Polynomial Structure of Topological String Partition Functions

Jie Zhou

Abstract We review the polynomial structure of the topological string partition functions as solutions to the holomorphic anomaly equations. We also explain the connection between the ring of propagators defined from special Kähler geometry and the ring of almost-holomorphic modular forms defined on modular curves.

1 Introduction

The last few decades have witnessed many exciting developments and applications of mirror symmetry [Gre96, CK99, HKK$^+$03, Ali12]. One of the prominent applications is using mirror symmetry to predict Gromov-Witten invariants initiated by the celebrated work [CdLOGP91].

Consider a smooth Calabi-Yau (CY) 3-fold $\check{X}$ sitting in a family of Calabi-Yau 3-folds $\check{\pi} : \check{\mathscr{X}} \to \check{\mathscr{M}}$, where $\check{\mathscr{M}}$ is the moduli space of complexified Kähler structures[1] of $\check{X}$ whose dimension is $h^{1,1}(\check{X})$. The generating function of genus g Gromov-Witten invariants gives the following quantity defined on the moduli space

$$\check{F}_g(\check{t}) = \sum_{\beta \in H_2(\check{X},\beta)} \langle e^{\omega} \rangle_{g,\beta} \, , \tag{1}$$

where

$$\omega = \sum_{i=1}^{h^{1,1}(\check{X})} \check{t}^i \omega_i \, , \quad \langle \omega_{i_1} \omega_{i_2} \cdots \omega_{i_k} \rangle_{g,\beta} = \prod_{j=1}^{k} ev_{i_j}^* \omega_{i_j} \cap \overline{[\mathscr{M}_{g,k}(\check{X},\beta)]}^{\mathrm{vir}} \, .$$

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[1] Throughout this note, we shall simply call them Kähler structures by abuse of language.

Here

- $\{\omega_i\}_{i=1}^{h^{1,1}(\check{X})}$ are the generators for the Kähler cone in the moduli space $\check{\mathscr{M}}$ of Kähler structure of $\check{X}$;
- $\check{t} = \{\check{t}^i\}_{i=1}^{h^{1,1}(\check{X})}$ are local coordinates on the moduli space $\check{\mathscr{M}}$;
- $[\mathscr{M}_{g,k}(\check{X}, \beta)]^{\mathrm{vir}}$ is the virtual fundamental class of the moduli space $\mathscr{M}_{g,k}(\check{X}, \beta)$ of stable maps of genus g and class β with k markings;
- $ev_{i_j}, j = 1, 2, \cdots k$ are the evaluation maps: $\mathscr{M}_{g,k}(\check{X}, \beta) \to \check{X}$.

The quantity defined in Eq. (1) is *a priori* only a formal series defined near the point $e^{\check{t}^i} = 0, i = 1, 2, \cdots, h^{1,1}(\check{X})$. That it represents a well-defined function is due to the mirror symmetry conjecture which we shall review below. An alternative way to write the above generating function $F_g(\check{t})$ in which the Gromov-Witten invariants appear naturally is the following

$$\check{F}_g(\check{t}) = \sum_{\beta \in H_2(\check{X}, \mathbb{Z})} N_{g,\beta}(\check{X}) e^{2\pi i \int_\beta \omega} \,. \tag{2}$$

Note that in this formula $N_{g,\beta}(\check{X})$ is independent of $\check{t}$ but depends only on the generic member $\check{X}$ in the family, this results from the fact that the Gromov-Witten invariants are deformation invariant.

The mirror symmetry conjecture predicts that for the CY 3-fold family (A-model) $\check{\pi} : \check{\mathscr{X}} \to \check{\mathscr{M}}$, there exists another family (B-model) of CY 3-folds $\pi : \mathscr{X} \to \mathscr{M}$ satisfying the following properties:

Mirror symmetry conjecture

- The moduli space $\check{\mathscr{M}}$ of Kähler structures of $\check{X}$ is identified with the moduli space $\mathscr{M}$ of complex structures of a smooth CY 3-fold X called the mirror manifold. This implies in particular that $h^{1,1}(\check{X}) = \dim \check{\mathscr{M}} = \dim \mathscr{M} = h^{2,1}(X)$.
- There exist distinguished coordinates $\check{t} = \{\check{t}^i\}_{i=1}^{h^{1,1}(\check{X})}$ and $t = \{t^i\}_{i=1}^{h^{2,1}(X)}$ called canonical coordinates (see [BCOV94]) so that the map $\check{t} = t$ gives the identification $\check{\mathscr{M}} \cong \mathscr{M}$. This map is called the mirror map. In practice, one first matches some distinguished singular points on the moduli spaces, for example, the large volume limit on $\check{\mathscr{M}}$ and the large complex structure limit on $\mathscr{M}$ (see Section 2.2), then one identifies neighborhoods of these singular points by matching the canonical coordinates on the moduli spaces.
- For each genus g, there is a function $F_g(t)$ defined on the moduli space $\mathscr{M}$ so that under the mirror map, it is identical to $\check{F}_g(\check{t})$.
- Moreover, topological string theory tells that the more natural objects one should be looking at on both sides are some non-holomorphic functions $\check{\mathscr{F}}^{(g)}(\check{t}, \bar{\check{t}})$ and $\mathscr{F}^{(g)}(t, \bar{t})$ which are again identical under the mirror map. These quantities are called topological string partition functions. The "holomorphic limits"

[BCOV93, BCOV94] of the normalized partition functions give rise to the quantities $\check{F}_g(\check{t})$ and $F_g(t)$, respectively.

Determining the function $\mathscr{F}^{(g)}(t,\bar{t})$ is usually much easier than computing the Gromov-Witten invariants appearing in the generating function $\check{F}_g(\check{t})$, since the latter requires a careful study of the moduli space of stable maps which is in general very complicated (see e.g., [Kon95, Giv98, LLY99]), while the former satisfies some recursive differential equations called holomorphic anomaly equations [BCOV94] which are relatively easier to handle, as will be discussed below. These differential equations and the corresponding boundary conditions were derived from physics, but can be formulated purely in terms of mathematical language.

The general idea of counting curves via mirror symmetry is as follows. First one takes the holomorphic anomaly equations with boundary conditions as the defining equations for the topological string partition functions $\mathscr{F}^{(g)}(t,\bar{t})$. Then one tries to solve for them from the equations. After that one normalizes them and takes the holomorphic limit at the large complex structure limit to get $F_g(t)$. Finally one uses the mirror map $\check{t} = t$ which matches the large volume limit with the large complex structure limit to obtain $\check{F}_g(\check{t})$, and thus extract the Gromov-Witten invariants $N_{g,\beta}(\check{X})$ from Eq. (2). In this way, via mirror symmetry, counting curves in the CY 3-fold $\check{X}$ is translated into solving the holomorphic anomaly equations on the moduli space $\mathscr{M}$. Interested readers are referred to [BCOV93, BCOV94] for details on this subject.[2]

This note aims to study some properties of the holomorphic anomaly equations and the solutions. The plan of this note is as follows. In Section 2, we review the basics of special Kähler geometry and holomorphic anomaly equations. In Section 3, we explain the polynomial recursion technique and show how to solve the topological string partition functions genus by genus recursively from the holomorphic anomaly equations. In Section 4, we discuss the similarity between the ring of propagators and the ring of almost-holomorphic modular forms. We conclude this note in Section 5.

This introductory note is based on the lectures that the author gave in the Concentrated Graduate Courses for the Fields thematic program Calabi-Yau Varieties: Arithmetic, Geometry and Physics at the Fields Institute in Toronto. None of the material presented in this note is original and the author apologizes in advance for everything that may have been left out or not been attributed correctly.

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[2] See also [MM99, KKV99, KZ99, KKRS05, YY04, KM08, HKQ09, ABK08, HK07, AL07, GKMW07, ALM10, HKR08, HK10, Sak11, AS12, KMW12, ASYZ13] for related works.

etry and Physics at the Fields Institute, and the Fields Institute for providing an excellent research atmosphere and partial financial support during his visiting.

2 Holomorphic anomaly equations

In this section, we shall first review briefly the special Kähler geometry on the deformation space of CY 3-folds and the basics of classical genus zero mirror symmetry. After that we shall display the holomorphic anomaly equations satisfied by the topological string partition functions.

2.1 Special Kähler geometry

Consider a family $\pi : \mathscr{X} \to \mathscr{M}$ of CY 3-folds $\mathscr{X} = \{\mathscr{X}_z\}$ over a variety $\mathscr{M}$ parametrized by the complex coordinate system $z = \{z^i\}_{i=1}^{\dim \mathscr{M}}$. For a generic $z \in \mathscr{M}$, the fiber $\mathscr{X}_z$ is a smooth CY 3-fold. We also assume that $\dim \mathscr{M} = h^1(\mathscr{X}_z, T\mathscr{X}_z)$ for a smooth $\mathscr{X}_z$, where $T\mathscr{X}_z$ is the holomorphic tangent bundle of $\mathscr{X}_z$. In the following, we shall use the notation X to denote a generic fiber $\mathscr{X}_z$ in the family without specifying the point z.

In the examples discussed in this note, the smooth CY 3-fold X is toric in nature, i.e., it is a hypersurface or complete intersection in a toric variety, and the variety $\mathscr{M}$ will be the moduli space of complex structure of X which can be constructed torically.

The variation of complex structure on X can be studied by looking at the periods according to the general theory of variation of Hodge structures. They are defined to be the integrals $\Pi = \int_C \Omega_z$, where $C \in H_3(\mathscr{X}_z, \mathbb{Z})$ and $\Omega = \{\Omega_z\}$ is a holomorphic section of the Hodge line bundle $\mathscr{L} = \mathscr{R}^0 \pi_* \Omega^3_{\mathscr{X}|\mathscr{M}}$ on $\mathscr{M}$. They satisfy a differential equation system $\mathscr{L}_{\mathrm{CY}} \Pi = 0$ called the Picard-Fuchs equations induced from the Gauss-Manin connection on the Hodge bundle

$$\mathscr{H} = \mathscr{R}^3 \pi_* \underline{\mathbb{C}} \otimes \mathscr{O}_{\mathscr{M}} = \mathscr{R}^3 \pi_* \Omega^\bullet_{\mathscr{X}|\mathscr{M}} = \mathscr{L} \oplus \mathscr{L} \otimes T\mathscr{M} \oplus \overline{\mathscr{L} \otimes T\mathscr{M}} \oplus \overline{\mathscr{L}}. \quad (3)$$

