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002

ON BOUNDARY ACCUMULATION POINTS OF A CONVEX DOMAIN IN $\mathbb{C}^n$*

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Abstract. In this paper we show that, for a smoothly bounded convex domain $\Omega \subset \mathbb{C}^n$, if there is $\{\phi_j\} \subset \mathrm{Aut}(\Omega)$ such that $\phi_j(z)$ converges to some boundary point non-tangentially for all $z \in \Omega$, then there does not exist a non-trivial analytic disc on $\partial\Omega$ through any boundary orbit accumulation points.

Key words. Automorphism Group, Convex Domains, Invariant Metrics/Measures.

AMS subject classifications. 32F18, 32F45.

1. Introduction. The study of biholomorphic automorphism groups, $\mathrm{Aut}(\Omega)$, of a domain $\Omega \subset \mathbb{C}^n$ is of major interest in various areas of research. The existence of an automorhpism reflects certain symmetry of the domain. It is a deep subject to study the discrete subgroups $\Gamma \subset \mathrm{Aut}(\Omega)$ such that Ω/Γ is a compact complex manifold. Although the construction of a cocompact lattice Γ in $\mathrm{Aut}(\Omega)$ is usually not straightforward, it is comparably easier to find a divergent sequence $\{\phi_j\} \subset \mathrm{Aut}(\Omega)$.

Let p be any point in Ω such that $\{\phi_j(p)\}$ converge to a boundary point $q \in \partial\Omega$. If we further assume $\partial\Omega$ is smooth, our knowledge of the biholomorphic invariants (i.e., Chern-Moser invariants, invariant Kähler metrics, intrinsic metrics/measures etc.) allows us to draw many interesting conclusions. For instance, if $q \in \partial\Omega$ is strongly pseudoconvex, the method in [9] can be used to show that Ω must be biholomorphic to the Euclidean ball.

In order to charaterize those smoothly bounded domains with non-compact automorhpism group, it is important to have a better understanding of the orbit accumulation points on the boundary. There has been recently a lot of research in this direction. One of the important conjectures in this regard is due to Greene and Krantz, which can be stated as follows.

CONJECTURE. *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. Suppose there exists $\{\phi_j\} \subset Aut(\Omega)$ such that $\{\phi_j(p)\}$ accumulates at a boundary point $q \in \partial\Omega$ for some $p \in \Omega$. Then $\partial\Omega$ is of finite type at q.*

In this paper we will prove the following result in support of the Greene/Krantz conjecture.

THEOREM. *Let Ω be a smoothly bounded convex domain in $\mathbb{C}^n$. Suppose that there is a sequence $\{\phi_j\} \subset Aut(\Omega)$ such that $\{\phi_j(p)\}$ accumulates non-tangentially at some boundary point for all $p \in \Omega$. Then, there does not exist a non-trivial analytic disc on $\partial\Omega$ passing through any orbit accumulation point on the boundary.*

In [2] this result was proved in $\mathbb{C}^2$ under a more general assumption that Ω is pseudoconvex. Earlier work in the convex setting in $\mathbb{C}^2$ was discussed in [5, 10]. For

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the pseudoconvex case, it is a highly non-trivial matter to generalize this result to higher dimensions since the geometry of the boundary of a pseudoconvex domain in $\mathbb{C}^n$, $n > 2$, is not as well understood as in $\mathbb{C}^2$. To overcome the technical difficulties generalizing the result in [2, 5, 9], we use the intrinsic measures defined with respect to $U = \mathbb{B}_{n-k} \times \Delta_k$, $0 \leq k \leq n$, where $\mathbb{B}_{n-k}$ is the unit ball in $\mathbb{C}^{n-k}$ and Δ_k is the unit polydisc in $\mathbb{C}^k$. We will prove that the orbit accumulation set on the boundary is actually biholomorphic to a euclidean ball, if it is not a point. This fact allows us to remove the obstacle of finding a higher dimensional analogue of the argument used in [2] for $\mathbb{C}^2$, which depends heavily on the classical result that a hyperbolic Riemann surface is covered by the unit disc.

A substantial portion of this paper can be found in [8]; this portion is a joint work of Lina Lee, Bradley Thomas, and Bun Wong.

2. Invariant metrics and invariant measures. Let $H(A, B)$ be the set of holomorphic mappings from A to B and Δ be the unit disc in $\mathbb{C}$. The Kobayashi and Carathéodory metrics are defined as follows.

Definition 1. The Kobayashi and Carathéodory metrics on $\Omega \subset \mathbb{C}^n$ at $p \in \Omega$ in the direction $\xi \in \mathbb{C}^n$, denoted as $F_K^\Omega(p,\xi)$ and $F_C^\Omega(p,\xi)$, respectively, are defined as follows:

$$F_K^\Omega(p,\xi) = \inf\left\{\frac{1}{\alpha} : \exists \phi \in H(\Delta,\Omega) \text{ s.t. } \phi(0) = p,\ \phi'(0) = \alpha\xi\right\} \tag{1}$$

$$F_C^\Omega(p,\xi) = \sup\left\{\left|\sum_{j=1}^{n} \frac{\partial f(p)}{\partial z_j}\xi_j\right| : \exists f \in H(\Omega,\Delta)\text{, s.t. } f(p) = 0\right\}. \tag{2}$$

If $z, w \in \Omega$, then the Kobayashi and Carathéodory pseudo-distance on Ω between z and w, denoted as $d_K^\Omega(z,w)$ and $d_C^\Omega(z,w)$, respectively, are given by

$$d_K^\Omega(z,w) = \inf_\gamma \int_0^1 F_K^\Omega(\gamma(t), \gamma'(t))\, dt, \tag{3}$$

$$d_C^\Omega(z,w) = \sup_f \rho(f(z), f(w)) \tag{4}$$

where $\gamma : [0,1] \longrightarrow \Omega$ is a piecewise C^1 curve connecting z and w and $\rho(p,q)$ is the Poincaré distance on Δ between $p, q \in \Delta$. The supremum in (4) is taken over all holomorphic mappings $f : \Omega \longrightarrow \Delta$.

Kobayashi originally defined the pseudo-distance on Ω using a chain of analytic discs as follows: for two given points $z, w \in \Omega$, consider a chain of analytic discs α that consists of $z_1, z_2, \ldots, z_n \in \Omega$, analytic discs $f_i : \Delta \longrightarrow \Omega$, and $n+1$ pairs of points $a_0, b_0, a_1, b_1, \ldots, a_n, b_n \in \Delta$ such that, for $0 \leq j \leq n$,

$$f_j(a_j) = z_j,\ f_j(b_j) = z_{j+1}, \text{ and } z_0 = z,\ z_{n+1} = w.$$

We define the length of the chain α as

$$\ell(\alpha) = \sum_{j=0}^{n} \rho(a_j, b_j).$$

Then the Kobayashi pseudo-distance between two points z, w is given as

$$d_K^\Omega(z,w) = \inf_\alpha \ell(\alpha). \tag{5}$$

It was Royden [7] who later proved that the definition given by (5) is equivalent to (3).

