GENUS ONE GW INVARIANTS OF QUINTIC THREEFOLDS VIA MSP LOCALIZATION

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ABSTRACT. The moduli stack of Mixed Spin P-fields (MSP) provides an effective algorithm to evaluate all genus Gromov-Witten invariants of quintic Calabi-Yau threefolds. This paper is to apply the algorithm in genus one case. We use the localization formula, the proposed algorithm in [CLLL1, CLLL2], and Zinger's packaging technique to compute the genus one Gromov-Witten invariants of quintic Calabi-Yau threefolds. New hypergeometric series identities are also discovered in the process.

1. INTRODUCTION

The genus one Gromov-Witten invariants of Calabi-Yau hypersurfaces were calculated by Zinger in [Zi2]. The method consists of two steps. The first step is the detailed study of the moduli space of genus one stable maps to CY hypersurfaces in [LZ, VZ]. The second step is the computation part using torus localization and some packaging techniques in [Zi2].

In this paper, we will use the MSP set-up to calculate the genus one Gromov-Witten invariants for quintic CY hypersurfaces in \mathbb{P}^4 . As a by product, we find integral relations for certain differential polynomials of I functions (e.g. section C.1.2).

Mixed-spin P-fields (MSP) are defined in [CLLL1] (see [CLLL3] for a survey). An MSP field is given by

$$\xi = (\mathfrak{C}, \Sigma^{\mathfrak{C}}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu)$$

with stability conditions, where \mathcal{C} is a twisted nodal curve (or an orbifold curve) with markings $\Sigma^{\mathcal{C}}$, \mathcal{L} and \mathcal{N} are invertible sheaves on \mathcal{C} , and φ, ρ, ν are sections of $\mathcal{L}^{\oplus 5}, \mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}}^{\log}, \mathcal{L} \otimes \mathcal{N} \oplus \mathcal{N}$ respectively. For simplicity, let's consider the case without markings. Then ξ has the numerical data (g, \mathbf{d}) where gis the genus of the curve \mathcal{C} and $\mathbf{d} = (d_0, d_\infty)$ with $d_0 = \deg(\mathcal{L} \otimes \mathcal{N})$ and $d_\infty = \deg(\mathcal{N})$. We use $\mathcal{W}_{g,\mathbf{d}}$ to denote the moduli stack of MSP fields with numerical data (g, \mathbf{d}) . The virtual dimension of $\mathcal{W}_{g,\mathbf{d}}$ is $d_0 + d_\infty - g + 1$.

MSP moduli stacks can be regarded as a platform interpolating the moduli of Gromov-Witten theory and the moduli of FJRW theory (see [FJR1, FJR2]). Using the *P*-field theory of [CL] and the cosection localization technique of [KL], a virtual cycle can be defined and the properness can be proved in [CLLL1]. The moduli stack admits a \mathbb{C}^* -action. Using this torus action, if the virtual dimension of $\mathcal{W}_{g,\mathbf{d}}$ is bigger than zero, the virtual cycle [$\mathcal{W}_{g,\mathbf{d}}$]^{vir} can give us the identity:

(1)
$$0 = \int_{[\mathcal{W}_{g,\mathbf{d}}]^{\mathrm{vir}}} \mathfrak{t} c_{\mathrm{top}-1} \big(R\pi_{\mathcal{W}*}(\mathcal{L}_{\mathcal{W}}\mathcal{N}_{\mathcal{W}}\mathcal{L}_{\mathfrak{t}}) \big),$$

where $\mathcal{L}_{\mathcal{W}}$ and $\mathcal{N}_{\mathcal{W}}$ are the universal line bundles on the universal curve \mathcal{C} over the moduli $\mathcal{W} := \mathcal{W}_{g,\mathbf{d}}$ with a projection $\pi_{\mathcal{W}} : \mathcal{C} \to \mathcal{W}$, top is the virtual rank of $R\pi_{\mathcal{W}*}(\mathcal{L}_{\mathcal{W}}\mathcal{N}_{\mathcal{W}})$, \mathfrak{t} is the generator of the equivariant cohomology $H^*_{\mathbb{C}^*}(B\mathbb{C}^*,\mathbb{Q}) = \mathbb{Q}[\mathfrak{t}]$, and $\mathcal{L}_{\mathfrak{t}}$ is the trivial line bundle with the weight-1 \mathbb{C}^* action on fibers.

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The identify (1) provides lots of equations among GW invariants, Hodge integrals, and FJRW invariants. This will potentially provide an effective algorithm to calculate GW invariants as well as FJRW invariants. In this paper, we will consider the virtual cycles $[\mathcal{W}_{1,(0,n)}]^{\text{vir}}$ for all n to compute genus one GW invariants of Fermat quintic CY three-folds.

Comparing with Zinger's work, MSP moduli replaces the detailed study of moduli spaces of stable maps, especially the complicated issue of the ghost component, by FJRW invariants. Since the study of the ghost components for higher genus is very complicated as it requires separation of components with different *P*-field (ghost) rank (c.f. [CL1]), MSP moduli provides a good way to work on higher genus GW invariants (see [MP] for another approach). For example, our method does not need genus zero two point functions. The g = 1 package in this paper is expected to be directly extended to higher genus package via MSP moduli.

The \mathbb{C}^* -action gives four types of graphs from the standard localization procedure, called type A, B, C and D (see Figure 2.2). The contribution from type A graphs involves genus one GW invariants of the quintic, while that from type D graphs involves FJRW invariants. Our packaging uses Zinger's technique ([Zi2]). We also discover (c.f. Section C.1.2) new hypergeometric series identities in the package of loop type (B) graphs.

The method of quasimaps also provides another way to calculate genus one GW invariants of CY hypersurfaces [CK1, CK2, KLh].

The organization of the paper is as follows. In §2, we review the graphs notations and their contributions in the localization of MSP theory in [CLLL1, CLLL2]. In §3, we review the mirror theorem for genus zero GW invariants of quintic threefolds [Gi1, Gi2] (also see [LLY]) and discuss how to transform Givental's set-up to our MSP set-up. In §4, we define \mathcal{K} -series and \mathcal{P} -series, which are essential for the calculations in later sections. We also calculate some \mathcal{K} -series. In §5, we compute the contribution of type A graphs to (1). In §6, we compute the contribution of type B graphs to (1). In §7, we compute the contribution of type C graphs to (1). In §8, we compute the contribution of type D graphs to (1). Many computations in the last four sections are contained in the Appendix.

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Notations: Throughout the paper we fix the notations $T = \mathbb{C}^*$, $\mathfrak{t} = -c_1(\mathfrak{O}_{\mathbb{CP}^{\infty}}(1))$ for $BT = \mathbb{CP}^{\infty}$, $q = e^t$, and $h \in H^2(\mathbb{P}^4, \mathbb{Z})$ always denotes the hyperplane class.

2. MSP THEORY AND LOCALIZATION SET-UP

In this section, we will review the localization scheme of the moduli stacks of mixed-spin P-fields in [CLLL1, CLLL2]. There are four types of graphs in genus one localization, labeled as type A, B, C and D.

2.1. Review of MSP theory. The MSP moduli $\mathcal{W} = \mathcal{W}_{g,(1,\rho)^n,\mathbf{d}}$ is constructed in [CLLL1]. It is indexed by genus g, the number n of markings labelled by $(1,\rho)$, and two integers $(d_0, d_\infty) = \mathbf{d}$. It is a separated DM stack locally of finite type equipped with a cosection σ and a $T = \mathbb{C}^*$ action. The cosection's zero locus $\mathcal{W}_{g,(1,\rho)^n,\mathbf{d}}^-$ is shown to be proper [CLLL1] and the torus action gives rise to an equivariant virtual cycle $[\mathcal{W}_{g,\gamma,\mathbf{d}}]_{\text{loc}}^{\text{vir}} \in A^T_*(\mathcal{W}_{g,\gamma,\mathbf{d}}^-)^T$. It has a localization formula (for $\delta := d_0 + d_\infty + 1 - g + n$)

(2)
$$[\mathcal{W}]_{\text{loc}}^{\text{vir}} = \sum_{\Gamma} j_{\Gamma*}^{-} \left(\frac{|\mathcal{W}_{\Gamma}]_{\text{loc}}^{\text{vir}}}{e(N_{\Gamma}^{\text{vir}})} \right) \in A_{\delta}^{T}(\mathcal{W}^{-})^{T}[\mathfrak{t}^{-1}],$$

where each connected component of fixed locus \mathcal{W}^T is indexed by a graph Γ and j^- is the natural inclusion from \mathcal{W}_{Γ} to \mathcal{W}^- . Each graph Γ is connected, and has all vertices labelled by 0, 1 or ∞ , namely

 $V(\Gamma) = V_0(\Gamma) \cup V_1(\Gamma) \cup V_\infty(\Gamma)$ according to $\nu_1 = 0, \nu_1 = 1 = \nu_2$ and $\nu_2 = 0$ respectively, and edges $E(\Gamma) = E_0(\Gamma) \cup E_\infty(\Gamma)$ where edges in $E_0(\Gamma)$ connect vertices $V_0(\Gamma)$ with $V_1(\Gamma)$, and edges in $E_\infty(\Gamma)$ connect vertices $V_0(\Gamma)$ with $V_1(\Gamma)$. Denote the set $F(\Gamma)$ of flags to be the set of $(e, v) \in E(\Gamma) \times V(\Gamma)$ such that e, v are adjacent. The graphs is required to possess decorations as follows:

- (a) (genus) $g_{\cdot}: V(\Gamma) \to \mathbb{Z}_{>0};$
- (b) (degree) $\vec{d} : E(\Gamma) \cup V(\Gamma) \to \mathbb{Q}^{\oplus 2}$ via $\vec{d}(a) = (d_{0a}, d_{\infty a})$, where $d_{0a} = \deg \mathcal{L} \otimes \mathbb{N}|_{\mathcal{C}_a}$ and $d_{\infty a} = \deg \mathbb{N}|_{\mathcal{C}_a}$ with \mathcal{C}_a being the curve associated to a.

(c) (marking) $S_{\cdot}: V(\Gamma) \to 2^{L(\Gamma)}$ via $v \mapsto S_v \subset L(\Gamma)$, where S_v is the subset of markings $\Sigma_i^{\mathcal{C}} \in \mathcal{C}_v$. We call a vertex $v \in V(\Gamma)$ unstable if $g_v = 0$, $d_v = \deg(\mathcal{L}|_{\mathcal{C}_v}) = 0$ and $|E_v \cup S_v| < 3$, otherwise stable. Thus $V(\Gamma) = V^S(\Gamma) \cup V^{US}(\Gamma)$. For any unstable vertex $v \in V^{US}(\Gamma)$, it must lie in $V^{0,1}$, $V^{1,1}$ or $V^{0,2}$, where

(3)
$$V^{a,b}(\Gamma) := \{ v \in V(\Gamma) - V^S(\Gamma) \mid |S_v| = a, |E_v| = b \}.$$

Let $V_c^{a,b}(\Gamma) = V_c(\Gamma) \cap V^{a,b}(\Gamma)$ and $\Delta_{q,\gamma,\mathbf{d}}$ denote the set of all (regular) graphs, up to graph isomorphisms preserving decorations. For each graph $\Gamma \in \Delta_{g,\gamma,\mathbf{d}}$ we use (Γ) to denote its class in $\Delta_{g,\gamma,\mathbf{d}}/\sim$, where \sim is by automorphisms of graphs. Then $\mathcal{W}_{g,\gamma,\mathbf{d}}^T$ is a disjoint union of $\mathcal{W}_{(\Gamma)}$, the locus whose curves are associated to the graph Γ (c.f. [CLLL2, Def 2.5]), over Γ 's in $\Delta_{q,\gamma,\mathbf{d}}$. In [CLLL2] the fixed locus of $\mathcal{W}_{g,\gamma,\mathbf{d}}$ decorated by Γ is denoted by \mathcal{W}_{Γ} . In this paper we will no distinguish \mathcal{W}_{Γ} from $\mathcal{W}_{(\Gamma)}$, and we always use $\pi_{\Gamma} : \mathcal{C}_{\mathcal{W}_{\Gamma}} \to \mathcal{W}_{\Gamma}, \mathcal{L}_{\Gamma}$ to denote its universal curve and universal line bundle.

For each $v \in V^{S}(\Gamma)$, there is a moduli stack \mathcal{W}_{v} ([CLLL2, Cor. 2.37]), where

$$\mathcal{W}_v \cong \overline{\mathcal{M}}_{g_v, E_v \cup S_v} (\mathbb{P}^4, d_v)^p, \quad \mathcal{W}_v \cong \overline{\mathcal{M}}_{g_v, E_v \cup S_v}, \quad \mathcal{W}_v \cong \overline{\mathcal{M}}_{g_v, \gamma}^{1/5.5p}$$

when $v \in V_0^S(\Gamma), V_1^S(\Gamma)$ and $V_{\infty}^S(\Gamma)$ respectively. Here $\overline{\mathcal{M}}_{g_v, E_v \cup S_v}(\mathbb{P}^4, d_v)^p$ is the moduli stack of degree d_v stable maps from genus g_v nodal curves with markings in $E_v \cup S_v$ to \mathbb{P}^4 with P-fields studied in [CL], and $\overline{M}_{g_v,\gamma}^{1/5.5p}$ is the moduli stack of genus $g_v \mu_5$ -twisted curve with a five-spin structure, markings labeled by γ and five *P*-fields studied in [CLL].

For $e \in E_0$, there is an MSP moduli $\mathcal{W}_e = \sqrt[d_e]{\mathcal{O}_{\mathbb{P}^4}(1)/\mathbb{P}^4}$ with $\pi_e : \mathcal{W}_e \to \mathbb{P}^4$ the map to its coarse moduli space. Let $h \in H^2(\mathbb{P}^4; \mathbb{Q})$ be the hyperplane class, $h_e = \pi_e^* h \in H^2(\mathcal{W}_e; \mathbb{Q})$, and d_e be the degree of $\mathcal{L}|_{\mathcal{C}_e}$.

For unstable $v \in V_0 - V_0^S$, we let $[\mathcal{W}_v]_{\text{loc}}^{\text{vir}} = -[Q_5]$, where $Q_5 \subset \mathbb{P}^4$ is the Fermat quintic.

Proposition 2.1. For each graph Γ appearing in localizing $\mathcal{W}_{q=1,(1,\rho)^n,(d,0)}$, we have

$$[\mathcal{W}_{(\Gamma)}]_{\text{loc}}^{\text{vir}} = \frac{1}{|\operatorname{Aut}(\Gamma)|} \frac{1}{\prod_{e \in E_0} d_e} \frac{1}{\prod_{e \in E_\infty} |5d_e|} \prod_{v \in V_0 \cup V_1^S \cup V_\infty^S} [\mathcal{W}_v]_{\text{loc}}^{\text{vir}}$$

Theorem 2.2.

$$\frac{1}{e_T(N_{\Gamma}^{\mathrm{vir}})} = \prod_{v \in V(\Gamma)} B_v \prod_{e \in E(\Gamma)} A_e,$$

where

$$B_{v} = \begin{cases} A_{v} \cdot \prod_{e \in E_{v}} \frac{1}{w_{(e,v)} - \psi_{(e,v)}}, & v \in V^{S}; \\ \frac{A_{v}}{w_{(e,v)} + w_{(e',v)}}, & v \in V^{0,2}, E_{v} = \{e, e'\}; \\ A_{v}, & v \in V^{1,1}; \\ A_{v} \cdot w_{(e,v)}, & v \in V^{0,1}, E_{v} = \{e\}. \end{cases}$$

The formulae for A_v, A_e and $w_{(e,v)}$ are provided below.

Lemma 2.1. Let v and v' be the two vertices of an edge e.

- (1) When $v \in V_0$, then $w_{(e,v)} = \frac{h_e + \mathfrak{t}}{d_e}$ and $w_{(e,v')} = -\frac{h_e + \mathfrak{t}}{d_e}$. (2) When $v \in V_\infty \setminus V_\infty^{0,1}$, then $w_{(e,v)} = \frac{\mathfrak{t}}{r_e d_e}$ and $w_{(e,v')} = -\frac{\mathfrak{t}}{d_e}$.

Lemma 2.2 (Contribution from edges). For the two cases (1) $e \in E_0$, (2) $e \in E_\infty$, $(e, v) \in F$, $v \in V_\infty \setminus V_\infty^{0,1}$, we have respectively

$$A_{e} = \frac{\prod_{j=1}^{5d_{e}-1} (-5h_{e} + \frac{j(h_{e}+\mathfrak{t})}{d_{e}})}{\prod_{j=1}^{d_{e}} (h_{e} - \frac{j(h_{e}+\mathfrak{t})}{d_{e}})^{5} \cdot \prod_{j=1}^{d_{e}} \frac{j(h_{e}+\mathfrak{t})}{d_{e}}}, \qquad \qquad \frac{\prod_{j=1}^{\lceil -d_{e}\rceil - 1} (-\mathfrak{t} - \frac{j\mathfrak{t}}{d_{e}})^{5}}{\prod_{j=1}^{\lceil -d_{e}\rceil} (-\mathfrak{t} - \frac{j\mathfrak{t}}{d_{e}})^{5}}.$$

Lemma 2.3 (Contribution from unstable vertices). If $v \in V^{0,2}$, then

$$A_{v} = \begin{cases} h_{e} + \mathfrak{t} = h_{e'} + \mathfrak{t}, & v \in V_{0}^{0,2} \text{ and } E_{v} = \{e, e'\}, \\ -5\mathfrak{t}^{6}, & v \in V_{1}^{0,2}, \end{cases}$$

If $v \in V_1^{1,1}$, $S_v \subset \Sigma_{(1,\rho)}^{\mathfrak{C}}$, then we have $A_v = 5\mathfrak{t}$. If $v \in V_0^{1,1}$, we have $A_v = 1$.

Let's introduce some notations.

- Given a stable vertex $v \in V^S$, let $\pi_v : \mathcal{C}_v \to \mathcal{W}_v$ be the universal curve, \mathcal{L}_v be the universal line bundle over \mathcal{C}_v .
- Given $v \in V_1^S$, let $\mathbb{E}_v := \pi_* \omega_{\pi_v}$ be the Hodge bundle, where $\omega_{\pi_v} \to \mathcal{C}_v$ is the relative dualizing sheaf. Then $\mathbb{E}_v^{\vee} = R^1 \pi_{v*} \mathcal{O}_{\mathcal{C}_v}$.
- Let \mathbf{L}_k be the one-dimensional weight k *T*-representation.

Lemma 2.4 (Contribution from all stable vertices). For $v \in V^S$, define

(4)
$$A_{v} := \begin{cases} \frac{1}{e_{T}(R\pi_{v*}\phi_{v}^{*}\mathcal{O}_{\mathbb{P}^{4}}(1)\otimes\mathbf{L}_{1})} \prod_{e\in E_{v}}(h_{e}+\mathfrak{t}), & v\in V_{0}^{S};\\ \left(\frac{e_{T}(\mathbb{E}_{v}^{\vee}\otimes\mathbf{L}_{-1})}{-\mathfrak{t}}\right)^{5} \cdot \frac{5\mathfrak{t}}{e_{T}(\mathbb{E}_{v}\otimes\mathbf{L}_{5})} \cdot (-\mathfrak{t}^{5})^{|E_{v}|} \cdot \left(\frac{-\mathfrak{t}^{4}}{5}\right)^{|S_{v}^{(1,\varphi)}|}, & v\in V_{1}^{S};\\ \frac{1}{e_{T}(R\pi_{v*}\mathcal{L}_{v}^{\vee}\otimes\mathbf{L}_{-1})}, & v\in V_{\infty}^{S}. \end{cases}$$

2.2. Types A, B, C and D. When the genus is one, we can divide the type of regular graphs associated to T-fixed points of W into four types A, B, C and D. They are given as follows.

Type A: This graph has one stable vertex $v \in V_0$ with genus $g_v = 1$.

Type B: This graph has a loop with \mathcal{C}_0 consisting of possibly many rational curves and/or points and \mathcal{C}_1 being rational curves and/or points. \mathcal{C}_∞ is empty. Here $\mathcal{C}_0 = \mathcal{C} \cap (\nu_1 = 0)$ and similarly for \mathcal{C}_1 and \mathcal{C}_∞ .

Type C: This graph has one stable vertex $v \in V_1$ with genus $g_v = 1$.

Type D: This graph has one stable vertex $v \in V_{\infty}$ with genus $g_v = 1$. All edges e in E_{∞} has $\deg(\mathcal{L}|_{\mathcal{C}_e}) = -\frac{1}{5}$.



Each circle represents all possible graphs (with one marking) for genus zero and $d_{\infty} = 0$ MSP moduli, where the marking lies on V_0 or V_1 accordingly.

3. Equivariant Mirror Theorem

3.1. Equivariant Mirror Theorem. Let $G = \mathbb{C}^*$ act on \mathbb{P}^5 as follows: $t \cdot [x_0, \ldots, x_5] = [x_0, \ldots, tx_5]$. Over

$$H^*_G(\mathbb{P}^5;\mathbb{Q}) = \mathbb{Q}[p,\mathfrak{t}]/\langle p^5(p+\mathfrak{t})\rangle$$

there is a G-equivariant pairing given by $(p^i, p^j) = (-\mathfrak{t})^{i+j-5}$ if $i+j \ge 5$ and 0 otherwise. The ordered basis

$$(\mathbb{T}^{0}, \mathbb{T}^{1}, \mathbb{T}^{2}, \mathbb{T}^{3}, \mathbb{T}^{4}, \mathbb{T}^{5}) := (p^{4}(p + \mathfrak{t}), p^{3}(p + \mathfrak{t}), p^{2}(p + \mathfrak{t}), p(p + \mathfrak{t}), (p + \mathfrak{t}), 1)$$

is dual to the ordered basis

$$(\mathbb{T}_0, \mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbb{T}_4, \mathbb{T}_5) := (1, p, p^2, p^3, p^4, p^5)$$

with respect to the G-equivariant Poincaré pairing, namely $(\mathbb{T}^i, \mathbb{T}_j) = \delta_{i,j}$.

Let $G' = \mathbb{C}^*$ acts on \mathbb{P}^5 trivially. Consider the $G \times G'$ equivariant bundle over \mathbb{P}^5

(5)
$$V = L_H^{\otimes 5} \oplus (L_H \otimes L_{\mathfrak{t}} \otimes L_{\mathfrak{t}'}),$$

where $\mathfrak{t}' = c_1(\mathcal{O}_{BG'}(-1))$ (with $BG' \cong \mathbb{CP}^{\infty}$) denotes the first Chern class of $L_{\mathfrak{t}'}$ and similarly for \mathfrak{t} with the group G' replaced by G. Let $(\pi_d, \mathfrak{f}_d) : \mathcal{C}_d \to \overline{M}_{0,1}(\mathbb{P}^5, d) \times \mathbb{P}^5$ be the universal family and $E_d = \pi_{d*} \mathfrak{f}_d^* V$.

Recall the conventions
$$q = e^t$$
, $f(q) = \sum_{d=0}^{\infty} \frac{(5d)!}{(d!)^5} q^d = I_0(t)$, $g_l(q) = \sum_{d=1}^{\infty} q^d \frac{(5d)!}{(d!)^5} \left(\sum_{m=1}^{ld} \frac{1}{m} \right)$,
 $I_1(t) := 5[g_5 - g_1](q) + tI_0(t)$, $T(t) := I_1(t)/I_0(t)$.

We set $T_0 = \hbar \log f + \mathfrak{t}' \frac{g_1}{f}$. Then the equivariant mirror theorem [Gi2] states that

$$I(q) = q^{p/\hbar} 5p(p + \mathfrak{t} + \mathfrak{t}') \sum_{d=0}^{\infty} q^d \frac{\prod_{m=1}^{5d} (5p + m\hbar) \prod_{m=1}^d (p + \mathfrak{t} + \mathfrak{t}' + m\hbar)}{\prod_{m=1}^d (p + m\hbar)^5 (p + \mathfrak{t} + m\hbar)}$$

is equal to, under the mirror map $Q(q) = e^{I_1(t)/I_0(t)} = e^T$,

$$J(\mathfrak{Q}) = e^{(T_0 + p \log \mathfrak{Q})/\hbar} \left(5p(p + \mathfrak{t} + \mathfrak{t}') + \sum_{d=1}^{\infty} \mathfrak{Q}^d \sum_{k=0}^5 \mathbb{T}_k \int_{\overline{\mathcal{M}}_{0,1}(\mathbb{P}^5, d)} \mathrm{ev}^* \mathbb{T}^k \frac{e(E_d)}{\hbar(\hbar - \psi)} \right).$$

3.2. Application to the MSP set-up. Recall $\mathcal{W}_d := \mathcal{W}_{g=0,(1,\rho),(d,0)} = \overline{M}_{0,1}(\mathbb{P}^5, d)$. Let

$$F_d = \pi_{d*} \mathfrak{f}_d^* L_H^{\otimes 5}, \qquad R_d = \pi_{d*} \mathfrak{f}_d^* (L_H \otimes L_{\mathfrak{t}}).$$

Then rank $R_d = d + 1$ and $E_d = F_d \oplus (R_d \otimes L_{\mathfrak{t}'})$. Thus

$$e(E_d) = e(F_d)e(R_d \otimes L_{\mathfrak{t}'}) = e(F_d)\sum_{k=0}^{\infty} c_{\operatorname{rank} R_d - k}(R_d)(\mathfrak{t}')^k.$$

With $[\overline{\mathcal{M}}_{0,1}(\mathbb{P}^5, d)] \cap e(F_d) = (-1)^{d+1} [\mathcal{W}_{g=0,(1,\rho),(d,0)}]^{\text{vir}}$, the mirror theorem is rephrased as follows. Corollary 3.1. The series $e^{-(T_0+p\log Q)/\hbar}I(q)$ is equal to

$$5p(p+\mathfrak{t}+\mathfrak{t}') - \sum_{d=1}^{\infty} (-\mathfrak{Q})^d \int_{[\mathcal{W}_{g=0,(1,\rho),(d,0)}]^{\mathrm{vir}}} \left(\sum_{k=0}^4 p^k \frac{e(R_d \otimes L_{\mathfrak{t}'})}{\hbar(\hbar-\psi)} \cup \mathrm{ev}_1^*[p^{4-k}(p+\mathfrak{t})] + p^5 \frac{e(R_d \otimes L_{\mathfrak{t}'})}{\hbar(\hbar-\psi)} \right).$$

For each $j \ge 0$, define

$$U_j := -\sum_{d=1}^{\infty} (-\mathcal{Q})^d \int_{[\mathcal{W}_{g=0,(1,\rho),(d,0)}]^{\operatorname{vir}}} \frac{e(R_d \otimes L_{\mathfrak{t}'})}{\hbar(\hbar - \psi)} \cup \operatorname{ev}_1^* p^j$$

Then $Y := Y(p,q) := e^{-(T_0 + p \log \Omega)/\hbar} I(q)$ in Corollary 3.1 equals

 $5p(p+\mathfrak{t}+\mathfrak{t}')+p^{0}(U_{5}+\mathfrak{t}U_{4})+p^{1}(U_{4}+\mathfrak{t}U_{3})+p^{2}(U_{3}+\mathfrak{t}U_{2})+p^{3}(U_{2}+\mathfrak{t}U_{1})+p^{4}(U_{1}+\mathfrak{t}U_{0})+p^{5}U_{0}.$

We also write $Y(p) = \sum_{j=0}^{5} (\operatorname{Coe}_{p^{j}} Y) p^{j}$. We define

$$\begin{split} \tilde{\mathcal{Z}}_{1,5} &:= -\sum_{d=1}^{\infty} (-\mathcal{Q})^d \sum_{(\Gamma)\in\Xi_d^1} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \frac{e(R\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}\mathcal{L}_{\mathfrak{t}'}(-D_{\Gamma})))}{e(N_{\mathcal{W}_{(\Gamma)}}^{vir})} \\ &= -\frac{1}{\mathfrak{t}'} \sum_{d=1}^{\infty} (-\mathcal{Q})^d \sum_{(\Gamma)\in\Xi_d^1} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \frac{e(R\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}\mathcal{L}_{\mathfrak{t}'}))}{e(N_{\mathcal{W}_{(\Gamma)}}^{vir}/\mathcal{W})}, \end{split}$$

where Ξ_n^1 denotes localization graphs with the first marking mapped to O and \mathcal{L}_{Γ} is the universal line bundle. Then, as $p^4|_{P_i} = 0$ for $0 \le i \le 4$ and $p|_O = -\mathfrak{t}$, by the localization formula, one has $U_j = (-\mathfrak{t})^j \cdot \mathfrak{t}' \cdot \tilde{\mathcal{Z}}_{1,5}$ whenever $j \ge 4$. Thus $\operatorname{Coe}_{p^0} Y = 0$ and

$$\begin{aligned} \mathfrak{t}^{4}\mathfrak{t}'\mathcal{Z}_{1,5} &= U_{4} = -5(\mathfrak{t} + \mathfrak{t}') + \operatorname{Coe}_{p^{1}}Y - \mathfrak{t}U_{3} = -5(\mathfrak{t} + \mathfrak{t}') + \operatorname{Coe}_{p^{1}}Y - \mathfrak{t}(\operatorname{Coe}_{p^{2}}Y - \mathfrak{t}U_{2} - 5) \\ &= -5\mathfrak{t}' + \operatorname{Coe}_{p^{1}}Y - \mathfrak{t}\operatorname{Coe}_{p^{2}}Y + \mathfrak{t}^{2}\operatorname{Coe}_{p^{3}}Y - \mathfrak{t}^{3}\operatorname{Coe}_{p^{4}}Y + \mathfrak{t}^{4}\operatorname{Coe}_{p^{5}}Y \end{aligned}$$

implies $-\mathfrak{t}^5\mathfrak{t}'\tilde{\mathcal{Z}}_{1,5} = 5\mathfrak{t}\mathfrak{t}' + Y|_{p\mapsto-\mathfrak{t}}$. Thus we have

Theorem 3.2. For MSP moduli $W_{g=0,(1,\rho),(d,0)}$,

$$\sum_{d=1}^{\infty} (-\mathfrak{Q})^d \sum_{(\Gamma)\in\Xi_d^1} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi_1)} \frac{e(\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\otimes\mathcal{L}_{\mathfrak{t}}\otimes\mathcal{L}_{\mathfrak{t}'}(-D_{\Gamma})))}{e(N_{\mathcal{W}_{\Gamma}}^{vir}/\mathcal{W})}$$

(here $-D_{\Gamma} \subset C_{W_{\Gamma}}$ is from the marking) is equal to

$$\frac{5}{\mathfrak{t}^4} + \frac{-5}{\mathfrak{t}^4} \frac{1}{I_0} \exp(\frac{\mathfrak{t}(T-\mathfrak{t})}{\hbar} - \frac{\mathfrak{t}'g_1}{I_0\hbar}) \cdot \sum_{d=0}^{\infty} q^d \frac{\prod_{m=1}^{5d} (-5\mathfrak{t} + m\hbar) \prod_{m=1}^d (\mathfrak{t}' + m\hbar)}{d!\hbar^d \prod_{m=1}^d (-\mathfrak{t} + m\hbar)^5} d!\hbar^d \prod_{m=1}^d (-\mathfrak{t} + m\hbar)^5 d!\hbar^d \prod_{m=1}^d (-\mathfrak{t} + m\hbar)^4 d!\hbar^d \prod_{m=1}^d (-\mathfrak{t} + m\hbar)^6 d!\hbar^d \prod_{m=1}^d (-\mathfrak{t} + m\hbar)^6$$

We then have the following package for c_i -capped invariants.

Theorem 3.3. For MSP moduli $\mathcal{W}_{g=0,(1,\rho),(d,0)}, \forall m \in \mathbb{N}_{\geq 1}$

$$\sum_{d=1}^{\infty} (-\mathfrak{Q})^d \sum_{(\Gamma)\in\Xi_d^1} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi_1)} \frac{c_{\mathrm{top}_{\Gamma}-m}(\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\otimes\mathcal{L}_{\mathfrak{t}}))}{e(N_{\mathcal{W}_{\Gamma}}^{vir}/\mathcal{W})}$$

is the $(\mathfrak{t}')^{m-1}$ coefficient of

$$\frac{5}{\mathfrak{t}^4} + \frac{-5}{\mathfrak{t}^4} \frac{1}{I_0} \exp(\frac{\mathfrak{t}(T-\mathfrak{t})}{\hbar} - \frac{\mathfrak{t}'g_1}{I_0\hbar}) \cdot \sum_{d=0}^{\infty} q^d \frac{\prod_{m=1}^{5d} (-5\mathfrak{t} + m\hbar) \prod_{m=1}^d (\mathfrak{t}' + m\hbar)}{d!\hbar^d \prod_{m=1}^d (-\mathfrak{t} + m\hbar)^5} d! \hbar^d \prod_{m=1}^d (-\mathfrak{t} + m\hbar)^5 d! \hbar^d \prod_{m=1}^d (-\mathfrak{t} + m\hbar)^6 d! \hbar$$

In the special case $\mathfrak{t}' = -\mathfrak{t}$ of Corollary 3.1, one has the following.