The base $\mathscr{M}$ of the family is equipped with the Weil-Petersson metric whose Kähler potential K is determined from

$$e^{-K(z,\bar{z})} = i \int_{\mathscr{X}_z} \Omega_z \wedge \overline{\Omega}_z, \quad (4)$$

where as above $\Omega = \{\Omega_z\}$ is a section of the Hodge line bundle $\mathscr{L}$. The metric $G_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K$ is the Hodge metric induced from the Hermitian metric $h(\Omega, \Omega) = i^3 \int \Omega \wedge \overline{\Omega}$ on the Hodge line bundle $\mathscr{L}$. This metric is called special Kähler metric [Str90, Fre99]. Among its other properties, it satisfies the following "special geom-

etry relation"

$$-R_{i\bar{j}\ l}^{\ \ k} = \partial_{\bar{j}}\Gamma_{il}^k = \delta_l^k G_{i\bar{j}} + \delta_i^k G_{l\bar{j}} - C_{ilm}\bar{C}_{\bar{j}}^{mk}, \quad i,\bar{j},k,l = 1,2,\cdots \dim\mathcal{M}, \quad (5)$$

where

$$C_{ijk}(z) = -\int_{\mathscr{X}_z} \Omega_z \wedge \partial_i\partial_j\partial_k\Omega_z \tag{6}$$

is the so-called Yukawa coupling and

$$\bar{C}_{\bar{j}}^{mk} = e^{2K}G^{k\bar{k}}G^{m\bar{m}}\bar{C}_{\bar{j}\bar{k}\bar{m}}. \tag{7}$$

Note that $C_{ijk} \in \Gamma(\mathcal{M}, \mathrm{Sym}^{\otimes 3}T^*\mathcal{M} \otimes \mathscr{L}^2)$ and it is symmetric in i,j,k by definition. Integrating Eq. (5), one then gets the "integrated special geometry relation"

$$\Gamma_{ij}^k = \delta_j^k K_i + \delta_i^k K_j - C_{ijm}S^{mk} + s_{ij}^k, \tag{8}$$

where S^{mk} is defined to be a solution to $\bar{\partial}_{\bar{n}}S^{mk} = \bar{C}_{\bar{n}}^{mk}$, and s_{ij}^k could be any holomorphic quantity. There is a natural covariant derivative D acting on sections of the Hodge bundle $\mathscr{H} = \mathscr{R}^3\pi_*\underline{\mathbb{C}} \otimes \mathscr{O}_{\mathcal{M}}$: it is induced from the Chern connection associated to the Weil-Petersson metric and the connection on $\mathscr{L}$ induced by the Hermitian metric $h = e^{-K}$. For example, on a section T_J^I of $\mathrm{Sym}^{\otimes k}T\mathcal{M} \otimes \mathrm{Sym}^{\otimes l}T^*\mathcal{M} \otimes \mathscr{L}^m \otimes \bar{\mathscr{L}}^n$, where $I = \{i_1,i_2,\cdots i_k\}, J = \{j_1,i_2,\cdots j_l\}$, one has

$$D_i T_J^I = \partial_i T_J^I + \sum_{i_r\in I}\Gamma_{ia}^{i_r}T_J^{ai_1i_2\cdots\hat{i}_r\cdots i_k} - \sum_{j_s\in J}\Gamma_{ib}^{j_s}T_J^{bj_1j_2\cdots\hat{j}_s\cdots j_l} + mK_iT_J^I,$$

$$\bar{D}_{\bar{i}}T_J^I = \bar{\partial}_{\bar{i}}T_J^I + nK_{\bar{i}}T_J^I,$$

where ˆ means the index is excluded. We have similar formulas for tensors with anti-holomorphic indices.

Eq. (5) implies that there exists a holomorphic quantity $\mathscr{F} \in \Gamma(\mathcal{M}, \mathscr{L}^2)$ called prepotential such that

$$C_{ijk} = D_iD_jD_k\mathscr{F}. \tag{9}$$

See [Str90, Fre99, Hos10] for details on this.

2.2 Variation of Hodge structures and genus zero mirror symmetry

Among the singular points on the moduli space $\mathcal{M}$, there is a distinguished point called the large complex structure limit. It is mirror to the large volume limit (in terms of the coordinates $\{\check{t}^i\}_{i=1}^{h^{1,1}(\check{X})}$ in Eq. (1), the point is given by setting all $\check{t}^i$s to infinity) on the mirror side (A-side) and is a maximally unipotent monodromy point [Mor93]. It plays a special role in genus zero mirror symmetry. The asymptotic behavior of the period map near this point can be studied using the theory of variation

of Hodge structures, see [Sch73] for the mathematical foundation and [CdLOGP91] (also [CK99, GHJ03] for nice reviews) for applications of this in mirror symmetry.

Assume the large complex structure limit is given by $z = 0$. Near this point, the solutions to the Picard-Fuchs system $\mathscr{L}_{CY}\Pi = 0$ could be obtained by the Frobenius method and have the following form

$$(X^0, X^a, P_a, P_0) = X^0(1, t^a, \partial_{t^a}F(t), 2F - t^a\partial_{t^a}F(t)), \quad a = 1, 2\cdots\dim\mathscr{M}, \quad (10)$$

where $X^0(z) \sim 1 + \mathscr{O}(z)$, while $t^a \sim (1/2\pi i)\log z^a + \text{regular}$ near $z = 0$ gives local coordinates on the punctured moduli space $\mathscr{M} - \{z = 0\}$. The existence of the holomorphic function $F(t)$, called prepotential, and the above particular structure of the periods result from the special Kähler geometry on $\mathscr{M}$. The coordinates $t = \{t^a\}_{a=1}^{\dim\mathscr{M}}$ are in fact the canonical coordinates based at the large complex structure limit, see [BCOV93, BCOV94] for details on this. The prepotential F here is related to the previously defined quantity $\mathscr{F}$ by $F = X_0^{-2}\mathscr{F} \in \Gamma(\mathscr{M}, \mathscr{L}^0)$.

One can easily solve for the Yukawa couplings $C_{ijk}(z)$ from the Picard-Fuchs equation, see e.g., [CK99, GHJ03]. The normalized (so that after normalization it gives a section of $\mathscr{L}^0$) Yukawa coupling in the t coordinates is then given by

$$C_{t^a t^b t^c} = \partial_{t^a t^b t^c}^3 F = \frac{1}{(X^0)^2}C_{z^i z^j z^k}\frac{\partial z^i}{\partial t^a}\frac{\partial z^j}{\partial t^b}\frac{\partial z^k}{\partial t^c} \in \Gamma(\mathscr{M}, \mathscr{L}^0). \quad (11)$$

By equating the normalized Yukawa couplings from both the A-side and the B-side, one can establish genus zero mirror symmetry under the mirror map $\check{t} = t$:

$$C_{t^a t^b t^c} = \kappa_{abc} + \sum_{\substack{\beta\in H_2(\check{X},\mathbb{Z})}}^{\infty} d_a d_b d_c N_{0,d_a d_b d_c}e^{2\pi i d_a t^a}, \quad (12)$$

where $\kappa_{abc} = \omega_a \cup \omega_b \cup \omega_c$ is the classical triple intersection number of the mirror manifold $\check{X}$ of X, $d_a = \int_\beta \omega_a$, and $N_{0,d_a d_b d_c}$ is the same as the quantity $N_{0,\beta}$ in Eq. (2). This prediction has been checked for many CY 3-fold families by directly computing $N_{0,\beta}$ using techniques from the A-side, e.g., the localization technique [Kon95].

2.3 Holomorphic anomaly equations

According to [BCOV93, BCOV94], the genus g topological string partition function[3] $\mathscr{F}^{(g)}$ is a (smooth) section of the line bundle $\mathscr{L}^{2-2g}$ over $\mathscr{M}$, it is shown to satisfy the following holomorphic anomaly equation [4]:

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[3] The quantity $\mathscr{F}^{(g)}$ is really a section rather than a function, but in the literature it is termed topological string partition function which we shall follow in this note.

[4] In this note, we shall use $\bar{\partial}_{\bar{i}}$ and $\partial_{\bar{i}}$ interchangeably to denote $\frac{\partial}{\partial\bar{z}^{\bar{i}}}$ for some local complex coordinates $z = \{z^i\}_{i=1}^{\dim\mathscr{M}}$ chosen on the moduli space $\mathscr{M}$.

$$\bar{\partial}_{\bar{i}}\partial_j\mathscr{F}^{(1)} = \frac{1}{2}C_{jkl}\overline{C}_{\bar{i}}^{kl} + (1 - \frac{\chi}{24})G_{j\bar{i}}, \tag{13}$$

$$\bar{\partial}_{\bar{i}}\mathscr{F}^{(g)} = \frac{1}{2}\overline{C}_{\bar{i}}^{jk}\left(\sum_{r=1}^{g-1}D_j\mathscr{F}^{(r)}D_k\mathscr{F}^{(g-r)} + D_jD_k\mathscr{F}^{(g-1)}\right), \quad g \geq 2, \tag{14}$$

where χ is the Euler characteristic of the mirror manifold $\check{X}$ of the CY 3-fold X.

As we shall see below, any genus $\mathscr{F}^{(g)}$ can be determined recursively from these equations up to addition by a holomorphic function $f^{(g)}$ called holomorphic ambiguity. Boundary conditions on the (global) moduli space are needed to fix the holomorphic ambiguity $f^{(g)}$. What are commonly used are the asymptotic behaviors of $\mathscr{F}^{(g)}$ at the singular points on the moduli space $\mathscr{M}$, see [BCOV93, BCOV94, GV95, HK07, HKQ09].

3 Polynomial structure of topological string partition functions

We now shall explain the polynomial recursion technique which was developed in [YY04, AL07] to solve the holomorphic anomaly equations in Eqs. (13, 14).

3.1 Propagators

First let us try to solve for genus one and two topological string partition function from the holomorphic anomaly equations, we shall see in the sequel why it is convenient to introduce the so-called propagators.

Consider the genus one topological string partition function, in order to solve

$$\bar{\partial}_{\bar{i}}\partial_j\mathscr{F}^{(1)} = \frac{1}{2}C_{jkl}\overline{C}_{\bar{i}}^{kl} + (1 - \frac{\chi}{24})G_{j\bar{i}},$$

one needs to turn the right hand side of the equation into a total derivative (with respect to $\bar{\partial}_{\bar{i}}$). This then leads to the following definition of the propagator S^{kl}:

$$\partial_{\bar{i}}S^{kl} = \overline{C}_{\bar{i}}^{kl}. \tag{15}$$

Note that this implies in particular that S^{kl} is a (non-holomorphic) section of $\mathscr{L}^{-2} \otimes \text{Sym}^2 T\mathscr{M}$. Then we get

$$\bar{\partial}_{\bar{i}}\partial_j\mathscr{F}^{(1)} = \bar{\partial}_{\bar{i}}\left(\frac{1}{2}C_{jkl}S^{kl} + (1 - \frac{\chi}{24})K_j\right). \tag{16}$$

From this one can see that there exists some holomorphic quantity $f_j^{(1)}$ called genus one holomorphic ambiguity so that

$$\partial_j \mathscr{F}^{(1)} = \frac{1}{2} C_{jkl} S^{kl} + (1 - \frac{\chi}{24}) K_j + f_j^{(1)}. \tag{17}$$

The above genus one result is proved rigorously in mathematics for a large class of Calabi-Yau manifolds, see [Fan08, Zin08, Zin09, Pop13] for some progress.