The metrics and distances given above are invariant under biholomorphic mappings since they satisfy the non-increasing property under holomorphic mappings, i.e., if $\Phi : \Omega_1 \longrightarrow \Omega_2$ is a holomorphic mapping between domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, and $p, q \in \Omega_1$, $\xi \in \mathbb{C}^n$, then we have

$$F^{\Omega_1}(p,\xi) \geq F^{\Omega_2}(\Phi(p), \Phi_*(p)\xi), \tag{6}$$

$$d^{\Omega_1}(p,q) \geq d^{\Omega_2}(\Phi(p), \Phi(q)), \tag{7}$$

where the metric F in (6) denotes either the Kobayashi or Carathéodory metric and the distance d in (7) is either the Kobayashi or Carathéodory distance.

We extend the definition of the metrics and define the Kobayashi and Carathéodory measures. Let $\mathbb{B}_k$ denote the complex k-dimensional unit ball and Δ_k the complex k-dimensional unit polydisc.

Definition 2. Let $\Omega \subset \mathbb{C}^n$ be a domain, $p \in \Omega$, and $\xi_1, \dots, \xi_m \in T_p^{\mathbb{C}}\Omega$, $1 \leq m \leq n$, be linearly independent vectors on the complex tangent space to Ω at p. One can find an (m,m) volume form M on Ω such that $M\left(\xi_1, \dots, \xi_m, \overline{\xi}_1, \dots, \overline{\xi}_m\right) = 1$. Let $U = \mathbb{B}_{m-j} \times \Delta_j$, $0 \leq j \leq m$, and $\mu_m = \prod_{j=1}^m \left(\frac{i}{2} dz_j \wedge d\overline{z}_j\right)$. We define the Kobayashi and Carathéodory m-measures with respect to U as follows:

$$K_U^{\Omega}(p;\xi_1,\dots\xi_m) = \inf\left\{\frac{1}{\alpha} : \exists\, \Phi \in H(U,\Omega), \text{s.t. } \Phi(0) = p, \right.$$
$$\left. \Phi^*(0) M = \alpha\mu_m, \text{ for some } \alpha > 0\right\},$$
$$C_U^{\Omega}(p;\xi_1,\dots,\xi_m) = \sup\Big\{\beta : \exists\, \Phi \in H(\Omega,U), \text{s.t. } \Phi(p) = 0,$$
$$\Phi^*(p)\mu_m = \beta M,\ \beta > 0\Big\}.$$

The Kobayashi and Carathéodory measures satisfiy the non-increasing property under holomorphic mappings.

Proposition 1. *Let $\Omega_1 \subset \mathbb{C}^n$, $\Omega_2 \subset \mathbb{C}^{n'}$ be domains and $U = \mathbb{B}_{m-j} \times \Delta_j$, $0 \leq j \leq m$, $m \leq \min\{n, n'\}$. Let $p \in \Omega_1$, $\xi_j \in T_p^{\mathbb{C}}\Omega_1$, $j = 1, \dots, m$, and ξ_j's be linearly independent. If $\phi \in H(\Omega_1, \Omega_2)$ is such that $\phi_*(p)\xi_j$'s are linearly independent, then*

$$K_U^{\Omega_1}(p;\xi_1,\dots\xi_m) \geq K_U^{\Omega_2}(\phi(p);\phi_*(p)\xi_1,\dots\phi_*(p)\xi_m), \quad \textit{and}$$
$$C_U^{\Omega_1}(p;\xi_1,\dots,\xi_m) \geq C_U^{\Omega_2}(\phi(p);\phi_*(p)\xi_1,\dots\phi_*(p)\xi_m).$$

Proof. Let M be an (m,m) volume form on Ω_1 such that $M\left(\xi_1, \dots, \xi_m, \overline{\xi}_1, \dots, \overline{\xi}_m\right) = 1$. Let $\Phi : U \longrightarrow \Omega_1$ be a holomorphic mapping such that $\Phi(0) = p$, $\Phi^*(0)M = \alpha\mu_m$. Consider $h = \phi \circ \Phi : U \longrightarrow \Omega_2$. Let M' be an (m,m) volume form on Ω_2 such that $\phi^*(p)M' = M$. Then $h(0) = \phi(p)$ and

$$h^*(0)M' = \Phi^*(0)(\phi^*(p)M') = \Phi^*(0)(M) = \alpha\mu_m.$$

Hence $1/\alpha \geq K_U^{\Omega_2}(\phi(p), M)$ and $\inf 1/\alpha \geq K_U^{\Omega_2}(\phi(p), M)$. One can show the second inequality in a similar way. ∎

Corollary 1. *Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be domains and $U = \mathbb{B}_{m-j} \times \Delta_j$, $0 \leq j \leq m$, $m \leq n$. Let $p \in \Omega_1$, $\xi_j \in T_p^{\mathbb{C}}\Omega_1$, $j = 1, \ldots, m$, and ξ_j's be linearly independent. If $\phi : \Omega_1 \longrightarrow \Omega_2$ is a biholomorphism, we have*

$$K_U^{\Omega_1}(p; \xi_1, \ldots \xi_m) = K_U^{\Omega_2}(\phi(p); \phi_*(p)\xi_1, \ldots \phi_*(p)\xi_m), \quad \text{and}$$
$$C_U^{\Omega_1}(p; \xi_1, \ldots, \xi_m) = C_U^{\Omega_2}(\phi(p); \phi_*(p)\xi_1, \ldots \phi_*(p)\xi_m).$$

Proof. Proposition 1 holds for ϕ and ϕ^{-1}. Therefore we have inequalities in both directions. □

Corollary 2. *Let $U = \mathbb{B}_{m-j} \times \Delta_j$, $p \in U$, $\xi_j \in T_p^{\mathbb{C}}U$, $1 \leq j \leq m$, and ξ_j's be linearly independent vectors. We have $K_U^U(p; \xi_1, \ldots \xi_m) = C_U^U(p; \xi_1, \ldots, \xi_m)$ for all $p \in U$.*

Proof. Since the automorphism group on U is transitive, we may assume $p = 0$. Also we may assume $\mu_m\left(\xi_1, \ldots, \xi_m, \overline{\xi}_1, \ldots, \overline{\xi}_m\right) = 1$. Let $f \in H(U, U)$ be such that $f(0) = 0$ and that $f^*(0)\mu_m = \alpha\mu_m$, $\alpha > 0$. By Carathéodory-Cartan-Kaup-Wu theorem, we have $\alpha \leq 1$. Since one can choose f as the identity mapping, we have $\inf 1/\alpha = 1 = \sup \alpha$. Therefore $K_U^U(0, \mu_m) = C_U^U(0, \mu_m) = 1$. The automorphism group on U is transitive. Hence by Corollary 1 we have $K_U^U(p, \mu_m) = C_U^U(p, \mu_m)$ for any $p \in U$. □

Proposition 2. *Let $\Omega \subset \mathbb{C}^n$, $p \in \Omega$ and $\xi_1, \ldots, \xi_m \in T_p^{\mathbb{C}}\Omega$, $1 \leq m \leq n$ be linearly independent vectors. If $U = \mathbb{B}_{m-j} \times \Delta_j$, $0 \leq j \leq m$, then*

$$\frac{C_U^{\Omega}(p, \xi_1, \ldots, \xi_m)}{K_U^{\Omega}(p, \xi_1, \ldots, \xi_m)} \leq 1. \tag{8}$$