Corollary 3.4. Write $\mathfrak{A}_d = \pi_{d*}\mathfrak{f}_d^*L_H$. It is a rank d+1 bundle over $\mathcal{W}_{0,(1,\rho),(d,0)}$. The series

$$5p^2 - \sum_{d=1}^{\infty} (-\mathfrak{Q})^d \sum_{k=0}^4 \int_{[\mathcal{W}_{g=0,(1,\rho),(d,0)}]^{\mathrm{vir}}} \left(p^k \frac{e(\mathfrak{A}_d)}{\hbar(\hbar-\psi)} \cup \mathrm{ev}_1^*[p^{4-k}(p+\mathfrak{t})] + p^5 \frac{e(\mathfrak{A}_d)}{\hbar(\hbar-\psi)} \right)$$

is equal to

(6)
$$\frac{5p^2}{I_0} e^{\frac{1}{\hbar}(\frac{g_1}{I_0}\mathfrak{t} + p(t-T))} \sum_{d=0}^{\infty} q^d \frac{\prod_{m=1}^{5d}(5p+m\hbar)}{\prod_{m=1}^d(p+m\hbar)^4(p+\mathfrak{t}+m\hbar)}$$

Taking coefficients of \mathfrak{t}' of the formula in Corollary 3.1, we obtain the following identity.

Corollary 3.5. The series

$$5p - \sum_{d=1}^{\infty} (-\Omega)^d \sum_{k=0}^{4} \int_{[\mathcal{W}_{g=0,(1,\rho),(d,0)}]^{\text{vir}}} \left(p^k \frac{c_d(R_d)}{\hbar(\hbar-\psi)} \cup \operatorname{ev}_1^*[p^{4-k}(p+\mathfrak{t})] + p^5 \frac{c_d(R_d)}{\hbar(\hbar-\psi)} \right)$$

is equal to the coefficient of t' in $e^{-(T_0 + p \log Q)/\hbar} I(0,q)$, which is

$$\frac{5p(p+\mathfrak{t})}{I_0}e^{\frac{p}{\hbar}(t-T)} \bigg[(\frac{1}{p+\mathfrak{t}} - \frac{g_1}{\hbar I_0}) \sum_{d=0}^{\infty} q^d \frac{\prod_{m=1}^{5d} (5p+m\hbar)}{\prod_{m=1}^d (p+m\hbar)^5} + \sum_{d=1}^{\infty} q^d \frac{\prod_{m=1}^{5d} (5p+m\hbar)(\sum_{m=1}^d \frac{1}{p+\mathfrak{t}+m\hbar})}{\prod_{m=1}^d (p+m\hbar)^5} \bigg]$$

3.3. From MSP to Givental I-functions. Consider

(7)
$$B_{d}(h) := \frac{\prod_{k=1}^{5d} (-5h + \frac{k(h+\mathfrak{t})}{d})}{\prod_{k=1}^{d} (h - \frac{k(h+\mathfrak{t})}{d})^{5}} = b_{0} + b_{1}\frac{h}{\mathfrak{t}} + b_{2}\frac{h^{2}}{\mathfrak{t}^{2}} + b_{3}\frac{h^{3}}{\mathfrak{t}^{3}} \in \mathbb{Q}[\mathfrak{t}, \mathfrak{t}^{-1}][h]/(h^{4})$$
$$A_{d}(h) := \frac{\prod_{k=1}^{5d} (5h+k)}{\prod_{k=1}^{d} (h+k)^{5}} = a_{0} + a_{1}h + a_{2}h^{2} + a_{3}h^{3} \in \mathbb{Q}[h]/(h^{4})$$

where both $a_i = a_i(d)$, $b_i = b_i(d)$ are rational numbers. Apply the map $\theta : \mathbb{Q}[\mathfrak{t}, \mathfrak{t}^{-1}][h]/(h^4) \to \mathbb{Q}[h]/(h^4)$ which sends \mathfrak{t} to (-1-h). Since (-1-h) is invertible in $\mathbb{Q}[h]/(h^4)$, this defines a ring homomorphism. Then $\theta(B_d(h)) = (-1)^{5d} A_d(dh)$ implies

$$b_0 = (-1)^d a_0$$
, $b_1 = (-1)^{d+1} a_1 d$, $b_2 = (-1)^d [a_2 d^2 + a_1 d]$, $b_3 = (-1)^{d+1} [a_3 d^3 + 2a_2 d^2 + a_1 d]$.
Recall the definition of Civental's I function

Recall the definition of Givental's I function

(8)
$$I_0(t) + I_1(t)h + I_2h^2 + \dots + I_3h^3 = e^{ht} \sum_{d=0}^{\infty} e^{dt} \frac{\prod_{r=1}^{5d} (5h+r)}{\prod_{r=1}^d (h+r)^5} = e^{ht} \sum_{d=0}^{\infty} e^{dt} A_d(h) \pmod{h^4}$$
.

Recall $D_t := q \frac{\partial}{\partial q} = \frac{\partial}{\partial t}$, and denote $D_t I_k$ as I'_k . The relations in (8) imply

$$(9) 1 + \sum_{d=1}^{\infty} b_0(-q)^d = 1 + \sum_{d=1}^{\infty} a_0(d)q^d = I_0,$$

$$\sum_{d=1}^{\infty} b_1(-q)^d = -\sum_{d=1}^{\infty} da_1(d)q^d = -D_t I_1 + D_t(tI_0),$$

$$\sum_{d=1}^{\infty} b_2(-q)^d = \sum_{d=1}^{\infty} [a_2d^2 + a_1d]q^d = D_t^2 I_2 - D_t(tI_1') + D_t(\frac{t^2}{2}I_0'),$$

$$\sum_{d=1}^{\infty} b_3(-q)^d = -\sum_{d=1}^{\infty} [a_3d^3 + 2a_2d^2 + a_1d]q^d = -D_t^3 I_3 + D_t(tI_2'') - D_t(\frac{t^2}{2}I_1'') + D_t(\frac{t^3}{6}I_0'').$$

4. *K*-series and *P*-series

4.1. Definitions of \mathcal{K} -series and \mathcal{P} -series. Consider the localization of $\mathcal{W}_{g=0,1_{\rho},(n,0)}$. Let Ξ_n be the set of flat g=0 graphs with one leg decorated by 1_{ρ} and $(d_0, d_{\infty}) = (n, 0)$, and let $\Xi_n^i \subset \Xi_n$, i=0 or 1, be the subset where the marking lies on vertices in V_i .

For c > 0 and $k = 0, \dots, 5$, we set a formal power series in $\mathbb{Q}(\mathfrak{t})[[\Omega]]$

(10)
$$\mathcal{K}_{c,k}(\mathfrak{t}) := \sum_{d=1}^{\infty} (-\mathfrak{Q})^d \sum_{(\Gamma) \in \hat{\Xi}_d^0} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\psi^c} \mathrm{ev}^* \mathbb{T}^k \frac{c_{\mathrm{top}_{\Gamma}-1}(R\pi_{\Gamma*}(\mathcal{L}_{\Gamma} \otimes \mathcal{L}_{\mathfrak{t}}(-D_{\Gamma})))}{e(N_{\mathcal{W}_{(\Gamma)}}^{vir}/\mathcal{W})} \in \mathbb{Q}[[\mathfrak{Q}]]\mathfrak{t}^{2-k-c},$$

where $\hat{\Xi}_d^0$ denotes the subset of graphs in Ξ_d^0 whose vertices contain an unstable 1_{ρ} -leg, $D_{\Gamma} \subset C_{W_{\Gamma}}$ is the divisor given by the marking, and top_{Γ} is the rank of the bundle $R\pi_{\Gamma*}(\mathcal{L}_{\Gamma} \otimes \mathcal{L}_{\mathfrak{t}}(-D_{\Gamma}))$.

For convenience, we also consider formal power series in $\mathbb{Q}(\mathfrak{t})[[\mathfrak{t}']][[\Omega]]$

$$\begin{split} \tilde{\mathcal{K}}_{c,k}(\mathfrak{t},\mathfrak{t}') &:= \sum_{d=1}^{\infty} (-\mathfrak{Q})^d \sum_{(\Gamma) \in \hat{\Xi}_d^0} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\psi^c} ev^* \mathbb{T}^k \frac{e(\pi_{\Gamma*} \mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}'}(-D_{\Gamma}))}{e(N_{\mathcal{W}_{(\Gamma)}}^{vir}/\mathcal{W})}, \\ \hat{\mathcal{K}}_k(w) &:= \sum_{c=1}^{\infty} \tilde{\mathcal{K}}_{c,k}(-w)^{c-1} \end{split}$$

$$= \sum_{d=1}^{\infty} (-\mathfrak{Q})^d \sum_{(\Gamma)\in\hat{\Xi}_d^0} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\psi_{\Gamma} + w} \mathrm{ev}^* \mathbb{T}^k \frac{e(R\pi_{\Gamma*}(\mathcal{L}_{\Gamma} \otimes \mathcal{L}_{\mathfrak{t}} \otimes \mathcal{L}_{\mathfrak{t}'}(-D_{\Gamma})))}{e(N_{\mathcal{W}_{(\Gamma)}}^{vir}/w)},$$

which is an expansion near w = 0. Note that

$$e(\pi_{\Gamma*}\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma})) = \sum_{\ell=0}^{\infty} c_{\operatorname{top}_{\Gamma}-\ell}(R\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\otimes\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma})))(\mathfrak{t}')^{\ell}.$$

By definition, $\mathcal{K}_{c,k}$ is the coefficient of $(\mathfrak{t}')^1$ in $\tilde{\mathcal{K}}_{c,k}$. Observe that the constant term of \mathfrak{t}' -expansion of $\tilde{\mathcal{K}}_{c,k}$ vanishes because ν_1 makes $e(R\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma})))$ vanishing. We call this *cap geometry*, and write the expansion

(11)
$$\tilde{\mathcal{K}}_{c,k} = \mathcal{K}_{c,k} \mathfrak{t}' + \mathcal{K}_{c,k,1} (\mathfrak{t}')^2 + \dots + \mathcal{K}_{c,k,\ell} (\mathfrak{t}')^{\ell+1} + \dots$$

We define \mathcal{P} -series as follows. For c > 0, we set a formal power series in $\mathbb{Q}(\mathfrak{t})[[\Omega]]$

$$\begin{aligned} \mathcal{P}_{c}(\mathfrak{t}) &:= \sum_{d=1}^{\infty} (-\mathcal{Q})^{d} \sum_{(\Gamma) \in \hat{\Xi}_{d}^{1}} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\psi^{c}} \frac{c_{\mathrm{top}_{\Gamma}}(R\pi_{\Gamma*}(\mathcal{L}_{\Gamma} \otimes \mathcal{L}_{\mathfrak{t}}(-D_{\Gamma})))}{e(N_{\mathcal{W}_{(\Gamma)}}^{vir}/\psi)} \\ \tilde{\mathcal{P}}_{c}(\mathfrak{t},\mathfrak{t}') &:= \sum_{d=1}^{\infty} (-\mathcal{Q})^{d} \sum_{(\Gamma) \in \hat{\Xi}_{d}^{1}} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\psi^{c}} \frac{e(\pi_{\Gamma*}\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}\mathcal{L}_{\mathfrak{t}'}(-D_{\Gamma}))}{e(N_{\mathcal{W}_{(\Gamma)}}^{vir}/\psi)}. \end{aligned}$$

Let $\hat{\Xi}_d^1$ be the subset of graphs in Ξ_d^1 whose vertices contain the unstable 1_{ρ} -leg. We also define

$$(12) \quad \hat{\mathcal{P}}(w) := \sum_{d=1}^{\infty} (-\mathcal{Q})^d \sum_{(\Gamma) \in \hat{\Xi}_d^1} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{w + \psi_{\Gamma}} \frac{e(\pi_{\Gamma*}\mathcal{L}_{\Gamma}\mathcal{L}_{t}\mathcal{L}_{\Gamma'}(-D_{\Gamma}))}{e(N_{\mathcal{W}_{(\Gamma)}}^{vir})} = \tilde{\mathcal{P}}_1 - w\tilde{\mathcal{P}}_2 + w^2\tilde{\mathcal{P}}_3 - \cdots,$$
$$\tilde{\mathcal{Z}}_{j,k} := -\sum_{d=1}^{\infty} (-\mathcal{Q})^d \sum_{(\Gamma) \in \Xi_d^j} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar - \psi)} \mathrm{ev}^* \mathbb{T}^k \frac{e(R\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{t}\mathcal{L}_{t'}(-D_{\Gamma})))}{e(N_{\mathcal{W}_{(\Gamma)}}^{vir}/\mathcal{W})}, \quad \text{for } j = 0, 1,$$

(13)
$$\mathcal{Z}_{j,k} := -\sum_{d=1}^{\infty} (-\mathcal{Q})^d \sum_{(\Gamma)\in\Xi_d^j} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \mathrm{ev}^* \mathbb{T}^k \frac{c_{\mathrm{top}-1}(R\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma})))}{e(N_{\mathcal{W}_{(\Gamma)}}^{vir}/\mathcal{W})}, \quad \text{for } j = 0, 1.$$

Lemma 4.1. For $c \ge 1$, we have

$$\tilde{\mathcal{K}}_{c,k} = \operatorname{Res}_{\hbar=0} \frac{\tilde{\mathcal{Z}}_{0,k}}{\hbar^{c-1}}, \quad \mathcal{K}_{c,k} = \operatorname{Res}_{\hbar=0} \frac{\mathcal{Z}_{0,k}}{\hbar^{c-1}}, \quad \tilde{\mathcal{P}}_c = \operatorname{Res}_{\hbar=0} \frac{\tilde{\mathcal{Z}}_{1,5}}{\hbar^{c-1}}, \quad \mathcal{P}_c = \operatorname{Res}_{\hbar=0} \frac{\mathcal{Z}_{1,5}}{\hbar^{c-1}}.$$

Proof. We prove the first two identities. In the case $(\Gamma) \in \hat{\Xi}_d^0$, its ψ is invertible and the expansion near $\hbar = 0$ gives $\frac{1}{\hbar - \psi} = \frac{-1}{\psi} (1 + \frac{\hbar}{\psi} + \frac{\hbar^2}{\psi^2} + \cdots)$. And in the case $(\Gamma) \in \Xi_d^0 - \hat{\Xi}_d^0$, its ψ^r will vanish on $[\mathcal{W}_{\Gamma}]^{\text{vir}}$ for some r. Thus we can write $\frac{1}{\hbar - \psi} = \frac{1}{\hbar} (1 + \frac{\psi}{\hbar} + \cdots + \frac{\psi^r}{\hbar^r}), \forall \hbar \neq 0$. Therefore such Γ doesn't contribute to the residue.

4.2. Calculations of $\tilde{\mathcal{Z}}_{1,k}$. For k < 5, one has $\mathbb{T}^k = p^{4-k}(p+\mathfrak{t})$ and thus $\mathbb{T}^k|_O = 0$ implies $\tilde{\mathcal{Z}}_{1,k} = 0$. Thus

$$-\sum_{d=1}^{\infty} (-\mathfrak{Q})^d \sum_{(\Gamma)\in\Xi_d^1} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \frac{e(R\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}\mathcal{L}_{\mathfrak{t}'}(-D_{\Gamma})))}{e(N_{\mathcal{W}_{(\Gamma)}}^{vir}/\mathcal{W})} \mathrm{ev}^* p^k = (-\mathfrak{t})^k \cdot \tilde{\mathcal{Z}}_{1,5}.$$

When k = 5, by Theorem 3.2, we have

$$\tilde{\mathcal{Z}}_{1,5} = -\sum_{d=1}^{\infty} (-\mathcal{Q})^d \sum_{(\Gamma)\in\Xi_d^1} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \frac{e(R\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}\mathcal{L}_{\mathfrak{t}'}(-D_{\Gamma})))}{e(N_{\mathcal{W}_{(\Gamma)}/\mathcal{W}}^{vir})} \\
(14) = -\frac{5}{\mathfrak{t}^4} + \frac{5}{\mathfrak{t}^4 I_0} \exp(\frac{\mathfrak{t}(T-t)}{\hbar} - \frac{\mathfrak{t}'g_1}{I_0\hbar}) \cdot \sum_{d=0}^{\infty} q^d \frac{\prod_{m=1}^{5d} (-5\mathfrak{t} + m\hbar) \prod_{m=1}^d (\mathfrak{t}' + m\hbar)}{d!\hbar^d \prod_{m=1}^d (-\mathfrak{t} + m\hbar)^5}.$$

We write

$$w := -\frac{\mathfrak{t}}{\hbar}, \ \mathcal{R}(w,t) := \sum_{d=0}^{\infty} q^d \frac{\prod_{k=1}^{5d} (5w+k)}{\prod_{k=1}^{d} (w+k)^5} \in \mathbb{Q}[[q,w]], \ \Lambda := \frac{1}{I_0} e^{(t-T)w} \mathcal{R}(w,t).$$

Corollary 4.1.

(15)
$$\mathcal{Z}_{1,5} = \operatorname{Coe}_{(\mathfrak{t}')^0} \tilde{\mathcal{Z}}_{1,5} = -\frac{5}{\mathfrak{t}^4} + \frac{5}{\mathfrak{t}^4} \Lambda.$$

$$\sum_{d=1}^{\infty} (-\Omega)^d \sum_{(\Gamma)\in\Xi_d^1} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \frac{c_d(R\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma})))}{e(N_{\mathcal{W}_{(\Gamma)}}^{vir})} ev^* p^k = 5(-\mathfrak{t})^{k-4} (1-\Lambda).$$

4.3. Calculation of $\mathcal{Z}_{0,k}$. Assume k < 5, then $\mathbb{T}^k = p^{4-k}(p+\mathfrak{t})$. We have for each $(\Gamma) \in \Xi^0_d$, $\pi_{\Gamma_0}(\mathcal{L}_{\Gamma_0}\mathcal{L}_{\ell}) = \pi_{\Gamma_0}(\mathcal{L}_{\Gamma_0}\mathcal{L}_{\ell}(-D_{\Gamma_0})) + \pi_{\Gamma_0}(\mathcal{L}_{\Gamma_0}\mathcal{L}_{\ell}(-D_{\Gamma_0})) = \pi_{\Gamma_0}(\mathcal{L}_{\Gamma_0}\mathcal{L}_{\ell}(-D_{\Gamma_0})) + e^{\chi t} L_{\Gamma_0}(\mathcal{L}_{\Gamma_0}\mathcal{L}_{\ell}) = \pi_{\Gamma_0}(\mathcal{L}_{\Gamma_0}\mathcal{L}_{\ell}) = \pi_{\Gamma_0}(\mathcal{L}_{\Gamma_0}\mathcal{L}) = \pi_{\Gamma_0}(\mathcal{L}_{\Gamma_0}\mathcal{L}) = \pi_{\Gamma_0}(\mathcal{L}_{\Gamma_0}\mathcal{L}) = \pi_{\Gamma_0}(\mathcal{L}_{\Gamma_0}\mathcal{L}) = \pi_{\Gamma_0}(\mathcal{L}_{\Gamma_0}\mathcal{L}) = \pi_{\Gamma_0}(\mathcal{L}_{\Gamma_0}\mathcal{L}) = \pi_{\Gamma_0}(\mathcal{L}) = \pi$

$$\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}) = \pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma})) + \pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}|_{D_{\Gamma}}) = \pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma})) + \operatorname{ev}^{*}L_{H}L_{\mathfrak{t}},$$

as classes in K-groups, and thus

$$c_d(\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}})) = e\big(\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma}))\big) + c_{d-1}\big(\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma}))\big) \cdot c_1(\operatorname{ev}^*L_HL_{\mathfrak{t}}),$$

$$\begin{aligned} \mathcal{Z}_{0,k} &= -\sum_{d=1}^{\infty} (-\Omega)^d \sum_{(\Gamma)\in\Xi_d^0} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \mathrm{ev}^* p^{4-k} (p+\mathfrak{t}) \frac{c_{d-1}(R\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma})))}{e_G(N_{\mathcal{W}_{(\Gamma)}/\mathcal{W}}^{vir})} \\ (16) &= -\sum_{d=1}^{\infty} (-\Omega)^d \sum_{(\Gamma)\in\Xi_d^0} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \mathrm{ev}^* p^{4-k} \frac{c_d(\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}))}{e_G(N_{\mathcal{W}_{(\Gamma)}/\mathcal{W}}^{vir})} \\ &+ \sum_{d=1}^{\infty} (-\Omega)^d \sum_{(\Gamma)\in\Xi_d^0} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \mathrm{ev}^* p^{4-k} \frac{e(\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma})))}{e_G(N_{\mathcal{W}_{(\Gamma)}/\mathcal{W}}^{vir})} \end{aligned}$$

The two integrals in (16) can be separately reduced as follows. [i]. The first integral in (16) can be calculated as

$$(17) \qquad \qquad -\sum_{d=1}^{\infty} (-Q)^{d} \sum_{(\Gamma)\in\Xi_{d}^{0}} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} ev^{*} p^{4-k} \frac{c_{d}(\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}))}{e_{G}(N_{\mathcal{W}_{(\Gamma)}/\mathcal{W}}^{vir})}$$
$$= -\sum_{d=1}^{\infty} (-Q)^{d} \operatorname{Coe}_{\mathfrak{t}'} \int_{[\mathcal{W}_{0,1_{\rho},(d,0)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} ev^{*} p^{4-k} \cdot e(\pi_{\mathcal{W}*}(\mathcal{L}_{\mathcal{W}}\mathcal{L}_{\mathfrak{t}}\mathcal{L}_{\mathfrak{t}'}))$$
$$+ \sum_{d=1}^{\infty} (-Q)^{d} \sum_{(\Gamma)\in\Xi_{d}^{1}} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} ev^{*} p^{4-k} \frac{c_{d}(\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}))}{e_{G}(N_{\mathcal{W}_{(\Gamma)}/\mathcal{W}}^{vir})},$$

where $\pi_{\mathcal{W}}$ is the projection from the universal family $\mathcal{C}_{\mathcal{W}}$ to \mathcal{W} and $\mathcal{L}_{\mathcal{W}}$ is the pull-back of $\mathcal{O}_{\mathbb{P}^5}(1)$ via the universal family map $\mathcal{C}_{\mathcal{W}} \to \mathbb{P}^5$. Denote the first term in (17) as

$$\mathfrak{B}_k := -\operatorname{Coe}_{\mathfrak{t}'} \sum_{d=1}^{\infty} (-\mathfrak{Q})^d \int_{[\mathcal{W}_{0,1_{\rho},(d,0)}]^{\operatorname{vir}}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^* p^{4-k} \cdot e(\pi_{\mathcal{W}*}(\mathcal{L}_{\mathcal{W}}\mathcal{L}_{\mathfrak{t}}\mathcal{L}_{\mathfrak{t}'})).$$

The second term in (17) is, by Corollary 4.1,

$$\sum_{d=1}^{\infty} (-\mathcal{Q})^d \sum_{(\Gamma)\in\Xi_d^1} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \mathrm{ev}^* p^{4-k} \frac{c_d(\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}))}{e_G(N_{\mathcal{W}_{(\Gamma)}}^{vir}/\mathcal{W})}$$
$$= 5(-\mathfrak{t})^{-k} \left(1 - \frac{1}{I_0} e^{\frac{\mathfrak{t}}{\hbar} \cdot (T-t)} \sum_{d=0}^{\infty} q^d \frac{\prod_{m=1}^{5d} (-5\mathfrak{t} + m\hbar)}{\prod_{m=1}^d (-\mathfrak{t} + m\hbar)^5}\right).$$

[ii]. The second integral in (16), denoted by \mathfrak{R}_k , is only contributed by those $\Gamma \in \Xi^0_d$ which have only one vertex v in V_0 (denote such graph as Γ_v), otherwise, ν_1 makes $e(\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma})))$ vanish. More precisely

$$\begin{aligned} \mathfrak{R}_{k} &= \sum_{d=1}^{\infty} (-\mathfrak{Q})^{d} \int_{(-1)^{d+1} [\overline{M}_{0,1}(Q,d)]^{\mathrm{vir}}} \frac{1}{\hbar^{2}} (1+\frac{\psi}{\hbar}) \cdot \mathrm{ev}^{*} h^{4-k} \cdot \frac{1}{e \left(\pi_{\Gamma*} (\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}|_{D_{\Gamma}}) \right)} \\ &= -\sum_{d=1}^{\infty} \mathfrak{Q}^{d} \int_{[\overline{M}_{0,1}(Q,d)]^{\mathrm{vir}}} \frac{1}{\hbar^{2}} (1+\frac{\psi}{\hbar}) \cdot \mathrm{ev}^{*} h^{4-k} \cdot \mathrm{ev}^{*} (1-\frac{h}{\mathfrak{t}}) \frac{1}{\mathfrak{t}}, \end{aligned}$$

where h is the hyperplane class in \mathbb{P}^4 . We calculate \mathfrak{R}_k for different k's.

$$\begin{aligned} \mathfrak{R}_{4} &= -\sum_{d=1}^{\infty} \mathfrak{Q}^{d} \int_{[\overline{M}_{0,1}(Q,d)]^{\mathrm{vir}}} \frac{1}{\hbar^{2}} \cdot \mathrm{ev}^{*}(-\frac{h}{\mathfrak{t}}) \frac{1}{\mathfrak{t}} - \sum_{d=1}^{\infty} \mathfrak{Q}^{d} \int_{[\overline{M}_{0,1}(Q,d)]^{\mathrm{vir}}} \frac{1}{\hbar^{2}} \frac{\psi}{\hbar} \cdot \frac{1}{\mathfrak{t}} \\ &= \sum_{d=1}^{\infty} \mathfrak{Q}^{d} \Big(\frac{1}{\hbar^{2} \mathfrak{t}^{2}} dN_{0,d} - \frac{1}{\hbar^{3} \mathfrak{t}} (-2) N_{0,d} \Big). \end{aligned}$$

~~

Let k < 4. The integral \Re_k becomes

$$-\sum_{d=1}^{\infty} \mathcal{Q}^d \int_{[\overline{M}_{0,1}(Q,d)]^{\mathrm{vir}}} \frac{1}{\hbar^2} \mathrm{ev}^* h^{4-k} \cdot \frac{1}{\mathfrak{t}} = \begin{cases} -\sum_{d=1}^{\infty} \mathcal{Q}^d \frac{1}{\hbar^2 \mathfrak{t}} dN_{0,d}, & k=3; \\ 0 & k=0,1,2. \end{cases}$$

Therefore we proved the following.

Lemma 4.2. For $k = 0, 1, \dots, 4$, we have

$$\mathcal{Z}_{0,k} = \mathfrak{B}_k + 5(-\mathfrak{t})^{-k} (1 - \Lambda) + \mathfrak{R}_k,$$

where $\Re_k = 0$ for k = 0, 1, 2 and

$$\Re_3 = -\sum_{d=1}^{\infty} \mathcal{Q}^d \frac{1}{\hbar^2 \mathfrak{t}} dN_{0,d}, \qquad \qquad \Re_4 = \sum_{d=1}^{\infty} \mathcal{Q}^d \Big(\frac{1}{\hbar^2 \mathfrak{t}^2} dN_{0,d} - \frac{1}{\hbar^3 \mathfrak{t}} (-2) N_{0,d} \Big).$$

We now calculate \mathfrak{B}_k for k = 0, 1, 2.

$$\begin{aligned} \mathfrak{B}_{0} &= -\frac{1}{\mathfrak{t}} \operatorname{Coe}_{\mathfrak{t}'} \sum_{d=1}^{\infty} (-\mathfrak{Q})^{d} \int_{[\mathcal{W}_{0,1_{\rho},(d,0)}]^{\operatorname{vir}}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} p^{4}(p+\mathfrak{t}) \cdot e(\pi_{\mathcal{W}*}(\mathcal{L}_{\mathcal{W}}\mathcal{L}_{\mathfrak{t}}\mathcal{L}_{\mathfrak{t}'})) \\ &+ \frac{1}{\mathfrak{t}} \operatorname{Coe}_{\mathfrak{t}'} \sum_{d=1}^{\infty} (-\mathfrak{Q})^{d} \int_{[\mathcal{W}_{0,1_{\rho},(d,0)}]^{\operatorname{vir}}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} p^{5} \cdot e(\pi_{\mathcal{W}*}(\mathcal{L}_{\mathcal{W}}\mathcal{L}_{\mathfrak{t}}\mathcal{L}_{\mathfrak{t}'})) \\ &= -\frac{\mathfrak{t}^{5}}{\mathfrak{t}} \sum_{d=1}^{\infty} (-\mathfrak{Q})^{d} \sum_{(\Gamma)\in\Xi_{d}^{1}} \int_{[\mathcal{W}_{(\Gamma)}]^{\operatorname{vir}}} \frac{1}{\hbar(\hbar-\psi)} \cdot \frac{c_{\operatorname{top}-1}(\pi_{\Gamma*}(\mathcal{L}_{\Gamma}\mathcal{L}_{\mathfrak{t}}))}{e(N_{\mathcal{W}_{\Gamma}/\mathcal{W}})} \\ &= -\mathfrak{t}^{4} \cdot \left(\frac{5}{\mathfrak{t}^{4}} + \frac{-5}{\mathfrak{t}^{4}} \frac{1}{I_{0}} \exp(\frac{\mathfrak{t}(T-t)}{\hbar}) \cdot \sum_{d=0}^{\infty} q^{d} \frac{\prod_{m=1}^{5d} (-5\mathfrak{t} + m\hbar)}{\prod_{m=1}^{d} (-\mathfrak{t} + m\hbar)^{5}} \right) = 5\Lambda(w, t) - 5 \end{aligned}$$

where the third identity is by G-localization and the last line comes from Theorem 3.3.

Taking Coe_p of the Corollary 3.5, one has

$$\mathfrak{B}_{1} = -\frac{1}{\mathfrak{t}} \sum_{d=1}^{\infty} (-\mathfrak{Q})^{d} \int_{[\mathcal{W}_{0,1_{\rho},(d,0)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \mathrm{ev}^{*} p^{3}(p+\mathfrak{t}) \cdot c_{\mathrm{top}-1}(\pi_{\mathcal{W}*}(\mathcal{L}_{\mathcal{W}}\mathcal{L}_{\mathfrak{t}}))$$
$$+ \frac{1}{\mathfrak{t}} \sum_{d=1}^{\infty} (-\mathfrak{Q})^{d} \int_{[\mathcal{W}_{0,1_{\rho},(d,0)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \mathrm{ev}^{*} p^{4} \cdot c_{\mathrm{top}-1}(\pi_{\mathcal{W}*}(\mathcal{L}_{\mathcal{W}}\mathcal{L}_{\mathfrak{t}}))$$

$$= -\frac{5}{\mathfrak{t}}\Lambda(w,t) + \frac{-5w}{\mathfrak{t}I_0} \bigg[(-\frac{I_0}{w} - g_1) + \sum_{d=1}^{\infty} q^d \frac{(5d)!}{(d!)^5} (\sum_{m=1}^d \frac{1}{m-w}) \bigg].$$

Taking Coe_{p^2} of Corollary 3.5, we get

$$\mathfrak{B}_{2} = -\frac{1}{\mathfrak{t}} \sum_{d=1}^{\infty} (-\mathfrak{Q})^{d} \int_{[\mathcal{W}_{0,1_{\rho},(d,0)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \mathrm{ev}^{*} p^{2}(p+\mathfrak{t}) \cdot c_{\mathrm{top}-1}(\pi_{\mathcal{W}*}(\mathcal{L}_{\mathcal{W}}\mathcal{L}_{\mathfrak{t}})) + \frac{1}{\mathfrak{t}} \sum_{d=1}^{\infty} (-\mathfrak{Q})^{d} \int_{[\mathcal{W}_{0,1_{\rho},(d,0)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \mathrm{ev}^{*} p^{3} \cdot c_{\mathrm{top}-1}(\pi_{\mathcal{W}*}(\mathcal{L}_{\mathcal{W}}\mathcal{L}_{\mathfrak{t}})) = -\frac{1}{\mathfrak{t}} \mathfrak{B}_{1} - \frac{5}{I_{0}} \cdot \frac{g_{1}}{\hbar\mathfrak{t}} + \frac{5}{I_{0}} \sum_{d=1}^{\infty} q^{d} \frac{(5d)!}{(d!)^{5}} \left(\left(\frac{1}{\mathfrak{t}} - \frac{T-t}{\hbar} + \sum_{m=d+1}^{5d} \frac{5}{m\hbar}\right) \sum_{m=1}^{d} \frac{1}{\mathfrak{t}+m\hbar} - \sum_{m=1}^{d} \frac{1}{(\mathfrak{t}+m\hbar)^{2}} \right)$$

Here we omit detailed calculations obtaining the second equality above.

