NONLINEAR DIFFERENTIAL GALOIS THEORY

JINZHI LEI

ABSTRACT. Differential Galois theory has played important roles in the theory of integrability of linear differential equation. In this paper we will extend the theory to nonlinear case and study the integrability of the first order nonlinear differential equation. We will define for the differential equation the differential Galois group, will study the structure of the group, and will prove the equivalent between the existence of the Liouvillian first integral and the solvability of the corresponding differential Galois group.

CONTENTS

2000 *Mathematics Subject Classification.* Primary 34A34; Secondary 13B05.

Key words and phrases. differential Galois theory, nonlinear differential equation, Liouvillian integrability.

Supported by the National Natural Science Foundation of China(10301006).

1. Introduction

1.1. **Historical background.** Despite the well studied of the existence of the solution of differential equation(s), it is usually impossible, except a very few types, to solve the differential equation(s) explicitly. The first rigorous proof of the non-solvability of a differential equation by method of quadrature was addressed by Liouville in 1840s[17]. Liouville's work was 'undoubtedly inspired by the results of Lagrange, Abel, and Galois on the non-solvability of algebraic equations by radicals'[10]. After the pioneer work of Liouville's, many approaches has been developing to study the insight of integrability of differential equation. The concerning approaches include Lie group[24], monodromy group[10, 34], holonomy group[2, 3, 4], differential Galois group[9, 12, 32], Galois groupoid [18, 19], *etc.*. Let us recall briefly the subject of differential Galois theory. For extensive survey, refer to [16, 29, 31].

At first, we recall Liouville's result. An exposition of Liouville's method was given in [33, pp.111-123]. Consider following second order linear differential equation

$$y'' + a(x)y = 0. \tag{1.1}$$

Liouville proved that the 'simple' equation (1.1) either has a solution of 'simple' type, or cannot be solved by quadrature. Explicitly, the equation (1.1) with $a(x)$ to be a rational function is solvable by quadrature if, and only if, it has a solution, $u(x)$ say, such that $u'(x)/u(x)$ is an algebraic function. This result proposed a scheme to determined whether the equation (1.1) is solvable through the algebraic function solution of the corresponding Riccati equation (let $z = -y'/y$)

$$z' = z^2 + a(x).$$

Algorithms for integrating (1.1) while $a(x)$ is a rational function can be referred to [14, 15, 30].

Liouville's method was analytic. Another approach to the problem of integrability of homogenous linear ordinary differential equation, now named as differential Galois theory, was developed by Picard and Vessiot at the end of 19'th century(see references of [12]). The firm footing step following this theory was presented by Kolchin in the middle of the 20'th century[9, 11]. This approach start with a differential field K containing the coefficients of the linear differential equation

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0. \tag{1.2}$$

Let M to be the smallest differential field that contains K and all of the solutions of the linear differential equation. Kolchin proved the existence and uniqueness of the extension M/K provided that K has characteristic zero and an algebraically closed field of constant. This extension is called by Kolchin as the Picard-Vessiot extension associated to the linear differential equation. As in the classical Galois theory of polynomials, the differential Galois group of (1.2), $G = \mathrm{Gal}(M/K)$, is defined as the set of all differential automorphisms of M that leaves K elementwise fixed. It has been proved by Kolchin that the differential Galois group is isomorphic to a linear algebraic group(see [9, Lemma 5.4]). An important said that the linear differential equation (1.2) is Liouvillian integrable (the general solution is obtained by a combination of algebraic functions, quadratures and exponential of quadratures) if, and only if, the identity component of the corresponding differential Galois group, which is a normal subgroup, is solvable. This result is similar to the Galois theorem for the solvability of a polynomial equation by radicals. Furthermore, we can reobtain Liouville's result for the second order linear differential equation when $n = 2$. In recent decades, there were many works by Kovacic, Magid, Mitschi, Singer, Ulmer, *et.al.* that address to both direct and inverse problems of linear differential Galois theory, for example, see [14, 20, 21, 22, 28, 29, 30, 31]. For detail following this approach, one can refer to [9, 13, 20, 29, 32] and the references.

Besides the linear systems, The solvability of first order nonlinear differential equation is also interested and will be the main subject of this paper. Consider the equation

$$\frac{dx_2}{dx_1} = \frac{X_2(x_1, x_2)}{X_1(x_1, x_2)} \tag{1.3}$$

where X_1 and X_2 are polynomials. The most profound result on the integrability of this equation was obtained by Singer[6, 27] in 1992. Singer proved that if (1.3) has a local Liouvillian first integral, then there is a Liouvillian first integral of the form

$$\omega(x_1, x_2) = \int_{(x_1^0, x_2^0)}^{(x_1, x_2)} RX_2 dx_1 - RX_1 dx_2 \tag{1.4}$$

where

$$R = \exp\left[\int_{(x_1^0, x_2^0)}^{(x_1, x_2)} U dx_1 + V dx_2\right], \tag{1.5}$$

with U and V are rational functions in x_1 and x_2 such that

$$\partial U/\partial x_2 = \partial V/\partial x_1.$$

The existence of the integrating factor R in form of (1.5) was proved by Christopher [6] to be equivalent to the existence of an integrating factor of form

$$\exp(D/E)\prod C_i^{l_1}$$

where D, E and the C_i are polynomials in x_1 and x_2. The profound result by Singer was proved by the method of differential algebra. The same result was reproved independently using Liouville's method by Guan & Lei in [7]. From this result, if (1.3) has local Liouvillian first integral $\omega(x_1, x_2)$ with form (1.4), then $\delta_i^2\omega/\delta_i\omega$ are rational functions in x_1 and x_2, here $\delta_i = \frac{\partial}{\partial x_i}$ $(i = 1, 2)$.

Besides the differential Galois group, monodromy group of linear differential equation is also essential for the integrability of the equation. The monodromy group of a multi-valued complex function was studied by Khovanskii in 1970s. Khovanskii proved that a function is representable by quadrature if and only if its monodromy group is solvable[10]. As application, monodromy group of a linear differential equation plays essential role for integrating the equation by quadrature[34, pp. 128-130, Khovanskiy's Theorem]. In fact, monodromy group has close connection with Galois group. It was shown in [26] that the monodromy group of a linear differential equation whose singular points are regular is Zariski dense in the Galois Group. In 1998, Żołądek extended the conception of monodromy group in 1998 to study the functions defined on $\mathbb{C}P^n$ with algebraic singular set [35]. The results were applied to study the structure of the monodromy group of the first integrals of a Liouvillian integrable Pfaff equation. Through these studies, the author was able to extend Singer's result partly to the integrable polynomial Pfaff equation[35, Multi-dimensional Singer's theorem].

In 1990s, the geometry methods were introduced by Camacho, Scárdua, *et. al.* to study the Liouvillian integrability of nonlinear differential equation. The geometry methods focus on the characters of the foliation associated with the equation. In series of their works, the holonomy group that induced by an invariant algebraic curve of a polynomial system was studied[2, 3, 4]. It was prove in [3] that under some mild restrictions, if equation (1.3) admits Liouvillian first integral, then the foliation is either Darboux foliation or rational pull-back of an exponent two Bernoulli foliation. This result indicated the characters of the foliations of the Liouvillian integrable differential equation.

In recent years, nonlinear differential Galois theory has become an active area of research. In 2001 Malgrange introduced the Galois groupoid associated to a foliation with meromorphic singularities [18, 19]. For linear differential equations, Malgrange showed that this groupoid coincides with the Galois group of the Picard-Vessiot theory and was able to proved the required results in the linear case[18]. However, the further development of groupoid theory to nonlinear differential equation is needed[16].

In this paper, we will propose a framework to address the nonlinear differential theory. We will focus on the differential Galois theory of the polynomial system. The basic ideal that we will propose is the view point of differential algebra that has been introduced by Picard and Vessiot and developed by Ritt and Kolchin for the differential Galois theory of linear differential equation.

1.2. **Preliminary knowledge of differential algebra.** Before further introduction, we give here a brief outline for the preliminary knowledge that concerning differential algebra. For detail, refer to [9] and [25].

Let A be a ring, by *derivation* of A we mean an additive mapping $a \mapsto \delta a$ of A into itself satisfying

$$\delta(ab) = (\delta a)b + a(\delta b).$$

We shall say δa the *derivative* of a. The *differential ring* A is a commutative ring with unit together with a derivation δ. If there are m derivations of A, $\delta_i,\ i = 1, 2 \cdots, m$, satisfying

$$\delta_i\delta_j a = \delta_j\delta_i a,\quad \forall i, j \in \{1, 2, \cdots, m\}, \forall a \in A,$$

we call A together with all the δ_is a *partial differential ring*. When A is a field, (*partial*) *differential field* can be defined similarly. In this paper, we will say differential ring (field) for brevity for both differential ring (field) and partial differential ring (field).

Let A be any ring, Y be a set of finite or infinite number of elements. We can form a ring $A[Y]$ of polynomials of the elements in Y with coefficients from A. In particular, when A is a differential ring with derivations $\delta_1, \cdots, \delta_m$, and $Y = \{y_{i_1,i_2,\cdots,i_m}\}$ $(i_j = 0, 1, \cdots)$ to be the ordinary indeterminates over A, we can extend the derivations of A to $A[Y]$ uniquely by assigning $y_{i_1\cdots i_{j+1}\cdots i_m}$ as $\delta_j y_{i_1\ldots i_j\cdots i_m}$. Rewrite the notations as following

$$y_{0\cdots 0} = y,\qquad y_{i_1\cdots i_m} = \delta_1^{i_1}\cdots\delta_m^{i_m} y.$$

Following above procedure, we have added a *differential indeterminate* y to a differential ring A. We will denote the result differential ring as $A\{y\}$. The elements of $A\{y\}$ are *differential polynomials* in y. Suppose that A is a differential field, then $A\{y\}$ is a differential integral domain, and its derivations can be extended uniquely to the quotient field. We write $A\langle y\rangle$ for this quotient field, and its elements are *differential rational function* of y. The notations $\{\ \}$ and $\langle\ \rangle$ will also be used when the adjoined elements are not differential indeterminates, but rather elements of a larger differential ring or field.

Let A be any differential ring, then all elements in A with derivatives 0 form a subring C. This ring is called the ring of *constants*. If A is a field, so is C. Note that C contains the subring that generated by the unit element of A.

Let A be a differential ring, with $\delta_i (i = 1, \cdots, m)$ the derivations. We say an ideal I in A to be a *differential ideal* if $a \in I$ implies $\delta_i a \in I$ $(\forall i)$. An ideal I is said to be a *prime ideal* if $AB \in I$ always implies that $A \in I$ or $B \in I$. Hereinafter, if not point out particularly, we use the term (prime) ideal in short for differential (prime) ideal.

Let A and B be two differential rings. A *differential homomorphism* from A to B is a homomorphism (purely algebraically) which furthermore commutes with derivatives. The terms *differential isomorphism* and *differential automorphism* are self-explanatory.

1.3. **Summary.** In this paper we will consider the nonlinear differential equation

$$\frac{dx_2}{dx_1} = \frac{X_2(x_1, x_2)}{X_1(x_1, x_2)} \tag{1.6}$$

where X_1 and X_2 are polynomials. Following form of second order polynomial differential equation is also used to indicate the equation (1.6)

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & X_1(x_1, x_2) \\ \dot{x}_2 & = & X_2(x_1, x_2) \end{array}\right. \tag{1.7}$$

Correspondingly, we have following first order differential operator X that will be used to indicate the equation (1.6) or (1.7),

$$X(\omega) = (X_1(x_1, x_2)\delta_1 + X_2(x_1, x_2)\delta_2)\omega = 0 \tag{1.8}$$

where $\delta_i = \partial/\partial x_i$. From the theory of differential equation[8, pp.510-513], for any non-critical point $\mathbf{x}^0 = (x_1^0, x_2^0) \in \mathbb{C}^2$, these exists a non-constant solution of (1.8) $\omega(x_1, x_2)$ that is analytic at x^0. The solution $\omega(x_1, x_2)$ is said a first integral of (1.6) at $\mathbf{x}^0$. Furthermore, following lemma can be derived directly from [1, Theorem 1, pp. 98]

Lemma 1.1. *Consider the differential equation* (1.6)*, if* $X_1(x_1^0, x_2^0) \neq 0$ *and* $f(x_2)$ *to be a function that analytic* $x_2 = x_2^0$*, then there exists unique first integral* $\omega(x_1, x_2)$ *of* (1.6)*, analytic at* $\mathbf{x}^0 = (x_1^0, x_2^0)$*, and*

$$\omega(x_1^0, x_2) = f(x_2)$$

for all x_2 *in a neighborhood of* x_2^0*.*

Through the existence of the first integrals of (1.6) at the regular point $\mathbf{x}^0$, we can define the Liouvillian integrability of (1.6) at $\mathbf{x}^0$ as following.

Definition 1.2. Let K be the differential field of rational functions of two variables with derivatives δ_1 and δ_2, we say M to be a *Liouvillian extension* of K, if there exists $r \geq 0$ and subfields $K_i (i = 0, 1, \cdots, r)$, such that:

$$K = K_0 \subset K_1 \subset \cdots \subset K_r = M,$$

with $K_{i+1} = K_i\langle u_i\rangle$, and $u_i \in K_{i+1}\backslash K_i$ satisfying one of the following:

(1) u_i is algebraic over K_i;
(2) $\delta_j u_i \in K_i \quad (j = 1, 2)$;
(3) $\delta_j u_i/u_i \in K_i \quad (j = 1, 2)$.

A function that contained in some Liouvillian extension of K is said a *Liouvillian function.*

Definition 1.3. Let K be the differential field of rational functions of two variables, X be defined as (1.8), then X is *Liouvillian integrable at* $\mathbf{x}^0$ if there exists a first integral ω of X at $\mathbf{x}^0$ such that $M = K\langle\omega\rangle$ is a Liouvillian extension of K.

If X is Liouvillian integrable at one point $\mathbf{x}^0 \in \mathbb{C}^2$, there exists a first integral obtained from rational functions by finite combination steps of algebraic functions, quadratures and exponential of quadratures. It is easy to show by induction that this first integral is analytic on a dense open set on $\mathbb{C}^2$ (refer [27]). And hence X it is also Liouvillian integrable on a dense open set on $\mathbb{C}^2$. Therefore, we can also say that X is Liouvillian integrable.

Definition 1.4. A group G is *solvable* if there exist a subgroup series

$$G = G_0 \supset G_1 \supset \cdots \supset G_m = \{e\}$$

such that for any $0 \le i \le m - 1$, either

(1) $|G_i/G_{i+1}|$ is a finite group; or
(2) G_{i+1} is a normal subgroup of G_i and G_i/G_{i+1} is Abelian.

Following result will be proved in this paper.

Main Theorem *Consider the differential equation* (1.6). *Assume that* $X_1(0, 0) \neq 0$. *Let* K *be the differential field of rational functions. Then* (1.6) *is Liouvillian integrable if, and only if, the differential Galois group of* (1.6) *over* K *at* $(0, 0)$ *is solvable.*

This paper is arranged as following. We will define the differential Galois group in Section 2. The structure of the Galois group will be discussed in Section 3. The main theorem will be proved at Section 4. As application, the differential Galois group of general Riccati and van der Pol equations are given in Section 4. Throughout this paper, the base field K will always means the field of all rational functions in x_1 and x_2 that contains all complex numbers $\mathbb{C}$ as the constant field.

2. Differential Galois group

In this section, we will present the definition of differential Galois group of the system (1.6) at the regular point $\mathbf{x}^0 = (x_1^0, x_2^0)$. To this end, we will at first define the group $\mathcal{G}[[\epsilon]]$ that acts at the first integrals of X at $\mathbf{x}^0$, secondly study the admissible differential isomorphism of (1.6) at $\mathbf{x}^0$ that is an element of $\mathcal{G}[[\epsilon]]$, and finally prove that all admissible differential isomorphisms form a subgroup of $\mathcal{G}[[\epsilon]]$, and this subgroup will be defined as the differential Galois group of (1.6) at $\mathbf{x}^0$. Hereinafter, we assume that $\mathbf{x}^0 = (0, 0)$ for short. When we mention a first integral, we will always mean a first integral that analytic at $(0, 0)$. Following notations will be used hereinafter. Let $\mathcal{A}_0$ denote the set of all functions $f(z)$ of one variable that analytic at $z = 0$, and

$$\begin{aligned} \mathcal{A}_0^0 &= \{f(z) \in \mathcal{A}_0 |\ \ f(0) = 0\}, \\ \mathcal{A}_0^1 &= \{f(z) \in \mathcal{A}_0^0 |\ \ f'(0) \neq 0\} \end{aligned}$$

Let $\Omega_{(0,0)}(X)$ denote the set of all first integral of (1.6) that analytic at $(0,0)$, and

$$\begin{aligned}\Omega^0_{(0,0)}(X) &= \{\omega(x_1,x_2)\in\Omega_{(0,0)}|\ \ \omega(0,0)=0\},\\ \Omega^1_{(0,0)}(X) &= \{\omega(x_1,x_2)\in\Omega^0_{(0,0)}|\ \ \delta_2\omega(0,0)\neq 0\}\end{aligned}$$

Therefore, $f(z):=\omega(0,z)\in\mathcal{A}^1_0$ for $\omega\in\Omega^1_{(0,0)}(X)$.

2.1. **Structure of first integrals at a regular point.** Since we will focus on the Liouvillian integrability of the polynomial system (1.6), it is enough to concentrate on the first integrals in $\Omega^1_{(0,0)}(X)$ by following lemma.

Lemma 2.1. *If X is Liouvillian integrable at $(0,0)$, then there exists a first integral $\omega\in\Omega^1_{(0,0)}(X)$, such that $M=K\langle\omega\rangle$ is a Liouvillian extension of K.*

Proof. Let u be a first integral such that $M\langle u\rangle$ is a Liouvillian extension of K. If $u\in\Omega^1_{(0,0)}(X)$, then the lemma has been concluded. If $\omega\notin\Omega^1_{(0,0)}(X)$, we can always assume that $u\in\Omega^0_{(0,0)}(X)$ by subtracting a constant $u(0,0)$ from u. Let

$$u(0,x_2)=\sum_{i\geq k}a_ix_2^i,\qquad k\geq 2, a_k\neq 0$$

and

$$f(x_2)=\sqrt[k]{\omega(0,x_2)}=\left(\sum_{i\geq k}a_ix_2^i\right)^{1/k}=x_2\left(\sum_{i\geq k}a_ix_2^{i-k}\right)^{1/k}$$

Then $f(x_2)\in\mathcal{A}^1_0$. From Lemma 1.1, there is unique first integral $\omega\in\Omega_{(0,0)}(X)$, such that $\omega(0,x_2)=f(x_2)$ and therewith $\omega\in\Omega^1_{(0,0)}(X)$. Moreover, it follows from $f(x_2)^k=u(0,x_2)$ that $\omega^k=u$, and hence $K\langle\omega\rangle$ is a Liouvillian extension of K. □

From Lemma 1.1, there is a one-to-one correspondence between $\Omega^1_{(0,0)}(X)$ and $\mathcal{A}^1_0$. Hence, the structure of $\Omega^1_{(0,0)}(X)$ can be described by that of $\mathcal{A}^1_0$. It is obvious that $\mathcal{A}^1_0$ contains the identity function $e(z)=z$, and for any $f(z),g(z)\in\mathcal{A}^1_0$, $f\circ g(z)\in\mathcal{A}^1_0$ and $f^{-1}(z)\in\mathcal{A}^1_0$. Hence, $\mathcal{A}^1_0$ is a group with the composition as the multiply operation. Furthermore, we have the following result.

Lemma 2.2. *For any $\omega\in\Omega^1_{(0,0)}(X)$, have*

$$\Omega^1_{(0,0)}(X)=\{f(\omega)|\ f\in\mathcal{A}^1_0\}=:\mathcal{A}^1_0(\omega)$$

Proof. It's obvious that $f(\omega)\in\Omega^0_{(0,0)}(X)$ for any $f\in\mathcal{A}^1_0$. Moreover,

$$\frac{\partial f(\omega)}{\partial x_2}(0,0)=f'(0)\delta_2\omega(0,0)\neq 0,$$

hence $f(\omega)\in\Omega^1_{(0,0)}(X)$ and therewith $\mathcal{A}^1_0(\omega)\subseteq\Omega^1_{(0,0)}(X)$.

For any $\omega,u\in\Omega^1_{(0,0)}(X)$, let

$$g(x_2)=\omega(0,x_2),\quad h(x_2)=u(0,x_2)$$

then $g,h\in\mathcal{A}^1_0$. Hence, $f=h\circ g^{-1}\in\mathcal{A}^1_0$, and $f(g(x_2))=h(x_2)$, i.e. $f(\omega(0,x_2))=u(0,x_2)$. By Lemma 1.1, $u=f(\omega)$ and hence $\Omega^1_{0,0}(X)\subseteq\mathcal{A}^1_0(\omega)$, the proposition is followed. □

The elements in $\mathcal{A}_0^1$ are map 0 to 0. To take the case that maps 0 to a nonzero value into account, we adjoin an infinitesimal variable ε to the constant field $\mathbb{C}$ and consider the ring of infinite series of ϵ with coefficients in $\mathbb{C}$. We denoted this extended constant ring as $\mathbb{C}[[\epsilon]]$. We have the following:

(1) A series $\sum_{i\geq 0} c_i\epsilon^i$ equal to 0 if and only if all coefficients c_i are equal to 0;
(2) $\delta_1\epsilon = \delta_2\epsilon = 0$

Consider the infinite power series

$$f(z;\epsilon) = \sum_{i,j\geq 0} f_{ij} z^i\epsilon^j \in \mathbb{C}[[z,\epsilon]].$$

The series $f(z;\epsilon)$ is said analytic if it is convergent when (z,ϵ) take values in neighborhood of $(0,0)$. We can also write an analytic series as

$$f(z;\epsilon) = \sum_{i=0}^{\infty} f_i(z)\epsilon^i \tag{2.1}$$

where $f_i(z) \in \mathcal{A}_0$. We will denote all analytic series as $\mathcal{A}_0[[\epsilon]]$. Let

$$\mathcal{G}[[\epsilon]] = \{f(z;\epsilon) = \sum_{i\geq 0} f_i(z)\epsilon^i \in \mathcal{A}_0[[\epsilon]] \;\Big|\; f_0(z) \in \mathcal{A}_0^1\}$$

and define the multiplication in $\mathcal{G}[[\epsilon]]$ as:

$$f(z;\epsilon)\cdot g(z;\epsilon) = f(g(z;\epsilon);\epsilon),\quad f(z;\epsilon), g(z;\epsilon) \in \mathcal{G}[[\epsilon]]$$

Then we have

Lemma 2.3. *$(\mathcal{G}[[\epsilon]],\cdot)$ is a group.*

Proof. At first, we will show that for any $f(z;\epsilon), g(z;\epsilon) \in \mathcal{G}[[\epsilon]]$, $f(z;\epsilon)\cdot g(z;\epsilon) \in \mathcal{G}[[\epsilon]]$. Let

$$f(z;\epsilon) = \sum_i f_i(z)\epsilon^j,\quad g(z;\epsilon) = \sum_i g_i(z)\epsilon^j.$$

Then $f(z;\epsilon)$ and $g(z;\epsilon)$ are analytic functions of z,ϵ at $(0,0)$. Since $g(0,0) = g_0(0) = 0$, $f(g(z;\epsilon);\epsilon)$ is also an analytic function at $(0,0)$, i.e., $f(g(z;\epsilon);\epsilon) \in \mathcal{A}_0[[\epsilon]]$. Moreover, $f(g(z;0);0) = f_0(g_0(z)) \in \mathcal{A}_0^1$ and therewith $f(z;\epsilon)\cdot g(z;\epsilon) \in \mathcal{G}[[\epsilon]]$.

It is easy to verify that

$$(f(z;\epsilon)\cdot g(z;\epsilon))\cdot h(z;\epsilon) = f(z;\epsilon)\cdot(g(z;\epsilon)\cdot h(z;\epsilon))$$

We can embed $\mathcal{A}_0^1$ into $\mathcal{A}_0^1[[\epsilon]]$ by identifying $f(z) \in \mathcal{A}_0^1$ with $f(z;0) = f(z) + \sum_{i\geq 1} 0\cdot\epsilon^i \in \mathcal{A}_0^1[[\epsilon]]$. Then $e(z;0) = e(z) \in \mathcal{G}[[\epsilon]]$, and for any $f(z;\epsilon) \in \mathcal{G}[[\epsilon]]$,

$$e(z;0)\cdot f(z;\epsilon) = e(f(z;\epsilon);0) = f(z;\epsilon),\quad f(z;\epsilon)\cdot e(z;0) = f(e(z;0);\epsilon) = f(z;\epsilon)$$

Thus, $e(z;0)$ is also an identity of $\mathcal{G}[[\epsilon]]$.

We have, for any $f(z;\epsilon) \in \mathcal{G}[[\epsilon]]$, $(\partial f/\partial z)(0,0) \neq 0$. Thus, the equation

$$u = f(z;\epsilon)$$

has unique solution $z = f^{-1}(u;\epsilon)$ in the neighborhood of $(0,0)$ such that

$$u = f(f^{-1}(u;\epsilon)).$$

In particular, $z = f(f^{-1}(z;\epsilon);\epsilon)$, and $z = f^{-1}(u;\epsilon) = f^{-1}(f(z;\epsilon);\epsilon)$. Thus, $f^{-1}(z;\epsilon)$ is the inverse element of $f(z;\epsilon)$. Furthermore, $f^{-1}(z;\epsilon)$ is analytic at

$(0,0)$, and $f^{-1}(z;0) \in \mathcal{A}_0^1$. Thus, we conclude that $f^{-1}(z;\epsilon) \in \mathcal{G}[[\epsilon]]$ and the lemma has been proved. □

For any $\sigma = f(z;\epsilon) \in \mathcal{G}[[\epsilon]]$ and $\omega \in \Omega^1_{(0,0)}$, we define the action of σ at ω as $\sigma\omega = f(\omega;\epsilon)$. This is well defined at the neighborhood of $(0,0)$. Taking account that

$$X(f(\omega;\epsilon)) = X(\sum_{i\geq 0} f_i(\omega)\epsilon^i) = \sum_{i\geq 0} X(f_i(\omega))\epsilon^i = 0$$

σ maps a first integral ω to another first integral $f(\omega;\epsilon)$.

Let $h(z;\epsilon) \in \mathcal{A}_0[[\epsilon]]$ and $f(z;\epsilon) \in \mathcal{G}[[\epsilon]]$, then

$$h(z;\epsilon)\cdot f(z;\epsilon) = h(f(z;\epsilon);\epsilon) \in \mathcal{A}_0[[\epsilon]]$$

is well defined, and $(h(z;\epsilon)\cdot f(z;\epsilon))\omega = h(f(\omega;\epsilon);\epsilon)$.

2.2. **Admissible differential isomorphism.** For any given $\omega \in \Omega^1_{(0,0)}(X)$, an extension field M of K is obtained by adjoining ω to K, and denoted as $M = K\langle\omega\rangle$. Throughout this paper, if not mentioned particularly, ω will always means a determinate first integral. In this subsection, we will define and study the admissible differential isomorphism that is an element of the group $\mathcal{G}[[\epsilon]]$ with additional conditions. Throughout this paper, we will call compactly a map $\sigma : \omega \mapsto \sigma(\omega)$ a differential isomorphism if there exists a differential isomorphism from $K\langle\omega\rangle$ to $K\langle\sigma(\omega)\rangle$ that maps ω to $\sigma(\omega)$ with elements in K fixed.

Definition 2.4. Let $M = K\langle\omega\rangle$ with $\omega \in \Omega^1_{(0,0)}(X)$. An *admissible differential isomorphism of M/K with respect to X at* $(0,0)$ (a.d.i., singular and plural) is a map σ that acts on M with the following properties:

(1) σ maps ω to $f(\omega;\epsilon)$ with $f(z;\epsilon) \in \mathcal{G}[[\epsilon]]$;
(2) $\sigma : \omega \mapsto f(\omega;\epsilon)$ is a differential isomorphism;
(3) for any $h_i(z;\epsilon) \in \mathcal{A}_0[[\epsilon]]$ $(1 \leq i \leq m < \infty)$, σ can be extended to a differential isomorphism of $K\langle\omega, h_1(\omega;\epsilon), \cdots, h_m(\omega;\epsilon)\rangle$ that maps $h_i(\omega;\epsilon)$ to $h_i(f(\omega;\epsilon);\epsilon)$, respectively, with K elementwise fixed.

We denote by $Ad(M/K,X)_{(0,0)}$ the set of all a.d.i. of M/K with respect to X at $(0,0)$.

Following two Lemmas show that $Ad(M/K,X)_{(0,0)}$ is a subgroup of $\mathcal{G}[[\epsilon]]$.

Lemma 2.5. *If $\sigma,\tau \in Ad(M/K,X)_{(0,0)}$, then $\sigma\cdot\tau \in Ad(M/K,X)_{(0,0)}$.*

Proof. For any $\varsigma = h(z;\epsilon) \in \mathcal{A}_0[[\epsilon]]$, we will prove that $\sigma\cdot\tau : \omega \mapsto (\sigma\cdot\tau)\omega$ can be extended to a differential isomorphism of $K\langle\omega, \varsigma\omega\rangle$ that maps $\varsigma\omega$ to $(\varsigma\cdot\sigma\cdot\tau)\omega$ with K elementwise fixed.

Since $\tau \in Ad(M/K,X)_{(0,0)}$, τ is well defined in $K\langle\omega, \sigma\omega, (\varsigma\cdot\sigma)\omega\rangle$. Hence, the restriction of τ at $K\langle\sigma\omega, (\varsigma\cdot\sigma)\omega\rangle$ is a differential isomorphism that maps $\sigma\omega$ and $(\varsigma\cdot\sigma)\omega$ to $(\sigma\cdot\tau)\omega$ and $(\varsigma\cdot\sigma\cdot\tau)\omega$, respectively, with K elementwise fixed. Consider the following

$$\begin{tikzcd}
K\langle \omega, \varsigma\omega\rangle \arrow[rr, "\sigma\cdot\tau"] \arrow[dr, "\sigma"'] & & K\langle (\sigma\cdot\tau)\omega, (\varsigma\cdot\sigma\cdot\tau)\omega\rangle \\
& K\langle \sigma\omega, (\varsigma\cdot\sigma)\omega\rangle \arrow[ur, "\tau|_{K\langle\sigma\omega,(\varsigma\cdot\sigma)\omega\rangle}"'] &
\end{tikzcd}$$

where σ and $\tau|_{K\langle\sigma\omega,(\varsigma\cdot\sigma)\omega\rangle}$ are differential isomorphisms with K elementwise fixed. It follows that $\sigma\cdot\tau$ is also a differential isomorphism with K elementwise fixed. The extension of $\sigma\cdot\tau$ to $K\langle\omega, h_1\omega,\cdots, h_m\omega\rangle$ can be proved similarly. Thus $\sigma\cdot\tau\in Ad(M/K,X)_{(0,0)}$. □

Lemma 2.6. *If $\sigma\in Ad(M/K,X)_{(0,0)}$, then $\sigma^{-1}\in Ad(M/K,X)_{(0,0)}$.*

Proof. Similarly, it's sufficient to prove that for any $\varsigma = h(z;\epsilon)\in\mathcal{A}_0[[\epsilon]]$, σ^{-1} can be extended to a differential isomorphism of $K\langle\omega,\varsigma\omega\rangle$ that maps $\varsigma\omega$ to $(\varsigma\cdot\sigma^{-1})\omega$ with K elementwise fixed.

Consider the extension of σ to $K\langle\omega,\sigma^{-1}\omega,(\varsigma\cdot\sigma^{-1})\omega\rangle$ that maps $\sigma^{-1}\omega$ and $(\varsigma\cdot\sigma^{-1})\omega$ to $(\sigma^{-1}\cdot\sigma)\omega=\omega$ and $(\varsigma\cdot\sigma^{-1}\cdot\sigma)\omega=\varsigma\omega$, respectively. Hence, the restricted map $\sigma|_{K\langle\sigma^{-1}\omega,(\varsigma\cdot\sigma^{-1})\omega\rangle}$ is a differential isomorphism that maps $K\langle\sigma^{-1}\omega,(\varsigma\cdot\sigma^{-1})\omega\rangle$ to $K\langle\omega,\varsigma\omega\rangle$ with K elementwise fixed. Let $\tau=\left(\sigma|_{K\langle\sigma^{-1}\omega,(\varsigma\cdot\sigma^{-1})\omega\rangle}\right)^{-1}$, then $\tau: K\langle\omega,\varsigma\omega\rangle\mapsto K\langle\sigma^{-1}\omega,(\varsigma\cdot\sigma^{-1})\omega\rangle$ is a differential isomorphism that maps ω and $\varsigma\omega$ to $\sigma^{-1}\omega$ and $(\varsigma\cdot\sigma^{-1})\omega$ respectively, with K elementwise fixed. Thus, we have $\sigma^{-1}=\tau\in Ad(M/K,X)_{(0,0)}$. □

2.3. **Differential Galois group.** From Lemma 2.5, 2.6, $Ad(M/K,X)_{(0,0)}$ is a subgroup of $\mathcal{G}[[\epsilon]]$. We will show that this subgroup is our desired differential Galois group.

Definition 2.7. Let X be defined as (1.8), K be the field of rational functions, $\omega\in\Omega^1_{(0,0)}(X)$ and $M=K\langle\omega\rangle$. The *differential Galois group of M/K with respect to X at* $(0,0)$, denoted as $\mathrm{Gal}(M/K,X)_{(0,0)}$, is defined as the subgroup of $\mathcal{G}[[\epsilon]]$ of all elements in $Ad(M/K,X)_{(0,0)}$.

Following two Lemmas show that the differential Galois group is determined 'uniquely' by the differential operator X (or the differential equation (1.6)).

Lemma 2.8. *Let $u\in\Omega^1_{(0,0)}(X)$ and $N=K\langle u\rangle$, then*

$$\mathrm{Gal}(N/K,X)_{(0,0)}\cong\mathrm{Gal}(M/K,X)_{(0,0)}.$$

Proof. By Lemma 2.2, $u\in\Omega^1_{(0,0)}(X)=\mathcal{A}^1_0(\omega)$. There is a function $h\in\mathcal{A}^1_0$ such that $u=h(\omega)$. Let $\tau=h(z;0)\in\mathcal{G}[[\epsilon]]$, then $u=\tau\omega$, i.e., $\omega=\tau^{-1}u$.

For any $\sigma\in\mathrm{Gal}(M/K,X)_{(0,0)}$, we will show that $\tau\cdot\sigma\cdot\tau^{-1}\in\mathrm{Gal}(N/K,X)_{(0,0)}$. To this end, we only need to show that for any $\varsigma\in\mathcal{A}_0[[\epsilon]]$, $\tau\cdot\sigma\cdot\tau^{-1}$ can be extended to a differential isomorphism of $K\langle u,\varsigma u\rangle$ that maps u and ςu to $(\tau\cdot\sigma\cdot\tau^{-1})u$ and $(\varsigma\cdot\tau\cdot\sigma\cdot\tau^{-1})u$ respectively, with K elementwise fixed.

