2^3 \frac{X_2}{X_1} - \delta_2(\frac{X_2}{X_1} a) - (\delta_2 \frac{X_2}{X_1}) a + u f(x_1, x_2, u) \\
&= \delta_1 a + u f(x_1, x_2, u) \\
&= (\frac{\partial}{\partial x_1} + f(x_1, x_2, u)\frac{\partial}{\partial u}) g(x_1, x_2, u)
\end{aligned}$$

and (36) is satisfied.

Now, assuming

$$u(x_1, x_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{i,j} (x_1 - x_1^0)^i (x_2 - x_2^0)^j \quad (u_{0,0} = u^0) \tag{37}$$

and applying the Method of Majorants, we can obtain the coefficients $u_{i,j}$ by induction, and the power series (37) is convergent in a neighborhood of (x_1^0, x_2^0) (refer to Lemma 28 in Appendix A for detail). Thus, the function (37) gives an analytic solution of (35).

Next, we construct a solution ω of (34) from the above solution u of (35). Let

$$v = -\delta_2 \frac{X_2}{X_1} - \frac{X_2}{X_1} u.$$

It is easy to verify $\delta_2 v = \delta_1 u$, and hence the 1-form $v dx_1 + u dx_2$ is closed. Let

$$\mu = \exp\left[\int_{(x_1^0, x_2^0)}^{(x_1, x_2)} v dx_1 + u dx_2\right], \tag{38}$$

then the function μ is well defined, non-zero, and analytic on a neighborhood of (x_1^0, x_2^0) (here we note that $X_1(x_1^0, x_2^0) \neq 0$), and

$$\mathscr{X}\mu = b_0\mu.$$

Following the proof of Lemma 13, there exists a non-constant function ω, analytic on a neighborhood of (x_1^0, x_2^0) (here we note that b_0 is analytic at (x_1^0, x_2^0)), such that

$$\mathscr{X}\omega = 0, \quad \delta_2\omega = \mu.$$

From (38) and (35), we have $\delta_2\mu = \mu u$, and

$$\mu\delta_2^2\mu = \mu\left((\delta_2\mu)u + \mu(a + \frac{1}{2}u^2)\right) = \frac{3}{2}(\delta_2\mu)^2 + a\mu^2.$$

Taking account of $\mu = \delta_2\omega$, we have

$$(\delta_2\omega)(\delta_2^3\omega) - \frac{3}{2}(\delta_2^2\omega)^2 - a(\delta_2\omega)^2 = 0.$$

Thus, ω satisfies (34) and the lemma is concluded. $\square$

Lemma 16. *Let* $[\delta_2, \mathscr{X}]$ *and* y_i *be defined as previously, then*

(1) $[\delta_2, \mathscr{X}] = (\frac{\delta_2 X_1}{X_1})\mathscr{X} - b_0\delta_2$;
(2) $\mathscr{X} y_j = \delta_2\mathscr{X} y_{j-1} - (\frac{\delta_2 X_1}{X_1})\mathscr{X} y_{j-1} + b_0 y_j$.

Proof. (1) is straightforward from

$$\begin{aligned}
[\delta_2, \mathscr{X}] &= (\delta_2 X_1)\delta_1 + (\delta_2 X_2)\delta_2 \\
&= \frac{\delta_2 X_1}{X_1}(X_1\delta_1 + X_2\delta_2) - \frac{X_2}{X_1}(\delta_2 X_1)\delta_2 + (\delta_2 X_2)\delta_2 \\
&= \frac{\delta_2 X_1}{X_1}\mathscr{X} + X_1\frac{X_1\delta_2 X_2 - X_2\delta_2 X_1}{X_1^2}\delta_2 \\
&= \frac{\delta_2 X_1}{X_1}\mathscr{X} - b_0\delta_2.
\end{aligned}$$

(2) can be obtained by direct calculations below:

$$\begin{aligned}
\mathscr{X} y_j &= \mathscr{X}\delta_2 y_{i-1} \\
&= \delta_2 \mathscr{X} y_{j-1} - [\delta_2, \mathscr{X}] y_{j-1} \\
&= \delta_2 \mathscr{X} y_{j-1} - (\frac{\delta_2 X_1}{X_1}\mathscr{X} - b_0\delta_2) y_{j-1} \\
&= \delta_2 \mathscr{X} y_{j-1} - \frac{\delta_2 X_1}{X_1}\mathscr{X} y_{j-1} + b_0\delta_2 y_{j-1} \\
&= \delta_2 \mathscr{X} y_{j-1} - (\frac{\delta_2 X_1}{X_1})\mathscr{X} y_{j-1} + b_0 y_j. \qquad \square
\end{aligned}$$

Lemma 17. *We have*

$$\mathscr{X} y_j \sim \sum_{i=0}^{j-1} c_{i,j}\, b_i\, y_{j-i} \quad (j \geq 1) \tag{39}$$

where $c_{i,j}$ are positive integers, and $c_{0,j} = j$.

Proof. From Lemma 16, when $j = 1$, we have

$$\mathscr{X} y_1 = \delta_2 \mathscr{X} y_0 - (\frac{\delta_2 X_1}{X_1})\mathscr{X} y_0 + b_0 y_1 \sim b_0 y_1.$$

Thus (39) holds for $j = 1$ with $c_{0,1} = 1$.

Assume that (39) is valid for $j = k$ with positive integer coefficients $c_{i,k}$, and $c_{0,k} = k$, applying Lemma 16, we have

$$\begin{aligned}
\mathscr{X} y_{k+1} &= \delta_2 \mathscr{X} y_k - (\frac{\delta_2 X_1}{X_1})\mathscr{X} y_k + b_0\, y_{k+1} \\
&\sim \delta_2(\sum_{i=0}^{k-1} c_{i,k} b_i\, y_{k-i}) - (\frac{\delta_2 X_1}{X_1})(\sum_{i=0}^{k-1} c_{i,k} b_i\, y_{k-i}) + b_0\, y_{k+1} \\
&= \sum_{i=0}^{k-1} c_{i,k}((\delta_2 b_i) y_{k-i} + b_i\, \delta_2 y_{k-i}) - \sum_{i=0}^{k-1} c_{i,k}\frac{\delta_2 X_1}{X_1} b_i\, y_{k-i} + b_0 y_{k+1} \\
&= \sum_{i=0}^{k-1} c_{i,k}\left((\delta_2 b_i - \frac{\delta_2 X_1}{X_1} b_i) y_{k-i} + b_i\, y_{k-i+1}\right) + b_0\, y_{k+1} \\
&= (c_{0,k}+1) b_0\, y_{k+1} + \sum_{i=0}^{k-2}\left(c_{i,k} X_1 \delta_2(\frac{b_i}{X_1}) + c_{i+1,k}\, b_{i+1}\right) y_{k-i} \\
&\quad + c_{k-1,k} X_1 \delta_2(\frac{b_{k-1}}{X_1}) y_1 \\
&= (c_{0,k}+1) b_0 y_{k+1} + \sum_{i=0}^{k-2}(c_{i,k} + c_{i+1,k}) b_{i+1} y_{k-i} + c_{k-1,k} b_k y_1.
\end{aligned}$$

Thus, let

$$\begin{cases} c_{0,k+1} = c_{0,k} + 1 = k+1, \\ c_{i,k+1} = c_{i-1,k} + c_{i,k}, \qquad (1 \le i \le k-1), \\ c_{k,k+1} = c_{k-1,k}, \end{cases}$$

which are positive integers, we have

$$\mathscr{X} y_{k+1} \sim \sum_{i=0}^{k} c_{i,k+1} b_i y_{k+1-i}.$$

The lemma is proved by induction. □

Lemma 18. *We have*

$$\mathscr{X} a_{\mathbf{m}} \mathbf{y}^{\mathbf{m}} \sim (\mathscr{X} a_{\mathbf{m}} + C(\mathbf{m}) b_0 a_{\mathbf{m}}) \mathbf{y}^{\mathbf{m}} + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} m_j c_{i,j} b_i a_{\mathbf{m}} \mathbf{y}^{\Delta_{i,j}(\mathbf{m})}, \tag{40}$$

where $a_{\mathbf{m}} \in K$, $\mathbf{y}^{\mathbf{m}} = y_0^{m_0} y_1^{m_1} \cdots y_r^{m_r}$, and $c_{i,j}$ is defined as in Lemma 17.