4.4. Formula of $\mathcal{K}_{1,2}$ and $\mathcal{K}_{2,1}$. For k = 1, 2, we can now apply $\mathcal{K}_{c,k} = \operatorname{Res}_{\hbar=0} \frac{\mathcal{Z}_{0,k}}{\hbar^{c-1}}$. Now we look at the case k = 1. The residue at $\hbar = 0$ of $5\mathfrak{t}^{-1}(1 - \Lambda) + \mathfrak{R}_1$ divided by \hbar is

$$\frac{5}{\mathfrak{t}}\left(1 - \operatorname{Res}_{\hbar=0}\frac{\Lambda}{\hbar}d\hbar\right) = \frac{5}{\mathfrak{t}}\left(1 - \operatorname{Res}_{w=\infty}\Lambda \cdot (-\mathfrak{t}) \cdot \frac{-dw}{w^2} \cdot \frac{-w}{\mathfrak{t}}\right) = \frac{5}{\mathfrak{t}}\left(1 + \operatorname{Res}_{w=\infty}\Lambda\frac{dw}{w}\right).$$

From

$$\mathfrak{B}_{1} = -\frac{5}{\mathfrak{t}}\Lambda(w,t) + \frac{-5w}{\mathfrak{t}I_{0}} \bigg[-g_{1}(q) + \sum_{d=1}^{\infty} q^{d} \frac{(5d)!}{(d!)^{5}} (\sum_{m=1}^{d} \frac{1}{m-w}) \bigg] + \frac{5}{\mathfrak{t}},$$

one calculates $\operatorname{Res}_{w=\infty} \frac{dw}{m-w} = 1$ for all $m \in \mathbb{Z}_{\geq 0}$ and thus

$$\operatorname{Res}_{w=\infty}\mathfrak{B}_1\frac{dw}{w} = -\frac{5}{\mathfrak{t}}\left(\operatorname{Res}_{w=\infty}\Lambda(w,t)\frac{dw}{w} + \frac{1}{I_0}\frac{dI_0}{dt}\right) - \frac{5}{\mathfrak{t}}$$

Thus we have

$$\mathcal{K}_{2,1} = \operatorname{Res}_{\hbar=0} \frac{\mathfrak{B}_1}{\hbar} - \frac{5}{\mathfrak{t}} \left(1 + \operatorname{Res}_{w=\infty} \Lambda \frac{dw}{w} \right) = \frac{5}{\mathfrak{t}} \frac{1}{I_0} \frac{dI_0}{dt}$$

For later purpose, we also use $\operatorname{Res}_{w=\infty} \frac{1}{m-w} \frac{dw}{w} = 0$ for all $m \in \mathbb{Z}_{\geq 0}$ to calculate

$$\operatorname{Res}_{\hbar=0}\mathfrak{B}_{1} = \operatorname{Res}_{\hbar=0}\mathfrak{B}_{1}d\hbar = \operatorname{tRes}_{w=\infty}\mathfrak{B}_{1}\frac{dw}{w^{2}}$$
$$= -5\operatorname{Res}_{w=\infty}\Lambda\frac{dw}{w^{2}} - \frac{5g_{1}}{I_{0}} - \frac{5}{I_{0}}\sum_{d=0}^{\infty}q^{d}\frac{(5d)!}{(d!)^{5}}\left(\sum_{m=0}^{d}\operatorname{Res}_{w=\infty}\frac{1}{m-w}\frac{dw}{w}\right)$$
$$= -5\operatorname{Res}_{w=\infty}\Lambda\frac{dw}{w^{2}} - \frac{5g_{1}}{I_{0}}.$$

We now calculate $\mathcal{K}_{1,2}$. Using $d\hbar = \mathfrak{t}\frac{dw}{w^2}$,

$$\begin{aligned} \operatorname{Res}_{\hbar=0}\mathfrak{B}_{2} \\ &= \frac{1}{\mathfrak{t}} \left(-5\operatorname{Res}_{w=\infty}\Lambda \frac{dw}{w^{2}} + \frac{5g_{1}}{I_{0}} \right) - \frac{5}{I_{0}} \cdot \frac{g_{1}}{\mathfrak{t}} + \frac{5}{I_{0}} \sum_{d=1}^{\infty} q^{d} \frac{(5d)!}{(d!)^{5}} \left(\frac{(t-T)d}{\mathfrak{t}} + \sum_{m=d+1}^{5d} \frac{5d}{m\mathfrak{t}} \right) \\ &= \frac{5}{\mathfrak{t}} \operatorname{Res}_{w=\infty}\Lambda \frac{dw}{w^{2}} + \frac{5}{I_{0}\mathfrak{t}} \sum_{d=1}^{\infty} dq^{d} \frac{(5d)!}{(d!)^{5}} \left(-(T-t) + \sum_{m=d+1}^{5d} \frac{5}{m} \right). \end{aligned}$$

By Lemma 4.2, $\mathcal{Z}_{0,2} = \mathfrak{B}_2 + 5\mathfrak{t}^{-2}(1-\Lambda)$. Thus $\mathcal{K}_{1,2} = \operatorname{Res}_{\hbar=0}\mathcal{Z}_{0,2}$ gives

$$\begin{aligned} \mathcal{K}_{1,2} &= \operatorname{Res}_{\hbar=0}\mathfrak{B}_2 + \frac{-5}{\mathfrak{t}} \bigg(\operatorname{Res}_{w=\infty} \Lambda \cdot \frac{dw}{w^2} \bigg) = \frac{5}{\mathfrak{t}I_0} \sum_{d=1}^{\infty} dq^d \frac{(5d)!}{(d!)^5} \bigg(t - T + \sum_{m=d+1}^{5d} \frac{5}{m} \bigg) \\ &= \frac{5}{\mathfrak{t}I_0} \big[q \frac{d}{dq} (I_1 - tI_0) - (T - t) \cdot q \frac{d}{dq} I_0 \big]. \end{aligned}$$

Lemma 4.3.

(19)
$$\mathcal{K}_{1,2} = \frac{5}{\mathfrak{t}}q\frac{d}{dq}(T-t) = \frac{5}{\mathfrak{t}}\frac{d}{dt}(T-t) \quad and \quad \mathcal{K}_{2,1} = \frac{5}{\mathfrak{t}}\frac{1}{I_0}\frac{dI_0}{dt}$$

4.5. Relations obtained by substituting \mathfrak{t}' by $-\mathfrak{t}$. The trick used in previous calulation of $\mathfrak{Z}_{0,k}$ may be used to calculate $\tilde{\mathfrak{Z}}_{0,k}(\mathfrak{t},-\mathfrak{t})$. For each $(\Gamma) \in \Xi_d^0$, we have

$$\pi_{\Gamma*}(\mathcal{L}_{\Gamma}) = \pi_{\Gamma*}(\mathcal{L}_{\Gamma}(-D_{\Gamma})) + \pi_{\Gamma*}(\mathcal{L}_{\Gamma}|_{D_{\Gamma}}) = \pi_{\Gamma*}(\mathcal{L}_{\Gamma}(-D_{\Gamma})) + \mathrm{ev}^*L_H,$$

and thus

$$e(\pi_{\Gamma*}(\mathcal{L}_{\Gamma})) = e(\pi_{\Gamma*}(\mathcal{L}_{\Gamma}(-D_{\Gamma}))) \cdot \mathrm{ev}^* H.$$

Suppose k < 4 and thus $\mathbb{T}^k = p^{4-k}(p + \mathfrak{t})$. Set

$$\begin{split} \tilde{\mathcal{Z}}_{0,k}(\mathfrak{t},-\mathfrak{t}) &:= -\sum_{d=1}^{\infty} (-\mathfrak{Q})^d \sum_{(\Gamma)\in\Xi_d^0} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} ev^* \mathbb{T}^k \frac{e(R\pi_{\Gamma*}(\mathcal{L}_{\Gamma}(-D_{\Gamma})))}{e(N_{\mathcal{W}_{(\Gamma)}}^{vir}/\mathcal{W})} \\ &= -\sum_{d=1}^{\infty} (-\mathfrak{Q})^d \sum_{(\Gamma)\in\Xi_d^0 \cup \Xi_d^1} \int_{[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} ev^* \mathbb{T}^{k+1} \frac{e(R\pi_{\Gamma*}\mathcal{L}_{\Gamma})}{e(N_{\mathcal{W}_{(\Gamma)}}^{vir}/\mathcal{W})} \\ &= -\sum_{d=1}^{\infty} (-\mathfrak{Q})^d \int_{[\mathcal{W}_{0,1_{\rho},(d,0)}]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} ev^* \mathbb{T}^{k+1} e(R\pi_{\Gamma*}\mathcal{L}_{\Gamma}). \end{split}$$

In the case k = 1, $\mathbb{T}^2 = p^2(p + \mathfrak{t})$. Taking coefficient of p^2 in (6), Corollary 3.4 implies

(20)
$$\tilde{\mathcal{Z}}_{0,1}(\mathfrak{t},-\mathfrak{t}) = -5 + \left[\frac{5}{I_0}e^{\frac{\mathfrak{t}}{\hbar}(\frac{g_1}{I_0})}\sum_{d=0}^{\infty}q^d\frac{(5d)!\hbar^d}{(d!)^4\prod_{m=1}^d(\mathfrak{t}+m\hbar)}\right]$$

5. Graph Type A

In genus one cap-MSP, we apply localization formula to

(21)
$$0 = \int_{[\mathcal{W}_{g=1,\emptyset,(n,0)}]^{\mathrm{vir}}} \mathfrak{t} c_{n-1}(R\pi_{\mathcal{W}*}\mathcal{L}_{\mathcal{W}}\mathcal{N}_{\mathcal{W}}\mathcal{L}_{\mathfrak{t}})$$

for all n. We say a graph Γ is of type A if for some $v \in V_0(\Gamma)$ we have $g_v = 1$.

5.1. Packaging graph A_+ and A_{single} using \mathcal{K} series. If Γ is of type A with $d_v > 0$, one can assume Γ has a single vertex at V_0 without edges, or Γ is as Figure 1 below. All other type of graphs have zero contribution to (21). This is because if Γ has more than one vertices at V_1 , one removes vfrom Γ to obtain connected graphs Γ_i 's for $i = 1, \dots, \ell$, with $\ell > 1$. This expresses $\mathcal{C}_{W_{\Gamma}}$ as a union of \mathcal{C}_{W_v} and a family of rational curves $\mathcal{C}_{W_{\Gamma_i}}$. Then $\{\nu_1|_{\mathcal{C}_{\Gamma_i}}\}_{i=1}^{\ell}$ are nowhere zero sections of the locally free sheaves $\{R\pi_{\mathcal{W}_{\Gamma_i}*}(\mathcal{L}_{\mathcal{W}}\mathcal{N}_{\mathcal{W}}\mathcal{L}_{\mathfrak{t}})|_{\mathcal{C}_{W_{\Gamma_i}}}\}_{i=1}^{\ell}$. Since

$$c(R\pi_{\mathcal{W}_v*}(\mathcal{L}_{\mathcal{W}}\mathcal{N}_{\mathcal{W}}\mathcal{L}_{\mathfrak{t}}))|_{\mathcal{C}_{\mathcal{W}_v}}\cap [\mathcal{W}_v]^{\mathrm{vir}}\cong c(\mathbb{C}^{d_v}\mathfrak{t})\cap [\mathcal{W}_v]^{\mathrm{vir}}$$

are chern classes of a bundle, the fact $n = \operatorname{rank} R\pi_{\mathcal{W}*}\mathcal{L}_{\mathcal{W}}\mathcal{N}_{\mathcal{W}}\mathcal{L}_{\mathfrak{t}}$ and the condition $\ell > 1$ imply $c_{n-1}(R\pi_{\mathcal{W}*}\mathcal{L}_{\mathcal{W}}\mathcal{N}_{\mathcal{W}}\mathcal{L}_{\mathfrak{t}})$ vanishes on \mathcal{W}_{Γ} .

(18)



FIGURE 1. The labelled graph $\Gamma_A(d_v, d_e, d_1, ..., d_k)$: $g_v = 1$; $g_{v_0} = 0$; for i = 1, ..., k, $d_i := d_{e_i}$; $v_i \in V_0$, $g_{v_i} = 0$ and $d_{v_i} \in \mathbb{Z}_{\geq 0}$.

In this subsection we calculate contributions of such graphs with $d_v > 0$. Let A_{single} be the graph type that no edges are attached to v, then $\text{Contri}(A_{\text{single}})$ is

(22)
$$\sum_{d_v=1}^{\infty} \mathcal{Q}^{d_v} \int_{[\overline{M}_1(Q,d_v)]^{\mathrm{vir}}} \frac{\mathrm{t}c_{d_v-1}(R\pi_{v*}\mathcal{L}_v\mathcal{L}_{\mathfrak{t}})}{e_T(R\pi_{v*}\mathcal{L}_v\mathcal{L}_{\mathfrak{t}})} = \sum_{d_v=1}^{\infty} \mathcal{Q}^{d_v} \int_{[\overline{M}_1(Q,d_v)]^{\mathrm{vir}}} \frac{\mathfrak{t}(d_v\mathfrak{t}^{d_v-1})}{\mathfrak{t}^{d_v}}$$
$$= \sum_{d_v=1}^{\infty} N_{1,d_v} d_v \mathcal{Q}^{d_v} = \frac{d}{dT} \tilde{F}_1(\mathcal{Q}).$$

Let A_+ be the graph type with $d_v > 0$ and $\sum_j (d_{e_j} + d_{v_j}) > 0$. For each such Γ we replace v by a $(1, \rho)$ -marking to obtain $\Gamma' \in \hat{\Xi}^0_m$ (see the sentences below (10)) for $m = d_{\Gamma} - d_v$. Using the Poincaré dual of the quintic diagonal $PD := 5^{-1} \sum_{j=0}^3 h^j \otimes h^{3-j}$ and $\deg(\psi_{(e,v)} \cap \overline{M}_{1,1}(Q, d_v)) = 0$, the contribution we want is

$$\begin{split} &\sum_{d_v,m=1}^{\infty} (-\mathbb{Q})^{d_v+m} \sum_{(\Gamma')\in \hat{\Xi}_m^0} \int_{[\overline{M}_{1,1}(\mathbb{P}^4, d_v)^p]^v \times [W_{(\Gamma')}]^v} \frac{\mathfrak{t} \cdot e_T(R\pi_{v*}\mathcal{L}_v \otimes \mathcal{L}_{\mathfrak{t}})}{e_T(R\pi_{v*}\mathcal{L}_v \otimes \mathcal{L}_{\mathfrak{t}})} \frac{(-1)PD(h_e + \mathfrak{t})}{w_{(e,v)} - \psi_{(e,v)}} \\ &\cdot \frac{c_{\mathrm{top}-1}(\pi_{\Gamma'*}\mathcal{L}_{\Gamma'}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma'}))}{e(N_{W_{\Gamma'}/W}^{\mathrm{vir}})} \\ &= \frac{1}{5} \sum_{j=0}^3 \sum_{d_v=1}^{\infty} (-\mathbb{Q})^{d_v} \int_{[\overline{M}_{1,1}(\mathbb{P}^4, d_v)^p]^v} \mathfrak{tev}^* h^j \\ &\cdot \sum_{m=1}^{\infty} (-\mathbb{Q})^m \sum_{(\Gamma')\in \hat{\Xi}_m^0} \int_{[W_{(\Gamma')}]^{\mathrm{vir}}} \frac{-\mathrm{ev}^* h^{3-j}(h+\mathfrak{t})}{w_{(e,v)}} \frac{c_{\mathrm{top}-1}(\pi_{\Gamma'*}\mathcal{L}_{\Gamma'}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma'}))}{e(N_{W_{\Gamma'}/W}^{\mathrm{vir}})} \\ &= \frac{1}{5} \left(\sum_{d_v=1}^{\infty} d_v N_{1,d_v} \mathbb{Q}^{d_v} \right) \sum_{m=1}^{\infty} (-\mathbb{Q})^m \sum_{(\Gamma')\in \hat{\Xi}_m^0} \int_{[W_{(\Gamma')}]^{\mathrm{vir}}} \frac{\mathfrak{t}}{\psi_{\Gamma}'} \mathrm{ev}^* (h^2(h+\mathfrak{t})) \frac{c_{\mathrm{top}-1}(\pi_{\Gamma'*}\mathcal{L}_{\Gamma'}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma'}))}{e(N_{W_{\Gamma'}/W}^{\mathrm{vir}})} \\ &= \frac{1}{5} \left(\sum_{d_v=1}^{\infty} d_v N_{1,d_v} \mathbb{Q}^{d_v} \right) \cdot \mathfrak{t} \mathcal{K}_{1,2} = \frac{\mathfrak{t}}{5} \cdot \mathcal{K}_{1,2} \cdot \frac{d}{dT} \mathcal{F}_1. \end{split}$$

where $\mathcal{F}_1 := \sum_{d=1}^{\infty} N_{1,d} \mathcal{Q}^d$ with $\mathcal{Q} = e^T$.

5.2. Contribution from type A graphs Γ with $d_v = 0, v \in V_0(\Gamma), g_v = 1$. Let A^0_+ be the graph type of $d_v = 0, v \in V_0(\Gamma), g_v = 1$. Such graphs are of the following shape.



In the above graph, Γ is decomposed uniquely to (i) a $\Gamma_0 = \{v\} \in \Xi_0$ with $d_v = 0$, and (ii) a sequence $\Gamma_1, \dots, \Gamma_k \in \hat{\Xi}^0$ where each edge containing the marking is denoted by e_i (for $i = 1, \dots, k$).

Denote $\psi_{(e_i,v)} = \psi_i$ and $w_{(e_i,v)} := -\psi_{\Gamma_i}$, where ψ_{Γ_i} is the ψ -class of \mathcal{C}_{e_i} . Then the contribution we want is

$$\begin{split} \mathfrak{t} &\sum_{m,k=1}^{\infty} \frac{(-\mathfrak{Q})^m}{k!} \sum_{\substack{(\Gamma_i) \in \hat{\Xi}_{m_i}^0 \\ \sum m_i = m}} \sum_{\substack{a+1=\sum a_i \\ a,a_i \ge 0}} \int_{[\mathcal{W}_{\Gamma_0}]^v \times \prod[\mathcal{W}_{(\Gamma_i)}]^v} \frac{c_a(R\pi_{\Gamma_0*}\mathcal{L}_{\Gamma_0}\mathcal{L}_{\mathfrak{t}})}{e_T(R\pi_{\Gamma_0*}\mathcal{L}_{\Gamma_0} \otimes \mathcal{L}_{\mathfrak{t}})} \\ \cdot \prod_{i=1}^k (-PD_i) \frac{h_{e_i} + \mathfrak{t}}{w_{(e_i,v)} - \psi_i} \frac{c_{m_i-a_i}(\pi_{\Gamma_i*}\mathcal{L}_{\Gamma_i}\mathcal{L}_{\mathfrak{t}}(-D_{\Gamma_i}))}{e(N_{\mathcal{W}_{\Gamma_i}}^{\mathrm{vir}})} \\ = & \mathfrak{t} \sum_{m,k=1}^{\infty} \frac{(-\mathfrak{Q})^m}{k!} \sum_{\substack{(\Gamma_i) \in \hat{\Xi}_{m_i}^0 \\ \sum m_i = m}} \sum_{a=0}^{\infty} \operatorname{Coe}_{(\mathfrak{t}')^{a+1}} \int_{[\mathcal{W}_{\Gamma_0}]^v \times \prod[\mathcal{W}_{(\Gamma_i)}]^v} \frac{c_a(R\pi_{\Gamma_0*}\mathcal{L}_{\Gamma_0}\mathcal{L}_{\mathfrak{t}})}{e_T(R\pi_{\Gamma_0*}\mathcal{L}_{\Gamma_0} \otimes \mathcal{L}_{\mathfrak{t}})} \\ \cdot \prod_{i=1}^k PD_i \frac{h_{e_i} + \mathfrak{t}}{\psi_{\Gamma_i} + \psi_i} \frac{e(\pi_{\Gamma_i*}\mathcal{L}_{\Gamma_i}\mathcal{L}_{\mathfrak{t}}\mathcal{L}_{\mathfrak{t}'}(-D_{\Gamma_i}))}{e(N_{\mathcal{W}_{\Gamma_i}}^{\mathrm{vir}}/\mathcal{W})}, \end{split}$$

where $PD_i = \frac{1}{5} \sum_{j=0}^3 h^j \otimes h_{e_i}^{3-j}$. Using $\lambda^2 = h^2 = \lambda h = 0$ over $[\mathcal{W}_{\Gamma_0}]^{\text{vir}} = (\overline{M}_{1,k} \times Q) \cap [-40h^3 - 10\lambda h^2]$, $\frac{c(R\pi_{\Gamma_0*}\mathcal{L}_{\Gamma_0}\mathcal{L}_{\mathfrak{t}})}{c(R\pi_{\Gamma_0*}\mathcal{L}_{\Gamma_0}\mathcal{L}_{\mathfrak{t}})} = \frac{1+h+\mathfrak{t}}{h+\mathfrak{t}} \frac{\mathfrak{t}+h-\lambda}{1+\mathfrak{t}+h-\lambda} = 1 - \frac{\lambda}{\mathfrak{t}(1+\mathfrak{t})} = 1 - \frac{\lambda}{\mathfrak{t}} + \lambda - \lambda \mathfrak{t} + \lambda \mathfrak{t}^2 - \lambda \mathfrak{t}^3 + \dots + (-1)^\ell \lambda \mathfrak{t}^\ell + \dots$

Hence we get

$$\frac{c_a(R\pi_{\Gamma_0*}\mathcal{L}_{\Gamma_0}\mathcal{L}_{\mathfrak{t}})}{e(R\pi_{\Gamma_0*}\mathcal{L}_{\Gamma_0}\mathcal{L}_{\mathfrak{t}})} = \begin{cases} (-1)^{a-1}\lambda\mathfrak{t}^{a-1} & \text{for } a \in \mathbb{N}, \\ 1-\lambda\mathfrak{t}^{-1} & \text{for } a = 0. \end{cases}$$

We express the contribution as the sum over $a = 0, 1, 2, 3, \cdots$ as follows. Case $a \in \mathbb{N}$: As $[-40h^3 - 10\lambda h^2]((-1)^{a-1}\lambda \mathfrak{t}^{a-1}) = (-1)^a 40h^3\lambda \mathfrak{t}^{a-1}$, the contribution is

$$\begin{aligned} \mathfrak{t} \sum_{k,m_{i}=1}^{\infty} \frac{(-\mathbb{Q})^{m_{1}+\cdots m_{k}}}{k!} \sum_{(\Gamma_{i})\in\hat{\Xi}_{m_{i}}^{0}} \operatorname{Coe}_{(\mathfrak{t}')^{a+1}} \int_{\overline{M}_{1,k}\times Q\times\prod[\mathcal{W}_{(\Gamma_{i})}]^{v}} (-1)^{a} 40h^{3}\lambda \mathfrak{t}^{a-1} \\ \cdot \prod_{i=1}^{k} \frac{h_{e_{i}}^{3}}{5} \frac{h_{e_{i}}+\mathfrak{t}}{\psi_{\Gamma_{i}}+\psi_{i}} \frac{e(\pi_{\Gamma_{i}*}\mathcal{L}_{\Gamma_{i}}\mathcal{L}_{\mathfrak{t}}\mathcal{L}_{\mathfrak{t}'}(-D_{\Gamma_{i}}))}{e(N_{\mathcal{W}_{\Gamma_{i}}}^{\mathrm{vir}}/\mathcal{W})} \\ = \mathfrak{t} \sum_{k=1}^{\infty} \frac{1}{k!} \operatorname{Coe}_{(\mathfrak{t}')^{a+1}} \int_{\overline{M}_{1,k}\times Q} \frac{(-1)^{a}}{5^{k}} 40h^{3}\lambda \mathfrak{t}^{a-1}\hat{\mathcal{K}}_{1}(\psi_{1})\hat{\mathcal{K}}_{1}(\psi_{2})\cdots\hat{\mathcal{K}}_{1}(\psi_{k}) \\ = (-1)^{a}\mathfrak{t}^{a} \sum_{k=1}^{\infty} \frac{200}{5^{k}k!} \operatorname{Coe}_{(\mathfrak{t}')^{a+1}} \int_{\overline{M}_{1,k}} \lambda\hat{\mathcal{K}}_{1}(\psi_{1})\hat{\mathcal{K}}_{1}(\psi_{2})\cdots\hat{\mathcal{K}}_{1}(\psi_{k}). \end{aligned}$$

Case a = 0: As $\left[-40h^3 - 10\lambda h^2\right]\left(1 - \frac{\lambda}{t}\right) = -40h^3 + 40h^3\frac{\lambda}{t} - 10\lambda h^2$, the contribution is the sum of:

$$\begin{aligned} (i) \quad \mathfrak{t} \sum_{k,m_{i}=1}^{\infty} \frac{(-\mathbb{Q})^{m_{1}+\cdots m_{k}}}{k!} \sum_{(\Gamma_{i})\in\hat{\Xi}_{m_{i}}^{0}} \operatorname{Coe}_{(\mathfrak{t}')^{1}} \int_{\overline{M}_{1,k}\times Q\times\prod[\mathcal{W}_{(\Gamma_{i})}]^{v}} 40(h^{3}\frac{\lambda}{\mathfrak{t}}-h^{3}) \\ & \cdot \prod_{i=1}^{k} PD_{i} \frac{h_{e_{i}}+\mathfrak{t}}{\psi_{\Gamma_{i}}+\psi_{i}} \frac{e(\pi_{\Gamma_{i}*}\mathcal{L}_{\Gamma_{i}}\mathcal{L}_{\mathfrak{t}}\mathcal{L}_{\mathfrak{t}'}(-D_{\Gamma_{i}}))}{e(N_{\mathcal{W}_{\Gamma_{i}}}^{\mathrm{vir}}/\mathcal{W})} \\ &= \mathfrak{t} \sum_{k=1}^{\infty} \frac{1}{5^{k}k!} \operatorname{Coe}_{(\mathfrak{t}')^{1}} \int_{\overline{M}_{1,k}\times Q} (-40h^{3}+40h^{3}\frac{\lambda}{\mathfrak{t}})\hat{\mathcal{K}}_{1}(\psi_{1})\cdots\tilde{\mathcal{K}}_{1}(\psi_{k}) \\ &= \mathfrak{t} \sum_{k=1}^{\infty} \frac{1}{5^{k}k!} \operatorname{Coe}_{(\mathfrak{t}')^{1}} \int_{\overline{M}_{1,k}} 200(\frac{\lambda}{\mathfrak{t}}-1)\hat{\mathcal{K}}_{1}(\psi_{1})\cdots\tilde{\mathcal{K}}_{1}(\psi_{k}), \end{aligned}$$

and (ii), recalling $PD_i = \frac{1}{5} \sum_{j=0}^3 h^j \otimes h_{e_i}^{3-j}$,

$$\begin{split} \mathfrak{t} & \sum_{k,m_{i}=1}^{\infty} \frac{(-\Omega)^{m_{1}+\cdots m_{k}}}{k!} \sum_{(\Gamma_{i})\in\hat{\Xi}_{m_{i}}^{0}} \operatorname{Coe}_{(\mathfrak{t}')^{1}} \int_{\overline{M}_{1,k}\times Q\times \prod[\mathcal{W}_{(\Gamma_{i})}]^{v}} (-10\lambda h^{2}) \\ & \cdot \prod_{i=1}^{k} PD_{i} \cdot \frac{h_{e_{i}} + \mathfrak{t}}{\psi_{\Gamma_{i}} + \psi_{i}} \frac{e(\pi_{\Gamma_{i}} \ast \mathcal{L}_{\Gamma_{i}} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}'} (-D_{\Gamma_{i}}))}{e(N_{\mathcal{W}_{\Gamma_{i}}}^{\operatorname{vir}} - \mathcal{W})} \\ &= \mathfrak{t} \sum_{k,m_{i}=1}^{\infty} \frac{(-\Omega)^{m_{1}+\cdots m_{k}}}{5^{k} k!} \sum_{(\Gamma_{i})\in\hat{\Xi}_{m_{i}}^{0}} \operatorname{Coe}_{(\mathfrak{t}')^{1}} \int_{\overline{M}_{1,k}\times Q\times \prod[\mathcal{W}_{(\Gamma_{i})}]^{v}} (-10\lambda h^{2}), \\ & \left(\sum_{j=1}^{k} (h^{0} \otimes h_{e_{1}}^{3}) \cdots (h^{0} \otimes h_{e_{j-1}}^{3})(h^{1} \otimes h_{e_{j}}^{2})(h^{0} \otimes h_{e_{j+1}}^{3}) \cdots (h^{0} \otimes h_{e_{k}}^{3})\right) \\ & \cdot \left(\prod_{i=1}^{k} \frac{h_{e_{i}} + \mathfrak{t}}{\psi_{\Gamma_{i}} + \psi_{i}} \frac{e(\pi_{\Gamma_{i}} \mathscr{L}_{\Gamma_{i}} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}'} (-D_{\Gamma_{i}}))}{e(N_{\mathcal{W}_{\Gamma_{i}}}^{\operatorname{vir}} - \mathcal{W})}\right) \\ &= \mathfrak{t} \sum_{k=1}^{\infty} \frac{1}{5^{k} k!} \operatorname{Coe}_{(\mathfrak{t}')^{1}} \int_{\overline{M}_{1,k}} (-50)\lambda \sum_{j=1}^{k} \hat{\mathcal{K}}_{1}(\psi_{1}) \hat{\mathcal{K}}_{1}(\psi_{2}) \cdots \hat{\mathcal{K}}_{1}(\psi_{j-1}) \hat{\mathcal{K}}_{2}(\psi_{j}) \hat{\mathcal{K}}_{1}(\psi_{j+1}) \cdots \hat{\mathcal{K}}_{1}(\psi_{k}). \end{split}$$

Therefore the contribution of graph type A^0_+ is

$$\begin{split} \mathfrak{t} &\sum_{k=1}^{\infty} \frac{(-50)}{5^{k} k!} \operatorname{Coe}_{(\mathfrak{t}')^{1}} \int_{\overline{M}_{1,k}} \lambda \sum_{j=1}^{k} \hat{\mathcal{K}}_{1}(\psi_{1}) \hat{\mathcal{K}}_{1}(\psi_{2}) \cdots \hat{\mathcal{K}}_{1}(\psi_{j-1}) \hat{\mathcal{K}}_{2}(\psi_{j}) \hat{\mathcal{K}}_{1}(\psi_{j+1}) \cdots \hat{\mathcal{K}}_{1}(\psi_{k}) \\ &- \mathfrak{t} \sum_{k=1}^{\infty} \frac{200}{5^{k} k!} \operatorname{Coe}_{(\mathfrak{t}')^{1}} \int_{\overline{M}_{1,k}} \hat{\mathcal{K}}_{1}(\psi_{1}) \cdots \hat{\mathcal{K}}_{1}(\psi_{k}) \\ &+ \sum_{a=0}^{\infty} (-1)^{a} \mathfrak{t}^{a} \sum_{k=1}^{\infty} \frac{200}{5^{k} k!} \operatorname{Coe}_{(\mathfrak{t}')^{a+1}} \int_{\overline{M}_{1,k}} \lambda \hat{\mathcal{K}}_{1}(\psi_{1}) \hat{\mathcal{K}}_{1}(\psi_{2}) \cdots \hat{\mathcal{K}}_{1}(\psi_{k}) \\ &= \frac{(-50)}{5} \mathfrak{t} \operatorname{Coe}_{(\mathfrak{t}')^{1}} \int_{\overline{M}_{1,1}} \lambda \hat{\mathcal{K}}_{2}(\psi_{1}) - \frac{200}{5} \mathfrak{t} \operatorname{Coe}_{(\mathfrak{t}')^{1}} \int_{\overline{M}_{1,1}} \hat{\mathcal{K}}_{1}(\psi_{1}) \\ &- \frac{1}{\mathfrak{t}} \sum_{k=1}^{\infty} \frac{200}{5^{k} k!} \sum_{a=0}^{\infty} (-\mathfrak{t})^{a+1} \operatorname{Coe}_{(\mathfrak{t}')^{a+1}} \int_{\overline{M}_{1,k}} \lambda \hat{\mathcal{K}}_{1}(\psi_{1}) \hat{\mathcal{K}}_{1}(\psi_{2}) \cdots \hat{\mathcal{K}}_{1}(\psi_{k}) \\ &= -\frac{10}{24} \mathfrak{t} \mathcal{K}_{1,2} + \frac{40}{24} \mathfrak{t} \mathcal{K}_{2,1} - \frac{1}{\mathfrak{t}} \sum_{k=1}^{\infty} \frac{200}{5^{k} k!} \int_{\overline{M}_{1,k}} \lambda \hat{\mathcal{K}}_{1}|_{\mathfrak{t}'=-\mathfrak{t}}(\psi_{1}) \hat{\mathcal{K}}_{1}|_{\mathfrak{t}'=-\mathfrak{t}}(\psi_{2}) \cdots \hat{\mathcal{K}}_{1}|_{\mathfrak{t}'=-\mathfrak{t}}(\psi_{k}), \end{split}$$

where the first equality comes from the fact that both $\tilde{\mathcal{K}}_{c,k}$ and $\hat{\mathcal{K}}_k$ for any c, k are formal power series of \mathfrak{t}' with coefficients of $(\mathfrak{t}')^0$ vanishing.