Since $\sigma\in\mathrm{Gal}(M/K,X)_{(0,0)}$ and $\tau,(\varsigma\cdot\tau)\in\mathcal{A}_0[[\epsilon]]$, σ can be extended to $K\langle\omega,\tau\omega,(\varsigma\cdot\tau)\omega\rangle$ that maps $\tau\omega=u$ and $(\varsigma\cdot\tau)\omega=\varsigma u$ to $(\tau\cdot\sigma)\omega=(\tau\cdot\sigma\cdot\tau^{-1})u$ and $(\varsigma\cdot\tau\cdot\sigma)\omega=(\varsigma\cdot\tau\cdot\sigma\cdot\tau^{-1})u$ respectively. Hence, $\sigma|_{K\langle u,\varsigma u\rangle}$, the restriction of σ to $K\langle u,\varsigma u\rangle$, is a differential isomorphism that maps u and ςu to $(\tau\cdot\sigma\cdot\tau^{-1})u$ and

$(\varsigma\cdot\tau\cdot\sigma\cdot\tau^{-1})u$ respectively, with K elements fixed. Thus, we have $\tau\cdot\sigma\cdot\tau^{-1}\in\mathrm{Gal}(N/K,X)_{(0,0)}$. In fact, we have further that $\tau\cdot\sigma\cdot\tau^{-1}|_{K\langle u\rangle}=\sigma|_{K\langle u\rangle}$.

Similarly, for any $\eta\in\mathrm{Gal}(N/K,X)_{(0,0)}$, $\tau^{-1}\cdot\eta\cdot\tau\in\mathrm{Gal}(M/K,X)_{(0,0)}$. Thus $\sigma\mapsto\tau^{-1}\cdot\sigma\cdot\tau$ is an isomorphism between $\mathrm{Gal}(M/K,X)_{(0,0)}$ and $\mathrm{Gal}(N/K,X)_{(0,0)}$. The Lemma has been proved. □

Lemma 2.8 shows that the differential Galois group of (1.6) at $(0,0)$ is independent to the choice of the first integral ω. The groups are different from a diffeomorphism for different choices of the first integrals. Following lemma will show that in particular cases, the group is also independent to the choices of regular points. We will prove latter (Theorem 3.9) that these are all possible cases given that the group is of finite order.

Lemma 2.9. *Assume that* $\omega\in\Omega^1_{(0,0)}(X)$, $M=K\langle\omega\rangle$, *and* $G=\mathrm{Gal}(M/K,X)_{(0,0)}$, *we have the following:*

1. *If* $\omega\in K$, *then* $\sigma\omega=\omega,\forall\sigma\in G$;
2. *If* $(\delta_2\omega)^n\in K$ *for some* $n\in\mathbb{N}$, *then* $\sigma\omega=\mu_n\omega+c(\epsilon),\forall\sigma\in G$, *where* μ_n *is a* n*-th root of unity;*
3. *If* $\delta_2^2\omega/\delta_2\omega\in K$, *then* $\sigma\omega=a(\epsilon)\omega+c(\epsilon),\ \forall\sigma\in G$;
4. *If* $(2(\delta_2\omega)(\delta_2^3\omega)-3(\delta_2^2\omega)^2)/(\delta_2\omega)^2\in K$, *then* $\sigma\omega=\dfrac{a(\epsilon)\omega}{1+b(\epsilon)\omega}+c(\epsilon)$, $\forall\sigma\in G$.

Here $a(\epsilon),b(\epsilon),c(\epsilon)\in\mathcal{A}_0[[\epsilon]]\cap\mathbb{C}[[\epsilon]]$, *and* $c(0)=0$. *Moreover, for any* (x_1^0,x_2^0) *such that* $X_1(x_1^0,x_2^0)\neq0$ *and* ω *is analytic at* (x_1^0,x_2^0) *with* $\delta_2\omega(x_1^0,x_2^0)\neq0$, *the first integral* u *that defined as* $u=\omega-\omega(x_1^0,x_2^0)$ *is contained in* $\Omega^1_{(x_1^0,x_2^0)}(X)$, *and above results* (1)-(4) *are also valid for all* $\sigma\in\mathrm{Gal}(K\langle u\rangle/K,X)_{(x_1^0,x_2^0)}$.

Proof. The first part is proved as follows.

(1) is evident.

(2). Let $(\delta_2\omega)^n=a\in K$, then

$$a-(\delta_2\omega)^n=0$$

Clearly, for any $\sigma=f(z;\epsilon)\in G$,

$$0=\sigma(a-(\delta_2\omega)^n)=a-(\delta_2(\sigma(\omega)))^n=a-(\delta_2(f(\omega;\epsilon)))^n=a-f'(\omega;\epsilon)^n(\delta_2\omega)^n$$

Hereinafter, f' means the derivative of $f(z;\epsilon)$ with respect to z. Thus, we have $f'(\omega;\epsilon)^n=1$ and therewith $f(\omega;\epsilon)=\mu_n\omega+c(\epsilon)$, where $c(\epsilon)\in\mathcal{A}_0[[\epsilon]]\cap\mathbb{C}[[\epsilon]]$ and μ_n is a n-th root of unity.

(3). Let $\delta_2^2\omega/\delta_2\omega=a\in K$, then

$$\delta_2^2\omega-a\delta_2\omega=0$$

For any $\sigma=f(z;\epsilon)\in G$,

$$\begin{aligned}0&=\sigma(\delta_2^2\omega-a\delta_2\omega)\\&=\delta_2^2(\sigma\omega)-a\delta_2(\sigma\omega)\\&=\delta_2^2(f(\omega;\epsilon))-a\delta_2(f(\omega;\epsilon))\\&=f''(\omega;\epsilon)(\delta_2\omega)^2+f'(\omega;\epsilon)\delta_2^2\omega-af'(\omega;\epsilon)\delta_2\omega\\&=f''(\omega;\epsilon)(\delta_2\omega)^2\end{aligned}$$

Hence, we have $f''(\omega;\epsilon)=0$ and therefore $\sigma\omega=f(\omega;\epsilon)=a(\epsilon)\omega+c(\epsilon)$ for some $a(\epsilon),c(\epsilon)\in\mathcal{A}_0[[\epsilon]]\cap\mathbb{C}[[\epsilon]]$.

(4). Let $(2(\delta_2\omega)(\delta_2^3\omega)-3(\delta_2^2\omega)^2)/(\delta_2\omega)^2 = a\in K$, then

$$2(\delta_2\omega)(\delta_2^3\omega)-3(\delta_2^2\omega)^2-a(\delta_2\omega)^2=0$$

For any $\sigma = f(z;\epsilon)\in G$,

$$\begin{aligned}
0 &= \sigma\left(2(\delta_2\omega)(\delta_2^3\omega)-3(\delta_2^2\omega)^2-a(\delta_2\omega)^2\right)\\
&= 2\delta_2(\sigma\omega)(\delta_2^3(\sigma\omega))-3(\delta_2^2(\sigma\omega))^2-a(\delta_2(\sigma\omega))^2\\
&= 2\delta_2(f(\omega;\epsilon))(\delta_2^3(f(\omega;\epsilon)))-3(\delta_2^2(f(\omega;\epsilon)))^2-a(\delta_2(f(\omega;\epsilon)))^2\\
&= 2f'(\omega;\epsilon)(\delta_2\omega)\left(f'''(\omega;\epsilon)(\delta_2\omega)^3+3f''(\omega;\epsilon)(\delta_2\omega)(\delta_2^2\omega)+f'(\omega;\epsilon)\delta_2^3\omega\right)\\
&\quad -3\left(f''(\omega;\epsilon)(\delta_2\omega)^2+f'(\omega;\epsilon)(\delta_2^2\omega)\right)^2-a\left(f'(\omega;\epsilon)(\delta_2\omega)\right)^2\\
&= 2f'(\omega;\epsilon)f''(\omega;\epsilon)(\delta_2\omega)^4+6f'(\omega;\epsilon)f''(\omega;\epsilon)(\delta_2\omega)^2(\delta_2^2\omega)+2(f'(\omega;\epsilon))^2(\delta_2\omega)(\delta_2^3\omega)\\
&\quad -3(f''(\omega;\epsilon))^2(\delta_2\omega)^4-6f'(\omega;\epsilon)f''(\omega;\epsilon)(\delta_2\omega)(\delta_2^2\omega)-3(f'(\omega;\epsilon))^2(\delta_2^2\omega)^2-a(f'(\omega;\epsilon))^2(\delta_2\omega)^2\\
&= \left(2f'(\omega;\epsilon)f''(\omega)-3(f''(\omega;\epsilon))^2\right)(\delta_2\omega)^4+(f'(\omega;\epsilon))^2(2(\delta_2\omega)(\delta_2^3\omega)-3(\delta_2^2\omega)^2-a(\delta_2\omega)^2)\\
&= (2f'(\omega;\epsilon)f''(\omega)-3(f''(\omega;\epsilon))^2)(\delta_2\omega)^4
\end{aligned}$$

Hence, we have

$$2f'(\omega;\epsilon)f''(\omega;\epsilon)-3(f''(\omega;\epsilon))^2=0$$

The general solution of this equation yields that

$$f(\omega;\epsilon)=\frac{a(\epsilon)\omega}{1+b(\epsilon)\omega}+c(\epsilon)$$

for some $a(\epsilon),b(\epsilon),c(\epsilon)\in\mathcal{A}_0[[\epsilon]]\cap\mathbb{C}[[\epsilon]]$.

Finally, taking account that $f(z;\epsilon)\in\mathcal{G}[[\epsilon]]$, we have $c(0)=0$.

For the second part, it's obvious that $u\in\Omega^1_{(x_1^0,x_2^0)}(X)$, and above discussions are also valid for u. The proof is complete. □

Similar to classical Galois theory, for any differential subfield L of M containing K, let

$$L' = \{\sigma\in \mathrm{Gal}(M/K,X)_{(0,0)}|\ \sigma a = a, \forall a\in L\}$$

be the subset of $\mathrm{Gal}(M/K,X)_{(0,0)}$ consisting of all a.d.i. leaving L elementwise fixed. For any subgroup H of G, let

$$H' = \{a\in M|\ \sigma a = a,\quad \forall\sigma\in H\}$$

be the set of all elements in M left fixed by H. We have the following result.

Lemma 2.10. *Let L, L_1, L_2 be subfields of M containing K, H, H_1, H_2 be subgroups of G, then*

(1) *L' is subgroup of G, H' is subfield of M;*
(2) *$L\subseteq L''$, $H\subseteq H''$;*
(3) *$L_1\supseteq L_2\Rightarrow L_1'\subseteq L_2'$;*
(4) *$H_1\supseteq H_2\Rightarrow H_1'\subseteq H_2'$.*

Let L to be a subfield of M that contains K. We can also consider M as an extension field of L by $M = K\langle\omega\rangle = L\langle\omega\rangle$ and the differential Galois group of M/L with respect to X at $(0,0)$ can be defined through the same procedure. We denote this Galois group as $\mathrm{Gal}(M/L,X)_{(0,0)}$.

Lemma 2.11. *Let L be the subfield of M containing K, then*

$$\mathrm{Gal}(M/L, X)_{(0,0)} = L'$$

In particular, $\mathrm{Gal}(M/K, X)_{(0,0)} = K'$.

Proof. It is easy to have $\mathrm{Gal}(M/L, X)_{(0,0)} \subseteq L'$. We will show that $L' \subseteq \mathrm{Gal}(M/L, X)_{(0,0)}$. For $\sigma \in L'$ and $\varsigma \in \mathcal{G}[[\epsilon]]$, upon $K \subseteq L \subseteq K\langle\omega, \varsigma\omega\rangle$ and $K \subseteq L \subseteq K\langle\sigma\omega, (\varsigma \cdot \sigma)\omega\rangle$, we have

$$L\langle\omega, \varsigma\omega\rangle = K\langle\omega, \varsigma\omega\rangle,\ L\langle\sigma\omega, (\varsigma \cdot \sigma)\omega\rangle = K\langle\sigma\omega, (\varsigma \cdot \sigma)\omega\rangle.$$

By definition 2.4, σ is a differential isomorphism of $K\langle\omega, \varsigma\omega\rangle$ onto $K\langle\sigma\omega, (\varsigma \cdot \sigma)\omega\rangle$. Hence, σ is a differential isomorphism of $L\langle\omega, \varsigma\omega\rangle$ onto $L\langle\sigma\omega, (\varsigma \cdot \sigma)\omega\rangle$, with L elementwise fixed. From which we conclude that $\sigma \in \mathrm{Gal}(M/L, X)_{(0,0)}$ and therewith $L' \subseteq \mathrm{Gal}(M/L, X)_{(0,0)}$. The Lemma has been proved. □

Lemma 2.12. [9, Lemma 3.1] *Let $M = K\langle\omega\rangle$, L and N be differential subfields of M containing K with $N \supset L$, $[N : L] = n$. Let L' and N' be the corresponding subgroups of* $\mathrm{Gal}(M/K, X)_{(0,0)}$. *Then the index of N' in L' is at most n.*

Lemma 2.13. [9, Lemma 3.2] *Let $M = K\langle\omega\rangle, G = \mathrm{Gal}(M/K, X)_{(0,0)}$ and H and J be subgroups of G with $H \supset J$ and J of index n in H. Let H' and J' be the corresponding intermediate differential fields. Then $[J' : H'] \leq n$.*

3. Structure of Differential Galois group

We will consider in this section the structure of the differential Galois group $\mathrm{Gal}(M/K, X)_{(0,0)}$. Firstly, we introduce several preliminary definitions for describing the structure of the differential Galois group.

3.1. **Generalized differential polynomial.** Let y be an indeterminate over K, and denote by $\mathcal{A}_0(y)$ the ring

$$\mathcal{A}_0(y) = \{f(y) \mid f \in \mathcal{A}_0\}.$$

Adjoining $\mathcal{A}_0(y)$ to K will result to a ring $K[\mathcal{A}_0(y)]$ with elements of form

$$\sum_{i=1}^{n} a_i f_i(y),$$

where $a_i \in K, f_i \in \mathcal{A}_0$. Furthermore, the ring $K[\mathcal{A}_0(y)]$ is able to be extended to a differential ring, denoted by $K\{\mathcal{A}_0(y)\}$, through the derivatives δ_1 and δ_2 by

$$\delta_i f(y) = f'(y)\delta_i y, \quad (i = 1, 2, f \in \mathcal{A}_0),$$
$$\delta_1(\delta_1^k\delta_2^l y) = \delta_1^{k+1}\delta_2^l y, \quad \delta_2(\delta_1^k\delta_2^l y) = \delta_1^k\delta_2^{l+1} y,$$

where f' is a derivative of f and contained in $\mathcal{A}_0$. It is easy to know that elements in $K\{\mathcal{A}_0(y)\}$ are polynomials in the derivatives $\delta_1^k\delta_2^l y$ $(k, l \in \mathbb{N})$, with coefficients in $K[\mathcal{A}_0(y)]$. Being distinguish with differential polynomial, the coefficients of the elements in $K\{\mathcal{A}_0(y)\}$ contain not only the polynomial of y, but also the terms of form $f(y)$ with $f \in \mathcal{A}_0$. We will say such a polynomial of the derivatives with coefficients consist of the combination of polynomials in y and functions $f(y)$ with $f \in \mathcal{A}_0$ to be a *quasi-differential polynomial* (QDP). A *proper quasi-differential polynomial* (PQDP) is a QDP that involves at least one proper derivatives of y. A *regular prime ideal* of $K\{\mathcal{A}_0(y)\}$ is a prime ideal $\Lambda \in K\{\mathcal{A}_0(y)\}$ that contains exclusively PQDP. It is the regular prime ideal $\Lambda \in K\{\mathcal{A}_0(y)\}$ that we will be interested in (see Theorem 3.7). The terms and results concerning differential polynomial are applicable to the PQDP. Let us recall a few basic facts on the differential polynomial, for detail, refer to [25].

Definition 3.1. Let

$$w_1 = \delta_1^{i_1}\delta_2^{i_2}y, \quad w_2 = \delta_1^{j_1}\delta_2^{j_2}y,$$

be proper derivatives of y, w_2 is higher then w_1 if $j_1 > i_1$ or $j_1 = i_1$ and $j_2 > i_2$. A proper derivative of y is always higher then y.

Definition 3.2. Let A be a QDP, if A involves proper derivative of y, by the *leader* of A, we mean the highest of those derivatives of y involved in A. If A involves y but no proper derivatives of y, then the leader of A is y. Let A_1 be a QDP, and A_2 be a PQDP, we say A_2 to be of *higher rank* than A_1, if either

(1). A_2 has a higher leader than A_1; or
(2). A_1 and A_2 have the same leader (which is a proper derivative of y), and the degree of A_2 in the leader exceeds that of A_1.

Two QDP for which no difference in the rank as created above will be said to be of the same rank.

Definition 3.3. Let A_1 be a PQDP, A_2 is said to be *reduced with respect to* A_1 if A_2 contains no proper derivative of the leader of A_1, and A_2 is either zero or of lower degree than A_1 in the leader of A_1. Consider a collection of PQDP

$$\Sigma = \{A_1, A_2, \cdots, A_r\}, \tag{3.1}$$

if a QDP B is reduced with respect to all the $A_i, (i = 1, \cdots, r)$, then B is said to be reduced with respect to Σ.

Definition 3.4. Let F be a PQDP with leader p, the QDP $\partial F/\partial p$ is said the *separant* of F. The coefficient of the highest power of p in F is said the *initial* of F.

Lemma 3.5. [25, pp. 6] *Let S_i and I_i be, respectively, the separant and initial of A_i in* (3.1) *and F be a QDP There exist nonnegative integers $s_i, t_i, i = 1, \cdots, r$, such that when a suitable linear combination of the A and their derivatives is subtracted from*

$$S_1^{s_1}\cdots S_r^{s_r} I_1^{t_1}\cdots I_r^{t_r} F,$$

the remainder is reduced with respect to (3.1).

Let $\Lambda \in K\{A_0(y)\}$ be a regular prime idea, and $X(y) = X_1\delta_1 y + X_2\delta_2 y \in \Lambda$. Let $A(y) \in \Lambda$ with the lowest rank and irreducible. If $\Lambda \supsetneqq \{X(y)\}$, here $\{X(y)\}$ is the differential ring generated by $X(y)$, then $A(y)$ involves no $\delta_1 y$ and its derivatives. Let $\delta_2^r y$ to be the leader of $A(y)$. Then $A(y)$ is a polynomial of the derivatives $\delta_2 y, \delta_2^2 y, \cdots, \delta_2^r y$, with coefficients $A_i(x_1, x_2, y) \in K[\mathcal{A}_0(y)]$. By Lemma 3.5, the regular prime idea Λ is the least regular prime idea containing $X(y)$ and $A(y)$. Thus, by [25], the *characteristic set* of Λ consist of $A(y)$ and $X(y)$. We will see latter that the number r is important to determine the structure of Λ, and named as the order of Λ, denoted by $\mathrm{ord}(\Lambda) = r$. If $\Lambda = \{X(y)\}$, then the characteristic set of Λ contains only one element $X(y)$ and the order is said to be ∞.

3.2. **Structure of the Differential Galois Group.**

Lemma 3.6. *If there exists $A(y) \in K[\mathcal{A}_0(y)]$ $(A(y) \not\equiv 0)$, and $\omega \in \Omega^1_{(0,0)}(X)$, such that $A(\omega(x_1, x_2)) = 0$ for all (x_1, x_2) in a neighborhood of $(0,0)$, then K contains a first integral of X.*

Proof. Hereinafter, we will write $A(\omega) = 0$ in short for $A(\omega(x_1, x_2)) = 0$ for all (x_1, x_2) in a neighborhood of $(0,0)$. Let

$$\Sigma_0 = \{A(y) \in K[\mathcal{A}_0(y)] |\ A(\omega) = 0, \quad A(y) \not\equiv 0\}.$$

Then $\Sigma_0 \neq \emptyset$. Write $A(y) \in \Sigma_0$ as

$$A(y) = \sum_{k=1}^{n} \alpha_k f_k(y),$$

with $\alpha_k \in K,\ f_k(y) \in \mathcal{A}_0(y)$. Among all such expressions, there exists one, of which the length n is the smallest. We denote by $n(A)$ the length and call it the length of $A(y)$. Let $A(y)$ to be the element in Σ_0 with the smallest length.

If $n(A) = 1$, then $A(y) = \alpha_1 f_1(y)$, and hence $f_1(\omega) = 0$, i.e., ω is a constant. Therefore, $n > 1$ since ω can not be a constant.

Upon $n > 1$, write

$$A(y) = \alpha_1 f_1(y) + \alpha_2 f_2(y) + \cdots + \alpha_n f_n(y), \quad (n = n(A))$$

then

$$A(\omega) = \alpha_1 f_1(\omega) + \alpha_2 f_2(\omega) + \cdots + \alpha_n f_n(\omega) = 0$$

and

$$X(A(\omega)) = X(\alpha_1) f_1(\omega) + X(\alpha_2) f_2(\omega) + \cdots + X(\alpha_n) f_n(\omega) = 0.$$

Therefore

$$\alpha_1 X(A(\omega)) - X(\alpha_1) A(\omega) = \sum_{i=2}^{n} (\alpha_1 X(\alpha_i) - X(\alpha_1)\alpha_i) f_i(\omega) = 0.$$

We have for any $i = 2, \cdots, n$,

$$\alpha_1 X(\alpha_i) - X(\alpha_1)\alpha_i = 0. \tag{3.2}$$

If on the contrary, $\alpha_1 X(\alpha_i) - X(\alpha_1)\alpha_i \neq 0$ for some i, then

$$B(y) = \sum_{i=2}^{n} (\alpha_1 X(\alpha_i) - X(\alpha_1)\alpha_i) f_i(y)$$

is contained in Σ_0 with smaller length then that of $A(y)$, contradict. By (3.2), we have $X(\alpha_2/\alpha_1) = 0$. It is evident that α_2/α_1 is not a constant, and hence α_2/α_1 is a first integral of X contained in K. □

From Lemma 3.6, if K contains no first integral of X, and a $A(y) \in K\{\mathcal{A}_0(y)\}$ such that $A(\omega) = 0$ for some $\omega \in \Omega^1_{(0,0)}(X)$, then $A(y)$ must involve some proper derivatives of y, i.e., $A(y)$ is a PQDP.

Theorem 3.7. *Let K be the differential field that contains no first integral of X. Let $M = K\langle\omega\rangle$ with $\omega \in \Omega^1_{(0,0)}(X)$. Then there exists a regular prime ideal Λ of PQDP, such that:*

(1). *For every $\sigma_f \in \mathrm{Gal}(M/K, X)_{(0,0)}$, let $\sigma_f \omega = f(\omega; \epsilon)$ $(f(z; \epsilon) \in \mathcal{G}[[\epsilon]])$, then $f(\omega(x_1, x_2); \epsilon)$ satisfies Λ.*

(2). *Given* $f(z;\epsilon)\in\mathcal{G}[[\epsilon]]$ *such that* $f(\omega(x_1,x_2);\epsilon)$ *satisfies* Λ*, there exists* $\sigma_f\in \mathrm{Gal}(M/K,X)_{(0,0)}$ *such that* $\sigma_f(\omega)=f(\omega;\epsilon)$*.*

Here $u(x_1,x_2;\epsilon)$ *satisfies* Λ *means that for any* $F(x_1,x_2,y,\delta_1 y,\delta_2 y,\cdots)\in\Lambda$*, while substitute* y *and the derivatives in* F *with* u *and the corresponding derivatives, the resulting expression is zero for all* x_1,x_2 *and* ϵ *small enough.*

Proof. Let y be a differential indeterminate over K, define the natural homomorphism from $K\{\mathcal{A}_0(y)\}$ to $K\{\mathcal{A}_0(\omega)\}$ that maps $h(y)$ to $h(\omega)$ ($\forall h\in\mathcal{A}_0$). Let Λ to be the kernel of the homomorphism, then Λ is a regular prime ideal of $K\{\mathcal{A}_0(y)\}$. We will prove that Λ fulfil the requirement of the Theorem.

(1). Let $\sigma_f\in\mathrm{Gal}(M/K,X)_{(0,0)}$ and $\sigma_f\omega=f(\omega;\epsilon)$. Then $f(z;\epsilon)\in\mathcal{G}[[\epsilon]]$. For any $F(y)\in\Lambda$, i.e., $F(\omega)=0$, there exist $h_i\in\mathcal{A}_0$ ($i=1,\cdots,m$), such that $F(y)\in K\{y,h_1(y),\cdots,h_m(y)\}$. Denote $F(y)$ as the differential polynomial of $h_1(y),\cdots,h_m(y)$

$$F(y)=F(y,h_1(y),\cdots,h_m(y))$$

and therefore

$$F(\omega,h_1(\omega),\cdots,h_m(\omega))=0$$

Since $\sigma_f\in\mathrm{Gal}(M/K,X)_{(0,0)}$, σ_f can be extended to a differential isomorphism of $K\{\omega,h_1(\omega),\cdots,h_m(\omega)\}$ that maps ω and $h_i(\omega)$ to $f(\omega;\epsilon)$ and $h_i(f(\omega;\epsilon))$ respectively. Thus, we have

$$F(f(\omega;\epsilon),h_1(f(\omega;\epsilon)),\cdots,h_m(f(\omega;\epsilon)))=0.$$

The requirement (1) has been proved.

(2). Now assume that $f(z;\epsilon)$ in $\mathcal{G}[[\epsilon]]$ such that $f(\omega;\epsilon)$ satisfies Λ, we will show that $\sigma_f\in\mathrm{Gal}(M/K,X)$. For any $h_1(z;\epsilon),\cdots,h_m(z;\epsilon)\in\mathcal{A}_0[[\epsilon]]$, consider the maps

$$\begin{array}{rccc}\pi: & K\langle y,h_1(y;\epsilon)),\cdots,h_m(y;\epsilon)\rangle & \mapsto & K\langle\omega,h_1(\omega;\epsilon),\cdots,h_m(\omega;\epsilon)\rangle\\ & y & \mapsto & \omega\\ & h(y;\epsilon) & \mapsto & h(\omega;\epsilon)\\ & \delta_j y & \mapsto & \delta_j\omega\end{array}$$

and

$$\begin{array}{rccc}\pi_\sigma: & K\langle y,h_1(y;\epsilon),\cdots,h_m(y;\epsilon)\rangle & \mapsto & K\langle f(\omega;\epsilon),h_1(f(\omega;\epsilon);\epsilon),\cdots,h_m(f(\omega;\epsilon);\epsilon)\rangle\\ & y & \mapsto & f(\omega;\epsilon)\\ & h(y;\epsilon) & \mapsto & h(f(\omega;\epsilon);\epsilon)\\ & \delta_j y & \mapsto & \delta_j\omega\end{array}$$

where $h\in\mathcal{A}_0[[\epsilon]]$ and $j=1,2$. Let the kernels of π and π_σ be Γ and Γ_σ, respectively. We will complete the proof by showing that $\Gamma=\Gamma_\sigma$.

We write $F(y,h_1(y;\epsilon),\cdots,h_m(y;\epsilon))\in\Gamma$ in form of the power series in ϵ

$$F(y,h_1(y;\epsilon),\cdots,h_m(y;\epsilon))=\sum_{i=0}^\infty F_i(y,h_{i,1}(y),\cdots,h_{i,m_i}(y))\epsilon^i\quad(h_{i,j}\in\mathcal{A}_0)$$

where F_i are differential polynomials. Then

$$0=F(\omega,h_1(\omega;\epsilon),\cdots,h_m(\omega;\epsilon))=\sum_{i=0}^\infty F_i(\omega,h_{i,1}(\omega),\cdots,h_{i,m_i}(\omega))\epsilon^i$$

i.e., $F_i(\omega,h_{i,1}(\omega),\cdots,h_{i,m_i}(\omega))=0$. Thus, the coefficients F_i are contained in Λ. Now, assume that $f(\omega;\epsilon)$ satisfies Λ, then

$$F_i(f(\omega;\epsilon),h_{i,1}(f(\omega;\epsilon)),\cdots,h_{i,m_i}(f(\omega;\epsilon)))=0,\quad(\forall i)$$

Thus,

$$F(f(\omega;\epsilon), h_1(f(\omega;\epsilon)), \cdots, h_m(f(\omega;\epsilon))) = 0$$

and hence $F(y, h_1(y;\epsilon), \cdots, h_m(y;\epsilon)) \in \Gamma_\sigma$. Therefore, $\Gamma \subseteq \Gamma_\sigma$.

We will prove that $\Gamma_\sigma \subseteq \Gamma$. If on the contrary, there exists $F(y;\epsilon) \in \Gamma_\sigma$ but $F(y;\epsilon) \notin \Gamma$, then

$$F(\omega;\epsilon) = F(\omega, h_1(\omega;\epsilon), \cdots, h_m(\omega;\epsilon)) \neq 0$$

and

$$F(f(\omega;\epsilon);\epsilon) = F(f(\omega;\epsilon), h(f(\omega;\epsilon);\epsilon), \cdots, h_m(f(\omega;\epsilon);\epsilon)) = 0$$

Write $F(y;\epsilon)$ as a power series in ϵ

$$F(y;\epsilon) = \sum_{i=0}^{\infty} F_i(y)\epsilon^i, \quad (F_i(y) \in K\{\mathcal{A}_0(y)\})$$

and let k the smallest index such that $F_i(y) \in \Lambda$ for any $0 \leq i \leq k-1$ and $F_k(y) \notin \Lambda$. By the assumption that $f(\omega;\epsilon)$ satisfies Λ, we have $F_i(f(\omega;\epsilon)) = 0$ for any $0 \leq i \leq k-1$. Thus, let $f_0(z) = f(z;0)$, we have

$$0 = F(f(\omega;\epsilon);\epsilon) = \epsilon^k F_k(f(\omega;\epsilon)) + \sum_{i \geq k+1} F_i(f(\omega;\epsilon))\epsilon^i = F_k(f_0(\omega))\epsilon^k + h.o.t.$$

and therewith $F_k(f_0(\omega)) = 0$.

Now, we obtain a $F_k(y) \notin \Lambda$, and $F_k(f_0(\omega)) = 0$. Let $A(y)$ in Λ with the lowest rank, and hence $A(y)$ and $X(y)$ make up the characteristic set of Λ. Let $S(y)$ and $I(y)$ the separant and initial of $A(y)$, respectively (If $\Lambda = \{X(y)\}$, we take $S(y) = I(y) = X_2(x_1, x_2)$). It is clear that $S(y), I(y) \notin \Lambda$. By Lemma 3.5, there exist nonnegative integrals s, t, and $R(y) \in K\{\mathcal{A}_0(y)\}$ that reduces with respect to Λ, such that

$$S(y)^s I(y)^t F_k(y) - R(y) \in \Lambda.$$

Since Λ is a prime ideal and $S(y), I(y), F_k(y) \notin \Lambda$, we have $S(y)^s I(y)^t F_k(y) \notin \Lambda$, and thus $R(y) \neq 0$. By above discussion, we have $f_0(\omega)$ satisfies both Λ and $F_k(y)$, and hence $R(f_0(\omega)) = 0$, which implies that $R(f_0(y)) \in \Lambda$. Whereas, simple computation shows that $R(f_0(y))$ has the same rank as $R(y)$, and therefore reduce with respect to Λ, which is contradict. Hence we have $\Gamma_\sigma \subseteq \Gamma$.

It follows from above discussion that $\Gamma = \Gamma_\sigma$. Consequently, $K\langle \omega, h_1(\omega;\epsilon), \cdots, h_m(\omega;\epsilon)\rangle$ is isomorphic to $K\langle f(\omega;\epsilon), h_1(f(\omega;\epsilon);\epsilon), \cdots, h_m(f(\omega;\epsilon);\epsilon)\rangle$ with the isomorphism $\sigma : \omega \mapsto f(\omega;\epsilon), h_i(\omega;\epsilon) \mapsto h_i(f(\omega;\epsilon);\epsilon)$. Therefore, σ is an admissible differential isomorphism. The theorem has been proved. □

Remark 3.8.
(1) Recall the order of Λ defined in Section 3.1. If $A(y)$ and $X(y)$ make up the characteristic set of Λ, and the highest derivative of $A(y)$ is $\delta_2^r y$ $(0 < r < +\infty)$, then $\mathrm{ord}(\Lambda) = r$. If $\Lambda = \{X(y)\}$, then $\mathrm{ord}(\Lambda) = \infty$. It's easy to derive from Theorem 3.7 that if $\mathrm{ord}(\Lambda) = \infty$, then $\mathrm{Gal}(M/K, X)_{(0,0)} = \mathcal{G}[[\epsilon]]$.
(2) If there exists $g \in \mathcal{A}_0^1$ such that $g(\omega) = u$ is contained in K, then $g(y) - u \in \Lambda$. We say in this case that the order of Λ is 0.
(3) We will also define the order of the differential Galois group to be the order of Λ.

Theorem 3.9. *Let $r = \mathrm{ord}(\Lambda)$ with Λ the prime ideal in Theorem 3.7, then either $0 \leq r \leq 3$ or $r = \infty$. Moreover, we have the following*

(1) *If $r=0$, then K contains a first integral of X, and*

$$\mathrm{Gal}(M/K,X)_{(0,0)} = \{e\} \tag{3.3}$$

(2) *If $r=1$, then there exists $\omega\in\Omega^1_{(0,0)}(X)$ such that $(\delta_2\omega)^n\in K$ for some $n\in\mathbb{N}$. Let $M=K\langle\omega\rangle$, then*

$$\mathrm{Gal}(M/K,X)_{(0,0)} = \{f(z;\epsilon)\in\mathcal{G}[[\epsilon]]\ \big|\ f(z;\epsilon)=\mu z+c(\epsilon),\quad c(0)=0,\mu^n=1\} \tag{3.4}$$

(3) *If $r=2$, then there exist $\omega\in\Omega^1_{(0,0)}(X)$ such that $\delta_2^2\omega/\delta_2\omega\in K$. Let $M=K\langle\omega\rangle$, then*

$$\mathrm{Gal}(M/K,X)_{(0,0)} = \{f(z;\epsilon)\in\mathcal{G}[[\epsilon]]\ \big|\ f(z;\epsilon)=a(\epsilon)z+c(\epsilon),\quad c(0)=0\} \tag{3.5}$$

(4) *If $r=3$, then there exists $\omega\in\Omega^1_{(0,0)}(X)$ such that*

$$\frac{\delta_2\omega\cdot\delta_2^3\omega-3\delta_2^2\omega}{(\delta_2\omega)^2}\in K$$

Let $M=K\langle\omega\rangle$, than

$$\mathrm{Gal}(M/K,X)_{(0,0)} = \{f(z;\epsilon)\in\mathcal{G}[[\epsilon]]\ \big|\ f(z;\epsilon)=\frac{a(\epsilon)z}{1+b(\epsilon)z}+c(\epsilon),\quad c(0)=0\} \tag{3.6}$$

(5) *If $r=\infty$, then $\mathrm{Gal}(M/K,X)_{(0,0)}=\mathcal{G}[[\epsilon]]$.*

In particular, if G is solvable, then $r\le 2$.