Proof. It is easy to have

$$\mathscr{X} a_{\mathbf{m}} \mathbf{y}^{\mathbf{m}} = (\mathscr{X} a_{\mathbf{m}}) \mathbf{y}^{\mathbf{m}} + a_{\mathbf{m}} \sum_{j=0}^{r} \frac{\partial \mathbf{y}^{\mathbf{m}}}{\partial y_j} \mathscr{X} y_j.$$

From Lemma 17, we have

$$\begin{aligned} \mathscr{X} a_{\mathbf{m}} \mathbf{y}^{\mathbf{m}} &= (\mathscr{X} a_{\mathbf{m}}) \mathbf{y}^{\mathbf{m}} + a_{\mathbf{m}} \sum_{j=0}^{r} m_j \mathbf{y}^{\mathbf{m} - \mathbf{e}_j} \mathscr{X} y_i \\ &\sim (\mathscr{X} a_{\mathbf{m}}) \mathbf{y}^{\mathbf{m}} + a_{\mathbf{m}} \sum_{j=1}^{r} m_j \mathbf{y}^{\mathbf{m} - \mathbf{e}_j} (\sum_{i=0}^{j-1} c_{i,j} b_i y_{j-i}) \\ &= (\mathscr{X} a_{\mathbf{m}}) \mathbf{y}^{\mathbf{m}} + a_{\mathbf{m}} b_0 (\sum_{j=1}^{r} c_{0,j} m_j) \mathbf{y}^{\mathbf{m}} + a_{\mathbf{m}} \sum_{j=1}^{r} \sum_{i=1}^{j-1} m_j c_{i,j} b_i \mathbf{y}^{\mathbf{m} + \mathbf{e}_{j-i} - \mathbf{e}_j} \\ &= (\mathscr{X} a_{\mathbf{m}} + C(\mathbf{m})\, b_0 a_{\mathbf{m}})\, \mathbf{y}^{\mathbf{m}} + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} m_j c_{i,j} b_i a_{\mathbf{m}}\, \mathbf{y}^{\Delta_{i,j}(\mathbf{m})}, \end{aligned}$$

and the lemma is concluded. □

Lemma 19. *Let Λ be a nontrivial extension of $\mathscr{X}$ and $\Lambda \neq \{X\}$, $A \in \Lambda$ with the lowest rank and $r = \mathrm{ord}(\Lambda)$ $(< \infty)$. Let $\mathbf{m}^* \in \mathcal{I}_A$ with the highest degree and assume that $a_{\mathbf{m}^*} = 1$, then for any $\mathbf{m} \in \mathbb{Z}^{r+1}$, $\mathbf{m} < \mathbf{m}^*$, we have*

$$\mathscr{X} a_{\mathbf{m}} = (C(\mathbf{m}^*) - C(\mathbf{m}))b_0 a_{\mathbf{m}} - \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} (m_j + 1) c_{i,j} b_i a_{\Delta_{i,j}^{-1}(\mathbf{m})}. \tag{41}$$

Here $a_{\mathbf{m}} = 0$ whenever $\mathbf{m} \notin \mathcal{I}_A$.

Proof. We can write

$$A = \sum_{\mathbf{m} \in \mathcal{I}_A} a_{\mathbf{m}} \mathbf{y}^{\mathbf{m}} = \sum_{\mathbf{m} \leq \mathbf{m}^*} a_{\mathbf{m}} \mathbf{y}^{\mathbf{m}}.$$

Hereinafter $a_{\mathbf{m}} = 0$ if $\mathbf{m} \notin \mathcal{I}_A$.

First, it is easy to have

$$\mathscr{X} A = X_1 \delta_1 A + X_2 \delta_2 A \in \Lambda.$$

On the other hand, from Lemma 18, we have

$$\begin{aligned}
\mathscr{X} A &= \sum_{\mathbf{m} \leq \mathbf{m}^*} \mathscr{X} a_{\mathbf{m}} \mathbf{y}^{\mathbf{m}} \\
&\sim \sum_{\mathbf{m} \leq \mathbf{m}^*} \left((\mathscr{X} a_{\mathbf{m}} + C(\mathbf{m}) b_0 a_{\mathbf{m}}) \mathbf{y}^{\mathbf{m}} + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} m_j c_{i,j} b_i a_{\mathbf{m}} \mathbf{y}^{\Delta_{i,j}(\mathbf{m})} \right) \\
&= \sum_{\mathbf{m} \leq \mathbf{m}^*} \left(\mathscr{X} a_{\mathbf{m}} + C(\mathbf{m}) b_0 a_{\mathbf{m}} + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} (m_j + 1) c_{i,j} b_i a_{\Delta_{i,j}^{-1}(\mathbf{m})} \right) \mathbf{y}^{\mathbf{m}}
\end{aligned}$$

Note that for any $j > i$, $\Delta_{i,j}^{-1}(\mathbf{m}^*) > \mathbf{m}^*$, and thus $\Delta_{i,j}^{-1}(\mathbf{m}^*) \notin \mathcal{I}_A$, *i.e.*, $a_{\Delta_{i,j}^{-1}(\mathbf{m}^*)} = 0$ for any $j > i$. Taking account of $a_{\mathbf{m}^*} = 1$, we have $\mathscr{X} a_{\mathbf{m}^*} = 0$, and hence

$$\begin{aligned}
\mathscr{X} A &\sim C(\mathbf{m}^*) b_0 y^{\mathbf{m}^*} \\
&\quad + \sum_{\mathbf{m} < \mathbf{m}^*} \left(\mathscr{X} a_{\mathbf{m}} + C(\mathbf{m}) b_0 a_{\mathbf{m}} + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} (m_j + 1) c_{i,j} b_i a_{\Delta_{i,j}^{-1}(\mathbf{m})} \right) \mathbf{y}^{\mathbf{m}}.
\end{aligned}$$

Therefore,

$$\mathscr{X} A - C(\mathbf{m}^*) b_0 A \sim R = \sum_{\mathbf{m} < \mathbf{m}^*} f_{\mathbf{m}} \mathbf{y}^{\mathbf{m}}, \tag{42}$$

where the coefficients $f_{\mathbf{m}}$ are

$$f_{\mathbf{m}} = \mathscr{X} a_{\mathbf{m}} + (C(\mathbf{m}) - C(\mathbf{m}^*))b_0 a_{\mathbf{m}} + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r-1} (m_j + 1)c_{i,j} b_i a_{\Delta_{i,j}^{-1}(\mathbf{m})}. \tag{43}$$

Now, we obtain a differential polynomial R that has lower rank than A and is contained in the differential ideal Λ. But A is an element in Λ with the lowest rank. Thus, we must have $R \equiv 0$. Therefore the coefficients (43) are zero, from which (41) is concluded. □

Note that $a_{\mathbf{m}^*} = 1$ and $\Delta_{i,j}^{-1}(\mathbf{m}^*) \notin \mathcal{I}_A$, the equation (41) is also valid for $a_{\mathbf{m}^*}$.

The equation (41) can be rewritten in another form below.