Using (11) and Lemma 4.1, the last term in the above formula equals

(23)
$$-\frac{200}{\mathfrak{t}} \cdot \left(\int_{\overline{M}_{1,1}} \lambda\right) \cdot \sum_{k=1}^{\infty} \sum_{\substack{i_1 + \dots + i_k = k-1 \\ i_1, \dots, i_k \ge 0}} \frac{1}{k} \cdot \prod_{j=1}^k \left(\frac{(-1)^{i_j}}{i_j!} \operatorname{Res}_{\hbar=0}\{\hbar^{-i_j} \frac{\tilde{\mathcal{Z}}_{0,1}(\mathfrak{t}, -\mathfrak{t})}{5}\}\right),$$

where we used $\int_{\overline{M}_{1,k}} \lambda \psi_1^{i_1} \cdots \psi_k^{i_k} = \frac{(k-1)!}{i_1! \cdots i_k!} \int_{\overline{M}_{1,1}} \lambda$ whenever $i_1 + \ldots + i_k = k - 1$. By (34) in the Appendix, we have (23) equals $-\frac{200}{24} \mathfrak{t}^{-1} \eta = -\frac{25}{3} \frac{g_1}{I_0}$. Thus the contribution of graph type A^0_+ equals

$$\frac{40\mathfrak{t}}{24} \cdot \mathcal{K}_{2,1} - \frac{10\mathfrak{t}}{24} \cdot \mathcal{K}_{1,2} - \frac{25}{3} \cdot \frac{g_1}{I_0} = \frac{25}{3I_0} \frac{\mathrm{d}I_0}{\mathrm{d}t} - \frac{25}{12} \cdot \frac{d}{dt} (T-t) - \frac{25}{3} \cdot \frac{g_1}{I_0}.$$

Thus type A contribution is, by (19),

$$\begin{aligned} (1 + \frac{\mathfrak{t}}{5}\mathcal{K}_{1,2})\frac{d\mathcal{F}_1}{dT} + \frac{25}{3I_0}\frac{dI_0}{dt} - \frac{25}{12}\cdot\frac{d(T-t)}{dt} - \frac{25}{3}\cdot\frac{g_1}{I_0} \\ = \frac{d}{dt}\mathcal{F}_1 - \frac{25}{12}\cdot(\frac{dT}{dt} - 1) + \frac{40}{24}\cdot5\frac{1}{I_0}\frac{dI_0}{dt} - \frac{25}{3}\cdot\frac{g_1}{I_0}. \end{aligned}$$

6. Graph Type B

The contribution from Graph B is the most involved. It splits into two parts. One part is in this section, another is in the Appendix C.

6.1. Notations for g = 0 two-point functions. We define in MSP weight a series

$$\tilde{\mathcal{Z}}_{66}^* := \frac{1}{2\hbar_1\hbar_2} \sum_{d=1}^{\infty} \mathcal{Q}^d \int_{\overline{M}_{0,2}(\mathbb{P}^5,d)} \frac{e(\tilde{E}_d)}{(\hbar_1 - \psi_1)(\hbar_2 - \psi_2)} ev_1^* H^5 ev_2^* H^5,$$

where $\tilde{E}_d := \tilde{\pi}_* \tilde{f}^* (L_H^{\otimes 5} \oplus (L_H L_t L_{t'}))$ is a rank 6d+2 bundle over $\overline{M}_{0,2}(\mathbb{P}^5, d)$, with $\tilde{\pi} : \tilde{\mathcal{C}}_d \to \overline{M}_{0,2}(\mathbb{P}^5, d)$ and $\tilde{f} : \mathcal{C}_d \to \overline{M}_{0,2}(\mathbb{P}^5, d)$ the universal family. Then

$$\begin{split} \tilde{\mathcal{Z}}_{66}^{*} &= \frac{-1}{2\hbar_{1}\hbar_{2}} \sum_{d=1}^{\infty} (-\mathfrak{Q})^{d} \int_{[\mathcal{W}_{0,(1,\rho)^{2},(d,0)}]^{\mathrm{vir}}} \frac{e\left(\tilde{\pi}_{*}\tilde{f}^{*}(L_{H}L_{\mathfrak{t}}L_{\mathfrak{t}}')\right)}{(\hbar_{1}-\psi_{1})(\hbar_{2}-\psi_{2})} ev_{1}^{*}H^{5}ev_{2}^{*}H^{5} \\ &= \frac{-1}{2\hbar_{1}\hbar_{2}} \sum_{d=1}^{\infty} (-\mathfrak{Q})^{d} \sum_{(\tilde{\Gamma})\in\tilde{\Delta}_{d}^{11}} \int_{[\mathcal{W}_{(\tilde{\Gamma})}]^{\mathrm{vir}}} \frac{e\left(\tilde{\pi}_{*}\tilde{f}^{*}(L_{\tilde{\Gamma}}L_{\mathfrak{t}}L_{\mathfrak{t}}')\right)}{e(N_{\mathcal{W}_{(\tilde{\Gamma})}}^{\mathrm{vir}}/\mathcal{W})} \frac{\mathfrak{t}^{10}}{(\hbar_{1}-\psi_{1})(\hbar_{2}-\psi_{2})}, \end{split}$$

where $\tilde{\Delta}_d^{ij}$ denotes the set of all (regular) graphs appearing in localizing \mathbb{C}^* -action on $\mathcal{W}_{0,(1,\rho)^2,(d,0)} = \overline{M}_{0,2}(\mathbb{P}^5, d)$ which have the first marking mapped to V_i and the second marking mapped to V_j , for $i, j \in \{0, 1\}$.

Let $\tilde{\Delta}_{d,Sep}^{11}$ denote the set of all graphs in $\tilde{\Delta}_d^{11}$ with two markings not on the same vertex.

$$\tilde{\mathcal{Z}}_{66,Sep}^{*} := \frac{-1}{2\hbar_{1}\hbar_{2}} \sum_{d=1}^{\infty} (-\mathcal{Q})^{d} \sum_{(\tilde{\Gamma})\in\tilde{\Delta}_{d,Sep}^{11}} \int_{[\mathcal{W}_{(\tilde{\Gamma})}]^{\mathrm{vir}}} \frac{e\left(\tilde{\pi}_{*}\tilde{f}^{*}(L_{\tilde{\Gamma}}L_{\mathfrak{t}}L_{\mathfrak{t}}')\right)}{e(N_{\mathcal{W}_{(\tilde{\Gamma})}}^{\mathrm{vir}}/\mathcal{W})} \frac{\mathfrak{t}^{10}}{(\hbar_{1}-\psi_{1})(\hbar_{2}-\psi_{2})}$$

Let $\mathfrak{D}_1, \mathfrak{D}_2 \subset \mathcal{C}_{\tilde{\Gamma}} \to \mathcal{W}_{(\Gamma)}$ be the universal family and divisors given by the two markings. We also have

$$\tilde{\mathcal{Z}}_{66,Sep}^{*} = \frac{-1}{2\hbar_{1}\hbar_{2}} \sum_{d=1}^{\infty} (-\mathcal{Q})^{d} \sum_{(\tilde{\Gamma})\in\tilde{\Delta}_{d,Sep}^{11}} \int_{[\mathcal{W}_{(\tilde{\Gamma})}]^{\mathrm{vir}}} (\mathfrak{t}')^{2} \cdot \frac{e\left(\tilde{\pi}_{*}\tilde{f}^{*}(L_{\tilde{\Gamma}}L_{\mathfrak{t}}L_{\mathfrak{t}'}(-\mathfrak{D}_{1}-\mathfrak{D}_{2}))\right)}{e(N_{\mathcal{W}_{(\tilde{\Gamma})}}^{\mathrm{vir}}/\mathcal{W})} \frac{\mathfrak{t}^{10}}{(\hbar_{1}-\psi_{1})(\hbar_{2}-\psi_{2})}$$

$$(24) \qquad = \quad \frac{-1}{2\hbar_{1}\hbar_{2}} \sum_{d=1}^{\infty} (-\mathfrak{Q})^{d} \sum_{(\tilde{\Gamma})\in\tilde{\Delta}_{d,Sep}^{11}} \int_{[\mathcal{W}_{(\tilde{\Gamma})}]^{\mathrm{vir}}} \frac{e\left(\tilde{\pi}_{*}\tilde{f}^{*}(L_{\tilde{\Gamma}}L_{\mathfrak{t}}(-\mathfrak{D}_{1}-\mathfrak{D}_{2}))\right)}{e(N_{\mathcal{W}_{(\tilde{\Gamma})}}^{\mathrm{vir}}/\mathcal{W})} \frac{\mathfrak{t}^{10}}{(\hbar_{1}-\psi_{1})(\hbar_{2}-\psi_{2})}.$$

6.2. Two types of contributions. Any graph Γ without positive genus vertex (called loop graph) that gives nonzero contribution to (21) necessarily has $|V_1(\Gamma)| = 1$, because of the integrand in (21). We call it *the geometry of cap*. We say such a graph is of type *B*, as associated to the left graph below. Let $\tilde{\Gamma}$ be the subgraph of Γ shown below on the right.



Here all vertices are of genus zero. Let Γ_0 be the graph of single vertex $\tilde{v} \in V_1$, $g_{\tilde{v}} = 0$, no edges, and with k + 2 many $(1, \rho)$ markings. The tail of \tilde{v} that contains e_i is denoted by Γ_i . Let the total degree of \mathcal{L} over objects in Γ_i be d_i . Then $d = d_{\tilde{\Gamma}} + d_1 + d_2 + \cdots + d_k$ is the degree \mathcal{L} . We divide into two cases.

- (1) the first case is when k = 0, \tilde{v} unstable. We leave its $Cont_I$ to next few subsections.
- (2) the other case is when k > 0 and \tilde{v} stable whose contribution is denoted by $Cont_{II}$.

Let's consider the second case. By [CLLL2],

$$[\mathcal{W}_{(\Gamma)}]^{\mathrm{vir}} = |Aut\tilde{\Gamma}| \cdot \frac{\prod |Aut\Gamma_i|}{|Aut\Gamma|} [\mathcal{W}_{(\Gamma_0)}]^{\mathrm{vir}} \times [\mathcal{W}_{(\tilde{\Gamma})}]^{\mathrm{vir}} \times \prod [\mathcal{W}_{(\Gamma_i)}]^{\mathrm{vir}}.$$

Thus the contribution of Γ is

$$Contri_{(\Gamma)} := \int_{[\mathcal{W}_{(\Gamma)}]^{v}} \frac{\mathbf{t}c_{d-1}R\pi_{\mathcal{W}_{\Gamma^{*}}}(\mathcal{L}_{\mathcal{W}_{\Gamma}}\mathcal{N}_{\mathcal{W}_{\Gamma}}\mathcal{L}_{\mathfrak{t}})}{e(N_{\mathcal{W}_{(\Gamma)}}^{\operatorname{vir}})}$$

$$= \frac{\mathbf{t}|Aut\tilde{\Gamma}|\prod|Aut\Gamma_{i}|}{|Aut\Gamma|} \sum_{\ell+\ell_{0}+\sum \ell_{i}=d-1} \int_{[\mathcal{W}_{(\Gamma_{0})}]^{v}} \frac{c_{\ell}(R\pi_{\mathcal{W}_{\Gamma_{0}}*}(\mathcal{L}_{\mathcal{W}_{\Gamma_{0}}}\mathcal{L}_{\mathfrak{t}}))}{e(N_{\mathcal{W}_{(\Gamma_{0})}}^{\operatorname{vir}})}$$

$$\cdot \int_{[\mathcal{W}_{(\tilde{\Gamma})}]^{v}} \frac{\mathbf{t}^{8}c_{\ell_{0}}(R\pi_{\mathcal{W}_{\tilde{\Gamma}}*}(\mathcal{L}_{\mathcal{W}_{\tilde{\Gamma}}}\mathcal{L}_{\mathfrak{t}}(-\mathfrak{D}_{1}-\mathfrak{D}_{2})))}{5^{2}(\psi_{1}+\psi_{1}')(\psi_{2}+\psi_{2}')e(N_{\mathcal{W}_{(\tilde{\Gamma})}}^{\operatorname{vir}})} \prod \int_{[\mathcal{W}_{(\Gamma_{i})}]^{v}} \frac{-\mathbf{t}^{4}c_{\ell_{i}}(R\pi_{\mathcal{W}_{\Gamma_{i}}*}(\mathcal{L}_{\mathcal{W}_{\Gamma_{i}}}\mathcal{L}_{\mathfrak{t}}(-D_{i})))}{5(-\psi_{\Gamma_{i}}-\hat{\psi}_{i})e(N_{\mathcal{W}_{(\Gamma_{i})}}^{\operatorname{vir}})}.$$

From the shape of the graph, one has $R\pi_{W_{\Gamma_0}*}(\mathcal{L}_{W_{\Gamma_0}}\mathcal{L}_{\mathfrak{t}}) \cong \mathcal{O}_{W_{\Gamma_0}}$. Both $R\pi_{W_{\tilde{\Gamma}}*}(\mathcal{L}_{W_{\tilde{\Gamma}}}\mathcal{L}_{\mathfrak{t}}(-\mathfrak{D}_1-\mathfrak{D}_2))$ and $\oplus_i R\pi_{W_{\Gamma_i}*}(\mathcal{L}_{W_{\Gamma_i}}\mathcal{L}_{\mathfrak{t}}(-D_i))$ are bundles, and the sum of their ranks is d-1. We see the only possible contribution to $Contri_{(\Gamma)}$ is from $\ell = 0$ and all c_{ℓ_i} are top Chern classes. Therefore, with $\frac{1}{e(N_{W_{(\Gamma_0)}}^{\mathrm{vir}})} = \frac{5\mathfrak{t}}{-\mathfrak{t}^5}, Contri_{(\Gamma)}$ equals

$$\begin{split} & \frac{-|Aut\tilde{\Gamma}|\prod|Aut\Gamma_{i}|}{5t^{5}|Aut\Gamma|} \sum_{a,b,b_{1},\cdots,b_{k}=0}^{\infty} \int_{\overline{M}_{0,2+k}} (-\psi_{1}')^{a} (-\psi_{2}')^{b} \prod_{i} (-\hat{\psi}_{i})^{b_{i}} \left(\int_{[\mathcal{W}_{(\tilde{\Gamma})}]^{\mathrm{vir}}} \frac{t^{10}}{(\psi_{1})^{a+1}(\psi_{2})^{b+1}} \right) \\ & \cdot \frac{e\left(\pi_{\mathcal{W}_{\tilde{\Gamma}}*}(\mathcal{L}_{\mathcal{W}_{\tilde{\Gamma}}}\mathcal{L}_{t}(-\mathfrak{D}_{1}-\mathfrak{D}_{2}))\right)}{e(N_{\mathcal{W}_{(\tilde{\Gamma})}}^{\mathrm{vir}})} \right) \prod \int_{[\mathcal{W}_{(\Gamma_{i})}]^{\mathrm{vir}}} \frac{t^{4}}{5} \frac{1}{(\psi_{\Gamma_{i}})^{b_{i}+1}} \frac{e\left(\pi_{\mathcal{W}_{\Gamma_{i}}*}(\mathcal{L}_{\mathcal{W}_{\Gamma_{i}}}\mathcal{L}_{t}(-D_{i}))\right)}{e(N_{\mathcal{W}_{(\Gamma_{i})}}^{\mathrm{vir}})} \\ &= \frac{-|Aut\tilde{\Gamma}|\prod|Aut\Gamma_{i}|}{5t^{5}|Aut\Gamma|} (k-1)! \sum_{a+b+b_{1}+\cdots+b_{k}=k-1}^{\infty} \frac{(-1)^{a+b}}{a!b!} \left(\int_{[\mathcal{W}_{(\tilde{\Gamma})}]^{\mathrm{vir}}} \frac{t^{10}}{(\psi_{1})^{a+1}(\psi_{2})^{b+1}} \right) \\ & \cdot \frac{e\left(\pi_{\mathcal{W}_{\tilde{\Gamma}}*}(\mathcal{L}_{\mathcal{W}_{\tilde{\Gamma}}}\mathcal{L}_{t}(-\mathfrak{D}_{1}-\mathfrak{D}_{2}))\right)}{e(N_{\mathcal{W}_{(\tilde{\Gamma})}}^{\mathrm{vir}})} \right) \prod_{i=1}^{k} \frac{(-1)^{b_{i}}}{b_{i}!} \frac{t^{4}}{5} \int_{[\mathcal{W}_{(\Gamma_{i})}]^{\mathrm{vir}}} \frac{1}{(\psi_{\Gamma_{i}})^{b_{i}+1}} \frac{e\left(\pi_{\mathcal{W}_{\Gamma_{i}}*}(\mathcal{L}_{\mathcal{W}_{\Gamma_{i}}}\mathcal{L}_{t}(-D_{i}))\right)}{e(N_{\mathcal{W}_{(\Gamma_{i})}}^{\mathrm{vir}})} . \end{split}$$

Let $\tilde{\Delta}_d^{11,\circ}$ denote the graphs in $\tilde{\Delta}_d^{11}$ which has only one vertex at V_0 and whose both markings are unstable. Using (24) and the geometry of cap, the sum of all above contributions is

$$Cont_{II} = \sum_{k=1}^{\infty} \frac{-(k-1)!}{5t^5 \cdot 2 \cdot k!} \sum_{a+b+b_1+\cdots b_k=k-1}^{\infty} \frac{(-1)^{a+b}}{a!b!} \sum_{(\tilde{\Gamma})\in\tilde{\Delta}_d^{11,\circ}} (-\mathfrak{Q})^{d_{\tilde{\Gamma}}} \left(\int_{[\mathcal{W}_{(\tilde{\Gamma})}]^{\mathrm{vir}}} \frac{\mathfrak{t}^{10}}{(\psi_1)^{a+1}(\psi_2)^{b+1}} \right) \frac{\mathfrak{t}^{10}}{\mathfrak{t}^{10}} \frac{\mathfrak{t}^{$$

where we used $\mathcal{P}_c = \operatorname{Res}_{\hbar=0} \frac{\mathcal{Z}_{1,5}}{\hbar^{c-1}}$ in Lemma 4.1. By (13), $\mathcal{Z}_{1,5} = \tilde{\mathcal{Z}}_{1,5}|_{\mathfrak{t}'=0}$ where $\tilde{\mathcal{Z}}_{1,5}$ is regular in \mathfrak{t}' . Applying (34), the above becomes $(\mathcal{Z}_6^* := \frac{\mathfrak{t}^4}{5} \tilde{\mathcal{Z}}_{1,5})$

$$Cont_{II} = \frac{1}{5t^5} \sum_{a,b\geq 0}^{\infty} \frac{(-1)^{a+b}}{a!b!} \Re_{\hbar_2=0} \Re_{\hbar_1=0} \left\{ \hbar_1^{-a} \hbar_2^{-b} Coe_{(t')^2} \tilde{\mathcal{Z}}^*_{66,Sep} \right\}$$
$$\cdot \sum_{m=1}^{\infty} \frac{1}{m} \left(\sum_{a+b+\sum_{b_\ell=m-1}^{\infty} \ell=1}^{\infty} \prod_{\ell=1}^{m} \frac{(-1)^{b_\ell}}{b_\ell!} \Re_{\hbar=0} \frac{\mathcal{Z}^*_6}{\hbar^{b_\ell}} \right) \Big|_{t'=0}$$
$$= \frac{1}{5t^5} \sum_{a,b\geq 0}^{\infty} \frac{(-1)^{a+b}}{a!b!} \Re_{\hbar_2=0} \Re_{\hbar_1=0} \left\{ \hbar_1^{-a} \hbar_2^{-b} Coe_{(t')^2} \tilde{\mathcal{Z}}^*_{66,Sep} \right\} \frac{\eta_0^{a+b+1}}{a+b+1},$$

where we used $\sum_{a,b\geq 0} \frac{y^{a+b+1}}{(a+b+1)a!b!\hbar_1^a\hbar_2^b} = \frac{\hbar_1\hbar_2}{\hbar_1+\hbar_2} \left(e^{y(\frac{1}{\hbar_1}+\frac{1}{\hbar_2})}-1\right)$, and we denote $\eta_0 := \eta|_{\mathfrak{t}'=0}$. For $Cont_I$, using $[\mathcal{W}_{(\Gamma)}]^{\operatorname{vir}} = \frac{1}{2}[\mathcal{W}_{(\tilde{\Gamma})}]^{\operatorname{vir}}$, MSP II gives us

$$Cont_{I} = \frac{1}{2} \sum_{d=1}^{\infty} (-\mathfrak{Q})^{d} \sum_{(\tilde{\Gamma}) \in \tilde{\Delta}_{d}^{11,\circ}} \int_{[\mathcal{W}_{(\tilde{\Gamma})}]^{\mathrm{vir}}} \frac{e\left(\tilde{\pi}_{*}\tilde{f}^{*}(L_{\tilde{\Gamma}}L_{\mathfrak{t}}(-\mathfrak{D}_{1}-\mathfrak{D}_{2}))\right)}{e(N_{\mathcal{W}_{(\tilde{\Gamma})}}^{\mathrm{vir}}/\mathcal{W})} \cdot \frac{\mathfrak{t}}{(5\mathfrak{t})^{2}} \frac{-5\mathfrak{t}^{6}}{-\psi_{1}-\psi_{2}}$$

6.3. Computations of Contri II. Let $\tilde{\Delta}_d^{11,\circ}$ denote the graphs in $\tilde{\Delta}_d^{11}$ which has only one vertex at V_0 and whose both markings are unstable. Using (24) and the geometry of cap, we can argue to obtain

$$\begin{aligned}
\mathfrak{X} &:= \mathfrak{R}_{\hbar_{2}=0} \mathfrak{R}_{\hbar_{1}=0} \left\{ \hbar_{1}^{-a} \hbar_{2}^{-b} Coe_{(\mathfrak{t}')^{2}} \mathcal{Z}_{66,Sep}^{*} \right\} \\
(25) &= \frac{-1}{2} \sum_{d=1}^{\infty} (-\mathfrak{Q})^{d} \sum_{(\tilde{\Gamma}) \in \tilde{\Delta}_{d}^{11,\circ}} \int_{[\mathcal{W}_{(\tilde{\Gamma})}]^{\mathrm{vir}}} \frac{e(\tilde{\pi}_{*} \tilde{f}^{*}(L_{\tilde{\Gamma}} L_{\mathfrak{t}}(-\mathfrak{D}_{1}-\mathfrak{D}_{2})))}{e(N_{\mathcal{W}_{(\tilde{\Gamma})}}^{\mathrm{vir}}/\mathcal{W})} \cdot \mathfrak{t}^{10} \frac{-1}{\psi_{1}^{a+1}} \cdot \frac{-1}{\psi_{2}^{b+1}}.
\end{aligned}$$

For each $\tilde{\Gamma} \in \tilde{\Delta}_d^{11,\circ}$, let $\bar{\psi}_i$ be the ψ -class of the marking (seperated from the node) on \mathcal{C}_v . For convenience, we introduce $W_{\mathfrak{e}_i} := Q$ with $[W_{\mathfrak{e}_i}]^{\mathrm{vir}} = -Q/d_{\mathfrak{e}_i}$, and we let $d_i = d_{\mathfrak{e}_i}$, and $h_{\mathfrak{e}_i}, h_i$ be the classes on $W_{\mathfrak{e}_i}, \overline{M}_{0,2}(\mathbb{P}^4, d_v)^p$ respectively induced by the hyperplane of \mathbb{P}^4 . Also let D_1, D_2 be divisors of $\mathcal{C}_{e_1}, \mathcal{C}_{e_2}$ given by the nodes. There are two cases, depending on $d_v > 0$ or $d_v = 0$, which we denote by $Cont_{II}^+$ and $Cont_{II}^0$ respectively. Then $Cont_{II} = Cont_{II}^+ + Cont_{II}^0$.

(1) Case one: $d_v > 0$. We have an exact sequence of sheaves

$$0 \to L_{\tilde{\Gamma}}L_{\mathfrak{t}}(-\mathfrak{D}_{1})|_{\mathcal{C}_{\mathfrak{e}_{1}}}(-D_{1}) \oplus L_{\tilde{\Gamma}}L_{\mathfrak{t}}(-\mathfrak{D}_{2})|_{\mathcal{C}_{\mathfrak{e}_{2}}}(-D_{2}) \to L_{\tilde{\Gamma}}L_{\mathfrak{t}}(-\mathfrak{D}_{1}-\mathfrak{D}_{2}) \to L_{\tilde{\Gamma}}L_{\mathfrak{t}}|_{\mathcal{C}_{v}} \to 0.$$

Thus, up to capping with $[\mathcal{W}_{\tilde{\Gamma}}]^{\mathrm{vir}}$, one has

$$(26) \quad e\big(\tilde{\pi}_*\tilde{f}^*(L_{\tilde{\Gamma}}L_{\mathfrak{t}}(-\mathfrak{D}_1-\mathfrak{D}_2))\big) = \prod_{a=1}^2 e\big(\tilde{\pi}_*\tilde{f}^*(L_{\mathcal{C}_{\mathfrak{c}_a}}L_{\mathfrak{t}}(-\mathfrak{D}_a-D_a))\big) \quad \cdot \quad e\big(\tilde{\pi}_*\tilde{f}^*L_{\tilde{\Gamma}}L_{\mathfrak{t}}|_{\mathcal{C}_v}\big)$$

$$= \prod_{j=1}^{d_1-1} \frac{j(h_{\mathfrak{e}_1}+\mathfrak{t})}{d_1} \cdot \prod_{j=1}^{d_2-1} \frac{j(h_{\mathfrak{e}_2}+\mathfrak{t})}{d_2} \cdot \mathfrak{t}^{d_v+1}.$$

Recall $\frac{1}{5} \sum_{j=0}^{3} h^{j} \otimes h_{\mathfrak{e}_{i}}^{3-j}$ is the Poincaré dual of the diagonal in $Q \times Q$. By [CLLL2], $e_{T}(R\pi_{v*}\mathcal{L}_{v} \otimes L_{\mathfrak{t}}) = \mathfrak{t}^{d_{v}+1}$,

$$\begin{split} [\mathcal{W}_{(\tilde{\Gamma})}]^{\mathrm{vir}} &= \Big[\prod_{i=1}^{2} \Big(\frac{1}{5} \sum_{j=0}^{3} h_{i}^{j} \otimes h_{\mathfrak{e}_{i}}^{3-j}\Big)\Big] \cap \bigg([\overline{M}_{0,2}(\mathbb{P}^{4}, d_{v})^{p}]^{\mathrm{vir}} \times \big([W_{\mathfrak{e}_{1}}]^{\mathrm{vir}} \times [W_{\mathfrak{e}_{2}}]^{\mathrm{vir}}\big)\bigg),\\ &\frac{1}{e(N_{\mathcal{W}_{(\tilde{\Gamma})}}^{\mathrm{vir}}/\mathcal{W})} = \frac{(5\mathfrak{t})^{2}(h_{\mathfrak{e}_{1}} + \mathfrak{t})(h_{\mathfrak{e}_{2}} + \mathfrak{t})}{e_{T}(R\pi_{v*}\mathcal{L}_{v} \otimes L_{\mathfrak{t}})} \frac{A_{\mathfrak{e}_{1}} \cdot A_{\mathfrak{e}_{2}}}{(\psi_{1} - \bar{\psi}_{1})(\psi_{2} - \bar{\psi}_{2})}. \end{split}$$

And for $i = 1, 2, \ \psi_i = \frac{h_i + \mathfrak{t}}{d_i}$ and $\bar{\psi_i}^3 = 0$ give $\frac{1}{\psi_i - \bar{\psi_i}} = \sum_{\ell_i=0}^2 \frac{\bar{\psi_i}^{\ell_i}}{\psi_i^{\ell_i+1}}$. The above relation and (26) gives

(27)
$$\frac{e\left(\tilde{\pi}_*\tilde{f}^*(L_{\tilde{\Gamma}}L_{\mathfrak{t}}(-\mathfrak{D}_1-\mathfrak{D}_2))\right)}{e(N_{\mathcal{W}_{(\tilde{\Gamma})}/\mathcal{W}}^{\mathrm{vir}})} = \frac{B_{d_1}(h_{\mathfrak{e}_1})B_{d_2}(h_{\mathfrak{e}_2})}{(\psi_1-\bar{\psi}_1)(\psi_2-\bar{\psi}_2)}$$

Thus contribution of case $d_v > 0$ to (25) is

$$\begin{aligned} \mathfrak{X}^{+} &= \frac{-1}{2} \sum_{j_{1}, j_{2}=0}^{3} \sum_{\ell_{1}, \ell_{2}=0}^{2} \left(\sum_{d_{v}=1}^{\infty} (-\mathfrak{Q})^{d_{v}} \int_{[\overline{M}_{0,2}(\mathbb{P}^{4}, d_{v})^{p}]^{\operatorname{vir}}} \bar{\psi}_{1}^{\ell_{1}} ev_{1}^{*} h_{1}^{j_{1}} \bar{\psi}_{2}^{\ell_{2}} ev_{2}^{*} h_{2}^{j_{2}} \frac{1}{\mathfrak{t}^{d_{v}+1}} \mathfrak{t}^{d_{v}+1} \right) \\ &\quad \cdot \left(\frac{5\mathfrak{t}}{5} \sum_{d_{1}=1}^{\infty} (-\mathfrak{Q})^{d_{1}} \int_{[W_{\mathfrak{e}_{1}}]^{\operatorname{vir}}} h_{\mathfrak{e}_{1}}^{3-j_{1}} \frac{h_{\mathfrak{e}_{1}} + \mathfrak{t}}{\psi_{1}^{\ell_{1}+1}} A_{\mathfrak{e}_{1}} \left(\prod_{j=1}^{d_{1}-1} \frac{j(h_{\mathfrak{e}_{1}} + \mathfrak{t})}{d_{1}} \right) \frac{\mathfrak{t}^{5}}{\psi_{1}^{a+1}} \right) \\ &\quad \cdot \left(\frac{5\mathfrak{t}}{5} \sum_{d_{2}=1}^{\infty} (-\mathfrak{Q})^{d_{2}} \int_{[W_{\mathfrak{e}_{2}}]^{\operatorname{vir}}} h_{\mathfrak{e}_{2}}^{3-j_{2}} \frac{h_{\mathfrak{e}_{2}} + \mathfrak{t}}{\psi_{2}^{\ell_{2}+1}} A_{\mathfrak{e}_{2}} \left(\prod_{j=1}^{d_{2}-1} \frac{j(h_{\mathfrak{e}_{2}} + \mathfrak{t})}{d_{2}} \right) \frac{\mathfrak{t}^{5}}{\psi_{2}^{b+1}} \right) \\ &= \frac{1}{2} \sum_{j_{1}, j_{2}=0}^{1} \sum_{\ell_{1}, \ell_{2}=0}^{2} \left(\sum_{d_{v}=1}^{\infty} \mathfrak{Q}^{d_{v}} < \tau_{\ell_{1}}(h^{j_{1}}) \tau_{\ell_{2}}(h^{j_{2}}) >_{g=0, d_{v}} \right) \\ &\quad \cdot \left(\frac{\mathfrak{t}^{5}}{5} \sum_{d_{1}=1}^{\infty} (-\mathfrak{Q})^{d_{1}} \int_{[W_{\mathfrak{e}_{1}}]^{\operatorname{vir}}} \frac{h_{\mathfrak{e}_{1}}^{3-j_{1}}}{\psi_{1}^{\ell_{1}+2+a}} B_{d_{1}}(h_{\mathfrak{e}_{1}}) \right) \frac{\mathfrak{t}^{5}}{5} \sum_{d_{2}=1}^{\infty} (-\mathfrak{Q})^{d_{2}} \int_{[W_{\mathfrak{e}_{2}}]^{\operatorname{vir}}} \frac{h_{\mathfrak{e}_{2}}^{3-j_{2}}}{\psi_{2}^{\ell_{2}+2+b}} B_{d_{2}}(h_{\mathfrak{e}_{2}}). \end{aligned}$$