Proof. Let M be an (m, m) volume form on Ω such that $M\left(\xi_1, \ldots, \xi_m, \overline{\xi}_1, \ldots, \overline{\xi}_m\right) = 1$. Let $\Phi : U \longrightarrow \Omega$ be a holomorphic mapping such that $\Phi(0) = p$, $\Phi^*(0)M = \alpha\mu_m$, $\alpha > 0$ and $\Psi : \Omega \longrightarrow U$ be a holomorphic mapping such that $\Psi(p) = 0$, $\Psi^*(p)\mu_m = \beta M$. Consider $h = \Psi \circ \Phi : U \longrightarrow U$. Then $h(0) = 0$ and $h^*(0)\mu_m = \alpha \cdot \beta \cdot \mu_m$. By Carathéodory-Cartan-Kaup-Wu theorem we have $\alpha \cdot \beta \leq 1$. Hence $\beta \leq 1/\alpha$. The inequality (8) follows after taking the infimum over α's and the supremum over β's. □

Lemma 1. *Let $\Omega \subset \mathbb{C}^n$, $p \in \Omega$ and $\xi_1, \ldots, \xi_m \in T_p^{\mathbb{C}}\Omega$, $1 \leq m \leq n$, be linearly independent vectors. Let $U = \mathbb{B}_{m-j} \times \Delta_j$. We have $\dfrac{C_U^{\Omega}(p; \xi_1, \ldots, \xi_m)}{K_U^{\Omega}(p; \xi_1, \ldots, \xi_m)} = 1$ if and only if Ω is biholomorphic to U.*

Proof. One can use a similar argument as in [9] (Theorem E). □

The Kobayashi m-measure is localizable near a strongly pseudocovnex boundary point. Refer to [6] for a detailed explanation. The Carathéodory m-measure is localizable near a boundary point p if one can find a global peak function that peaks at p. Hence we have the following Lemma.

Lemma 2. *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain and $p \in \partial\Omega$ be a strongly covnex boundary point. Let V be a neighborhood of p. Then we have*

$$\frac{K_U^{\Omega}(z; \xi_1, \ldots, \xi_m)}{K_U^{\Omega \cap V}(z; \xi_1, \ldots, \xi_m)} \to 1, \quad \frac{C_U^{\Omega}(z; \xi_1, \ldots, \xi_m)}{C_U^{\Omega \cap V}(z; \xi_1, \ldots, \xi_m)} \to 1, \quad \text{as } z \to p.$$

Remark 1. Let Ω be a smoothly bounded convex domain. The domain Ω near a strongly convex boundary point can be approximated by ellipsoids which are biholomorphic to balls. Since $\mathbb{B}_m$ and $\mathbb{B}_{m-j} \times \Delta_j$, $j \geq 1$, are not biholomorphic and the Kobayashi and Carathéodory measures are localizable near a strongly convex boundary point by Lemma 2, we have

$$\frac{C_U^{\Omega}(z;\xi_1,\ldots,\xi_m)}{K_U^{\Omega}(z;\xi_1,\ldots,\xi_m)} < c < 1, U = \mathbb{B}_{m-j} \times \Delta_j,\ j \geq 1$$
$$\frac{C_U^{\Omega}(z;\xi_1,\ldots,\xi_m)}{K_U^{\Omega}(z;\xi_1,\ldots,\xi_m)} \to 1, \quad U = \mathbb{B}_m$$

as z approaches a strongly convex boundary point.

3. Geometry of a convex domain.

3.1. Non-tangential convergence. Let $\Omega \subset \mathbb{C}^n$ be a domain with a C^1-boundary. Let $\{q_j\} \subset \Omega$ be a sequence of points. We say $q_j \to q \in \partial\Omega$ *non-tangentially* for some boundary point q if

$$q_j \in \Gamma_\alpha(q) = \{z \in \Omega : |z - q| < \alpha \mathrm{dist}(z, \partial\Omega)\} \tag{9}$$

for all j large enough for some $\alpha > 1$ and we say $q_j \to q \in \partial\Omega$ *normally* if q_j's approach q along the real normal line to the boundary through q for all j large enough.

Lemma 3. *Let $\Omega \subset \mathbb{C}^n$ be a convex domain with C^1 boundary and $q \in \partial\Omega$. Let ν be the outward unit normal vector to $\partial\Omega$ at q and $q' = q - t\nu \in \Omega$ for some small $t > 0$. Then we have*

$$\Gamma_\alpha(q) \subset \{z \in \Omega : 0 \leq \angle zqq' < \arccos(1/\alpha)\}.$$

Proof. Let $q = 0$ and $\nu = (0,\ldots,0,1)$. Then $\Omega \subset H = \{\mathrm{Re} z_n < 0\}$. Therefore $\mathrm{dist}(z,\partial\Omega) \leq \mathrm{dist}(z,\partial H) = |\mathrm{Re} z_n|$. Hence $|z - q| < \alpha|\mathrm{Re} z_n| = \alpha|(0,\ldots,0,\mathrm{Re} z_n)|$. Therefore $\angle zqq' < \arccos(1/\alpha)$. □

Lemma 4. *Let $\Omega \subset \mathbb{C}^n$ be a convex domain with C^1 boundary. Suppose $\{\phi_j\} \subset Aut(\Omega)$ and $\phi_j(p) \to q \in \partial\Omega$ non-tangentially for some $p \in \Omega$. Then there exists $\{p_j\} \subset \Omega$ such that $\phi_j(p_j) \to q$ normally and that $d_K^{\Omega}(p,p_j) \leq r$ for some $r > 0$.*

Proof. Let $\phi_j(p) = q_j$. Since $q_j \to q$ non-tangentially, one can find $\alpha > 1$ such that $q_j \in \Gamma_\alpha(q)$ for all j large enough.

Let ν be the outward unit normal vector to $\partial\Omega$ at q and ℓ_q be the real normal line to $\partial\Omega$ through q, i.e., $\ell_q = \{q + t\nu : t \in \mathbb{R}\}$. Define the mapping $\pi : \Omega \longrightarrow \ell_q$ as the projection of Ω onto ℓ_q. Let $\tilde{q}_j = \pi(q_j)$. Then we have $|q_j - \tilde{q}_j| \leq |q - q_j| < \alpha \mathrm{dist}(q_j, \partial\Omega)$. Let $p_j = \phi_j^{-1}(\tilde{q}_j)$. Then we have $\phi_j(p_j) = \tilde{q}_j \to q$ normally after taking a subsequence if necessary.