Lemma 20. *In Lemma 19, for any $\mathbf{m} \le \mathbf{m}^*$, let $k = \#(\mathbf{m})$ and $\mathcal{P}(\mathbf{m}) = \{\mathbf{p}_1, \cdots, \mathbf{p}_k\}$, and assume $\Delta_{i_l, j_l}(\mathbf{p}_l) = \mathbf{m}$ $(l = 1, 2, \cdots, k)$, then the coefficients $a_{\mathbf{p}_l}, a_{\mathbf{m}}$ satisfy*

$$\mathscr{X} a_{\mathbf{m}} = (C(\mathbf{m}^*) - C(\mathbf{m}))b_0 a_{\mathbf{m}} - \sum_{l=1}^{\#(\mathbf{m})} (m_{j_l} + 1)c_{i_l, j_l} b_{i_l} a_{\mathbf{p}_l}. \tag{44}$$

Lemma 21. *Let Λ be a nontrivial extension of $\mathscr{X}$ and $\Lambda = \{X\}$, $A \in \Lambda$ with the lowest rank and $r = \operatorname{ord}(\Lambda) > 1$. Let $\mathbf{m}^* \in \mathcal{I}_A$ with the highest degree. Then for any $\mathbf{m} \in \mathcal{I}_A$, $\#(\mathbf{m}) = 0$ if and only if $C(\mathbf{m}) = C(\mathbf{m}^*)$. Furthermore, if $\#(\mathbf{m}) = 0$, then $a_{\mathbf{m}}$ is a constant.*

Proof. First, we prove that if $\#(\mathbf{m}) = 0$, then $C(\mathbf{m}) = C(\mathbf{m}^*)$.

If $\#(\mathbf{m}) = 0$, Lemma 20 yields

$$\mathscr{X} a_{\mathbf{m}} = (C(\mathbf{m}^*) - C(\mathbf{m}))b_0 a_{\mathbf{m}}.$$

If otherwise $C(\mathbf{m}) \neq C(\mathbf{m}^*)$, then $n = C(\mathbf{m}^*) - C(\mathbf{m})$ is a non-zero integer, and $a_{\mathbf{m}} \neq 0$ such that

$$\mathscr{X} a_{\mathbf{m}} = n b_0 a_{\mathbf{m}}.$$

From Lemma 13, let

$$A' = (\delta_2 y)^{|n|} - a_{\mathbf{m}}{}^{|n|/n},$$

then the differential ideal $\Lambda' = \{X, A'\}$ is a nontrivial extension of $\mathscr{X}$ and with order ≤ 1. This contradicts with the assumption that Λ is an essential extension with order > 1. Thus, we have concluded that $C(\mathbf{m}) = C(\mathbf{m}^*)$.

Next, we prove that if $C(\mathbf{m}) = C(\mathbf{m}^*)$, then $\#(\mathbf{m}) = 0$.

If on the contrary, $C(\mathbf{m}) = C(\mathbf{m}^*)$ but $\#(\mathbf{m}) > 0$, there exists $\mathbf{m}_1 \in \mathcal{P}(\mathbf{m})$. From (24), we have $C(\mathbf{m}_1) > C(\mathbf{m}) = C(\mathbf{m}^*)$. Applying the previous part of the proof to $\mathbf{m}_1$, we have $\#(\mathbf{m}_1) > 0$. Thus, we can repeat the above process, and obtain $\mathbf{m}_2 \in \mathcal{P}(\mathbf{m}_1)$ such that $C(\mathbf{m}_2) > C(\mathbf{m}_1) > C(\mathbf{m}^*)$ and $\#(\mathbf{m}_2) > 0$. This procedure can be continued to obtain an infinite sequence $\{\mathbf{m}_k\}_{k=1}^{\infty} \subseteq \mathcal{I}_A$ such that $\#(\mathbf{m}_k) > 0$ and $C(\mathbf{m}_{k+1}) > C(\mathbf{m}_k) > C(\mathbf{m}^*)$. But $\mathcal{I}_A$ is a finite set. Thus, we come to a contradiction, and therefore $\#(\mathbf{m}) = 0$.

Now, we have proved that $\#(\mathbf{m}) = 0$ if and only if $C(\mathbf{m}) = C(\mathbf{m}^*)$.

If $\#(\mathbf{m}) = 0$, then $C(\mathbf{m}^*) = C(\mathbf{m})$, and therefore (44) yields $\mathscr{X} a_{\mathbf{m}} = 0$. But $\mathrm{ord}(\Lambda) > 1$, thus $a_{\mathbf{m}}$ must be a constant according to Lemma 12. $\square$

Lemma 22. *Let Λ be a nontrivial extension of $\mathscr{X}$ and $\Lambda \neq \{X\}$, $A \in \Lambda$ with the lowest rank and $r = \mathrm{ord}(\Lambda) > 1$. Let $\mathbf{m}^* \in \mathcal{I}_A$ with the highest degree. Then for any $\mathbf{m} \in \mathcal{I}_A$, $C(\mathbf{m}) \le C(\mathbf{m}^*)$.*

Proof. If otherwise, there is $\mathbf{m} \in \mathcal{I}_A$ such that $C(\mathbf{m}) > C(\mathbf{m}^*)$, then $\#(\mathbf{m}) \ge 1$ by Lemma 21. Thus, there is a $\mathbf{m}_1 \in \mathcal{P}(\mathbf{m})$, and $C(\mathbf{m}_1) > C(\mathbf{m}) > C(\mathbf{m}^*)$. Thus, we can repeat the procedure to obtain an infinite sequence $\{\mathbf{m}_k\}_{k=1}^{\infty} \subseteq \mathcal{I}_A$. This is contradiction to the fact that $\mathcal{I}_A$ is a finite set, and the lemma is concluded. $\square$

Lemma 23. *Assume $3 \le r = \mathrm{ord}(\mathscr{X}) < \infty$. Let Λ be an essential extension of $\mathscr{X}$ ($\Lambda \neq \{X\}$), $A \in \Lambda$ with the lowest rank, $\mathbf{m}^* = (m_0^*, m_1^*, \cdots, m_r^*) \in \mathcal{I}_A$ with the highest degree and $a_{\mathbf{m}^*} = 1$, then $m_1^* > 0$ and $m_2^* = 0$.*

Proof. (1) If $m_1^* = 0$, we can write $\mathbf{m}^*$ as

$$\mathbf{m}^* = (m_0^*, 0, \cdots, 0, m_k^*, \cdots, m_r^*),$$

where $1 < k \le r$ and $m_k^* > 0$. Let

$$\mathbf{m} = \Delta_{1,k}(\mathbf{m}^*) = (m_0^*, 0, \cdots, 0, 1, m_k^* - 1, m_{k+1}^*, \cdots, m_r^*).$$

It is easy to have $\mathcal{P}(\mathbf{m}) = \{\mathbf{m}^*\}$. Hence,

$$\mathscr{X} a_{\mathbf{m}} = b_0 a_{\mathbf{m}} - m_k^* c_{1,k} b_1$$

from Lemma 20. Here we have applied $C(\mathbf{m}^*) - C(\mathbf{m}) = 1$ and $a_{\mathbf{m}^*} = 1$. Let

$$a = -\frac{a_{\mathbf{m}}}{m_k^* c_{1,k}},$$

then

$$\mathscr{X} a = b_0 a + b_1.$$

Thus, we have $\mathrm{ord}(\mathscr{X}) \le 2$ from Lemma 14, which contradicts with the fact $r \ge 3$.

(2) If $m_2^* > 0$, let

$$\mathbf{p} = \Delta_{1,2}(\mathbf{m}^*) = (m_0^*, m_1^* + 1, m_2^* - 1, m_3^*, \cdots, m_r^*).$$

It is easy to verify $\mathcal{P}(\mathbf{p}) = \{\mathbf{m}^*\}$ as follows. 1. Since $\Delta_{1,2}(\mathbf{m}^*) = \mathbf{p}$, we have $\mathbf{m}^* \in \mathcal{P}(\mathbf{p})$. 2. If there is any other $\mathbf{m}' \in \mathcal{P}(\mathbf{p})$, then $\Delta_{i,j}(\mathbf{m}') = \mathbf{p}$ for some $(i, j) \neq (1, 2)$. Thus, we always have $j > 2$, which yields $\mathbf{m}' > \mathbf{m}^*$, and hence contradicts with the assumption that $\mathbf{m}^*$ is the highest.