Thus the contribution from Case one is

$$Cont_{II}^{+} = \frac{1}{5\mathfrak{t}^{5}} \sum_{a,b\geq 0}^{\infty} \frac{(-1)^{a+b}}{a!b!} \mathfrak{X}^{+} \frac{\eta_{0}^{a+b+1}}{a+b+1} = \frac{1}{2} \frac{\mathfrak{t}^{5}}{5} \sum_{j_{1},j_{2}=0}^{1} \sum_{\ell_{1},\ell_{2}=0}^{\infty} \sum_{d_{v}=1}^{\infty} \mathfrak{Q}^{d_{v}} < \tau_{\ell_{1}}(h^{j_{1}})\tau_{\ell_{2}}(h^{j_{2}}) >_{g=0,d_{v}} + \left(\frac{1}{5^{2}} \sum_{d_{1},d_{2}=1}^{\infty} (-\mathfrak{Q})^{d_{1}+d_{2}} \int_{[W_{\mathfrak{e}_{1}}]^{\mathrm{vir}} \times [W_{\mathfrak{e}_{2}}]^{\mathrm{vir}}} \frac{h_{\mathfrak{e}_{1}}^{3-j_{1}}}{\psi_{1}^{\ell_{1}+1}} \frac{h_{\mathfrak{e}_{2}}^{3-j_{2}}}{\psi_{2}^{\ell_{2}+1}} B_{d_{1}}(h_{\mathfrak{e}_{1}}) B_{d_{2}}(h_{\mathfrak{e}_{2}}) \frac{e^{(-\eta_{0})(\frac{1}{\psi_{1}}+\frac{1}{\psi_{2}})}-1}{\psi_{1}+\psi_{2}}\right).$$

(2) Case two: $d_v = 0$. Then v is unstable. Similarly by [CLLL2], $[\mathcal{W}_{(\tilde{\Gamma})}]^{\text{vir}} = \frac{[-Q]}{d_1 d_2}, \ \psi_i = \frac{h_{e_i} + \mathfrak{t}}{d_i},$

$$\frac{1}{e(N_{\mathcal{W}(\tilde{\Gamma})}^{\mathrm{vir}}/\mathcal{W})} = (5\mathfrak{t})^2 \frac{(h+\mathfrak{t})A_{\mathfrak{e}_1}A_{\mathfrak{e}_2}}{\psi_1 + \psi_2}, \ e\big(\tilde{\pi}_*\tilde{f}^*(L_{\tilde{\Gamma}}L\mathfrak{t}(-\mathfrak{D}_1 - \mathfrak{D}_2))\big)$$
$$= (h+\mathfrak{t})\prod_{j=1}^{d_1-1}\frac{j(h_{\mathfrak{e}_1}+\mathfrak{t})}{d_1}\prod_{j=1}^{d_2-1}\frac{j(h_{\mathfrak{e}_2}+\mathfrak{t})}{d_2}.$$

Thus contribution of Case two to (25) is,

$$\mathfrak{X}^{0} := \frac{-1}{2} \sum_{d_{1}, d_{2}=1}^{\infty} \frac{(-\mathcal{Q})^{d_{1}+d_{2}}}{d_{1}d_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{1}+\psi_{2}} A_{\mathfrak{e}_{1}} A_{\mathfrak{e}_{2}} \bigg)^{d_{1}+d_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{1}+\psi_{2}} A_{\mathfrak{e}_{1}} A_{\mathfrak{e}_{2}} \bigg)^{d_{1}+d_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{1}+\psi_{2}} A_{\mathfrak{e}_{2}} A_{\mathfrak{e}_{2}} \bigg)^{d_{1}+d_{2}} \bigg)^{d_{1}+d_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{1}+\psi_{2}} A_{\mathfrak{e}_{2}} \bigg)^{d_{1}+d_{2}} \bigg)^{d_{1}+d_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{1}+\psi_{2}} A_{\mathfrak{e}_{2}} \bigg)^{d_{1}+d_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{1}+\psi_{2}} A_{\mathfrak{e}_{2}} \bigg)^{d_{1}+d_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{1}+\psi_{2}} A_{\mathfrak{e}_{2}} \bigg)^{d_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{1}+\psi_{2}} A_{\mathfrak{e}_{2}} \bigg)^{d_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{1}+\psi_{2}} A_{\mathfrak{e}_{2}} \bigg)^{d_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{1}+\psi_{2}} A_{\mathfrak{e}_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{2}} A_{\mathfrak{e}_{2}} \bigg)^{d_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{2}} A_{\mathfrak{e}_{2}} \bigg)^{d_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{2}} A_{\mathfrak{e}_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{2}} A_{\mathfrak{e}_{2}} \bigg)^{d_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{2}} A_{\mathfrak{e}_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{2}} A_{\mathfrak{e}_{2}} \bigg(\int_{[-Q]} (5\mathfrak{t})^{2} \frac{h$$

$$\cdot \Big(\prod_{j=1}^{d_1-1} \frac{j(h_{\mathfrak{e}_1}+\mathfrak{t})}{d_1}\Big)\Big(\prod_{j=1}^{d_2-1} \frac{j(h_{\mathfrak{e}_2}+\mathfrak{t})}{d_2}\Big)(h+\mathfrak{t})\mathfrak{t}^{10}\frac{1}{\psi_1^{a+1}}\frac{1}{\psi_2^{b+1}}\Big) \\ = \frac{1}{2}\sum_{d_1,d_2=1}^{\infty} \frac{(-\mathfrak{Q})^{d_1+d_2}}{d_1+d_2} \int_Q (5\mathfrak{t})^2 A_{\mathfrak{e}_1} A_{\mathfrak{e}_2}\Big(\prod_{j=1}^{d_1-1} \frac{j(h_{\mathfrak{e}_1}+\mathfrak{t})}{d_1}\Big)\Big(\prod_{j=1}^{d_2-1} \frac{j(h_{\mathfrak{e}_2}+\mathfrak{t})}{d_2}\Big)(h+\mathfrak{t})\mathfrak{t}^{10}\frac{1}{\psi_1^{a+1}}\frac{1}{\psi_2^{b+1}}$$

We see that the contribution of Case 2 to $Contri_{II}$ is

$$Cont_{II}^{0} = \frac{1}{5t^{5}} \sum_{a,b=0}^{\infty} \frac{(-1)^{a+b}}{a!b!} \mathfrak{X}^{0} \frac{\eta_{0}^{a+b+1}}{a+b+1}.$$

6.4. Computations of Contri I. The graph of the contribution type I is a type B graph Γ with the unique vertex $\tilde{v} \in V_1(\Gamma)$ unstable. Let $\tilde{\Gamma}$ be the graph obtained by resolving the node at V_1 of Γ shown in the Figure Graph $\tilde{\Gamma}$. Then $\tilde{\Gamma}$ is a graph in $\tilde{\Delta}_d^{11,\circ}$. We have $[\mathcal{W}_{(\Gamma)}]^{\text{vir}} = \frac{1}{2}[\mathcal{W}_{(\tilde{\Gamma})}]^{\text{vir}}$. The contribution of Γ is

$$\frac{1}{2}\sum_{d=1}^{\infty}(-\mathcal{Q})^d\sum_{(\tilde{\Gamma})\in\tilde{\Delta}_d^{11,\circ}}\int_{[\mathcal{W}_{(\tilde{\Gamma})}]^{\mathrm{vir}}}\frac{e\big(\tilde{\pi}_*\tilde{f}^*(L_{\tilde{\Gamma}}L_{\mathfrak{t}}(-\mathfrak{D}_1-\mathfrak{D}_2))\big)}{e(N_{\mathcal{W}_{(\tilde{\Gamma})}}^{\mathrm{vir}}/\mathcal{W})}\cdot\frac{\mathfrak{t}}{(5\mathfrak{t})^2}\frac{-5\mathfrak{t}^6}{-\psi_1-\psi_2}.$$

There are two cases, depending on $d_v > 0$ or $d_v = 0$.

(1) Case one: $d_v > 0$. The factors are the same as those in $Cont_{II}^+$. By (27), one has

$$Cont_{I}^{+} = \frac{-1}{2} \frac{\mathfrak{t}^{5}}{5} \sum_{j_{1}, j_{2}=0}^{1} \sum_{\ell_{1}, \ell_{2}=0}^{2} \left(\sum_{d_{v}=1}^{\infty} \mathcal{Q}^{d_{v}} < \tau_{\ell_{1}}(h^{j_{1}})\tau_{\ell_{2}}(h^{j_{2}}) >_{g=0, d_{v}} \right) \\ \left(\frac{1}{5^{2}} \sum_{d_{1}, d_{2}=1}^{\infty} (-\mathcal{Q})^{d_{1}+d_{2}} \int_{[W_{\mathfrak{e}_{1}}]^{\mathrm{vir}} \times [W_{\mathfrak{e}_{2}}]^{\mathrm{vir}}} \frac{h_{\mathfrak{e}_{1}}^{3-j_{1}}}{\psi_{1}^{\ell_{1}+1}} B_{d_{1}}(h_{\mathfrak{e}_{1}}) \frac{h_{\mathfrak{e}_{2}}^{3-j_{2}}}{\psi_{2}^{\ell_{2}+1}} B_{d_{2}}(h_{\mathfrak{e}_{2}}) \frac{1}{\psi_{1}+\psi_{2}} \right).$$

(2) Case two: $d_v = 0$. The factors are the same as those in $Cont_{II}^0$. Thus contribution of this case is, noting $h_{\mathfrak{e}_1} = h_{\mathfrak{e}_2} = h$ over the diagonal $Q \subset Q \times Q$,

$$Cont_{I}^{0} = \frac{1}{2} \sum_{d_{1},d_{2}=1}^{\infty} \frac{(-\Omega)^{d_{1}+d_{2}}}{d_{1}d_{2}} \int_{[-Q]} (5\mathfrak{t})^{2} \frac{(h+\mathfrak{t})A_{\mathfrak{e}_{1}}A_{\mathfrak{e}_{2}}}{\psi_{1}+\psi_{2}} \Big(\prod_{j=1}^{d_{1}-1} \frac{j(h_{\mathfrak{e}_{1}}+\mathfrak{t})}{d_{1}}\Big) \Big(\prod_{j=1}^{d_{2}-1} \frac{j(h_{\mathfrak{e}_{2}}+\mathfrak{t})}{d_{2}}\Big) \frac{\mathfrak{t}(h+\mathfrak{t})}{\psi_{1}+\psi_{2}} \frac{\mathfrak{t}^{4}}{5}$$
$$= -\frac{1}{2} \frac{\mathfrak{t}^{5}}{5} \sum_{d_{1},d_{2}=1}^{\infty} \frac{(-\Omega)^{d_{1}+d_{2}}}{d_{1}+d_{2}} \int_{Q} (5\mathfrak{t})^{2} A_{\mathfrak{e}_{1}} A_{\mathfrak{e}_{2}} \Big(\prod_{j=1}^{d_{1}-1} \frac{j(h_{\mathfrak{e}_{1}}+\mathfrak{t})}{d_{1}}\Big) \Big(\prod_{j=1}^{d_{2}-1} \frac{j(h_{\mathfrak{e}_{2}}+\mathfrak{t})}{d_{2}}\Big) \frac{(h+\mathfrak{t})}{\psi_{1}+\psi_{2}}.$$

6.5. Summations $Cont_{II}^+ + Cont_I^+$ and $Cont_{II}^0 + Cont_I^0$. Using

$$\sum_{a,b=0}^{\infty} \frac{(-1)^{a+b}}{a!b!} \frac{1}{\psi_1^{a+1}} \frac{1}{\psi_2^{b+1}} \frac{\eta_0^{a+b+1}}{a+b+1} = \frac{-1}{\psi_1 + \psi_2} \left(e^{(-\eta_0)(\frac{1}{\psi_1} + \frac{1}{\psi_2})} - 1 \right),$$

we see the sum $Cont_{II}^+ + Cont_I^+$ equals

$$Cont^{+} := Cont_{II}^{+} + Cont_{I}^{+} = \frac{1}{2} \frac{\mathfrak{t}^{5}}{5} \sum_{j_{1}, j_{2}=0}^{1} \sum_{\ell_{1}, \ell_{2}=0}^{2} \left(\sum_{d_{v}=1}^{\infty} \mathcal{Q}^{d_{v}} < \tau_{\ell_{1}}(h^{j_{1}}) \tau_{\ell_{2}}(h^{j_{2}}) >_{g=0, d_{v}} \right)$$

$$(28) \qquad \cdot \left(-\frac{1}{5^{2}} \sum_{d_{1}, d_{2}=1}^{\infty} (-\mathcal{Q})^{d_{1}+d_{2}} \int_{[W_{\mathfrak{e}_{1}}]^{\mathrm{vir}} \times [W_{\mathfrak{e}_{2}}]^{\mathrm{vir}}} \frac{h_{\mathfrak{e}_{1}}^{3-j_{1}}}{\psi_{1}^{\ell_{1}+1}} \frac{h_{\mathfrak{e}_{2}}^{3-j_{2}}}{\psi_{2}^{\ell_{2}+1}} B_{d_{1}}(h_{\mathfrak{e}_{1}}) B_{d_{2}}(h_{\mathfrak{e}_{2}}) \frac{e^{(-\eta_{0})(\frac{1}{\psi_{1}}+\frac{1}{\psi_{2}})}}{\psi_{1}+\psi_{2}} \right)$$

A similar combination holds for $Cont_{II}^0 + Cont_I^0$. Recall $h_{\mathfrak{e}_1} = h = h_{\mathfrak{e}_2}$ over diagonal $Q \subset Q \times Q$. We have

(29)
$$Cont^{0} := Cont_{II}^{0} + Cont_{I}^{0} = -\frac{1}{2}\frac{\mathfrak{t}^{5}}{5}\sum_{d_{1},d_{2}=1}^{\infty}\frac{(-\mathfrak{Q})^{d_{1}+d_{2}}}{d_{1}+d_{2}}\bigg(\int_{Q}(5\mathfrak{t})^{2}A_{\mathfrak{e}_{1}}A_{\mathfrak{e}_{2}}\right) \\ \cdot \big(\prod_{j=1}^{d_{1}}\frac{j(h_{\mathfrak{e}_{1}}+\mathfrak{t})}{d_{1}}\big)\big(\prod_{j=1}^{d_{2}}\frac{j(h_{\mathfrak{e}_{2}}+\mathfrak{t})}{d_{2}}\big)\frac{1}{d_{1}d_{2}}\frac{1}{\psi_{1}}\frac{1}{\psi_{2}}\frac{h+\mathfrak{t}}{\psi_{1}+\psi_{2}}e^{(-\eta_{0})(\frac{1}{\psi_{1}}+\frac{1}{\psi_{2}})}\bigg).$$

6.6. S-series and I-functions. In this subsection we introduce a S-series which is a generating function of MSP's A_e for all edges e connecting V_0 with V_1 . We express S-series using Givental's I-unctions. This will enable us to express the two series $Cont_{II}^+ + Cont_{II}^0$ and $Cont_I^+ + Cont_I^0$ in previous subsections using Givental's I-functions.

subsections using Givental's *I*-functions. For $j \in \mathbb{Z}_{\leq 3}$, $\psi = \frac{h_{\mathfrak{e}} + \mathfrak{t}}{d_{\mathfrak{e}}}$, and for arbitrary $m \in \mathbb{Z}$, we formulate

$$T_{j,m} := \frac{\mathfrak{t}^5}{5} \sum_{d=1}^{\infty} (-\mathfrak{Q})^d \int_{[W_{\mathfrak{e}}]^{\mathrm{vir}}} \frac{h^{3-j}}{\psi^m} B_d(h) := S_{j,m-2}$$

Note that this implies $S_{j,m} = 0$ whenever j < 0. Also, by definition, we have

$$\int_{[W_{\epsilon}]^{\text{vir}}} d\frac{h^{3-j}}{\psi^{m+2}} B_d(h) = \int_{[W_{\epsilon}]^{\text{vir}}} (h+\mathfrak{t}) \frac{h^{3-j}}{\psi^{m+3}} B_d(h) = \int_{[W_{\epsilon}]^{\text{vir}}} \frac{h^{3-(j-1)}}{\psi^{m+3}} B_d(h) + \mathfrak{t} \int_{[W_{\epsilon}]^{\text{vir}}} \frac{h^{3-j}}{\psi^{m+3}} B_d(h).$$

This implies directly

$$\partial_T S_{j,m} = S_{j-1,m+1} + \mathfrak{t} S_{j,m+1}, \quad \forall j \in \mathbb{Z}_{\leq 3}, m \in \mathbb{Z}.$$

We also have

$$S_{j,m}|_{T=-\infty} = 0, \qquad T^k S_{j,m}|_{T=-\infty} = 0, \forall k \in \mathbb{R},$$

where $-\infty$ refers to the limit of the point on the real line with the real part approaching the negative infinity.

On the other hand, denote $I_{\ell} = 0 = b_{\ell}(d)$ whenever $\ell < 0$. By (7), for $j = 0, \dots, 3$, and $m \in \mathbb{Z}$, we have

$$T_{j,m} = \mathfrak{t}^{5-j-m} \sum_{d=1}^{\infty} d^{m-1} (-\mathfrak{Q})^d \bigg[-b_j(d) + mb_{j-1}(d) - \binom{m+1}{2} b_{j-2}(d) + \binom{m+2}{3} b_{j-3}(d) \bigg].$$

We denote $I_j(y) = I_j(t)|_{t \mapsto y}$, and $I_j^{(k)} = \partial_y^k I_j(y)$. As an example, we have

$$S_{0,-1} = T_{0,1} = -\mathfrak{t}^5 \sum_{d=1}^{\infty} (-\Omega)^d b_0(d) \frac{1}{\mathfrak{t}} = -\mathfrak{t}^4 (I_0(y) - 1)|_{y \mapsto T}$$
$$S_{1,-1} = T_{1,1} = \mathfrak{t}^3 I_1'|_{t \mapsto T} - \mathfrak{t}^3 T I_0'(y) |_{y \mapsto T} - \mathfrak{t}^3.$$

Applying the formulae in (9), one directly computes

Lemma 6.1. Denote $I_{\ell} = 0$ whenever $\ell < 0$. Then for m > 0 and $j = 0, \dots, 3$, we have

$$T_{j,m} = (-1)^{j+1} \mathfrak{t}^{5-j-m} \left(I_j^{(m+j-1)} - t I_{j-1}^{(m+j-1)} + \frac{t^2}{2} I_{j-2}^{(m+j-1)} - \frac{t^3}{6} I_{j-3}^{(m+j-1)} \right) |_{t \mapsto T} + (-1)^j \mathfrak{t}^{5-j-m} \binom{m+j-1}{j} \delta_{m,1.}$$

Lemma 6.2. For arbitrary $\ell \in \mathbb{Z}_{\geq -1}$, $j = 0, \dots, 3$, we have

$$\sum_{a=0}^{\infty} \frac{(-1)^a}{a!} S_{j,a+\ell} (-\mathfrak{t}(y-T))^a = (-1)^{j+1} \mathfrak{t}^{3-j-\ell} \left(I_j^{(\ell+j+1)}(y) - T I_{\ell+j}^{(\ell+j+1)}(y) + \frac{T^2}{2} I_{\ell+j-1}^{(\ell+j+1)}(y) - \frac{T^3}{6} I_{\ell+j-2}^{(\ell+j+1)}(y) \right) + (-1)^j \mathfrak{t}^{3-j-\ell} \binom{\ell+j+1}{j} \delta_{\ell,-1}.$$

6.7. Expression of $Cont^+$ by *I*-functions. Let us define a function, for $j_1, j_2 \in \mathbb{Z}_{\leq 3}, \ell_1, \ell_2 \in \mathbb{Z}$,

$$\begin{split} M_{j_1,j_2,\ell_1,\ell_2}(x,T) \\ &:= -\frac{1}{2} \frac{\mathfrak{t}^6}{5} \sum_{d_1,d_2=1}^{\infty} \frac{(-e^T)^{d_1+d_2}}{d_1+d_2} \int_{[W_{\mathfrak{e}_1}]^{\mathrm{vir}} \times [W_{\mathfrak{e}_2}]^{\mathrm{vir}}} \frac{1}{5} \frac{h_{\mathfrak{e}_1}^{3-j_1}}{\psi_1^{\ell_1+1}} \frac{h_{\mathfrak{e}_2}^{3-j_2}}{\psi_2^{\ell_2+1}} B_{d_1}(h_{\mathfrak{e}_1}) B_{d_2}(h_{\mathfrak{e}_2}) \frac{e^{(\mathfrak{t}x)(\frac{1}{\psi_1}+\frac{1}{\psi_2})}}{\psi_1+\psi_2} \end{split}$$

Then

$$\partial_T M_{j_1,j_2,\ell_1,\ell_2}(x,T) = -\frac{1}{2} \frac{\mathfrak{t}^6}{5} \sum_{d_1,d_2=1}^{\infty} (-e^T)^{d_1+d_2} \int_{[W_{\mathfrak{e}_1}]^{\mathrm{vir}} \times [W_{\mathfrak{e}_2}]^{\mathrm{vir}}} \frac{1}{5} \frac{h_{\mathfrak{e}_1}^{3-j_1}}{\psi_1^{\ell_1+1}} \frac{h_{\mathfrak{e}_2}^{3-j_2}}{\psi_2^{\ell_2+1}} B_{d_1}(h_{\mathfrak{e}_1}) B_{d_2}(h_{\mathfrak{e}_2}) \frac{e^{(\mathfrak{t}x)(\frac{1}{\psi_1}+\frac{1}{\psi_2})}}{\psi_1+\psi_2}.$$

We have, when x or T approaches real $-\infty$,

 $M_{j_1,j_2,\ell_1,\ell_2}(x,T)|_{x\to-\infty} = 0, \quad \partial_T M_{j_1,j_2,\ell_1,\ell_2}(x,T)|_{x\to-\infty} = 0, \quad M_{j_1,j_2,\ell_1,\ell_2}(x,T)|_{T\to-\infty} = 0.$ This implies

$$\partial_T M_{j_1, j_2, \ell_1, \ell_2}(z, T) = \int_{-\infty}^z \partial_x \partial_T M_{j_1, j_2, \ell_1, \ell_2}(x, T) dx,$$
$$M_{j_1, j_2, \ell_1, \ell_2}(x, u) = \int_{-\infty}^u \partial_w M_{j_1, j_2, \ell_1, \ell_2}(x, w) dw = \int_{-\infty}^u \int_{-\infty}^x \partial_z \partial_w M_{j_1, j_2, \ell_1, \ell_2}(z, w) dz dw.$$

We have a closed formula

$$\partial_x \partial_T M_{j_1, j_2, \ell_1, \ell_2}(x, T) = -\frac{1}{2} \frac{1}{\mathfrak{t}^3} \sum_{a, b=0}^{\infty} S_{j_1, \ell_1 + a} S_{j_2, \ell_2 + b} \frac{(\mathfrak{t}x)^a}{a!} \frac{(\mathfrak{t}x)^b}{b!}.$$

The definition of M enables us to simplify, by (28), (29) and the above formula of $\partial_x \partial_T M$,

$$\begin{split} Cont^{+} &:= Cont_{II}^{+} + Cont_{I}^{+} \\ &= \frac{1}{5t} \sum_{j_{1}, j_{2}=0}^{1} \sum_{\ell_{1}, \ell_{2}=0}^{2} \left(\sum_{d_{v}=1}^{\infty} \mathcal{Q}^{d_{v}} < \tau_{\ell_{1}}(h^{j_{1}})\tau_{\ell_{2}}(h^{j_{2}}) >_{g=0,d_{v}} \right) \left(\partial_{T} M_{j_{1}, j_{2}, \ell_{1}, \ell_{2}}(x, T) \right) |_{x=t-T} \\ &= \frac{1}{5t} \partial_{T}^{2} F_{0} \left(\partial_{T} M_{1,1,0,0}(x, T) \right) |_{x=t-T} \\ &+ \frac{1}{5t} 2 \partial_{T} F_{0} \left(\left(\partial_{T} M_{1,0,1,0}(x, T) \right) \right) |_{x=t-T} - \left(\partial_{T} M_{1,0,0,1}(x, T) \right) |_{x=t-T} \right) \\ &+ \frac{1}{5t} 2F_{0} \left(\left(\partial_{T} M_{0,0,1,1}(x, T) \right) |_{x=t-T} - 2 \left(\partial_{T} M_{0,0,2,0}(x, T) \right) |_{x=t-T} \right) \\ &= -\frac{1}{10} \partial_{T}^{2} F_{0} \int_{-\infty}^{t} (I_{1}^{(2)} - T I_{0}^{(2)})^{2} (y) dy \\ &- \frac{1}{10} 2 \partial_{T} F_{0} \left(- \int_{-\infty}^{t} (I_{1}^{(3)} - T I_{0}^{(3)}) (y) I_{0}^{(1)} (y) dy + \int_{-\infty}^{t} (I_{1}^{(2)} - T I_{0}^{(2)}) (y) I_{0}^{(2)} (y) \right) \\ &- \frac{1}{10} 2F_{0} \left(\int_{-\infty}^{t} (I_{0}^{(2)})^{2} (y) - 2 \int_{-\infty}^{t} I_{0}^{(3)} (y) I_{0}^{(1)} (y) dy \right), \end{split}$$

where $I_a^{(k)}(y) := \frac{d^k}{dy^k} (I_a(t)|_{t \mapsto y})$, and in last identity we used Lemma 6.2. Furthermore, using $F_0 = \frac{5}{2} (J_1 J_2 - J_3) - \frac{5}{6} T^3, \qquad \partial_T F_0 = 5J_2 - 5 \cdot \frac{T^2}{2}, \qquad \partial_T^2 F_0 = 5J_2' - 5T,$

one easily calculates

$$Cont^{+} = -\frac{1}{2} \partial_{T} J_{2} \int_{-\infty}^{t} (I_{1}^{(2)} - TI_{0}^{(2)})^{2} dy - \frac{1}{2} 2J_{2} \left(\int_{-\infty}^{t} (I_{1}^{(2)} - TI_{0}^{(2)})I_{0}^{(2)} - (I_{1}^{(3)} - TI_{0}^{(3)})I_{0}^{\prime} dy \right) \\ -\frac{1}{2} \partial_{T}^{-1} (2J_{2}) \int_{-\infty}^{t} (I_{0}^{(2)})^{2} - 2I_{0}^{(3)}I_{0}^{(1)} dy + \frac{1}{2} T \int_{-\infty}^{t} (I_{1}^{(2)})^{2} dy - \frac{1}{2} T^{2} (I_{1}^{\prime\prime}(t) - \frac{T}{3}I_{0}^{\prime\prime}(t))I_{0}^{\prime}(t).$$

6.8. Expression of $Cont^0$ by *I*-functions. In this subsection we use I_r to mean Givental's *I*-functions $I_r(t)$, which is a power series in $q = e^t$ as usual. The difference of the I_r notation from before is just the variable y changed to variable t here.

$$Cont_{II}^{0} + Cont_{I}^{0} = -\frac{1}{2} \frac{\mathfrak{t}^{5}}{5} \sum_{d_{1}, d_{2}=1}^{\infty} \frac{(-1)^{d_{1}+d_{2}}}{(d_{1}+d_{2})^{2}} \int_{Q} B_{d_{1}}(h) \cdot B_{d_{2}}(h) \frac{1}{\psi_{1}} \frac{1}{\psi_{2}} q^{d_{1}+d_{2}}$$
$$\cdot \left(1 + (d_{1}+d_{2})\frac{(T-t)h}{h+\mathfrak{t}} + \frac{1}{2}(d_{1}+d_{2})^{2}(T-t)^{2}\frac{h^{2}}{\mathfrak{t}^{2}}\right) + \frac{1}{2}(T-t)^{2}(I_{0}')^{2} - \frac{1}{12}(T-t)^{3}\partial_{t}\left(I_{0}' \cdot I_{0}'\right).$$

Let us calculate the three terms appeared above respectively.

$$\begin{split} &-\frac{1}{2}\frac{\mathfrak{t}^5}{5}\sum_{d_1,d_2=1}^{\infty}\frac{(-1)^{d_1+d_2}}{(d_1+d_2)^2}\int_Q B_{d_1}(h)\cdot B_{d_2}(h)\frac{1}{\psi_1}\frac{1}{\psi_2}q^{d_1+d_2}\cdot \left(\frac{1}{2}(d_1+d_2)^2(T-t)^2\frac{h^2}{\mathfrak{t}^2}\right)\frac{(h+\mathfrak{t})^2}{(h+\mathfrak{t})^2}\\ &=-\frac{\mathfrak{t}^5}{20}(T-t)^2\sum_{d_1,d_2=1}^{\infty}(-1)^{d_1+d_2}\int_Q \frac{B_{d_1}(h)\cdot B_{d_2}(h)}{\psi_1\psi_2}\frac{1}{(h+\mathfrak{t})^2}q^{d_1+d_2}\cdot h^2\\ &-\frac{\mathfrak{t}^5}{10}(T-t)^2\sum_{d_1,d_2=1}^{\infty}(-1)^{d_1+d_2}\int_Q \frac{B_{d_1}(h)\cdot B_{d_2}(h)}{\psi_1\psi_2}\frac{1}{(h+\mathfrak{t})^2}q^{d_1+d_2}\cdot \frac{h^3}{\mathfrak{t}}\\ &=-\frac{\mathfrak{t}^5}{100}(T-t)^2\sum_{j=0}^{1}\sum_{d_1,d_2=1}^{\infty}(-1)^{d_1+d_2}\int_{[W_{\mathfrak{t}_1}]^{\mathrm{vir}}\times[W_{\mathfrak{t}_2}]^{\mathrm{vir}}}h_1^{j+2}h_2^{3-j}\frac{B_{d_1}(h_1)\cdot B_{d_2}(h_2)}{\psi_1^2\psi_2^2}q^{d_1+d_2}\\ &-\frac{1}{2}(T-t)^2\sum_{d_1,d_2=1}^{\infty}(-1)^{d_1+d_2}d_1d_2b_0(d_1)b_0(d_2)q^{d_1+d_2}\\ &=-\frac{\mathfrak{t}^5}{100}(T-t)^2\sum_{j=0}^{1}\frac{5^2}{\mathfrak{t}^{10}}(S_{1-j,0}S_{j,0})|_{T\mapsto t}-\frac{1}{2}(T-t)^2(I_0')^2&=\frac{1}{2}(T-t)^2\Big((I_1''-tI_0'')I_0'-(I_0')^2\Big). \end{split}$$