Since Ω is convex, by Lemma 3, we have $0 \leq \angle q_j q \tilde{q}_j \leq \arccos(1/\alpha)$. Therefore

$$\cos(\angle q_j q \tilde{q}_j) = \frac{|\tilde{q}_j - q|}{|q_j - q|} \geq \frac{1}{\alpha}.$$

Let $\gamma(t) = (1-t)\, q_j + t\tilde{q}_j$. Then we have

$$\begin{aligned} d_K^{\Omega}(p, p_j) = d_K^{\Omega}(q_j, \tilde{q}_j) &\le \int_0^1 F_K(\gamma(t), \gamma'(t))\, dt \\ &\le \int_0^1 |\gamma'(t)| \frac{1}{\operatorname{dist}(\gamma(t), \partial\Omega)}\, dt \le \int_0^1 |\gamma'(t)| \frac{\alpha}{|\gamma(t) - q|}\, dt \\ &\le \frac{\alpha |q_j - \tilde{q}_j|}{|\tilde{q}_j - q|} \le \frac{\alpha |q_j - q|}{|\tilde{q}_j - q|} \le \alpha^2. \end{aligned}$$

We let $r = \alpha^2$. □

LEMMA 5. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded complete hyperbolic domain with a C^2 boundary and $p \in \partial\Omega$ be a strongly convex boundary point. Then for any fixed $r > 0$, the Euclidean diam $\beta_K^{\Omega}(z, r) \longrightarrow 0$ as $z \to p$, where*

$$\beta_K^{\Omega}(z, r) = \left\{ w \in \Omega : d_K^{\Omega}(z, w) < r \right\} \subset \Omega.$$

Proof. Let $\delta(z) = \operatorname{dist}(z, \partial\Omega)$ and $z' \in \Omega$ be the boundary point that satisfies $|z - z'| = \delta(z)$. It is a well-known fact that for $z \in \Omega$ close to a strongly pseudoconvex boundary point the Kobayashi metric estimate is given as follows (refer to [1, 3]):

$$F_K^{\Omega}(z, \xi) \approx \frac{1}{\delta(z)} \xi_N + \frac{1}{\sqrt{\delta(z)}} \xi_T,$$

where ξ_T and ξ_N are the tangential and normal components of ξ at z', respectively. The assertion can be derived from the above fact and the complete hyperbolicity. □

3.2. Maximal chain of analytic discs. Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain and V be a connected subset of $\partial\Omega$. We say $\partial\Omega$ is *geometrically flat* along V if the direction of the gradient vector of $\partial\Omega$ does not change along V.

The following proposition is the generalization of Lemma 3.2 in [10]. The proof is basically the same.

PROPOSITION 3. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded convex domain. If $\phi : \Delta \longrightarrow \partial\Omega$ is a holomorphic mapping, then $\partial\Omega$ is geometrically flat along $\phi(\Delta)$.*

Proof. Let $\Omega = \{\rho < 0\}$ and $p = \phi(0) \in \partial\Omega$. Let $H = \{\operatorname{Re} h = 0\}$ be the real tangent plane to $\partial\Omega$ at p, where h is a linear holomorpic function. Since Ω is convex, we have $\overline{\Omega} \subset \{\operatorname{Re} h \le 0\}$. Consider $f(\zeta) = h \circ \phi(\zeta)$. Then f is a holomorphic function on Δ and satisfies $\operatorname{Re} f(\zeta) \le 0$ for all $\zeta \in \Delta$ and that $\operatorname{Re} f(0) = 0$. By the maximum principle for harmonic functions, we have $\operatorname{Re} f(\zeta) = 0$ for all $\zeta \in \Delta$. Therefore $f \equiv 0$ on Δ and hence $h \equiv 0$ on $\phi(\Delta)$. □

DEFINITION 3. Let $H \subset \mathbb{C}^n$ be a subset of $\mathbb{C}^n$ and $q \in H$. We define the maximal chain of analytic discs on H through q, denoted as Δ_q^H, as follows:

$$\Delta_q^H = \{z \in H : \text{there exists a finite chain of analytic discs joining } z \text{ and } q\},$$

i.e., there exists holomorphic maps $\phi_1, \phi_2, \ldots, \phi_k : \Delta \longrightarrow \mathbb{C}^n$ such that $\phi_j(\Delta) \subset H$, $1 \le j \le k$, and $z_i \in H$, $a_i, b_i \in \Delta$, $1 \le i \le k$, such that $\phi_j(a_j) = z_{j-1}$, $\phi_j(b_j) = z_j$, where $z_0 = q$ and $z_k = z$. Note that $\Delta_q^H = \Delta_z^H$, if $z \in \Delta_q^H$. We say Δ_q^H is trivial if $\Delta_q^H = \{q\}$.

REMARK 2. If $V \subset H$ is a complex variety through q, then $V \subset \Delta_q^H$.

The following Corollary follows immediately from Proposition 3.

COROLLARY 3. *If $\Omega \subset\subset \mathbb{C}^n$ is a smoothly bounded convex domain, then $\partial\Omega$ is geometrically flat along $\Delta_q^{\partial\Omega}$ for all $q \in \partial\Omega$.*

In the following theorem we show that a maximal chain of analytic discs on the boundary of a smoothly bounded convex domain is linearly convex.

THEOREM 1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded convex domain. Then $\Delta_q^{\partial\Omega}$ is linearly convex for all $q \in \partial\Omega$, i.e., if $z, w \in \Delta_q^{\partial\Omega}$, then $t \cdot z + (1-t)\, w \in \Delta_q^{\partial\Omega}$ for all $t \in [0,1]$.*

Proof. We first show that if $z, w \in \Delta_q^{\partial\Omega}$, then $t \cdot z + (1-t)\, w \in \partial\Omega$ for all $t \in [0,1]$. Since $\partial\Omega$ is geometrically flat along $\Delta_q^{\partial\Omega}$, we may assume $\Delta_q^{\partial\Omega} \subset \{\text{Re } z_n = 0\}$. We have $t \cdot z + (1-t)\, w \in \overline{\Omega}$ since Ω is convex. Also Re $(t \cdot z + (1-t)\, w)_n = t \cdot$ Re $z_n + (1-t)\,\text{Re } w_n = 0$ for all $t \in [0,1]$. Since Re $z_n < 0$ for all $z \in \Omega$, we have $t \cdot z + (1-t)\, w \in \partial\Omega$.

We use induction on the length of the chain (i.e. number of analytic discs) joining two points $z, w \in \Delta_q^{\partial\Omega}$.

Suppose $z, w \in \Delta_q^{\partial\Omega}$ and z, w both lie on the same analytic disc, then $t \cdot z + (1-t)\, w \in \Delta_q^{\partial\Omega}$. Let $z = \phi(a)$ and $w = \phi(b)$ for some analytic disc $\phi : \Delta \longrightarrow \partial\Omega$ and $a, b \in \Delta$ and define an analytic disc $\tilde{\phi}_t$ as follows:

$$\tilde{\phi}_t(\zeta) = t \cdot \phi(\zeta) + (1-t)\, \phi(b).$$

Then $\tilde{\phi}_t(\zeta) \in \partial\Omega$ for all $\zeta \in \Delta$ and for any fixed $t \in [0,1]$, and $\tilde{\phi}_t(b) = \phi(b) \in \Delta_q^{\partial\Omega}$. Hence $\tilde{\phi}_t(\zeta) \in \Delta_q^{\partial\Omega}$ for all $\zeta \in \Delta$. Therefore $\tilde{\phi}_t(a) = t \cdot \phi(a) + (1-t)\, \phi(b) \in \Delta_q^{\partial\Omega}$ for all $t \in [0,1]$.