Hence, we have

$$\mathscr{X} a_{\mathbf{p}} = (C(\mathbf{m}^*) - C(\mathbf{p})) b_0 a_{\mathbf{p}} - c_{1,2} m_2^* b_1 a_{\mathbf{m}^*}$$

from Lemma 20. Similar to the above argument as in (1), we have $\mathrm{ord}(\mathscr{X}) \leq 2$, which is contradiction to the assumption. Thus, we must have $m_2^* = 0$. $\square$

Here we give two examples to show the reduction procedure in Lemma 23.

Example 1. Assume

$$A = yy_3 + ayy_2 + \cdots,$$

then $\mathbf{m}^* = (1, 0, 0, 1)$ with $m_1 = 0$. First, we have

$$\mathscr{X} A = \mathscr{X} yy_3 + \mathscr{X} ayy_2 + \cdots,$$

and from Lemma 18

$$\mathscr{X} yy_3 \sim 3b_0 yy_3 + c_{1,3} b_1 yy_2 + c_{2,3} b_2 yy_1,$$
$$\mathscr{X} ayy_2 \sim (\mathscr{X} a + 2b_0 a) yy_2 + c_{1,2} b_1 ayy_1.$$

Thus,

$$\begin{aligned} \mathscr{X} A &\sim 3b_0 yy_3 + (\mathscr{X} a + 2b_0 a + c_{1,3} b_1) yy_2 + \cdots \\ &= 3b_0 A + (\mathscr{X} a - b_0 a + c_{1,3} b_1) yy_2 + \cdots. \end{aligned}$$

Here the dots represent terms with lower rank than yy_2. Hence, we must have

$$\mathscr{X} a - b_0 a + c_{1,3} b_1 = 0,$$

which implies $\mathrm{ord}(\mathscr{X}) \leq 2$ from Lemma 14. If otherwise, we have a non-zero polynomial

$$B = (\mathscr{X} a - b_0 a + c_{1,3} b_1) yy_2 + \cdots$$

that is contained in Λ but with lower rank than A.

Example 2. Assume

$$A = y_2 y_3 + ay_1 y_3 + \cdots$$

then $\mathbf{m}^* = (0, 0, 1, 1)$ with $m_2 > 0$. Then

$$\begin{aligned} \mathscr{X} A &= \mathscr{X} y_2 y_3 + \mathscr{X} ay_1 y_3 + \cdots \\ &= 5b_0 y_2 y_3 + c_{1,2} b_1 y_1 y_3 + (\mathscr{X} a + 4b_0 a_{1,3}) y_1 y_3 + \cdots \\ &= 5b_0 y_2 y_3 + (\mathscr{X} a + 4b_0 a + c_{1,2} b_1) y_1 y_3 + \cdots. \end{aligned}$$

Thus, similar to the above argument, we have

$$\mathscr{X} a - b_0 a + c_{1,2} b_1 = 0,$$

which again implies $\mathrm{odd}(\mathscr{X}) \leq 2$ from Lemma 14.

2.4. Proof of Theorem 9

Now, we are ready to prove our main theorem.

Proof of Theorem 9. Let Λ be a nontrivial extension of $\mathscr{X}$. We only need to show that if $\Lambda \neq \{X\}$, then $\mathrm{ord}(\Lambda) \leq 3$. Thus, we assume $\Lambda \neq \{X\}$ and $A \in \Lambda$ with the lowest rank, $\mathbf{m}^* \in \mathcal{I}_A$ with the highest degree, and $a_{\mathbf{m}^*} = 1$.

(1) If $r = 0$, let $n = \mathbf{m}^*$, we can write A as

$$A = y^n + a_1 y^{n-1} + \cdots + a_n \quad (a_i \in K, i = 1, \cdots, n),$$

with at least one coefficient $a_i \in K\backslash\mathbb{C}$. Thus, the equation (44) implies

$$\mathscr{X} a_i = 0.$$

Let

$$B = y - a_i,$$

then a_i satisfies equations

$$\mathscr{X} y = 0, \quad \mathscr{D}_B y = 0. \tag{45}$$

Hence, $\{X, B\}$ is a nontrivial extension of $\mathscr{X}$ with order 0, and (1) is concluded.

(2) If $r = 1$, we argue that there exists $\mathbf{m} \in \mathcal{I}_A$, with $\mathbf{m} < \mathbf{m}^*$, such that $C(\mathbf{m}) \neq C(\mathbf{m}^*)$. If otherwise, for any $\mathbf{m} \in \mathcal{I}_A$, $C(\mathbf{m}) = C(\mathbf{m}^*)$, then A must have form $A = (\delta_2 y)^n p(y)$, where $n = C(\mathbf{m}^*)$ and $p(y)$ is a polynomial of y with coefficients in K. Thus, let ω be a non-constant solution of

$$\mathscr{X} y = 0, \quad \mathscr{D}_A y = 0,$$

then either

$$\mathscr{X} \omega = 0, \quad \delta_2 \omega = 0, \quad \text{or} \quad \mathscr{X} \omega = 0, \quad p(\omega) = 0.$$

But these are not possible because the former case implies $X_1 \equiv 0$, and the latter case implies $\mathrm{ord}(\Lambda) = 0$, both are in contradiction to our assumptions.

Now, let $\mathbf{m}$ be such that $C(\mathbf{m}) \neq C(\mathbf{m}^*)$. We note that $\#(\mathbf{m}) = 0$, thus, the equation (44) yields

$$\mathscr{X} a_{\mathbf{m}} = (C(\mathbf{m}^*) - C(\mathbf{m})) b_0 a_{\mathbf{m}}.$$

From Lemma 13, let $n = C(\mathbf{m}^*) - C(\mathbf{m})$, $a = a_{\mathbf{m}}{}^{|n|/n}$, and

$$B = (\delta_2 y)^{|n|} - a,$$

then $\{X, B\}$ is a nontrivial extension of $\mathscr{X}$, and hence (2) is proved.

(3) If $r=2$, let $\mathbf{m}^*=(m_0^*,m_1^*,m_2^*)$ and $\mathbf{m}=\Delta_{1,2}(\mathbf{m})=(m_0^*,m_1^*+1,m_2^*-1)$. It is easy to verify $\mathcal{P}(\mathbf{m})=\{\mathbf{m}^*\}$. Thus, from Lemma 20, we have

$$\mathscr{X} a_{\mathbf{m}}=b_0a_{\mathbf{m}}-m_2^*c_{1,2}b_1.$$

Here, we note $C(\mathbf{m}^*)-C(\mathbf{m})=1$ and $a_{\mathbf{m}^*}=1$. Let

$$a=-\frac{a_{\mathbf{m}}}{m_2^*c_{1,2}},$$

then a satisfies

$$\mathscr{X} a=b_0a+b_1.$$

From Lemma 14, let

$$B=\delta_2^2y-a\delta_2y,$$

then the differential ideal $\{X,B\}$ is a nontrivial extension of $\mathscr{X}$.

(4) If $r=3$, we write $\mathbf{m}=(m_0^*,m_1^*,m_2^*,m_3^*)$. From Lemma 23, we have $m_1^*>0$ and $m_2^*=0$, *i.e.*, $\mathbf{m}^*=(m_0^*,m_1^*,0,m_3^*)$. Let

$$\begin{aligned}\mathbf{p}&=\Delta_{1,3}(\mathbf{m}^*)=(m_0^*,m_1^*,1,m_3^*-1),\\ \mathbf{m}&=\Delta_{1,2}^{-1}(\mathbf{p})=(m_0^*,m_1^*-1,2,m_3^*-1),\\ \mathbf{q}&=\Delta_{1,2}(\mathbf{p})=(m_0^*,m_1^*+1,0,m_3^*-1).\end{aligned}$$

It is easy to have $C(\mathbf{m})=C(\mathbf{m}^*)$. Therefore, from Lemma 21, $a_{\mathbf{m}}$ is a constant. Furthermore, we have $\mathcal{P}(\mathbf{p})\subseteq\{\mathbf{m}^*,\mathbf{m}\}$ and $\mathcal{P}(\mathbf{q})\subseteq\{\mathbf{m}^*,\mathbf{p}\}$.

