# GENUS ONE GW INVARIANTS OF QUINTIC THREEFOLDS VIA MSP LOCALIZATION 

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#### Abstract

The moduli stack of Mixed Spin P-fields (MSP) provides an effective algorithm to evaluate all genus Gromov-Witten invariants of quintic Calabi-Yau threefolds. This paper is to apply the algorithm in genus one case. We use the localization formula, the proposed algorithm in CLLL1, CLLL2, and Zinger's packaging technique to compute the genus one Gromov-Witten invariants of quintic Calabi-Yau threefolds. New hypergeometric series identities are also discovered in the process.


## 1. Introduction

The genus one Gromov-Witten invariants of Calabi-Yau hypersurfaces were calculated by Zinger in [Zi2]. The method consists of two steps. The first step is the detailed study of the moduli space of genus one stable maps to CY hypersurfaces in [LZ, VZ. The second step is the computation part using torus localization and some packaging techniques in [Zi2].

In this paper, we will use the MSP set-up to calculate the genus one Gromov-Witten invariants for quintic CY hypersurfaces in $\mathbb{P}^{4}$. As a by product, we find integral relations for certain differential polynomials of $I$ functions (e.g. section C.1.2).

Mixed-spin $P$-fields (MSP) are defined in CLLL1 (see CLLL3 for a survey). An MSP field is given by

$$
\xi=\left(\mathcal{C}, \Sigma^{\mathcal{C}}, \mathcal{L}, \mathcal{N}, \varphi, \rho, \nu\right)
$$

with stability conditions, where $\mathcal{C}$ is a twisted nodal curve (or an orbifold curve) with markings $\Sigma^{\mathcal{C}}, \mathcal{L}$ and $\mathcal{N}$ are invertible sheaves on $\mathcal{C}$, and $\varphi, \rho, \nu$ are sections of $\mathcal{L}^{\oplus 5}, \mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}}^{\log }, \mathcal{L} \otimes \mathcal{N} \oplus \mathcal{N}$ respectively. For simplicity, let's consider the case without markings. Then $\xi$ has the numerical data ( $g$, d) where $g$ is the genus of the curve $\mathcal{C}$ and $\mathbf{d}=\left(d_{0}, d_{\infty}\right)$ with $d_{0}=\operatorname{deg}(\mathcal{L} \otimes \mathcal{N})$ and $d_{\infty}=\operatorname{deg}(\mathcal{N})$. We use $\mathcal{W}_{g, \mathbf{d}}$ to denote the moduli stack of MSP fields with numerical data $(g, \mathbf{d})$. The virtual dimension of $\mathcal{W}_{g, \mathbf{d}}$ is $d_{0}+d_{\infty}-g+1$.

MSP moduli stacks can be regarded as a platform interpolating the moduli of Gromov-Witten theory and the moduli of FJRW theory (see [FJR1, FJR2]). Using the $P$-field theory of CL and the cosection localization technique of [KL], a virtual cycle can be defined and the properness can be proved in CLLL1. The moduli stack admits a $\mathbb{C}^{*}$-action. Using this torus action, if the virtual dimension of $\mathcal{W}_{g, \mathbf{d}}$ is bigger than zero, the virtual cycle $\left[\mathcal{W}_{g, \mathbf{d}}\right]^{\text {vir }}$ can give us the identity:

$$
\begin{equation*}
0=\int_{\left[\mathcal{W}_{g, \mathbf{d}}\right]^{\mathrm{vir}}} \mathfrak{t} c_{\text {top }-1}\left(R \pi_{\mathcal{W} *}\left(\mathcal{L}_{\mathcal{W}} \mathcal{N}_{\mathcal{W}} \mathcal{L}_{\mathfrak{t}}\right)\right) \tag{1}
\end{equation*}
$$

where $\mathcal{L}_{\mathcal{W}}$ and $\mathcal{N}_{\mathcal{W}}$ are the universal line bundles on the universal curve $\mathcal{C}$ over the moduli $\mathcal{W}:=\mathcal{W}_{g, \mathrm{~d}}$ with a projection $\pi_{\mathcal{W}}: \mathcal{C} \rightarrow \mathcal{W}$, top is the virtual rank of $R \pi_{\mathcal{W} *}\left(\mathcal{L}_{\mathcal{W}} \mathcal{N}_{\mathcal{W}}\right), \mathfrak{t}$ is the generator of the equivariant cohomology $H_{\mathbb{C}^{*}}^{*}\left(B \mathbb{C}^{*}, \mathbb{Q}\right)=\mathbb{Q}[\mathfrak{t}]$, and $\mathcal{L}_{\mathfrak{t}}$ is the trivial line bundle with the weight- $1 \mathbb{C}^{*}$ action on fibers .