Similar calculations give

$$\begin{split} &-\frac{1}{2}\frac{\mathfrak{t}^{5}}{5}\sum_{d_{1},d_{2}=1}^{\infty}\frac{(-1)^{d_{1}+d_{2}}}{(d_{1}+d_{2})^{2}}\int_{Q}B_{d_{1}}(h)\cdot B_{d_{2}}(h)\frac{1}{\psi_{1}}\frac{1}{\psi_{2}}q^{d_{1}+d_{2}}\cdot\left((d_{1}+d_{2})\frac{(T-t)h}{h+\mathfrak{t}}\right)\\ &=-\frac{1}{2}(T-t)\partial_{t}^{-1}\left(\left(I_{1}^{\prime\prime\prime}(t)-tI_{0}^{\prime\prime\prime}(t)\right)^{2}+2\left(I_{2}^{\prime\prime\prime\prime}(t)-tI_{1}^{\prime\prime\prime\prime}(t)+\frac{t^{2}}{2}I_{0}^{\prime\prime\prime\prime}(t)\right)I_{0}^{\prime}(t)\right)\\ &+(T-t)\partial_{t}^{-1}\left(I_{1}^{\prime\prime\prime}(t)-tI_{0}^{\prime\prime\prime}(t)\right)I_{0}^{\prime}(t),\\ &-\frac{1}{2}\frac{\mathfrak{t}^{5}}{5}\sum_{d_{1},d_{2}=1}^{\infty}\frac{(-1)^{d_{1}+d_{2}}}{(d_{1}+d_{2})^{2}}\int_{Q}B_{d_{1}}(h)\cdot B_{d_{2}}(h)\frac{1}{\psi_{1}}\frac{1}{\psi_{2}}q^{d_{1}+d_{2}}.\\ &=\partial_{t}^{-2}\Big(\left(I_{1}^{\prime\prime\prime}-tI_{0}^{\prime\prime\prime}\right)I_{0}^{\prime}\Big)-\partial_{t}^{-2}\Big(\left(I_{1}^{\prime\prime\prime}-tI_{0}^{\prime\prime\prime}\right)^{2}+2\left(I_{2}^{\prime\prime\prime\prime}-tI_{1}^{\prime\prime\prime\prime}+\frac{t^{2}}{2}I_{0}^{\prime\prime\prime\prime}\right)I_{0}^{\prime}\Big)\\ &+\partial_{t}^{-2}\Big(\left(I_{3}^{\prime\prime\prime\prime\prime}-tI_{2}^{\prime\prime\prime\prime\prime}+\frac{t^{2}}{2}I_{1}^{\prime\prime\prime\prime\prime}-\frac{t^{3}}{6}I_{0}^{\prime\prime\prime\prime\prime}\right)I_{0}^{\prime}+\left(I_{2}^{\prime\prime\prime\prime}-tI_{1}^{\prime\prime\prime\prime}+\frac{t^{2}}{2}I_{0}^{\prime\prime\prime\prime}\right)\left(I_{1}^{\prime\prime}-tI_{0}^{\prime\prime\prime}\right)\Big),\end{split}$$

where I_k, I'_k, I''_k, \cdots denotes $I_k(t), I'_k(t), I''_k(t), \cdots$. We then conclude

$$Cont_{II}^{0} + Cont_{I}^{0} = -\frac{1}{12}(T-t)^{3}\partial_{t}\left(I_{0}' \cdot I_{0}'\right) + \frac{1}{2}(T-t)^{2}\left(I_{1}''-tI_{0}''\right)I_{0}'$$

$$-\frac{1}{2}(T-t)\partial_{t}^{-1}\left(\left(I_{1}''-tI_{0}''\right)^{2} + 2\left(I_{2}'''-tI_{1}'''+\frac{t^{2}}{2}I_{0}'''\right)I_{0}'\right) + (T-t)\partial_{t}^{-1}\left(\left(I_{1}''-tI_{0}''\right)I_{0}'\right)$$

$$+\partial_{t}^{-2}\left(\left(I_{1}''-tI_{0}''\right)I_{0}'\right) - \partial_{t}^{-2}\left(\left(I_{1}''-tI_{0}''\right)^{2} + 2\left(I_{2}'''-tI_{1}'''+\frac{t^{2}}{2}I_{0}'''\right)I_{0}'\right)$$

$$+\partial_{t}^{-2}\left(\left(I_{3}'''-tI_{2}''''+\frac{t^{2}}{2}I_{1}''''-\frac{t^{3}}{6}I_{0}''''\right)I_{0}'+\left(I_{2}'''-tI_{1}'''+\frac{t^{2}}{2}I_{0}'''\right)\left(I_{1}''-tI_{0}''\right)\right).$$

One can combine a few terms to write in terms of total derivatives,

$$Cont^{0} := Cont_{II}^{0} + Cont_{I}^{0} = -\frac{1}{12}(T-t)^{3}\partial_{t}\left(I_{0}'\cdot I_{0}'\right) + \frac{1}{2}(T-t)^{2}\left(I_{1}''-tI_{0}''\right)I_{0}'$$

$$-\frac{1}{2}(T-t)\partial_{t}^{-1}\left(\left(I_{1}''-tI_{0}''\right)^{2} + 2\left(I_{2}''-tI_{1}''+\frac{t^{2}}{2}I_{0}''\right)'I_{0}'\right)$$

$$+\partial_{t}^{-2}\left(\left(I_{3}'''-tI_{2}'''+\frac{t^{2}}{2}I_{1}'''-\frac{t^{3}}{6}I_{0}''\right)'I_{0}'+\left(I_{2}''-tI_{1}''+\frac{t^{2}}{2}I_{0}''\right)'\left(I_{1}'-tI_{0}'\right)'\right).$$

After simplifications one has

$$Cont^{0} = \left(\frac{T^{2}}{2}I_{1}'' - \frac{T^{3}}{6}I_{0}''\right)I_{0}'(t) - \frac{1}{2}T\partial_{t}^{-1}(I_{1}'')^{2} - (T-t)\partial_{t}^{-1}(I_{2}'''I_{0}') + \partial_{t}^{-2}\left(\left(I_{3}''' - tI_{2}'''\right)'I_{0}' + I_{2}'''(I_{1}' - tI_{0}')'\right).$$

Combining previous subsection and using Proposition C.4, the type B contribution is (30)

 $Cont^+ + Cont^0$

$$= -\frac{1}{2} \partial_T J_2 \int_{-\infty}^t (I_1^{(2)} - TI_0^{(2)})^2 dy - J_2 \Big(\int_{-\infty}^t (I_1^{(2)} - TI_0^{(2)}) I_0^{(2)} - (I_1^{(3)} - TI_0^{(3)}) I_0^{(1)} dy \Big) \\ - \partial_T^{-1} (J_2) \int_{-\infty}^t (I_0^{(2)})^2 - 2I_0^{(3)} I_0^{(1)} dy + \partial_t^{-2} \Big((I_3''' - tI_2''')' I_0' + I_2''' (I_1' - tI_0')' \Big) - (T - t) \partial_t^{-1} (I_2''' I_0') \\ = \frac{1}{2} \frac{T''}{T'} + 2 \frac{I_0'}{I_0} + \frac{\ln(1 - 5^5 e^t)}{5}.$$

7. Graph Type C

A type C graph Γ is of the following form:

$$v \frac{\psi_1 \qquad \psi_k}{\psi_{\Gamma_1 \dots}} \qquad g_v = 1$$

Separating all black nodes, Γ is decomposed into v and rational tails $\{\Gamma_i\}_{i=1}^k$. Let $d_i = \deg \Gamma_i$. Then $d_1 + d_2 + \cdots + d_k = n$. Thus the contribution of Γ to (1) is

$$\frac{\mathfrak{t}\prod|Aut\Gamma_{i}|}{|Aut\Gamma|}\sum_{a+\sum a_{i}=n-1}\int_{[\mathcal{W}_{v}]^{\mathrm{vir}}}\frac{c_{a}\left(R\pi_{\mathcal{W}_{v}*}(\mathcal{L}_{\mathcal{W}_{v}}\mathcal{L}_{\mathfrak{t}})\right)}{e(N_{\mathcal{W}_{v}}^{\mathrm{vir}})}\prod_{i=1}^{k}\int_{[\mathcal{W}_{(\Gamma_{i})}]^{\mathrm{vir}}}\frac{-\mathfrak{t}^{4}c_{a_{i}}\left(\pi_{\mathcal{W}_{\Gamma_{i}}*}(\mathcal{L}_{\mathcal{W}_{\Gamma_{i}}}\mathcal{L}_{\mathfrak{t}}(-D_{i}))\right)}{5(-\psi_{\Gamma_{i}}-\psi_{i})e(N_{\mathcal{W}_{(\Gamma_{i})}}^{\mathrm{vir}})}$$

$$= \frac{\prod |\operatorname{Aut}\Gamma_{i}|}{|\operatorname{Aut}\Gamma|} \frac{\mathfrak{t}^{4k+1}}{5^{k}} \sum_{a=0}^{n-1} \int_{[\mathcal{W}_{v}]^{\operatorname{vir}}} \frac{c_{a} \left(\pi_{\mathcal{W}_{v}} (\mathcal{L}_{\mathcal{W}_{v}} \mathcal{L}_{\mathfrak{t}})\right)}{e(N_{\mathcal{W}_{v}}^{\operatorname{vir}})}$$
$$\cdot \operatorname{Coe}_{(\mathfrak{t}')^{a+1}} \prod \int_{[\mathcal{W}_{(\Gamma_{i})}]^{\operatorname{vir}}} \frac{e \left(\pi_{\mathcal{W}_{\Gamma_{i}}} (\mathcal{L}_{\mathcal{W}_{\Gamma_{i}}} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}'}(-D_{i}))\right)}{(\psi_{\Gamma_{i}} + \psi_{i}) e(N_{\mathcal{W}_{(\Gamma_{i})}}^{\operatorname{vir}})}.$$

Since $\mathcal{W}_v = \overline{\mathcal{M}}_{1,k}$ and, over its universal curve, the section ν_1 trivializes $\mathcal{L}_{\mathcal{W}_v}\mathcal{L}_t$, $R\pi_{\mathcal{W}_v*}(\mathcal{L}_{\mathcal{W}_v}\mathcal{L}_t) = R\pi_{\mathcal{W}_v*}\mathcal{O}_{\mathcal{W}_v}$ and $c(R\pi_{\mathcal{W}_v*}(\mathcal{L}_{\mathcal{W}_v}\mathcal{L}_t)) = \frac{1}{1-\lambda} = 1 + \lambda$. Also by (4), over $[\mathcal{W}_v]^{\text{vir}}$ one has

$$\frac{1}{e(N_{\mathcal{W}_{(v)}}^{\mathrm{vir}})} = \left(\frac{\lambda + \mathfrak{t}}{\mathfrak{t}}\right)^5 \cdot \left(\frac{5\mathfrak{t}}{\lambda + 5\mathfrak{t}}\right) = \left(1 + \frac{5\lambda}{\mathfrak{t}}\right) \left(1 - \frac{\lambda}{5\mathfrak{t}}\right) = 1 + \frac{24}{5}\frac{\lambda}{\mathfrak{t}}$$

Let Δ_C be the subset of type C graphs with degree (d, 0) for $d = 1, 2, \cdots$.

Then the total contribution of type C graphs is, using (12),

$$\operatorname{Contri}(C) = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{\mathfrak{t}^4}{5}\right)^k \sum_{a=0}^{1} \int_{\overline{M}_{1,k}} c_a \left(R\pi_{\mathcal{W}_v} \mathscr{L}_{\mathfrak{t}}\right) \left(\mathfrak{t} + \frac{24}{5}\lambda\right) \operatorname{Coe}_{(\mathfrak{t}')^{a+1}} \prod_{i=1}^k \hat{\mathcal{P}}(\psi_i)$$
$$= \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{\mathfrak{t}^4}{5}\right)^k \int_{\overline{M}_{1,k}} \left[\frac{24}{5}\lambda \operatorname{Coe}_{(\mathfrak{t}')^1} \prod_{i=1}^k \hat{\mathcal{P}}(\psi_i) + \lambda \mathfrak{t} \operatorname{Coe}_{(\mathfrak{t}')^2} \prod_{i=1}^k \hat{\mathcal{P}}(\psi_i) + \mathfrak{t} \operatorname{Coe}_{(\mathfrak{t}')^1} \prod_{i=1}^k \hat{\mathcal{P}}(\psi_i)\right].$$

Since $\mathcal{Z}_6^* := \frac{t^4}{5} \tilde{\mathcal{Z}}_{1,5}(\hbar, \mathbb{Q})$ is regularizable(c.f. Lemma A.2), by (12) we have

$$\sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{t^4}{5}\right)^k \int_{\overline{M}_{1,k}} \lambda \prod_{i=1}^k \hat{\mathcal{P}}(\psi_i) = \frac{1}{24} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \left(\frac{t^4}{5}\right)^k \sum_{\substack{\sum_{\ell=1}^k a_\ell = k-1 \\ a_\ell \ge 0}} \binom{k-1}{a_1, \cdots, a_k} \prod_{\ell=1}^k \tilde{\mathcal{P}}_{1+a_\ell}$$
$$= \frac{1}{24} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{\sum_{\ell=1}^k a_\ell = k-1 \\ a_\ell \ge 0}} \prod_{\ell=1}^k \frac{(-1)^{a_\ell}}{a_\ell!} \operatorname{Res}_{\hbar=0} \{\hbar^{-a_\ell} \mathcal{Z}_6^*(\hbar, \Omega)\} = \frac{1}{24} \operatorname{Res}_{\hbar=0} \ln \left(1 + Z_6^*(\hbar, \Omega)\right).$$

Using (36) in Appendix A, the last term in (31) becomes

$$\sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{\mathfrak{t}^4}{5}\right)^k \int_{\overline{M}_{1,k}} \mathfrak{t} \operatorname{Coe}_{\mathfrak{t}'} \hat{\mathcal{P}}(\psi_1) \hat{\mathcal{P}}(\psi_2) \cdots \hat{\mathcal{P}}(\psi_k) = -\frac{\mathfrak{t}}{24} \operatorname{Coe}_{\mathfrak{t}'} \operatorname{Res}_{\hbar=0} \left(\ln(1+\bar{\mathcal{Z}}_6^*))/\hbar \right).$$

Thus, the type C contribution is

$$\operatorname{Contri}(C) = \left(\frac{24}{5}\operatorname{Coe}_{\mathfrak{t}'} + \mathfrak{t}\operatorname{Coe}_{(\mathfrak{t}')^2}\right) \frac{1}{24}\operatorname{Res}_{\hbar=0}\ln(1 + \mathcal{Z}_6^*(\hbar)) - \frac{\mathfrak{t}}{24}\operatorname{Coe}_{\mathfrak{t}'}\operatorname{Res}_{\hbar=0}\left(\hbar^{-1}\ln(1 + \mathcal{Z}_6^*)\right)\right).$$

Using (37), the type C contribution is

$$(\frac{24}{5}\operatorname{Coe}_{\mathfrak{t}'} + \mathfrak{t}\operatorname{Coe}_{(\mathfrak{t}')^2})\frac{\eta}{24} - \frac{\mathfrak{t}}{24}\operatorname{Coe}_{\mathfrak{t}'}\operatorname{Res}_{\hbar=0}(\hbar^{-1}\log(1+\mathfrak{Z}_6^*)))$$

= $-\frac{1}{5}\frac{g_1}{I_0} - \frac{1}{5}\ln(1-5^5q) - \frac{1}{12}\cdot\frac{5^5q}{1-5^5q}.$

8. Graph Type D

$$v \frac{\psi_1}{1} \frac{\psi_\ell}{1} g_v = 1$$

$$-\frac{1}{5} \left| \begin{array}{c} & & \\ & -\frac{1}{5} \\ \psi_1 \\ \psi_\ell \\ \psi_$$

Let $d_i = \deg \Gamma_i$ and $d = d_1 + \ldots + d_\ell$. Each Γ_i is in $\Xi^1_{d_i}$. The $\tilde{\psi}_i$ denotes the ψ -class of the marking on \mathcal{C}_{Γ_i} and ψ_i is the ψ -class of the *i*-th marking on the curve \mathcal{C}_v . The contribution from type D graphs is

$$(31) \sum_{\ell=1} \sum_{d_i>0, 1\leq i\leq \ell} \sum_{(\Gamma_i)\in\hat{\Xi}_{d_i}^1} \frac{(-Q)^{d_1+\dots+d_\ell}}{|\operatorname{Aut}(\{\Gamma_1,\dots,\Gamma_\ell\})|} \int_{\overline{M}_{1,\ell}^{1/5,5p}\times[\mathcal{W}_{(\Gamma_1)}]^{vir}\times\dots\times[\mathcal{W}_{(\Gamma_\ell)}]^{vir}} \mathfrak{t} \cdot \prod_{i=1}^{\ell} \frac{1}{-\mathfrak{t}-\psi_i} \cdot \frac{1}{e_T(R\pi_v(\mathcal{L}_v^{\vee}\mathcal{L}_\mathfrak{t}))} \cdot \frac{(-\mathfrak{t}^5)^{\ell}}{(5\mathfrak{t})^{\ell}} \cdot \prod_{i=1}^{\ell} \left(\frac{1}{5\mathfrak{t}-\tilde{\psi}_i} \cdot \frac{1}{e_T(N_{\mathcal{W}_{\Gamma_1}}^{vir})}\right) \cdot \left(\frac{1}{1-\lambda}c(R\pi_{\mathcal{W}_{\Gamma}}(\mathcal{L}_{\mathcal{W}_{\Gamma}}\mathcal{N}_{\mathcal{W}_{\Gamma}}\mathcal{L}_\mathfrak{t}))\right)_{d-1},$$

where $\overline{M}_{1,\ell}^{1/5,5p}$ is the moduli of 5-spin curves of genus 1 with five *P*-fields and $(\cdots)_{d-1}$ is the degree d-1 component.

Using the variable t', we can rewrite $\left(\frac{1}{1-\lambda}c\left(R\pi_{\mathcal{W}_{\Gamma}*}(\mathcal{L}_{\mathcal{W}_{\Gamma}}\mathcal{N}_{\mathcal{W}_{\Gamma}}\mathcal{L}_{\mathfrak{t}})\right)\right)_{d-1}$ as $\left(\operatorname{Coe}_{\mathfrak{t}'}+\lambda\operatorname{Coe}_{\mathfrak{t}'^{2}}\right)\prod_{i=1}^{\ell}e\left(R\pi_{\mathcal{W}_{\Gamma_{i}}*}(\mathcal{L}_{\mathcal{W}_{\Gamma_{i}}}\mathcal{L}_{\mathfrak{t}}\mathcal{L}_{\mathfrak{t}'})\right).$

Let $h_{\ell} \colon \overline{M}_{1,1^{\ell}}^{1/5,5p} \to \overline{M}_{1,1^{\ell-1}}^{1/5,5p}$ be the contracting map forgetting the last marking point. Let D_i be the boundary divisor of $\overline{M}_{1,1^{\ell}}^{1/5,5p}$ which is the graph of the section of h_{ℓ} induced by ℓ -th marking. For $i \leq \ell - 1$, we have the identity $\psi_i^k = h_{\ell}^* \psi_i^k + (D_i/5)h_{\ell}^* \psi_i^{k-1}$. This implies

$$\frac{1}{\mathfrak{t} + \psi_i} = \frac{1}{\mathfrak{t}} \left(1 - \frac{h_\ell^* \psi_i + D_i/5}{\mathfrak{t}} + \frac{h_\ell^* (\psi_i)^2 + h_\ell^* \psi_i \cdot D_i/5}{\mathfrak{t}^2} + \dots \right) = \frac{1 - D_i/5\mathfrak{t}}{\mathfrak{t} + h_\ell^* \psi_i}$$

We denote $e_T(R\pi_{v*}ev^*\mathcal{L}^{\vee}\mathcal{L}_{\mathfrak{t}})^{-1} = \mathfrak{e}_{\ell}$. It is direct to check $\mathfrak{e}_{\ell} = (-\mathfrak{t} - \psi_{\ell})h_{\ell}^*\mathfrak{e}_{\ell-1}$. Therefore

$$\begin{split} &\int_{\overline{M}_{1,1^{\ell}}^{1/5,5p}} \frac{1}{-\mathfrak{t}-\psi_{1}} \cdots \frac{1}{-\mathfrak{t}-\psi_{\ell}} \mathfrak{e}_{\ell} = \int_{\overline{M}_{1,1^{\ell}}^{1/5,5p}} \frac{1-D_{1}/5\mathfrak{t}}{-\mathfrak{t}-h_{\ell}^{*}\psi_{1}} \cdots \frac{1-D_{\ell-1}/5\mathfrak{t}}{-\mathfrak{t}-h_{\ell}^{*}\psi_{\ell-1}} h_{\ell}^{*}\ell_{\ell-1} \\ &= -\frac{\ell-1}{5\mathfrak{t}} \int_{\overline{M}_{1,1^{\ell-1}}^{1/5,5p}} \frac{1}{-\mathfrak{t}-\psi_{1}} \cdots \frac{1}{-\mathfrak{t}-\psi_{\ell-1}} \mathfrak{e}_{\ell-1} = (-1)^{\ell-1} \frac{(\ell-1)!}{(5\mathfrak{t})^{\ell-1}} \int_{\overline{M}_{1,1}^{1/5,5p}} \frac{1}{-\mathfrak{t}-\psi_{1}} \mathfrak{e}_{1} \\ &= (-1)^{\ell} \frac{(\ell-1)!}{(5\mathfrak{t})^{\ell}} \cdot \frac{128}{3}. \end{split}$$

Here we used the fact that $D_i \cdot D_j = 0$ if $i \neq j$ and a reduction Lemma about FJRW invariants. Using $\lambda = 5\psi_1$ and a computation of a FJRW invariant, we obtain similarly

$$\int_{\overline{M}_{1,1^{\ell}}^{1/5,5p}} \frac{1}{-\mathfrak{t}-\psi_{1}} \cdots \frac{1}{-\mathfrak{t}-\psi_{\ell}} \mathfrak{e}_{\ell} = (-1)^{\ell-1} \frac{(\ell-1)!}{(5\mathfrak{t})^{\ell-1}} \int_{\overline{M}_{1,1}^{1/5,5p}} \frac{-1}{\mathfrak{t}} \Big(1-\frac{\psi_{1}}{\mathfrak{t}}\Big) 5\psi_{1}\mathfrak{e}_{1} = (-1)^{\ell} \frac{(\ell-1)!}{(5\mathfrak{t})^{\ell-1}} \cdot \frac{25}{3} \cdot \frac{(\ell-1)!}{(5\mathfrak{t})^{\ell-1}} + \frac{(\ell-1)!}{(5\mathfrak{t})^{\ell-1}} \cdot \frac{(\ell-1)!}{(5\mathfrak{t})^{\ell-1}} + \frac{(\ell-1)!}{(5\mathfrak{t})^{\ell-1}} \cdot \frac{(\ell-1)!}{(5\mathfrak{t})^{\ell-1}} + \frac{(\ell-1)!}{(5\mathfrak{t})^{\ell-1}} \cdot \frac{(\ell-1)!}{(5\mathfrak{t})^{\ell-1}} \cdot \frac{(\ell-1)!}{(5\mathfrak{t})^{\ell-1}} + \frac$$

Using (37), the type D contribution (31) is

$$\begin{split} &\sum_{\ell=1} \sum_{d_i > 0, 1 \le i \le \ell} \sum_{(\Gamma_i) \in \hat{\Xi}_{d_i}^1} (-\mathfrak{Q})^{d_1 + \ldots + d_\ell} \frac{\mathfrak{t}}{|\operatorname{Aut}(\{\Gamma_1, \ldots, \Gamma_\ell\})|} \cdot \frac{(-\mathfrak{t}^5)^\ell}{(5\mathfrak{t})^\ell} \left((-1)^\ell \frac{(\ell-1)!}{(5\mathfrak{t})^\ell} \cdot \frac{128}{3} \operatorname{Coe}_{\mathfrak{t}} \right) \\ &+ (-1)^\ell \frac{(\ell-1)!}{(5\mathfrak{t})^{\ell-1}} \cdot \frac{25}{3} \operatorname{Coe}_{\mathfrak{t}'^2} \right) \prod_{i=1}^\ell \left(\int_{[\mathcal{W}_{(\Gamma_i)}]^{vir}} \frac{1}{5\mathfrak{t} - \tilde{\psi}_i} \frac{e(R\pi_{\Gamma_i} \ast \mathcal{L}_{\Gamma_i}(-D_{\Gamma_i}) \otimes \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}'})}{e_T(N_{\mathcal{W}_{\Gamma_i}/\mathcal{W}}^{vir})} \right) \\ &= \mathfrak{t} \cdot \frac{128}{3} \sum_{\ell=1}^\infty \frac{1}{\ell} \left(\sum_{d=1}^\infty \sum_{\Gamma \in \Sigma_d^1} (-\mathfrak{Q})^d \cdot \frac{\mathfrak{t}^4}{5} \cdot \int_{[\mathcal{W}_{(\Gamma)}]^{vir}} \frac{1}{5\mathfrak{t} \cdot (5\mathfrak{t} - \tilde{\psi})} \frac{e(R\pi_{\Gamma_i} (\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}'}(-D_{\Gamma)}))}{e_T(N_{\mathcal{W}_{\Gamma}/\mathcal{W}}^{vir})} \right)^\ell |_{\mathfrak{t}'} \\ &+ \frac{125}{3} \mathfrak{t}^2 \cdot \sum_{\ell=1}^\infty \frac{1}{\ell} \left(\sum_{d=1}^\infty \sum_{\Gamma \in \Sigma_d^1} (-\mathfrak{Q})^d \frac{\mathfrak{t}^4}{5} \cdot \int_{[\mathcal{W}_{(\Gamma)}]^{vir}} \frac{1}{5\mathfrak{t} \cdot (5\mathfrak{t} - \tilde{\psi})} \frac{e(R\pi_{\Gamma_i} (\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}'}(-D_{\Gamma})))}{e_T(N_{\mathcal{W}_{\Gamma}/\mathcal{W}}^{vir})} \right)^\ell |_{\mathfrak{t}'^2} \end{split}$$

$$= -\mathfrak{t} \cdot \frac{128}{3} \operatorname{Coe}_{\mathfrak{t}'} \ln(1 + \frac{\mathfrak{t}^4}{5} \tilde{\mathcal{Z}}^*_{1,5}(5\mathfrak{t})) - \frac{125}{3} \mathfrak{t}^2 \operatorname{Coe}_{\mathfrak{t}'^2} \ln(1 + \frac{\mathfrak{t}^4}{5} \tilde{\mathcal{Z}}^*_{1,5}(5\mathfrak{t})) \\ = -\frac{128}{3} \mathfrak{t} \left(-\frac{g_1}{5\mathfrak{t}I_0} \right) = \frac{128}{3} \cdot \frac{g_1}{5I_0}.$$
9. A+B+C+D

Let's recall the results from previous sections about the contributions of graph type A, B, C, and D.

$$\begin{array}{lll} \text{Graphs A:} & \mathcal{F}_1' - \frac{25}{12} \cdot (T'-1) & + \frac{40}{24} \cdot 5\frac{I_0'}{I_0} - \frac{25}{3} \cdot \frac{g_1}{I_0};\\ \text{Graphs B:} & 2\frac{I_0'}{I_0} + \frac{\ln(1-5^5q)}{5} + \frac{1}{2}\frac{T''(t)}{T'(t)};\\ \text{Graphs C:} & -\frac{1}{5}\frac{g_1}{I_0} - \frac{1}{5}\ln(1-5^5e^t) - \frac{1}{12} \cdot \frac{5^5e^t}{1-5^5e^t};\\ \text{Graphs D:} & \frac{128}{3}\frac{g_1}{5I_0}. \end{array}$$

Here $\frac{d}{dt}$ is replaced by '.

Taking the summation of the four contributions above, we get

$$\mathcal{F}_1' - \frac{25}{12} \cdot (T'-1) + \frac{31}{3} \frac{I_0'}{I_0} + \frac{1}{2} \frac{T''(t)}{T'(t)} - \frac{1}{12} \cdot \frac{5^5 e^t}{1 - 5^5 e^t} = 0$$

This implies the formula for $\mathcal{F}_1 = \sum_{d=1}^{\infty} N_{1,d} e^{Td}$ as follows:

(32)
$$\mathcal{F}_1 = \frac{25}{12} \cdot (T-t) - \ln\left(I_0^{\frac{31}{3}} \cdot T'(t)^{\frac{1}{2}} \cdot (1-5^5 e^t)^{\frac{1}{12}}\right)$$

since both side of (32) are power series in q with no q^0 term. It is exactly what A. Zinger obtained in [Zi2].

A. Appendix: Regularity and some identities

A.1. **Regularity.** We prove all the results involving regularities of series here.

Recall the definition of regularizable power series introduced by Zinger in [Zi2]. Let $\mathbb{Q}_t = \mathbb{Q}[t, t^{-1}, t']$.

Definition A.1. A power series $\mathcal{Z}^* = \mathcal{Z}^*(\hbar, q) \in \mathbb{Q}_t(\hbar)[[q]]$ is regularizable at $\hbar = 0$ if there exist power series

$$p = \eta(q) \in \mathbb{Q}_{\mathfrak{t}}[[q]] \quad \text{and} \quad \bar{\mathcal{Z}}^* = \bar{\mathcal{Z}}^*(\hbar, q) \in \mathbb{Q}_{\mathfrak{t}}(\hbar)[[q]]$$

with no degree-zero term in q such that \overline{Z}^* is regular at $\hbar = 0$ and

$$1 + \mathcal{Z}^*(\hbar, q) = e^{\eta(q)/\hbar} \left(1 + \bar{\mathcal{Z}}^*(\hbar, q) \right)$$

If \mathcal{Z}^* is regularizable at $\hbar = 0$, then \mathcal{Z}^* has no degree-zero term in u and the regularizing pair $(\eta, \overline{\mathcal{Z}}^*)$ is unique. It is determined by $\eta(u) = \operatorname{Res}_{\hbar=0} \ln (1 + \mathcal{Z}^*(\hbar, u))$.

Lemma A.1. $\frac{\tilde{\mathcal{Z}}_{0,1}(\mathfrak{t},-\mathfrak{t})}{5} \in \mathbb{Q}_{\mathfrak{t}}(\hbar)[[q]]$ is regularizable.

Proof. From (20), we have

$$1 + \frac{\tilde{\mathcal{Z}}_{0,1}(\mathfrak{t}, -\mathfrak{t})}{5} = e^{\frac{g_1\mathfrak{t}}{I_0\hbar}} \cdot \frac{1}{I_0} \cdot \sum_{d=0}^{\infty} q^d \frac{\hbar^d (5d)!}{(d!)^4 \prod_{m=1}^d (\mathfrak{t} + m\hbar)} := e^{\frac{g_1\mathfrak{t}}{I_0\hbar}} (1 + \overline{\mathcal{Z}}_{0,1}).$$

Then $\frac{g_1 \mathbf{t}}{I_0}$ and $\overline{\mathcal{Z}}_{0,1}$ has no degree-zero terms in q, and $\overline{\mathcal{Z}}_{0,1}$ is regular at $\hbar = 0$.

Lemma A.2. $\mathcal{Z}_6^* := \frac{\mathfrak{t}^4}{5} \tilde{\mathcal{Z}}_{1,5}(\hbar, \mathbb{Q}) \in \mathbb{Q}(\mathfrak{t}, \mathfrak{t}')(\hbar)[[\mathbb{Q}]]$ is regularizable at $\hbar = 0$ *Proof.* The argument in [Zi2, lemm 2.3] applies here.