Applying Lemma 20 to $a_{\mathbf{p}}$ and $a_{\mathbf{q}}$, respectively, and noticing that $C(\mathbf{m}^*)-C(\mathbf{p})=1$ and $C(\mathbf{m}^*)-C(\mathbf{q})=2$, we have

$$\mathscr{X} a_{\mathbf{p}}=b_0a_{\mathbf{p}}-(m_3^*c_{1,3}a_{\mathbf{m}^*}+2c_{1,2}a_{\mathbf{m}})b_1,\tag{46}$$

and

$$\mathscr{X} a_{\mathbf{q}}=2b_0a_{\mathbf{q}}-(m_3^*c_{2,3}b_2a_{\mathbf{m}^*}+c_{1,2}b_1a_{\mathbf{p}}).\tag{47}$$

Since $a_{\mathbf{m}^*}$ and $a_{\mathbf{m}}$ are constants, we must have $m_3^*c_{1,3}a_{\mathbf{m}^*}+2c_{1,2}a_{\mathbf{m}}=0$ and $a_{\mathbf{p}}=0$. If otherwise, we should have $r\le 2$ from Lemma 13 or Lemma 14.

From (47), let $a_{\mathbf{p}}=0$, $a_{\mathbf{m}^*}=1$, and let

$$a=-\frac{a_{\mathbf{q}}}{m_3^*c_{2,3}},$$

then a satisfies

$$\mathscr{X} a=2b_0a+b_2.$$

From Lemma 15 and letting

$$B = 2(\delta_2 y)(\delta_2^3 y) - 3(\delta_2^2 y)^2 - a(\delta_2 y)^2,$$

the differential ideal $\{X, B\}$ is a nontrivial extension of $\mathscr{X}$, and (4) is proved.

(5) If $r > 3$, we show that $r = \infty$. If otherwise, r is finite, then Lemma 23 yields $m_1^* > 0$ and $m_2^* = 0$, and therefore $\mathbf{m}^*$ can be written as

$$\mathbf{m}^* = (m_0^*, m_1^*, 0, \cdots, 0, m_k^*, \cdots, m_r^*),$$

where $2 < k \leq r$ and $m_1^*, m_k^* > 0$.

(a) If $k = 3$, then

$$\mathbf{m}^* = (m_0^*, m_1^*, 0, m_3^*, m_4^* \cdots, m_r^*).$$

Let

$$\begin{aligned} \mathbf{p} &= \Delta_{1,3}(\mathbf{m}^*) = (m_0^*, m_1^*, 1, m_3^* - 1, m_4^*, \cdots, m_r^*) \\ \mathbf{m} &= \Delta_{1,2}^{-1}(\mathbf{p}) = (m_0^*, m_1^* - 1, 2, m_3^* - 1, m_4^* \cdots, m_r^*) \\ \mathbf{q} &= \Delta_{1,2}(\mathbf{p}) = (m_0^*, m_1^* + 1, 0, m_3^* - 1, m_4^* \cdots, m_r^*). \end{aligned}$$

Then $\#(\mathbf{m}) = 0$, $\mathcal{P}(\mathbf{p}) \subseteq \{\mathbf{m}^*, \mathbf{m}\}$ and $\mathcal{P}(\mathbf{q}) \subseteq \{\mathbf{m}^*, \mathbf{p}\}$. Following the discussions as in (4), we have $\mathrm{ord}(\mathscr{X}) \leq 3$, which contradicts with $r > 3$.

(b) If $k > 3$, let

$$\begin{aligned} \mathbf{p} &= \Delta_{1,k}(\mathbf{m}^*) = (m_0^*, m_1^*, 0, \cdots, 1, m_k^* - 1, \cdots, m_r^*) \\ \mathbf{m} &= \Delta_{k-2,k-1}^{-1}(\mathbf{p}) = (m_0^*, m_1^* - 1, 0, \cdots, 2, m_k^* - 1, m_r^*). \end{aligned}$$

Then $\mathcal{P}(\mathbf{p}) \subseteq \{\mathbf{m}^*, \mathbf{m}\}$. Therefore,

$$\mathscr{X} a_{\mathbf{p}} = b_0 a_{\mathbf{p}} - (m_k^* c_{1,k} b_1 + 2c_{k-2,k-1} b_{k-2} a_{\mathbf{m}}).$$

Thus, we have $a_{\mathbf{m}} \neq 0$, *i.e.*, $\mathbf{m} \in \mathcal{I}_A$. If otherwise, we should have $\mathrm{ord}(\mathscr{X}) \leq 2$ as previous. Furthermore, we have

$$C(\mathbf{m}) = C(\mathbf{m}^*) + k - 3 > C(\mathbf{m}^*),$$

which is in contradiction to Lemma 22.

Thus, the above arguments conclude that r must be ∞, and the theorem has been proved. $\square$

Here we give an example to show the reduction procedure in the proof of point (4).

Example 3. Assume

$$A = y_1 y_3 + a_p y_1 y_2 + a_m y_2^2 + a_q y_1^2,$$

then

$$\mathbf{m}^* = (0,1,0,1), \mathbf{p} = (0,1,1,0), \mathbf{m} = (0,0,2,0), \mathbf{q} = (0,2,0,0),$$

and

$$\mathcal{P}(\mathbf{p}) \subseteq \{\mathbf{m}^*, \mathbf{m}\}, \quad \mathcal{P}(\mathbf{q}) \subseteq \{\mathbf{m}^*, \mathbf{p}\}.$$

Now, we have

$$\begin{aligned}
\mathscr{X} A &= \mathscr{X} y_1 y_3 + \mathscr{X} a_p y_1 y_2 + \mathscr{X} a_m y_2^2 + \mathscr{X} a_q y_1^2 \\
&\sim 4b_0 y_1 y_3 + c_{1,3} b_1 y_1 y_2 \\
&\quad + (\mathscr{X} a_p + 3b_0 a_p) y_1 y_2 + c_{1,2} b_1 a_p y_1^2 + c_{2,3} b_2 y_1^2 \\
&\quad + (\mathscr{X} a_m + 4b_0 a_m) y_2^2 + 2c_{1,2} b_1 a_m y_1 y_2 \\
&\quad + (\mathscr{X} a_q + 2b_0 a_q) y_1^2 \\
&= 4b_0 y_1 y_3 + (c_{1,3} b_1 + \mathscr{X} a_p + 3b_0 a_p + 2c_{1,2} b_1 a_m) y_1 y_2 \\
&\quad + (\mathscr{X} a_m + 4b_0 a_m) y_2^2 + (c_{1,2} b_1 a_p + c_{2,3} b_2 + \mathscr{X} a_q + 2b_0 a_q) y_1^2 \\
&= 4b_0 A + (\mathscr{X} a_p - b_0 a_p + 2c_{1,2} b_1 a_m + c_{1,3} b_1) y_1 y_2 \\
&\quad + (\mathscr{X} a_m) y_2^2 + (\mathscr{X} a_q - 2b_0 a_q + c_{2,3} b_2 + c_{1,2} b_1 a_p) y_1^2.
\end{aligned}$$

Hence, we have

$$\mathscr{X} a_p = b_0 a_p - (2c_{1,2} a_m + c_{1,3}) b_1 \tag{48}$$

$$\mathscr{X} a_m = 0 \tag{49}$$

$$\mathscr{X} a_q = 2b_0 a_q - c_{2,3} b_2 - c_{1,2} b_1 a_p. \tag{50}$$

Thus, we obtain the equations of form (46) and (47) in the proof.

3. Applications

In this section, we apply the previous results to study the classification of polynomial differential equations (3) and give some examples.

First, from the proof of Lemmas 13–15, an explicit method to determine the class of a polynomial differential equation (3) is given below.