[^0]The identify (11) provides lots of equations among GW invariants, Hodge integrals, and FJRW invariants. This will potentially provide an effective algorithm to calculate GW invariants as well as FJRW invariants. In this paper, we will consider the virtual cycles $\left[\mathcal{W}_{1,(0, n)}\right]^{\text {vir }}$ for all $n$ to compute genus one GW invariants of Fermat quintic CY three-folds.

Comparing with Zinger's work, MSP moduli replaces the detailed study of moduli spaces of stable maps, especially the complicated issue of the ghost component, by FJRW invariants. Since the study of the ghost components for higher genus is very complicated as it requires separation of components with different $P$-field (ghost) rank (c.f. CL1), MSP moduli provides a good way to work on higher genus GW invariants (see MP for another approach). For example, our method does not need genus zero two point functions. The $g=1$ package in this paper is expected to be directly extended to higher genus package via MSP moduli.

The $\mathbb{C}^{*}$-action gives four types of graphs from the standard localization procedure, called type A, B, C and D (see Figure 2.2). The contribution from type A graphs involves genus one GW invariants of the quintic, while that from type D graphs involves FJRW invariants. Our packaging uses Zinger's technique ( $(\underline{\mathrm{Zi} 2]})$. We also discover (c.f. Section C.1.2) new hypergeometric series identities in the package of loop type (B) graphs.

The method of quasimaps also provides another way to calculate genus one GW invariants of CY hypersurfaces CK1, CK2, KLh].

The organization of the paper is as follows. In §2, we review the graphs notations and their contributions in the localization of MSP theory in CLLL1, CLLL2. In $\S 3$, we review the mirror theorem for genus zero GW invariants of quintic threefolds [Gi1, Gi2] (also see [LLY]) and discuss how to transform Givental's set-up to our MSP set-up. In $\S 4$, we define $\mathcal{K}$-series and $\mathcal{P}$-series, which are essential for the calculations in later sections. We also calculate some $\mathcal{K}$-series. In $\S 5$, we compute the contribution of type A graphs to (11). In $\S 6$, we compute the contribution of type B graphs to (11). In $\S 7$, we compute the contribution of type C graphs to (1). In $\S 8$, we compute the contribution of type D graphs to (11). Many computations in the last four sections are contained in the Appendix.

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Notations: Throughout the paper we fix the notations $T=\mathbb{C}^{*}, \mathfrak{t}=-c_{1}\left(\mathcal{O}_{\mathbb{C P P}^{\infty}}(1)\right)$ for $B T=\mathbb{C P} \mathbb{P}^{\infty}$, $q=e^{t}$, and $h \in H^{2}\left(\mathbb{P}^{4}, \mathbb{Z}\right)$ always denotes the hyperplane class.

## 2. MSP THEORY AND LOCALIZATION SET-UP

In this section, we will review the localization scheme of the moduli stacks of mixed-spin P-fields in CLLL1, CLLL2. There are four types of graphs in genus one localization, labeled as type A, B, C and D.
2.1. Review of MSP theory. The MSP moduli $\mathcal{W}=\mathcal{W}_{g,(1, \rho)^{n}, \mathbf{d}}$ is constructed in CLLL1. It is indexed by genus $g$, the number $n$ of markings labelled by $(1, \rho)$, and two integers $\left(d_{0}, d_{\infty}\right)=\mathbf{d}$. It is a separated DM stack locally of finite type equipped with a cosection $\sigma$ and a $T=\mathbb{C}^{*}$ action. The cosection's zero locus $\mathcal{W}_{g,(1, \rho)^{n}, \mathbf{d}}^{-}$is shown to be proper CLLL1 and the torus action gives rise to an equivariant virtual cycle $\left[\mathcal{W}_{g, \gamma, \mathbf{d}}\right]_{\mathrm{loc}}^{\mathrm{vir}} \in A_{*}^{T}\left(\mathcal{W}_{g, \gamma, \mathbf{d}}^{-}\right)^{T}$. It has a localization formula (for $\delta:=$ $\left.d_{0}+d_{\infty}+1-g+n\right)$