Thus we can express

(33)
$$1 + \mathcal{Z}_6^*(\hbar) = e^{\eta/\hbar} (1 + \bar{\mathcal{Z}}_6^*(\hbar)), \qquad \eta = \operatorname{Res}_{\hbar=0} \ln(1 + \mathcal{Z}_6^*(\hbar)),$$

where $\bar{\mathbb{Z}}_{6}^{*} \in \mathbb{Q}(\mathfrak{t},\mathfrak{t}')(\hbar)[[\Omega]]\Omega$ is regular at $\hbar = 0$. Mimicking [Zi2, Lemm 2.2(i)(ii)], one can show

Lemma A.3. If (η, \overline{Z}^*) is the regularizing pair for Z^* at $\hbar = 0$, then for every $a \ge 0$,

(34)
$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{\sum \\ \ell=1 \\ a_{\ell} \ge 0}} \left(\prod_{\ell=1}^{m} \frac{(-1)^{a_{\ell}}}{a_{\ell}!} \operatorname{Res}_{\hbar=0} \{ \hbar^{-a_{\ell}} Z^{*}(\hbar, q) \} \right) = \frac{\eta^{a+1}}{a+1}.$$

A.2. Some identities I. Using methods in Zinger's [Zi2, proof of (2.4)], we want to calculate

$$\mathbf{X} := \sum_{m=1}^{\infty} \frac{1}{m!} (\frac{t^4}{5})^m \int_{\overline{M}_{1,m}} \hat{\mathcal{P}}(\psi_1) \hat{\mathcal{P}}(\psi_2) \cdots \hat{\mathcal{P}}(\psi_m) = \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\vec{a}=(a_1,\cdots,a_m)} \vec{a}! \int_{\overline{M}_{1,m}} \psi_1^{a_1} \cdots \psi_m^{a_m} \prod_{\ell=1}^m \frac{(-1)^{a_\ell}}{a_\ell!} \operatorname{Res}_{\hbar=0} \{\hbar^{-a_\ell} \mathcal{Z}_6^*(\hbar,q)\}.$$

Writing $\bar{\mathcal{Z}}_6^* = \sum_{m=0}^{\infty} C_m \hbar^m$, one has

$$\operatorname{Res}_{\hbar=0}\hbar^{a}\mathcal{Z}_{6}^{*} = \sum_{\substack{p-m=1+a\\p,m\geq 0}} \frac{\eta^{p}}{p!}C_{m} + \begin{cases} \frac{\eta^{a+1}}{(a+1)!} & a\geq 0,\\ 0 & \text{otherwise} \end{cases}$$

For fixed $k \ge 0$, let $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$, where $N = \alpha_1 + \dots + \alpha_k := |\alpha|$. Pick a subset of $\mathbb{Z}_{\ge 0}$

(35)
$$m = \{m_1 = \dots = m_{\alpha_1}, m_{\alpha_1+1} = \dots = m_{\alpha_1+\alpha_2}, \dots, m_{\alpha_1+\dots+\alpha_{k-1}+1} = \dots = m_N\}$$

be a sequence of N distinct numbers. We look at the coefficient of $C_m^{\alpha} := C_{m_1}^{\alpha_1} C_{m_{\alpha_1+\alpha_2}}^{\alpha_2} \cdots C_{m_N}^{\alpha_N}$ in **X**. For each $\beta : \{1, \cdots, N\} \to \mathbb{Z}_{\geq 0}$ and a choice of α above, the term C_m^{α} appears in the $n = |\beta| := \beta_1 + \cdots + \beta_N$ summand in **X** with the coefficient

$$\eta^{n-N} \prod_{\ell} (\beta_{\ell}!) \int_{\overline{M}_{1,|\beta|}} (\prod_{s=1}^{N} \psi_{s}^{\beta_{s}}) \prod_{\ell=1}^{N} \frac{(-1)^{\beta_{\ell}}}{\beta_{\ell}!} \frac{\eta^{m_{\ell}+1-\beta_{\ell}}}{(m_{\ell}+1-\beta_{\ell})!}$$
$$= \frac{\eta^{\alpha \cdot m}}{(m+1)!^{\alpha}} \prod_{\ell} (\beta_{\ell}!) \int_{\overline{M}_{1,|\beta|}} (\prod_{s=1}^{N} \psi_{s}^{\beta_{s}}) \prod_{\ell=1}^{N} (-1)^{\beta_{\ell}} \binom{m_{\ell}+1}{\beta_{\ell}},$$

where $\alpha \cdot m := \sum_{\ell} m_{\ell}$ and $(m+1)!^{\alpha} := \prod_{\ell=1}^{N} (m_{\ell}+1)!$. The number of such choices of α is $\binom{|\beta|}{\alpha, |\beta| - \alpha}$. Set $b = |\beta|$ and sum over it. We then have

$$\operatorname{Coe}_{C_m^{\alpha}} \mathbf{X} = \frac{\eta^{\alpha \cdot m}}{(m+1)!^{\alpha}} \sum_{b=0}^{\infty} \frac{(-1)^b}{b!} {b \choose \alpha, b-N} \sum_{\substack{\beta_1 + \dots + \beta_N = b \\ \beta_\ell \ge 0}} \prod_{\ell} (\beta_\ell!) \int_{\overline{M}_{1,b}} \psi_1^{\beta_1} \cdots \psi_N^{\beta_N} \prod_{\ell=1}^N {m_\ell + 1 \choose \beta_\ell}$$
$$= \frac{\eta^{\alpha \cdot m}}{(m+1)!^{\alpha}} \frac{N!}{\alpha!} \sum_{b=N}^{\infty} \frac{(-1)^b}{b!} {b \choose N} \Lambda_b(m_1 + 1, \cdots, m_N + 1),$$

where for $b \ge N$, $r_1, \cdots, r_N \ge 0$, we set

$$\Lambda_b(r_1,\cdots,r_N) := \sum_{\substack{\beta_1+\cdots+\beta_N=b\\r_\ell\geq\beta_\ell\geq 0}} \int_{\overline{M}_{1,b}} \psi_1^{\beta_1}\cdots\psi_N^{\beta_N}\prod_{\ell=1}^N \frac{r_\ell!}{(r_\ell-\beta_\ell)!}.$$

Lemma A.4. Set $\Lambda_b(r_1, \dots, r_N) := 0$ if some $r_i < 0$. Then $\forall b \ge N$, the series $\Lambda_{b+1}(r_1, \dots, r_N)$ is equal to

$$r_1\Lambda_b(r_1-1,r_2,\cdots,r_N)+r_2\Lambda_b(r_1,r_2-1,r_3,\cdots,r_N)+\cdots+r_N\Lambda_b(r_1,r_2,\cdots,r_{N-1},r_N-1).$$

The identity is a simple consequence of the string equation. Using the Lemma, we obtain

$$\begin{split} \frac{N!}{\alpha!} \sum_{b=N}^{\infty} \frac{(-1)^b}{b!} {k \choose N} \Lambda_b(r_1, \cdots, r_N) &= \frac{1}{\alpha!} \sum_{b=N}^{\infty} \frac{(-1)^b}{(b-N)!} \Lambda_b(r_1, \cdots, r_N) \\ &= \frac{(-1)^N}{\alpha!} \sum_{b=0}^{\infty} \frac{(-1)^b}{b!} \Lambda_{b+N}(r_1, \cdots, r_N) \\ &= \frac{(-1)^N}{\alpha!} \sum_{0 \le a_i \le r_i} \frac{(-1)^{|r|-|a|}}{(|r|-|a|)!} {r_1 - a_1, \cdots, r_N - a_N} \frac{r_1!}{a_1!} \cdots \frac{r_N!}{a_N!} \Lambda_N(a_1, \cdots, a_N) \\ &= \frac{(-1)^N}{\alpha!} \sum_{\substack{\beta_1 + \cdots + \beta_N = N \\ r_\ell \ge \beta_\ell \ge 0}} \sum_{\beta_i \le a_i \le r_i} \left((-1)^{|r|-|a|} \frac{1}{(r_1 - a_1)! \cdots (r_N - a_N)!} \right) \\ &\cdot \frac{r_1!}{(a_1 - \beta_1)!} \cdots \frac{r_N!}{(a_N - \beta_N)!} \cdot \int_{\overline{M}_{1,N}} \psi_1^{\beta_1} \cdots \psi_N^{\beta_N} \right) \\ &= \frac{(-1)^N}{\alpha!} \sum_{\substack{\beta_1 + \cdots + \beta_N = N \\ r_\ell \ge \beta_\ell \ge 0}} (-1)^{|\beta| - |r|} \prod_{i=1}^N \frac{r_i!}{(r_i - \beta_i)!} \sum_{\beta_i \le a_i \le r_i} (-1)^{|a| - |\beta|} \prod_{i=1}^N \binom{r_i - \beta_i}{a_i - \beta_i} \int_{\overline{M}_{1,N}} \psi_1^{\beta_1} \cdots \psi_N^{\beta_N} \\ &= \begin{cases} 0 & \text{if } r_1 + r_2 + \cdots + r_N \ne N, \\ \frac{(-1)^N}{\alpha!} r_1! \cdots r_N! \int_{\overline{M}_{1,N}} \psi_1^{r_1} \cdots \psi_N^{r_N} & \text{if } r_1 + r_2 + \cdots + r_N \ne N, \end{cases}$$

where we used $\sum_{\beta_i \leq a_i \leq r_i} (-1)^{a_i - \beta_i} {r_i - \beta_i \choose a_i - \beta_i} = 0$ because $\sum_{k=0}^n (-1)^n {n \choose k} = 0$. The second case implies $r_1 = r_2 = \cdots = r_N = 1, \ k = 1$ and $N = \alpha_1 \in \mathbb{N}$ by (35). Therefore we have

$$\operatorname{Coe}_{C_m^{\alpha}} \mathbf{X} = \frac{(-1)^N}{N!} \int_{\overline{M}_{1,N}} \psi_1 \cdots \psi_N = \frac{(-1)^N}{24N},$$

whenever k = 1 $(N = \alpha_1)$ and |m| = 0, and $\operatorname{Coe}_{C_m^{\alpha}} \mathbf{X} = 0$ otherwise. Taking $\operatorname{Res}_{\hbar=0} \ln$ to (33) we have

(36)
$$\mathbf{X} = \sum_{N=1}^{\infty} \frac{(-1)^N}{24N} C_0^N = -\frac{1}{24} \ln\left(1 + C_0\right) = -\frac{1}{24} \ln(1 + \bar{\mathcal{Z}}_6^*|_{\hbar=0}) = -\frac{1}{24} \operatorname{Res}_{\hbar=0} \left(\ln(1 + \bar{\mathcal{Z}}_6^*))/\hbar\right).$$

A.3. Some identities II. We now determine $\operatorname{Coe}_{(\mathfrak{t}')^i} \ln(1 + \mathfrak{Z}_6^*)$ for i = 1, 2. By (14), we have

$$\mathcal{Z}_{6}^{*} = \frac{\mathfrak{t}^{4}}{5}\tilde{\mathcal{Z}}_{1,5}(\hbar) = -1 + \frac{1}{I_{0}}exp(-\frac{\mathfrak{t}}{\hbar}(t-T) - \frac{\mathfrak{t}'g_{1}}{\hbar I_{0}}) \cdot \sum_{d=0}^{\infty} q^{d} \frac{\prod_{m=1}^{5d}(-5\mathfrak{t} + m\hbar)\prod_{m=1}^{d}(m + \frac{\mathfrak{t}'}{\hbar})}{d!\prod_{m=1}^{d}(-\mathfrak{t} + m\hbar)^{5}}.$$

As \mathcal{Z}_6^* has no *q*-degree zero term, so is $\ln(1 + \mathcal{Z}_6^*)$. Together with the fact that \mathcal{Z}_6^* is a formal power series in \mathfrak{t}' , we know $\ln(1 + \mathcal{Z}_6^*)$ is also a formal power series in \mathfrak{t}' .

Lemma A.5. Let *R* be a ring and $A_j \in R[[q]]q^j$, $j = 0, 1, 2, 3, \cdots$. Then $\ln(1 + A_0 + A_1 \mathfrak{t}' + A_2(\mathfrak{t}')^2 + \cdots)$ $= \ln(1 + A_0) + \frac{A_1}{1 + A_0} \mathfrak{t}' + \left(\frac{A_2}{1 + A_0} - \frac{A_1^2}{2(1 + A_0)^2}\right)(\mathfrak{t}')^2 + higher \text{ powers of } \mathfrak{t}'$ Introduce $\Re(\hbar, \mathfrak{t}) := \sum_{d=0}^{\infty} q^d \frac{\prod_{m=1}^{5d} (-5\mathfrak{t} + m\hbar)}{\prod_{m=1}^d (-\mathfrak{t} + m\hbar)^5}$ and observe first $\sum_{d=1}^{\infty} q^d \frac{\prod_{m=1}^{5d} (-5\mathfrak{t} + m\hbar) \prod_{m=1}^d (m + \frac{\mathfrak{t}'}{\hbar})}{d! \prod_{m=1}^d (-\mathfrak{t} + m\hbar)^5} = (\Re - 1) + C_1 \frac{\mathfrak{t}'}{\hbar} + C_2 \frac{(\mathfrak{t}')^2}{\hbar^2} + \text{higher powers of } \mathfrak{t}',$

where

$$C_{1} = \sum_{d=1}^{\infty} q^{d} \frac{\prod_{m=1}^{5d} (-5\mathfrak{t} + m\hbar) \sum_{m=1}^{d} \frac{1}{m}}{\prod_{m=1}^{d} (-\mathfrak{t} + m\hbar)^{5}} \quad \text{and} \quad C_{2} = \sum_{d=2}^{\infty} q^{d} \frac{\prod_{m=1}^{5d} (-5\mathfrak{t} + m\hbar) \sum_{1 \le a < b \le d} \frac{1}{ab}}{\prod_{m=1}^{d} (-\mathfrak{t} + m\hbar)^{5}}.$$

Applying Lemma A.5 and above observations to the last term of

$$\ln(1+\mathcal{Z}_{6}^{*}) = -\ln I_{0} - \frac{\mathbf{t}}{\hbar}(t-T) - \frac{\mathbf{t}'g_{1}}{\hbar I_{0}} + \ln\left(1 + \sum_{d=1}^{\infty} q^{d} \frac{\prod_{m=1}^{5d} (-5\mathbf{t} + m\hbar) \prod_{m=1}^{d} (m + \frac{\mathbf{t}'}{\hbar})}{d! \prod_{m=1}^{d} (-\mathbf{t} + m\hbar)^{5}}\right),$$

we obtain

$$\operatorname{Coe}_{(\mathfrak{t}')^1} \ln(1 + \mathcal{Z}_6^*) = -\frac{g_1}{\hbar I_0} + \frac{C_1}{\hbar \mathcal{R}}, \quad \operatorname{Coe}_{(\mathfrak{t}')^2} \ln(1 + \mathcal{Z}_6^*) = \frac{1}{\hbar^2} \left(\frac{C_2}{\mathcal{R}} - \frac{C_1^2}{2\mathcal{R}^2}\right).$$

For simplicity, denote and find

$$A := \sum_{d=1}^{\infty} (5^5 q)^d \sum_{m=1}^d \frac{1}{m}, \text{ and } B := \sum_{d=1}^{\infty} (5^5 q)^d \sum_{1 \le a < b \le d} \frac{1}{ab}.$$
$$A(1-5^5 q) = -\ln(1-5^5 q) \text{ and } B(1-5^5 q) = \sum_{d=2}^{\infty} (5^5 q)^d \frac{1}{d} \sum_{a=1}^{d-1} \frac{1}{a}.$$

Then

$$C_1 = A + \frac{\hbar}{\mathfrak{t}} \cdot 2\frac{d}{dt}A + \text{higher powers of }\hbar, \text{ and } C_2 = B + \frac{\hbar}{\mathfrak{t}} \cdot 2\frac{d}{dt}B + \text{higher powers of }\hbar,$$

$$\mathcal{R}(\hbar,\mathfrak{t}) = \sum_{d=0}^{\infty} (5^5q)^d + \frac{\hbar}{\mathfrak{t}} \sum_{d=0}^{\infty} 2d(5^5q)^d + \text{higher powers of } \hbar = \frac{1}{1-5^5q} + \frac{\hbar}{\mathfrak{t}} \frac{d}{dt} \frac{2}{1-5^5q} (\bmod \hbar^2),$$

where the expansion is near $\hbar = 0$. We can now calculate the followings:

(37)

$$Coe_{(t')^{0}}Res_{\hbar=0}\ln(1+\mathcal{Z}_{6}^{*}) = Res_{\hbar=0}\left(Coe_{(t')^{0}}\ln(1+\mathcal{Z}_{6}^{*})\right) = -\mathfrak{t}(t-T),$$

$$Coe_{(t')^{1}}Res_{\hbar=0}\frac{\ln(1+\mathcal{Z}_{6}^{*})}{\hbar} = Res_{\hbar=0}\frac{C_{1}}{\hbar^{2}\mathcal{R}} = \frac{2}{\mathfrak{t}}\frac{d}{dt}(A(1-5^{5}q)) = \frac{2}{\mathfrak{t}} \cdot \frac{5^{5}q}{1-5^{5}q},$$

$$Coe_{(t')^{1}}\eta = Res_{\hbar=0}Coe_{(t')^{1}}\ln(1+\mathcal{Z}_{6}^{*}) = Res_{\hbar=0}\left(-\frac{g_{1}}{\hbar I_{0}} + \frac{C_{1}}{\hbar\mathcal{R}}\right)$$

$$\begin{aligned} &= -\frac{g_1}{I_0} + A(1-5^5q) = -\frac{g_1}{I_0} - \ln(1-5^5q), \\ &\quad \operatorname{Coe}_{(\mathfrak{t}')^2}\eta = \operatorname{Res}_{\hbar=0}\operatorname{Coe}_{(\mathfrak{t}')^2}\ln(1+\mathcal{Z}_6^*) = \operatorname{Res}_{\hbar=0}(\frac{1}{\hbar^2}(\frac{C_2}{\mathcal{R}} - \frac{C_1^2}{2\mathcal{R}^2})) \\ &= \frac{2}{\mathfrak{t}}\frac{dB(1-5^5q)}{dt} + \frac{1}{2}\frac{4}{\mathfrak{t}}A(1-5^5q)\frac{dA(1-5^5q)}{dt} \\ &= \frac{2}{\mathfrak{t}}\sum_{d=2}^{\infty}(5^5q)^d\sum_{a=1}^{d-1}\frac{1}{a} + \frac{2}{\mathfrak{t}}\frac{5^5q}{1-5^5q}\ln(1-5^5q) = 0. \end{aligned}$$

B. APPENDIX: DIFFERENTIAL RELATIONS ARISING FROM GEOMETRY

We shall derive some differential relations among periods from geometry in such a way that we keep track of the arithmetic origins of the coefficients in the relations. These relations are needed to prove Proposition C.4 for type B contribution studied in g = 1 MSP localization. We separate the discussions into this appendix since the relations obtained seem to be of independent interest.

The discussion applies to any one-parameter family of Calabi-Yau threefolds, but the focus of this work is on the quintic/mirror quintic family. The corresponding differential operator is

$$\mathcal{L} = \theta^4 - \alpha \prod_{k=1}^4 (\theta + \frac{k}{5}),$$

where $\alpha = 5^5 q = 5^5 e^t$ and $\theta = \alpha \frac{\partial}{\partial \alpha}$. This can be regarded as the Picard-Fuchs operator for the mirror quintic family, or the quantum differential equation for the quintic family itself. The properties are about the differential operator itself, regardless of whether we are looking at the A-model or B-model. However, we shall use the language for the B-model, which is more convenient.

We first expand the above operator as

$$\mathcal{L}_{\mathrm{hyp}} = \theta^4 - \alpha \sum_{k=0}^3 \sigma_{4-k} \theta^k,$$

where σ_k means the kth fundamental symmetric polynomials of the numbers 1/5, 2/5, 3/5, 4/5. For the quintic/mirror quintic case in consideration, one has

(38)
$$\sigma_1 = 2, \quad \sigma_2 = \frac{7}{5}, \quad \sigma_3 = \frac{2}{5}, \quad \sigma_4 = \frac{24}{5^4}.$$

Introducing the coordinate $t = \ln q$, then the above differential operator becomes

(39)
$$\mathcal{L}_{\text{hyp}} = (1-\alpha)\partial_t^4 - \alpha \sum_{k=0}^3 \sigma_{4-k}\partial_t^k \, .$$

Introducing the notation (recall $\alpha = 5^5 q = 5^5 e^t$)

(40)
$$C_{ttt} = \frac{1}{1-\alpha} = \frac{1}{1-5^5q}, \quad C = \partial_t \ln C_{ttt} = \frac{\alpha}{1-\alpha} = \frac{5^5q}{1-5^5q}$$

then we can normalize the leading term of the differential operator to 1 and get

$$\mathcal{L} := \frac{1}{1 - \alpha} \mathcal{L}_{\text{hyp}} = \partial_t^4 + \sum_{k=0}^3 a_k \partial_t^k, \quad a_k = -\sigma_{4-k} C.$$

The 4 solutions of the corresponding differential equation obtained by the Frobenius method near the singular point $\alpha = 0$ are denoted by (I_0, I_1, I_2, I_3) . In particular, one has

(41)
$$I_0 = {}_4F_3(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; 1, 1, 1; \alpha), \quad I_1 = \log(\frac{\alpha}{5^5}) \cdot I_0 + \cdots.$$

Then one has

$$T := \frac{I_1}{I_0} = t + \cdots .$$

In the B-model, the operator \mathcal{L}_{hyp} is the Picard-Fuchs operator, and the four solutions above are the periods of the mirror quintic. In the A-model, they are the first four coefficients in the *H*-expansion of the *I*-function $e^{Hq}J$, where *H* is the hyperplane class of \mathbb{P}^4 and *J* is the *J*-function.

B.1. Special geometry relation. The set of periods has the special geometry structure [S, F] which implies that there exists a holomorphic function F (called the prepotential) of $T = I_1/I_0$, such that the set of periods above has the following structure

(42)
$$(I_0, I_1, I_2, I_3) = I_0(1, T, F_T, TF_T - 2F).$$

This fact can be easily proved by using Griffiths transversality. This structure in fact holds everywhere, but we shall only focus on a neighborhood near the point $\alpha = 0$ near which we have

(43)
$$F(T) = \frac{1}{3!}T^3 + \text{quadratic polynomial in } T + \mathcal{O}(e^T)$$

It follows from this structure (called the special geometry structure) that the 4 solutions I_k , k = 0, 1, 2, 3 above, after normalization by the fundamental period I_0 , are solutions to the following equation (see [CDFLL, CDFLLR])

$$\partial_T^2 F_{TTT}^{-1} \partial_T^2 (\frac{I_k}{I_0}) = 0, \quad k = 0, 1, 2, 3$$

By comparing the leading terms, we can see that the Picard-Fuchs operator (39) must satisfy the relation

$$(44) \qquad \qquad \mathcal{L} = \mathcal{D}$$

where

$$\mathcal{D} := \left(\frac{1}{I_0} F_{TTT}^{-1} (\frac{\partial t}{\partial T})^4\right)^{-1} \circ \partial_T^2 F_{TTT}^{-1} \partial_T^2 \circ \frac{1}{I_0}.$$

We now study the consequences of this identity by using the "computing twice" trick. We shall see that this identity (44) gives rise to some differential relations.

First, it is easy to see that we can rewrite \mathcal{D} as

$$\mathcal{D} = \left(\partial_t + \partial_t \log F_{TTT}^{-1} + \partial_t \log(\frac{\partial t}{\partial T})^3 + \partial_t \log\frac{1}{I_0}\right) \circ \left(\partial_t + \partial_t \log F_{TTT}^{-1} + \partial_t \log(\frac{\partial t}{\partial T})^2 + \partial_t \log\frac{1}{I_0}\right)$$

$$(45) \quad \circ \left(\partial_t + \partial_t \log(\frac{\partial t}{\partial T}) + \partial_t \log\frac{1}{I_0}\right) \circ \left(\partial_t + \partial_t \log\frac{1}{I_0}\right).$$

For convenience, we introduce the following notations

$$c_{1} = \partial_{t} \log \frac{1}{I_{0}}, \quad c_{2} = \partial_{t} \log(\frac{\partial t}{\partial T}) + \partial_{t} \log \frac{1}{I_{0}},$$

$$(46) \qquad c_{3} = \partial_{t} \log F_{TTT}^{-1} + \partial_{t} \log(\frac{\partial t}{\partial T})^{2} + \partial_{t} \log \frac{1}{I_{0}}, \quad c_{4} = \partial_{t} \log F_{TTT}^{-1} + \partial_{t} \log(\frac{\partial t}{\partial T})^{3} + \partial_{t} \log \frac{1}{I_{0}}$$

They satisfy the obvious relation

$$(47) c_2 - c_1 = c_4 - c_3 \, .$$

Straightforward computations show that¹

$$\mathcal{D} = \partial_t^4 + b_3 \partial_t^3 + b_2 \partial_t^2 + b_1 \partial_t + b_0 \,,$$

¹Hereafter ' means ∂_t and similar convention is used for " etc.

where

$$b_{3} = \sum_{k} c_{k}, \quad b_{2} = \sum_{k \neq l} c_{k}c_{l} + 3c'_{1} + 2c'_{2} + c'_{3},$$

$$b_{1} = \sum_{j \neq k \neq l} c_{j}c_{k}c_{l} + (2c_{2} + 2c_{3} + 2c_{4})c'_{1} + (2c_{1} + c_{3} + c_{4})c'_{2} + (c_{1} + c_{2})c'_{3} + 3c''_{1} + c''_{2},$$

$$b_{0} = c_{1}c_{2}c_{3}c_{4} + (c_{2}c_{3} + c_{2}c_{4} + c_{3}c_{4})c'_{1} + (c_{1}c_{3} + c_{1}c_{4})c'_{2} + 2c'_{1}c'_{2} + c_{1}c_{2}c'_{3}$$

$$(48) \qquad \qquad +c'_{1}c'_{3} + c_{2}c''_{1} + c_{3}c''_{1} + c_{4}c''_{1} + c_{1}c''_{2} + c'''_{1}.$$

B.1.1. Matching degree three terms: Yukawa coupling F_{TTT} in terms of period integrals. The matching of the coefficients of ∂_t^3 yields $b_3 = a_3$ which according to (48) is

$$c_1 + c_2 + c_3 + c_4 = a_3$$

Plugging in the relation $a_k = -\sigma_{4-k}C$ and (46), we get the relation

(49)
$$\partial_t \log F_{TTT} + 2\partial_t \log I_0 + 3\partial_t \log \frac{\partial T}{\partial t} = -\frac{1}{2}a_3 = \sigma_1 C$$

This gives a relation between the transcendental series $I_0, \frac{\partial T}{\partial t}$ etc., and the rational function C.

For later use, we introduce furthermore

(50)
$$A = \partial_t \log \frac{\partial T}{\partial t}, \quad B = \partial_t \log I_0, \quad Y = \partial_t \log F_{TTT}.$$

The above relation is then (recall (40))

$$Y + 2B + 3A = \frac{1}{2}\sigma_1 C \,.$$

That is

$$F_{TTT} = \frac{1}{(1-\alpha)^{\frac{\sigma_1}{2}}} \frac{1}{I_0^2} \frac{1}{T'^3} \cdot const$$

for some constant *const* which can be fixed to be 1 by using the boundary conditions in (41) and the known asymptotic behaviour of the the Yukawa coupling F_{TTT} following from (43).

B.1.2. Matching degree two terms: flatness of Gauss-Manin. For the next coefficients, we have $b_2 = a_2$. By (48) this is

$$\sum_{k \neq l} c_k c_l + 3c'_1 + 2c'_2 + c'_3 = -\sigma_2 C \,.$$

Using the b_3 -relation to eliminate c_3 , we can see that the above relation becomes

(51)
$$-A' - 4B' - C' - A^2 - 2B^2 + 2BC + C^2 - 2AB + AC = -\sigma_2 C + C^2 - 2AB + C^2 - 2AB + AC = -\sigma_2 C + C^2 - 2AB + AC = -\sigma_2 C + C^2 - 2AB + C^2 + C^2 - 2AB + C^2 + C$$

This relation is discussed in [LY] by using the Wronskian method. It is further studied in [YY, H, Z]. In particular, as pointed out explicitly in [YY, Z], this relation is equivalent to the flatness of the Gauss-Manin connection on the Hodge bundle. Note that the above differential relation about A, B, C is in fact an identity without referring to the prepotential, although in eliminating c_3 above we have used the b_3 -relation whose definition does require the existence of prepotential. In the next section, we shall see that this relation can be naturally rephrased in terms of Wronskians, again without using the prepotential, which offers a more intrinsic way of explaining what this relation is. B.1.3. *Matching degree one terms: symplectic structure.* We now match the degree one terms in (44). By straightforward calculation from (48), this is equivalent to

(52)
$$b_1 = b'_2 + \frac{1}{2}b_2b_3 - \frac{1}{8}b_3^3 - \frac{1}{2}b''_3 - \frac{3}{4}b_3b'_3.$$

This relation is studied in details in [CDFLL, CDFLLR, LY, AZ]. In particular, as pointed out in [CDFLL, CDFLLR], this relation means that the Gauss-Manin connection is symplectic. Note that, in this relation, the c_k 's are combined in such a way that they do not appear individually. Hence this relation does not give new differential relations among them.

B.1.4. Matching degree zero terms: Picard-Fuchs for fundamental period. Consider the degree zero terms in (44). By using the relation $b_3 = a_3$ and (48), we get an ODE for c_1 :

(53)
$$b_0 = c_1''' - c_1^4 + 6c_1'c_1^2 - 4c_1c_1'' - 3c_1'c_1' + b_3(c_1'' - 3c_1c_1' + c_1^3) + b_2(c_1' - c_1^2) + b_1c_1.$$

On the other hand, the Picard-Fuchs equation for the fundamental period I_0 reads (recall $B = \partial_t \log I_0$)

$$0 = \frac{I_0''''}{I_0} + a_3 \frac{I_0''}{I_0} + a_2 \frac{I_0''}{I_0} + a_1 \frac{I_0'}{I_0} + a_0.$$

Recall that $c_1 = -B$, we can see that the Picard-Fuchs equation for I_0 above is exactly the relation $b_0 = a_0$.

B.1.5. A summary. We remark that the b_0 -relation (53) is always true, independent of the geometry structure. The b_1 -structure (52) is the symplectic structure, which can be further translated into relations on the periods, see the discussion in the next section below. It does not refer to the prepotential, as we have pointed out earlier that it is not a relation on the individual c_k 's. The b_2 -relation (51), the flatness condition, can also be phrased without referring to the prepotential, which can also be seen from the explicit expression in terms of Wronskians discussed in the next section. Finally, the b_3 -relation (49) gives an identity that the prepotential satisfies, in case the special geometry structure is present.

B.2. Wronskian method. The Wronskian method and the exterior square structure are useful in simplifications and in thinking about what the quadratic differential polynomials in MSP calculation are. A computational evidence is that the derivative of a normalized period is naturally expressed in terms of Wronskians.

It is shown in [AZ] that for a generic order 4 differential equation \mathcal{L} , the Wronskians

$$M_{ab} = \det \begin{pmatrix} I_a & I_b \\ I'_a & I'_b \end{pmatrix}$$

satisfies an order 6 differential equation. This is the exterior square of \mathcal{L} . We now review the results in [AZ], which are important for the proof of the main result in this work. We denote

$$u_{1} = M_{ab}, \quad u_{2} = M'_{ab} = \det \begin{pmatrix} I_{a} & I_{b} \\ I''_{a} & I''_{b} \end{pmatrix},$$
$$u_{3} = \det \begin{pmatrix} I_{a} & I_{b} \\ I'''_{a} & I''_{b} \end{pmatrix}, \quad u_{4} = \det \begin{pmatrix} I'_{a} & I'_{b} \\ I''_{a} & I''_{b} \end{pmatrix},$$
$$u_{5} = \det \begin{pmatrix} I'_{a} & I'_{b} \\ I'''_{a} & I''_{b} \end{pmatrix}, \quad u_{6} = \det \begin{pmatrix} I''_{a} & I''_{b} \\ I'''_{a} & I'''_{b} \end{pmatrix},$$

Recall that the I_a 's are annihilated by the order 4 differential operator

$$\mathcal{L} = \partial^4 + a_3 \partial^3 + a_2 \partial^2 + a_1 \partial + a_0$$
.

Then the set $\{u_k, k = 1, 2, \dots 6\}$ satisfy the following relations

$$\begin{array}{rcl} u_1' &=& u_2\,, & u_2' = u_3 + u_4\,, & u_3' = u_5 - a_1 u_1 - a_2 u_2 - a_3 u_3\,, \\ (54) & u_4' &=& u_5\,, & u_5' = u_6 + a_0 u_1 - a_2 u_4 - a_3 u_5\,, & u_6' = a_0 u_2 + a_1 u_4 - a_3 u_6\,. \end{array}$$

B.2.1. Symplectic structure. It is shown in [AZ] that the above system gives rise to an order 6 differential equation for u_1 . The solutions are M_{ab} , $a, b = 1, 2, \dots 4$. When the symplectic structure (52)

(55)
$$a_1 = \frac{1}{2}a_2a_3 - \frac{1}{8}a_3^3 + a_2' - \frac{3}{4}a_3a_3' - \frac{1}{2}a_3''$$

is present, the order 6 differential equation becomes an order 5 differential equation. We now recall the computations therein. Define

$$U = u_1''' + a_3 u_1'' + a_2 u_1' + a_1 u_1,$$

then

(56)
$$U - a_3 u_4 = 2u_5 \,.$$

Taking the derivative, one gets

$$U' - a'_{3}u_{4} - a_{3}u_{5} = 2u_{6} + 2a_{0}u_{1} - 2a_{2}u_{4} - 2a_{3}u_{5}.$$

Eliminating u_5 using (56), we obtain

(57)
$$U' + \frac{1}{2}a_3U = u_4(a'_3 + \frac{1}{2}a_3^2 - 2a_2) + 2a_0u_1 + 2u_6 := C_1u_4 + 2a_0u_1 + 2u_6.$$

Taking one more derivative and eliminating u_5 using (56), we obtain

$$U'' + \frac{1}{2}a_3'U + \frac{1}{2}a_3U' = C_1'u_4 + \frac{1}{2}C_1(U - a_3u_4) + 2a_0'u_1 + 2a_0u_1' + 2(a_0u_1' + a_1u_4 - a_3u_6).$$

Using (57) to eliminate u_6 , we obtain

$$U'' + \frac{1}{2}a'_{3}U + \frac{1}{2}a_{3}U'$$

= $C'_{1}u_{4} + \frac{1}{2}C_{1}(U - a_{3}u_{4}) + 2a'_{0}u_{1} + 4a_{0}u'_{1} + 2a_{1}u_{4} - a_{3}(U' + \frac{1}{2}a_{3}U - C_{1}u_{4} - 2a_{0}u_{1})$

Simplifying, one has

$$W := U'' + \frac{1}{2}a'_{3}U + \frac{3}{2}a_{3}U' - \frac{1}{2}C_{1}U + \frac{1}{2}a_{3}U - 2a'_{0}u_{1} - 4a_{0}u'_{1} - 2a_{0}a_{3}u_{1}$$
$$= C'_{1}u_{4} + \frac{3}{2}C_{1}a_{3}u_{4} + 2a_{1}u_{4} := C_{2}u_{4}$$

If the condition $C_2 = 0$ is satisfied, then we are done. It turns out that this condition is exactly the symplectic structure relation (55). Otherwise, we get

$$W' = C_2 u_4' + C_2' u_4 = C_2 u_5 + C_2' u_4 = \frac{1}{2} C_2 (U - a_3 u_4) + C_2' u_4 = \frac{1}{2} C_2 U + (C_2' - \frac{1}{2} C_2 a_3) u_4.$$

Hence

(58)
$$W' = \frac{1}{2}C_2U + (C'_2 - \frac{1}{2}C_2a_3)\frac{W}{C_2}.$$

This gives an order 6 ODE as desired.