Theorem 24. *Consider the polynomial differential equation (3), let*

$$b_i = -X_1 \delta_2^{i+1}\left(\frac{X_2}{X_1}\right) \quad (i = 0, 1, 2) \tag{51}$$

and r be the order of the corresponding differential operator (1), then

(1) $r = 0$ *if, and only if,* K *contains a first integral of* (3).

(2) $r = 1$ *if, and only if,* K *contains no first integral of* (3)*, and there exists* $a \in K \backslash \{0\}$*, and* $n \in \mathbb{Z}\backslash\{0\}$*, such that*

$$\mathscr{X} a = n b_0 a. \tag{52}$$

In this case, (3) *has an integrating factor*

$$\eta = \frac{a^{1/n}}{X_1}. \tag{53}$$

(3) $r = 2$ *if, and only if,* (52) *is not satisfied by any* $a \in K\backslash\{0\}$ *and* $n \in \mathbb{N}$*, and there exists* $a \in K$*, such that*

$$\mathscr{X} a = b_0 a + b_1. \tag{54}$$

In this case, (3) *has an integrating factor of form*

$$\eta = \frac{1}{X_1} \exp\left[\int\limits_{(x_1^0, x_2^0)}^{(x_1, x_2)} \frac{a}{X_1} (X_1 dx_2 - (X_2 a + b_0) dx_1) \right]. \tag{55}$$

(4) $r = 3$ *if, and only if,* (54) *is not satisfied by any* $a \in K$*, and there exists* $a \in K$*, such that*

$$\mathscr{X} a = 2 b_0 a + b_2. \tag{56}$$

In this case, (3) *has an integrating factor of form*

$$\eta = \frac{1}{X_1} \exp\left[\int\limits_{(x_1^0, x_2^0)}^{(x_1, x_2)} (-\delta_2 \frac{X_2}{X_1} - \frac{X_2}{X_1} u) dx_1 + u dx_2 \right], \tag{57}$$

where u *is a solution of the following partial differential equations*

$$\begin{cases} \delta_1 u &= -\delta_2^2 \dfrac{X_2}{X_1} - \dfrac{X_2}{X_1} a - (\delta_2 \dfrac{X_2}{X_1}) u - \dfrac{1}{2} (\dfrac{X_2}{X_1}) u^2 \\ \delta_2 u &= a + \dfrac{1}{2} u^2. \end{cases} \tag{58}$$

(5) $r = \infty$ *if, and only if,* (56) *is not satisfied by any* $a \in K$.

The proof is straightforward from previous sections, and is omitted here.

From Theorem 24, the order of the differential operator (1) is determined by the function type of its first integrals in the sense of Liouvillian extension. The function type of the lowest order is an intrinsic property of the operator, and therefore our classification is invariant under linear transformations in (x_1, x_2). Thus, our assumption $X_1 \not\equiv 0$ does not lose the generality after a linear transformation. In particular, if $X_1 \equiv 0$ (or $X_2 \equiv 0$), there is a first integral $\omega = x_1$ (or $\omega = x_2$) in K, and hence the order $r = 0$.

Below we give examples for each of the classes in Theorem 24.

It is easy to see that all equations

$$\frac{dx_1}{dt} = 1, \quad \frac{dx_2}{dt} = p(x_1), \tag{59}$$

with $p(x_1)$ a polynomial, have order $r = 0$. The general homogeneous linear equations[2]

$$\frac{dx_1}{dt} = 1, \quad \frac{dx_2}{dt} = p(x_1)x_2, \tag{60}$$

with $p(x_1)$ a rational function, have order $r = 1$. The general non-homogeneous linear equations

$$\frac{dx_1}{dt} = 1, \quad \frac{dx_2}{dt} = p(x_1)x_2 + q(x_1), \tag{61}$$

where $p(x_1)$ and $q(x_1)$ are rational functions, have order $r = 2$.

In the proposition below, we show that the general Riccati equation is an example of order $r = 3$.

Proposition 25. *The general Riccati equations*

$$\frac{dx_1}{dt} = 1, \quad \frac{dx_2}{dt} = p_2(x_1)x_2^2 + p_1(x_1)x_2 + p_0(x_1), \tag{62}$$

where $p_i(x)$ $(i = 0, 1, 2)$ are rational functions, have order $r = 3$.

Proof. We have known that the general Riccati equation (62) does not have Liouvillian first integral (refer to [5] and [3]), and hence the order r is either 3 or ∞ according to [3].

From the equation (62), we have $X_1 = 1$ and $X_2 = p_2(x_1)x_2^2 + p_1(x_1)x_2 + p_0(x_1)$. Thus, we have $b_2 = 0$ from (51), and the equation (56) has a solution $a = 0$, therefore the order is 3. $\square$

Finally, we show an example of differential equation with order $r = \infty$, which is given by the van der Pol equation

$$\begin{cases} \dot{x}_1 &= x_2 - \mu(\frac{x_1^3}{3} - x_1) \\ \dot{x}_2 &= -x_1 \end{cases} \quad (\mu \neq 0). \tag{63}$$

The van der Pol equation is well known for its existence of a limit cycle. The following lemma was proved independently by Cheng et al. [6] and Odani [7], respectively.

Lemma 26. *(See [6] and [7].) The system of the van der Pol equation (63) has no algebraic solution curves. In particular, the limit cycle is not algebraic.*

2 Here in general we mean most equations of this form.

Proposition 27. *The order of the van der Pol equation (63) is $r = \infty$.*

Proof. Let

$$X_1(x_1, x_2) = x_2 - \mu(\frac{x_1^3}{3} - x_1), \qquad X_2(x_1, x_2) = -x_1,$$

the equation (56) for the van der Pol equation (63) reads

$$X_1^3 \mathscr{X} a + 2x_1 X_1^2 a + 6x_1 = 0. \tag{64}$$

We only need to show that (64) has no rational function solution a.

If on the contrary, (64) has a rational function solution $a = a_1/a_2$, where a_1, a_2 are relatively prime polynomials, then a_1 and a_2 satisfy

$$X_1^3(a_2 \mathscr{X} a_1 - a_1 \mathscr{X} a_2) + 2x_1 X_1^2 a_1 a_2 + 6x_1 a_2^2 = 0,$$

i.e.,

$$a_2(X_1^3 \mathscr{X} a_1 + 2x_1 X_1^2 a_1 + 6x_1 a_2) = a_1 X_1^3 \mathscr{X} a_2.$$

Hence, there exists a polynomial $c(x_1, x_2)$, such that

$$X_1^3 \mathscr{X} a_2 = c a_2, \tag{65}$$

$$X_1^3 \mathscr{X} a_1 = (c - 2x_1 X_1^2) a_1 - 6x_1 a_2. \tag{66}$$

Let $a_2 = X_1^k p$, where k is the maximum integer such that the polynomial p does not contain X_1 as a factor. Substituting a_2 into (65), we have

$$X_1^3 \mathscr{X} p = p(c - k X_1^2 \mathscr{X} X_1).$$

Thus, $p|(X_1^3 \mathscr{X} p)$, and therewith $p|\mathscr{X} p$ because X_1 is a prime polynomial and p does not contain X_1 as a factor. Therefore, either p is a constant or the planar curve defined by $p(x_1, x_2) = 0$ is an algebraic invariant curve of the van der Pol equation (63). However, Lemma 26 has excluded the latter case. Therefore, p must be a constant.

We can let $p = 1$ without loss of generality, and therefore

$$a_2 = X_1^k, \quad c = k X_1^2 \mathscr{X} X_1. \tag{67}$$

Substituting (67) into (66), we have

$$X_1^3 \mathscr{X} a_1 = (k \mathscr{X} X_1 - 2x_1) X_1^2 a_1 - 6x_1 X_1^k \quad (k \geq 0). \tag{68}$$

Noting that

$$(k \mathscr{X} X_1 - 2x_1) = -k\mu(x_1^2 - 1) X_1 - (k+2) x_1,$$

(68) can be rewritten as

$$X_1^3 \mathscr{X} a_1 = -k\mu(x_1^2 - 1)X_1^3 a_1 - (k+2)x_1 X_1^2 a_1 - 6x_1 X_1^k. \tag{69}$$

From (69), we claim $k = 2$. If otherwise, we should have $X_1 | (k+2)x_1 a_1$ if $k > 2$, or $X_1 | 6x_1$ if $k < 2$, which are not possible.