$$
\begin{equation*}
[\mathcal{W}]_{\mathrm{loc}}^{\mathrm{vir}}=\sum_{\Gamma} J_{\Gamma *}^{-}\left(\frac{\left[\mathcal{W}_{\Gamma}\right]_{\mathrm{loc}}^{\mathrm{vir}}}{e\left(N_{\Gamma}^{\mathrm{vir}}\right)}\right) \in A_{\delta}^{T}\left(\mathcal{W}^{-}\right)^{T}\left[\mathfrak{t}^{-1}\right] \tag{2}
\end{equation*}
$$

where each connected component of fixed locus $\mathcal{W}^{T}$ is indexed by a graph $\Gamma$ and $\jmath^{-}$is the natural inclusion from $\mathcal{W}_{\Gamma}$ to $\mathcal{W}^{-}$. Each graph $\Gamma$ is connected, and has all vertices labelled by 0,1 or $\infty$, namely
$V(\Gamma)=V_{0}(\Gamma) \cup V_{1}(\Gamma) \cup V_{\infty}(\Gamma)$ according to $\nu_{1}=0, \nu_{1}=1=\nu_{2}$ and $\nu_{2}=0$ respectively, and edges $E(\Gamma)=E_{0}(\Gamma) \cup E_{\infty}(\Gamma)$ where edges in $E_{0}(\Gamma)$ connect vertices $V_{0}(\Gamma)$ with $V_{1}(\Gamma)$, and edges in $E_{\infty}(\Gamma)$ connect vertices $V_{0}(\Gamma)$ with $V_{1}(\Gamma)$. Denote the set $F(\Gamma)$ of flags to be the set of $(e, v) \in E(\Gamma) \times V(\Gamma)$ such that $e, v$ are adjacent. The graphs is required to possess decorations as follows:
(a) (genus) $g .: V(\Gamma) \rightarrow \mathbb{Z}_{\geq 0}$;
(b) (degree) $\vec{d}: E(\Gamma) \cup V(\Gamma) \rightarrow \mathbb{Q}^{\oplus 2}$ via $\vec{d}(a)=\left(d_{0 a}, d_{\infty a}\right)$, where $d_{0 a}=\operatorname{deg} \mathcal{L} \otimes \mathcal{N} \mid \mathcal{C}_{a}$ and $d_{\infty a}=\left.\operatorname{deg} \mathcal{N}\right|_{\mathcal{C}_{a}}$ with $\mathcal{C}_{a}$ being the curve associated to $a$.
(c) (marking) $S$. : $V(\Gamma) \rightarrow 2^{L(\Gamma)}$ via $v \mapsto S_{v} \subset L(\Gamma)$, where $S_{v}$ is the subset of markings $\Sigma_{i}^{\mathcal{C}} \in \mathcal{C}_{v}$. We call a vertex $v \in V(\Gamma)$ unstable if $g_{v}=0, d_{v}=\operatorname{deg}\left(\mathcal{L} \mid \mathcal{C}_{v}\right)=0$ and $\left|E_{v} \cup S_{v}\right|<3$, otherwise stable. Thus $V(\Gamma)=V^{S}(\Gamma) \cup V^{U S}(\Gamma)$. For any unstable vertex $v \in V^{U S}(\Gamma)$, it must lie in $V^{0,1}, V^{1,1}$ or $V^{0,2}$, where

$$
\begin{equation*}
V^{a, b}(\Gamma):=\left\{v \in V(\Gamma)-V^{S}(\Gamma)| | S_{v}\left|=a,\left|E_{v}\right|=b\right\}\right. \tag{3}
\end{equation*}
$$