Now let us proceed to genus two holomorphic anomaly equation. Recall that $\mathscr{F}^{(1)}$ is a section of $\mathscr{L}^0$, we know $D_j \mathscr{F}^{(1)} = \partial_j \mathscr{F}^{(1)}$. Hence the holomorphic anomaly equation for genus two simplifies to

$$\bar{\partial}_{\bar{i}} \mathscr{F}^{(2)} = \frac{1}{2} \overline{C}_{\bar{i}}^{jk} \left(\partial_j \mathscr{F}^{(1)} \partial_k \mathscr{F}^{(1)} + D_j \partial_k \mathscr{F}^{(1)} \right)$$
$$= \frac{1}{2} \overline{C}_{\bar{i}}^{jk} \left(\partial_j \mathscr{F}^{(1)} \partial_k \mathscr{F}^{(1)} + \partial_j \partial_k \mathscr{F}^{(1)} - \Gamma_{jk}^l \partial_l \mathscr{F}^{(1)} \right).$$

Again one needs to turn the right hand side of the equation into a total derivative. Note that S^{jk} is symmetric in j, k, using Eq. (15) and integration by parts, we then get

$$\bar{\partial}_{\bar{i}} \mathscr{F}^{(2)} = \frac{1}{2} \partial_{\bar{i}} \left(S^{jk} (\partial_j \mathscr{F}^{(1)} \partial_k \mathscr{F}^{(1)} + \partial_j \partial_k \mathscr{F}^{(1)} - \Gamma_{jk}^l \partial_l \mathscr{F}^{(1)}) \right)$$
$$+ \underbrace{-S^{jk} \partial_{\bar{i}} \partial_j \mathscr{F}^{(1)} \partial_k \mathscr{F}^{(1)}}_{\text{I}} + \underbrace{-\frac{1}{2} S^{jk} \partial_{\bar{i}} \partial_j \partial_k \mathscr{F}^{(1)}}_{\text{II}}$$
$$+ \underbrace{\frac{1}{2} S^{jk} \partial_{\bar{i}} \Gamma_{jk}^l \partial_l \mathscr{F}^{(1)}}_{\text{III}} + \underbrace{\frac{1}{2} S^{jk} \Gamma_{jk}^l \partial_{\bar{i}} \partial_l \mathscr{F}^{(1)}}_{\text{IV}}.$$

Then we can plug in the special geometry relation Eq. (5), the integrated special geometry relation Eq. (8), the holomorphic anomaly equation for $\mathscr{F}^{(1)}$ given in Eq. (13) and Eq. (17) to simplify the above expressions $\text{I} - \text{IV}$. For simplicity, we shall denote $c_\chi := 1 - \frac{\chi}{24}$. The terms $\text{I}, \text{II}, \text{III}, \text{IV}$ are given by

$$\text{I} = -S^{jk} (\frac{1}{2} C_{jmn} \partial_{\bar{i}} S^{mn} + c_\chi \partial_{\bar{i}} K_j)(\frac{1}{2} C_{krs} S^{rs} + c_\chi K_k + f_k^{(1)})$$
$$= -\frac{1}{4} C_{jmn} C_{krs} \partial_{\bar{i}} S^{mn} S^{jk} S^{rs} - \frac{1}{2} c_\chi C_{jmn} \partial_{\bar{i}} S^{mn} S^{jk} K_k - \frac{1}{2} f_k^{(1)} C_{jmn} \partial_{\bar{i}} S^{mn} S^{jk}$$
$$- \frac{1}{2} c_\chi C_{krs} \partial_{\bar{i}} K_j S^{jk} S^{rs} - c_\chi^2 \partial_{\bar{i}} K_j S^{jk} K_k - c_\chi f_k^{(1)} S^{jk} \partial_{\bar{i}} K_j$$
$$\text{II} = -\frac{1}{2} S^{jk} \partial_k (\frac{1}{2} C_{jmn} \partial_{\bar{i}} S^{mn} + c_\chi \partial_{\bar{i}} K_j),$$
$$\text{III} = \frac{1}{2} S^{jk} (\delta_j^l \partial_{\bar{i}} K_k + \delta_k^l \partial_{\bar{i}} K_j - C_{jkp} \partial_{\bar{i}} S^{pl})(\frac{1}{2} C_{lmn} S^{mn} + c_\chi K_l + f_l^{(1)}),$$
$$\text{IV} = \frac{1}{2} S^{jk} (\delta_j^l K_k + \delta_k^l K_j - C_{jkp} S^{pl} + s_{jk}^l)(\frac{1}{2} C_{lmn} \partial_{\bar{i}} S^{mn} + c_\chi \partial_{\bar{i}} K_l).$$

Now we think of all the terms as formal polynomials in S^{jk}, K_j and their holomorphic and anti-holomorphic derivatives. In the sum $\text{I} + \text{II} + \text{III} + \text{IV}$ there is no mono-

mial of the form $\partial_{\bar{i}} S^{jk} K_j$. Moreover, one can show that the coefficient of the mono-
mial $S^{jk} \partial_{\bar{i}} K_j$ which involves $f_k^{(1)}, s_{jk}^j$ is not vanishing for general geometries. It fol-
lows that terms of the form $S^{jk} \partial_{\bar{i}} K_j$ must be total derivatives with respect to $\partial_{\bar{i}}$. This
leads to the following definition of the propagator S^j which is a non-holomorphic
section of $\mathscr{L}^{-2} \otimes T\mathscr{M}$.:

$$\partial_{\bar{i}} S^j = G_{\bar{i}k} S^{jk}. \tag{18}$$

Then we can further simplify the quantities $\mathrm{I} - \mathrm{IV}$ as follows:

$$\mathrm{I} = -\frac{1}{4} C_{jmn} C_{krs} \partial_{\bar{i}} S^{mn} S^{jk} S^{rs} - \frac{1}{2} c_\chi C_{jmn} \partial_{\bar{i}} S^{mn} S^{jk} K_k - \frac{1}{2} f_k^{(1)} C_{jmn} \partial_{\bar{i}} S^{mn} S^{jk}$$
$$- \frac{1}{2} c_\chi C_{krs} \partial_{\bar{i}} K_j S^{jk} S^{rs} - c_\chi^2 \partial_{\bar{i}} S^j K_j - c_\chi f_k^{(1)} \partial_{\bar{i}} S^k,$$

$$\mathrm{II} = -\frac{1}{4} S^{jk} \partial_k \partial_{\bar{i}} (C_{jmn} S^{mn}) - \frac{1}{2} c_\chi S^{jk} \partial_k \partial_{\bar{i}} K_j,$$

$$\mathrm{III} = \frac{1}{2} \partial_{\bar{i}} K_k S^{jk} C_{jmn} S^{mn} + c_\chi K_j \partial_{\bar{i}} S^j + f_j^{(1)} \partial_{\bar{i}} S^j$$
$$- \frac{1}{4} C_{jkp} C_{lmn} S^{jk} S^{mn} \partial_{\bar{i}} S^{pl} - \frac{1}{2} c_\chi C_{jkp} S^{jk} \partial_{\bar{i}} S^{pl} K_l - \frac{1}{2} C_{jkp} f_l^{(1)} S^{jk} \partial_{\bar{i}} S^{pl},$$

$$\mathrm{IV} = \frac{1}{2} C_{jmn} S^{jk} K_k \partial_{\bar{i}} S^{mn} - \frac{1}{4} C_{lmn} C_{jkp} S^{jk} S^{pl} \partial_{\bar{i}} S^{mn} + \frac{1}{4} C_{lmn} s_{jk}^l S^{jk} \partial_{\bar{i}} S^{mn}$$
$$+ c_\chi K_j \partial_{\bar{i}} S^j - \frac{1}{2} c_\chi C_{jkp} S^{jk} \partial_{\bar{i}} K_l S^{pl} + \frac{1}{2} c_\chi S^{jk} s_{jk}^l \partial_{\bar{i}} K_l.$$

Again to turn $K_j \partial_{\bar{i}} S^j = \partial_{\bar{i}} (K_j S^j) - S^j \partial_{\bar{i}} K_j$ into a total derivative, we need to define
a propagator S which is a non-holomorphic section of $\mathscr{L}^{-2}$ so that

$$\partial_{\bar{i}} S = G_{\bar{i}j} S^j. \tag{19}$$

Now the terms I, II become

$$\widetilde{\mathrm{I}} = -\frac{1}{4} C_{jmn} C_{krs} \partial_{\bar{i}} S^{mn} S^{jk} S^{rs} - \frac{1}{2} c_\chi C_{jmn} \partial_{\bar{i}} S^{mn} S^{jk} K_k - \frac{1}{2} f_k^{(1)} C_{jmn} \partial_{\bar{i}} S^{mn} S^{jk}$$
$$- \frac{1}{2} c_\chi C_{krs} \partial_{\bar{i}} K_j S^{jk} S^{rs} - c_\chi^2 \partial_{\bar{i}} (S^k K_k) + c_\chi^2 \partial_{\bar{i}} S - c_\chi \partial_{\bar{i}} (f_k^{(1)} S^k),$$

$$\widetilde{\mathrm{II}} = -\frac{1}{4} S^{jk} \partial_k \partial_{\bar{i}} (C_{jmn} S^{mn}) - \frac{1}{2} c_\chi S^{jk} \partial_k \partial_{\bar{i}} K_j.$$

It turns out that after introducing these propagators, the quantity $\widetilde{\mathrm{I}} + \widetilde{\mathrm{II}} + \mathrm{III} + \mathrm{IV}$
becomes a total derivative. Before we show this, we make a pause and study the
derivatives of quantities S^{jk}, S^j, S, K_i. They will be needed later, for example, to
simplify $\widetilde{\mathrm{II}}$.

3.2 Differential ring of propagators

We use the definition of the propagators Eqs. (15, 18, 19) and the relations Eqs. (5, 8) to derive the differential ring structure. The results in this section follow the presentation in [AL07].