Assume $t \cdot z + (1-t)\, w \in \Delta_q^{\partial\Omega}$ for all $t \in [0,1]$ if z, w can be joined by a chain of length less than or equal to n. Suppose $z, w \in \Delta_q^{\partial\Omega}$ can be joined by $n+1$ number of analytic discs, i.e., there exists analytic discs $\phi_j : \Delta \longrightarrow \partial\Omega$, $a_j, b_j \in \Delta$ and $z_j \in \partial\Omega$, $1 \le j \le n+1$, such that $\phi_j(a_j) = z_{j-1}$, $\phi_j(b_j) = z_j$ and $z = z_0$, $w = z_{n+1}$. Define an analytic disc $\tilde{\phi}_t$ as follows:

$$\tilde{\phi}_t(\zeta) = t \cdot \phi_1(\zeta) + (1-t)\, \phi_{n+1}(b_{n+1}), \quad t \in [0,1].$$

Then $\tilde{\phi}_t(\zeta) \in \partial\Omega$ for all $\zeta \in \Delta$ and for all $t \in [0,1]$. We have

$$\begin{aligned} \tilde{\phi}_t(b_1) &= t \cdot \phi_1(b_1) + (1-t)\, \phi_{n+1}(b_{n+1}) \\ &= t \cdot \phi_2(a_2) + (1-t)\, \phi_{n+1}(b_{n+1}) \end{aligned}$$

and hence $\tilde{\phi}_t(b_1) \in \Delta_q^{\partial\Omega}$ for all $t \in [0,1]$ since $\phi_2(a_2)$ and $\phi_{n+1}(b_{n+1})$ are joined by n analytic discs. Therefore $\tilde{\phi}_t(\zeta) \in \Delta_q^{\partial\Omega}$ for all $\zeta \in \Delta$ and hence $\tilde{\phi}_t(a_1) = t \cdot z + (1-t)\, w \in \Delta_q^{\partial\Omega}$ for all $t \in [0,1]$. □

4. Normal convergence.

PROPOSITION 4. *Let Ω be a smoothly bounded convex domain in $\mathbb{C}^n$. Suppose $\{\phi_j\} \subset Aut(\Omega)$ and $\phi_j(p) \to q \in \partial\Omega$ non-tangentially for some $p \in \Omega$ and that $\Delta_q^{\partial\Omega}$ is not trivial. Then there exists a non-constant holomorphic onto mapping $\phi : \Omega \longrightarrow \Delta_q^{\partial\Omega}$ such that $\phi_j \to \phi$ after taking a subsequence if necessary.*

Proof. Since $\phi_j(p) \to q \in \partial\Omega$, we konw that $\phi_j \to \phi$ locally uniformly (after taking a subsequence if necessary) where $\phi : \Omega \longrightarrow \partial\Omega$ is a holomorphic mapping by a normal family argument.

We shall show that $\phi(\Omega) = \Delta_q^{\partial\Omega}$. Since $\phi(\Omega) \subset \Delta_q^{\partial\Omega}$ is clear, we need only to show that $\Delta_q^{\partial\Omega} \subset \phi(\Omega)$.

Let $q' \in \Delta_q^{\partial\Omega}$ and $q' \neq q$. By Corollary 3, $\partial\Omega$ is geometrically flat along $\Delta_q^{\partial\Omega}$. Let ν be the constant outward unit normal vector to $\partial\Omega$ along $\Delta_q^{\partial\Omega}$. By Lemma 4, there exists $\{p_j\} \subset \beta_K^\Omega(p,r)$ for some $r > 0$ such that $\phi_j(p_j) \to q$ normally. Let δ_j's be such that

$$\phi_j(p_j) = q - \delta_j \nu.$$

Then we have

$$d_K^\Omega(q - \delta_j\nu, q' - \delta_j\nu) < r' < \infty,$$

for all j for some $r' > 0$. Hence if we let $p'_j = \phi_j^{-1}(q' - \delta_j\nu)$, then

$$\begin{aligned} d_K^\Omega(p'_j, p) &\le d_K^\Omega(p_j, p'_j) + d_K^\Omega(p_j, p) \\ &= d_K^\Omega(q - \delta_j\nu, q' - \delta_j\nu) + r < r + r' < \infty, \quad \forall j. \end{aligned}$$

Since $\overline{\beta_K^\Omega(p, r + r')}$ is compact in Ω, one can find $p' \in \Omega$ such that $p'_j \to p'$ and that $\phi(p') = q'$. Therefore $\Delta_q^{\partial\Omega} \subset \phi(\Omega)$. □

Corollary 4. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded convex domain and $\{\phi_j\} \subset Aut(\Omega)$. If $\phi_j(p) \to q \in \partial\Omega$ non-tangentially and $\Delta_q^{\partial\Omega}$ is not trivial, then $\Delta_q^{\partial\Omega}$ is an open convex set contained in a complex m-dimensional plane, where $m = dim_{\mathbb{C}}\Delta_q^{\partial\Omega}$.*

Proof. By Theorem 1, $\Delta_q^{\partial\Omega}$ is convex. Hence it is contained in a complex m-dimensional plane, where $m = \dim_{\mathbb{C}}\Delta_q^{\partial\Omega}$. Suppose $\Delta_q^{\partial\Omega}$ is not open and $w \in \partial\Delta_q^{\partial\Omega}$ is a boundary point. By Proposition 4, one can find $z \in \Omega$ such that $\phi(z) = w$, where ϕ is the limit of $\{\phi_j\}$. One can find a germ of complex m-dimensional manifold, say M, near z such that $\dim_{\mathbb{C}}\phi(M) = m$. Let H be the complex $m-1$ dimensional subspace of the real supporting plane to $\Delta_q^{\partial\Omega}$ at $w = \phi(z)$. By the maximum principle argument used in Proposition 3, we have that $\phi(M) \subset H$. But dim $H < m$. Hence a contradiction. □

Theorem 2. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded domain. Suppose $\Delta_q^{\partial\Omega}$ is not trivial for some $q \in \partial\Omega$ and that $\phi : \Omega \longrightarrow \Delta_q^{\partial\Omega}$ is a surjective holomorphic mapping. Then there exists a sequence of points $\{p_j\} \subset \Omega$ such that $p_j \to p \in \partial\Omega$ and that $\{\phi(p_j)\} \subset \Delta_q^{\partial\Omega}$ converge to a point in $\Delta_q^{\partial\Omega}$ for some strongly pseudoconvex boundary point $p \in \partial\Omega$.*

Proof. Since Ω is smoothly bounded, there exists a strongly pseudoconvex boundary point $p \in \partial\Omega$. Let ν be the outward unit normal vector to $\partial\Omega$ at p. One can find a holomorphic support function h of $\partial\Omega$ at p such that, for a small neighborhood U of p, we have $\{h = 0\} \cap \overline{\Omega} \cap U = \{p\}$. Let $H = \{h = 0\}$ and let H_n be the translation of H in the direction of $-\nu$ by the length of $1/n$, i.e.,

$$H_n = \left\{ z - \nu\frac{1}{n} : z \in H \right\}, \quad n \in \mathbb{N}.$$

One can find a small neighborhood U of p and $N > 0$ large enough such that $\partial\Omega \cap U$ is strongly pseudoconvex and that $H_n \cap \Omega \subset U \cap \Omega$ for all $n > N$.