For proof of Theorem 3.9, refer to the appendix. From Theorem 3.9 and Lemma 2.9, the differential Galois group is independent to the choice of the point $(0,0)$. Moreover, from Lemma 2.8, the structure of the group is also independent to the choice of the first integral ω. Hence, we can omit the point $(0,0)$ and the particular extension M, and simply say $\mathrm{Gal}(M/K,X)$ the differential Galois group of X over K. This group is determined by the equation (1.6) of the operator X uniquely and will tel the insight of the integrability of the differential equation.

4. Liouvillian integrability of the nonlinear differential equation

We are now at the point of proving the main theorem of this paper.

4.1. **Preliminary results of Galois theory.**

Definition 4.1. Let K be a (differential) field, M be an extension field of K, G be a set of isomorphisms of M, with K elementwise fixed. M is *normal* over K with respect to G if there are no elements of $M\backslash K$ that are fixed by all members of G.

Obviously, we have

Lemma 4.2. *Let G be the differential Galois group of $M=K\langle\omega\rangle$ over K with respect to X at $(0,0)$, and H be a subgroup of G. Let*

$$H'=\{a\in M|\ \sigma a=a,\forall\sigma\in H\},$$

then M is normal over H' with respect to H.

Lemma 4.3. *If K contains no first integral of X, then for any $\omega\in\Omega^1_{(0,0)}(X)$, $M=K\langle\omega\rangle$ is normal over K with respect to $\mathrm{Gal}(M/K,X)_{(0,0)}$.*

Proof. We only need to prove that for any $\alpha \in M\backslash K$, there exists $\sigma \in \mathrm{Gal}(M/K, X)_{(0,0)}$, such that $\sigma\alpha \neq \alpha$.

Let $\alpha = p(\omega)/q(\omega)$, with $p(y), q(y) \in K\{\mathcal{A}_0(y)\}$. Non lost the generality, we assume further that $p(y), q(y)$ are reduce with respect to the prime ideal Λ in Theorem 3.7. Therefore, $p(\omega) \neq 0, q(\omega) \neq 0$. Write $p(y)$ and $q(y)$ explicitly as

$$p(y) = p(\mathbf{x}, y, \delta_2 y, \cdots, \delta_2^{r-1} y), \quad q(y) = q(\mathbf{x}, y, \delta_2 y, \cdots, \delta_2^{r-1} y)$$

where $\mathbf{x} = (x_1, x_2), r = \mathrm{ord}(\Lambda)$, and let

$$A(\mathbf{x}, y) = p(\mathbf{x}, y, \delta_2 y, \cdots, \delta_2^{r-1} y) - \alpha(\mathbf{x}) q(\mathbf{x}, y, \delta_2 y, \cdots, \delta_2^{r-1} y)$$

Taking account that $q(\omega) \neq 0$, there exists $\mathbf{x}^0 \in \mathbb{C}^2$ such that $\omega(\mathbf{x})$ is analytic at $\mathbf{x}^0$, and $q(\omega(\mathbf{x}^0)) \neq 0$. Let $\mathbf{c} = (\omega(\mathbf{x}^0), \cdots, \delta_2^{r-1}\omega(\mathbf{x}^0))$ and $\mathbf{x}^*$ in a neighborhood U of $\mathbf{x^0}$ such that

$$q(\mathbf{x}^*, \mathbf{c}) \neq 0 \quad \text{and} \quad A(\mathbf{x}^*, \mathbf{c}) \neq 0$$

We claim that such $\mathbf{x}^*$ always exists. If on the contrary, for any $\mathbf{x} \in U$, such that

$$q(\mathbf{x}, \mathbf{c}) \neq 0$$

always have

$$A(\mathbf{x}, \mathbf{c}) = p(\mathbf{x}, \mathbf{c}) - \alpha(\mathbf{x}) q(\mathbf{x}, \mathbf{c}) = 0$$

then

$$\alpha(\mathbf{x}) = \frac{p(\mathbf{x}, \mathbf{c})}{q(\mathbf{x}, \mathbf{c})} \in K$$

which is contradict to the fact that $\alpha \in M\backslash K$.

Let $\mathbf{c}^* = (\omega(\mathbf{x}^*, \cdots, \delta_2^{r-1}\omega(\mathbf{x}^*))$, and ϵ to be an infinitesimal parameter, then

$$q(\mathbf{x}^*, \mathbf{c}^* + \epsilon(\mathbf{c} - \mathbf{c}^*)) \neq 0$$

and

$$A(\mathbf{x}^*, \mathbf{c}^* + \epsilon(\mathbf{c} - \mathbf{c}^*)) \neq 0$$

And hence

$$\frac{p(\mathbf{x}^*, \mathbf{c}^* + \epsilon(\mathbf{c} - \mathbf{c}^*))}{q(\mathbf{x}^*, \mathbf{c}^* + \epsilon(\mathbf{c} - \mathbf{c}^*))} \neq \alpha(x_1^*, x_2^*).$$

By Theorem 3.9, it's not difficult to verify that when $r \neq 0$, there exists $\sigma \in \mathrm{Gal}(M/K, X)_{(0,0)}$ such that

$$(\delta_i \sigma\omega)(\mathbf{x}^*) = c_i^* + (c_i - c_i^*)\epsilon \quad (i = 0, 1, \cdots, r-1)$$

and therefore

$$(\sigma\alpha)(\mathbf{x}^*) = \frac{p(\mathbf{x}^*, \mathbf{c}^* + \epsilon(\mathbf{c} - \mathbf{c}^*))}{q(\mathbf{x}^*, \mathbf{c}^* + \epsilon(\mathbf{c} - \mathbf{c}^*))} \neq \alpha(\mathbf{x}^*)$$

i.e., $\sigma\alpha \neq \alpha$. The Lemma has been proved. □

In following Lemmas, we let L, N, M be extension fields of K, with $M = K\langle\omega\rangle$, $K \subset L \subset N \subset M$, and $G = \mathrm{Gal}(M/K, X)_{(0,0)}$. Assume that $N = L\langle u\rangle$ with u satisfying $\delta_i u \in L$ or $\delta_i u/u \in L$, $(i = 1, 2)$. Then L' and N' are subgroups of G, and N'' is a subfield of M.

Lemma 4.4. *Any $\sigma \in L'$ maps N into N''.*

Proof. We have $\sigma a = a$ for any σ in L' and a in L. At first, assume that $N = \langle u\rangle$. If $\delta_i u = a_i \in L, (i = 1,2)$, then $\delta_i(\sigma u) = a_i,\ (i = 1,2)$. Thus $\sigma u = u + c(\epsilon)$ with $c(\epsilon) \in \mathbb{C}[\epsilon]$, and therewith $\sigma u \in N''$, which implies that $\sigma N \subseteq N''$. The proof for the case $\delta_i u/u \in K$ is similar by the fact that $\sigma u = c(\epsilon)u$ with $c(\epsilon) \in \mathbb{C}[[\epsilon]]$ for any σ in L'. □

Lemma 4.5. *N' is a normal subgroup of L', and L'/N' is Abelian.*

Proof. Let $\sigma \in L', \tau \in N'$, then $\sigma a \in N''$ for any $a \in N$, and therewith $\tau(\sigma a) = \sigma a$. Thus $(\sigma^{-1}\cdot\tau\cdot\sigma)a = \sigma^{-1}(\sigma a) = a$, and hence $\sigma^{-1}\cdot\tau\cdot\sigma \in N'$. This implies that N' is a normal subgroup of L'.

Assume that $N = L\langle u\rangle$ with $\delta_i u \in L\ (i = 1,2)$. By the proof of Lemma 4.4, for any $\sigma \in L'$, $\sigma u = u + c(\epsilon)$ for some $c(\epsilon) \in \mathbb{C}[[\epsilon]]$. Thus, the subgroup $H = \{\sigma|_N \mid \sigma \in L'\}$ is isomorphic to a subgroup of the addition group $\mathbb{C}[[\epsilon]]$ and hence Abelian. Consider the homomorphism from L' to H that maps σ to $\sigma|_N$. The kernel of the map is N', and the image is H. Thus, L'/N' is isomorphic to H and therefore Abelian.

The case that $N = L\langle u\rangle$ with $\delta_i u/u \in L\ (i = 1,2)$ can be proved similarly. □

From Lemma 2.12 and 4.5, we have:

Lemma 4.6. *Let L, N, M be extension fields of K, with $M = K\langle\omega\rangle, N = L\langle u\rangle$, and $K \subset L \subset N \subset M$. Assume further that K, L, N contain no first integral of X. We have*

(1). *if u is algebraic over L, then $|L'/N'| \leq [N : L]$;*
(2). *if $\delta_i u \in L$ or $\delta_i u/u \in L$ $(i = 1,2)$, then N' is a normal subgroup of L', and L'/N' is Abelian,*

where the subgroups L' and N' are defined as previous.

Lemma 4.7. *Assume that $M = K\langle\omega\rangle$ is normal over L with respect to G. If for every $\sigma \in G$, there exist $c(\epsilon) \in \mathbb{C}[[\epsilon]]$, such that*

$$\sigma\omega = \omega + c(\epsilon),$$

then M is a Liouvillian extension of K.

Proof. For any $\sigma \in G$, we have

$$\sigma(\delta_i\omega) = \delta_i\omega, \quad (i = 1,2).$$

Since M is normal over L with respect to G, we have $\delta_i\omega \in K, (i = 1,2)$, and M is a Liouvillian extension of K. □

Lemma 4.8. *Assume that $M = K\langle\omega\rangle$ is normal over K with respect to G. If for every $\sigma \in G$, there exist $a(\epsilon), c(\epsilon) \in \mathbb{C}[[\epsilon]]$, such that*

$$\sigma\omega = a(\epsilon)\omega + c(\epsilon),$$

then M is a Liouvillian extension of K.

Proof. For any $\sigma \in G$, we have

$$\sigma(\delta_i^2\omega/\delta_i\omega) = \delta_i^2\omega/\delta_i\omega, \quad (i = 1,2).$$

Since M is normal over K with respect to G, there exist $a_i \in K$, such that

$$\delta_i^2\omega = a_i\delta_i\omega, \quad (i = 1,2).$$

Taking account that $X_1\delta_1\omega + X_2\delta_2\omega = 0$, there exists $\mu \in M$ such that $\delta_1\omega = \mu X_2, \delta_2\omega = -\mu X_1$, and hence

$$\frac{\delta_1\mu}{\mu} = a_1 - \frac{\delta_1 X_2}{X_2} \in K, \quad \frac{\delta_2\mu}{\mu} = a_2 - \frac{\delta_2 X_1}{X_1} \in K.$$

Thus M is a Liouvillian extension of K, and

$$K \subseteq K\langle\mu\rangle \subseteq K\langle\omega\rangle = M.$$

□

4.2. **Proof of the Main Result and Applications.**

Theorem 4.9. *Consider the differential equation*

$$\frac{dx_2}{dx_1} = \frac{X_2(x_1,x_2)}{X_1(x_1,x_2)} \tag{4.1}$$

where $X_1(x_1,x_2), X_2(x_1,x_2)$ are polynomials, and $X_1(0,0) \neq 0$. Let K be the differential field of rational functions, with constant field $\mathbb{C}$. Then (4.1) *is Liouvillian integrable if, and only if, the differential Galois group of* (4.1) *over K at* $(0,0)$ *is solvable.*

Proof. 1). If K contains a first integral of X, then G contains exclusively the identity mapping, and is solvable.

Now, we assume that K contains no first integral of X, and X is Liouvillian integrable. Then there exists $\omega \in \Omega^1_{(0,0)}(X)$ such that $M = K\langle\omega\rangle$ is a Liouvillian extension of K. Upon the definition 1.2, suppose that

$$K = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_m = M,$$

with $K_{i+1} = K_i\langle u_i\rangle$, where either u_i is algebraic over K_i or $\delta_j u_i \in K_i$ or $\delta_j u_i/u_i \in K_i$ $(j = 1,2)$. Let $G_0 = \mathrm{Gal}(M/K,X)(=K'), G_i = K_i'$, $(i = 1,2,\cdots,m)$, then

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_m = \{e\}.$$

By Lemma 4.6, either $|G_i/G_{i+1}| \leq [K_{i+1} : K_i] < \infty$ or G_{i+1} is a normal subgroup of G_i, and G_i/G_{i+1} is Abelian. Thus, G is solvable by definition 1.4.

2). If the differential Galois group of (4.1) over K at $(0,0)$ is solvable, by Theorem 3.9, either K contains a first integral of X, or the Galois group has order $r = 1$ or $r = 2$.

By Theorem 3.9, if $r = 1$, then there exists $\omega \in \Omega^1_{(0,0)}(X)$ and $n \in \mathbb{N}$ such that, while let $M = K\langle\omega\rangle$ and $G = \mathrm{Gal}(M/K,X)_{(0,0)}$,

$$G = \{f(z;\epsilon) \in \mathcal{G}[[\epsilon]] \mid f(z;\epsilon) = \mu z + c(\epsilon), \quad c(\epsilon) \in \mathbb{C}[[\epsilon]], c(0) = 0,\ \mu^n = 1\}$$

Let

$$G_0 = \{\sigma \in G |\ \sigma\omega = \omega + c(\epsilon), \quad c(\epsilon) \in \mathbb{C}[[\epsilon]], c(0) = 0\}$$

then G_0 is a subgroup of G, and $|G/G_0| = n$. By Lemma 4.3, $K = G'$. Hence, according to Lemma 2.13,

$$[G_0' : K] = [G_0' : G'] \leq [G/G_0] = n,$$

it follows that G_0' is an algebraic extension of K. By Lemma 4.2, M is normal over G_0' with respect to G_0. Hence Lemma 4.7 is applicable and yields that M is a Liouvillian extension of G_0', and consequently a Liouvillian extension of K.

If $r = 2$, then there exists a first integral $\omega \in \Omega^1_{(0,0)}$ such that, while let $M = K\langle\omega\rangle$, and $G = \mathrm{Gal}(M/K, X)_{(0,0)}$,

$$G = \{f(z;\epsilon) \in \mathcal{G}[[\epsilon]] \mid f(z;\epsilon) = a(\epsilon)z + c(\epsilon),\ c(0) = 0\}$$

Again, by Lemma 4.3, $K = G'$. Hence, Lemma 4.8 is applicable, and M is a Liouvillian extension of K. The Theorem has been proved. □

From the proof of Lemma 5.1–5.3 in next section, the explicit method to determine the differential Galois group is given as follows.

Theorem 4.10. *Consider the differential equation* (1.6)*, let*

$$B_i = -X_1\delta_2^{i+1}(\frac{X_2}{X_1}), \quad (i = 0, 1, 2)$$

and r to be the order of the corresponding differential Galois group,

(1) *If $r = 0$, then K contains a first integral of X;*

(2) *If $r = 1$, then there exists $a \in K, a \neq 0$, and $n \in \mathbb{N}$, such that*

$$X(a) = nB_0 a \tag{4.2}$$

(3) *If $r = 2$, then there exists $a \in K$, such that*

$$X(a) = B_0 a + B_1 \tag{4.3}$$

(4) *If $r = 3$, then there exists $a \in K$, such that*

$$X(a) = 2B_0 a + B_2 \tag{4.4}$$

(5) *If (4.4) has no rational solution, than $r = \infty$.*

Following result is the immediate consequence of Theorem 4.10.

Theorem 4.11. *The differential Galois group of the Riccati equation*

$$\frac{dx_2}{dx_1} = p(x_1)x_2^2 + q(x_1)x_2 + r(x_1)$$

has order $r = 3$.

Theorem 4.11 presents another point of view that the Riccati equation is in general unsolvable by the quadrature method. On the other hand, from the proof of Lemma 5.3, if the order of the differential Galois group of a differential equation is 3, the first integral of the equation can be obtained by solving (5.7), which have the form of Riccati equation. Hence, Riccati equation is important for integrating the differential equation.

Lemma 4.12. *Consider the van der Pol equation*

$$\left\{\begin{array}{rcl}\dot{x}_1 &=& x_2 - \mu(\frac{x_1^3}{3} - x_1),\\ \dot{x}_2 &=& -x_1\end{array}\right. \qquad (\mu \neq 0). \tag{4.5}$$

The order of the differential Galois group of (4.5) *is infinity.*

Proof. Let

$$X_1(x_1,x_2) = x_2 - \mu(\frac{x_1^3}{3} - x_1), \quad X_2(x_1,x_2) = -x_1,$$

the equation (4.4) for the van der Pol equation (4.5) reads

$$X_1^3 X(a) + 2x_1X_1^2 a + 6x_1 = 0 \tag{4.6}$$

We will only need to prove the (4.6) has no rational function solution. If (4.6) has a rational function solution $a = a_1/a_2$, where a_1, a_2 are relatively prime polynomials, then a_1, a_2 satisfy

$$X_1^3(a_2X(a_1) - a_1X(a_2)) - 2x_1X_1^2a_1a_2 + 6x_1a_2^2 = 0.$$

Hence, there exist a polynomial $c(x_1,x_2)$, such that

$$\begin{aligned} X_1^3X(a_2) &= ca_2 && (4.7)\\ X_1^3X(a_2) &= (c - 2x_1X_1^2)a_1 - 6x_1a_2 && (4.8)\end{aligned}$$

Let $a_2 = X_1^k b_2$, where $k \geq 0$, b_2 is a nonzero polynomial and $(b_2, X_1) = 1$. Substitute a_2 into (4.7) yields

$$X_1^3X(b_2) + kX_1^2X(X_1)b_2 = cb_2.$$

Thus, $b_2|X_1^3X(b_2)$ and therewith $b_2|X(b_2)$, i.e., b_2 is either a constant or an invariant algebraic solution of (4.5). However, it had known that the van der Pol equation has no invariant algebraic solution([5, 23]), and therefore b_2 must be a constant. Let $b_2 = 1$ and consequently

$$a_2 = X_1^k, \quad c = kX_1^2X(X_1). \tag{4.9}$$

Substitute (4.9) into (4.8) yields

$$X_1^3X(a_1) = (kX_1^2X(X_1) - 2x_1X_1^2)a_1 - 6xX_1^k, \quad (k \geq 0) \tag{4.10}$$

Note that

$$(kX(X_1) - 2x_1) = -k\mu(x_1^2 - 1)X_1 - (k+2)x_1$$

and (4.10) become

$$X_1^3X(a_1) = X_1^2(-k\mu(x_1^2-1)X_1 - (k+2)x_1)a_1 - 6x_1X_1^k.$$

If $k \neq 2$, then $X_1|(k+2)x_1$ for $k > 2$ or $X_1|6x_1$ for $k < 2$, which are impossible. Hence, we conclude that $k = 2$.

Let $k = 2$, (4.10) become

$$\begin{aligned} &\left(x_2 - \mu(\tfrac{x_1^3}{3} - x_1)\right)\left((x_2 - \mu(\tfrac{x_1^3}{3} - x_1))\frac{\partial a_1}{\partial x_1} - x_1\frac{\partial a_1}{\partial x_2}\right)\\ = &\left(-2\mu(x_1^2-1)(x_2 - \mu(\tfrac{x_1^3}{3} - x_1)) - 4x_1\right)a_1 - 6x_1.\end{aligned} \tag{4.11}$$

Let

$$a_1(x_1,x_2) = \sum_{i=0}^m h_i(x_2)x_1^i,$$

where $h_i(x_2)$ are polynomials and $h_m(x_2) \neq 0$. Substitute $a_1(x_1,x_2)$ into (4.11), and comparing the coefficient of x^{m+5}, we have

$$\frac{1}{9}\mu^2 m h_m(x_2) = \frac{2}{3}\mu^2 h_m(x_2),$$

and hence $m = 6$. Comparing the coefficients of x_1^i $(0 \leq i \leq 10)$, we obtain the equations satisfied by $h_i(x_2), i = 0, \cdots, 6$:

$$\begin{aligned}
0 &= x_2(-2\mu h_0(x_2) + x_2 h_1(x_2)) \\
0 &= 6 - 2(-2+\mu^2)h_0(x_2) + 2x_2^2 h_2(x_2) - x_2 h_0'(x_2) \\
0 &= 2\mu x_2 h_0(x_2) - (-4+\mu^2)h_1(x_2) + 2\mu x_2 h_2(x_2) + 3x_2^2 h_3(x_2) - \mu h_0'(x_2) - x_2 h_1'(x_2) \\
0 &= \frac{8\mu^2}{3}h_0(x_2) + \frac{4\mu x_2}{3}h_1(x_2) + 4h_2(x_2) + 4\mu x_2 h_3(x_2) + 4x_2^2 h_4(x_2) - \mu h_1'(x_2) - x_2 h_2'(x_2) \\
0 &= 2\mu^2 h_1(x_2) + \frac{2\mu x_2}{3}h_2(x_2) + 4h_3(x_2) + \mu^2 h_3(x_2) + 6\mu x_2 h_4(x_2) + 5x_2^2 h_5(x_2) \\
&\quad + \frac{\mu}{3}h_0'(x_2) - \mu h_2'(x_2) - x_2 h_3'(x_2) \\
0 &= \frac{1}{3}(-2\mu^2 h_0(x_2) + 4\mu^2 h_2(x_2) + 12h_4(x_2) + 6\mu^2 h_4(x_2) + 24\mu x_2 h_5(x_2) + 18x_2^2 h_6(x_2) \\
&\quad + \mu h_1'(x_2) - 3\mu h_3'(x_2) - 3x_2 h_4'(x_2)) \\
0 &= \frac{1}{9}(-5\mu^2 h_1(x_2) + 6\mu^2 h_3(x_2) - 6\mu x_2 h_4(x_2) + 36h_5(x_2) + 27\mu^2 h_5(x_2) + 90\mu x_2 h_6(x_2) \\
&\quad + 3\mu h_2'(x_2) - 9\mu h_4'(x_2) - 9x_2 h_5'(x_2)) \\
0 &= -\frac{4\mu^2}{9}h_2(x_2) - \frac{4\mu x_2}{3}h_5(x_2) + 4h_6(x_2) + 4\mu^2 h_6(x_2) + \frac{\mu}{3}h_3'(x_2) - \mu h_5'(x_2) - x_2 h_6'(x_2) \\
0 &= -\frac{\mu}{3}(\mu h_3(x_2) + 2\mu h_5(x_2) + 6x_2 h_6(x_2) - h_4'(x_2) + 3h_6'(x_2)) \\
0 &= -\frac{\mu}{9}(2\mu h_4(x_2) + 12\mu h_6(x_2) - 3h_5'(x_2)) \\
0 &= -\frac{\mu}{9}(\mu h_5(x_2) - 3h_6'(x_2))
\end{aligned}$$

The equations reduce to

$$x_2(3x_2 h_5'(x_2) - 2\mu h_4'(x_2)) = 2\mu^3,$$

which is impossible since $h_4(x_2)$ and $h_5(x_2)$ are polynomials. The contradiction conclude that (4.6) has no rational function solution, and hence the order of the differential Galois group of the van der Pol equation is infinity. □

5. Proof of Theorem 3.9

Before giving the proof of Theorem 3.9, we introduce some notations as following. Let $\delta_1 = \frac{\partial}{\partial x_1}$, $\delta_2 = \frac{\partial}{\partial x_2}$, y be an indeterminate over K, and denote $\delta_2^i y$ by y_i $(y_0 = y)$. Let

$$X = X_1\delta_1 + X_2\delta_2, \qquad \delta_2 X = (\delta_2 X_1)\delta_1 + (\delta_2 X_2)\delta_2$$

$$\hat{X} = X_1\delta_1 + X_2\delta_2 + \sum_{i\geq 0} X(y_i)\frac{\partial}{\partial y_i}$$

$$B_0 = -X_1\delta_2(\frac{X_2}{X_1}), \quad B_i = X_1\delta_2(\frac{B_{i-1}}{X_1}) = -X_1\delta_2^{i+1}(\frac{X_2}{X_1}), \quad i = 1, 2, \cdots$$

Let $F(y) \in K\{\mathcal{A}_0(y)\}$, we write

$$F(y) \sim R(y)$$

if $R(y) \in K\{\mathcal{A}_0(y)\}$ such that $F(y) - R(y)$ is contained in $\{X(y)\}$, the differential ideal generated by $X(y)$.

Let $\mathbf{n} = (n_1, n_2, \cdots, n_r) \in \mathbb{Z}^{*r}$, define d_i^j and b_i^j for $1 < i < j$ by

$$\begin{array}{rcl} d_i^j(\mathbf{n}) & = & (n_1, \cdots, n_{j-i}+1, \cdots, n_j - 1, \cdots, n_r) \\ b_i^j(\mathbf{n}) & = & (n_1, \cdots, n_{j-i}-1, \cdots, n_j + 1, \cdots, n_r) \end{array}$$

Then $b_i^j(d_i^j(\mathbf{n})) = \mathbf{n}$. Let $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^{*r}$, the *degree* of $\mathbf{n}$ is higher than that of $\mathbf{m}$, denoted by $\mathbf{n} > \mathbf{m}$, if there exists $1 \leq k \leq r$ such that $n_k > m_k$ and

$$n_i = m_i, \quad i = k+1, \cdots, r.$$

We say $\mathbf{n} \succ \mathbf{m}$ if there exist $1 < i < j$ such that

$$d_i^j(\mathbf{n}) = \mathbf{m}$$

It is evident that when $1 > i > j$,

$$b_i^j(\mathbf{n}) > \mathbf{n} > d_i^j(\mathbf{n}) \tag{5.1}$$

$$d_i^j(\mathbf{n}) \succ \mathbf{n} \succ b_i^j(\mathbf{n}) \tag{5.2}$$

Let Λ be the regular prime ideal of QDP that corresponds to the differential Galois group in Theorem 3.7, and assume that $\mathrm{ord}(\Lambda) = r$. Let A in Λ with the lowest rank and therewith irreducible. Denoted A as

$$A(x_1, x_2, y, y_1, \cdots, y_r) = \sum_{\mathbf{m}} A_{\mathbf{m}}(x_1, x_2, y) y_1^{m_1} \cdots y_r^{m_r}$$

and let

$$\mathcal{I}_A = \{\mathbf{m} \in \mathbb{Z}^{*r} | \;\; A_{\mathbf{m}} \neq 0\}.$$

By $\mathbf{n}$ we will always denote the element in $\mathcal{I}_A$ with the highest degree. We can assume further that $A_{\mathbf{n}} = 1$ and therefore the coefficients $A_{\mathbf{m}}$ are rational functions. For any $\mathbf{m} \in \mathcal{I}_A$, let

$$\mathcal{P}(\mathbf{m}) = \{\mathbf{p} \in \mathcal{I}_A \mid d_i^j(\mathbf{p}) = \mathbf{m} \text{ for some } 1 < i < j\} \tag{5.3}$$

and $\#(\mathbf{m}) = |\mathcal{P}(\mathbf{m})|$. A subset $\mathcal{J}_A \subseteq \mathcal{I}_A$ is *closed* if for every $\mathbf{m} \in \mathcal{J}_A$, $\mathbf{p} \succ \mathbf{m}$ implies $\mathbf{p} \in \mathcal{J}_A$. It's obvious that $\mathcal{I}_A$ and $\{\mathbf{n}\}$ are closed.

The complete proof of Theorem 3.9 is followed from several Lemmas. The proof will be done by showing that all possible structures of $\mathcal{I}_A$ are, besides the cases $r = 0$ and $r = \infty$,

(a). $r = 1$, and $\mathcal{I}_A = \{n, 0\}$;

(b). $r = 2$, and $\mathcal{I}_A = \{(0,1), (1,0)\}$, with

$$(0,1) \succ (1,0)$$

(c). $r = 3$, and

$$\mathcal{I}_A = \{(1,0,1), (0,2,0), (2,0,0)\}$$

with

$$\begin{array}{cl} & (1,0,1) \\ & \quad\curlyvee \\ \curlyvee & (1,1,0) \prec (0,2,0) \\ & \quad\curlyvee \\ & (2,0,0) \end{array}$$

Here $(1,1,0)$ is an auxiliary index with $A_{(1,1,0)} = 0$.

The flow chart of the proof is given at Figure 1.

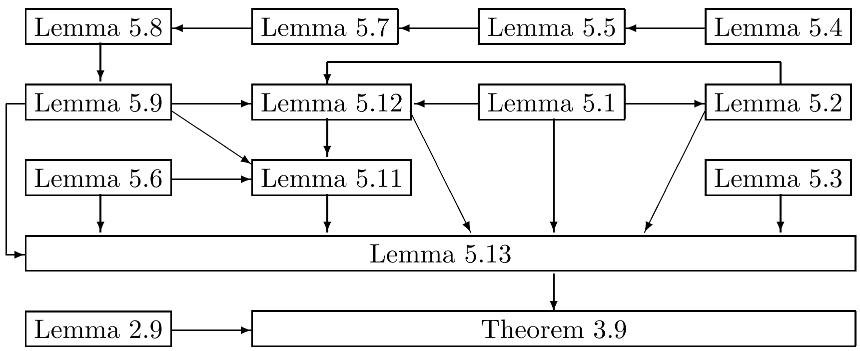

FIGURE 1. Flow chart of the proof of Theorem 3.9

Lemma 5.1. *If $u \neq 0$ satisfies*

$$Xu = B_0 u \tag{5.4}$$

then there exists a first integral ω of X such that

$$\delta_2 \omega = u.$$

Proof. Upon (5.4), we have

$$\begin{aligned}
X_1\delta_1 u + X_2\delta_2 u &= B_0 u = -X_1\delta_2(\frac{X_2}{X_1})u \\
\delta_1 u + \frac{X_2}{X_1}\delta_2 u &= -\delta_2(\frac{X_2}{X_1}u) \\
\delta_1 u &= -\delta_2(\frac{X_2}{X_1}u - \frac{X_2}{X_1}\delta_2 u) \\
&= \delta_2(-\frac{X_2}{X_1}u)
\end{aligned}$$

Let $v = -\frac{X_2}{X_1}u$, then the 1-form $vdx_1 + udx_2$ is closed, and

$$\omega(x_1, x_2) = \int_{(0,0)}^{(x_1,x_2)} vdx_1 + udx_2$$

is a first integral of X, with $\delta_2\omega = u$. □

Lemma 5.2. *If there exists u satisfying*

$$Xu = B_0 u + B_1 \tag{5.5}$$

then X has a first integral ω such that

$$\frac{\delta_2^2\omega}{\delta_2\omega} = u$$

Proof. From (5.5), we have

$$\begin{aligned} X_1\delta_1 u + X_2\delta_2 u &= -X_1\delta_2(\frac{X_2}{X_1})u + X_1\delta_2(\frac{B_0}{X_1}) \\ \delta_1 u &= -\frac{X_2}{X_1}\delta_2 u - \delta_2(\frac{X_2}{X_1})u + \delta_2(\frac{B_0}{X_1}) \\ &= \delta_2(-\frac{X_2}{X_1}u + \frac{B_0}{X_1}) \end{aligned}$$

Thus, let

$$v = -\frac{X_2}{X_1}u + \frac{B_0}{X_1}$$

then $vdx_1 + udx_2$ is a closed 1-form. Let

$$\eta(x_1, x_2) = \exp\left[\int_{(0,0)}^{(x_1,x_2)} vdx_1 + udx_2\right],$$

then

$$X(\eta) = \eta(X_1 v + X_2 u) = \eta(X_1(-\frac{X_2}{X_1}u + \frac{B_0}{X_1}) + X_2 u) = B_0\eta$$

By Lemma 5.1, there exists a first integral ω of X such that

$$\delta_2\omega = \eta,$$

and therewith

$$\frac{\delta_2^2\omega}{\delta_2\omega} = u.$$

The Lemma is concluded. □

Lemma 5.3. *If there exists u satisfying*

$$Xu = 2B_0 u + B_2 \tag{5.6}$$

then X has a first integral ω of X such that

$$\frac{2\delta_2\omega\cdot\delta_2^3\omega - 3(\delta_2^2\omega)^2}{(\delta_2\omega)^2} = u$$

Proof. From (5.6), we have

$$\begin{aligned} X_1\delta_1 u + X_2\delta_2 u &= -2X_1\delta_2(\frac{X_2}{X_1})u - X_1\delta_2^3(\frac{X_2}{X_1}) \\ \delta_1 u + \frac{X_2}{X_1}\delta_2 u &= -2\delta_2(\frac{X_2}{X_1})u - \delta_2^3(\frac{X_2}{X_1}) \\ \delta_1 u &= -2\delta_2(\frac{X_2}{X_1})u - \frac{X_2}{X_1}\delta_2 u - \delta_2^3(\frac{X_2}{X_1}) \\ &= -\delta_2(\frac{X_2}{X_1}u) - \frac{X_2}{X_1}\delta_2 u - \delta_2^3(\frac{X_2}{X_1}) \end{aligned}$$

Consider the partial differential equations:

$$\left\{\begin{array}{rcl} \delta_2 w &=& u + \frac{1}{2}w^2 \\ \delta_1 w &=& -\delta_2^2(\frac{X_2}{X_1}) - \frac{X_2}{X_1}u - \delta_2(\frac{X_2}{X_1})w - \frac{1}{2}(\frac{X_2}{X_1})w^2 \end{array}\right. \tag{5.7}$$

We have

$$\begin{aligned}
\delta_1\delta_2 w &= \delta_1(u+\frac{1}{2}w^2)\\
&= \delta_1 u + w\delta_1 w\\
&= -\delta_2(\frac{X_2}{X_1}u) - \frac{X_2}{X_1}\delta_2 u - \delta_2^3(\frac{X_2}{X_1}) + w\left(-\delta_2^2(\frac{X_2}{X_1}) - \frac{X_2}{X_1}u - \delta_2(\frac{X_2}{X_1})w - \frac{1}{2}(\frac{X_2}{X_1})w^2\right)\\
&= -\delta_2(\frac{X_2}{X_1}u) - \frac{X_2}{X_1}\delta_2 u - \delta_2^3(\frac{X_2}{X_1}) - \left(\delta_2^2(\frac{X_2}{X_1}) + \frac{X_2}{X_1}u\right)w - \delta_2(\frac{X_2}{X_1})w^2 - \frac{1}{2}(\frac{X_2}{X_1})w^3\\
\delta_2\delta_1 w &= \delta_2\left(-\delta_2^2(\frac{X_2}{X_1}) - \frac{X_2}{X_1}u - \delta_2(\frac{X_2}{X_1})w - \frac{1}{2}(\frac{X_2}{X_1})w^2\right)\\
&= -\delta_2^3(\frac{X_2}{X_1}) - \delta_2(\frac{X_2}{X_1}u) - \delta_2^2(\frac{X_2}{X_1})w - \delta_2(\frac{X_2}{X_1})\delta_2 w - \frac{1}{2}\delta_2(\frac{X_2}{X_1})w^2 - \frac{X_2}{X_1}w\delta_2 w\\
&= -\delta_2^3(\frac{X_2}{X_1}) - \delta_2(\frac{X_2}{X_1}u) - \delta_2^2(\frac{X_2}{X_1})w - \delta_2(\frac{X_2}{X_1})(u+\frac{1}{2}w^2) - \frac{1}{2}\delta_2(\frac{X_2}{X_1})w^2 - \frac{X_2}{X_1}w(u+\frac{1}{2}w^2)\\
&= -\delta_2^3(\frac{X_2}{X_1}) - \delta_2(\frac{X_2}{X_1}u) - \delta_2(\frac{X_2}{X_1})u - \left(\delta_2^2(\frac{X_2}{X_1}) + \frac{X_2}{X_1}u\right)w - \delta_2(\frac{X_2}{X_1})w^2 - \frac{1}{2}(\frac{X_2}{X_1})w^3
\end{aligned}$$

Therefore, $\delta_1\delta_2 w = \delta_2\delta_1 w$, and the equations (5.7) have a solution w that is analytic at $(0,0)$. Let

$$v = -\delta_2(\frac{X_2}{X_1}) - (\frac{X_2}{X_1})w$$

then

$$\begin{aligned}
\delta_2 v &= \delta_2(-\delta_2(\frac{X_2}{X_1}) - (\frac{X_2}{X_1})w)\\
&= -\delta_2^2(\frac{X_2}{X_1}) - \delta_2(\frac{X_2}{X_1})w - (\frac{X_2}{X_1})\delta_2 w\\
&= -\delta_2^2(\frac{X_2}{X_1}) - \delta_2(\frac{X_2}{X_1})w - (\frac{X_2}{X_1})(u+\frac{1}{2}w^2)\\
&= -\delta_2^2(\frac{X_2}{X_1}) - (\frac{X_2}{X_1})u - \delta_2(\frac{X_2}{X_1})w - \frac{1}{2}(\frac{X_2}{X_1})w^2\\
&= \delta_1 w
\end{aligned}$$

Therefore, $vdx_1 + wdx_2$ is a closed 1-form. Let

$$\omega_2 = \exp\left[\int_{(0,0)}^{(x_1,x_2)} vdx_1 + wdx_2\right], \quad \omega_1 = -\frac{X_2}{X_1}\omega_2,$$

than

$$\begin{aligned}
\delta_1\omega_2 &= \omega_2 v\\
\delta_2\omega_1 &= -\delta_2(\frac{X_2}{X_1})\omega_2 - (\frac{X_2}{X_1})\delta_2\omega_2\\
&= \omega_2(-\delta_2(\frac{X_2}{X_1}) - \frac{X_2}{X_1}\frac{\delta_2\omega_2}{\omega_2})\\
&= \omega_2(-\delta_2(\frac{X_2}{X_1}) - \frac{X_2}{X_1}w)\\
&= \omega_2 v = \delta_1\omega_2
\end{aligned}$$

Hence, $\omega = \int_{(0,0)}^{(x_1,x_2)} \omega_1 dx_1 + \omega_2 dx_2$ is well defined, and a first integral of X at $(0,0)$. It is easy to verify that

$$\frac{2\delta_2\omega \cdot \delta_2^3\omega - 3(\delta_2^2\omega)^2}{(\delta_2\omega)^2} = u$$

The proof is completed. □

Lemma 5.4. (1) $\delta_2 X = (\frac{\delta_2 X_1}{X_1})X - B_0\delta_2$;
(2) $X(y_j) = \delta_2(X(y_{j-1})) - (\frac{\delta_2 X_1}{X_1})X(y_{j-1}) + B_0 y_j$;

Proof. (1).

$$\begin{aligned}
\delta_2 X &= (\delta_2 X_1)\delta_1 + (\delta_2 X_2)\delta_2 \\
&= \frac{\delta_2 X_1}{X_1}(X_1\delta_1 + X_2\delta_2) - \frac{X_2}{X_1}(\delta_2 X_1)\delta_2 + (\delta_2 X_2)\delta_2 \\
&= \frac{\delta_2 X_1}{X_1}X + X_1\frac{X_1\delta_2 X_2 - X_2\delta_2 X_1}{X_1^2}\delta_2 \\
&= \frac{\delta_2 X_1}{X_1}X - B_0\delta_2
\end{aligned}$$

(2).

$$\begin{aligned}
X(y_j) &= \delta_2(X(y_{j-1})) - (\delta_2 X)y_{j-1} \\
&= \delta_2(X(y_{j-1})) - (\frac{\delta_2 X_1}{X_1}X - B_0\delta_2)(y_{j-1}) \\
&= \delta_2(X(y_{j-1})) - \frac{\delta_2 X_1}{X_1}X(y_{j-1}) + B_0\delta_2(y_{j-1}) \\
&= \delta_2(X(y_{j-1})) - (\frac{\delta_2 X_1}{X_1})X(y_{j-1}) + B_0 y_j
\end{aligned}$$

□

Lemma 5.5.

$$X(y_j) \sim \sum_{i=0}^{j-1} a_{ji}B_i y_{j-i} \tag{5.8}$$

where a_{ji} are constants, with $a_{j0} = j$.