To obtain an expression for $D_i S^{jk}$ in terms of known quantities, we compute $\bar\partial_{\bar l} D_i S^{jk}$ by commuting the derivatives, then we integrate what we get to obtain the desired expression. Since by definition S^{jk} is a section of $\mathscr{L}^{-2} \otimes \mathrm{Sym}^{\otimes 2} T\mathscr{M}$, we know when acting on S^{jk} the covariant derivative is given by $D_i = \partial_i + \Gamma_i + (-2)K_i$, hence

$$\bar\partial_{\bar l} D_i S^{jk} = \bar\partial_{\bar l}(\partial_i S^{jk} + \Gamma^j_{im} S^{mk} + \Gamma^k_{in} S^{jn} - 2K_i S^{jk}) := A + B + C + D\,.$$

Then

$$A = \partial_i \bar C^{jk}_{\bar l} = \partial_i(e^{2K} G^{j\bar j} G^{k\bar k} \bar C_{\bar j \bar k \bar l})$$
$$= 2K_i \bar C^{jk}_{\bar l} + e^{2K} \partial_i G^{j\bar j} G^{k\bar k} \bar C_{\bar j \bar k \bar l} + e^{2K} G^{j\bar j} \partial_i G^{k\bar k} \bar C_{\bar j \bar k \bar l} + e^{2K} G^{j\bar j} G^{k\bar k} \partial_i \bar C_{\bar j \bar k \bar l}\,.$$

Since C_{ijk} is holomorphic, $\bar C_{\bar j \bar k \bar l}$ is anti-holomorphic, the last term above is 0. It follows that

$$A = 2K_i \bar C^{jk}_{\bar l} + e^{2K} \partial_i G^{j\bar j} G^{k\bar k} \bar C_{\bar j \bar k \bar l} + e^{2K} G^{j\bar j} \partial_i G^{k\bar k} \bar C_{\bar j \bar k \bar l} + e^{2K} G^{j\bar j} G^{k\bar k} \partial_i \bar C_{\bar j \bar k \bar l}$$
$$= 2K_i \bar C^{jk}_{\bar l} + e^{2K}(-\Gamma^j_{ir} G^{r\bar j}) G^{k\bar k} \bar C_{\bar j \bar k \bar l} + e^{2K} G^{j\bar j}(-\Gamma^k_{is} G^{s\bar k}) \bar C_{\bar j \bar k \bar l} + 0\,.$$

Similarly,

$$B = \bar\partial_{\bar l}\Gamma^j_{im} S^{mk} + \Gamma^j_{im}\bar\partial_{\bar l} S^{mk} = (\delta^j_i G_{m\bar l} + \delta^j_m G_{i\bar l} - C_{imt}\bar C^{tj}_{\bar l}) S^{mk} + \Gamma^j_{im} \bar C^{mk}_{\bar l}\,,$$
$$C = \bar\partial_{\bar l}\Gamma^k_{in} S^{nj} + \Gamma^k_{in}\bar\partial_{\bar l} S^{nj} = (\delta^k_i G_{n\bar l} + \delta^k_n G_{i\bar l} - C_{int}\bar C^{tk}_{\bar l}) S^{nj} + \Gamma^k_{in} \bar C^{nj}_{\bar l}\,,$$
$$D = -2G_{i\bar l} S^{jk} - 2K_i \bar C^{jk}_{\bar l}\,.$$

It follows that

$$A + B + C + D$$
$$= K_i \bar C^{jk}_{\bar l} + e^{2K}(-\Gamma^j_{ir} G^{r\bar j}) G^{k\bar k} \bar C_{\bar j \bar k \bar l} + e^{2K} G^{j\bar j}(-\Gamma^k_{is} G^{s\bar k}) \bar C_{\bar j \bar k \bar l} + 0$$
$$+ \delta^j_i G_{m\bar l} S^{mk} + \delta^j_m G_{i\bar l} S^{mk} - C_{imt}\bar C^{tj}_{\bar l} S^{mk} + \Gamma^j_{im} \bar C^{mk}_{\bar l}$$
$$+ \delta^k_i G_{n\bar l} S^{nj} + \delta^k_n G_{i\bar l} S^{nj} - C_{int}\bar C^{tk}_{\bar l} S^{nj} + \Gamma^k_{in} \bar C^{nj}_{\bar l} \quad - 2G_{i\bar l} S^{jk} - 2K_i \bar C^{jk}_{\bar l}$$
$$= K_i \bar C^{jk}_{\bar l} + e^{2K}(-\Gamma^j_{ir} G^{r\bar j}) G^{k\bar k} \bar C_{\bar j \bar k \bar l} + e^{2K} G^{j\bar j}(-\Gamma^k_{is} G^{s\bar k}) \bar C_{\bar j \bar k \bar l} + 0$$
$$+ \delta^j_i G_{m\bar l} S^{mk} + \delta^j_m G_{i\bar l} S^{mk} - C_{imt}\bar C^{tj}_{\bar l} S^{mk} + \Gamma^j_{im} \bar C^{mk}_{\bar l}$$
$$+ \delta^k_i G_{n\bar l} S^{nj} + \delta^k_n G_{i\bar l} S^{nj} - C_{int}\bar C^{tk}_{\bar l} S^{nj} + \Gamma^{ki}_{in} \bar C^{nj}_{\bar l} \quad - 2G_{i\bar l} S^{jk} - 2K_i \bar C^{jk}_{\bar l}\,.$$

Using the definition $G_{m\bar l} S^{mk} = S^k_{\bar l} = \bar\partial_{\bar l} S^k$, we get

$$A + B + C + D$$
$$= K_i \bar{C}_{\bar{l}}^{jk} + e^{2K}(-\Gamma_{ir}^j G^{r\bar{j}})G^{k\bar{k}}\bar{C}_{\bar{j}k\bar{l}} + e^{2K}G^{j\bar{j}}(-\Gamma_{is}^k G^{s\bar{k}})\bar{C}_{\bar{j}k\bar{l}} + 0$$
$$+ \delta_i^j \bar{\partial}_{\bar{l}} S^k + \delta_m^j G_{i\bar{l}} S^{mk} - C_{imt}\bar{C}_{\bar{l}}^{tj} S^{mk} + \Gamma_{im}^j \bar{C}_{\bar{l}}^{mk}$$
$$+ \delta_i^k \bar{\partial}_{\bar{l}} S^j + \delta_n^k G_{i\bar{l}} S^{nj} - C_{int}\bar{C}_{\bar{l}}^{tk} S^{nj} + \Gamma_{in}^k \bar{C}_{\bar{l}}^{nj} \quad -2G_{i\bar{l}}S^{jk} - 2K_i\bar{C}_{\bar{l}}^{jk}$$
$$= e^{2K}(-\Gamma_{ir}^j G^{r\bar{j}})G^{k\bar{k}}\bar{C}_{\bar{j}k\bar{l}} + e^{2K}G^{j\bar{j}}(-\Gamma_{is}^k G^{s\bar{k}})\bar{C}_{\bar{j}k\bar{l}}$$
$$+ \delta_i^j \bar{\partial}_{\bar{l}} S^k - C_{imt}\bar{C}_{\bar{l}}^{tj} S^{mk} + \Gamma_{im}^j \bar{C}_{\bar{l}}^{mk} + \delta_i^k \bar{\partial}_{\bar{l}} S^j - C_{int}\bar{C}_{\bar{l}}^{tk} S^{nj} + \Gamma_{in}^k \bar{C}_{\bar{l}}^{nj}.$$

According to $-\bar{C}_{\bar{l}}^{tj} S^{mk} = -\bar{\partial}_{\bar{l}} S^{tj} S^{mk} = -\bar{\partial}_{\bar{l}}(S^{tj} S^{mk}) + S^{tj}\bar{\partial}_{\bar{l}} S^{mk}$, we then obtain

$$A + B + C + D$$
$$= e^{2K}(-\Gamma_{ir}^j G^{r\bar{j}})G^{k\bar{k}}\bar{C}_{\bar{j}k\bar{l}} + e^{2K}G^{j\bar{j}}(-\Gamma_{is}^k G^{s\bar{k}})\bar{C}_{\bar{j}k\bar{l}}$$
$$+ \bar{\partial}_{\bar{l}}(\delta_i^j S^k + \delta_i^k S^j - C_{imt} S^{tj} S^{mk}) + C_{imt} S^{tj}\bar{\partial}_{\bar{l}} S^{mk} - C_{int}\bar{C}_{\bar{l}}^{tk} S^{nj} + \Gamma_{im}^j \bar{C}_{\bar{l}}^{mk} + \Gamma_{in}^k \bar{C}_{\bar{l}}^{nj}$$
$$= \bar{\partial}_{\bar{l}}(\delta_i^j S^k + \delta_i^k S^j - C_{imt} S^{tj} S^{mk}).$$

Therefore,

$$D_i S^{jk} = \delta_i^j S^k + \delta_i^k S^j - C_{imn} S^{jn} S^{mk} + h_i^{jk} \tag{20}$$

for some holomorphic quantity h_i^{jk}.

The quantity $D_i S^j$ can be obtained in the same manner:

$$\bar{\partial}_{\bar{l}}(D_i S^j)$$
$$= \bar{\partial}_{\bar{l}}(\partial_i S^j - 2K_i S^j + \Gamma_{ir}^j S^r)$$
$$= \partial_i S_{\bar{l}}^j + \bar{\partial}_{\bar{l}}(-2K_i S^j) + \bar{\partial}_{\bar{l}}(\Gamma_{ir}^j S^r)$$
$$= \partial_i(G_{l\bar{l}} S^{lj}) - 2G_{i\bar{l}} S^j - 2K_i\bar{\partial}_{\bar{l}} S^j + \bar{\partial}_{\bar{l}}\Gamma_{ir}^j S^r + \Gamma_{ir}^j \bar{\partial}_{\bar{l}} S^r$$
$$= \partial_i G_{l\bar{l}} S^{lj} + G_{l\bar{l}}\partial_i S^{lj} - 2G_{i\bar{l}} S^j - 2K_i\bar{\partial}_{\bar{l}} S^j + \bar{\partial}_{\bar{l}}\Gamma_{ir}^j S^r + \Gamma_{ir}^j \bar{\partial}_{\bar{l}} S^r$$
$$= \partial_i G_{l\bar{l}} S^{lj} + G_{l\bar{l}}\partial_i S^{lj} - 2G_{i\bar{l}} S^j - 2K_i\bar{\partial}_{\bar{l}} S^j + (\delta_i^j G_{r\bar{l}} + \delta_r^j G_{i\bar{l}} - C_{irt}\bar{C}_{\bar{l}}^{tj})S^r + \Gamma_{ir}^j \bar{\partial}_{\bar{l}} S^r$$
$$= \partial_i G_{l\bar{l}} S^{lj} + G_{l\bar{l}}(D_i S^{lj} + 2K_i S^{lj} - \Gamma_{im}^l S^{mj} - \Gamma_{in}^j S^{ln}) - 2G_{i\bar{l}} S^j - 2K_i\bar{\partial}_{\bar{l}} S^j$$
$$\quad + (\delta_i^j G_{r\bar{l}} + \delta_r^j G_{i\bar{l}} - C_{irt}\bar{C}_{\bar{l}}^{tj})S^r + \Gamma_{ir}^j \bar{\partial}_{\bar{l}} S^r$$
$$= \Gamma_{il}^m G_{m\bar{l}} S^{lj} + G_{l\bar{l}}(D_i S^{lj} - \Gamma_{im}^l S^{mj}) - 2G_{i\bar{l}} S^j + (\delta_i^j G_{r\bar{l}} + \delta_r^j G_{i\bar{l}} - C_{irt}\bar{C}_{\bar{l}}^{tj})S^r$$
$$= G_{l\bar{l}} D_i S^{lj} - 2G_{i\bar{l}} S^j + (\delta_i^j G_{r\bar{l}} + \delta_r^j G_{i\bar{l}} - C_{irt}\bar{C}_{\bar{l}}^{tj})S^r$$
$$= G_{l\bar{l}}(\delta_i^l S^j + \delta_i^j S^l - C_{itm} S^{tl} S^{mj} + h_i^{lj}) - 2G_{i\bar{l}} S^j + (\delta_i^j G_{r\bar{l}} + \delta_r^j G_{i\bar{l}} - C_{irt}\bar{C}_{\bar{l}}^{tj})S^r$$
$$= 2\delta_i^j \bar{\partial}_{\bar{l}} S - G_{l\bar{l}} C_{itm} S^{tl} S^{mj} - C_{irt}\bar{C}_{\bar{l}}^{tj} S^r + G_{l\bar{l}} h_i^{lj}.$$