Let $\dim_{\mathbb{C}} \Delta_q^{\partial\Omega} = m$. Choose a complex m-dimensional closed analytic subset of H_n through $p - \nu \cdot \frac{1}{n}$ and perturb it at $p - \nu \cdot \frac{1}{n}$, call it H_n', so that the rank of the restriction mapping of ϕ on H_n', say $\phi_n : H_n' \longrightarrow \Delta_q^{\partial\Omega}$, has rank m generically and that $\partial H_n' \subset \partial\Omega$. One can make the perturbation small enough that $H_n' \subset U \cap \Omega$ for all n. Suppose ϕ_n is not proper for some $n > N$. Then one can find a compact set $K \subset\subset \Delta_q^{\partial\Omega}$ such that the preimage of K is not compact in H_n'. Hence one can find $\{p_j\} \subset H_n'$ such that $\phi(p_j)$'s lie in K for all j and p_j's approach a boundary point of H_n', which is strongly pseudoconvex.

If ϕ_n is proper for all $n > N$, then they are surjective because the rank of ϕ_n is equal to m. One can find $p_n \in H_n'$ for $n > N$, arbitrarily close to p, which is a strongly pseudoconvex point. Moreover $\{\phi(p_n)\}$ converge to a point in $\Delta_q^{\partial\Omega}$. □

LEMMA 6. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose there exists $\{\phi_j\} \subset Aut(\Omega)$ such that $\phi_j(z)$ converges to some boundary point non-tangentially for all $z \in \Omega$ for some fixed α in (9) and that $\Delta_q^{\partial\Omega}$ is not trivial for some orbit accumulation point $q \in \partial\Omega$. Then for any $\epsilon > 0$ one can find $\{p_j\} \subset \Omega$ such that $\phi_j(p_j) \to q' \in \Delta_q^{\partial\Omega}$ normally for some point q' and that $p_j \in B(p', \epsilon) \cap \Omega$ for some strongly convex boundary point $p' \in \partial\Omega$.*

Proof. By Proposition 4, ϕ_j's converge locally uniformly to a non-constant holomorphic mapping $\phi : \Omega \longrightarrow \Delta_q^{\partial\Omega}$. By Theorem 2, one can find a point z close enough to some strongly pseudoconvex boundary point p' such that $\phi(z) = q'$ for some $q' \in \Delta_q^{\partial\Omega}$. We have $\phi_j(z) \to q'$ non-tangentially as $j \to \infty$. Therefore by Lemma 4, one can find $r > 0$ and $\{p_j\} \subset \beta_K^{\Omega}(z, r)$ such that $\phi_j(p_j) \to q'$ normally as $j \to \infty$. As shown in the proof of Lemma 4, r depends on α, $r = \alpha^2$, to be precise. Since we assume $\alpha > 0$ is fixed, by Lemma 5 one can choose z close enough to p' such that $\beta_K^{\Omega}(z, r) \subset B(p', \epsilon)$. □

5. Boundary accumulation points.

PROPOSITION 5. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose $\Delta_q^{\partial\Omega}$ is not trivial for some $q \in \partial\Omega$. If there exists $\{\phi_j\} \subset Aut(\Omega)$ such that $\phi_j(z) \to \Delta_q^{\partial\Omega}$ nontangentially for all $z \in \Omega$, then $\Delta_q^{\partial\Omega}$ is biholomorphic to a complex m-ball, where m is the complex dimension of $\Delta_q^{\partial\Omega}$ (i.e., real $2m$ dimensional ball).*

Proof. Let $p \in \Omega$ be arbitrarily close to a strongly pseudoconvex boundary point and let $\phi(p) = q \in \Delta_q^{\partial\Omega}$. Also denote $p_j = \phi_j(p)$ and $V = \Delta_q^{\partial\Omega}$.

Let $\xi_1, \ldots, \xi_m$ be m linearly independent complex tangent vectors to V and use the intrinsic measure defined with respect to the complex unit m-ball, i.e., U is the complex unit m-ball in Definition 2. We may assume V lies in the $z_2 \ldots z_{m+1}$ plane, where $\operatorname{Re} z_1$ is the outward normal direction. Let π be the projection mapping of $\mathbb{C}^n$ onto the $z_1 \ldots z_{m+1}$ plane and $\tilde{p}_j = \pi(p_j)$. For j large enough, one can find V' such that $q \in V' \subset\subset V$ and that one can move V' into Ω using the translation mapping that maps q to $\tilde{p}_j$. Let V_j' be the image of such translation mapping of V'.

We may assume $q = 0$. Suppose $p_j = (a_1, \ldots, a_n), \tilde{p}_j = (a_1, \ldots, a_{m+1}, 0, \ldots, 0)$. Consider the holomorphic mapping $f_j : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ defined as
$f_j(z) = (h_1(z), \ldots, h_n(z))$, where

$$h_k = \begin{cases} z_k, & k = 1, \ldots, m+1 \\ \dfrac{a_k \cdot \overline{a_1}}{|a_1|^2} z_1, & k = m+2, \ldots, n. \end{cases}$$

Then $f_j(0) = 0$ and $f_j(\tilde{p}_j) = p_j$.

We have

$$\frac{C_U^{\Omega}\left(p; \left(\phi_j^{-1} \circ f_j\right)_* (\tilde{p}_j)\, \xi_l\right)}{K_U^{\Omega}\left(p; (\phi^{-1})_* (q)\, \xi_l\right)} \leq \frac{C_U^{f_j(V_j')}\left(p_j; (f_j)_* (\tilde{p}_j)\, \xi_l\right)}{K_U^{\Omega}\left(p; (\phi^{-1})_* (q)\, \xi_l\right)}$$
$$\leq \frac{C_U^{V_j'}\left(\tilde{p}_j; \xi_l\right)}{K_U^{\Omega}\left(p; (\phi^{-1})_* (q)\, \xi_l\right)} \leq \frac{C_U^{V'}\left(q; \xi_l\right)}{K_U^{V}\left(q; \xi_l\right)},$$

where ξ_l stands for the set of m-vectors, $\xi_1, \dots, \xi_m$. Note that $(\phi^{-1})_* \xi_j$ should be interpreted as the pre image vector of ξ_j, which is well-defined since the rank of ϕ is m along $\Delta_q^{\partial\Omega}$.

As $j \to \infty$, one can let $V' \to V$. Then the left hand side approaches 1, whereas the right hand side is always less than or equal to 1.

Therefore we have

$$\frac{C_U^{V}\left(q; \xi_l\right)}{K_U^{V}\left(q; \xi_l\right)} = 1$$

and hence V is biholomorphic to a complex m-dimensional ball. □

In the following Theorem, we assume that there exists $\alpha < \infty$ such that (9) holds for all z and in Theorem 4, we will give a proof without the assumption on α. The proof of Theorem 3 has its own merit, since is uses the invariant measures to compare the domain Ω near a strongly convex boundary point and a flat boundary point.