Proof. By Lemma 5.4, when $j = 1$,

$$X(y_1) = \delta_2(X(y_0)) - (\frac{\delta_2 X_1}{X_1})X(y_0) + B_0 y_1 \sim B_0 y_1$$

which is the desired (5.8) with $a_{10}=1$. Assume that (5.8) is valid for $j=k$, and $a_{k0}=k$, then by Lemma 5.4,

$$\begin{aligned}
X(y_{k+1}) &= \delta_2(X(y_k)) - (\frac{\delta_2 X_1}{X_1})X(y_k) + B_0 y_{k+1}\\
&\sim \delta_2(\sum_{i=0}^{k-1} a_{ki}B_i y_{k-i}) - (\frac{\delta_2 X_1}{X_1})(\sum_{i=0}^{k-1} a_{ki}B_i y_{k-i}) + B_0 y_{k+1}\\
&= \sum_{i=0}^{k-1} a_{ki}((\delta_2 B_i)y_{k-i} + B_i\delta_2 y_{k-i}) - \sum_{i=0}^{k-1} a_{ki}\frac{\delta_2 X_1}{X_1}B_i y_{k-i} + B_0 y_{k+1}\\
&= \sum_{i=0}^{k-1} a_{ki}\left((\delta_2 B_i - \frac{\delta_2 X_1}{X_1}B_i)y_{k-i} + B_i y_{k-i+1}\right) + B_0 y_{k+1}\\
&= (a_{k0}+1)B_0 y_{k+1} + \sum_{i=0}^{k-2}(a_{ki}X_1\delta_2(\frac{B_i}{X_1}) + a_{k(i+1)}B_{i+1})y_{k-i} + a_{k(k-1)}X_1\delta_2(\frac{B_{k-1}}{X_1})y_1\\
&= (a_{k0}+1)B_0 y_{k+1} + \sum_{i=0}^{k-2}(a_{ki} + a_{k(i+1)})B_{i+1}y_{k-i} + a_{k(k-1)}B_k y_1\\
&= \sum_{i=0}^{k} a_{(k+1)i}B_i y_{k+1-i}
\end{aligned}$$

where

$$a_{(k+1)0} = a_{k0}+1 = k+1,\ \ a_{(k+1)i} = a_{k(i-1)}+a_{ki},\ \ (1\le i\le k-1),\ \ \ a_{(k+1)k} = a_{k(k-1)}$$

Therefore, the Lemma has been proved. □

Lemma 5.6. *Let $\mathbf{m}\in\mathbb{Z}^{*r}$, and define*

$$C(\mathbf{m}) = m_1 + 2m_2 + \cdots + rm_r \tag{5.9}$$

If $\mathbf{m}\succ\mathbf{p}$, then $C(\mathbf{m}) > C(\mathbf{p})$. In particularly, if $d_i^j(\mathbf{m}) = \mathbf{p}$, than $C(\mathbf{m}) - C(\mathbf{p}) = i$.

Proof. Let $d_i^j(\mathbf{m}) = \mathbf{p}$, then

$$C(\mathbf{m}) - C(\mathbf{p}) = (j-i)m_{j-i} + jm_j - ((j-i)(m_{j-i}+1) + j(m_j-1)) = i.$$

□

Lemma 5.7. *Let $P = A_{\mathbf{m}}y_1^{m_1}\cdots y_r^{m_r}$, then*

$$\hat{X}(P) \sim (X(A_{\mathbf{m}}) + C(\mathbf{m})B_0A_{\mathbf{m}})\mathbf{y}^{\mathbf{m}} + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r} m_j a_{ji}B_iA_{\mathbf{m}}\mathbf{y}^{d_i^j(\mathbf{m})}$$

where $\mathbf{y}^{\mathbf{m}} = y^{m_1}\cdots y_r^{m_r}$.

Proof. By Lemma 5.5, we have

$$\begin{aligned}
\hat{X}(P) &= X(A_{\mathbf{m}})\mathbf{y}^{\mathbf{m}} + \frac{\partial A_{\mathbf{m}}}{\partial y}\mathbf{y}^{\mathbf{m}}X(y) + A_{\mathbf{m}}\sum_{j=1}^{r} m_j y_1^{m_1}\cdots y_j^{m_j-1}\cdots y_r^{m_r}X(y_j)\\
&\sim X(A_{\mathbf{m}})\mathbf{y}^{\mathbf{m}} + A_{\mathbf{m}}\sum_{j=1}^{r} m_j y_1^{m_1}\cdots y_j^{m_j-1}\cdots y_r^{m_r}(\sum_{i=0}^{j-1} a_{ji}B_i y_{j-i})\\
&= X(A_{\mathbf{m}})\mathbf{y}^{\mathbf{m}} + A_{\mathbf{m}}B_0(\sum_{j=1}^{r} m_j a_{j0})\mathbf{y}^{\mathbf{m}}\\
&\quad + A_{\mathbf{m}}\sum_{j=1}^{r}\sum_{i=1}^{j-1} m_j a_{ji}B_i y_1^{m_1}\cdots y_{j-i}^{m_{j-i}+1}\cdots y_j^{m_j-1}\cdots y_r^{m_r}\\
&= (X(A_{\mathbf{m}}) + C(\mathbf{m})B_0A_{\mathbf{m}})\mathbf{y}^{\mathbf{m}} + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r} m_j a_{ji}B_iA_{\mathbf{m}}\mathbf{y}^{d_i^j(\mathbf{m})}
\end{aligned}$$

and the Lemma is concluded. □

Lemma 5.8. *Let $A \in \Lambda$ with the lowest rank and $r = \mathrm{ord}(\Lambda) \geq 1$. Let $\mathbf{n} \in \mathcal{I}_A$ with the highest degree and $A_{\mathbf{n}} = 1$, then for any $\mathbf{m} < \mathbf{n}$,*

$$X(A_{\mathbf{m}}) = (C(\mathbf{n}) - C(\mathbf{m}))B_0A_{\mathbf{m}} - \sum_{i=1}^{r-1}\sum_{j=i+1}^{r}(m_j+1)a_{ji}B_iA_{b_i^j(\mathbf{m})} \tag{5.10}$$

where $A_{\mathbf{m}} = 0$ if $\mathbf{m} \notin \mathcal{I}_A$.

Proof. Let

$$A = \sum_{\mathbf{m}\in\mathcal{I}_A} A_{\mathbf{m}}\mathbf{y}^{\mathbf{m}}.$$

By Lemma 5.7, we have

$$\begin{aligned}
X(A) &= \sum_{\mathbf{m}\in\mathcal{I}_A} X(A_{\mathbf{m}}\mathbf{y}^{\mathbf{m}})\\
&\sim \sum_{\mathbf{m}\in\mathcal{I}_A}\left((X(A_{\mathbf{m}}) + C(\mathbf{m})B_0A_{\mathbf{m}})\mathbf{y}^{\mathbf{m}} + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r} m_j a_{ji}B_iA_{\mathbf{m}}\mathbf{y}^{d_i^j(\mathbf{m})}\right)\\
&= \sum_{\mathbf{m}}\left((X(A_{\mathbf{m}}) + C(\mathbf{m})B_0A_{\mathbf{m}}) + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r}(m_j+1)a_{ji}B_iA_{b_i^j(\mathbf{m})}\right)\mathbf{y}^{\mathbf{m}}
\end{aligned}$$

Note that for any $j > i$, $b_i^j(\mathbf{n}) > \mathbf{n}$ and thus $b_i^j(\mathbf{n}) \notin \mathcal{I}_A$, i.e., $A_{b_i^j(\mathbf{n})} = 0$. Taking account that $A_{\mathbf{n}} = 1$, we have

$$\begin{aligned}
X(A) - C(\mathbf{n})B_0A &= \sum_{\mathbf{m}<\mathbf{n}}\Big(X(A_{\mathbf{m}} + (C(\mathbf{m}) - C(\mathbf{n}))B_0A_{\mathbf{m}})\\
&\quad + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r-1}(m_j+1)a_{ji}B_iA_{b_i^j(\mathbf{m})}\Big)\mathbf{y}^{\mathbf{m}}
\end{aligned}$$

Thus, we obtain $X(A) - C(\mathbf{n})B_0A$ that is contained in Λ and has lower rank than A. But A is the element in Λ with the lowest rank, as we have assumed, therefore $X(A) - C(\mathbf{n})B_0A \equiv 0$. Thus (5.10) is followed for all $\mathbf{m} < \mathbf{n}$. □

From Lemma 5.8, we have

Lemma 5.9. *If* $\mathcal{P}(\mathbf{m}) = \{\mathbf{p}_1, \cdots, \mathbf{p}_k\}$*, and* $d_{i_l}^{j_l}(\mathbf{p}_l) = \mathbf{m}$*,* $(l = 1, 2, \cdots, k)$*, then the coefficients* $A_{\mathbf{p}_l}, A_{\mathbf{m}}$ *satisfy*

$$X(A_{\mathbf{m}}) = (C(\mathbf{n}) - C(\mathbf{m}))B_0 A_{\mathbf{m}} - \sum_{l=1}^{k}(m_{j_l} + 1)a_{j_l i_l} B_{i_l} A_{\mathbf{p}_l} \tag{5.11}$$

Remark 5.10. Since the differential operator X is independent to y, we can always assign y in (5.11) as a constant and assume that the coefficients $A_{\mathbf{p}_l}, A_{\mathbf{m}} \in K$. Hence, we will always assume that the coefficients $A_{\mathbf{m}} \in K$.

Lemma 5.11. *Let* $A \in \Lambda$ *with the lowest rank and* $r = \mathrm{ord}(\Lambda) \geq 1$*. Let* $\mathbf{n} \in \mathcal{I}_A$ *with highest degree. Then for any* $\mathbf{m} \in \mathcal{I}_A$*,* $\#(\mathbf{m}) = 0$ *if, and only if,* $C(\mathbf{m}) = C(\mathbf{n})$*. Furthermore, if* $\#(\mathbf{m}) = 0$*, then* $A_{\mathbf{m}}$ *is a constant.*

Proof. At first we will prove that if $\#(\mathbf{m}) = 0$, then $C(\mathbf{m}) = C(\mathbf{n})$. If $\#(\mathbf{m}) = 0$, by Lemma 5.9, we have

$$X(A_{\mathbf{m}}) = (C(\mathbf{n}) - C(\mathbf{m}))B_0 A_{\mathbf{m}}$$

If $C(\mathbf{m}) \neq C(\mathbf{n})$, let $n = C(\mathbf{n}) - C(\mathbf{m})$, then

$$X(A_{\mathbf{m}}^{1/n}) = B_0 A_{\mathbf{m}}^{1/n}$$

From Lemma 5.1, there exists a first integral ω of X such that

$$\delta_2 \omega = A_{\mathbf{m}}^{1/n}$$

And hence

$$y_1^n - A_{\mathbf{m}} \in \Lambda$$

i.e., $r = 1$, contradict. Thus, we have prove that $C(\mathbf{m}) = C(\mathbf{n})$.

Now, we will prove that if $C(\mathbf{p}) = C(\mathbf{n})$, then $\#(\mathbf{m}) = 0$. If on the contrary, $\#(\mathbf{m}) > 0$, then there exists $\mathbf{p} \in \mathcal{P}(\mathbf{m})$, and by (5.1), $C(\mathbf{p}) > C(\mathbf{m}) = C(\mathbf{n})$. On the other hand, there exist $\mathbf{p}_1, \cdots, \mathbf{p}_k$, such that

$$\mathbf{p}_1 \succ \cdots \succ \mathbf{p}_k = \mathbf{p}$$

and $\#(\mathbf{p}_1) = 0$. Therefore $C(\mathbf{p}) = C(\mathbf{n})$. It is easy to have $C(\mathbf{p}) = C(\mathbf{p}_k) < \cdots < C(\mathbf{p}_1) = C(\mathbf{n})$, which is contradict, and the statement is concluded.

If $\#(\mathbf{m}) = 0$, then $C(\mathbf{n}) = C(\mathbf{m})$, and by (5.11), $X(A_{\mathbf{m}}) = 0$. But $\mathrm{ord}(\Lambda) > 0$, it follows that $A_{\mathbf{m}}$ is a constant. ◻

Lemma 5.12. *Let* $A \in \Lambda$ *with the lowest rank and* $r = \mathrm{ord}(\Lambda) \geq 3$*. Assume that* $\mathcal{J}_A \subseteq \mathcal{I}_A$ *is closed, and* $\mathbf{m} = (m_1, m_2, \cdots, m_r) \in \mathcal{J}_A$ *with the highest degree, then* $m_2 = 0$*.*

Proof. If on the contrary, $m_2 > 0$, let $\mathbf{p} = d_1^2(\mathbf{m}) = (m_1 + 1, m_2 - 1, m_3, \cdots, m_r)$. It is easy to verify that $\mathcal{P}(\mathbf{p}) = \{\mathbf{m}\}$ by (1) $d_1^2(\mathbf{m}) = \mathcal{P}$ and (2) if any other $\mathbf{m}'$ such that $d_i^j(\mathbf{m}') = p$, then $\mathbf{m}' > \mathbf{m}$. Hence, we have

$$\left\{ \begin{array}{rcl} X(A_{\mathbf{m}}) & = & (C(\mathbf{n}) - C(\mathbf{m}))B_0 A_{\mathbf{m}} \\ X(A_{\mathbf{p}}) & = & B_0 A_{\mathbf{p}} - m_2 a_{21} B_1 A_{\mathbf{m}} \end{array} \right. \tag{5.12}$$

We conclude from (5.12) and Lemmas 5.1 and 5.2 that either $r = 0$ (if $C_{\mathbf{n}} = C_{\mathbf{m}}$ and $A_{\mathbf{m}}$ is not a constant) or $r = 1$ (if $C_{\mathbf{n}} \neq C_{\mathbf{m}}$) or $r = 2$ (if $C(\mathbf{n}) - C(\mathbf{m}) = 0$ and $A_{\mathbf{m}}$ is constant), contradict with the assumption $r \geq 3$. ◻

Lemma 5.13. *Assume that K contains no first integral of X, and let r to be the order of the differential Galois group of X,*

(1) *If $r = 1$, then there exists a first integral ω of X, and $n \in \mathbb{N}$, such that*

$$(\delta_2\omega)^n \in K$$

(2) *If $r = 2$, then there exists a first integral ω of X, such that*

$$\delta_2^2\omega/\delta_2\omega \in K$$

(3) *If $r = 3$, then there exists a first integral ω of X, such that*

$$\frac{2\delta_2\omega \cdot \delta_2^3\omega - 3(\delta_2^2\omega)^2}{(\delta_2\omega)^2} \in K$$

(4) *If $r \neq \infty$, then $r \leq 3$.*

Proof. Let Λ to be the regular prime ideal of QDP that corresponds to the differential Galois of X, and $A \in \Lambda$ with the lowest rank, $\mathbf{n} \in \mathcal{I}_A$ with the highest degree.

(1). If $r = 1$, assume that

$$A = y_1^n + A_1 y_1^{n-1} + \cdots + A_n$$

From Lemma 5.9,

$$X(A_n) = nB_0A_n$$

i.e.,

$$X(A_n^{1/n}) = B_0A_n^{1/n}.$$

By Lemma 5.1 and $A_n \neq 0$ (because A is irreducible), there exists a first integral ω of X such that

$$\delta_2\omega = A_n^{1/n}$$

i.e.,

$$(\delta_2\omega)^n = A_n \in K.$$

We have applied the Remark 5.10 to assume that $A_n \in K$.

(2). If $r = 2$, let $\mathbf{n} = (n_1, n_2)$ and $\mathbf{m} = d_1^2(\mathbf{n}) = (n_1+1, n_2-1)$, then $\mathcal{P}(\mathbf{m}) = \{\mathbf{n}\}$. Thus, by Lemma 5.9 and Lemma 5.6, we have

$$X(A_{\mathbf{m}}) = B_0A_{\mathbf{m}} - n_2a_{21}B_1$$

i.e.,

$$X(-\frac{A_{\mathbf{m}}}{n_2a_{21}}) = B_0(-\frac{A_{\mathbf{m}}}{-n_2a_{21}}) + B_1$$

Apply Lemma 5.2, there exists a first integral ω, such that

$$\frac{\delta_2^2\omega}{\delta_2\omega} = -\frac{A_{\mathbf{m}}}{n_2a_{21}} \in K.$$

(3). If $r = 3$, and let $\mathbf{n} = (n_1, n_2, n_3)$. By Lemma 5.12, we have $\mathbf{n} = (n_1, 0, n_3)$. Let

$$\mathbf{p} = d_1^3(\mathbf{n}) = (n_1, 1, n_3 - 1),$$
$$\mathbf{q} = d_2^3(\mathbf{n}) = (n_1 + 1, 0, n_3 - 1),$$
$$\mathbf{m} = b_1^2(\mathbf{p}) = (n_1 - 1, 2, n_3 - 1).$$

It is easy to have $C(\mathbf{m}) = C(\mathbf{n})$. Therefore, by Lemma 5.11, $A_{\mathbf{m}}$ is a constant. Furthermore, we have $\mathcal{P}(\mathbf{p}) = \{\mathbf{n}, \mathbf{m}\}$ and $\mathcal{P}(\mathbf{q}) = \{\mathbf{n}, \mathbf{p}\}$. By Lemma 5.9,

$$X(A_{\mathbf{p}}) = B_0 A_{\mathbf{p}} - (n_3 a_{31} A_{\mathbf{n}} + 2a_{21} A_{\mathbf{m}}) B_1 \tag{5.13}$$

$$X(A_{\mathbf{q}}) = 2B_0 A_{\mathbf{q}} - n_3 a_{32} A_{\mathbf{n}} B_2 - a_{21} A_{\mathbf{p}} \tag{5.14}$$

Taking account that $A_{\mathbf{n}}$ and $A_{\mathbf{m}}$ are constants, and $r = 3$, we conclude that $n_3 a_{31} A_{\mathbf{n}} + 2a_{21} A_{\mathbf{m}} = 0$ and $A_{\mathbf{p}} = 0$. Or else, we should have $r = 2$ by the similar discussion in (2). Let $A_{\mathbf{p}} = 0$ and $A_{\mathbf{n}} = 1$ in (5.14), we have

$$X(-\frac{A_{\mathbf{q}}}{n_3 a_{32}}) = 2B_0(-\frac{A_{\mathbf{q}}}{n_3 a_{32}}) + B_2$$

By Lemma 5.3, there exists a first integral ω of X such that

$$\frac{2\delta_2\omega \cdot \delta_2^3\omega - 3(\delta_2^2\omega)^2}{(\delta_2\omega)^2} = -\frac{A_{\mathbf{q}}}{n_3 a_{32}} \in K$$

and (3) has been proved.

(4). If on the contrary, assume that $3 < r < \infty$. Upon Lemma 5.12, we can write $\mathbf{n} = (n_1, 0, n_3, \cdots, n_r)$. Let

$$\begin{aligned} \mathbf{m} &= d_1^r(\mathbf{n}) = (n_1, 0, n_3, \cdots, n_{r-1} + 1, n_r - 1) \\ \mathbf{p} &= b_1^2(\mathbf{m}) = (n_1 - 1, 1, n_3, \cdots, n_{r-1} + 1, n_r - 1) \\ \mathbf{q} &= d_1^{r-1}(\mathbf{p}) = (n_1 - 1, 1, n_3, \cdots, n_{r-2} + 1, n_{r-1}, n_r - 1) \end{aligned}$$

then $C(\mathbf{p}) = C(\mathbf{n})$ and therewith $\#(\mathbf{p}) = 0$ by Lemma 5.11. Let $\mathcal{J}_A$ to be the minimal closed subsystem of $\mathcal{I}_A$ containing $\mathbf{q}$, then $\mathbf{p} \in \mathcal{J}_A$ with the highest degree. However, $p_2 = 1 \neq 0$, which is contradict to Lemma 5.12. Thus, we concluded that either $r = \infty$ or $r \leq 3$. $\square$

Finally, Theorem 3.9 is concluded from Lemma 5.13 and Lemma 2.9.

Acknowledgements

The author is highly grateful to Professor Keying Guan who lead him to the field of differential Galois theory. The author wishes to express his hearty thanks to Professor Claude Mitschi, Université Louis Pasteur, Professor Michael F. Singer, North Carolina State University, and Professor Colin Christopher, University of Plymouth, for their great interested in reading the early version of the manuscript, for their much valuable advice and helpful discussions.

References

1. A. D. Bruno, *Local Methods in Nonlinear Differential Equations*, Springer-Verlag, Berlin, 1989.
2. C. Camacho, A. B. Scárdua, *Complex foliations with algebraic limit sets*, Astérisque, **261**(2000), 57–88.
3. C. Camacho, C., A. B. Scárdua, *Holomorphic foliations with Liouvillian first integrals*, Ergodic Theory Dynam. Systems, **21**(2001), 717-756.
4. D. Cerveau, P. Sad, *Liouvillian integration and Bernoulli foliations*, *Trans. Amer. Math. Soc.*, **350**(8)(1998), 3065-3081.
5. R. Cheng, K. Guan, S. Zhang, *On the judement of the existence of algebraic curve solution to the second order polyomial autonomous system*, J. Beijing Univ. Aeronaut. Astronaut., **21**(1)(1995), 109-115 (Chinese).
6. C. Christopher, *Liouvillian first integrals of second order polynomial differential equations*, Electron. J. Differential Equations, **1999**(49)(1999), 1-7.

7. K. Guan, J. Lei, *Integrability of second order autonmous system*, Ann. Differential Equations, **18**(2)(2002), 117-135.
8. W. Kaplan, *Ordinary Differential Equations*, Reading Mass.: Addision-Wesley. 1958.
9. I. Kaplansky, *An Introduction to Differential Algebra*. 2nd ed. Hermann. Paris, 1976.
10. A. G. Khovanskii, *Topological obstructions to the representability of functions by quadratures*, J. Dynam. Control Systems, **1**(1)(1995), 91-123.
11. E. R. Kolchin, *Existence theorems connected with the Picard-Vessiot theory of homogeneous linear ordinary differential equations*, Bull. Amer. Math. Soc., **54**(1948), 927-932.
12. E. R. Kolchin, *Algebraic matric groups and the Picard-Vessiot theory of homogeneous lnear ordinary differential equations*, Ann. of Math., **49**(1)(1948), 1-42.
13. E. R. Kolchin, *Differential algebra and algebraic groups*, Academic Press, NewYork, 1973.
14. J. Kovacic, *An algorithm for solving second order linear homogeneous differential equations*, J. Symbolic Comput., **2**(1986), 3-43.
15. J. Lei, K. Guan, *An algorithm to judge the integrability of second order linear differential euqation with rational coefficients*, Neural Parallel Sci. Comput, **8**(3&4)(2000), 243 - 252.
16. F. Loray, *Towards the Galois groupoid of non linear first order O.D.E.*, in B. L. J. Braaksma, et. al., (eds), *Differential Equations and the Stokes Phenomenon*, World Scientific, Singapore, 2002.
17. J. Liouville, *Remarques nouvelles sur l'équation de Riccati*, Journal de mathématiques, pures et appliquées, VI(1841), 1-13.
18. B. Malgrange, *Le groupoïd de Galois d'un feuilletage*, Enseignement Math., **38**(2001). 465-501.
19. B. Malgrange, *On nonlinear Differential Galois Theory*, Chinese Ann. Math. Ser. B, **23**(2)(2002), 219-226.
20. A. R. Magid, *Lectures on Differential Galois Theory*, University Lecture Series, Vol. 7, Amer. Math. Soc., Providence, RI, 1994.
21. C. Mitschi, *Differential Galois of confluent generalized hypergeometric equations: an approach using stokes multipliers*, Pacific J. Math., **176**(2)(1998), 365-405.
22. C. Mitschi, M. F. Singer, *The Inverse Problem in Differential Galois Theory*, in : B.l.J. Braaksma, et. al., (eds),*The Stokes Phenomenon and Hilbert's 16th Problem*, World Scientific, Singapore, 1996, 185-196.
23. K. Odani, *The limit cycle of the van der Pol equation is not algebraic*, J. Differential Equations, **115**(1995), 146-152.
24. P. J. Olver, *Applications of Lie groups to differential equations*, 2nd ed. Springer-Verlag, Berlin, 1999.
25. J. F. Ritt, *Differential Algebra*, Amer. Math. Soc., Providence, RI, 1950. http://www.ams.org/online_bks/coll33/.
26. L. Schlesinger, *Handbuch der Theorie der Linearen Differentialgleichungen*, Teubner, Leipzig, 1887.
27. M. F. Singer, *Liouvillian first integrals of differential equations*, Trans. Amer. Math. Soc., **333**(2)(1992), 673-688.
28. M. F. Singer, F. Ulmer, *Galois groups of second and third order linear differential euations*, J. Symbolic Comput., **16**(1993), 1-30.
29. M. F. Singer, *Direct and Inverse Problems in Differential Galois Theory*, in: B.B.Cassidy, (eds), *Selected Works of Ellis Kolchin with Commentary*, Amer. Math. Soc., Providence, RI, 1999, 527-554.
30. M. F. Singer, F. Ulmer, *A Kovacic-style algorithm for liouvillian solutions of third order differential equations*, Prepublication IRMAR 01-12, 2001, URL: http://citeseer.csail.mit.edu/446364.html.
31. M. Van Der Put, M. F. Singer, *Galois theory of linear differential equations*, Springer-Verlag, Berlin, 2003.
32. M. Van Der Put, M. F. Singer, *Galois theory of linear differential equations*. Grundlehren der Mathematischen Wissenschaften. 328. Berlin, 2003.
33. G. N. Waston, *A Treatise on the Theory of Bessel Functions*, 2nd ed, Gambridge, 1944.
34. V. I. Arnold, Yu. S. II'yashenko, *Ordinary Differential Equation*, In *Dynamical System I*, D. V. Anosov, V. I. Arnold (Eds), Springer-Verlag, Berlin, 1988.
35. H. Żołądek, *The extended monodromy group and Liouvillian first integral*, J. Dynam. Control Systems, **4**(1998), 1-28.

Zhou Pei-Yuan Center for Applied Mathematics, Tsinghua University, Beijing, 100084, P.R.China

E-mail address: jzlei@mail.tsinghua.edu.cn, jin_zhi_lei@yahoo.com

URL: http://zcam.tsinghua.edu.cn/~jzlei

Nonlinear Differential Galois Theory

Jinzhi Lei

Zhou Pei-Yuan Center for Applied Mathematics, Tsinghua University, Beijing, 100084, P.R.China

Abstract

Differential Galois theory had played important roles in the integrability theory of linear differential equation. In this paper we will extend the theory to nonlinear differential equation and study the integrability of second order polynomial system. We will propose a definition of the differential Galois group of a differential equation, will study the structure of the group, and will prove the equivalence between the existence of the Liouvillian first integral for the differential equation and the solvability of the corresponding differential Galois group.

Keywords: differential Galois theory, nonlinear differential equation, Liouvillian integrability
2000 MSC: 34A05, 08C99

1. Introduction

In this paper, we will establish the nonlinear differential Galois theory to study the Liouvillian integrability of following nonlinear differential equation

$$\frac{dx_2}{dx_1} = \frac{X_2(x_1, x_2)}{X_1(x_1, x_2)}, \tag{1.1}$$

where X_1 and X_2 are polynomials. We will propose a definition of the differential Galois group with respect to (1.1), will study the structure of the group, and willl prove the equivalence between the exstience of the Liouillian first integral and the solvability of the group.

Email address: jzlei@mail.tsinghua.edu.cn (Jinzhi Lei)

Preprint submitted to Annales de l'Institut Henri Poincaré(C) Analyse Non Linéaire *April 26, 2011*

Before we state the main theorem, we give here a brief outline for the preliminary knowledge of differential algebra. For detail, refer to [13] and [30].

1.1. Preliminary knowledge of differential algebra

Let A be a ring, by *derivation* of A we mean an additive mapping $a \mapsto \delta a$ of A into itself satisfying

$$\delta(ab) = (\delta a)b + a(\delta b).$$

We shall say δa the *derivative* of a. The *differential ring* A is a commutative ring with unit together with a derivation δ. If there are m derivations of A, $\delta_i,\ i = 1, 2 \cdots, m$, satisfying

$$\delta_i \delta_j a = \delta_j \delta_i a, \quad \forall i, j \in \{1, 2, \cdots, m\}, \forall a \in A,$$

we call A together with all the δ_is a *partial differential ring.* When A is a field, the *(partial) differential field* can be defined similarly. In this paper, we will say differential ring (field) for brevity for both differential ring (field) and partial differential ring (field).

Let A be any ring, Y be a set of finite or infinite number of elements. We can generate a ring $A[Y]$ of polynomials of the elements in Y with coefficients in A. In particular, when A is a differential ring with derivations $\delta_1, \cdots, \delta_m$, and $Y = \{y_{i_1,i_2,\cdots,i_m}\}$ $(i_j = 0, 1, \cdots)$ to be the ordinary indeterminates over A, we can extend the derivations of A to $A[Y]$ uniquely by assigning $y_{i_1 \cdots i_{j+1} \cdots i_m}$ as $\delta_j y_{i_1 \ldots i_j \cdots i_m}$, and rewriting the notations as following

$$y_{0 \cdots 0} = y, \qquad y_{i_1 \cdots i_m} = \delta_1^{i_1} \cdots \delta_m^{i_m} y.$$

Following the above procedure, we had added a *differential indeterminate* y to a differential ring A. We will denote the resulting differential ring as $A\{y\}$. The elements of $A\{y\}$ are *differential polynomials* in y. Suppose that A is a differential field, then $A\{y\}$ is a differential integral domain, and its derivations can be extended uniquely to the quotient field. We write $A\langle y\rangle$ for this quotient field, and its elements are *differential rational function* of y. The notations $\{\ \}$ and $\langle\ \rangle$ will also be used when the adjoined elements are not differential indeterminates, but rather elements of a larger differential ring or field.

Let A be any differential ring, then all elements in A with derivatives 0 form a subring C. This ring is called the ring of *constants.* If A is a field, so is C. Note that C contains the subring that is generated by the unit element of A.