Since

$$G_{l\bar{l}} C_{itm} S^{tl} S^{mj} = \bar{\partial}_{\bar{l}} K_l C_{imt} S^{tl} S^{mj} = \bar{\partial}_{\bar{l}} S^t C_{imt} S^{mj} = \bar{\partial}_{\bar{l}}(S^t C_{imt} S^{mj}) - S^t C_{imt}\bar{\partial}_{\bar{l}} S^{mj},$$

we then get

$$\bar{\partial}_{\bar{l}}(D_i S^j) = 2\delta_i^j \bar{\partial}_{\bar{l}} S - \bar{\partial}_{\bar{l}}(S^t C_{imt} S^{mj}) + G_{l\bar{l}} h_i^{lj}.$$

Hence

$$D_i S^j = 2\delta_i^j S - C_{imn} S^m S^{nj} + K_l h_i^{lj} + h_i^j \tag{21}$$

for some holomorphic quantity h_i^j.

In the following we shall calculate $\bar{\partial}_{\bar{l}} D_i S$:

$$\bar{\partial}_{\bar{l}} D_i S$$
$$= \bar{\partial}_{\bar{l}}(\partial_i S - 2K_i S)$$
$$= \partial_i(G_{l\bar{l}} S^l) + \bar{\partial}_{\bar{l}}(-2K_i S)$$
$$= \partial_i G_{l\bar{l}} S + G_{l\bar{l}} \partial_i S - 2\bar{\partial}_{\bar{l}}(K_i S)$$
$$= \partial_i G_{l\bar{l}} S + G_{l\bar{l}}(D_i S^l + 2K_i S - \Gamma_{im}^l S^m) - 2\bar{\partial}_{\bar{l}}(K_i S)$$
$$= G_{m\bar{l}} \Gamma_{il}^m S^l + G_{l\bar{l}}(2\delta_i^l S - C_{imn} S^m S^{nl} + h_i^{lk} K_k + h_i^l) + G_{l\bar{l}}(2K_i S^l - \Gamma_{im}^l S^m) + \bar{\partial}_{\bar{l}}(-2K_i S)$$
$$= G_{l\bar{l}}(-C_{imn} S^m S^{nl} + h_i^{lk} K_k + h_i^l)$$
$$= -C_{imn} \bar{\partial}_{\bar{l}} S^n S^m + \bar{\partial}_{\bar{l}} K_l h_i^{lk} K_k + \bar{\partial}_{\bar{l}} K_l h_i^l$$
$$= -\bar{\partial}_{\bar{l}}(\frac{1}{2} S^n C_{imn} S^m) + \bar{\partial}_{\bar{l}}(\frac{1}{2} h_i^{lk} K_k K_l) + \bar{\partial}_{\bar{l}}(K_l h_i^l).$$

So we get

$$D_i S = -\frac{1}{2} C_{imn} S^m S^n + \frac{1}{2} h_i^{kl} K_k K_l + K_l h_i^l + h_i. \tag{22}$$

for some holomorphic quantity h_i.

Now we calculate $D_i K_j$ as follows. According to $D_i K_j = \partial_i K_j - \Gamma_{ji}^m K_m$, it follows that

$$\bar{\partial}_{\bar{l}}(D_i K_j) = \bar{\partial}_{\bar{l}}(\partial_i K_j - \Gamma_{ji}^m K_m)$$
$$= \partial_i K_{j\bar{l}} - \bar{\partial}_{\bar{l}}(\Gamma_{ji}^m) K_m - \Gamma_{ji}^m K_{m\bar{l}}$$
$$= \Gamma_{ij}^m G_{m\bar{l}} - (\delta_j^m G_{i\bar{l}} + \delta_i^m G_{j\bar{l}} - C_{jin} \bar{C}_{\bar{l}}^{mn}) K_m - \Gamma_{ji}^m K_{m\bar{l}}$$
$$= -G_{i\bar{l}} K_j - G_{j\bar{l}} K_i + C_{jin} \bar{C}_{\bar{l}}^{mn} K_m$$
$$= -\bar{\partial}_{\bar{l}}(K_i K_j) + C_{jin} \bar{\partial}_{\bar{l}} S^{mn} K_m$$
$$= -\bar{\partial}_{\bar{l}}(K_i K_j) + \bar{\partial}_{\bar{l}}(C_{jin} S^{mn} K_m) - C_{jin} S^{mn} \bar{\partial}_{\bar{l}} K_m$$
$$= -\bar{\partial}_{\bar{l}}(K_i K_j) + \bar{\partial}_{\bar{l}}(C_{jin} S^{mn} K_m) - C_{jin} S^{mn} G_{m\bar{l}}$$
$$= -\bar{\partial}_{\bar{l}}(K_i K_j) + \bar{\partial}_{\bar{l}}(C_{jin} S^{mn} K_m) - C_{jin} \bar{\partial}_{\bar{l}} S^n$$
$$= \bar{\partial}_{\bar{l}}(-K_i K_j + C_{jin} S^{mn} K_m - C_{jin} S^n).$$

Therefore,

$$D_i K_j = -K_i K_j + C_{ijn} S^{mn} K_m - C_{ijn} S^n + h_{ij} \tag{23}$$

for some holomorphic quantity h_{ij}.

We want to point out that the holomorphic limit of the differential ring of generators S^{jk}, S^j, S, K_i also satisfies similar equations, with everything replaced by their holomorphic limits [ALM10].

Some of the holomorphic quantities $h_i^{jk}, h_i^j, h_i, h_{ij}$ can not be uniquely determined [ALM10], as briefly discussed in Section 3.4 below, since the above equations are derived by integrating equations. Eqs. (20, 21, 22, 23) do not actually give a differential ring due to the existence of these holomorphic quantities and their derivatives. To make it a genuine ring, one needs to include all of the derivatives of these holomorphic quantities [Hos10]. In [YY04, Hos10, ASYZ13], it is shown that for some special CY 3-fold families, all of the holomorphic quantities and their derivatives are packaged together by making use of the special Kähler geometry on the moduli space, and are in fact Laurent polynomials of the Yukawa couplings. Then one gets a differential ring with finitely many generators, including the non-holomorphic generators S^{jk}, S^j, S, K_i and the holomorphic Yukawa couplings.

Now we shall focus on the cases $h^{2,1}(X) = \dim \mathcal{M} = 1$ and consider the differential ring structure of the generators with derivatives taken in the $\tau = (1/2\pi i)\kappa^{-1} F_{tt}$ coordinate, where κ is the classical triple intersection (of the A-model CY) and t is the coordinate defined in Eq. (10). This definition was introduced [ABK08] to match the known modularity for the moduli space of some non-compact CY 3-folds whose geometries are completely determined by the mirror curves sitting inside them, see also [ASYZ13, Zho13, Ali14] for related works. First one makes the following change of generators [AL07]:

$$\begin{aligned}
\tilde{S}^{tt} &= S^{tt}, \\
\tilde{S}^t &= S^t - S^{tt} K_t, \\
\tilde{S} &= S - S^t K_t + \frac{1}{2} S^{tt} K_t K_t, \\
\tilde{K}_t &= K_t.
\end{aligned}$$

Then one defines $\tau = (1/2\pi i)\kappa^{-1}\partial_t F_t$ which gives $\frac{\partial \tau}{\partial t} = (1/2\pi i)\kappa^{-1} C_{ttt}$. After that one forms the following quantities [ASYZ13] on the deformation space $\mathcal{M}$:

$$\begin{aligned}
K_0 &= \kappa C_{ttt}^{-1}(\theta t)^{-3}, \\
G_1 &= \theta t, \\
K_2 &= \kappa C_{ttt}^{-1} \tilde{K}_t, \\
T_2 &= \tilde{S}^{tt}, \\
T_4 &= C_{ttt}^{-1} \tilde{S}^t, \\
T_6 &= C_{ttt}^{-2} \tilde{S}, \tag{24}
\end{aligned}$$

where $\theta = z\frac{\partial}{\partial z}$ and the propagators $\tilde{S}^{tt}, \tilde{S}^t, \tilde{S}$ are the normalized (by suitable powers of X^0 so that they become sections of $\mathscr{L}^0$) propagators. It follows that the derivatives

of the generators given in Eqs. (20, 21, 22, 23) now become [ASYZ13] the following
differential equations satisfied by $K_0, G_1, K_2, T_2, T_4, T_6$:

$$\partial_\tau K_0 = -2K_0 K_2 - K_0^2 G_1^2 (\tilde{h}_{zzz}^z + 3(s_{zz}^z + 1)),$$

$$\partial_\tau G_1 = 2G_1 K_2 - \kappa G_1 T_2 + K_0 G_1^3 (s_{zz}^z + 1),$$

$$\partial_\tau K_2 = 3K_2^2 - 3\kappa K_2 T_2 - \kappa^2 T_4 + K_0^2 G_1^4 k_{zz} - K_0 G_1^2 K_2 \tilde{h}_{zzz}^z,$$

$$\partial_\tau T_2 = 2K_2 T_2 - \kappa T_2^2 + 2\kappa T_4 + \kappa^{-1} K_0^2 G_1^4 \tilde{h}_{zz}^z,$$

$$\partial_\tau T_4 = 4K_2 T_4 - 3\kappa T_2 T_4 + 2\kappa T_6 - K_0 G_1^2 T_4 \tilde{h}_{zzz}^z - \kappa^{-1} K_0^2 G_1^4 T_2 k_{zz} + \kappa^{-2} K_0^3 G_1^6 \tilde{h}_{zz}^z,$$

$$\partial_\tau T_6 = 6K_2 T_6 - 6\kappa T_2 T_6 + \frac{\kappa}{2} T_4^2 - \kappa^{-1} K_0^2 G_1^4 T_4 k_{zz}$$
$$+ \kappa^{-3} K_0^4 G_1^8 \tilde{h}_z - 2K_0 G_1^2 T_6 \tilde{h}_{zzz}^z, \tag{25}$$

where $\partial_\tau = \frac{1}{2\pi i} \frac{\partial}{\partial \tau}$ and the quantities $\tilde{h}_{zzz}^z, s_{zz}^z, k_{zz}, \tilde{h}_{zz}^z, \tilde{h}_{zz}, \tilde{h}_z$ are some holomorphic
functions. It turns out that they are polynomials of the quantity $C_0 = \theta \log(z^3 C_{zzz})$
which satisfies

$$\partial_\tau C_0 = C_0(C_0 + 1)G_1^2. \tag{26}$$

For special CY 3-fold families these explicit polynomials could be found in [ASYZ13,
Ali14] and will be discussed later in Section 4. In the rest of the note, we shall call
this particular form of the differential ring the special polynomial ring.