B.2.2. Simplication of Wronskians in the presence of symplectic structure. As shown in [AZ], the symplectic structure condition (55) is equivalent to either of the following relations

$$\begin{array}{rcl} \displaystyle \frac{\partial^2}{\partial T^2} \frac{I_3}{I_0} &=& T \frac{\partial^2}{\partial T^2} \frac{I_2}{I_0} \,, \\ \\ \displaystyle \frac{\partial^2}{\partial T^2} \frac{I_2}{I_0} &=& \displaystyle \frac{1}{I_0^2 (\partial_t T)^3} \exp(-\frac{1}{2} \int a_3) \,. \end{array}$$

This condition is evidently satisfied if the stronger special geometry structure (42) holds, as it should be. In this case, the second condition above condition is nothing but the b_3 -relation in (49). Note that this should be thought of as a relation on b_1 instead of one on b_3 , since in general there is no notion of prepotential but one can always talk about the condition above. In terms of Wronskians, this is equivalent to

$$W = U'' + \frac{1}{2}a_3'U + \frac{3}{2}a_3U' - \frac{1}{2}C_1U + \frac{1}{2}a_3U - 2a_0'u_1 - 4a_0u_1' - 2a_0a_3u_1 = 0,$$

where

$$C_1 = (a'_3 + \frac{1}{2}a_3^2 - 2a_2), \quad U - a_3u_4 = 2u_5.$$

We now focus on the relation

$$\frac{\partial^2}{\partial T^2} \frac{I_3}{I_0} = T \frac{\partial^2}{\partial T^2} \frac{I_2}{I_0} \,.$$

above. Integrating, one gets

$$\frac{\partial}{\partial T}\frac{I_3}{I_0} = T\frac{\partial}{\partial T}\frac{I_2}{I_0} - \frac{I_2}{I_0} + const = T^2\frac{\partial}{\partial T}(\frac{1}{T}\frac{I_2}{I_0}) + const,$$

for some constant *const*. Using the boundary conditions given in (41), (42), (43) (in fact, only the log α terms) for the periods I_2, I_3 one can see that const = 0.

Now since

$$\frac{\partial}{\partial T} = \frac{1}{T'} \frac{\partial}{\partial t} \,,$$

the above is equivalent to

$$\frac{\partial}{\partial t} \left(\frac{I_3}{I_0} \right) = T^2 \frac{\partial}{\partial t} \left(\frac{1}{T} \frac{I_2}{I_0} \right) = T^2 \frac{\partial}{\partial t} \left(\frac{I_2}{I_1} \right).$$

 $I_0 I'_3 - I'_0 I_3 = I_1 I'_2 - I'_1 I_2$.

This gives

(59)

That is,

$$\det \begin{pmatrix} I_0 & I_3 \\ I'_0 & I'_3 \end{pmatrix} = \det \begin{pmatrix} I_1 & I_2 \\ I'_1 & I'_2 \end{pmatrix} .$$

This is why the order 6 differential equation (58) for the Wronskians, which has 6 solutions M_{ab} , becomes an order 5 differential equation.

B.2.3. Flatness condition in terms of Wronskians in the present of symplectic structure. The exterior square structure above, which is the underlying geometry of Wronskians, provides a nice way to express the identities we found above by matching coefficients in (44).

We have done this for the symplectic structure relation in terms of

$$\frac{\partial^2}{\partial T^2} \frac{I_3}{I_0} = T \frac{\partial^2}{\partial T^2} \frac{I_2}{I_0},$$

or equivalently in terms of the Wronskians

$$I_0 I'_3 - I'_0 I_3 = I_1 I'_2 - I'_1 I_2$$
.

We now do this for the flatness condition without assuming the symplectic structure. We specialize to M_{ab} with a = 0, b = 1 and again use the following convention $u_k, k = 1, 2, \dots 6$ for the Wronskians. One has the relations

$$u_2' = u_3 + u_4$$
,

and also

$$\frac{u_2'}{u_1} = \left(\frac{u_2}{u_1}\right)' + \left(\frac{u_2}{u_1}\right)^2.$$

The latter is due to $u_2 = u'_1$. We write them in terms of the coefficients c_k in (46) and the Wronskians. From $c_1 = -B, c_2 = -A - B$, it is easy to see that

$$\frac{u_2}{u_1} = -(c_1 + c_2), \quad \left(\frac{u_2}{u_1}\right)' = -(c_1' + c_2'), \quad \frac{u_3}{u_1} = -2c_1' - c_2' + 3c_1c_2 + (c_1 - c_2)^2, \quad \frac{u_4}{u_1} = c_1' + c_1c_2, \\ \frac{u_4 + u_3}{u_1} = \frac{u_2'}{u_1} = \left(\frac{u_2}{u_1}\right)' + \left(\frac{u_2}{u_1}\right)^2 = -(c_1' + c_2') + (c_1 + c_2)^2, \quad \frac{u_4 - u_3}{u_1} = 3c_1' + c_2' - (c_1^2 + c_2^2).$$

Recall that the b_2 , b_3 -terms in (48) are

$$b_2 = \frac{1}{2}((\sum c_k)^2 - \sum c_k^2) + 3c_1' + 2c_2' + c_3', \quad b_3 = \sum c_k.$$

Using the relation (47), we can solve for c_3, c_4 as follows

$$c_3 = \frac{1}{2}b_3 - c_2, \quad c_4 = \frac{1}{2}b_3 - c_1$$

Plugging these into the b_2 -coefficient, we have

$$b_2 = \frac{1}{2}b_3^2 - \frac{1}{2}(c_1^2 + c_2^2 + c_3^2 + c_4^2) + 3c_1' + 2c_2' + c_3'$$

= $\frac{1}{4}b_3^2 - (c_1^2 + c_2^2) + \frac{1}{2}b_3(c_1 + c_2) + \frac{1}{2}b_3' + 3c_1' + c_2'$.

In terms of Wronskians, we have

$$c'_1 = \frac{u_4}{u_1} - c_1 c_2$$
, $2c'_1 + c'_2 = -\frac{u_3}{u_1} + 3c_1 c_2 + (c_1 - c_2)^2$.

Eliminating the derivatives from the b_2 -expression, we get

$$b_{2} = \frac{1}{4}b_{3}^{2} - (c_{1}^{2} + c_{2}^{2}) + \frac{1}{2}b_{3}(c_{1} + c_{2}) + \frac{1}{2}b_{3}' + 3c_{1}' + c_{2}'$$

$$= \frac{1}{4}b_{3}^{2} - (c_{1}^{2} + c_{2}^{2}) + \frac{1}{2}b_{3}(c_{1} + c_{2}) + \frac{1}{2}b_{3}' + \frac{u_{4}}{u_{1}} - \frac{u_{3}}{u_{1}} + 2c_{1}c_{2} + (c_{1} - c_{2})^{2}$$

$$= \frac{1}{4}b_{3}^{2} + \frac{1}{2}b_{3}(c_{1} + c_{2}) + \frac{1}{2}b_{3}' + \frac{u_{4} - u_{3}}{u_{1}}.$$

Now using the relation $u_2/u_1 = -(c_1 + c_2)$ above, this gives

$$-\frac{1}{2}b_3\frac{u_2}{u_1} + \frac{u_4 - u_3}{u_1} = b_2 - \frac{1}{2}b_3' - \frac{1}{4}b_3^2$$

We hence get the flatness condition in terms of Wronskians u_1, u_2, u_3, u_4 :

(60)
$$(u_4 - u_3) = (b_2 - \frac{1}{2}b'_3 - \frac{1}{4}b^2_3)u_1 + \frac{1}{2}b_3u_2 + \frac{1}{2}b_3u_3 + \frac$$

Substituting the relations $b_3 = a_3 = -\sigma_1 C = -2C$ and $b_2 = a_2 = -\sigma_2 C$, recall that $C = \frac{\alpha}{1-\alpha}$ satisfies $C' = C^2 + C$.

we then get

$$(u_4 - u_3) = -C\left((\sigma_2 - 1)u_1 + \frac{1}{2}\sigma_1 u_2\right).$$

That is,

$$5(1-\alpha)(u_4-u_3) = -\alpha (2u_1+5u_2),$$

as $\sigma_2 = 7/5$ from (38). It is easy to check that this equation is the same as (51), as it should be. Note that here we did not use the b_1 -relation (52), hence no symplectic structure is used.

Remark B.1. The quantity $C = \partial_t \ln C_{ttt}$ is independent of the σ_k 's. This is in contrast with the value of the Yukawa coupling $C_{ttt} := \int \Omega \wedge \partial_t^3 \Omega$ which is computationally solved from

$$\partial_t \log C_{ttt} = -\frac{a_3}{2}$$

and is given by

$$C_{ttt} = const \cdot \exp(\int -\frac{a_3}{2}) = const \cdot \exp(\int \frac{\sigma_1}{2}C)$$

for some constant const. Hence it depends on the specific value of a_3 or equivalently σ_1 for the present hypergeometric case.

C. Appendix: Antiderivatives of differential polynomials, and its application

C.1. **Preparation: anti-derivatives.** The structure in the relation (44) has other interesting consequences besides the differential relations we obtain in Appendix B. This structure is written as

(61)
$$\mathcal{L} = \mathcal{D},$$

where as in (45)

(62)
$$\mathcal{L} = \partial_t^4 + \sum_{k=0}^3 (-\sigma_{4-k}C)\partial_t^k ,$$

and as in (46)

(63)
$$\mathcal{D} = (\partial_t + c_4)(\partial_t + c_3)(\partial_t + c_2)(\partial_t + c_1).$$

For later use, we shall denote

$$\beta = 1 - \alpha$$
, $\mathcal{D}_k = (\partial_t + c_k)$, $k = 1, 2, 3, 4$.

C.1.1. Commuting differential operator actions on periods. Some first properties of the relation (44) are as follows.

Lemma C.1. For the four solutions I_k , k = 0, 1, 2, 3 to the Picard-Fuchs equation, one has

$$\mathcal{L}_{\mathrm{hyp}} \circ \partial_t I_k = \partial_t^4 I_k$$

Proof. By definition,

$$\mathcal{L}_{ ext{hyp}} \circ \partial_t I_k = [\mathcal{L}_{ ext{hyp}}, \partial_t] I_k + \partial_t \circ \mathcal{L}_{ ext{hyp}} I_k = -[\partial_t, \mathcal{L}_{ ext{hyp}}] I_k$$
 .

Now using (62) one has

$$[\partial_t, \mathcal{L}_{hyp}] = (\partial_t \beta)\partial_t^4 + \sum_{k=0}^3 (-\sigma_{4-k}\partial_t \alpha)\partial_t^k = (\beta - 1)\partial_t^4 + \sum_{k=0}^3 (-\sigma_{4-k}\alpha)\partial_t^k = -\partial_t^4 + \mathcal{L}_{hyp}.$$

Therefore, one obtains

$$\mathcal{L}_{\mathrm{hyp}} \circ \partial_t I_k = \partial_t^4 I_k - \mathcal{L}_{\mathrm{hyp}} I_k = \partial_t^4 I_k \,.$$

Lemma C.2. The following relation for the differential operators holds

$$I_0 \mathcal{L}_{hyp} = \partial_t \circ (I_0 \beta) \circ \mathcal{D}_3 \circ \mathcal{D}_2 \circ \mathcal{D}_1 \,.$$

Proof. The relation (61) reads that

$$\mathcal{L}_{\text{hyp}} = \beta \mathcal{D}_4 \circ \mathcal{D}_3 \circ \mathcal{D}_2 \circ \mathcal{D}_1, \quad \mathcal{D}_4 = \partial_t + c_4 = \partial_t - Y - B - 3A.$$

where the quantities C, A, B, Y are as defined in (40) and (50). According to the b_3 -relation in (49) and (40), we have

$$Y - B - 3A = B - C = \partial_t \ln I_0 - \partial_t \ln C_{ttt} = \partial_t \ln(I_0\beta).$$

Therefore, we obtain

$$\mathcal{D}_4 = \partial_t + \partial_t \log(I_0\beta)$$

It follows that

$$(I_0\beta)\circ\mathcal{D}_4=\partial_t\circ(I_0\beta)$$

According to (61), we obtain

$$I_0 \mathcal{L}_{hyp} = \partial_t \circ (I_0 \beta) \circ \mathcal{D}_3 \circ \mathcal{D}_2 \circ \mathcal{D}_1$$

Proposition C.1. For any solution I_k , k = 0, 1, 2, 3 to the Picard-Fuchs equation, one has

$$\int I_0 \partial_t^4 I_k = (I_0 \beta) \circ \mathcal{D}_3 \circ \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_k + C_k \,,$$

for some constant C_k .

Proof. Combining Lemma C.1 and Lemma C.2, we obtain

$$I_0\partial_t^4 I_k = I_0\mathcal{L} \circ \partial_t I_k = \partial_t \circ (I_0\beta) \circ \mathcal{D}_3 \circ \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_k.$$

The claim then follows.

Proposition C.2. For any solution I_k , k = 0, 1, 2, 3 to the Picard-Fuchs equation, one has

$$\int T' \int I_0 \partial_t^4 I_k = (I_0 \beta T') \circ \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_k + C_k T + D_k \,,$$

for some constants C_k, D_k .

Proof. Since

$$I_0\beta T'\mathcal{D}_3 = I_0\beta T' \circ (\partial_t + \partial_t \log(\beta I_0 T')) = \partial_t \circ (I_0\beta T')$$

the claim then follows from Proposition C.1.

For simplicity in what follows we shall omit the integration constants, equipped with the known boundary conditions provided in (41), (42), (43).

By integration by parts, one has

Proposition C.3. One has

$$\int I_k I_0^{\prime\prime\prime\prime} = \int J_k I_0 I_0^{\prime\prime\prime\prime} = J_k \int I_0 I_0^{\prime\prime\prime\prime} - \int J_k^{\prime} \int I_0 I_0^{\prime\prime\prime\prime}, \quad J_k = \frac{I_k}{I_0}$$

In particular, when k = 1, this is Proposition C.2. When k = 2, one has

$$\int J'_k \int I_0 I_0^{\prime\prime\prime\prime\prime} = \int F_{TT} \cdot T' \int I_0 I_0^{\prime\prime\prime\prime} = F_{TT} \cdot \int T' \int I_0 I_0^{\prime\prime\prime\prime} - \int F'_{TT} \int T' \int I_0 I_0^{\prime\prime\prime\prime} = F_{TT} \cdot (T'I_0\beta)\mathcal{D}_2 \circ \mathcal{D}_1 \partial_t I_0 - (F_{TTT}T'^2I_0\beta)\mathcal{D}_1 \partial_t I_0.$$

C.1.2. Anti-derivatives of degree four differential polynomials of periods. One can now obtain the antiderivatives of degree four differential polynomials of periods. We now compute some which shall be used later. We have

(64)
$$\int I_0'' I_0'' = I_0'' I_0' - I_0''' I_0 + \int I_0''' I_0 = I_0'' I_0' - I_0''' I_0 + (I_0\beta) \mathcal{D}_3 \circ \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_0.$$

(65)
$$\int I_1'' I_0'' = I_0' I_1'' - I_0 I_1''' + \int I_0 I_1''' = I_0' I_1'' - I_0 I_1''' + (I_0 \beta) \mathcal{D}_3 \circ \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_1.$$

For the anti-derivative of $I_1''I_1''$, one has

(66)
$$\int I_1'' I_1'' = I_1'' I_1' - I_1''' I_1 + \int T I_1''' I_0 = I_1'' I_1' - I_1''' I_1 + T \int I_0 I_1''' - \int T' \int I_0 I_1'''' = I_1'' I_1' - I_1''' I_1 + T (I_0 \beta) \mathcal{D}_3 \circ \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_1 - (T' I_0 \beta) \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_1.$$

C.1.3. *Relations arising from integration by parts.* There are sometimes multiple ways to compute the anti-derivatives. One can simply apply Proposition C.1, or alternatively combine Proposition C.1 and Proposition C.3. For example, we have besides (65) the following

$$\int I_1'' I_0'' = I_0'' I_1' - I_0''' I_1 + \int I_1 I_0''' = I_0'' I_1' - I_0''' I_1 + \int T I_0 I_0'''$$

= $I_0'' I_1' - I_0''' I_1 + T \int I_0 I_0''' - \int T' \int I_0 I_0'''$
= $I_0'' I_1' - I_0''' I_1 + T (I_0 \beta) \mathcal{D}_3 \circ \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_0 - (T' I_0 \beta) \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_0.$

Taking the difference between these two equations, we have

$$I_{0}'I_{1}'' - I_{0}I_{1}''' - (I_{0}''I_{1}' - I_{0}'''I_{1}) = \int (I_{1}I_{0}''' - I_{1}'''I_{0})$$

= $T(I_{0}\beta)\mathcal{D}_{3}\circ\mathcal{D}_{2}\circ\mathcal{D}_{1}\circ\partial_{t}I_{0} - (T'I_{0}\beta)\mathcal{D}_{2}\circ\mathcal{D}_{1}\circ\partial_{t}I_{0} - (I_{0}\beta)\mathcal{D}_{3}\circ\mathcal{D}_{2}\circ\mathcal{D}_{1}\circ\partial_{t}I_{1}.$

Similarly, in computing the anti-derivative of $I_1''I_0'$, one can get the same identity. While directly computing the right hand side in the above seems complicated, the left hand side is easy. That is, by integration by parts and the integration formulas in Proposition C.1 and Proposition C.3, we can easily produce nontrivial relations.

C.2. **Proof of an identity.** The antiderivatives of $I_j^{(a)}I_k^{(b)}$'s in previous section admits an application in packaging loop contribution in g = 1 MSP. We shall prove the following identity.

Proposition C.4. In the following, $I_j = I_j(y)$ in integrand and ' means derivative with respect to y.

$$\begin{split} &-\frac{1}{2}\;\partial_{T}J_{2}\;\int_{-\infty}^{t}(I_{1}^{(2)}-TI_{0}^{(2)})^{2}dy-J_{2}\Big(\int_{-\infty}^{t}(I_{1}^{(2)}-TI_{0}^{(2)})I_{0}^{(2)}-(I_{1}^{(3)}-TI_{0}^{(3)})I_{0}^{(1)}dy\Big)\\ &-(\partial_{T}^{-1}J_{2})\int_{-\infty}^{t}(I_{0}^{(2)})^{2}-2I_{0}^{(3)}I_{0}^{(1)}dy+\partial_{t}^{-2}\Big(\Big(I_{3}^{\prime\prime\prime\prime}-tI_{2}^{\prime\prime\prime\prime}\Big)'I_{0}'+I_{2}^{\prime\prime\prime}\Big(I_{1}'-tI_{0}'\Big)'\Big)-(T-t)\partial_{t}^{-1}(I_{2}^{\prime\prime\prime}I_{0}')\\ &=\frac{1}{2}\frac{T^{\prime\prime}}{T^{\prime}}+2\frac{I_{0}'}{I_{0}}+\frac{\ln(1-5^{5}e^{t})}{5}. \end{split}$$

By (42), we have $J_2 = I_2/I_0 = F_T$ and $\partial_T J_2 = F_{TT}$, $\partial_T^{-1} J_2 = F$. By (40) and (50), the R.H.S. of the above identity is $\frac{1}{2}A + 2B + \frac{1}{5} \ln C_{ttt}^{-1}$.

The rest of this section will be devoted to proving Proposition C.4. In what follows, we denote the left and right hand sides of the identity in Proposition C.4 by *LHS*, *RHS* respectively.

C.2.1. *First simplification: anomalous term.* We first focus on the anomalous term on the left hand side:

$$\begin{split} &\int \int (I_3'''(y) - tI_2'''(y))' I_0'(y) dy + \int \int I_2'''(y) (I_1'(y) - tI_0'(y))' dy - (T-t) \int^t I_2'''(y) I_0'(y) dy \\ &= \int \int (I_3'''(y) I_0'(y) + I_2'''(y) I_1''(y)) dy - T \int^t I_2'''(y) I_0'(y) dy \,. \end{split}$$

By integration by parts, one has

$$\int I_{3}^{\prime\prime\prime\prime}I_{0}^{\prime} = I_{3}^{\prime\prime\prime}I_{0}^{\prime} - \int I_{3}^{\prime\prime\prime}I_{0}^{\prime\prime}.$$

This gives

$$\int \int (I_{3}'''(y)I_{0}'(y) + I_{2}'''(y)I_{1}''(y))dy - T \int^{t} I_{2}'''(y)I_{0}'(y)dy$$

=
$$\int \int (-I_{3}'''(y)I_{0}''(y) + I_{2}'''(y)I_{1}''(y))dy + \int^{t} I_{3}'''(y)I_{0}'(y)dy - T \int^{t} I_{2}'''(y)I_{0}'(y)dy.$$

Now applying integration by parts again, we get

$$\int I_3'' I_0'' = I_3'' I_0'' - \int I_3'' I_0'''.$$

Hence

$$\int I_3''' I_0'' = \frac{1}{2} I_3'' I_0'' + \frac{1}{2} \int (I_3''' I_0'' - I_3'' I_0''')$$

Similarly,

$$\int I_2^{\prime\prime\prime} I_1^{\prime\prime} = \frac{1}{2} I_2^{\prime\prime} I_1^{\prime\prime} + \frac{1}{2} \int (I_2^{\prime\prime\prime} I_1^{\prime\prime} - I_2^{\prime\prime} I_1^{\prime\prime\prime})$$

Therefore,

$$\int (-I_3''I_0'' + I_2'''I_1'') = -\frac{1}{2}I_3''I_0'' + \frac{1}{2}I_2''I_1'' - \frac{1}{2}\int (I_3'''I_0'' - I_3''I_0'') + \frac{1}{2}\int (I_2'''I_1'' - I_2''I_1'').$$

To compute

$$-\frac{1}{2}(I_3'''I_0'' - I_3''I_0''') + \frac{1}{2}(I_2'''I_1'' - I_2''I_1''') = -\frac{1}{2}u_6^{(03)} + \frac{1}{2}u_6^{(12)}$$

we use the Wronskian method discussed in Section B.2.1 above. Here $u_6^{(03)}$ means u_6 with (a, b) = (0, 3)and $u_6^{(12)}$ is similar. We recall that the two u_1 's are equal, hence so are

$$u_2 = u_1', \quad u_2' = u_3 + u_4$$

and

$$U = a_3 u_4 + 2u_5, \quad C_1 u_4 + 2a_0 u_1 + 2u_6,$$

where

$$C_1 = a'_3 + \frac{1}{2}a_3^2 - 2a_2 = \frac{4}{5}C$$
.

The difference

$$\Delta u_6 = u_6^{(03)} - u_6^{(12)} = M_{03} - M_{12}$$

can be found by using the Wronskian method as follows. Observe that all of the equations involved are linear. To figure out Δu_6 , we recall

$$u_6' = a_0 u_2 + a_1 u_4 - a_3 u_6 \,.$$

This gives

$$\Delta(C_1u_4 + 2a_0u_1 + 2u_6) = C_1\Delta u_4 + 2\Delta u_6 = 0$$

and

$$\Delta(u_6' - a_0 u_2 - a_1 u_4 + a_3 u_6) = (\Delta u_6)' - a_1(\Delta u_4) + a_3 \Delta u_6 = 0$$

It follows that

$$(\Delta u_6)' + a_1 \frac{2}{C_1} \Delta u_6 + a_3 \Delta u_6 = 0.$$

Plugging in $C_1 = \frac{4}{5}C, a_1 = -\frac{2}{5}C, a_3 = -2C$, we obtain

$$\Delta u_6 = kC' = kC(C+1).$$

for some constant k. This constant can be fixed by looking at the boundary conditions of $-\frac{1}{2}(I_3''I_0'' - I_3''I_0'') + \frac{1}{2}(I_2'''I_1'' - I_2''I_1'')$ obtained by the first few terms of the series obeying (42) and (43). This fixes $k = -\frac{1}{5}$.

Remark C.5. To minimize the computations, we can also prove the identity as follows. Note that the same reasoning as above tells us that

$$\Delta u_3 + \Delta u_4 = 0,$$

(\Delta u_3)' = (\Delta u_4)' - a_3 \Delta u_3 = (-\Delta u_3)' - a_3 \Delta u_3.

Solving the latter differential equation, we see that $\Delta u_3 = a/\beta$ for some constant a. Then we get $\Delta u_4 = -a/\beta$ and hence

$$\Delta u_6 = -\frac{C_1}{2}\Delta u_4 = \frac{C_1}{2}\frac{a}{\beta} = \frac{2}{5}C\frac{a}{\beta}.$$

Hence the constant k in the previous discussion is related to a by $k = \frac{2}{5}a$. But now computing Δu_3 or Δu_4 is much easier: one has

$$\Delta u_3 = -\frac{1}{2} (I_3''' I_0 - I_3 I_0'') + \frac{1}{2} (I_2''' I_1 - I_2 I_1''),$$

$$\Delta u_4 = -\frac{1}{2} (I_3'' I_0' - I_3' I_0'') + \frac{1}{2} (I_2'' I_1' - I_2' I_1'').$$

Due to the asymptotic behaviour of $1/\beta$, this amounts to computing the constant term. We consider $\Delta u_4 = -a/\beta = -a + \mathcal{O}(q)$ for simplicity. Now knowing that the first term I_0 is 1 from (41) is enough to fix -a = 1/2. This gives k = -1/5.

Therefore, we obtain

$$\int (-I_3'''I_0'' + I_2'''I_1'') = -\frac{1}{2}I_3''I_0'' + \frac{1}{2}I_2''I_1'' - \frac{1}{5}C$$

It follows then the anomalous term considered above is given by

$$\int \left(-\frac{1}{2}I_3''(y)I_0''(y) + \frac{1}{2}I_2''(y)I_1''(y)\right)dy + \int^t I_3'''(y)I_0'(y)dy - T\int^t I_2'''(y)I_0'(y)dy.$$

Now the left hand side of the desired identity in Proposition C.4 is simplified into

$$LHS = -\frac{1}{2}F_{TT} \int^{t} (I_{1}''(y) - T(t)I_{0}''(y))^{2} dy$$

$$-F_{T} \int^{t} (I_{1}''(y) - T(t)I_{0}''(y))I_{0}''(y)dy + F_{T} \int^{t} (I_{1}'''(y) - T(t)I_{0}'''(y))I_{0}'(y)dy$$

$$-F \int^{t} I_{0}''(y)^{2} dy + 2F \int^{t} I_{0}'''(y)I_{0}'(y)dy$$

$$+ \int (-\frac{1}{2}I_{3}''I_{0}'' + \frac{1}{2}I_{2}''I_{1}'')' + \int^{t} I_{3}''I_{0}' - T \int^{t} I_{2}'''I_{0}' + \int -\frac{1}{5}C$$

$$= -\frac{1}{2}F_{TT} \int^{t} (I_{1}''(y) - T(t)I_{0}''(y))I_{0}''(y)dy - 3F \int^{t} I_{0}''(y)^{2}dy$$

$$+F_{T}(I_{1}''I_{0}' - TI_{0}''I_{0}') + 2FI_{0}''I_{0}'$$

$$+ \int (-\frac{1}{2}I_{3}''I_{0}'' + \frac{1}{2}I_{2}''I_{1}'') + \int^{t} I_{3}'''I_{0}' - T \int^{t} I_{2}'''I_{0}' + \int -\frac{1}{5}C.$$

C.2.2. Taking the derivative. We can apply Proposition C.1, Proposition C.3 and integral by parts to simply the single integrals involving I_0 in the quantity *LHS*. But the double integral seems to be out of reach. Both this and the $\log C_{ttt}^{-1} = \int C$ term on the right hand side of the expected identity in Proposition C.4 suggest that one takes the derivative of the identity.

The derivative is then computed to be

$$\partial_t LHS = -\frac{1}{2} F_{TTT} T' \int^t (I_1''(y) - T(t) I_0''(y))^2 dy$$

$$-F_{TT}T'\int^{t} (I_{1}''(y) - T(t)I_{0}''(y))I_{0}''(y)dy - F_{T}T'\int^{t} I_{0}''(y)^{2}dy$$

$$-T'\int I_{2}'''I_{0}' - \frac{1}{2}F_{TT}(I_{1}'' - TI_{0}'')^{2} - F_{T}(I_{1}'' - TI_{0}'')I_{0}'' - FI_{0}''^{2} + I_{0}'(F_{T}(I_{1}''I_{0}' - TI_{0}'') + 2FI_{0}'')'$$

$$+(-\frac{1}{2}I_{3}''I_{0}'' + \frac{1}{2}I_{2}''I_{1}'') + I_{3}'''I_{0}' - TI_{2}'''I_{0}' - \frac{1}{5}C.$$

After plugging in (64),(65),(66), the first two terms above reduce to expressions only involving $\mathcal{D}_2, \mathcal{D}_1$ thanks to Proposition C.1, Proposition C.3:

$$\int (I_1''(y) - T(t)I_0''(y))^2 dy = T'I_0(I_1'' - TI_0'') - (T'I_0\beta)\mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_1 + T(T'I_0\beta)\mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_0,$$

$$\int (I_1''(y) - T(t)I_0''(y))I_0''(y)dy = I_0I_0''T' - (T'I_0\beta)\mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_0.$$

We also apply integration by parts, Proposition C.1 and Proposition C.3 to simplify the other integrals. This then gives a differential polynomial in the generators A, B, C displayed in (50). Below are the details.

$$-F_T T' \int^t I_0''(s)^2 ds = -F_T T' (I_0'' I_0' - I_0''' I_0) - F_T T' (I_0 \beta) \mathcal{D}_3 \circ \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_0,$$

$$-T' \int I_2''' I_0' = T' [-I_2'' I_0' + I_2' I_0'' - I_2 I_0''' + F_T (I_0 \beta) \mathcal{D}_3 \circ \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_0$$

$$-F_{TT} (T' I_0 \beta) \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_0 + (F_{TTT} T'^2 I_0 \beta) \mathcal{D}_1 \circ \partial_t I_0].$$

C.2.3. Completion of the proof of Proposition C.4. Plugging in all of the above expressions for the anti-derivatives, we obtain

$$\begin{aligned} \partial_t LHS &= -\frac{1}{2} F_{TTT} T'T' I_0 (I_1'' - TI_0'') - F_{TT} T' I_0 I_0'' T' - F_T T' (I_0'' I_0' - I_0''' I_0) \\ &- \frac{1}{2} F_{TT} (I_1'' - TI_0'')^2 - F_T (I_1'' - TI_0'') I_0'' - FI_0''^2 \\ &+ I_0' F_{TT} T' (I_1'' - TI_0'') + I_0' F_T (I_1'' - TI_0'')' + 2F_T T' I_0' I_0'' + 2FI_0' I_0''' \\ &+ T' (I_2''' I_0 - I_2'' I_0' + I_2' I_0'' - I_2 I_0''') + (-\frac{1}{2} I_3'' I_0'' + \frac{1}{2} I_2'' I_1'') + (I_3''' I_0' - I_2''' I_1') \\ &+ \frac{1}{2} F_{TTT} T' (T' I_0 \beta) \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_1 \\ &- \frac{1}{2} F_{TTT} T' T (T' I_0 \beta) \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_0 + T' (F_{TTT} T'^2 I_0 \beta) \mathcal{D}_1 \circ \partial_t I_0 - \frac{1}{5} C \\ &= 0 + \frac{1}{2} F_{TTT} T' (T' I_0 \beta) \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_1 - \frac{1}{2} F_{TTT} T' T (T' I_0 \beta) \mathcal{D}_2 \circ \mathcal{D}_1 \circ \partial_t I_0 \\ &+ T' (F_{TTT} T'^2 I_0 \beta) \mathcal{D}_1 \circ \partial_t I_0 - \frac{1}{5} C . \end{aligned}$$

Keeping simplifying, we obtain

$$\begin{aligned} \partial_t LHS &= +\frac{1}{2} F_{TTT} T'(T'I_0\beta) [T'''I_0 + 3T''I_0' + 3T'I_0'' \\ &- (A+2B)(T''I_0 + 2T'I_0') + (AB + B^2 - B')T'I_0] \\ &+ T'(F_{TTT}T'^2\beta)(I_0''I_0 - I_0'I_0') - \frac{1}{5}C \\ &= (\frac{1}{2}A' + B') + B' - \frac{1}{5}C \,. \end{aligned}$$

This is identical to

$$\partial_t RHS = \partial_t \left(\frac{1}{2}A + 2B + \frac{1}{5} \ln C_{ttt}^{-1} \right) = \frac{1}{2}A' + 2B' - \frac{1}{5}C.$$

The identity LHS = RHS then follows from the above equation $\partial_t LHS = \partial_t RHS$, and the fact that they have the same boundary conditions which follow from (41),(42) and (43).

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