Let A be a differential ring, with $\delta_i (i = 1, \cdots, m)$ the derivations. We say an ideal I in A to be a *differential ideal* if $a \in I$ implies $\delta_i a \in I$ $(\forall i)$. An ideal I is said to be a *prime ideal* if $A\,B \in I$ always implies either $A \in I$ or $B \in I$. Hereinafter, if not point out particularly, we use the term (prime) ideal in short for differential (prime) ideal.

Let A and B be two differential rings. A *differential homomorphism* from A to B is a homomorphism (purely algebraically) which furthermore commutes with derivatives. The terms *differential isomorphism* and *differential automorphism* are self-explanatory.

1.2. Definitions and statements

Following first order differential operator X accosiated with the equation (1.1) is convenient in the study,

$$X(\omega) = (X_1(x_1, x_2)\delta_1 + X_2(x_1, x_2)\delta_2)\omega = 0, \tag{1.2}$$

where $\delta_i = \partial/\partial x_i$. From the theory of differential equation[12, pp.510-513], for any non-critical point $\mathbf{x}^0 = (x_1^0, x_2^0) \in \mathbb{C}^2$, the equation (1.2) has non-constant solution $\omega(x_1, x_2)$ that is analytic at x^0. The solution $\omega(x_1, x_2)$ is said to be a *first integral* of (1.1) at $\mathbf{x}^0$. Furthermore, following lemma can be derived directly from [2, Theorem 1, pp. 98]

Lemma 1.1 *Consider the differential equation* (1.1)*, if* $X_1(x_1^0, x_2^0) \neq 0$*, and* $f(x_2)$ *is a function analytic at* $x_2 = x_2^0$*, then there exists a unique first integral* $\omega(x_1, x_2)$ *of* (1.1)*, analytic at* $\mathbf{x}^0 = (x_1^0, x_2^0)$*, and*

$$\omega(x_1^0, x_2) = f(x_2)$$

for all x_2 *in a neighborhood of* x_2^0*.*

From the existence of the first integrals of (1.1) at the regular point $\mathbf{x}^0$, we can define the Liouvillian integrability of (1.1) at $\mathbf{x}^0$ as follows.

Definition 1.2 *Let* K *be the differential field of rational functions of two variables with derivatives* δ_1 *and* δ_2*, we say* M *to be a Liouvillian extension of* K *if there exist* $r \geq 0$ *and subfields* $K_i (i = 0, 1, \cdots, r)$ *such that:*

$$K = K_0 \subset K_1 \subset \cdots \subset K_r = M,$$

where $K_{i+1} = K_i\langle u_i\rangle$*, and* $u_i \in K_{i+1}\backslash K_i$ *satisfy one of the following:*

1. *u_i is algebraic over K_i; or*
2. *$\delta_j u_i \in K_i \quad (j = 1, 2)$; or*
3. *$\delta_j u_i / u_i \in K_i \quad (j = 1, 2)$.*

A function that is contained in some Liouvillian extension of K is said a Liouvillian function.

Definition 1.3 *Let K be the differential field of rational functions of two variables, X be defined as* (1.2)*, then X is Liouvillian integrable at $\mathbf{x}^0$ if there exists a first integral ω of X at $\mathbf{x}^0$, such that $M = K\langle\omega\rangle$ is a Liouvillian extension of K.*

If X is Liouvillian integrable at one point $\mathbf{x}^0 \in \mathbb{C}^2$, there exists a first integral that can be obtained from the rational functions by finite steps of solving algebraic equations, integrals, and exponents of integrals. It is easy to prove by induction that this first integral is analytic in a dense open set in $\mathbb{C}^2$ [33]. And hence X it is also Liouvillian integrable in a dense open set in $\mathbb{C}^2$. Therefore, we can also say that X is Liouvillian integrable.

Definition 1.4 *A group G is solvable if there exist a subgroup series*

$$G = G_0 \supset G_1 \supset \cdots \supset G_m = \{e\}$$

such that for any $0 \leq i \leq m - 1$, either

1. *$|G_i/G_{i+1}|$ is a finite group; or*
2. *G_{i+1} is a normal subgroup of G_i and G_i/G_{i+1} is an Abelian group.*

Following theorem will be proved in this paper.

Theorem 1.5 (Main Theorem) *Consider the differential equation* (1.1)*. Assume that $X_1(0, 0) \neq 0$. Let K be the differential field of rational functions. Then* (1.1) *is Liouvillian integrable if and only if the differential Galois group of* (1.1) *over K at $(0, 0)$ is solvable.*

The keynote in out study is that we don't need to restrict the elements in the differential Galois group to be the automorphoisms. Instead, they are isomorphisms between different extension fields.

1.3. Historical background

The first rigorous proof of the non-solvable of a differential equation by quadrature method was given by Liouville in 1840s[22]. Liouville's work was 'undoubtedly inspired by the results of Lagrange, Abel, and Galois on the non-solvability of algebraic equations by radicals'[14]. Since Liouville's pioneer work, many approaches have been developing toward the integrability theory of differential equation. The concerning approaches include Lie group[29], monodromy group[14, 39], holonomy group[4, 5, 6], differential Galois group[13, 16, 37], Galois groupoid [23, 24], *etc.*. Let us first recall briefly the subject of differential Galois theory. For extensive survey, refer to [20, 35].

At first, we recall Liouville's result. Consider following second order linear differential equation

$$y'' + a(x)y = 0. \tag{1.3}$$

Liouville proved that the 'simple' equation (1.3) either has a solution of 'simple' type, or cannot be solved by quadrature[21]. An exposition of Liouville's proof was given in [38, pp.111-123]. Explicitly, we have the following result when $a(x)$ is a rational function.

Theorem 1.6 (Liouville's Theorem) *[22]If $a(x)$ is a rational function, the equation* (1.3) *is solvable by quadrature if and only if it has a solution, $u(x)$ say, such that $u'(x)/u(x)$ is an algebraic function.*

From the Liouville's Theorem, to determined the integrability of (1.3) with $a(x)$ a rational function, we only need to study the algebraic function solution of the corresponding Riccati equation (by letting $z = -y'/y$)

$$z' = z^2 + a(x).$$

One can refer [18, 19, 36] for the algorithms to find the Liouvillian solution of (1.3) with $a(x)$ a rational function.

Liouville's result was obtained by analytic method. Another approach to the problem of the integrability of homogenous linear ordinary differential equation, now known as differential Galois theory, or differential algebra, was established by Picard and Vessiot at the end of the 19'th century, and well developed by Ritt and Kochin in the next 50 years (see [17, 30] and the references therein). The firm footing step throughout this approach was established by Kolchin in 1948[15, 16]. For a self-contained exposition of Kochin's work, refer the little book by Kaplansky[13].

Consider the linear homogeneous differential equation

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{1.4}$$

with coefficients in a differential K. Let M be an extension of K that contants n linearly independent solutions of (1.4), and has the same field of constants as K. Kolchin proved the existence and uniqueness of the extension M/K if K is of characteristic 0 and has an algebraically closed field of constants. This extension is named by Kolchin as the *Picard-Vessiot extension* associated to the linear differential equation. As in th eclassical Galois theory, the differential Galois group of (1.4) is defined as the subgroup of all differential automorphisms of M that leaves K elementwise fixed. Kolchin established an isomorphosim between the differential Galois group and an algebraic matric group of degree n over the field of constants (refer [16] or [13, Lemma 5.4]). Accordingly, Kolchin was able to prove following connection between the existence of Liouvillian extension and the solvability of the differential Galois group (refer [16] or [13, Theorem 5.11-5.12]).

Theorem 1.7 (Kolchin's Theorem) *[13, 16] The linear differential equation* (1.4) *is Liouvillian integrable (the general solution can be obtained by a combination of algebraic functions, integrals and exponentials of integrals) if and only if the identity component of the corresponding differential Galois group, which is a normal subgroup, is solvable.*

Kolchin's Theorem is similar to the Galois theory for solvability a polynomial equation by radicals. As application, the previous Liouville's Theorem is easy to be concluded[13, Theorem 6.4]. In recent decades, many works by Kovacic, Magid, Mitschi, Singer, Ulmer, *et al.* try to address to both direct and inverse problems of linear differential Galois theory, for example, see [18, 25, 26, 27, 34, 35, 36, 37]. The differential Galois theory is also applied to study the non-integrability of Hamiltonian systems[31]. For details following this approach, one can refer to [1, 13, 17, 31, 35, 37] and the references therein.

Besides the linear systems, the solvability of first order nonlinear differential equation is also interested and will be the main topic of this paper. Consider the equation

$$\frac{dx_2}{dx_1} = \frac{X_2(x_1, x_2)}{X_1(x_1, x_2)}, \tag{1.5}$$

where X_1 and X_2 are polynomials. The most profound result for the integrability of this system was obtained by Singer[33] in 1992. Singer proved the following result.

Theorem 1.8 (Singer's Theorem) *[33] If* (1.5) *has a local Liouvillian first integral, then there is a Liouvillian first integral of the form*

$$\omega(x_1, x_2) = \int_{(x_1^0, x_2^0)}^{(x_1, x_2)} R\, X_2\, dx_1 - R\, X_1\, dx_2 \tag{1.6}$$

where

$$R = \exp\left[\int_{(x_1^0, x_2^0)}^{(x_1, x_2)} U \mathrm{d}x_1 + V \mathrm{d}x_2\right], \tag{1.7}$$

with U and V are rational functions in x_1 and x_2 such that

$$\partial U / \partial x_2 = \partial V / \partial x_1.$$

Christopher proved that the existence of R in the form (1.7) is equivalent to the existence of an integrating factor of form

$$\exp(D/E) \prod C_i^{l_1},$$

where D, E and the C_i are polynomials in x_1 and x_2[8]. Singer's Theorem was obtained through the method of differential algebra. The same result was obtained independently by Guan and Lei through Liouville's approach [10]. Guan and Lei shown that if (1.5) has local Liouvillian first integral $\omega(x_1, x_2)$ in the form of (1.6), then $\delta_i^2 \omega / \delta_i \omega$ $(i = 1, 2)$ are rational functions in x_1 and x_2, here $\delta_i = \frac{\partial}{\partial x_i}$ $(i = 1, 2)$.

Monodromy group of linear differential equation is also important for the integrability of differential equation. In 1970s, Khovanskii proved that a function is representable by quadrature if and only if its monodromy group is solvable[14]. As application of this result, Khovanskii claimed that monodromy group of a linear differential equation is essential for integrating the equation by quadrature[39, pp. 128-130, Khovanskiy's Theorem].

Theorem 1.9 (Khovanskiy's Theorem) *[39] If the monodromy group of a fuchsian system has a solvable normal divisor of finite index, then the system is integrable in quadratures. If the monodromy group does not have this property, then the system is not even integrable by "generalized quadrature".*

This means that the solution of the system cannot be expressed in terms of the coefficients by solving algebraic equations, integration, and composition with entire functions of any number of variables.

Monodromy group closely relates with the Galois group. The monodromy group of a linear differential equations with regular singular points is Zariski dense in the associated Galois Group[32]. Żołądek extended the conception of monodromy group to study the functions defined on $\mathbb{C}P^n$ with algebraic singular set, and to investigate the structure of the monodromy group of the first integrals of a Liouvillian integrable Pfaff equation[40]. Through these studies, Żołądek was able to extend Singer's Theorem partly to the integrable polynomial Pfaff equation[40].

Theorem 1.10 (Multi-dimensional Singer's theorem) *[40] If an integrable polynomial Pfaff equation has a generalized Liouvillian first integral, then it has an integrating factor whose logarithmic differential is a closed rational 1-form.*

In 1990s, the geometry approaches were introduced by Camacho, Scárdua, *et. al.* to study the Liouvillian integrability of a nonlinear differential equation. The geometry approaches focus on the characters of the foliation associated with the equation. In a series of their works, the holonomy group that is induced by an invariant algebraic curve of a polynomial system was studied[4, 5, 6]. Camacho and Scárdua studied the structure of the foliation of a polynomial system with Liouvian first integral. Following result shows such foliations must have simple forms.

Theorem 1.11 (Camacho-Scárdua Theorem) *[5] Let $\mathcal{F}$ be a codimension-one holomorphic foliation on $\mathbb{C}P^n$ admitting a Liouvillian first integral. Assume that one of the algebraic leaves of $\mathcal{F}$ has only non-dicritical infinitesimally hyperbolic singularities. Then either $\mathcal{F}$ is a Darboux foliation or an exponent two Bernoulli foliation after some rational pull-back.*

In 2001, Malgrange published two papers to introduce the Galois groupoid associated with a foliation with meromorphic singularities [23, 24]. For linear differential equations, Malgrange has showed that this groupoid coincides with the Galois group of the Picard-Vessiot theory and has proved the required results in the linear case[23]. But further development of the groupoid theory is need to established the theory for nonlinear differential equation[20].

The rest of the paper is organized as follows. Section 2 will give the definition of the differential Galois group, with the discussion of the structure in Section 3, and leave the proof of Theorem 3.9 to Section 5. Section 4 will prove the above main theorem. As application, the differential Galois group of general Riccati and van der Pol equations will also be discussed in Section 4. Throughout this paper, the base field K always means the field of all rational functions in x_1 and x_2 with $\mathbb{C}$ the constant field.

2. Differential Galois group

This section will give the definition of the differential Galois group of equation (1.1) at a regular point $\mathbf{x}^0 = (x_1^0, x_2^0)$. To this end, we will first define the group $\mathcal{G}[[\epsilon]]$ that acts at all first integrals of X at $\mathbf{x}^0$, next studied the admissible differential isomorphism of (1.1) at x^0 that is an element of $\mathcal{G}[[\epsilon]]$, and finally prove that all admissible differential isomorphisms form a subgroup of $\mathcal{G}[[\epsilon]]$, which is defined as the differential Galois group of (1.1) at $\mathbf{x}^0$.

Hereinafter, we will assume $\mathbf{x}^0 = (0,0)$ for short. When we mention a first integral, we will always mean a first integral that is analytic at $(0,0)$. Following notations will be used hereinafter.

Let $\mathcal{A}_0$ denote the set of all functions $f(z)$ of one variable that are analytic at $z = 0$, and

$$\begin{aligned}\mathcal{A}_0^0 &= \{f(z) \in \mathcal{A}_0 |\ \ f(0) = 0\},\\ \mathcal{A}_0^1 &= \{f(z) \in \mathcal{A}_0^0 |\ \ f'(0) \neq 0\}.\end{aligned}$$

Let $\Omega_{(0,0)}(X)$ denote the set of all first integrals of (1.1) that are analytic at $(0,0)$, and

$$\begin{aligned}\Omega_{(0,0)}^0(X) &= \{\omega(x_1,x_2) \in \Omega_{(0,0)} |\ \ \omega(0,0) = 0\},\\ \Omega_{(0,0)}^1(X) &= \{\omega(x_1,x_2) \in \Omega_{(0,0)}^0 |\ \ \delta_2\omega(0,0) \neq 0\}.\end{aligned}$$

It is easy to have $f(z) := \omega(0,z) \in \mathcal{A}_0^1$ for any $\omega \in \Omega_{(0,0)}^1(X)$.

2.1. The space of all first integrals at a regular point

Since we will focus on the Liouvillian integrability of the polynomial system (1.1), it is enough to concentrate on the first integrals in $\Omega_{(0,0)}^1(X)$ according to the following lemma.

Lemma 2.1 *If X is Liouvillian integrable at $(0,0)$, then there exists a first integral $\omega \in \Omega^1_{(0,0)}$ such that $M = K\langle\omega\rangle$ is a Liouvillian extension of K.*

Proof. Let u be a first integral such that $M\langle u\rangle$ is a Liouvillian extension of K. If $u \in \Omega^1_{(0,0)}(X)$, then the Lemma has been concluded. If $u \notin \Omega^1_{(0,0)}(X)$, we can always assume that $u \in \Omega^0_{(0,0)}(X)$ by subtracting $u(0,0)$ from u. Let

$$u(0,x_2) = \sum_{i\geq k} a_i x_2^i \qquad (k \geq 2, a_k \neq 0)$$

and

$$f(x_2) = \sqrt[k]{u(0,x_2)} = \left(\sum_{i\geq k} a_i x_2^i\right)^{1/k} = x_2 \left(\sum_{i\geq k} a_i x_2^{i-k}\right)^{1/k}.$$

Then $f(x_2) \in \mathcal{A}^1_0$. From Lemma 1.1, there is a unique first integral $\omega \in \Omega_{(0,0)}$, such that $\omega(0,x_2) = f(x_2)$ and therewith $\omega \in \Omega^1_{(0,0)}(X)$. Moreover, $f(x_2)^k = u(0,x_2)$ implies $\omega^k = u$, and hence $K\langle\omega\rangle$ is a Liouvillian extension of K. $\square$

From Lemma 1.1, there is a one-to-one correspondence between $\Omega^1_{(0,0)}(X)$ and $\mathcal{A}^1_0$. Hence, it is enough to study the structure of $\mathcal{A}^1_0$. It is obvious that $\mathcal{A}^1_0$ contains the identity function $e(z) = z$, and for any $f(z), g(z) \in \mathcal{A}^1_0$, $f \circ g(z) := f(g(z)) \in \mathcal{A}^1_0$ and $f^{-1}(z) \in \mathcal{A}^1_0$. Hence $(\mathcal{A}^1_0, \circ)$ is a group. Furthermore, we have the following result.

Lemma 2.2 *For any $\omega \in \Omega^1_{(0,0)}(X)$, let $\mathcal{A}^1_0(\omega) = \{f\omega) | f \in \mathcal{A}^1_0\}$, then*

$$\Omega^1_{(0,0)}(X) = \mathcal{A}^1_0(\omega).$$

Proof. It is easy to see that for any $f \in \mathcal{A}^1_0$, $f(\omega) \in \Omega^0_{(0,0)}(X)$. Moreover,

$$\frac{\partial f(\omega)}{\partial x_2}(0,0) = f'(0)\, \delta_2\omega(0,0) \neq 0.$$

Hence $f(\omega) \in \Omega^1_{(0,0)}(X)$ and therefore $\mathcal{A}^1_0(\omega) \subseteq \Omega^1_{(0,0)}(X)$.

For any $\omega, u \in \Omega^1_{(0,0)}(X)$, let

$$g(x_2) = \omega(0,x_2), \quad h(x_2) = u(0,x_2),$$

then $g, h \in \mathcal{A}^1_0$. Hence $f = h \circ g^{-1} \in \mathcal{A}^1_0$ and $f(g(x_2)) = h(x_2)$, i.e. $f(\omega(0,x_2)) = u(0,x_2)$. Lemma 1.1 yields $u = f(\omega)$ and hence $\Omega^1_{0,0}(X) \subseteq \mathcal{A}^1_0(\omega)$, the Lemma is proved. $\square$

All elements in $\mathcal{A}_0^1$ map 0 to 0. To take the functions that map 0 to a nonzero value into account, we adjoin an infinitesimal variable ε to the constant field $\mathbb{C}$ and consider the ring of infinite series of ϵ with coefficients in $\mathbb{C}$. Denote this extension constant ring as $\mathbb{C}[[\epsilon]]$. Then we have the following:

1. A series $\sum_{i\geq 0} c_i\epsilon^i \in \mathbb{C}[[\epsilon]]$ equals 0 if and only if all coefficients c_i are 0;
2. The derivations δ_i can be extended to $\mathbb{C}[[\epsilon]]$ by setting $\delta_1\epsilon = \delta_2\epsilon = 0$.

Consider the infinite power series

$$f(z;\epsilon) = \sum_{i,j\geq 0} f_{i,j} z^i \epsilon^j \in \mathbb{C}[[z,\epsilon]].$$

The series $f(z;\epsilon)$ is analytic if it is convergent for any (z,ϵ) in a neighborhood of $(0,0)$. We can also write an analytic series $f(z,\epsilon)$ in the form of power series of ϵ as

$$f(z;\epsilon) = \sum_{i=0}^{\infty} f_i(z)\,\epsilon^i,$$

where $f_i(z) \in \mathcal{A}_0$. We denote all analytic series as $\mathcal{A}_0[[\epsilon]]$. Let

$$\mathcal{G}[[\epsilon]] = \{f(z;\epsilon)|\ f_0(z) \in \mathcal{A}_0^1\},$$

and define the multiplication in $\mathcal{G}[[\epsilon]]$ as:

$$f(z;\epsilon)\cdot g(z;\epsilon) = f(g(z;\epsilon);\epsilon)$$

for any $f(z;\epsilon), g(z;\epsilon) \in \mathcal{G}[[\epsilon]]$. Then we have

Lemma 2.3 *$(\mathcal{G}[[\epsilon]],\cdot)$ is a group.*

Proof. First, we will show that $(\mathcal{G}[[\epsilon]],\cdot)$ is closure, i.e., for any $f(z;\epsilon), g(z;\epsilon) \in \mathcal{G}[[\epsilon]]$, $f(z;\epsilon)\cdot g(z;\epsilon) \in \mathcal{G}[[\epsilon]]$. Let

$$f(z;\epsilon) = \sum_i f_i(z)\,\epsilon^j,\quad g(z;\epsilon) = \sum_i g_i(z)\,\epsilon^j.$$

Then $f(z;\epsilon)$ and $g(z;\epsilon)$ are analytic functions of (z,ϵ) at $(0,0)$. Since $g(0,0) = g_0(0) = 0$, $f(g(z;\epsilon);\epsilon)$ is also an analytic function at $(0,0)$, i.e., $f(g(z;\epsilon);\epsilon) \in \mathcal{A}_0[[\epsilon]]$. Moreover, $f(g(z;0);0) = f_0(g_0(z)) \in \mathcal{A}_0^1$ and therefore $f(z;\epsilon)\cdot g(z;\epsilon) \in \mathcal{G}[[\epsilon]]$.

It is easy to verify that

$$(f(z;\epsilon)\cdot g(z;\epsilon))\cdot h(z;\epsilon) = f(z;\epsilon)\cdot (g(z;\epsilon)\cdot h(z;\epsilon)),$$

Thus $(\mathcal{G}[[\epsilon]],\cdot)$ is associativity.

To prove the identity, we embed $\mathcal{A}_0^1$ into $\mathcal{A}_0^1[[\epsilon]]$ by identifying $f(z)\in\mathcal{A}_0^1$ with $f(z;0) = f(z) + \sum_{i\geq 0} 0\cdot \epsilon^i \in \mathcal{A}_0^1[[\epsilon]]$. Then $e(z;0) = e(z)\in\mathcal{G}[[\epsilon]]$, and for any $f(z;\epsilon)\in\mathcal{G}[[\epsilon]]$,

$$e(z;0)\cdot f(z;\epsilon) = e(f(z;\epsilon);0) = f(z;\epsilon),$$

and

$$f(z;\epsilon)\cdot e(z;0) = f(e(z;0);\epsilon) = f(z;\epsilon).$$

Thus, $e(z;0)$ is also an identity of $(\mathcal{G}[[\epsilon]],\cdot)$.

Finally, we only need to prove the invertibility. For any $f(z;\epsilon)\in\mathcal{G}[[\epsilon]]$, we have $(\partial f/\partial z)(0,0)\neq 0$. Thus, the equation $u = f(z;\epsilon)$ has a unique solution $z = f^{-1}(u;\epsilon)$ in the neighborhood of $(0,0)$ such that $u = f(f^{-1}(u;\epsilon))$. Moreover, we have $z = f(f^{-1}(z;\epsilon);\epsilon)$ and $z = f^{-1}(u;\epsilon) = f^{-1}(f(z;\epsilon);\epsilon)$. Thus, $f^{-1}(z;\epsilon)$ is the inverse element of $f(z;\epsilon)$. Furthermore, $f^{-1}(z;\epsilon)$ is analytic at $(0,0)$ and $f^{-1}(z;0)\in\mathcal{A}_0^1$. Thus, we conclude that the inverse element $f^{-1}(z;\epsilon)\in\mathcal{G}[[\epsilon]]$ and the invertibility is concluded. $\square$

For any $\sigma = f(z;\epsilon)\in\mathcal{G}[[\epsilon]]$ and $\omega\in\Omega^1_{(0,0)}$, we define the action of σ at ω as $\sigma\omega = f(\omega;\epsilon)$. This is well defined at the neighborhood of $(0,0)$. Taking account that

$$X(f(\omega;\epsilon)) = X(\sum_{i\geq 0} f_i(\omega)\,\epsilon^i) = \sum_{i\geq 0} X(f_i(\omega))\,\epsilon^i = 0,$$

σ maps a first integral ω to another first integral $f(\omega;\epsilon)$.

Let $h(z;\epsilon)\in\mathcal{A}_0[[\epsilon]]$ and $f(z;\epsilon)\in\mathcal{G}[[\epsilon]]$, then

$$h(z;\epsilon)\cdot f(z;\epsilon) = h(f(z;\epsilon);\epsilon)\in\mathcal{A}_0[[\epsilon]]$$

is well defined, and $(h(z;\epsilon)\cdot f(z;\epsilon))\omega = h(f(\omega;\epsilon);\epsilon)$.

2.2. Admissible differential isomorphism

For any $\omega\in\Omega^1_{(0,0)}(X)$, an extension field M of K is obtained by adjoining ω to K, and denoted as $M = K\langle\omega\rangle$. In this paper, if not mentioned particularly, ω will always means a determinate first integral. In this subsection, we will define and study the admissible differential isomorphism that is an element of the group $\mathcal{G}[[\epsilon]]$ with additional restrictions. Throughout this paper, we will call compactly a map $\sigma : \omega \mapsto \sigma(\omega)$ a differential isomorphism if there exists a differential isomorphism from $K\langle\omega\rangle$ to $K\langle\sigma(\omega)\rangle$ that maps ω to $\sigma(\omega)$ with elements in K fixed.

Definition 2.4 *Let $M = K\langle\omega\rangle$ with $\omega \in \Omega^1_{(0,0)}(X)$. An admissible differential isomorphism of M/K with respect to X at $(0,0)$ (a.d.i., singular and plural) is a map σ that acts on M with the following properties:*

1. *σ maps ω to $f(\omega;\epsilon)$ with some $f(z;\epsilon) \in \mathcal{G}[[\epsilon]]$;*
2. *$\sigma : \omega \mapsto f(\omega;\epsilon)$ is a differential isomorphism;*
3. *for any $h_i(z;\epsilon) \in \mathcal{A}_0[[\epsilon]]$ $(0 \le i \le m < \infty)$, σ can be extended to a differential isomorphism of $K\langle\omega, h_1(\omega;\epsilon), \cdots, h_m(\omega;\epsilon)\rangle$ that maps $h_i(\omega;\epsilon)$ to $h_i(f(\omega;\epsilon);\epsilon)$, respectively, with K elementwise fixed.*

It is obvious that the identity element of $\mathcal{G}[[\epsilon]]$ is an a.d.i.. The following two Lemmas show that the set of all a.d.i. is closure under multiplication and inverse operation.

Lemma 2.5 *If σ, τ are a.d.i., then $\sigma \cdot \tau$ is an a.d.i..*

Proof. For any $\varsigma = h(z;\epsilon) \in \mathcal{A}_0[[\epsilon]]$, we will prove that $\sigma \cdot \tau : \omega \mapsto (\sigma \cdot \tau)\omega$ can be extended to a differential isomorphism of $K\langle\omega, \varsigma\omega\rangle$ that maps $\varsigma\omega$ to $(\varsigma \cdot \sigma \cdot \tau)\omega$ with K elementwise fixed.

Since τ is an a.d.i., τ is well defined in $K\langle\omega, \sigma\omega, (\varsigma \cdot \sigma)\omega\rangle$. Hence, the restriction of τ at $K\langle\sigma\omega, (\varsigma \cdot \sigma)\omega\rangle$ is a differential isomorphism that maps $\sigma\omega$ and $(\varsigma \cdot \sigma)\omega$ to $(\sigma \cdot \tau)\omega$ and $(\varsigma \cdot \sigma \cdot \tau)\omega$, respectively, with K elementwise fixed. Consider the following

$$\begin{tikzcd} K\langle\omega, \varsigma\omega\rangle \arrow[rr, "\sigma \cdot \tau"] \arrow[dr, "\sigma"'] & & K\langle(\sigma \cdot \tau)\omega, (\varsigma \cdot \sigma \cdot \tau)\omega\rangle \\ & K\langle\sigma\omega, (\varsigma \cdot \sigma)\omega\rangle \arrow[ur, "\tau|_{K\langle\sigma\omega, (\varsigma\cdot\sigma)\omega\rangle}"'] & \end{tikzcd}$$

where σ and $\tau|_{K\langle\sigma\omega, (\varsigma\cdot\sigma)\omega\rangle}$ are differential isomorphisms with K elementwise fixed. Thus $\sigma \cdot \tau$ is also a differential isomorphism with K elementwise fixed. The extension of $\sigma \cdot \tau$ to $K\langle\omega, h_1\omega, \cdots, h_m\omega\rangle$ can be proved similarly. Hence $\sigma \cdot \tau$ is also an a.d.i.. $\square$

Lemma 2.6 *If σ is an a.d.i., then the inverse σ^{-1} is alo an a.d.i..*

Proof. Similar to the proof of the previous Lemma, it is sufficient to show that for any $\varsigma = h(z;\epsilon) \in \mathcal{A}_0[\epsilon]$, σ^{-1} can be extended to a differential isomorphism of $K\langle \omega, \varsigma\omega\rangle$ that maps ς to $(\varsigma\cdot\sigma^{-1})\omega$ with K elementwise fixed.

Consider the extension of σ to $K\langle \omega, \sigma^{-1}\omega, (\varsigma\cdot\sigma^{-1})\omega\rangle$ that maps $\sigma^{-1}\omega$ and $(\varsigma\cdot\sigma^{-1})\omega$ to $(\sigma^{-1}\cdot\sigma)\omega = \omega$ and $(\varsigma\cdot\sigma^{-1}\cdot\sigma)\omega = \varsigma\omega$ respectively. Hence, the restricted map $\sigma|_{K\langle\sigma^{-1}\omega,(\varsigma\cdot\sigma^{-1})\omega\rangle}$ is a differential isomorphism that maps $K\langle\sigma^{-1}\omega,(\varsigma\cdot\sigma^{-1})\omega\rangle$ to $K\langle\omega,\varsigma\omega\rangle$ with K elementwise fixed. Let $\tau = \left(\sigma|_{K\langle\sigma^{-1}\omega,(\varsigma\cdot\sigma^{-1})\omega\rangle}\right)^{-1}$, then $\tau : K\langle\omega,\varsigma\omega\rangle \mapsto K\langle\sigma^{-1}\omega,(\varsigma\cdot\sigma^{-1})\omega\rangle$ is a differential isomorphism that maps ω and $\varsigma\omega$ to $\sigma^{-1}\omega$ and $(\varsigma\cdot\sigma^{-1})\omega$, respectively, with K elementwise fixed. Thus, we have $\sigma^{-1} = \tau$ is an a.d.i.. $\square$

2.3. Differential Galois group

From the previous discussion, the set of all a.d.i. contains identity element, and satisfies the closure and invertibility, and therefore form a subgroup of $\mathcal{G}[[\epsilon]]$. This subgroup is our desired differential Galois group.

Definition 2.7 *Let X be defined as* (1.2)*, K be the field of rational functions, $\omega \in \Omega^1_{(0,0)}(X)$ and $M = K\langle\omega\rangle$. The differential Galois group of M/K with respect to X at $(0,0)$, denoted as $\mathrm{Gal}(M/K,X)_{(0,0)}$, is defined as the subgroup of $\mathcal{G}[[\epsilon]]$ with all elements are admissible differential isomorphism of M/K with respect to X at $(0,0)$.*

Following two Lemmas show that the differential Galois group is determined 'uniquely' by the differential operator X (or the differential equation (1.1)).

Lemma 2.8 *Let $u \in \Omega^1_{(0,0)}(X)$ and $N = K\langle u\rangle$, then*

$$\mathrm{Gal}(N/K,X)_{(0,0)} \cong \mathrm{Gal}(M/K,X)_{(0,0)}.$$

Proof. From Lemma 2.2, $u \in \Omega^1_{(0,0)}(x) = \mathcal{A}^1_0(\omega)$. There is a function $h \in \mathcal{A}^1_0$ such that $u = h(\omega)$. Let $\tau = h(z;0) \in \mathcal{G}[[\epsilon]]$, then $u = \tau\omega$, i.e., $\omega = \tau^{-1}u$.

For any $\sigma \in \mathrm{Gal}(M/K,X)_{(0,0)}$, we will show that $\tau\cdot\sigma\cdot\tau^{-1} \in \mathrm{Gal}(N/K,X)_{(0,0)}$. To this end, we only need to show that for any $\varsigma \in \mathcal{A}_0[[\epsilon]]$, $\tau\cdot\sigma\cdot\tau^{-1}$ can be extended to a differential isomorphism of $K\langle u,\varsigma u\rangle$ that maps u and ςu to $(\tau\cdot\sigma\cdot\tau^{-1})u$ and $(\varsigma\cdot\tau\cdot\sigma\cdot\tau^{-1})u$, respectively, with K elementwise fixed.

Since $\sigma \in \mathrm{Gal}(M/K,X)_{(0,0)}$ and $\tau, (\varsigma\cdot\tau) \in \mathcal{A}_0[[\epsilon]]$, σ can be extended to $K\langle \omega, \tau\omega, (\varsigma\cdot\tau)\omega\rangle$ and maps $\tau\omega = u$ and $(\varsigma\cdot\tau)\omega = \varsigma u$ to $(\tau\cdot\sigma)\omega = (\tau\cdot\sigma\cdot\tau^{-1})u$ and $(\varsigma\cdot\tau\cdot\sigma)\omega = (\varsigma\cdot\tau\cdot\sigma\cdot\tau^{-1})u$, respectively. Hence, $\sigma|_{K\langle u,\varsigma u\rangle}$, the restriction of σ to $K\langle u, \varsigma u\rangle$, is a differential isomorphism that maps u and ςu to $(\tau\cdot\sigma\cdot\tau^{-1})u$ and $(\varsigma\cdot\tau\cdot\sigma\cdot\tau^{-1})u$, respectively, with K elements fixed. Thus, we have $\tau\cdot\sigma\cdot\tau^{-1} \in \mathrm{Gal}(N/K,X)_{(0,0)}$. In fact, we have further $\tau\cdot\sigma\cdot\tau^{-1} = \sigma|_{K\langle u\rangle}$.

Similarly, for any $\eta \in \mathrm{Gal}(N/K,X)_{(0,0)}$, $\tau^{-1}\cdot\eta\cdot\tau \in \mathrm{Gal}(M/K,X)_{(0,0)}$. Thus $\sigma \mapsto \tau^{-1}\cdot\sigma\cdot\tau$ is an isomorphism between $\mathrm{Gal}(M/K,X)_{(0,0)}$ and $\mathrm{Gal}(N/K,X)_{(0,0)}$. The Lemma has been proved. $\square$

Lemma 2.8 shows that the structure of the differential Galois group of (1.1) at $(0,0)$ is independent to the choice of the first integral ω. For different choices of first integrals, the corresponding Galois groups are different by a diffeomorphism. Following Lemma will show that with mild restriction on the regular point, the group is also independent to the choice of the regular points. We will prove latter (Theorem 3.9) that these are all possible cases when the group is of finite order.