3.3 Polynomial structure

Let us resume the discussion on genus two holomorphic anomaly equation. Direct
computation shows that

$$\widetilde{\Pi} = -\frac{1}{4}\partial_k C_{jmn} S^{jk} \bar{\partial}_{\bar{i}} S^{mn} - \frac{1}{4} C_{jmn} S^{jk} \bar{\partial}_{\bar{i}} \partial_k S^{mn} - \frac{1}{2} c_\chi S^{jk} \bar{\partial}_{\bar{i}} \partial_k K_j$$

$$= -\frac{1}{4}\partial_k C_{jmn} S^{jk} \bar{\partial}_{\bar{i}} S^{mn} - \frac{1}{4} C_{jmn} S^{jk} \bar{\partial}_{\bar{i}} (D_k S^{mn} + 2K_k S^{mn} - \Gamma_{kp}^m S^{pn} - \Gamma_{kp}^n S^{pm})$$

$$- \frac{1}{2} c_\chi S^{jk} \bar{\partial}_{\bar{i}} (D_k K_j + \Gamma_{kj}^l K_l)$$

$$= -\frac{1}{4}\partial_k C_{jmn} S^{jk} \bar{\partial}_{\bar{i}} S^{mn}$$

$$- \frac{1}{4} C_{jmn} S^{jk} \bar{\partial}_{\bar{i}} (\delta_k^m S^n + \delta_k^n S^m - C_{kpq} S^{mq} S^{nq}) - \frac{1}{4} C_{jmn} S^{jk} \bar{\partial}_{\bar{i}} (2K_k S^{mn} - \Gamma_{kp}^m S^{pn} - \Gamma_{kp}^n S^{pm})$$

$$- \frac{1}{2} c_\chi S^{jk} \bar{\partial}_{\bar{i}} (-K_j K_k + C_{jkp} S^{pq} K_q - C_{jkp} S^p) - \frac{1}{2} c_\chi S^{jk} \bar{\partial}_{\bar{i}} (\Gamma_{kj}^l K_l).$$

It follows then that

$$\widetilde{I} + \widetilde{II} + III + IV$$

$$= -\frac{1}{4}C_{jmn}C_{krs}\partial_{\bar{i}}S^{mn}S^{jk}S^{rs} - \frac{1}{2}c_\chi C_{jmn}\partial_{\bar{i}}S^{mn}S^{jk}K_k - \frac{1}{2}f_k^{(1)}C_{jmn}\partial_{\bar{i}}S^{mn}S^{jk}$$

$$-\frac{1}{2}c_\chi C_{krs}\partial_{\bar{i}}K_jS^{kj}S^{rs} - c_\chi^2\partial_{\bar{i}}(S^kK_k) + c_\chi^2\partial_{\bar{i}}S - c_\chi\partial_{\bar{i}}(f_k^{(1)}S^k)$$

$$+ -\frac{1}{4}\partial_kC_{jmn}S^{jk}\bar{\partial}_{\bar{i}}S^{mn}$$

$$-\frac{1}{4}C_{jmn}S^{jk}\bar{\partial}_{\bar{i}}(\delta_k^mS^n + \delta_k^nS^m - C_{kpq}S^{pm}S^{qn}) - \frac{1}{4}C_{jmn}S^{jk}\bar{\partial}_{\bar{i}}(2K_kS^{mn} - \Gamma_{kp}^mS^{pn} - \Gamma_{kp}^nS^{pm})$$

$$-\frac{1}{2}c_\chi S^{jk}\bar{\partial}_{\bar{i}}(-K_jK_k + C_{jkp}S^{pq}K_q - C_{jkp}S^p) - \frac{1}{2}c_\chi S^{jk}\bar{\partial}_{\bar{i}}\Gamma_{kj}^lK_l - \frac{1}{2}c_\chi S^{jk}\Gamma_{kj}^l\bar{\partial}_{\bar{i}}K_l$$

$$+ \frac{1}{4}C_{lmn}S^{jk}\partial_{\bar{i}}\Gamma_{jk}^lS^{mn} + \frac{1}{2}c_\chi S^{jk}\partial_{\bar{i}}\Gamma_{jk}^lK_l + \frac{1}{2}f_l^1S^{jk}\partial_{\bar{i}}\Gamma_{jk}^l$$

$$+ \frac{1}{4}C_{lmn}S^{jk}\Gamma_{jk}^l\partial_{\bar{i}}S^{mn} + \frac{1}{2}c_\chi S^{jk}\Gamma_{jk}^l\partial_{\bar{i}}K_l$$

$$= -\frac{1}{4}C_{jmn}C_{krs}\partial_{\bar{i}}S^{mn}S^{jk}S^{rs} - \frac{1}{4}\partial_kC_{jmn}S^{jk}\bar{\partial}_{\bar{i}}S^{mn}$$

$$-\frac{1}{4}C_{jmn}S^{jk}\bar{\partial}_{\bar{i}}(\delta_k^mS^n + \delta_k^nS^m - C_{kpq}S^{pm}S^{qn}) - \frac{1}{4}C_{jmn}S^{jk}\bar{\partial}_{\bar{i}}(2K_kS^{mn} - \Gamma_{kp}^mS^{pn} - \Gamma_{kp}^nS^{pm})$$

$$+ \frac{1}{4}C_{lmn}S^{jk}\partial_{\bar{i}}\Gamma_{jk}^lS^{mn} + \frac{1}{4}C_{lmn}S^{jk}\Gamma_{jk}^l\partial_{\bar{i}}S^{mn}$$

$$-\frac{1}{2}c_\chi C_{jkp}\partial_{\bar{i}}(K_qS^{jk}S^{pq}) - \frac{1}{2}f_l^{(1)}C_{jkm}\partial_{\bar{i}}(S^{jk}S^{ml})$$

$$+ f_j^{(1)}\partial_{\bar{i}}S^j - c_\chi^2\partial_{\bar{i}}(S^kK_k) + c_\chi^2\partial_{\bar{i}}S - c_\chi\partial_{\bar{i}}(f_k^{(1)}S^k) + c_\chi\partial_{\bar{i}}(S^jK_j) - c_\chi\partial_{\bar{i}}S.$$

Now we make use of the fact that $D_kC_{jmn} = D_kD_jD_mD_n\mathscr{F}$ is symmetric in j,k,m,n to simplify the above expression. By definition, we have

$$\partial_kC_{jmn} = D_kC_{jmn} - 2K_kC_{jmn} + \Gamma_{kj}^lC_{lmn} + \Gamma_{km}^lC_{jln} + \Gamma_{kn}^lC_{jml}. \qquad (27)$$

Then

$$\partial_kC_{jmn}S^{jk}\bar{\partial}_{\bar{i}}S^{mn}$$

$$= D_kC_{jmn}S^{jk}\bar{\partial}_{\bar{i}}S^{mn} - 2K_kC_{jmn}S^{jk}\bar{\partial}_{\bar{i}}S^{mn}$$

$$+ \Gamma_{kj}^lC_{lmn}S^{jk}\bar{\partial}_{\bar{i}}S^{mn} + \Gamma_{km}^lC_{jln}S^{jk}\bar{\partial}_{\bar{i}}S^{mn} + \Gamma_{kn}^lC_{jml}S^{jk}\bar{\partial}_{\bar{i}}S^{mn}$$

$$= \frac{1}{2}(\bar{\partial}_{\bar{i}}(D_kC_{jmn}S^{jk}S^{mn}) - \bar{\partial}_{\bar{i}}D_kC_{jmn}S^{jk}S^{mn})$$

$$-2K_kC_{jmn}S^{jk}\bar{\partial}_{\bar{i}}S^{mn} + \Gamma_{kj}^lC_{lmn}S^{jk}\bar{\partial}_{\bar{i}}S^{mn} + \Gamma_{km}^lC_{jln}S^{jk}\bar{\partial}_{\bar{i}}S^{mn} + \Gamma_{kn}^lC_{jml}S^{jk}\bar{\partial}_{\bar{i}}S^{mn}$$

$$= \frac{1}{2}\bar{\partial}_{\bar{i}}(D_kC_{jmn}S^{jk}S^{mn})$$

$$-\frac{1}{2}(2C_{jmn}\bar{\partial}_{\bar{i}}K_kS^{jk}S^{mn} - C_{lmn}\bar{\partial}_{\bar{i}}\Gamma_{kj}^lS^{jk}S^{mn} - C_{jln}\bar{\partial}_{\bar{i}}\Gamma_{km}^lS^{jk}S^{mn} - C_{jml}\bar{\partial}_{\bar{i}}\Gamma_{kn}^lS^{jk}S^{mn})$$

$$-2K_kC_{jmn}S^{jk}\bar{\partial}_{\bar{i}}S^{mn} + \Gamma_{kj}^lC_{lmn}S^{jk}\bar{\partial}_{\bar{i}}S^{mn} + \Gamma_{km}^lC_{jln}S^{jk}\bar{\partial}_{\bar{i}}S^{mn} + \Gamma_{kn}^lC_{jml}S^{jk}\bar{\partial}_{\bar{i}}S^{mn}.$$

Therefore, we obtain

$$
\begin{aligned}
&\widetilde{\mathrm{I}} + \widetilde{\mathrm{II}} + \mathrm{III} + \mathrm{IV} \\
&= -\frac{1}{8}(\bar{\partial}_{\bar{i}} D_k C_{jmn}) S^{jk} S^{mn} + \frac{1}{4} C_{jmn} S^{jk} C_{kpq} S^{pm} \bar{\partial}_{\bar{i}} S^{qn} \\
&\quad -\frac{1}{4} C_{jpq} C_{krs} \partial_{\bar{i}} S^{pq} S^{jk} S^{rs} - \frac{1}{8} C_{lmn} C_{jkp} S^{jk} \partial_{\bar{i}} S^{pl} S^{mn} \\
&\quad -\frac{1}{2} c_\chi C_{jkp} \partial_{\bar{i}} (K_q S^{jk} S^{pq}) - \frac{1}{2} f_l^{(1)} C_{jkp} \partial_{\bar{i}} (S^{jk} S^{pl}) \\
&\quad + f_j^{(1)} \partial_{\bar{i}} S^j - c_\chi^2 \partial_{\bar{i}} (S^k K_k) + c_\chi^2 \partial_{\bar{i}} S - c_\chi \partial_{\bar{i}} (f_k^{(1)} S^k) + c_\chi \partial_{\bar{i}} (S^j K_j) - c_\chi \partial_{\bar{i}} S \\
&= -\frac{1}{8}(\bar{\partial}_{\bar{i}} D_k C_{jmn}) S^{jk} S^{mn} + \frac{1}{12} C_{jmn} C_{kpq} \bar{\partial}_{\bar{i}} (S^{jk} S^{pm} S^{qn}) - \frac{1}{8} C_{jpq} C_{krs} \partial_{\bar{i}} (S^{pq} S^{jk} S^{rs}) \\
&\quad -\frac{1}{2} c_\chi C_{jkp} \partial_{\bar{i}} (K_q S^{jk} S^{pq}) - \frac{1}{2} f_l^{(1)} C_{jkp} \partial_{\bar{i}} (S^{jk} S^{pl}) \\
&\quad + f_j^{(1)} \partial_{\bar{i}} S^j - c_\chi^2 \partial_{\bar{i}} (S^k K_k) + c_\chi^2 \partial_{\bar{i}} S - c_\chi \partial_{\bar{i}} (f_k^{(1)} S^k) + c_\chi \partial_{\bar{i}} (S^j K_j) - c_\chi \partial_{\bar{i}} S.
\end{aligned}
$$