Theorem 3. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose there exists $\{\phi_j\} \subset Aut(\Omega)$ such that $\phi_j(z)$ converges nontangentially to some boundary point for all $z \in \Omega$. We also assume there exists $\alpha < \infty$ such that (9) holds for all $z \in \Omega$. If $q \in \partial\Omega$ is an orbit accumulation point, then $\Delta_q^{\partial\Omega}$ is trivial and hence there does not exist a complex variety on $\partial\Omega$ passing through q.*

Proof. Suppose $\Delta_q^{\partial\Omega}$ is not trivial and let $V = \Delta_q^{\partial\Omega}$. Let m be the complex dimension of V. Since V is convex by Theorem 1, we may assume V lies on a complex m-dimensional plane.

We may assume $\nu = (1, 0, \dots, 0)$ is the constant outward unit normal vector along V and V lies in $z_2 z_3 \cdots z_{m+1}$ plane after a linear change of coordinates. Let $\pi : \Omega \to \{z_{m+2} = z_{m+3} = \cdots = z_n = 0\}$ be the projection mapping.

By Lemma 6, one can find a strongly convex boundary point $p' \in \partial\Omega$ such that for any $\epsilon > 0$, there exists $\{p_j\} \subset B(p', \epsilon) \cap \Omega$ such that $\phi_j(p_j) = q_j \to q' \in V$ normally for some $q' \in \Delta_q^{\partial\Omega}$. Choose Ω_j's, as a relatively compact exhaustion of Ω, such that $\Omega_j \nearrow \Omega$ and that $p_j \in \Omega_j$ for all j. Let $U = \Delta \times \mathbb{B}_m$ and choose m linearly independent vectors $\xi_1, \dots, \xi_m \in T_{q'}^{\mathbb{C}} V$. Since $\partial\Omega$ is geometrically flat along V, we have $\xi_j \in T_{q'-\nu\epsilon}^{\mathbb{C}}(V - \nu\epsilon)$. Hence for j large enough $\xi_j \in T_{q_j}^{\mathbb{C}}(V - \nu\, |q_j - q'|)$. Let $\xi_j' = \left(\phi_j^{-1}\right)_* (q_j)\, \xi_j$ and $\nu' = \left(\phi_j^{-1}\right)_* (q_j)\, \nu$

We let $\Gamma_\epsilon = \left\{z \in \mathbb{C} : \frac{\pi}{2} + \epsilon < \arg z < \frac{3\pi}{2} - \epsilon\right\}$ and $H = \{z \in \mathbb{C} : \text{Re } z < 0\}$. Then $\Gamma_\epsilon \to H$ as $\epsilon \to 0$. Let V_ϵ be a subset of $\partial\Omega$ such that $V_\epsilon \searrow V$ as $\epsilon \to 0$. Then we have

$$\frac{C_U^{\Omega_j}(p_j;\nu',\xi_1',\ldots,\xi_m')}{K_U^{\Omega}(p_j;\nu',\xi_1',\ldots,\xi_m')} \geq \frac{C_U^{\phi_j(\Omega_j)}(q_j;\nu,\xi_1,\ldots,\xi_m)}{K_U^{\Omega}(q_j;\nu,\xi_1,\ldots,\xi_m)} \tag{10}$$

$$\geq \frac{C_U^{\pi(\phi_j(\Omega_j))}(q_j;\nu,\xi_1,\ldots,\xi_m)}{K_U^{\Omega}(q_j;\nu,\xi_1,\ldots,\xi_m)}$$

$$\geq \frac{C_U^{(H\times V_\epsilon)\cap W'}(q_j;\nu,\xi_1,\ldots,\xi_m)}{K_U^{(\Gamma_\epsilon\times V)\cap W'}(q_j;\nu,\xi_1,\ldots,\xi_m)}, \tag{11}$$

where $W' = W\cap\Omega$, W an open neighborhood of V. In the last inequality we used the inclusion mapping $i : \pi(\phi_j(\Omega_j)) \longrightarrow (H\times V_\epsilon)\cap W'$ for the numerator and another inclusion mapping $\tilde{i} : (\Gamma_\epsilon \times V_\epsilon)\cap W' \longrightarrow \Omega$ for the denominator. The left hand side of (10) is strictly less than 1 since we may assume p_j is arbitrarily close to a strongly convex boundary point and j is large enough, whereas the right hand side of (11) approaches 1 since one can let $\epsilon \to 0$ as $j\to\infty$, choose W small enough, and V is biholomorphic to a ball by Proposition 5, which leads to a contradition. □

REMARK 3. In the proof of Theorem 3, one can let $U = \mathbb{B}^{m+1}$ instead of $\Delta\times\mathbb{B}^m$. In this case we should consider the ratio K^Ω/C^Ω. The left hand side of (10) approaches 1 as p_j's approach a strongly pseudoconvex boundary point, whereas the right hand side of (11) is strictly greater than 1 as q_j's approach a flat boundary point. Hence it gives rise to a contradiction.

Additionally, we prove a lemma that shows that if a point converges non-tangentially then all the other points must converge non-tangentially in the normal direction.

LEMMA 7. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose $\Delta_q^{\partial\Omega}$ is not trivial for some $q\in\partial\Omega$ and that there exists $p\in\Omega$ and $\{\phi_j\}\subset Aut(\Omega)$ such that $\phi_j(p)\to q\in\Delta_q^{\partial\Omega}$ non-tangentially. Then $\phi_j(a)\to b\in\Delta_q^{\partial\Omega}$ non-tangentially in the normal direction for all $a\in\Omega$ for some $b\in\Delta_q^{\partial\Omega}$.*

Proof. Let $a\in\Omega$. Since Ω is complete hyperbolic, we have $d_K^\Omega(p,a) = r < \infty$ for some $r > 0$.

We may assume $q = 0$ and the outward normal vector to $\partial\Omega$ along $\Delta_q^{\partial\Omega}$ is in the direction of $\mathrm{Re}z_n$-axis. Let $p_j = \phi_j(p)$ and $a_j = \phi_j(a)$. By Proposition 4, a_j's converge to $b\in\Delta_q^{\partial\Omega}$ for some $b\in\Delta_q^{\partial\Omega}$. Let p_j' and a_j' be the projection of p_j and a_j onto z_n-axis. Then $p_j' = (0,\ldots,0,s_j)$ and $a_j' = (0,\ldots,0,t_j)$ for some $s_j, t_j\in\mathbb{C}$. Let $t_j = A_j e^{i\alpha_j}$ and $s_j = P_j e^{i\theta_j}$. Since Ω is convex we have Re s_j, Re $t_j < 0$.