Lemma 2.9 *Assume that* $\omega \in \Omega^1_{(0,0)}(X)$, $M = K\langle\omega\rangle$, *and* $G = \mathrm{Gal}(M/K,X)_{(0,0)}$, *we have the following:*

(1). If $\omega \in K$, *then* $\sigma\omega = \omega, \forall \sigma \in G$;

(2). If $(\delta_2\omega)^n \in K$ *for some* $n \in \mathbb{N}$, *then* $\sigma\omega = \mu_n\omega + c(\epsilon), \forall\sigma \in G$, *where* μ_n *is a n-th root of unity;*

(3). If $\delta_2^2\omega/\delta_2\omega \in K$, *then* $\sigma\omega = a(\epsilon)\,\omega + c(\epsilon)$, $\forall \sigma \in G$;

(4). If $(2\,(\delta_2\omega)\,(\delta_2^3\omega) - 3\,(\delta_2^2\omega)^2)/(\delta_2\omega)^2 \in K$, *then* $\sigma\omega = \frac{a(\epsilon)\omega}{1+b(\epsilon)\,\omega} + c(\epsilon)$, $\forall \sigma \in G$.

Here $a(\epsilon), b(\epsilon), c(\epsilon) \in \mathbb{C}[[\epsilon]]$, *and* $c(0) = 0$. *Moreover, for any* (x_1^0, x_2^0) *such that* $X_1(x_1^0, x_2^0) \neq 0$ *and* ω *is analytic at* (x_1^0, x_2^0) *and* $\delta_2\omega(x_1^0, x_2^0) \neq 0$, *the first integral* u *that is defined as* $u = \omega - \omega(x_1^0, x_2^0)$ *is contained in* $\Omega^1_{(x_1^0, x_2^0)}(X)$, *and the above results are also valid for all* $\sigma \in \mathrm{Gal}(K\langle u\rangle/K, X)_{(x_1^0,x_2^0)}$.

Proof. The first part is proved as follows.

First, (1) is obvious.

(2). Let $(\delta_2\omega)^n = a \in K$, then

$$a - (\delta_2\omega)^n = 0$$

It is easy to see that for any $\sigma = f(z;\epsilon) \in G$,

$$0 = \sigma(a - (\delta_2\omega)^n) = a - (\delta_2(\sigma(\omega)))^n = a - (\delta_2(f(\omega;\epsilon)))^n = a - f'(\omega;\epsilon)^n(\delta_2\omega)^n$$

Hereinafter, $'$ means the derivative with respect to z. Thus, we have $f'(\omega;\epsilon)^n = 1$ for any ω in the neighborhood of $\omega(0,0)=0$, and hence $f(\omega;\epsilon)=\mu_n\omega+c(\epsilon)$, where $c(\epsilon)\in\mathbb{C}[[\epsilon]]$ and μ_n is a n-th root of unity.

(3). Let $\delta_2^2\omega/\delta_2\omega = a \in K$, then

$$\delta_2^2\omega - a\,\delta_2\omega = 0.$$

For any $\sigma = f(z;\epsilon)\in G$, we have

$$\begin{aligned}
0 &= \sigma(\delta_2^2\omega - a\,\delta_2\omega)\\
&= \delta_2^2(\sigma\omega) - a\,\delta_2(\sigma\omega)\\
&= \delta_2^2(f(\omega;\epsilon)) - a\,\delta_2(f(\omega;\epsilon))\\
&= f''(\omega;\epsilon)\,(\delta_2\omega)^2 + f'(\omega;\epsilon)\,\delta_2^2\omega - a\,f'(\omega;\epsilon)\,\delta_2\omega\\
&= f''(\omega;\epsilon)\,(\delta_2\omega)^2.
\end{aligned}$$

Hence, we have $f''(\omega;\epsilon)=0$ and therefore $\sigma\omega = f(\omega;\epsilon) = a(\epsilon)\,\omega + c(\epsilon)$ for some $a(\epsilon), c(\epsilon)\in\mathbb{C}[[\epsilon]]$.

(4). Let $(2\,(\delta_2\omega)\,(\delta_2^3\omega) - 3\,(\delta_2^2\omega)^2)/(\delta_2\omega)^2 = a\in K$, then

$$2\,(\delta_2\omega)\,(\delta_2^3\omega) - 3\,(\delta_2^2\omega)^2 - a\,(\delta_2\omega)^2 = 0.$$

For any $\sigma = f(z;\epsilon)\in G$, we have

$$\begin{aligned}
0 &= \sigma\left(2\,(\delta_2\omega)\,(\delta_2^3\omega) - 3\,(\delta_2^2\omega)^2 - a\,(\delta_2\omega)^2\right)\\
&= 2\,\delta_2(\sigma\omega)\,(\delta_2^3(\sigma\omega)) - 3\,(\delta_2^2(\sigma\omega))^2 - a\,(\delta_2(\sigma\omega))^2\\
&= 2\,\delta_2(f(\omega;\epsilon))\,(\delta_2^3(f(\omega;\epsilon))) - 3\,(\delta_2^2(f(\omega;\epsilon)))^2 - a\,(\delta_2(f(\omega;\epsilon)))^2\\
&= 2\,f'(\omega;\epsilon)\,(\delta_2\omega)\,\left(f'''(\omega;\epsilon)\,(\delta_2\omega)^3 + 3\,f''(\omega;\epsilon)\,(\delta_2\omega)\,(\delta_2^2\omega) + f'(\omega;\epsilon)\,\delta_2^3\omega\right)\\
&\quad - 3\,\left(f''(\omega;\epsilon)\,(\delta_2\omega)^2 + f'(\omega;\epsilon)\,(\delta_2^2\omega)\right)^2 - a\,\left(f'(\omega;\epsilon)\,(\delta_2\omega)\right)^2\\
&= 2\,f'(\omega;\epsilon)\,f''(\omega;\epsilon)\,(\delta_2\omega)^4 + 6\,f'(\omega;\epsilon)\,f''(\omega;\epsilon)\,(\delta_2\omega)^2\,(\delta_2^2\omega)\\
&\quad + 2\,(f'(\omega;\epsilon))^2\,(\delta_2\omega)\,(\delta_2^3\omega) - 3\,(f''(\omega;\epsilon))^2\,(\delta_2\omega)^4\\
&\quad - 6\,f'(\omega;\epsilon)\,f''(\omega;\epsilon)\,(\delta_2\omega)\,(\delta_2^2\omega) - 3\,(f'(\omega;\epsilon))^2\,(\delta_2^2\omega)^2 - a\,(f'(\omega;\epsilon))^2\,(\delta_2\omega)^2\\
&= \left(2\,f'(\omega;\epsilon)\,f''(\omega) - 3\,(f''(\omega;\epsilon))^2\right)\,(\delta_2\omega)^4\\
&\quad + (f'(\omega;\epsilon))^2\,(2\,(\delta_2\omega)\,(\delta_2^3\omega) - 3\,(\delta_2^2\omega)^2 - a\,(\delta_2\omega)^2)\\
&= (2\,f'(\omega;\epsilon)\,f''(\omega) - 3\,(f''(\omega;\epsilon))^2)\,(\delta_2\omega)^4.
\end{aligned}$$

Hence, $f(\omega,\epsilon)$ satisfies

$$2\,f'(\omega;\epsilon)\,f''(\omega;\epsilon) - 3\,(f''(\omega;\epsilon))^2 = 0,\quad f(0;\epsilon)=0. \tag{2.1}$$

The general solution of (2.1) is given by

$$f(\omega;\epsilon) = \frac{a(\epsilon)\,\omega}{1 + b(\epsilon)\,\omega} + c(\epsilon),$$

with $a(\epsilon), b(\epsilon), c(\epsilon) \in \mathbb{C}[[\epsilon]]$.

Finally, since $f(z;\epsilon) \in \mathcal{G}[[\epsilon]]$, we have $f(0;0) = 0$, and hence $c(0) = 0$ in all cases.

For the second part, it is obvious that $u \in \Omega^1_{(x_1^0, x_2^0)}(X)$, and the above discussions are also valid for u. The proof is complete. $\square$

Similar to classical Galois theory, for any differential subfield L of M containing K, let

$$L' = \{\sigma \in \mathrm{Gal}(M/K, X)_{(0,0)} |\ \sigma a = a, \forall a \in L\}$$

to be the subset of $\mathrm{Gal}(M/K, X)_{(0,0)}$ consisting all a.d.i. leaving L element-wise fixed. For any subgroup H of G, let

$$H' = \{a \in M |\ \sigma a = a, \quad \forall \sigma \in H\}$$

to be the set of all elements in M left fixed by H. Following lemma is obvious from the above definitions. following results.

Lemma 2.10 *Let L, L_1, L_2 be subfields of M containing K, and H, H_1, H_2 be the subgroups of G, then*

(1). L' is a subgroup of G, and H' is a subfield of M;
(2). $L \subseteq L''$, $H \subseteq H''$;
(3). $L_1 \supseteq L_2 \Rightarrow L_1' \subseteq L_2'$;
(4). $H_1 \supseteq H_2 \Rightarrow H_1' \subseteq H_2'$.

Let L to be a subfield of M that contains K. We can also consider M as an extension field of L by setting $M = K\langle\omega\rangle = L\langle\omega\rangle$, and the differential Galois group of M/L with respect to X at $(0,0)$ can be defined through the same procedure. We denote this Galois group as $\mathrm{Gal}(M/L, X)_{(0,0)}$.

Lemma 2.11 *Let L be the subfield of M containing K, then*

$$\mathrm{Gal}(M/L, X)_{(0,0)} = L'.$$

In particular, $\mathrm{Gal}(M/K, X)_{(0,0)} = K'$.

Proof. It is easy to have $\mathrm{Gal}(M/L, X)_{(0,0)} \subseteq L'$. We will only need to show that $L' \subseteq \mathrm{Gal}(M/L, X)_{(0,0)}$.

For $\sigma \in L'$ and $\varsigma \in \mathcal{G}[[\epsilon]]$, since $K \subseteq L \subseteq K\langle \omega, \varsigma\omega\rangle$ and $K \subseteq L \subseteq K\langle \sigma\omega, (\varsigma \cdot \sigma)\omega\rangle$, we have

$$L\langle \omega, \varsigma\omega\rangle = K\langle \omega, \varsigma\omega\rangle, \; L\langle \sigma\omega, (\varsigma \cdot \sigma)\omega\rangle = K\langle \sigma\omega, (\varsigma \cdot \sigma)\omega\rangle.$$

From definition 2.4, σ is a differential isomorphism that maps $K\langle \omega, \varsigma\omega\rangle$ onto $K\langle \sigma\omega, (\varsigma \cdot \sigma)\omega\rangle$. Hence, σ is also a differential isomorphism that maps $L\langle \omega, \varsigma\omega\rangle$ onto $L\langle \sigma\omega, (\varsigma \cdot \sigma)\omega\rangle$, with L elementwise fixed. From which we conclude that $\sigma \in \mathrm{Gal}(M/L, X)_{(0,0)}$ and therewith $L' \subseteq \mathrm{Gal}(M/L, X)_{(0,0)}$. The Lemma has been proved. $\square$

Lemma 2.12 *[13, Lemma 3.1] Let $M = K\langle \omega\rangle$, L and N be differential subfields of M containing K with $N \supset L$, $[N : L] = n$. Let L' and N' be the corresponding subgroups of* $\mathrm{Gal}(M/K, X)_{(0,0)}$*. Then the index of N' in L' is at most n.*

Lemma 2.13 *[13, Lemma 3.2] Let $M = K\langle \omega\rangle, G = \mathrm{Gal}(M/K, X)_{(0,0)}$ and H and J be subgroups of G with $H \supset J$ and J of index n in H. Let H' and J' be the corresponding intermediate differential fields. Then $[J' : H'] \leq n$.*

3. Structure of Differential Galois group

This section will study the structure of the differential Galois group $\mathrm{Gal}(M/K, X)_{(0,0)}$. First, we introduce the preliminary concepts in order to describe the structure of the differential Galois group.

3.1. Quasi-differential polynomial

Let y be an indeterminate over K, and denot by $\mathcal{A}_0(y)$ the ring

$$\mathcal{A}_0(y) = \{f(y) \mid f \in \mathcal{A}_0\}.$$

Adjoining $\mathcal{A}_0(y)$ to K results to a ring $K[\mathcal{A}_0(y)]$ with all elements of form

$$\sum_{i=1}^{n} a_i\, f_i(y),$$

where $a_i \in K, f_i \in \mathcal{A}_0$. The ring $K[\mathcal{A}_0(y)]$ can be extended to a differential ring through the derivatives δ_1 and δ_2 by the rules

$$\delta_i f(y) = f'(y)\delta_i y, \quad (i = 1, 2, f \in \mathcal{A}_0),$$

$$\delta_1(\delta_1^k \delta_2^l y) = \delta_1^{k+1}\delta_2^l y, \quad \delta_2(\delta_1^k \delta_2^l y) = \delta_1^k \delta_2^{l+1} y,$$

where $f' \in \mathcal{A}_0$ is a derivative of f. Hereinafter, we denote this ring as $K\{\mathcal{A}_0(y)\}$.

It is easy to know that all elements in $K\{\mathcal{A}_0(y)\}$ are polynomials in the derivatives $\delta_1^k \delta_2^l y \ \ (k, l \in \mathbb{N}_0, k + l > 0)$ with coefficients in $K[\mathcal{A}_0(y)]$. The elements in $K\{\mathcal{A}_0(y)\}$ differ from differential polynomials in the coefficients that contain not only the polynomials of y, but also the terms of form $f(y)$ with $f \in \mathcal{A}_0$. We will call such polynomials of the derivatives with coefficients in $K[\mathcal{A}_0(y)]$ *quasi-differential polynomials* (QDP, singular and plural). A *proper quasi-differential polynomial* (PQDP, singular and plural) is a QDP that involves at least one proper derivative of y. A *regular prime ideal* of $K\{\mathcal{A}_0(y)\}$ is a prime ideal $\Lambda \subset K\{\mathcal{A}_0(y)\}$ that contains exclusively PQDP. In this study, we will interest at the regular prime ideal $\Lambda \subset K\{\mathcal{A}_0(y)\}$ (see Theorem 3.7).

The terminologies and results for differential polynomials are applicable to PQDP. Let us recall some basic facts of differential polynomials. For detail, refer to [30].

Definition 3.1 *Let*

$$w_1 = \delta_1^{i_1}\delta_2^{i_2} y, \quad w_2 = \delta_1^{j_1}\delta_2^{j_2} y,$$

be proper derivatives of y, w_2 is higher then w_1 if $j_1 > i_1$ or $j_1 = i_1$ and $j_2 > i_2$. A proper derivative of y is always higher then y.

Definition 3.2 *Let A be a QDP, if A involves proper derivatives of y, by the leader of A, we mean the highest of those derivatives of y involved in A. If A involves y but no proper derivatives of y, then the leader of A is y. Let A_1 be a QDP, and A_2 be a PQDP, we say A_2 to be of higher rank than A_1, if either*

1. *A_2 has a higher leader than A_1; or*
2. *A_1 and A_2 have the same leader (which is a proper derivative of y), and the degree of A_2 in the leader exceeds that of A_1.*

Two QDP for which no difference in the rank as created above will be said to be of the same rank.

Definition 3.3 *Let A_1 be a PQDP, A_2 is said to be reduced with respect to A_1 if A_2 contains no proper derivative of the leader of A_1, and A_2 is either* *zero or of lower degree than A_1 in the leader of A_1. Consider a collection of PQDP*

$$\Sigma = \{A_1, A_2, \cdots, A_r\}, \tag{3.1}$$

if a QDP B is reduced with respect to all the $A_i, (i = 1, \cdots, r)$, then B is said to be reduced with respect to Σ.

Definition 3.4 *Let F be a PQDP with leader p, the QDP $\partial F/\partial p$ is said the separant of F. The coefficient of the highest power of p in F is said the initial of F.*

Lemma 3.5 *[30, pp.6] Let S_i and I_i be, respectively, the separant and initial of A_i in* (3.1)*, and F be a QDP. There exist nonnegative integers $s_i, t_i, i = 1, \cdots, r$, such that when a suitable linear combination of the A and their derivatives is subtracted from*

$$S_1^{s_1} \cdots S_r^{s_r} I_1^{t_1} \cdots I_r^{t_r} F,$$

the remainder is reduced with respect to (3.1)*.*

Let $\Lambda \in K\{A_0(y)\}$ be a regular prime idea, $X(y) = X_1\delta_1 y + X_2\,\delta_2 y \in \Lambda$, and $\{X(y)\}$ is the differential ring that generated by $X(y)$. Let $A(y) \in \Lambda$ with the lowest rank and irreducible. If $\Lambda \supsetneqq \{X(y)\}$, it is easy to see that $A(y)$ involves no $\delta_1 y$ and its derivatives. Let $\delta_2^r y$ be the leader of $A(y)$. Then $A(y)$ is a polynomial of the derivatives $\delta_2 y, \delta_2^2 y, \cdots, \delta_2^r y$, with coefficients $A_i(x_1, x_2, y) \in K[\mathcal{A}_0(y)]$. From Lemma 3.5, the regular prime idea Λ is the least regular prime idea containing $X(y)$ and $A(y)$. Thus, according to [30, pp. 4-5], the *characteristic set* of Λ consists of $A(y)$ and $X(y)$. We will see latter that the number r is important to determine the structure of Λ, and named as the *order* of Λ, denoted by $\mathrm{ord}(\Lambda) = r$. If $\Lambda = \{X(y)\}$, then the characteristic set of Λ contains only one element $X(y)$, and the order is said to be ∞.

3.2. Structure of Differential Galois Group

Lemma 3.6 *If there exists $A(y) \in K[\mathcal{A}_0(y)]$ $(A(y) \not\equiv 0)$, and $\omega \in \Omega^1_{(0,0)}(X)$, such that $A(\omega(x_1, x_2)) = 0$ for all (x_1, x_2) in a neighborhood of $(0, 0)$, then K contains a first integral of X.*

Proof. Hereinafter, we will write $A(\omega) \equiv 0$ in short for $A(\omega(x_1, x_2)) = 0$ for all (x_1, x_2) in a neighborhood of $(0, 0)$.

Let

$$\Sigma_0 = \{A(y) \in K[\mathcal{A}_0(y)] |\ A(\omega) \equiv 0, \quad A(y) \not\equiv 0\}.$$

Then $\Sigma_0 \neq \emptyset$. For any $A(y) \in \Sigma_0$, we can write $A(y)$ in the form as

$$A(y) = \sum_{k=1}^{n} \alpha_k f_k(y), \tag{3.2}$$

with $\alpha_k \in K,\ f_k(y) \in \mathcal{A}_0(y)$. It is not unique to express $A(y)$ with the form (3.2). Within all possible expressions, there is one with the shortest length n. We call this shortest length n the length of $A(y)$, and denote by $n(A)$.

Let $A(y)$ to be the element in Σ_0 with the smallest length. If $n(A) = 1$, then $A(y) = \alpha_1 f_1(y)$, and hence $f_1(\omega) \equiv 0$, i.e., ω is a constant. Therefore we have $n > 1$ since the first integral ω cannot be a constant.

When $n > 1$, write

$$A(y) = \alpha_1 f_1(y) + \alpha_2 f_2(y) + \cdots + \alpha_n f_n(y)$$

with $n = n(A)$, then

$$A(\omega) = \alpha_1 f_1(\omega) + \alpha_2 f_2(\omega) + \cdots + \alpha_n f_n(\omega) \equiv 0,$$

and hence

$$X(A(\omega)) = X(\alpha_1) f_1(\omega) + X(\alpha_2) f_2(\omega) + \cdots + X(\alpha_n) f_n(\omega) \equiv 0.$$

Therefore

$$\alpha_1 X(A(\omega)) - X(\alpha_1) A(\omega) = \sum_{i=2}^{n} (\alpha_1 X(\alpha_i) - X(\alpha_1)\alpha_i) f_i(\omega) \equiv 0. \tag{3.3}$$

If $\alpha_1 X(\alpha_i) - X(\alpha_1)\alpha_i \neq 0$ for some i, then (3.3) implies that

$$B(y) = \sum_{i=2}^{n} (\alpha_1 X(\alpha_i) - X(\alpha_1)\alpha_i) f_i(y)$$

is an element in Σ_0 with smaller length then $A(y)$, which is contradict. Thus, we have

$$\alpha_1 X(\alpha_i) - X(\alpha_1)\alpha_i = 0, \quad (i = 2, 3, \cdots, n). \tag{3.4}$$

Moreover, it is easy to see that α_2/α_1 is not a constant and $X(\alpha_2/\alpha_1) = 0$ by (3.4). Hence α_2/α_1 is a first integral of X contained in K, and the Lemma is proved. $\square$

From Lemma 3.6, if K contains no first integral of X, and there is a $A(y) \in K\{\mathcal{A}_0(y)\}$ such that $A(\omega) \equiv 0$ for some $\omega \in \Omega^1_{(0,0)}(X)$, then $A(y) \notin K[\mathcal{A}_0(y)]$, i.e., $A(y)$ is a PQDP that involves some proper derivatives of y.

Let Λ be e regular prime ideal of PQDP, we say a function $u(x_1, x_2; \epsilon)$ satisfies Λ if for any $F(x_1, x_2, y, \delta_1 y, \delta_2 y, \cdots) \in \Lambda$, while substitute y and the derivatives in F with u and the corresponding derivatives, the resulting expression is zero for all x_1, x_2 and ϵ that are small enough.

Theorem 3.7 *Let K and X be defined as previous. Assume that K contains no first integral of X. Let $M = K\langle\omega\rangle$ with $\omega \in \Omega^1_{(0,0)}(X)$. Then there exists a regular prime ideal Λ of PQDP such that:*

(1). For every $\sigma_f \in \mathrm{Gal}(M/K, X)_{(0,0)}$, let $\sigma_f\omega = f(\omega;\epsilon)$ $(f(z;\epsilon) \in \mathcal{G}[[\epsilon]])$, then $f(\omega(x_1, x_2);\epsilon)$ satisfies Λ.

(2). Given $f(z;\epsilon) \in \mathcal{G}[[\epsilon]]$ such that $f(\omega(x_1, x_2);\epsilon)$ satisfies Λ, there exists $\sigma_f \in \mathrm{Gal}(M/K, X)_{(0,0)}$ such that $\sigma_f(\omega) = f(\omega;\epsilon)$.

Proof. Let y be a differential indeterminate over K, define the natural homomorphism from $K\{\mathcal{A}_0(y)\}$ to $K\{\mathcal{A}_0(\omega)\}$ that maps $h(y)$ to $h(\omega)$ $(\forall h \in \mathcal{A}_0)$. Let Λ to be the kernel of the homomorphism, then Λ is a regular prime ideal of $K\{\mathcal{A}_0(y)\}$. We will prove that Λ fulfil the requirement of the Theorem.

(1). Let $\sigma_f \in \mathrm{Gal}(M/K, X)_{(0,0)}$ and $\sigma_f\omega = f(\omega;\epsilon)$. Then $f(z;\epsilon) \in \mathcal{G}[[\epsilon]]$. For any $F(y) \in \Lambda$, i.e., $F(\omega) \equiv 0$, there exist $h_i \in \mathcal{A}_0$ $(i = 1, \cdots, m)$, such that $F(y) \in K\{y, h_1(y), \cdots, h_m(y)\}$. Write $F(y)$ as a differential polynomial of $h_1(y), \cdots, h_m(y)$, i.e.,

$$F(y) = F(y, h_1(y), \cdots, h_m(y)),$$

then

$$F(\omega, h_1(\omega), \cdots, h_m(\omega)) \equiv 0.$$

Since $\sigma_f \in \mathrm{Gal}(M/K, X)_{(0,0)}$, σ_f can be extended to a differential isomorphism of $K\{\omega, h_1(\omega), \cdots, h_m(\omega)\}$ that maps ω and $h_i(\omega)$ to $f(\omega;\epsilon)$ and $h_i(f(\omega;\epsilon))$, respectively. Thus, we have

$$F(f(\omega;\epsilon), h_1(f(\omega;\epsilon)), \cdots, h_m(f(\omega;\epsilon))) \equiv 0.$$

The requirement (1) has been proved.

(2). Now, consider $f(z;\epsilon)$ in $\mathcal{G}[[\epsilon]]$ such that $f(\omega;\epsilon)$ satisfies Λ, we will show that $\sigma_f \in \mathrm{Gal}(M/K, X)$. For any $h_1(z;\epsilon), \cdots, h_m(z;\epsilon) \in \mathcal{A}_0[[\epsilon]]$, consider the maps

$$\begin{array}{rccc} \pi: & K\langle y, h_1(y;\epsilon)), \cdots, h_m(y;\epsilon)\rangle & \mapsto & K\langle \omega, h_1(\omega;\epsilon), \cdots, h_m(\omega;\epsilon)\rangle \\ & y & \mapsto & \omega \\ & h(y;\epsilon) & \mapsto & h(\omega;\epsilon) \\ & \delta_j y & \mapsto & \delta_j \omega \end{array}$$

and

$$\begin{array}{rccc} \pi_\sigma: & K\langle y, h_1(y;\epsilon), \cdots, h_m(y;\epsilon)\rangle & \mapsto & K\langle f(\omega;\epsilon), h_1(f(\omega;\epsilon);\epsilon), \cdots, h_m(f(\omega;\epsilon);\epsilon)\rangle \\ & y & \mapsto & f(\omega;\epsilon) \\ & h(y;\epsilon) & \mapsto & h(f(\omega;\epsilon);\epsilon) \\ & \delta_j y & \mapsto & \delta_j \omega \end{array}$$

where $h \in \mathcal{A}_0(y)$ and $j = 1, 2$. Let the kernels of π and π_σ be Γ and Γ_σ, respectively.

We will show that $\Gamma = \Gamma_\sigma$. Then $K\langle \omega, h_1(\omega;\epsilon), \cdots, h_m(\omega;\epsilon)\rangle$ is isomorphic to $K\langle f(\omega;\epsilon), h_1(f(\omega;\epsilon);\epsilon), \cdots, h_m(f(\omega;\epsilon);\epsilon)\rangle$ with the isomorphism $\sigma: \omega \mapsto f(\omega;\epsilon), h_i(\omega;\epsilon) \mapsto h_i(f(\omega;\epsilon);\epsilon)$. Therefore, σ is an admissible differential isomorphism, and the Theorem is proved.

First, we will prove $\Gamma \subseteq \Gamma_\sigma$. For any $F(y, h_1(y;\epsilon), \cdots, h_m(y;\epsilon)) \in \Gamma$, we write F in the form of the power series in ϵ

$$F(y, h_1(y;\epsilon), \cdots, h_m(y;\epsilon)) = \sum_{i=0}^{\infty} F_i(y, h_{i,1}(y), \cdots, h_{i,m_i}(y))\,\epsilon^i \quad (h_{i,j} \in \mathcal{A}_0)$$

where F_i are differential polynomials. Then

$$F(\omega, h_1(\omega;\epsilon), \cdots, h_m(\omega;\epsilon)) = \sum_{i=0}^{\infty} F_i(\omega, h_{i,1}(\omega), \cdots, h_{i,m_i}(\omega))\,\epsilon^i \equiv 0$$

i.e., $F_i(\omega, h_{i,1}(\omega), \cdots, h_{i,m_i}(\omega)) \equiv 0$ for all i. Thus, the coefficients F_i are contained in Λ. Now, assume that $f(\omega;\epsilon)$ satisfies Λ, then

$$F_i(f(\omega;\epsilon), h_{i,1}(f(\omega;\epsilon)), \cdots, h_{i,m_i}(f(\omega;\epsilon))) \equiv 0, \quad (\forall i).$$

Thus,

$$F(f(\omega;\epsilon), h_1(f(\omega;\epsilon)), \cdots, h_m(f(\omega;\epsilon))) \equiv 0,$$

and hence $F(y, h_1(y;\epsilon), \cdots, h_m(y;\epsilon)) \in \Gamma_\sigma$. Therefore, $\Gamma \subseteq \Gamma_\sigma$.

Next, we will show that $\Gamma_\sigma \subseteq \Gamma$. If on the contrary, there exists $F(y;\epsilon) \in \Gamma_\sigma$ but $F(y;\epsilon) \notin \Gamma$, then

$$F(f(\omega;\epsilon);\epsilon) = F(f(\omega;\epsilon), h(f(\omega;\epsilon);\epsilon), \cdots, h_m(f(\omega;\epsilon);\epsilon)) \equiv 0,$$

but

$$F(\omega;\epsilon) = F(\omega, h_1(\omega;\epsilon), \cdots, h_m(\omega;\epsilon)) \not\equiv 0.$$

Write $F(y;\epsilon)$ as a power series in ϵ

$$F(y;\epsilon) = \sum_{i=0}^{\infty} F_i(y)\,\epsilon^i, \quad (F_i(y) \in K\{\mathcal{A}_0(y)\}),$$

and let k the smallest index such that $F_i(y) \in \Lambda$ for any $0 \le i \le k-1$ and $F_k(y) \notin \Lambda$. From the assumption that $f(\omega;\epsilon)$ satisfies Λ, we have $F_i(f(\omega;\epsilon)) = 0$ for any $0 \le i \le k-1$. Thus, let $f_0(z) = f(z;0)$, we have

$$F(f(\omega;\epsilon);\epsilon) = \epsilon^k\,F_k(f(\omega;\epsilon)) + \sum_{i \ge k+1} F_i(f(\omega;\epsilon))\,\epsilon^i = F_k(f_0(\omega))\,\epsilon^k + h.o.t. \equiv 0$$

and therefore $F_k(f_0(\omega)) \equiv 0$.

Now, we obtain a $F_k(y) \notin \Lambda$, and $F_k(f_0(\omega)) \equiv 0$. Let $A(y)$ in Λ with the lowest rank, and hence $A(y)$ and $X(y)$ make up the characteristic set of Λ. Let $S(y)$ and $I(y)$ to be the separant and initial of $A(y)$, respectively (If $\Lambda = \{X(y)\}$, we take $S(y) = I(y) = X_2(x_1, x_2)$). It is clear that $S(y), I(y) \notin \Lambda$. Lemma 3.5 yields that there exist nonnegative integrals s, t, and $R(y) \in K\{\mathcal{A}_0(y)\}$ that is reduced with respect to Λ, such that

$$S(y)^s I(y)^t F_k(y) - R(y) \in \Lambda.$$

Since Λ is a prime ideal and $S(y), I(y), F_k(y) \notin \Lambda$, we have $S(y)^s I(y)^t F_k(y) \notin \Lambda$, and thus $R(y) \neq 0$. From the above discussion, $f_0(\omega)$ satisfies both Λ and $F_k(y)$, and hence $R(f_0(\omega)) = 0$, which implies $R(f_0(y)) \in \Lambda$. Whereas, simple computation shows that $R(f_0(y))$ has the same rank as $R(y)$, and therefore is reduced with respect to Λ, which is contradict. Hence we have proved $\Gamma_\sigma \subseteq \Gamma$.

Now, we have concluded $\Gamma = \Gamma_\sigma$, and the Theorem has been proved.
□

Remark 3.8 *Theorem 3.7 indicates that when X has no first integral in K, the regular prime ideal Λ is essential to determine the differential Galois group. Here we associate the order of the differential Galois group with that of Λ as following.*

1. *Recall the order of Λ that was defined in Section 3.1. If $A(y)$ and $X(y)$ make up the characteristic set of Λ, and the highest derivative of $A(y)$ is $\delta_2^r y$ ($0 < r < +\infty$), then $\mathrm{ord}(\Lambda) = r$. If $\Lambda = \{X(y)\}$, then $\mathrm{ord}(\Lambda) = \infty$. It is easy to obtain from Theorem 3.7 that if $\mathrm{ord}(\Lambda) = \infty$, then $\mathrm{Gal}(M/K, X)_{(0,0)} = \mathcal{G}[[\epsilon]]$.*
2. *If there exists $g \in \mathcal{A}_0^1$ such that $g(\omega) = u$ is contained in K, then $g(y) - u \in \Lambda$. In this case, we say the order of Λ is 0.*
3. *We will also define the order of the differential Galois group as the order of Λ.*

Theorem 3.9 *Let Λ the prime ideal in Theorem 3.7 and $r = \mathrm{ord}(\Lambda)$, then either $0 \leq r \leq 3$ or $r = \infty$. Moreover, let $G = \mathrm{Gal}(M/K, X)_{(0,0)}$, we have the following*

(1). If $r = 0$, then K contains a first integral of X, and

$$G = \{e\}. \tag{3.5}$$

(2). If $r = 1$, then there exists $\omega \in \Omega^1_{(0,0)}(X)$ such that $(\delta_2\omega)^n \in K$ for some $n \in \mathbb{N}$. Let $M = K\langle\omega\rangle$, then

$$G = \{f(z;\epsilon) \in \mathcal{G}[[\epsilon]] \;\Big|\; f(z;\epsilon) = \mu\, z + c(\epsilon),\quad c(0) = 0, \mu^n = 1\}. \tag{3.6}$$

(3). If $r = 2$, then there exist $\omega \in \Omega^1_{(0,0)}(X)$ such that $\delta_2^2\omega/\delta_2\omega \in K$. Let $M = K\langle\omega\rangle$, then

$$G = \{f(z;\epsilon) \in \mathcal{G}[[\epsilon]] \;\Big|\; f(z;\epsilon) = a(\epsilon)\, z + c(\epsilon),\quad c(0) = 0\}. \tag{3.7}$$

(4). If $r = 3$, then there exists $\omega \in \Omega^1_{(0,0)}(X)$ such that

$$\frac{\delta_2\omega \cdot \delta_3\omega - 3\,\delta_2^2\omega}{(\delta_2\omega)^2} \in K.$$

Let $M = K\langle\omega\rangle$, than

$$G = \{f(z;\epsilon) \in \mathcal{G}[[\epsilon]] \;\Big|\; f(z;\epsilon) = \frac{a(\epsilon)\, z}{1 + b(\epsilon)\, z} + c(\epsilon),\quad c(0) = 0\}. \tag{3.8}$$

(5). If $r = \infty$, then $G = \mathcal{G}[[\epsilon]]$.

Here $a(\epsilon), b(\epsilon), c(\epsilon) \in \mathbb{C}[[\epsilon]]$. In particular, if G is solvable, then $r \leq 2$.

We leave the proof of Theorem 3.9 to Section 5. From Theorem 3.9 and Lemma 2.9, the differential Galois group is independent to the choice of the point $(0, 0)$. Moreover, from Lemma 2.8, the structure of the group is also independent to the choice of the first integral ω. Hence, we can omit the point $(0, 0)$ and the particular extension M, and simply say $\mathrm{Gal}(M/K, X)$ the differential Galois group of X over K. This group is determined uniquely by the equation (1.1) or the operator X and will tell the insight of the integrability of the differential equation.

4. Liouvillian integrability of the polynomial system

We are now ready to prove the main theorem of this paper.