Let $k = 2$, then equation (68) becomes

$$X_1 \mathscr{X} a_1 = (2\mathscr{X} X_1 - 2x_1)a_1 - 6x_1, \tag{70}$$

which gives

$$\begin{aligned}&\left(x_2 - \mu(\frac{x_1^3}{3} - x_1)\right)\left((x_2 - \mu(\frac{x_1^3}{3} - x_1))\frac{\partial a_1}{\partial x_1} - x_1 \frac{\partial a_1}{\partial x_2}\right)\\ &= \left(-2\mu(x_1^2 - 1)(x_2 - \mu(\frac{x_1^3}{3} - x_1)) - 4x_1\right) a_1 - 6x_1.\end{aligned} \tag{71}$$

Let

$$a_1(x_1, x_2) = \sum_{i=0}^{m} h_i(x_2) x_1^i, \tag{72}$$

where $h_i(x_2)$ are polynomials and $h_m(x_2) \neq 0$. Substituting (72) into (71), and comparing the coefficients of x_1^{m+5}, we have

$$\frac{1}{9}\mu^2 m\, h_m(x_2) = \frac{2}{3}\mu^2 h_m(x_2),$$

which implies $m = 6$. Hence, we have 7 coefficients $h_i(x_2)$ $(i = 0, \cdots, 6)$ to be determined, which are all polynomials of x_2. Next, comparing the coefficients of x_1^i $(0 \le i \le 10)$, we obtain the following 11 differential-algebra equations for the coefficients:

$$\begin{aligned}
0 &= x_2(-2\mu h_0(x_2) + x_2 h_1(x_2))\\
0 &= 6 - 2(-2 + \mu^2)h_0(x_2) + 2x_2^2 h_2(x_2) - x_2 h_0'(x_2)\\
0 &= 2\mu x_2 h_0(x_2) - (-4 + \mu^2)h_1(x_2) + 2\mu x_2 h_2(x_2) + 3x_2^2 h_3(x_2)\\
&\quad - \mu h_0'(x_2) - x_2 h_1'(x_2)\\
0 &= \frac{8\mu^2}{3} h_0(x_2) + \frac{4\mu x_2}{3} h_1(x_2) + 4h_2(x_2) + 4\mu x_2 h_3(x_2) + 4x_2^2 h_4(x_2)\\
&\quad - \mu h_1'(x_2) - x_2 h_2'(x_2)\\
0 &= 2\mu^2 h_1(x_2) + \frac{2\mu x_2}{3} h_2(x_2) + 4h_3(x_2) + \mu^2 h_3(x_2) + 6\mu x_2 h_4(x_2)\\
&\quad + 5x_2^2 h_5(x_2) + \frac{\mu}{3} h_0'(x_2) - \mu h_2'(x_2) - x_2 h_3'(x_2)
\end{aligned}$$

$$0=\frac{1}{3}(-2\mu^2h_0(x_2)+4\mu^2h_2(x_2)+12h_4(x_2)+6\mu^2h_4(x_2)+24\mu x_2h_5(x_2)$$
$$+18x_2^2h_6(x_2)+\mu h_1'(x_2)-3\mu h_3'(x_2)-3x_2h_4'(x_2))$$
$$0=\frac{1}{9}(-5\mu^2h_1(x_2)+6\mu^2h_3(x_2)-6\mu x_2h_4(x_2)+36h_5(x_2)+27\mu^2h_5(x_2)$$
$$+90\mu x_2h_6(x_2)+3\mu h_2'(x_2)-9\mu h_4'(x_2)-9x_2h_5'(x_2))$$
$$0=-\frac{4\mu^2}{9}h_2(x_2)-\frac{4\mu x_2}{3}h_5(x_2)+4h_6(x_2)+4\mu^2h_6(x_2)+\frac{\mu}{3}h_3'(x_2)$$
$$-\mu h_5'(x_2)-x_2h_6'(x_2)$$
$$0=-\frac{\mu}{3}(\mu h_3(x_2)+2\mu h_5(x_2)+6x_2h_6(x_2)-h_4'(x_2)+3h_6'(x_2))$$
$$0=-\frac{\mu}{9}(2\mu h_4(x_2)+12\mu h_6(x_2)-3h_5'(x_2))$$
$$0=-\frac{\mu}{9}(\mu h_5(x_2)-3h_6'(x_2))$$

The above equations yield the following

$$x_2(3x_2h_5'(x_2)-2\mu h_4'(x_2))=2\mu^3. \tag{73}$$

But (73) cannot be satisfied because $h_4(x_2)$ and $h_5(x_2)$ are polynomials, and the left hand side contains a factor x_2, while the right hand side does not. Thus, we conclude that (64) has no rational function solution, and hence the order of the van der Pol equation is infinity from Theorem 24. □

4. Connection with Godbillon–Vey sequences

In previous sections, we have made connections between the proposed classification with the Liouvillian integrability. Here, we show the connection with a Galois reducibility of a codimension one foliation defined by a germ of holomorphic one-form ω_0, the Godbillon–Vey sequences [8]. A Godbillon–Vey sequence for a holomorphic 1-form ω_0 is a family of meromorphic 1-forms (ω_k) such that

$$\Omega=dz+\sum_{k=0}^{\infty}\frac{z^k}{k!}\omega_k$$

is an integrable 1-form, *i.e.*, $\Omega\wedge d\Omega=0$. This condition is equivalent to

$$d\omega_k=\omega_0\wedge\omega_{k+1}+\sum_{l=1}^{k}\binom{l}{k}\omega_l\wedge\omega_{k+1-l}. \tag{74}$$

The length of a Godbillon–Vey sequence is the minimum N such that $\omega_k=0$ for $k\ge N$. For a sequence of length three, the 1-forms ω_0,ω_1 and ω_2 satisfy

$$d\omega_0=\omega_0\wedge\omega_1,\ d\omega_1=\omega_0\wedge\omega_2,\ d\omega_2=\omega_1\wedge\omega_2. \tag{75}$$

Here we show that the differential operators $\mathscr{X}$ with order $r \leq 2$ give a sequence of length 2, and those operators with $r = 3$ give a sequence of length 3.

The 1-form ω_0 corresponding to the operator (1) is given by

$$\omega_0 = X_2 dx_1 - X_1 dx_2. \tag{76}$$

Let η be an integrating factor, *i.e.*, $d(\eta\omega_0) = 0$, then

$$d\omega_0 = \omega_0 \wedge \eta^{-1} d\eta. \tag{77}$$

When the order $r \leq 2$, from Theorem 24, there is an integrating factor η so that $\omega_1 = \eta^{-1} d\eta$ is a meromorphic one-form. Thus, we obtain a Godbillon–Vey sequence of length two ($\omega_2 = 0$ in (75)).

When the order $r = 3$, there is an integrating factor η given by (57), and hence

$$\eta^{-1} d\eta = -\delta_2 \frac{X_2}{X_1} dx_1 - \frac{dX_1}{X_1} - \frac{u}{X_1}\omega_0,$$

where u is a solution of (58). Let

$$\omega_1 = -\delta_2 \frac{X_2}{X_1} dx_1 - \frac{dX_1}{X_1}, \tag{78}$$

then ω_1 is a meromorphic 1-form, and

$$d\omega_0 = \omega_0 \wedge \omega_1. \tag{79}$$

Next, define a meromorphic 1-form

$$\omega_2 = \frac{1}{X_1} \delta_2^2 \frac{X_2}{X_1} dx_1 + \frac{a}{X_1^2}\omega_0, \tag{80}$$

where a is a rational function satisfying (56). We note (56), (58), and

$$\omega_1 = \eta^{-1} d\eta + \frac{u}{X_1}\omega_0;$$

it is not difficult to verify

$$d\omega_1 = \omega_0 \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_2. \tag{81}$$

Thus, the order $r = 3$ yields a Godbillon–Vey sequence of length three.