It follows that

$$
\partial_{\bar{i}} \mathscr{F}^{(2)} \tag{28}
$$

$$
\begin{aligned}
&= \partial_{\bar{i}} [-\frac{1}{8}(D_k C_{jmn}) S^{jk} S^{mn} + \frac{1}{12} C_{jmn} C_{kpq} S^{jk} S^{pm} S^{qn} - \frac{1}{8} C_{jpq} C_{krs} S^{pq} S^{jk} S^{rs} \\
&\quad -\frac{1}{2} c_\chi C_{jkp} K_q S^{jk} S^{pq} - \frac{1}{2} f_l^{(1)} C_{jkp} S^{jk} S^{pl} \\
&\quad + f_j^{(1)} S^j - c_\chi^2 S^k K_k + c_\chi^2 S - c_\chi f_k^{(1)} S^k + c_\chi S^j K_j - c_\chi S].
\end{aligned}
$$

Therefore, we can see up to the addition by a holomorphic ambiguity $f^{(2)}$, the genus two topological string partition function $\mathscr{F}^{(2)}$ is a polynomial of the propagators S^{jk}, S^j, S and the generators K_i.

In fact, it was originally shown in [BCOV94] that a solution of the recursion holomorphic anomaly equations is given in terms of Feynman rules. The propagators S^{ij}, S^i, S for these Feynman rules were defined in Eqs. (15,18, 19):

$$
\partial_{\bar{i}} S^{jk} = \overline{C}_{\bar{i}}^{jk}, \quad \partial_{\bar{i}} S^j = G_{\bar{i}k} S^{jk}, \quad \partial_{\bar{i}} S = G_{\bar{i}k} S^k. \tag{29}
$$

The vertices of the Feynman rules are given by the functions $\mathscr{F}_{i_1 \cdots i_n}^{(g)} = D_{i_1} \cdots D_{i_n} \mathscr{F}^{(g)}$. For example, for genus two, the above topological string partition function in Eq. (28) has the form

$$
\begin{aligned}
\mathscr{F}^{(2)} &= \frac{1}{2} S^{jk} D_j D_k \mathscr{F}^{(1)} + \frac{1}{2} S^{jk} D_j \mathscr{F}^{(1)} D_k \mathscr{F}^{(1)} - \frac{1}{8} S^{jk} S^{mn} D_j D_k D_m D_n \mathscr{F} \\
&\quad -\frac{1}{2} S^{jk} C_{jkm} S^{mn} D_n \mathscr{F}^{(1)} \frac{\chi}{24} S^j D_j \mathscr{F}^{(1)} + \frac{1}{8} S^{jk} C_{jkp} S^{pq} C_{qmn} S^{mn} \\
&\quad +\frac{1}{12} S^{jk} S^{pq} S^{mn} C_{jpm} C_{kqn} - \frac{\chi}{48} S^j C_{jkl} S^{kl} + \frac{\chi}{24}(\frac{\chi}{24} - 1) S + f^{(2)},
\end{aligned}
$$

for some holomorphic function $f^{(2)}$. Each of the terms involving the propagators S^{jk}, S^j, S has a diagrammatic interpretation and corresponds to the contribution of a specific boundary component of the moduli space of genus two stable curves, see [BCOV94] for details.

Motivated by [BCOV94], in [YY04, AL07] it was proved, using Eqs. (5, 8, 16, 17), that the holomorphic anomaly equations Eq. (14) for $g \geq 2$ can be put into the following form

$$\bar{\partial}_{\bar{i}} \mathscr{F}^{(g)} = \bar{\partial}_{\bar{i}} \mathscr{P}^{(g)}, \tag{30}$$

where $\mathscr{P}^{(g)}$ is a polynomial in the generators S^{jk}, S^j, S, K_i with the coefficients being holomorphic quantities which might have poles. The proof relies on the fact that these generators form a differential ring [AL07] as displayed in Eqs. (20, 21, 22, 23) and recalled below:

$$\begin{aligned}
D_i S^{jk} &= \delta_i^j S^k + \delta_i^k S^j - C_{imn} S^{mj} S^{nk} + h_i^{jk}, \\
D_i S^j &= -C_{imn} S^m S^{jn} + 2\delta_i^j S + h_i^{jk} K_k + h_i^j, \\
D_i S &= -\frac{1}{2} C_{imn} S^m S^n + \frac{1}{2} h_i^{mn} K_m K_n + h_i^m K_m + h_i, \\
D_i K_j &= -K_i K_j + C_{ijm} S^{mn} K_n - C_{ijm} S^m + h_{ij}, \tag{31}
\end{aligned}$$

where $h_i^{jk}, h_i^j, h_i, h_{ij}$ are holomorphic quantities.

Now we justify the structure in Eq. (30) by induction, following [AL07]. Note that the non-holomorphicity of the topological string partition functions only comes from the non-holomorphic generators S^{ij}, S^i, S, K_i and thus the anti-holomorphic derivative on the left-hand side of the holomorphic anomaly equations can be replaced by derivatives with respect to these generators. Furthermore, one can make a change of generators [AL07]:

$$\begin{aligned}
\tilde{S}^{ij} &= S^{ij}, \\
\tilde{S}^i &= S^i - S^{ij} K_j, \\
\tilde{S} &= S - S^i K_i + \frac{1}{2} S^{ij} K_i K_j, \\
\tilde{K}_i &= K_i. \tag{32}
\end{aligned}$$

The differential ring structure among these new non-holomorphic generators follows from Eq. (31) easily. Replacing the $\bar{\partial}_{\bar{i}}$ derivative in the holomorphic anomaly equations by derivatives with respect to the new non-holomorphic generators and using the definitions Eq. (29), one then gets,

$$\bar{\partial}_{\bar{i}}\mathscr{F}^{(g)}$$

$$= \bar{C}_{\bar{i}}^{jk}\left(\frac{\partial\mathscr{F}^{(g)}}{\partial S^{jk}} - \frac{1}{2}\frac{\partial\mathscr{F}^{(g)}}{\partial\tilde{S}^{k}}\tilde{K}_{j} - \frac{1}{2}\frac{\partial\mathscr{F}^{(g)}}{\partial\tilde{S}^{j}}\tilde{K}_{k} + \frac{1}{2}\frac{\partial\mathscr{F}^{(g)}}{\partial\tilde{S}}\tilde{K}_{j}\tilde{K}_{k}\right) + G_{\bar{i}j}\frac{\partial\mathscr{F}^{(g)}}{\partial\tilde{K}_{j}}$$

$$= \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\left(\sum_{r=1}^{g-1}D_{j}\mathscr{F}^{(r)}D_{k}\mathscr{F}^{(g-r)} + D_{j}D_{k}\mathscr{F}^{(g-1)}\right).$$

Assuming the independence[5] of $\bar{C}_{\bar{i}}^{jk}$ and $G_{\bar{i}j}$, then one gets two sets of equations:

$$\frac{\partial\mathscr{F}^{(g)}}{\partial\tilde{S}^{jk}} - \frac{1}{2}\frac{\partial\mathscr{F}^{(g)}}{\partial\tilde{S}^{k}}\tilde{K}_{j} - \frac{1}{2}\frac{\partial\mathscr{F}^{(g)}}{\partial\tilde{S}^{j}}\tilde{K}_{k} + \frac{1}{2}\frac{\partial\mathscr{F}^{(g)}}{\partial\tilde{S}}\tilde{K}_{j}\tilde{K}_{k}$$

$$= \sum_{r=1}^{g-1}D_{j}\mathscr{F}^{(r)}D_{k}\mathscr{F}^{(g-r)} + D_{j}D_{k}\mathscr{F}^{(g-1)},$$

$$\frac{\partial\mathscr{F}^{(g)}}{\partial\tilde{K}_{j}} = 0. \tag{33}$$

Eq. (30) then follows from the above equations and Eq. (31).

The polynomial structure given in Eq. (30) also allows to determine the non-holomorphic part $\mathscr{P}^{(g)}$ of $\mathscr{F}^{(g)}$ genus by genus recursively from Eq. (33) as polynomials of the new non-holomorphic generators $\tilde{S}^{ij}, \tilde{S}^{i}, \tilde{S}, \tilde{K}_{i}$ and thus of the old ones S^{ij}, S^{i}, S, K_{i}:

$$\mathscr{P}^{(g)} = \mathscr{P}^{(g)}(S^{ij}, S^{i}, S, K_{i}). \tag{34}$$

Moreover, the coefficients of the monomials in these non-holomorphic generators are explicit Laurent polynomials in the holomorphic generators, with the coefficients of the monomials in the non-holomorphic and holomorphic generators being universal constants. These constants come from the Feynman diagram interpretation [BCOV94], or equivalently, the combinatorics from recursion. They are independent of the geometry under consideration. For example, for any geometry, the highest power of S^{ij} in the genus two partition function in Eq. (28) always takes the form $\frac{1}{12}C_{ipm}C_{jqn}S^{ij}S^{pq}S^{mn} + \cdots$.