4.1. Preliminary results of Galois theory

Definition 4.1 *Let K be a (differential) field, M be an extension field of K, G be a set of isomorphisms of M, with K elementwise fixed. M is normal over K with respect to G if there is no element in $M \backslash K$ that is fixed by all elements in G.*

Obviously, we have

Lemma 4.2 *Let $G = \mathrm{Gal}(M/K, X)_{(0,0)}$ and H be a subgroup of G. Let*

$$H' = \{a \in M |\ \sigma a = a, \forall \sigma \in H\},$$

then M is normal over H' with respect to H.

Lemma 4.3 *If K contains no first integral of X, then for any $\omega \in \Omega^1_{(0,0)}(X)$, $M = K\langle\omega\rangle$ is normal over K with respect to $\mathrm{Gal}(M/K, X)_{(0,0)}$.*

Proof. We only need to prove that for any $\alpha \in M\backslash K$, there exists $\sigma \in \mathrm{Gal}(M/K, X)_{(0,0)}$, such that $\sigma\alpha \neq \alpha$.

Let $\alpha = p(\omega)/q(\omega)$, with $p(y), q(y) \in K\{\mathcal{A}_0(y)\}$. Without loss of generality, we assume further that $p(y), q(y)$ are reduced with respect to the prime ideal Λ given by Theorem 3.7. Therefore, $p(\omega) \neq 0$ and $q(\omega) \neq 0$. Write $p(y)$ and $q(y)$ explicitly as

$$p(y) = p(\mathbf{x}, y, \delta_2 y, \cdots, \delta_2^{r-1} y), \quad q(y) = q(\mathbf{x}, y, \delta_2 y, \cdots, \delta_2^{r-1} y)$$

where $\mathbf{x} = (x_1, x_2)$, $r = \mathrm{ord}(\Lambda)$, and let

$$A(\mathbf{x}, y) = p(\mathbf{x}, y, \delta_2 y, \cdots, \delta_2^{r-1} y) - \alpha(\mathbf{x})\, q(x_1, x_2, y, \delta_2 y, \cdots, \delta_2^{r-1} y).$$

Since $q(\omega) \not\equiv 0$, there exists $\mathbf{x}^0 = (x_1^0, x_2^0)$ such that $\omega(\mathbf{x})$ is analytic at $\mathbf{x}^0$, and $q(\omega(\mathbf{x}^0)) \neq 0$. Let $\mathbf{c} = (\omega(\mathbf{x}^0), \cdots, \delta_2^{r-1}\omega(\mathbf{x}^0))$ and $\mathbf{x}^*$ in a neighborhood U of $\mathbf{x}^0$ such that

$$q(\mathbf{x}^*, \mathbf{c}) \neq 0 \quad \text{and} \quad A(\mathbf{x}^*, \mathbf{c}) \neq 0.$$

We claim that such $\mathbf{x}^*$ always exists. If on the contrary, we have

$$A(\mathbf{x}, \mathbf{c}) = p(\mathbf{x}, \mathbf{c}) - \alpha(\mathbf{x})\, q(\mathbf{x}, \mathbf{c}) = 0$$

for any $\mathbf{x} \in U$ such that

$$q(\mathbf{x}, \mathbf{c}) \neq 0,$$

then

$$\alpha(\mathbf{x}) = \frac{p(\mathbf{x}, \mathbf{c})}{q(\mathbf{x}, \mathbf{c})} \in K,$$

which is contradict to the fact that $\alpha \in M \backslash K$.

Let $\mathbf{c}^* = (\omega(\mathbf{x}^*), \cdots, \delta_2^{r-1}\omega(\mathbf{x}^*))$, and ϵ to be an infinitesimal parameter, then

$$q(\mathbf{x}^*, \mathbf{c}^* + \epsilon\,(\mathbf{c} - \mathbf{c}^*)) \neq 0$$

and

$$A(\mathbf{x}^*, \mathbf{c}^* + \epsilon\,(\mathbf{c} - \mathbf{c}^*)) \neq 0.$$

And hence

$$\frac{p(x_1^*, x_2^*, \mathbf{c}^* + \epsilon\,(\mathbf{c} - \mathbf{c}^*))}{q(x_1^*, x_2^*, \mathbf{c}^* + \epsilon\,(\mathbf{c} - \mathbf{c}^*))} \neq \alpha(\mathbf{x}^*).$$

By Theorem 3.9, it is easy to verify that when the order $r \neq 0$, there exists $\sigma \in \mathrm{Gal}(M/K, X)_{(0,0)}$ such that

$$(\delta_i \sigma \omega)(\mathbf{x}^*) = c_i^* + (c_i - c_i^*)\epsilon \quad (i = 0, 1, \cdots, r-1).$$

Therefore

$$(\sigma\alpha)(\mathbf{x}^*) = \frac{p(\mathbf{x}^*, \mathbf{c}^* + \epsilon\,(\mathbf{c} - \mathbf{c}^*))}{q(\mathbf{x}^*, \mathbf{c}^* + \epsilon\,(\mathbf{c} - \mathbf{c}^*))} \neq \alpha(\mathbf{x}^*),$$

i.e., $\sigma\alpha \neq \alpha$. The Lemma has been proved. $\square$

In following Lemmas, we let L, N, M be the extension fields of K, with $M = K\langle\omega\rangle$, $K \subset L \subset N \subset M$, and $G = \mathrm{Gal}(M/K, X)_{(0,0)}$. Assume that $N = L\langle u\rangle$ with u satisfying $\delta_i u \in L$ or $\delta_i u/u \in L$, $(i = 1, 2)$. Then L' and N' are subgroups of G, and N'' is a subfield of M.

Lemma 4.4 *Any $\sigma \in L'$ maps N to N''.*

Proof. We have $\sigma a = a$ for any σ in L' and a in L. At first, assume that $N = \langle u\rangle$. If $\delta_i u = a_i \in L, (i = 1, 2)$, then $\delta_i(\sigma u) = a_i$, $(i = 1, 2)$. Thus $\sigma u = u + c(\epsilon)$ with $c(\epsilon) \in \mathbb{C}[\epsilon]$, and therewith $\sigma u \in N''$, which implies $\sigma N \subseteq N''$.

The proof for the case $\delta_i u/u \in K$ is similar by the fact that $\sigma u = c(\epsilon)\, u$ with $c(\epsilon) \in \mathbb{C}[[\epsilon]]$ for any σ in L'. $\square$

Lemma 4.5 *N' is a normal subgroup of L', and L'/N' is Abelian.*

Proof. Let $\sigma \in L', \tau \in N'$, then $\sigma a \in N''$ for any $a \in N$, and therefore $\tau(\sigma a) = \sigma a$. Thus $(\sigma^{-1} \cdot \tau \cdot \sigma)a = \sigma^{-1}(\sigma a) = a$, and hence $\sigma^{-1} \cdot \tau \cdot \sigma \in N'$. This implies that N' is a normal subgroup of L'.

Assume that $N = L\langle u\rangle$ with $\delta_i u \in L$ $(i = 1, 2)$. From the proof of Lemma 4.4, for any $\sigma \in L'$, we have $\sigma u = u + c(\epsilon)$ for some $c(\epsilon) \in \mathbb{C}[[\epsilon]]$. Thus, the subgroup $H = \{\sigma|_N \mid \sigma \in L'\}$ is isomorphic to a subgroup of the addition group $\mathbb{C}[[\epsilon]]$ and hence is Abelian. Consider the homomorphism from L' to H that maps σ to $\sigma|_N$. The kernel of the map is N', and the image is H. Thus, L'/N' is isomorphic to H and is Abelian.

The case that $N = L\langle u\rangle$ with $\delta_i u/u \in L$ $(i = 1, 2)$ can be proved similarly. $\square$

From Lemma 2.12 and 4.5, we have:

Lemma 4.6 *Let L, N, M be the extension fields of K, with $M = K\langle\omega\rangle, N = L\langle u\rangle$, and $K \subset L \subset N \subset M$. Assume further that K, L, N contain no first integral of X. We have*

(1). if u is algebraic over L, then $|L'/N'| \leq [N : L]$; and

(2). if $\delta_i u \in L$ or $\delta_i u/u \in L$ $(i = 1, 2)$, then N' is a normal subgroup of L', and L'/N' is Abelian.

Lemma 4.7 *Assume that $M = K\langle\omega\rangle$ is normal over L with respect to G. If for every $\sigma \in G$, there exist $c(\epsilon) \in \mathbb{C}[[\epsilon]]$, such that*

$$\sigma\omega = \omega + c(\epsilon),$$

then M is a Liouvillian extension of L.

Proof. For any $\sigma \in G$, we have

$$\sigma(\delta_i\omega) = \delta_i\omega, \quad (i = 1, 2).$$

Since M is normal over L with respect to G, we have $\delta_i\omega \in L, (i = 1, 2)$, and M is a Liouvillian extension of L. □

Lemma 4.8 *Assume that $M = K\langle\omega\rangle$ is normal over K with respect to G. If for every $\sigma \in G$, there exist $a(\epsilon), c(\epsilon) \in \mathbb{C}[[\epsilon]]$, such that*

$$\sigma\omega = a(\epsilon)\omega + c(\epsilon),$$

then M is a Liouvillian extension of K.

Proof. For any $\sigma \in G$, we have

$$\sigma(\delta_i^2\omega/\delta_i\omega) = \delta_i^2\omega/\delta_i\omega, \quad (i = 1, 2).$$

Since M is normal over K with respect to G, there exist $a_i \in K$, such that

$$\delta_i^2\omega = a_i\delta_i\omega, \quad (i = 1, 2).$$

Taking account that $X_1\delta_1\omega + X_2\delta_2\omega = 0$, there exists $\mu \in M$ such that $\delta_1\omega = \mu X_2, \delta_2\omega = -\mu X_1$, and hence

$$\frac{\delta_1\mu}{\mu} = a_1 - \frac{\delta_1 X_2}{X_2} \in K, \quad \frac{\delta_2\mu}{\mu} = a_2 - \frac{\delta_2 X_1}{X_1} \in K.$$

Thus M is a Liouvillian extension of K, and

$$K \subseteq K\langle\mu\rangle \subseteq K\langle\omega\rangle = M.$$

□

4.2. Proof of the Main Theorem

Proof of Theorem 1.5. (1). Let G to be the differential Galois group of (1.1) over K at $(0,0)$. If K contains a first integral of X, then G contains exclusively the identity mapping, and is solvable.

Now, we assume that K contains no first integral of X, and X is Liouvillian integrable. Then there exists $\omega \in \Omega^1_{(0,0)}(X)$ such that $M = K\langle\omega\rangle$ is a Liouvillian extension of K. From definition 1.2, suppose that:

$$K = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_m = M,$$

with $K_{i+1} = K_i\langle u_i\rangle$, where either u_i is algebraic over K_i or $\delta_j u_i \in K_i$ or $\delta_j u_i/u_i \in K_i$ $(j = 1,2)$. Let $G_0 = \mathrm{Gal}(M/K, X)(= K'), G_i = K_i'$, $(i = 1, 2, \cdots, m)$, then

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_m = \{e\}.$$

By Lemma 4.6, either $|G_i/G_{i+1}| \leq [K_{i+1} : K_i] < \infty$ or G_{i+1} is a normal subgroup of G_i, and G_i/G_{i+1} is Abelian. Thus, G is solvable by definition 1.4.

(2). If the differential Galois group of (1.1) over K at $(0,0)$ is solvable, by Theorem 3.9, either K contains a first integral of X, or the Galois group has order $r = 1$ or $r = 2$.

By Theorem 3.9, if $r = 1$, then there exists $\omega \in \Omega^1_{(0,0)}(X)$ and $n \in \mathbb{N}$ such that

$$G = \{f(z;\epsilon) \in \mathcal{G}[[\epsilon]] \mid f(z;\epsilon) = \mu\, z + c(\epsilon),\ c(\epsilon) \in \mathbb{C}[[\epsilon]], c(0) = 0,\ \mu^n = 1\},$$

where $M = K\langle\omega\rangle$ and $G = \mathrm{Gal}(M/K, X)_{(0,0)}$. Let

$$G_0 = \{\sigma \in G |\ \sigma\omega = \omega + c(\epsilon),\ c(\epsilon) \in \mathbb{C}[[\epsilon]], c(0) = 0\},$$

then G_0 is a subgroup of G, and $|G/G_0| = n$. By Lemma 4.3, $K = G'$. Hence, according to Lemma 2.13,

$$[G_0' : K] = [G_0' : G'] \leq [G/G_0] = n,$$

which means that G_0' is an algebraic extension of K. By Lemma 4.2, M is normal over G_0' with respect to G_0. Hence Lemma 4.7 is applicable and yields that M is a Liouvillian extension of G_0', and consequently a Liouvillian extension of K.

If $r = 2$, then there exists a first integral $\omega \in \Omega^1_{(0,0)}$ such that

$$G = \{f(z;\epsilon) \in \mathcal{G}[[\epsilon]] \mid f(z;\epsilon) = a(\epsilon)\, z + c(\epsilon),\ c(0) = 0\},$$

where $M = K\langle\omega\rangle$ and $G = \mathrm{Gal}(M/K, X)_{(0,0)}$ as in previous. Again, by Lemma 4.3, $K = G'$. Hence Lemma 4.8 is applicable, and M is a Liouvillian extension of K. The Theorem has been proved. $\square$

4.3. Applications

From the proof of Lemmas 5.1-5.3 in Section 5, the explicit method to determine the differential Galois group can be given as follows.

Theorem 4.9 *Consider the differential equation* (1.1), *let*

$$B_i = -X_1 \delta_2^{i+1}(\frac{X_2}{X_1}), \quad i = 0, 1, 2 \tag{4.1}$$

and r to be the order of the corresponding differential Galois group, then

(1). $r = 0$ if and only if K contains a first integral of X;

(2). $r = 1$ if and only if K contains no first integral of X, and there exists $a \in K, a \neq 0$, and $n \in \mathbb{N}$, such that

$$X(a) = n\, B_0\, a. \tag{4.2}$$

(3). $r = 2$ is and only if (4.2) *is not satisfied by all $a \in K$ and $n \in \mathbb{N}$, and there exists $a \in K$, such that*

$$X(a) = B_0\, a + B_1. \tag{4.3}$$

(4). $r = 3$ if and only if (4.3) *is not satisfied by all $a \in K$, and there exists $a \in K$, such that*

$$X(a) = 2\, B_0\, a + B_2. \tag{4.4}$$

(5). $r = \infty$ if and only if (4.4) *is not satisfied by all $a \in K$.*

It is easy to see that the equation

$$\frac{dy}{dx} = p(x) \tag{4.5}$$

with $p(x)$ a polynomial has the order of the differential Galois group $r = 0$. The general homogenous linear equation

$$\frac{dy}{dx} = p(x)y \tag{4.6}$$

has the order $r = 1$, and the general nonhomoegenous linear equation

$$\frac{dy}{dx} = p(x)y + q(x) \tag{4.7}$$

has the order $r = 2$. Following result shows that the general Riccati equation is an example with order $r = 3$.

Theorem 4.10 *The differential Galois group of the general Riccati equation*

$$\frac{dy}{dx} = p_2(x)\, y^2 + p_1(x)\, y + p_0(x) \tag{4.8}$$

has order $r = 3$.

Proof. We have known that the general Riccati equation (4.8) does not have Liouvillian first intergral[22], and hence the Theorems 1.5 and 3.9 indicate that either the order $r = 3$ or $r = \infty$.

From the equation (4.8), we have $X_1 = 1$ and $X_2 = p_2(x)\, y^2 + p_1(x)\, y + p_0(x)$ and $\delta_2 = \partial/\partial y$, and $B_2 = 0$ from (4.1). Thus, the equation (4.4) has solution $a = 0$, and the order is 3. □

Next, we will show an example with the order of the differential Galois group to be infinity.

Consider the van der Pol equation

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2 - \mu(\frac{x_1^3}{3} - x_1), \\ \dot{x}_2 & = & -x_1 \end{array}\right. \qquad (\mu \neq 0). \tag{4.9}$$

The van der Pol equation is well known for the existence of a limit cycle. Following Lemma was proved independently by Cheng et al.[7] and Odani[28], respectively, at almost the same time.

Lemma 4.11 *[7, 28] The system of the van der Pol equation* (4.9) *has no algebraic solution curves. In particular, the limit cycle of it is not algebraic.*

Theorem 4.12 *The order of the differential Galois group of the van der Pol equation* (4.9) *is infinity.*

Proof. Let

$$X_1(x_1, x_2) = x_2 - \mu(\frac{x_1^3}{3} - x_1), \quad X_2(x_1, x_2) = -x_1,$$

the equation (4.4) for the van der Pol equation (4.9) becomes

$$X_1^3\, X(a) + 2x_1 X_1^2\, a + 6x_1 = 0. \tag{4.10}$$

We will only need to prove that (4.10) has no rational function solution.

If (4.10) has a rational function solution $a = a_1/a_2$, where a_1, a_2 are relatively prime polynomials, then a_1 and a_2 satisfy

$$X_1^3\,(a_2\, X(a_1) - a_1\, X(a_2)) - 2x_1\, X_1^2\, a_1\, a_2 + 6\, x_1\, a_2^2 = 0.$$

Hence, there exist a polynomial $c(x_1, x_2)$, such that

$$\begin{aligned} X_1^3\, X(a_2) &= c\, a_2, & (4.11)\\ X_1^3\, X(a_2) &= (c - 2\, x_1\, X_1^2)\, a_1 - 6x_1\, a_2. & (4.12) \end{aligned}$$

Let $a_2 = X_1^k\, b_2$, where $k \geq 0$, b_2 is a nonzero polynomial and $(b_2, X_1) = 1$. Substitute a_2 into (4.11) yields

$$X_1^3\, X(b_2) + k\, X_1^2\, X(X_1)\, b_2 = c\, b_2.$$

Thus, $b_2 | X_1^3\, X(b_2)$ and therewith $b_2 | X(b_2)$, i.e., either b_2 is a constant or $b_2(x_1, x_2) = 0$ is an algebraic solution curve of (4.9). However, Lemma 4.11 has shown that (4.9) has no algebraic solution curves. Thus b_2 must be a constant. Let $b_2 = 1$ without loss of generality , we have

$$a_2 = X_1^k, \quad c = k\, X_1^2\, X(X_1). \tag{4.13}$$

Substitute (4.13) into (4.12) yields

$$X_1^3\, X(a_1) = (k\, X_1^2\, X(X_1) - 2\, x_1\, X_1^2) a_1 - 6\, x\, X_1^k, \quad (k \geq 0). \tag{4.14}$$

Note that

$$(k\, X(X_1) - 2x_1) = -k\, \mu\, (x_1^2 - 1)\, X_1 - (k + 2)\, x_1,$$

(4.14) becomes

$$X_1^3\,X(a_1) = X_1^2\,(-k\,\mu\,(x_1^2-1)\,X_1 - (k+2)\,x_1)\,a_1 - 6\,x_1\,X_1^k.$$

If $k \neq 2$, then either $X_1|(k+2)\,x_1$ for $k > 2$ or $X_1|6\,x_1$ for $k < 2$, which are impossible. Hence, we should have $k = 2$.

Let $k = 2$, (4.14) becomes

$$\begin{aligned} & (x_2 - \mu\,(\tfrac{x_1^3}{3} - x_1))\,((x_2 - \mu\,(\tfrac{x_1^3}{3} - x_1))\,\frac{\partial a_1}{\partial x_1} - x_1\,\frac{\partial a_1}{\partial x_2}) \\ = \;& (-2\,\mu\,(x_1^2-1)\,(x_2 - \mu(\tfrac{x_1^3}{3} - x_1)) - 4\,x_1)\,a_1 - 6\,x_1. \end{aligned} \tag{4.15}$$

Let

$$a_1(x_1,x_2) = \sum_{i=0}^{m} h_i(x_2)\,x_1^i,$$

where $h_i(x_2)$ are polynomials and $h_m(x_2) \neq 0$. Substitute $a_1(x_1,x_2)$ into (4.15), and comparing the coefficient of x^{m+5}, we have

$$\frac{1}{9}\,\mu^2\,m\,h_m(x_2) = \frac{2}{3}\,\mu^2\,h_m(x_2),$$

which implies $m = 6$. Comparing the coefficients of x_1^i $(0 \leq i \leq 10)$, we obtain the equations that are satisfied by $h_i(x_2), i = 0, \cdots, 6$:

$$\begin{aligned}
0 &= x_2\,(-2\,\mu\,h_0(x_2) + x_2\,h_1(x_2)) \\
0 &= 6 - 2\,(-2+\mu^2)\,h_0(x_2) + 2\,x_2^2\,h_2(x_2) - x_2\,h_0'(x_2) \\
0 &= 2\,\mu\,x_2\,h_0(x_2) - (-4+\mu^2)\,h_1(x_2) + 2\,\mu\,x_2\,h_2(x_2) + 3\,x_2^2\,h_3(x_2) \\
&\quad - \mu\,h_0'(x_2) - x_2\,h_1'(x_2) \\
0 &= \frac{8\,\mu^2}{3}\,h_0(x_2) + \frac{4\,\mu\,x_2}{3}\,h_1(x_2) + 4\,h_2(x_2) + 4\,\mu\,x_2\,h_3(x_2) + 4\,x_2^2\,h_4(x_2) \\
&\quad - \mu\,h_1'(x_2) - x_2\,h_2'(x_2) \\
0 &= 2\,\mu^2\,h_1(x_2) + \frac{2\,\mu\,x_2}{3}\,h_2(x_2) + 4\,h_3(x_2) + \mu^2\,h_3(x_2) + 6\,\mu\,x_2\,h_4(x_2) \\
&\quad + 5\,x_2^2\,h_5(x_2) + \frac{\mu}{3}\,h_0'(x_2) - \mu\,h_2'(x_2) - x_2\,h_3'(x_2) \\
0 &= \frac{1}{3}(-2\,\mu^2\,h_0(x_2) + 4\,\mu^2\,h_2(x_2) + 12\,h_4(x_2) + 6\,\mu^2\,h_4(x_2) + 24\,\mu\,x_2\,h_5(x_2) \\
&\quad + 18\,x_2^2\,h_6(x_2) + \mu\,h_1'(x_2) - 3\,\mu\,h_3'(x_2) - 3\,x_2\,h_4'(x_2))
\end{aligned}$$

$$
\begin{aligned}
0 &= \frac{1}{9}\,(-5\,\mu^2\,h_1(x_2)+6\,\mu^2\,h_3(x_2)-6\,\mu\,x_2\,h_4(x_2)+36\,h_5(x_2)+27\,\mu^2\,h_5(x_2)\\
&\quad +90\,\mu x_2\,h_6(x_2)+3\,\mu\,h_2'(x_2)-9\,\mu\,h_4'(x_2)-9\,x_2\,h_5'(x_2))\\
0 &= -\frac{4\,\mu^2}{9}\,h_2(x_2)-\frac{4\,\mu\,x_2}{3}\,h_5(x_2)+4\,h_6(x_2)+4\,\mu^2\,h_6(x_2)+\frac{\mu}{3}\,h_3'(x_2)\\
&\quad -\mu\,h_5'(x_2)-x_2\,h_6'(x_2)\\
0 &= -\frac{\mu}{3}\,(\mu\,h_3(x_2)+2\,\mu\,h_5(x_2)+6\,x_2\,h_6(x_2)-h_4'(x_2)+3\,h_6'(x_2))\\
0 &= -\frac{\mu}{9}\,(2\,\mu\,h_4(x_2)+12\,\mu\,h_6(x_2)-3\,h_5'(x_2))\\
0 &= -\frac{\mu}{9}\,(\mu\,h_5(x_2)-3\,h_6'(x_2))
\end{aligned}
$$

The above equations can be reduced to

$$x_2\,(3\,x_2\,h_5'(x_2)-2\,\mu\,h_4'(x_2))=2\,\mu^3,$$

which is impossible since $h_4(x_2)$ and $h_5(x_2)$ are polynomials. The contradiction concludes that (4.10) has no rational function solution, and hence the order of the differential Galois group of the van der Pol equation is infinity. $\square$

5. Proof of Theorem 3.9

Before proving Theorem 3.9, we introduce some notations as following. Let $\delta_1=\frac{\partial}{\partial x_1}$, $\delta_2=\frac{\partial}{\partial x_2}$, y be an indeterminate over K, and denote $\delta_2^i y$ by y_i ($y_0=y$). Let

$$X=X_1\,\delta_1+X_2\,\delta_2,\qquad \delta_2X=(\delta_2X_1)\,\delta_1+(\delta_2X_2)\,\delta_2,$$

$$\hat{X}=X_1\delta_1+X_2\delta_2+\sum_{i\geq 0}X(y_i)\frac{\partial}{\partial y_i},$$

$$B_0=-X_1\,\delta_2(\frac{X_2}{X_1}),\quad B_i=X_1\,\delta_2(\frac{B_{i-1}}{X_1})=-X_1\,\delta_2^{i+1}(\frac{X_2}{X_1}),\quad i=1,2,\cdots$$

For $F(y)\in K\{\mathcal{A}_0(y)\}$, and $\{X(y)\}$ be the differential ideal that is generated by $X(y)$, we write

$$F(y)\sim R(y)\tag{5.1}$$

if $R(y)\in K\{\mathcal{A}_0(y)\}$ such that $F(y)-R(y)$ is contained in $\{X(y)\}$.

Let $\mathbf{n} = (n_1, n_2, \cdots, n_r) \in \mathbb{Z}^{*r}$, define the operators d_i^j and b_i^j for $1 < i < j$ by

$$d_i^j(\mathbf{n}) = (n_1, \cdots, n_{j-i} + 1, \cdots, n_j - 1, \cdots, n_r) \tag{5.2}$$

and

$$b_i^j(\mathbf{n}) = (n_1, \cdots, n_{j-i} - 1, \cdots, n_j + 1, \cdots, n_r), \tag{5.3}$$

respectively. Then $b_i^j(d_i^j(\mathbf{n})) = \mathbf{n}$.

Let $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^{*r}$, the *degree* of $\mathbf{n}$ is higher than that of $\mathbf{m}$, denoted by $\mathbf{n} > \mathbf{m}$, if there exists $1 \le k \le r$ such that $n_k > m_k$ and

$$n_i = m_i, \quad i = k+1, \cdots, r.$$

We say $\mathbf{n} \succ \mathbf{m}$ if there exist $1 < i < j$ such that

$$d_i^j(\mathbf{n}) = \mathbf{m}.$$

It is obvious that when $1 > i > j$,

$$b_i^j(\mathbf{n}) > \mathbf{n} > d_i^j(\mathbf{n}) \tag{5.4}$$

and

$$d_i^j(\mathbf{n}) \succ \mathbf{n} \succ b_i^j(\mathbf{n}). \tag{5.5}$$

Let Λ be the regular prime ideal of QDP corresponding to the differential Galois group in Theorem 3.7, and assume that $\mathrm{ord}(\Lambda) = r$. Let A in Λ with the lowest rank and therefore irreducible. Denoted A as

$$A(x_1, x_2, y, y_1, \cdots, y_r) = \sum_{\mathbf{m}} A_{\mathbf{m}}(x_1, x_2, y) y_1^{m_1} \cdots y_r^{m_r}$$

and let

$$\mathcal{I}_A = \{\mathbf{m} \in \mathbb{Z}^{*r} | \ \ A_{\mathbf{m}} \neq 0\}.$$

By $\mathbf{n}$ we will always denote the element in $\mathcal{I}_A$ with the highest degree. We assume further that that $A_{\mathbf{n}} = 1$ and the coefficients $A_{\mathbf{m}}$ are rational functions in $K(\mathcal{A}_0(y))$. For any $\mathbf{m} \in \mathcal{I}_A$, let

$$\mathcal{P}(\mathbf{m}) = \{\mathbf{p} \in \mathcal{I}_A \mid d_i^j(\mathbf{p}) \succ \mathbf{m} \text{ for some } 1 < i < j\} \tag{5.6}$$

and $\#(\mathbf{m}) = |\mathcal{P}(\mathbf{m})|$. A subset $\mathcal{J}_A \subseteq \mathcal{I}_A$ is *closed* if for every $\mathbf{m} \in \mathcal{J}_A$, $\mathbf{p} \succ \mathbf{m}$ implies $\mathbf{p} \in \mathcal{J}_A$. It is easy to see that $\mathcal{I}_A$ and $\{\mathbf{n}\}$ are closed.

The proof will be done by showing that all possible structures of $\mathcal{I}_A$ are, besides the cases with $r = 0$ and $r = \infty$,

1. $r = 1$, and $\mathcal{I}_A = \{n, 0\}$; or
2. $r = 2$, and $\mathcal{I}_A = \{(0,1), (1,0)\}$, with

$$(0,1) \succ (1,0);$$

   or
3. $r = 3$, and $\mathcal{I}_A = \{(1,0,1), (0,2,0), (2,0,0)\}$, with relations

$$\begin{array}{cc} & (1{,}0{,}1) \\ & \curlyvee \\ \curlyvee & (1{,}1{,}0) \prec (0{,}2{,}0) \\ & \curlyvee \\ & (2{,}0{,}0) \end{array}$$

   Here $(1,1,0)$ is an auxiliary index with $A_{(1,1,0)} = 0$.

The proof will be complete following the flow chart in Figure 1.

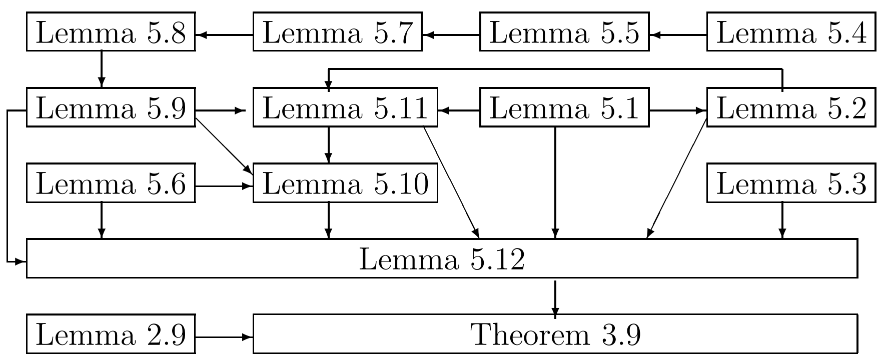

Figure 1: Flow chart of the proof of Theorem 3.9

Lemma 5.1 *If $u \neq 0$ satisfies*

$$X\,u = B_0\,u \tag{5.7}$$

then there exists a first integral ω of X such that

$$\delta_2 \omega = u.$$

Proof. From (5.7), we have

$$\begin{aligned}
X_1\,\delta_1 u + X_2\,\delta_2 u &= B_0\,u = -X_1\,\delta_2(\frac{X_2}{X_1})\,u \\
\delta_1\,u + \frac{X_2}{X_1}\,\delta_2 u &= -\delta_2(\frac{X_2}{X_1}\,u) \\
\delta_1 u &= -\delta_2(\frac{X_2}{X_1}\,u - \frac{X_2}{X_1}\,\delta_2 u) \\
&= \delta_2(-\frac{X_2}{X_1}\,u)
\end{aligned}$$

Let $v = -\frac{X_2}{X_1}\,u$, then the 1-form $vdx_1 + udx_2$ is closed, and

$$\omega(x_1,x_2) = \int_{(0,0)}^{(x_1,x_2)} vdx_1 + udx_2$$

is a first integral of X, with $\delta_2\omega = u$. $\square$

Lemma 5.2 *If there exists u satisfying*

$$Xu = B_0\,u + B_1 \tag{5.8}$$

then X has a first integral ω such that

$$\frac{\delta_2^2\omega}{\delta_2\omega} = u.$$

Proof. From (5.8), we have

$$\begin{aligned}
X_1\,\delta_1 u + X_2\,\delta_2 u &= -X_1\,\delta_2(\frac{X_2}{X_1})\,u + X_1\,\delta_2(\frac{B_0}{X_1}) \\
\delta_1 u &= -\frac{X_2}{X_1}\,\delta_2 u - \delta_2(\frac{X_2}{X_1})\,u + \delta_2(\frac{B_0}{X_1}) \\
&= \delta_2(-\frac{X_2}{X_1}\,u + \frac{B_0}{X_1}).
\end{aligned}$$

Thus, let

$$v = -\frac{X_2}{X_1}\,u + \frac{B_0}{X_1},$$

the 1-form $v\,dx_1 + u\,dx_2$ is a closed. Let

$$\eta(x_1, x_2) = \exp\left[\int_{(0,0)}^{(x_1,x_2)} v\,dx_1 + u\,dx_2\right],$$

then

$$X(\eta) = \eta\,(X_1 v + X_2 u) = \eta\,(X_1(-\frac{X_2}{X_1}\,u + \frac{B_0}{X_1}) + X_2 u) = B_0\,\eta.$$

From Lemma 5.1, there exists a first integral ω of X such that

$$\delta_2\omega = \eta,$$

and therewith

$$\frac{\delta_2^2\omega}{\delta_2\omega} = u.$$

The Lemma is concluded. $\square$

Lemma 5.3 *If there exists u satisfying*

$$Xu = 2\,B_0\,u + B_2, \tag{5.9}$$

then X has a first integral ω of X such that

$$\frac{2\,\delta_2\omega\cdot\delta_2^3\omega - 3(\delta_2^2\omega)^2}{(\delta_2\omega)^2} = u.$$

Proof. From (5.9), we have

$$\begin{aligned}
X_1\,\delta_1 u + X_2\,\delta_2 u &= -2\,X_1\,\delta_2(\frac{X_2}{X_1})u - X_1\,\delta_2^3(\frac{X_2}{X_1}),\\
\delta_1 u + \frac{X_2}{X_1}\delta_2 u &= -2\,\delta_2(\frac{X_2}{X_1})u - \delta_2^3(\frac{X_2}{X_1})\\
\delta_1 u &= -2\,\delta_2(\frac{X_2}{X_1})\,u - \frac{X_2}{X_1}\,\delta_2 u - \delta_2^3(\frac{X_2}{X_1}),\\
&= -\delta_2(\frac{X_2}{X_1}\,u) - \frac{X_2}{X_1}\,\delta_2 u - \delta_2^3(\frac{X_2}{X_1}).
\end{aligned}$$

Consider the partial differential equations:

$$\left\{\begin{aligned}
\delta_2 w &= u + \tfrac{1}{2}\,w^2\\
\delta_1 w &= -\delta_2^2(\tfrac{X_2}{X_1}) - \tfrac{X_2}{X_1}\,u - \delta_2(\tfrac{X_2}{X_1})\,w - \tfrac{1}{2}(\tfrac{X_2}{X_1})\,w^2.
\end{aligned}\right. \tag{5.10}$$

We have

$$\begin{aligned}
\delta_1\delta_2 w &= \delta_1(u + \frac{1}{2}w^2) \\
&= \delta_1 u + w\,\delta_1 w \\
&= -\delta_2(\frac{X_2}{X_1}u) - \frac{X_2}{X_1}\,\delta_2 u - \delta_2^3(\frac{X_2}{X_1}) \\
&\quad + w\left(-\delta_2^2(\frac{X_2}{X_1}) - \frac{X_2}{X_1}\,u - \delta_2(\frac{X_2}{X_1})\,w - \frac{1}{2}(\frac{X_2}{X_1})\,w^2\right) \\
&= -\delta_2(\frac{X_2}{X_1}u) - \frac{X_2}{X_1}\,\delta_2 u - \delta_2^3(\frac{X_2}{X_1}) - \left(\delta_2^2(\frac{X_2}{X_1}) + \frac{X_2}{X_1}\,u\right)\,w \\
&\quad - \delta_2(\frac{X_2}{X_1})\,w^2 - \frac{1}{2}(\frac{X_2}{X_1})\,w^3
\end{aligned}$$

and

$$\begin{aligned}
\delta_2\delta_1 w &= \delta_2\left(-\delta_2^2(\frac{X_2}{X_1}) - \frac{X_2}{X_1}\,u - \delta_2(\frac{X_2}{X_1})\,w - \frac{1}{2}(\frac{X_2}{X_1})\,w^2\right) \\
&= -\delta_2^3(\frac{X_2}{X_1}) - \delta_2(\frac{X_2}{X_1}u) - \delta_2^2(\frac{X_2}{X_1})w - \delta_2(\frac{X_2}{X_1})\,\delta_2 w \\
&\quad - \frac{1}{2}\delta_2(\frac{X_2}{X_1})\,w^2 - \frac{X_2}{X_1}\,w\,\delta_2 w \\
&= -\delta_2^3(\frac{X_2}{X_1}) - \delta_2(\frac{X_2}{X_1}u) - \delta_2^2(\frac{X_2}{X_1})w - \delta_2(\frac{X_2}{X_1})\,(u + \frac{1}{2}\,w^2) \\
&\quad - \frac{1}{2}\delta_2(\frac{X_2}{X_1})\,w^2 - \frac{X_2}{X_1}\,w\,(u + \frac{1}{2}\,w^2) \\
&= -\delta_2^3(\frac{X_2}{X_1}) - \delta_2(\frac{X_2}{X_1}u) - \delta_2(\frac{X_2}{X_1})\,u - \left(\delta_2^2(\frac{X_2}{X_1}) + \frac{X_2}{X_1}u\right)\,w \\
&\quad - \delta_2(\frac{X_2}{X_1})\,w^2 - \frac{1}{2}(\frac{X_2}{X_1})\,w^3
\end{aligned}$$

Therefore, $\delta_1\delta_2 w = \delta_2\delta_1 w$, and the equations (5.10) have a solution w that is analytic at $(0,0)$. Let

$$v = -\delta_2(\frac{X_2}{X_1}) - (\frac{X_2}{X_1})\,w,$$

then

$$\begin{aligned}
\delta_2 v &= \delta_2(-\delta_2(\frac{X_2}{X_1}) - (\frac{X_2}{X_1})\, w) \\
&= -\delta_2^2(\frac{X_2}{X_1}) - \delta_2(\frac{X_2}{X_1})\, w - (\frac{X_2}{X_1})\, \delta_2 w \\
&= -\delta_2^2(\frac{X_2}{X_1}) - \delta_2(\frac{X_2}{X_1})\, w - (\frac{X_2}{X_1})\, (u + \frac{1}{2}\, w^2) \\
&= -\delta_2^2(\frac{X_2}{X_1}) - (\frac{X_2}{X_1})\, u - \delta_2(\frac{X_2}{X_1}) w - \frac{1}{2}(\frac{X_2}{X_1})\, w^2 \\
&= \delta_1 w.
\end{aligned}$$

Therefore, the 1-form $v dx_1 + w dx_2$ is a closed. Let

$$\omega_2 = \exp\left[\int_{(0,0)}^{(x_1,x_2)} v dx_1 + w dx_2\right], \quad \omega_1 = -\frac{X_2}{X_1}\omega_2,$$

than

$$\begin{aligned}
\delta_1\omega_2 &= \omega_2\, v \\
\delta_2\omega_1 &= -\delta_2(\frac{X_2}{X_1})\, \omega_2 - (\frac{X_2}{X_1})\, \delta_2\omega_2 \\
&= \omega_2(-\delta_2(\frac{X_2}{X_1}) - \frac{X_2}{X_1}\, \frac{\delta_2\omega_2}{\omega_2}) \\
&= \omega_2(-\delta_2(\frac{X_2}{X_1}) - \frac{X_2}{X_1}\, w) \\
&= \omega_2\, v = \delta_1\, \omega_2.
\end{aligned}$$

Hence, $\omega = \int_{(0,0)}^{(x_1,x_2)} \omega_1 dx_1 + \omega_2 dx_2$ is well defined, and is a first integral of X at $(0,0)$. It is easy to verify that

$$\frac{2\delta_2\omega \cdot \delta_2^3\omega - 3(\delta_2^2\omega)^2}{(\delta_2\omega)^2} = u.$$

The proof is completed. $\square$

Lemma 5.4 *We have*

(1). $\delta_2 X = (\frac{\delta_2 X_1}{X_1})\, X - B_0\, \delta_2$*;*
(2). $X(y_j) = \delta_2(X(y_{j-1})) - (\frac{\delta_2 X_1}{X_1})\, X(y_{j-1}) + B_0\, y_j$.