Since $p_j\to q$ non-tangentially, $p_j\in\Gamma_\alpha(q)$ for some α for all j large enough. By Lemma 3, we have $\pi - \theta_j < \arccos(1/\alpha)$ for all j large enough. Hence

$$\cos\theta_j < -1/\alpha. \tag{12}$$

We have

$$\infty > r = d_K^\Omega(p,a) = d_K^\Omega(p_j,a_j) \geq d_K^\Omega(p_j',a_j') \geq d_K^H(s_j,t_j), \tag{13}$$

where $H = \{z\in\mathbb{C} : \mathrm{Re}z < 0\}$. Using the Poincaré distance between two points $z,w\in\Delta$ given by $\ln\left(\frac{|1-w\overline{z}|+|w-z|}{|1-w\overline{z}|-|w-z|}\right)$ and the biholomorphic mapping $f(z) = (z +$

$1)/(z-1)$ that maps H to Δ, we get

$$d_K^H(s_j,t_j)=\ln\left(\frac{\frac{|t_j+\overline{s}_j|}{|\overline{s}_j-1|}+\frac{|t_j-s_j|}{|s_j-1|}}{\frac{|t_j+\overline{s}_j|}{|\overline{s}_j-1|}-\frac{|t_j-s_j|}{|s_j-1|}}\right).$$

We may assume $|s_j|,|t_j|<1/2$. Then we have

$$\begin{aligned} d_K^H(s_j,t_j) &\geq \ln\left(\frac{1}{3}\frac{|t_j+\overline{s}_j|+|t_j-s_j|}{|t_j+\overline{s}_j|-|t_j-s_j|}\right)\\ &\geq \ln\frac{1}{3}+\ln\left(\frac{\sqrt{1+\cos(\theta_j+\alpha_j)}+\sqrt{1-\cos(\theta_j-\alpha_j)}}{\sqrt{1+\cos(\theta_j+\alpha_j)}-\sqrt{1-\cos(\theta_j-\alpha_j)}}\right)\\ &\to\infty, \end{aligned}\tag{14}$$

if $\alpha_j\to\pi/2$. From (12), (13), and (14), we conclude that $a_j'\to b\in\Delta_q^{\partial\Omega}$ non-tangentially for some $b\in\Delta_q^{\partial\Omega}$. □

Remark 4. From Lemma 7, it is not hard to see counting the dimensions involved that if there exists a point $p\in\Omega$ such that $\{\phi_j(p)\}$ converges non-tangentially to a boundary point $q\in\partial\Omega$, then $\dim\Delta_q^{\partial\Omega}<n-1$, where $n=\dim\Omega$.

In the following theorem we give another proof of Theorem 3 without using the assumption that there exists $\alpha<\infty$ such that (9) holds for all $z\in\Omega$.

Theorem 4. *Let $\Omega\subset\subset\mathbb{C}^n$ be a smoothly bounded convex domain. Suppose there exists $\{\phi_j\}\subset Aut(\Omega)$ such that $\phi_j(z)$ converges nontangentially to some boundary point for all $z\in\Omega$. If $q\in\partial\Omega$ is an orbit accumulation point, then $\Delta_q^{\partial\Omega}$ is trivial and hence there does not exist a complex variety on $\partial\Omega$ passing through q.*

Proof. As in the proof of Theorem 3, one can assume $V=\Delta_q^{\partial\Omega}$ lies on a complex m-dimensional plane, where m is the complex dimension of V.

Let the Re z_1-direction be the outward normal direction along V and V lies on the complex $z_2z_3\cdots z_{m+1}$ plane.

Let $\Gamma_{\epsilon,r}$ be a wedge domain with radius less than r in $\mathbb{C}$ defined as $\Gamma_{\epsilon,r}=\left\{z\in\mathbb{C}:\frac{\pi}{2}+\epsilon<\arg z<\frac{3\pi}{2}-\epsilon,\ |z|<r\right\}$. Choose $p\in\Omega$ close to a strongly pseudoconvex boundary point. Then $\phi_j(p)\to q\in V$ non-tangentially for some q. Let $V'\subset\subset V$ and $q\in V'$. Consider the product domain $\Gamma_{\epsilon,r}\times V'\subset\overline{\Omega}$. Let $A_{\epsilon,r}$ be the interior of $\Gamma_{\epsilon,r}\times V'$. Let $q=0$, $p_j=\phi_j(p)$ and $\tilde{p}_j$ be the projection of p_j onto the $z_1z_2\cdots z_{m+1}$-plane, i.e. if $p_j=(a_1,\dots,a_n)$, then $\tilde{p}_j=(a_1,a_2,\dots,a_{m+1},0,\cdots,0)$. Then $\tilde{p}_j\to q$ nontangentially.

Consider the holomorphic mapping $f_j:\mathbb{C}^n\longrightarrow\mathbb{C}^n$ defined as
$f_j(z)=(h_1(z),\dots,h_n(z))$, where

$$h_k=\begin{cases} z_k, & k=1,\dots,m+1\\ \dfrac{a_k\cdot\overline{a_1}}{|a_1|^2}z_1, & k=m+2,\dots,n.\end{cases}$$

Note that f_j is the identity mapping when restricted to V and $f_j(\tilde{p}_j)=p_j$. Since $p_j\to q$ non-tangentially, one can find $\epsilon,r>0$ such that $f_j(A_{\epsilon,r})\subset\Omega$ assuming j is large enough.

Let $U = \Delta\times\mathbb{B}_m$, ξ_j be the unit vector in the z_j-direction and Ω_k be the exhaustion of Ω, i.e, $\Omega_k \nearrow \Omega$. Then we have

$$\frac{C_U^{\Omega_k}\left(p; \left(\phi_j^{-1}\right)_* (p_j)\, \xi_l\right)}{K_U^{\Omega}\left(p; \left(\phi_j^{-1} \circ f_j\right)_* (\tilde{p}_j)\, \xi_l\right)} \geq \frac{C_U^{\phi_j(\Omega_k)}\left(p_j; \xi_l\right)}{K_U^{\Omega}\left(p_j; (f_j)_* (\tilde{p}_j)\, \xi_l\right)} \geq \frac{C_U^{A_{\epsilon,r}}\left(\tilde{p}_j; \xi_l\right)}{K_U^{A_{\epsilon,r}}\left(\tilde{p}_j; \xi_l\right)}, \tag{15}$$

where ξ_l stands for the set of $m+1$ vectors $\xi_2, \ldots, \xi_{m+1}$. Note that the first $(m+1)$ by $(m+1)$ complex Jacobian of f_j is the identity and hence $(f_j)_*\xi_l$ is well-defined for $l = 1, \ldots, m+1$. The second inequality for the Carathéodory measure is derived using the projection mapping of $\mathbb{C}^n$ onto the $z_1 z_2 \ldots z_{m+1}$ plane. For j and k large enough we may assume the projection of $\phi_j(\Omega_k)$ is inside $A_{\epsilon,r}$ for some ϵ and r. Note that the Jacobian matrix of the projection is identity along $z_1 \ldots z_{m+1}$ direction, hence $\xi_1, \ldots, \xi_{m+1}$ remain unchanged.

Since f_j is the identity along $z_1, \ldots, z_{m+1}$ directions, letting $j, k \to \infty$, we see that the left side of (15) is strictly less than 1, whereas the right hand side converges to 1 as one can let $\epsilon \to 0$ and $V' \to V$. Hence a contradiction. □

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