3.4 Solutions of propagators

To obtain explicit results for $\mathscr{P}^{(g)}$, one needs to get formulas for the propagators S^{ij}, S^{i}, S, K_{i}. The generator K_{i} could be obtained by using its definition as the Kähler potential for the Weil-Petersson geometry Eq. (4) and the periods of the CY family Eq. (10). The generators S^{ij}, S^{i}, S can be solved from Eq. (29), up to addition by holomorphic quantities. A special set of solutions whose holomorphic limits are

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[5] This assumption is reasonable since these quantities have different singular behaviors when written in the canonical coordinates at the large complex structure.

vanishing was given by [ALM10, Hos10] in terms of geometric quantities. Alternatively, they could be determined [ALM10] from the differential structure Eq. (31):

$$S^{kl} = (C_*^{-1})^{kj}(\delta_*^l K_j + \delta_j^l K_* - \Gamma_{*j}^l + s_{*j}^l),$$
$$S^i = \frac{1}{2}(D_i S^{ii} + C_{imn} S^{mi} S^{ni} - h_i^{ii}),$$
$$S = \frac{1}{2}(D_i S^i + C_{imn} S^m S^{ni} - h_i^{ik} K_k - h_i^i), \tag{35}$$

or equivalently

$$S^{kl} = (C_*^{-1})^{kj}(\delta_*^l K_j + \delta_j^l K_* - \Gamma_{*j}^l + s_{*j}^l),$$
$$S^i = (C_*^{-1})^{ij}(-D_* K_j - K_* K_j + C_{*jk} S^{kl} K_l + h_{*j})$$
$$= (C_*^{-1})^{ij}(K_* K_j - K_{*j} + h_{*j}^l K_l + h_{*j}),$$
$$S = \frac{1}{2}(D_i S^i + C_{imn} S^m S^{ni} - h_i^{ik} K_k - h_i^i), \tag{36}$$

where the sub-index $*$ is chosen so that the matrix $((C_*)_{ij})$ is invertible. These formulas are useful [ALM10] in analyzing the degrees of freedom of the holomorphic quantities $h_i^{jk}, h_i^j, h_i, h_{ij}$. Moreover, in the one-modulus case for which the dimension of $\mathcal{M}$ is one, they also tell that the set of non-holomorphic generators S^{zz}, S^z, S, K_z is equivalent to the set of generators $\Gamma_{zz}^z, K_{zzz}, K_{zz}, K_z$ considered in [YY04] and the differential rings are thus identical. By using the latter set of generators, the differential structure follows from the Picard-Fuchs equation for the quantity e^{-K} and the special geometry relation Eq. (5), see [YY04] for details.

For non-compact CY 3-folds, one can choose a very simple set of generators due to the existence of a constant period. More precisely, according to Eqs. (4, 10), one can see that the holomorphic limit of K_i is zero. Then from Eq. (36) it follows that by choosing vanishing h_{ij}, h_i^i we can arrange so that the holomorphic limits of S^i, S are zero, see [ALM10] for more detailed discussions on this. This sometimes makes the calculations easier when computing the quantity $\mathcal{P}^{(g)}$ from recursion.

4 Connection to modular forms

After determining $\mathcal{P}^{(g)}$ from recursion by using the polynomial structure, one can then try to fix the holomorphic ambiguity $f^{(g)}$ by using the boundary conditions. We are not going to explain this due to limit of space, but we refer the interested reader to e.g., [BCOV93, BCOV94].[6]

In this section we shall emphasize the connection ([HK07, ABK08, Hos10, ASYZ13, Zho13, Ali14]) between the differential ring generated by S^{ij}, S^i, S, K_i

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[6] See also [MM99, KKV99, KZ99, KKRS05, YY04, KM08, HKQ09, ABK08, HK07, AL07, GKMW07, ALM10, HKR08, HK10, Sak11, AS12, KMW12, ASYZ13] for related works.

and the differential ring of modular forms. The reason for this expectation is that in some nicest cases, the topological string partition functions are known to be almost-holomorphic modular forms (see for example [Dij95, KZ95, MR11, Li12]) and are thus polynomials in the generators of the ring of almost-holomorphic modular forms. The similarity between the differential ring structure in the propagators/holomorphic limit of propagators of the special polynomial ring Eq. (25) and the polynomial structure of the ring of almost-holomorphic/quasi-modular forms seems to suggest a connection between them.

For the $K_{\mathbb{P}^2}$ family (each member of the family is the total space of $K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$, but the Kähler structure varies), the mirror family can be constructed following the lines in [CKYZ99] using Batyrev toric duality [Bat94], or using the Hori-Vafa construction [HV00]. For definiteness, the equation for the mirror family $\mathscr{X} \to \mathscr{M}$ obtained by the Hori-Vafa method is displayed below:

$$uv - H(y_1, y_2; z) = 0, \quad (u, v, y_1, y_2) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2,$$

where $H(y_1, y_2; z) = y_1 y_2(-z + y_1 + y_2) + 1$ and z is the parameter for the base $\mathscr{M}$. The mirror family $\mathscr{X} \to \mathscr{M}$ comes with the following Picard-Fuchs equation:

$$\left(\theta^3 - 27z\theta(\theta + \frac{1}{3})(\theta + \frac{2}{3}) \right) \phi = 0 \text{ for a period } \phi, \theta = z\frac{\partial}{\partial z}.$$

The coordinate z on $\mathscr{M}$ is chosen so that $z = 0$ is the large complex structure limit and $\Delta = 1 - 27z$ is the discriminant. For this case, the quantity κ which appears in Eq. (25) is $-\frac{1}{3}$. Then in Eq. (24) we have[7] $\tilde{K}_t = 0$, we can arrange so that

$$T_4 = T_6 = 0,$$

by choosing

$$s_{zz}^z = -\frac{4}{3} + \frac{1}{6\Delta}, \quad \tilde{h}_{zz}^z = \frac{1}{36\Delta^2}, \quad \tilde{h}_{zzz}^z = \frac{1}{2\Delta}.$$

It follows that the special polynomial ring in Eq. (25) becomes [ASYZ13]:

$$\partial_\tau C_0 = G_1^2 C_0,$$
$$\partial_\tau G_1 = \frac{1}{6}\left(2G_1 T_2 + G_1^3 \left(\frac{C_0 - 1}{C_0 + 1} \right) \right),$$
$$\partial_\tau T_2 = \frac{T_2^2}{3} - \frac{G_1^4}{12}.$$

Moreover, in [ASYZ13] it was shown that for this case the moduli space $\mathscr{M}$ can be identified with the modular curve $\Gamma_0(3) \backslash \mathscr{H}^*$. Under this identification, the points $z = 0, 1/27$ are identified with the cusp classes $[\tau] = [i\infty], [0]$ respectively. The gen-

---

[7] This is due to properties of special Kähler geometry and the particular form for the Picard-Fuchs equation, see [BCOV94] for details.

erators given in Eq. (24) which can be computed by using Eq. (35) or Eq. (36) are as follows:

$$T_2 = \frac{1}{8}(E_2(\tau) + 3E_2(3\tau)),$$
$$G_1 = \theta_2(2\tau)\theta_2(6\tau) + \theta_3(2\tau)\theta_3(6\tau),$$
$$C_0 = 27\left(\frac{\eta(3\tau)}{\eta(\tau)}\right)^{12}.$$

They are essentially the generators for the ring [KZ95] of quasi-modular forms $\widehat{M}(\Gamma_0(3), \chi_{-3})$ for the modular group $\Gamma_0(3)$, with the sub-indices corresponding to the modular weights. The differential ring structure in Eq. (25) corresponds to the Ramanujan-like identities among these quasi-modular forms, see [ABK08, Zag08, Mai09, ASYZ13, Zho13] and references therein for more details on this ring and its application in studying the modularity of topological string partition functions.

Similarly, for certain mirror families of $K_{dP_n}, n = 5, 6, 7, 8$, one can identify [ASYZ13] the moduli spaces with certain modular curves and show that the differential rings Eq. (25) are identical to the differential rings of quasi-modular forms with respect to corresponding modular groups.

Assuming the validity of mirror symmetry for these families of CY 3-folds, we can then show that if the solutions to the holomorphic anomaly equations for the B-model CY 3-fold family exist and are unique, then the generating functions of Gromov-Witten invariants for the A-model CY 3-fold family are quasi-modular forms [ASYZ13]. In fact, the existence and uniqueness for the $K_{\mathbb{P}^2}$ case are proved in [Zho14] which imply a version of integrality for the sequence of Gromov-Witten invariants $\{N_{g,d}\}_{d=1}^{\infty}$ for any (fixed) g.

For the mirror quintic family[CdLOGP91] $\pi : \mathscr{X} \to \mathscr{M}$, the Picard-Fuchs equation is given by

$$\left(\theta^4 - 5^5 z(\theta + \frac{1}{5})(\theta + \frac{2}{5})(\theta + \frac{3}{5})(\theta + \frac{4}{5})\right)\phi = 0 \text{ for a period } \phi, \ \theta = z\frac{\partial}{\partial z}.$$

We choose the usual z coordinate as the local coordinate on the moduli space $\mathscr{M}$ so that [8] $z = 0$ gives the large complex structure limit and the discriminant is $\Delta = 1 - 5^5 z$. In this case, the classical triple intersection in Eq. (25) is $\kappa = 5$. We then choose the following ambiguities [YY04, Hos10, Ali12, ASYZ13]:

$$s_{zz}^z = -\frac{8}{5}, \tilde{h}_{zz}^z = \frac{1}{5\Delta}, \tilde{h}_{zz} = -\frac{1}{5^2\Delta}, \tilde{h}_z = \frac{2}{5^3\Delta}, k_{zz} = \frac{2}{5^2}.$$

Now the holomorphic quantities in the ring Eq. (25) are polynomials in $\frac{1}{\Delta}$. Moreover, we also have $C_0 = \frac{5^5 z}{1 - 5^5 z} = \frac{1}{\Delta} - 1$ with $\partial_\tau C_0 = C_0(C_0 + 1)K_0 G_1^2$ which tells that

---

[8] This is related to the ψ coordinate in [CdLOGP91] by $z = (5\psi)^{-5}$.

the derivative of $\frac{1}{\Delta}$ sits inside the ring generated by $K_0, G_1, K_2, T_2, T_4, T_6, C_0$. More examples on compact CY geometries can be found in [Ali14].

For compact CY 3-folds, the period domain is in general not Hermitian symmetric and the suitable theory of almost-holomorphic modular forms and quasi-modular forms is not known. Hence one can not say much about the connection between this ring and the ring of modular objects. Nevertheless, the rings of quantities generated by $K_0, G_1, K_2, T_2, T_4, T_6, C_0$, which are defined from the special Kähler geometry on the moduli spaces of complex structures of CY 3-folds, share very similar properties to those of the rings of almost-holomorphic modular forms defined on modular curves, see [Hos10, Zho13] for more discussions on this.

5 Conclusions

We first showed how to solve for the non-holomorphic part of $\mathscr{F}^{(2)}$ by introducing the propagators S^{ij}, S^i, S. We then derived the differential ring structure of the ring generated by these propagators and K_i. After that we proved by induction that for any g, the non-holomorphic part $\mathscr{P}^{(g)}$ is a polynomial of the generators and can be solved recursively genus by genus. For some special non-compact CY 3-fold families, we pointed out that after the identification between the moduli spaces of complex structures with modular curves, the generators for the special polynomial ring become the generators for the ring of quasi-modular forms, and the differential ring structure is identified with the Ramanujan-like identities for the quasi-modular forms.

It would be interesting to see whether these rings could help construct ring of modular objects (see for example [Mov11]), and how the global properties of the generators could help solve for the topological string partition functions from the holomorphic anomaly equations with boundary conditions for more general CY 3-fold families.

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