Proof. The proof is straightforward from (1)

$$\begin{aligned}
\delta_2 X &= (\delta_2 X_1)\,\delta_1 + (\delta_2 X_2)\,\delta_2 \\
&= \frac{\delta_2 X_1}{X_1}\,(X_1\,\delta_1 + X_2\,\delta_2) - \frac{X_2}{X_1}(\delta_2 X_1)\,\delta_2 + (\delta_2 X_2)\,\delta_2 \\
&= \frac{\delta_2 X_1}{X_1}\,X + X_1\,\frac{X_1\,\delta_2 X_2 - X_2\,\delta_2 X_1}{X_1^2}\delta_2 \\
&= \frac{\delta_2 X_1}{X_1}\,X - B_0\,\delta_2,
\end{aligned}$$

and (2)

$$\begin{aligned}
X(y_j) &= \delta_2(X(y_{j-1})) - (\delta_2 X) y_{j-1} \\
&= \delta_2(X(y_{j-1})) - (\frac{\delta_2 X_1}{X_1}\,X - B_0\,\delta_2)(y_{j-1}) \\
&= \delta_2(X(y_{j-1})) - \frac{\delta_2 X_1}{X_1}\,X(y_{j-1}) + B_0\,\delta_2(y_{j-1}) \\
&= \delta_2(X(y_{j-1})) - (\frac{\delta_2 X_1}{X_1})\,X(y_{j-1}) + B_0\,y_j.
\end{aligned}$$

□

Lemma 5.5 *We have*

$$X(y_j) \sim \sum_{i=0}^{j-1} a_{j,i}\,B_i\,y_{j-i} \tag{5.11}$$

where $a_{j,i}$ are constants, with $a_{j,0} = j$.

Proof. From Lemma 5.4, when $j = 1$, we have

$$X(y_1) = \delta_2(X(y_0)) - (\frac{\delta_2 X_1}{X_1})\,X(y_0) + B_0\,y_1 \sim B_0\,y_1,$$

which is the desired (5.11) with $a_{1,0} = 1$.

Assume that (5.11) is valid for $j = k$, and $a_{k,0} = k$, then by Lemma 5.4,

$$\begin{aligned}
X(y_{k+1}) &= \delta_2(X(y_k)) - (\frac{\delta_2 X_1}{X_1})\, X(y_k) + B_0\, y_{k+1} \\
&\sim \delta_2(\sum_{i=0}^{k-1} a_{k,i} B_i y_{k-i}) - (\frac{\delta_2 X_1}{X_1})\, (\sum_{i=0}^{k-1} a_{k,i} B_i y_{k-i}) + B_0\, y_{k+1} \\
&= \sum_{i=0}^{k-1} a_{k,i}\, ((\delta_2 B_i)\, y_{k-i} + B_i\, \delta_2 y_{k-i}) - \sum_{i=0}^{k-1} a_{k,i} \frac{\delta_2\, X_1}{X_1}\, B_i\, y_{k-i} + B_0 y_{k+1} \\
&= \sum_{i=0}^{k-1} a_{k,i} \left((\delta_2 B_i - \frac{\delta_2 X_1}{X_1}\, B_i)\, y_{k-i} + B_i\, y_{k-i+1} \right) + B_0\, y_{k+1} \\
&= (a_{k,0} + 1)\, B_0\, y_{k+1} + \sum_{i=0}^{k-2} (a_{k,i} X_1\, \delta_2(\frac{B_i}{X_1}) + a_{k,i+1}\, B_{i+1}) y_{k-i} \\
&\quad + a_{k,k-1}\, X_1 \delta_2(\frac{B_{k-1}}{X_1})\, y_1 \\
&= (a_{k,0} + 1)\, B_0\, y_{k+1} + \sum_{i=0}^{k-2} (a_{k,i} + a_{k,i+1})\, B_{i+1} y_{k-i} + a_{k,k-1}\, B_k\, y_1 \\
&= \sum_{i=0}^{k} a_{k+1,i}\, B_i\, y_{k+1-i},
\end{aligned}$$

where

$$\begin{cases} a_{k+1,0} = a_{k,0} + 1 = k + 1, & \\ a_{k+1,i} = a_{k,i-1} + a_{k,i}, & (1 \le i \le k-1), \\ a_{k+1,k} = a_{k,k-1}. & \end{cases}$$

Thus, the Lemma has been proved. □

Lemma 5.6 *Let* $\mathbf{m} \in \mathbb{Z}^{*r}$*, and define*

$$C(\mathbf{m}) = m_1 + 2\, m_2 + \cdots + r\, m_r. \tag{5.12}$$

If $\mathbf{m} \succ \mathbf{p}$*, then* $C(\mathbf{m}) > C(\mathbf{p})$*. In particularly, if* $d_i^j(\mathbf{m}) = \mathbf{p}$*, then* $C(\mathbf{m}) - C(\mathbf{p}) = i$*.*

Proof. Let $d_i^j(\mathbf{m}) = \mathbf{p}$, then

$$C(\mathbf{m}) - C(\mathbf{p}) = (j-i)\, m_{j-i} + j\, m_j - ((j-i)\, (m_{j-i} + 1) + j\, (m_j - 1)) = i.$$

□

Lemma 5.7 *Let* $P = A_{\mathbf{m}} y_1^{m_1} \cdots y_r^{m_r}$*, then*

$$\hat{X}(P) \sim (X(A_{\mathbf{m}}) + C(\mathbf{m})\, B_0\, A_{\mathbf{m}})\, \mathbf{y}^{\mathbf{m}} + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} n_j\, a_{j,i}\, B_i\, A_{\mathbf{m}}\, \mathbf{y}^{d_i^j(\mathbf{m})},$$

where $\mathbf{y}^{\mathbf{m}} = y^{m_1} \cdots y_r^{m_r}$.

Proof. By Lemma 5.5, we have

$$\begin{aligned}
\hat{X}(P) &= X(A_{\mathbf{m}})\mathbf{y}^{\mathbf{m}} + \frac{\partial A_{\mathbf{m}}}{\partial y}\, \mathbf{y}^{\mathbf{m}}\, X(y) + A_{\mathbf{m}} \sum_{j=1}^{r} m_j y_1^{m_1} \cdots y_j^{m_j - 1} \cdots y_r^{m_r}\, X(y_i) \\
&\sim X(A_{\mathbf{m}})\mathbf{y}^{\mathbf{m}} + A_{\mathbf{m}} \sum_{j=1}^{r} m_j y_1^{m_1} \cdots y_j^{m_j - 1} \cdots y_r^{m_r}\, (\sum_{i=0}^{j-1} a_{j,i}\, B_i\, y_{j-i}) \\
&= X(A_{\mathbf{m}})\mathbf{y}^{\mathbf{m}} + A_{\mathbf{m}}\, B_0 (\sum_{j=1}^{r} m_j\, a_{j,0}) \mathbf{y}^{\mathbf{m}} \\
&\quad + A_{\mathbf{m}} \sum_{j=1}^{r} \sum_{i=1}^{j-1} m_j a_{j,i} B_i\, y_1^{m_1} \cdots y_{j-i}^{m_{j-i}+1} \cdots y_j^{m_j - 1} \cdots y_r^{m_r} \\
&= (X(A_{\mathbf{m}}) + C(\mathbf{m})\, B_0\, A_{\mathbf{m}})\, \mathbf{y}^{\mathbf{m}} + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} m_j\, a_{j,i}\, B_i\, A_{\mathbf{m}}\, \mathbf{y}^{d_i^j(\mathbf{m})}
\end{aligned}$$

and the Lemma is concluded. □

Lemma 5.8 *Let* $A \in \Lambda$ *with the lowest rank and* $r = \mathrm{ord}(\Lambda) > 1$*. Let* $\mathbf{n} \in \mathcal{I}_A$ *with the highest degree and* $A_{\mathbf{n}} = 1$*, then for any* $\mathbf{m} < \mathbf{n}$*,*

$$X(A_{\mathbf{m}}) = (C(\mathbf{n}) - C(\mathbf{m}))\, B_0\, A_{\mathbf{m}} - \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} (m_j + 1)\, a_{j,i}\, B_i\, A_{b_i^j(\mathbf{m})} \quad (5.13)$$

where $A_{\mathbf{m}} = 0$ *if* $\mathbf{m} \notin \mathcal{I}_A$.

Proof. Let

$$A = \sum_{\mathbf{m} \in \mathcal{I}_A} A_{\mathbf{m}} \mathbf{y}^{\mathbf{m}}.$$

By Lemma 5.7, we have

$$\begin{aligned}
X(A) &= \sum_{\mathbf{m}\in\mathcal{I}_A} X(A_{\mathbf{m}}\mathbf{y}^{\mathbf{m}})\\
&\sim \sum_{\mathbf{m}\in\mathcal{I}_A}\left((X(A_{\mathbf{m}}) + C(\mathbf{m})\,B_0\,A_{\mathbf{m}})\,\mathbf{y}^{\mathbf{m}} + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r} m_j\,a_{j,i}B_i\,A_{\mathbf{m}}\mathbf{y}^{d_i^j(\mathbf{m})}\right)\\
&= \sum_{\mathbf{m}\in\mathcal{I}_A}\left(X(A_{\mathbf{m}}) + C(\mathbf{m})\,B_0\,A_{\mathbf{m}} + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r}(m_j+1)\,a_{j,i}B_i\,A_{b_i^j(\mathbf{m})}\right)\mathbf{y}^{\mathbf{m}}
\end{aligned}$$

Note that for any $j > i$, $b_i^j(\mathbf{n}) > \mathbf{n}$, and thus $b_i^j(\mathbf{n}) \notin \mathcal{I}_A$, i.e., $A_{b_i^j(\mathbf{n})} = 0$. Taking account $A_{\mathbf{n}} = 1$, we have

$$\begin{aligned}
X(A) - C(\mathbf{n})\,B_0\,A &= \sum_{\mathbf{m}<\mathbf{n}}\Big(X(A_{\mathbf{m}}) + (C(\mathbf{m}) - C(\mathbf{n}))\,B_0\,A_{\mathbf{m}}\\
&\quad + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r-1}(m_j+1)\,a_{j,i}\,B_i\,A_{b_i^j(\mathbf{m})}\Big)\mathbf{y}^{\mathbf{m}} \qquad (5.14)
\end{aligned}$$

Thus, $X(A) - C(\mathbf{n})\,B_0\,A$ contains in Λ and has lower rank than A. But A is the element in Λ with the lowest rank, thus $X(A) - C(\mathbf{n})\,B_0\,A \equiv 0$. Therefore the coefficients in (5.14) are zero, i.e.

$$X(A_{\mathbf{m}}) + (C(\mathbf{m}) - C(\mathbf{n}))\,B_0\,A_{\mathbf{m}} + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r-1}(m_j+1)\,a_{j,i}B_i\,A_{b_i^j(\mathbf{m})} = 0,\ \ (\forall \mathbf{m} < \mathrm{n})$$

from which (5.13) is concluded. $\square$

From Lemma 5.8, we have

Lemma 5.9 *If $\mathcal{P}(\mathbf{m}) = \{\mathbf{p}_1, \cdots, \mathbf{p}_k\}$, and $d_{i_l}^{j_l}(\mathbf{p}_l) = \mathbf{m}$, $(l = 1, 2, \cdots, k)$, then the coefficients $A_{\mathbf{p}_l}, A_{\mathbf{m}}$ satisfy*

$$X(A_{\mathbf{m}}) = (C(\mathbf{n}) - C(\mathbf{m}))\,B_0\,A_{\mathbf{m}} - \sum_{l=1}^{k}(m_{j_l}+1)\,a_{j_l,i_l}\,B_{i_l}\,A_{\mathbf{p}_l}. \qquad (5.15)$$

Lemma 5.10 *Let $A \in \Lambda$ with the lowest rank and $r = \mathrm{ord}(\Lambda) > 1$. Let $\mathbf{n} \in \mathcal{I}_A$ with the highest degree. Then for any $\mathbf{m} \in \mathcal{I}_A$, $\#(\mathbf{m}) = 0$ if and only if $C(\mathbf{m}) = C(\mathbf{n})$. Furthermore, if $\#(\mathbf{m}) = 0$, then $A_{\mathbf{m}}$ is a constant.*

Proof. At first we will prove that if $\#(\mathbf{m}) = 0$, then $C(\mathbf{m}) = C(\mathbf{n})$.

When $\#(\mathbf{m}) = 0$, by Lemma 5.9, we have

$$X(A_{\mathbf{m}}) = (C(\mathbf{n}) - C(\mathbf{m}))\, B_0\, A_{\mathbf{m}}.$$

If $C(\mathbf{m}) \neq C(\mathbf{n})$, let $n = C(\mathbf{n}) - C(\mathbf{m})$, then

$$X(A_{\mathbf{m}}^{1/n}) = B_0\, A_{\mathbf{m}}^{1/n}.$$

From Lemma 5.1, there exists a first integral ω of X such that

$$\delta_2 \omega = A_{\mathbf{m}}^{1/n},$$

and hence

$$y_1^n - A_{\mathbf{m}} \in \Lambda,$$

which yields $r = 1$ and contradict. Thus, we have concluded that $C(\mathbf{m}) = C(\mathbf{n})$.

Now, we will prove that if $C(\mathbf{m}) = C(\mathbf{n})$, then $\#(\mathbf{m}) = 0$. If on the contrary, $\#(\mathbf{m}) > 0$, then there exists $\mathbf{p} \in \mathcal{P}(\mathbf{m})$, and from (5.4), $C(\mathbf{p}) > C(\mathbf{m}) = C(\mathbf{n})$. On the other hand, there exist $\mathbf{p}_1, \cdots, \mathbf{p}_k$, such that

$$\mathbf{p}_1 \succ \cdots \succ \mathbf{p}_k = \mathbf{p}$$

and $\#(\mathbf{p}_1) = 0$. Therefore $C(\mathbf{p}) = C(\mathbf{n})$. It is easy to have $C(\mathbf{p}) = C(\mathbf{p}_k) < \cdots < C(\mathbf{p}_1) = C(\mathbf{n})$, which is contradict, and the statement is concluded.

Now, we have proved that $\#(\mathbf{m})$ if and only if $C(\mathbf{m}) = C(\mathbf{n})$.

If $\#(\mathbf{m}) = 0$, then $C(\mathbf{n}) = C(\mathbf{m})$, and by (5.15), $X(A_{\mathbf{m}}) = 0$. But $\mathrm{ord}(\Lambda) > 0$, thus $A_{\mathbf{m}}$ is a constant. $\square$

Lemma 5.11 *Let $A \in \Lambda$ with the lowest rank and $r = \mathrm{ord}(\Lambda) \geq 3$. Assume that $\mathcal{J}_A \subseteq \mathcal{I}_A$ is closed, and $\mathbf{m} = (m_1, m_2, \cdots, m_r) \in \mathcal{J}_A$ with the highest degree, then $m_2 = 0$.*

Proof. If on the contrary, $m_2 > 0$, let $\mathbf{p} = d_1^2(\mathbf{m}) = (m_1 + 1, m_2 - 1, m_3, \cdots, m_r)$. It is easy to verify $\mathcal{P}(\mathbf{p}) = \{\mathbf{m}\}$ by (1) $d_1^2(\mathbf{m}) = \mathcal{P}$, and (2) if any other $\mathbf{m}'$ such that $d_i^j(\mathbf{m}') = \mathcal{P}$, then $\mathbf{m}' > \mathbf{m}$. Hence, we have

$$\left\{\begin{array}{rcl} X(A_{\mathbf{m}}) & = & (C(\mathbf{n}) - C(\mathbf{m}))\, B_0\, A_{\mathbf{m}} \\ X(A_{\mathbf{p}}) & = & B_0\, A_{\mathbf{p}} - m_2\, a_{2,1}\, B_1\, A_{\mathbf{m}}. \end{array}\right. \tag{5.16}$$

From (5.16) and Lemmas 5.1 and 5.2, either $r = 0$ (if $C_{\mathbf{n}} = C_{\mathbf{m}}$ and $A_{\mathbf{m}}$ is not a constant), or $r = 1$ (if $C_{\mathbf{n}} \neq C_{\mathbf{m}}$), or $r = 2$ (if $C(\mathbf{n}) - C(\mathbf{m}) = 0$ and $A_{\mathbf{m}}$ is constant), contradict with the assumption $r \geq 3$. $\square$

Lemma 5.12 *Assume that K contains no first integral of X, and let r to be the order of the differential Galois group of X.*

(1). If $r = 1$, there exists a first integral ω of X, and $n \in \mathbb{N}$, such that

$$(\delta_2\omega)^n \in K.$$

(2). If $r = 2$, there exists a first integral ω of X, such that

$$\delta_2^2\omega/\delta_2\omega \in K.$$

(3). If $r = 3$, there exists a first integral ω of X, such that

$$\frac{2\,\delta_2\omega \cdot \delta_2^3\omega - 3(\delta_2^2\omega)^2}{(\delta_2\omega)^2} \in K.$$

(4). If $r \neq \infty$, then $r \leq 3$.

Proof. Let Λ to be the regular prime ideal of QDP corresponding to the differential Galois of X, and $A \in \Lambda$ with the lowest rank, $\mathbf{n} \in \mathcal{I}_A$ with the highest degree.

(1). If $r = 1$, we can write A as

$$A = y_1^n + A_1\, y_1^{n-1} + \cdots + A_n$$

with $A_i \in K[\mathcal{A}_0(y)]$ and $A_n \neq 0$ since A is irreducible. From Lemma 5.9, we have

$$X(A_n) = n\, B_0\, A_n,$$

i.e.,

$$X(A_n^{1/n}) = B_0\, A_n^{1/n}.$$

By Lemma 5.1 and $A_n \neq 0$, there exists a first integral ω of X such that

$$\delta_2\omega = A_n^{1/n},$$

i.e.,

$$(\delta_2\omega)^n = A_n \in K[\mathcal{A}_0(y)].$$

But $\delta_2\omega$ is independent to y, hence A_n does not involves y, and therefore $(\delta_2\omega)^n \in K$.

(2). If $r=2$, let $\mathbf{n}=(n_1,n_2)$ and $\mathbf{m}=d_1^2(\mathbf{n})=(n_1+1,n_2-1)$, then $\mathcal{P}(\mathbf{m})=\{\mathbf{n}\}$. Thus, by Lemma 5.9 and Lemma 5.6, we have

$$X(A_{\mathbf{m}})=B_0\,A_{\mathbf{m}}-n_2\,a_{2,1}\,B_1,$$

i.e.,

$$X(-\frac{A_{\mathbf{m}}}{n_2\,a_{2,1}})=B_0\,(-\frac{A_{\mathbf{m}}}{-n_2\,a_{2,1}})+B_1.$$

From Lemma 5.2, there exists a first integral ω such that

$$\frac{\delta_2^2\omega}{\delta_2\omega}=-\frac{A_{\mathbf{m}}}{n_2\,a_{2,1}}\in K[\mathcal{A}_0(y)].$$

Similar to the above argument, we have $\delta_2^2\omega/\delta_2\omega\in K$ and (2) is proved.

(3). If $r=3$, we can write $\mathbf{n}=(n_1,n_2,n_3)$. From Lemma 5.11, we have $\mathbf{n}=(n_1,0,n_3)$. Let

$$\begin{aligned}\mathbf{p}&=d_1^3(\mathbf{n})=(n_1,1,n_3-1),\\ \mathbf{q}&=d_2^3(\mathbf{n})=(n_1+1,0,n_3-1),\\ \mathbf{m}&=b_1^2(\mathbf{p})=(n_1-1,2,n_3-1).\end{aligned}$$

It is easy to have $C(\mathbf{m})=C(\mathbf{n})$. Therefore, by Lemma 5.10, $A_{\mathbf{m}}$ is a constant. Furthermore, we have $\mathcal{P}(\mathbf{p})=\{\mathbf{n},\mathbf{m}\}$ and $\mathcal{P}(\mathbf{q})=\{\mathbf{n},\mathbf{p}\}$. Lemma 5.9, yields

$$X(A_{\mathbf{p}})=B_0\,A_{\mathbf{p}}-(n_3\,a_{3,1}\,A_{\mathbf{n}}+2\,a_{2,1}\,A_{\mathbf{m}})\,B_1,\tag{5.17}$$

and

$$X(A_{\mathbf{q}})=2\,B_0\,A_{\mathbf{q}}-n_3\,a_{3,2}\,A_{\mathbf{n}}\,B_2-a_{2,1}\,A_{\mathbf{p}}.\tag{5.18}$$

Since $A_{\mathbf{n}}$ and $A_{\mathbf{m}}$ are constants, and $r=3$, we conclude that $n_3\,a_{3,1}\,A_{\mathbf{n}}+2\,a_{2,1}\,A_{\mathbf{m}}=0$ and $A_{\mathbf{p}}=0$. Otherwise, we should have $r=2$ following the similar discussion in (2). Let $A_{\mathbf{p}}=0$ and $A_{\mathbf{n}}=1$ in (5.18), we have

$$X(-\frac{A_{\mathbf{q}}}{n_3\,a_{3,2}})=2\,B_0\,(-\frac{A_{\mathbf{q}}}{n_3\,a_{3,2}})+B_2.$$

From Lemma 5.3, there exists a first integral ω of X such that

$$\frac{2\delta_2\omega\cdot\delta_2^3\omega-3\,(\delta_2^2\omega)^2}{(\delta_2\omega)^2}=-\frac{A_{\mathbf{q}}}{n_3\,a_{3,2}}\in K[\mathcal{A}_0(y)],$$

which implies (3) following the argument similar to the previous discussion.

Finally, we will show that it is impossible to have $3 < r < \infty$. If on the contrary, we have $3 < r < \infty$, from Lemma 5.11, we can write $\mathbf{n} = (n_1, 0, n_3, \cdots, n_r)$. Let

$$\begin{aligned}
\mathbf{m} &= d_1^r(\mathbf{n}) = (n_1, 0, n_3, \cdots, n_{r-1}+1, n_r - 1),\\
\mathbf{p} &= b_1^2(\mathbf{m}) = (n_1 - 1, 1, n_3, \cdots, n_{r-1}+1, n_r - 1),\\
\mathbf{q} &= d_1^{r-1}(\mathbf{p}) = (n_1 - 1, 1, n_3, \cdots, n_{r-2}+1, n_{r-1}, n_r - 1),
\end{aligned}$$

then $C(\mathbf{p}) = C(\mathbf{n})$ and therefore $\#(\mathbf{p}) = 0$ according to Lemma 5.10. Let $\mathcal{J}_A$ to be the minimal closed subsystem of $\mathcal{I}_A$ that contains $\mathbf{q}$, then $\mathbf{p} \in \mathcal{J}_A$ and has the highest degree. However, $p_2 = 1 \neq 0$, which is contradict to Lemma 5.11. Thus, we concluded that either $r = \infty$ or $r \leq 3$. $\square$

Proof of Theorem 3.9. The first part of Theorem 3.9 is concluded from Lemma 5.12 and Lemma 2.9.

It is easy to verify that when $r \leq 2$, the group G is solvable, and when $r = 3$, G is unsolvable. Here we omit the detail calculations. In fact, we can consider G as a multi-parameter group of transformation on $z \in \mathbb{C}$, and therefore the arguments for the corresponding multi-parameter Lie group are applicable. It is known that the transformation Lie groups with 1 or 2 parameters are solvable, while the three-parameter group is unsolvable[3, pp. 85-86]. When $r = \infty$, G contains an unsolvable subgroup that is defined by (3.8), and therefore is unsolvable. $\square$

References

[1] Magid, A.R., *Lectures on Differential Galois Theory*, Amer. Math. Soc. 1994.

[2] Bruno, A.D., *Local Methods in Nonlinear Differential Equations*, Springer-Verlag, Berlin, 1989.

[3] Bluman, G. W., Kumei, S., *Symmetries and Differential Equations*, Springer-Verlag, Berlin, 1989.

[4] Camacho, C., Scárdua, A.B., Complex foliations with algebraic limit sets, *Astérisque*, **261**(2000), 57-88.

[5] Camacho, C., Scárdua, A.B., Holomorphic foliations with Liouvillian first integrals, *Ergod. Th. & Dynam. Sys.*, **21**(2001), 717-756.

[6] Cerveau, D., Sad, P., Liouvillian integration and Bernoulli foliations, *Trans. Amer. Math. Soc.*, **350**(8)(1998), 3065-3081.

[7] Cheng Ruyi, Guan Keying, Zhang Shaofei, On the judement of the existence of algebraic curve solution to the second order polyomial autonomous system, *Journal of Beijing University of Aeronautics and Astronautics*, **21**(1)(1995), 109-115 (Chinese).

[8] Christopher, C., Liouvillian first integrals of second order polynomial differential equations, *Ele. J. Diff. Eqs.*, **1999**(1999)(49), 1-7.

[9] Glubev,V.V., *Lecture on Analytic Theory of Differential Equation*, 2nd ed., Moscow-Leningrad, Gos. Izd. Tekh. Teor. Lit., 1950, (Russian).

[10] Guan Keying, Lei Jinzhi, Integrability of second order autonmous system, *Ann. of Diff. Eqs.*, **18**(2)(2002), 117-135.

[11] Kaplan, W., *Advanced Calcalus*, Reading, Mass.: Addison-Wesley, 1952.

[12] Kaplan, W., *Ordinary Differential Equations*, Reading Mass. : Addision-Wesley. 1958.

[13] Kaplansky, I., *An Introduction to Differential Algebra.* 2nd ed. Hermann. Paris, 1976.

[14] Khovanskii, A.G., Topological obstructions to the representability of functions by quadratures, *J. Dynam. and Contr. Syst.*, **1**(1)(1995), 91-123.

[15] Kolchin, E.R., Existence theorems connected with the Picard-Vessiot theory of homogeneous linear ordinary differential equations. *Bull. Amer. Math. Soc.* **54**(1948), 927-932.

[16] Kolchin, E.R., Algebraic matric groups and the Picard-Vessiot theory of homogeneous lnear ordinary differential equations, *Ann. Math.*, **49**(1)(1948), 1-42.

[17] Kolchin, E.R., *Differential algebra and algebraic groups*, Academic Press. NewYork, 1973.

[18] Kovacic, J., An algorithm for solving second order linear homogeneous differential equations, *J. Symb. Comp.*, **2**(1986), 3-43.

[19] Lei Jinzhi, Guan Keying, An algorithm to judge the integrability of second order linear differential euqation with rational coefficients, *Nerual, Parallel & Scientific Computations*, **8**(3& 4)(2000), 243 - 252.

[20] Loray,F., Towards the Galois groupoid of non linear first order O.D.E., *Preprint*. 2001.

[21] Liouville, J., Mémorie sur I'intégration d'une classe d'équations différentielles du second order en quantités finies explicites. *Journal de Mathématiques Pures et Appliquées*, IV(1839), 423-456.

[22] Liouville, J., Remarques nouvelles sur l'équation de Riccati, *Journal de Mathématiques Pures et Appliquées*, VI(1841), 1-13.

[23] Malgrange,B., Le groupoïd de Galois d'un feuilletage, *Enseignement Math.*, **38**(2001). 465-501.

[24] Malgrange,B., On nonlinear Differential Galois Theory, *Chinese Ann. Math.*, **23**(2)(2002), Ser. B, 219-226.

[25] Magid, A.R., *Lectures on differential galois theory*, University Lecture Series, Vol. 7, AMS, Providence, 1994.

[26] Mitschi,C., Differential Galois of confluent generalized hypergeometric equations: an approach using stokes multipliers, *Pacific J. Math.*, **176**(2)(1998), 365-405.

[27] Mitschi,C., Singer, M.F., The Inverse Problem in Differential Galois Theory, in : B.l.J. Braaksma, et. al., (eds),*The Stokes Phenomenon and Hilbert's 16th Problem*, World Scientific, Singapore, 1996, 185-196.

[28] Odani,K., The limit cycle of the van der Pol equation is not algebraic, *J. Diff. Eqs.*, **115**(1995), 146-152.

[29] Olver,P.J., *Applications of Lie groups to differential equations*, 2nd ed. Springer-Verlag, New York, Berlin, Heidelvberg, 1999.

[30] Ritt,J.F, *Differential Algbra*, Amer. Math. Soc. Coll. Pub., New-York, 1950.

[31] Ruiz, J. M., *Differential Galois theory and non-integrability of Hamiltonian systems*, Birkh̀'auser Basel, Boston, 1999.

[32] Schlesinger, L., *Handbuch der Theorie der Linearen Differentialgleichungen*, Teubner, Leipzig, 1887.

[33] Singer, M.F., Liouvillian first integrals of differential equations, *Trans. Amer. Math. Soc.*, **333**(2)(1992), 673-688.

[34] Singer, M.F., Ulmer, F., Galois groups of second and third order linear differential euations, *J. Symb. Comp.*, **16**(1993), 1-30.

[35] Singer, M.F., Direct and Inverse Problems in Differential Galois Theory, in: B.B.Cassidy, (eds), *Selected Works of Ellis Kolchin with Commentary*, Amer. Math. Soc., 1999, 527-554.

[36] Michael F.S., Ulmer,F., A Kovacic-style algorithm for liouvillian solutions of third order differential equations, Prepublication IRMAR 01-12, 2001, URL: http://citeseer.csail.mit.edu/446364.html.

[37] van der Put, M., Singer, M.F., *Galois theory of linear differential equations*, Springer-Verlag, Berlin, 2003.

[38] Waston, G.N., *A Treatise on the Theory of Bessel Functions*, 2nd ed, Gambridge, 1944.

[39] Arnold, V.I., II'yashenko, Yu. S., Ordinary Differential Equation, In *Dynamical System*, I. Anosov, D.V., Arnold, V.I. (Eds), Springer-Verlag, Berlin, Heidelberg, 1988.

[40] Żołądek,H., The extended monodromy group and Liouvillian first integral, *J. Dynam. and Contr. Syst.*, **4**(1998), 1-28.

Acknowledgements

The author is highly grateful to Keying Guan who leads him to the field of differential Galois theory. The author wishes to express his hearty thanks to Claude Mitschi, Université Louis Pasteur, Michael F. Singer, North Carolina State University, and Colin Christopher, University of Plymouth, for their great interested in reading the early versions of the manuscript, and for their valuable advices and helpful discussions.