In [9], the authors pointed out that "we still do not know any example of foliation having finite length > 4". The results present here ($\mathrm{ord}(\mathscr{X}) > 3$ implies $\mathrm{ord}(\mathscr{X}) = \infty$) might imply that such foliations do not exist.

Appendix A

Lemma 28. *Consider the following partial differential equations*

$$\begin{cases} \dfrac{\partial u}{\partial x_1} = f(x_1, x_2, u) \\ \dfrac{\partial u}{\partial x_2} = g(x_1, x_2, u) \end{cases} \tag{82}$$

Let

$$D_1 = \frac{\partial}{\partial x_1} + f(x_1, x_2, u)\frac{\partial}{\partial u}, \quad D_2 = \frac{\partial}{\partial x_2} + g(x_1, x_2, u)\frac{\partial}{\partial u}.$$

If the functions f and g are analytic, and satisfy

$$D_2 f(x_1, x_2, u) \equiv D_1 g(x_1, x_2, u) \tag{83}$$

in a neighborhood of $(0, 0, 0)$, *then the equation* (82) *has a unique solution $u = u(x_1, x_2)$ that is analytic on a neighborhood of* $(0, 0)$ *and* $u(0, 0) = 0$.

Proof. Without loss of generality, we assume that f and g are analytic in

$$\Omega = \{(x, y, u) \in \mathbb{C}^3 \,\big|\, |x_1| + |x_2| + |u| \le \rho\},$$

where $\rho > 0$. We can write $f(x_1, x_2, u)$ and $g(x_1, x_2, u)$ as power series

$$f(x_1, x_2, u) = \sum_{i,j,k} f_{i,j,k} x_1^i x_2^j u^k \tag{84}$$

and

$$g(x_1, x_2, u) = \sum_{i,j,k} g_{i,j,k} x_1^i x_2^j u^k, \tag{85}$$

respectively, and these series are convergent in Ω.

Letting

$$u(x_1, x_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{i,j} x_1^i x_2^j \quad (u_{0,0} = 0), \tag{86}$$

and substituting it into (82), we obtain the following equations

$$\sum_{i,j} i u_{i,j} x_1^{i-1} x_2^j = \sum_{i,j,k} f_{i,j,k} x_1^i x_2^j \Big(\sum_{p,q} u_{p,q} x_1^p x_2^q\Big)^k \tag{87}$$

$$\sum_{i,j} j u_{i,j} x_1^i x_2^{j-1} = \sum_{i,j,k} g_{i,j,k} x_1^i x_2^j \Big(\sum_{p,q} u_{p,q} x_1^p x_2^q\Big)^k. \tag{88}$$

First, from (87) and comparing the coefficients of the same degrees of x_1^m $(m \geq 1)$, we have

$$u_{1,0} = f_{0,0,0}, \tag{89}$$

and

$$u_{m,0} = \frac{1}{m!} D_1^{m-1} f(x_1, x_2, u(x_1, x_2))|_{(x_1,x_2)=(0,0)}. \tag{90}$$

Next, from (88) and comparing the coefficients of the same degrees of $x_1^m x_2^n$ $(n \geq 1)$, we have

$$u_{0,1} = g_{0,0,0}, \tag{91}$$

and

$$u_{m,n} = \frac{1}{m!n!} D_1^m D_2^{n-1} g(x_1, x_2, u(x_1, x_2))|_{(x_1,x_2)=(0,0)}. \tag{92}$$

The right hand side of (90) is a polynomial of $u_{i,0}$ with $i < m$. Thus, the coefficients $u_{m,0}$ $(m > 0)$ are well defined by (89) and (90) step by step. Similarly, the right hand side of (92) is a polynomial of the coefficients $u_{i,j}$ with $i < n$, $j \leq m$ and $i + j \leq m + n - 1$. Thus, the coefficients of form $u_{m,n}$ $(n \geq 1)$ can be determined by (91), (92), and the coefficients $u_{m,0}$ obtained previously. Thus, the coefficients in the power series (86) are well defined and unique.

Convergence of the power series (86) can be proved by the Method of Majorants as follows. Let

$$M = \max_{(x,y,u)\in\Omega} \{|f(x_1, x_2, u)|, |g(x_1, x_2, u)|\},$$

then

$$F(x, y, u) = \frac{M}{1 - \dfrac{x_1 + x_2 + u}{\rho}} \tag{93}$$

is a majorant function of both $f(x_1, x_2, u)$ and $g(x_1, x_2, u)$. Thus, the following equations

$$\begin{cases} \dfrac{\partial u}{\partial x_1} &= F(x_1, x_2, u) \\ \dfrac{\partial u}{\partial x_2} &= F(x_1, x_2, u) \end{cases} \tag{94}$$

majorize the equation (82). It is easy to verify that the equation (94) has an analytic solution $u(x_1, x_2) = U(x_1 + x_2)$, with $U(z)$ the analytic solution of

$$\frac{dU}{dz} = \frac{M}{1 - \dfrac{z + U}{\rho}}, \quad U(0) = 0. \tag{95}$$

Thus, the convergence of (86) is concluded by the Method of Majorants. Therefore, the function $u(x_1, x_2)$ given by (86) is well defined in Ω.

Finally, we need to show that the function $u(x_1, x_2)$ obtained above satisfies (82), *i.e.*, all coefficients in both sides of (87) and (88) are consistent. To this end, we only need to verify the coefficients of $x_1^m x_2^n$ ($n \geq 1$) in (87), and of x_1^m ($m \geq 1$) in (88), which gives

$$u_{m,n} = \frac{1}{m!n!} D_1^{m-1} D_2^n f(x_1, x_2, u(x_1, x_2))|_{(x_1,x_2)=(0,0)} \tag{96}$$

and

$$u_{0,n} = \frac{1}{n!} D_2^{n-1} g(x_1, x_2, u(x_1, x_2))|_{(x_1,x_2)=(0,0)}. \tag{97}$$

We note (92) automatically yields (96). When $m \geq 1$, the condition (83) yields

$$D_1^m D_2^{n-1} g = D_1^{m-1} D_2^{n-1} D_1 g = D_1^{m-1} D_2^{n-1} D_2 f = D_1^{m-1} D_2^n f,$$

and hence (92) is equivalent to (96). Thus, the function $u(x_1, x_2)$ satisfies both equations in (82), and the lemma is proved. □

References

[1] I. Kaplansky, An Introduction to Differential Algebra, 2nd edition, Hermann, Paris, 1976.
[2] J.F. Ritt, Differential Algebra, Amer. Math. Soc. Colloq. Publ., Amer. Math. Soc., New York, 1950.
[3] M.F. Singer, Liouvillian first integrals of differential equations, Trans. Amer. Math. Soc. 333 (2) (1992) 673–688.
[4] K. Guan, J. Lei, Integrability of second order autonomous system, Ann. Differential Equations 18 (2) (2002) 117–135.
[5] J. Liouville, Remarques nouvelles sur l'équation de Riccati, J. Math. Pures Appl. VI (1841) 1–13.
[6] R. Cheng, K. Guang, S. Zhang, On the judgment of the existence of algebraic curve solution to the second order polynomial autonomous system, J. Beijing Univ. Aeronautics Astronautics 21 (1) (1995) 109–115.
[7] K. Odani, The limit cycle of the van der Pol equation is not algebraic, J. Differential Equations 115 (1995) 146–152.
[8] C. Godbillon, J. Vey, Un invariant des feuilletages de condimension un, C. R. Acad. Sci. Paris 273 (1971) 92–95.
[9] D. Cerveau, A. Lins-Neto, F. Loray, J. Pereira, F. Touzet, Complex codimension one singular foliations and Godbillon–Vey sequences, Mosc. Math. J. 7 (1) (2007) 21–54.