Let $V_{c}^{a, b}(\Gamma)=V_{c}(\Gamma) \cap V^{a, b}(\Gamma)$ and $\Delta_{g, \gamma, \mathbf{d}}$ denote the set of all (regular) graphs, up to graph isomorphisms preserving decorations. For each graph $\Gamma \in \Delta_{g, \gamma, \mathbf{d}}$ we use $(\Gamma)$ to denote its class in $\Delta_{g, \gamma, \mathbf{d}} / \sim$, where $\sim$ is by automorphisms of graphs. Then $\mathcal{W}_{g, \gamma, \mathbf{d}}^{T}$ is a disjoint union of $\mathcal{W}_{(\Gamma)}$, the locus whose curves are associated to the graph $\Gamma$ (c.f. CLLL2, Def 2.5]), over $\Gamma$ 's in $\Delta_{g, \gamma, \mathrm{~d}}$. In [LLL2] the fixed locus of $\mathcal{W}_{g, \gamma, \mathrm{~d}}$ decorated by $\Gamma$ is denoted by $\mathcal{W}_{\Gamma}$. In this paper we will no distinguish $\mathcal{W}_{\Gamma}$ from $\mathcal{W}_{(\Gamma)}$, and we always use $\pi_{\Gamma}: \mathcal{C}_{\mathcal{W}_{\Gamma}} \rightarrow \mathcal{W}_{\Gamma}, \mathcal{L}_{\Gamma}$ to denote its universal curve and universal line bundle.

For each $v \in V^{S}(\Gamma)$, there is a moduli stack $\mathcal{W}_{v}$ ([LLL2, Cor. 2.37]), where

$$
\mathcal{W}_{v} \cong \overline{\mathcal{M}}_{g_{v}, E_{v} \cup S_{v}}\left(\mathbb{P}^{4}, d_{v}\right)^{p}, \quad \mathcal{W}_{v} \cong \overline{\mathcal{M}}_{g_{v}, E_{v} \cup S_{v}}, \quad \mathcal{W}_{v} \cong \bar{M}_{g_{v}, \gamma}^{1 / 5.5 p}
$$

when $v \in V_{0}^{S}(\Gamma), V_{1}^{S}(\Gamma)$ and $V_{\infty}^{S}(\Gamma)$ respectively. Here $\overline{\mathcal{M}}_{g_{v}, E_{v} \cup S_{v}}\left(\mathbb{P}^{4}, d_{v}\right)^{p}$ is the moduli stack of degree $d_{v}$ stable maps from genus $g_{v}$ nodal curves with markings in $E_{v} \cup S_{v}$ to $\mathbb{P}^{4}$ with P-fields studied in CL, and $\bar{M}_{g_{v}, \gamma}^{1 / 5.5 p}$ is the moduli stack of genus $g_{v} \mu_{5}$-twisted curve with a five-spin structure, markings labeled by $\gamma$ and five $P$-fields studied in CLL.

For $e \in E_{0}$, there is an MSP moduli $\mathcal{W}_{e}=\sqrt[d_{e}]{\mathcal{O}_{\mathbb{P}^{4}}(1) / \mathbb{P}^{4}}$ with $\pi_{e}: \mathcal{W}_{e} \rightarrow \mathbb{P}^{4}$ the map to its coarse moduli space. Let $h \in H^{2}\left(\mathbb{P}^{4} ; \mathbb{Q}\right)$ be the hyperplane class, $h_{e}=\pi_{e}^{*} h \in H^{2}\left(\mathcal{W}_{e} ; \mathbb{Q}\right)$, and $d_{e}$ be the degree of $\mathcal{L} \mid \mathcal{C}_{e}$.

For unstable $v \in V_{0}-V_{0}^{S}$, we let $\left[\mathcal{W}_{v}\right]_{\text {loc }}^{\text {vir }}=-\left[Q_{5}\right]$, where $Q_{5} \subset \mathbb{P}^{4}$ is the Fermat quintic.
Proposition 2.1. For each graph $\Gamma$ appearing in localizing $\mathcal{W}_{g=1,(1, \rho)^{n},(d, 0)}$, we have

$$
\left[\mathcal{W}_{(\Gamma)}\right]_{\text {loc }}^{\mathrm{vir}}=\frac{1}{|\operatorname{Aut}(\Gamma)|} \frac{1}{\prod_{e \in E_{0}} d_{e}} \frac{1}{\prod_{e \in E_{\infty}}\left|5 d_{e}\right|} \prod_{v \in V_{0} \cup V_{1}^{S} \cup V_{\infty}^{S}}\left[\mathcal{W}_{v}\right]_{\mathrm{loc}}^{\mathrm{vir}}
$$

## Theorem 2.2.

$$
\frac{1}{e_{T}\left(N_{\Gamma}^{\text {vir }}\right)}=\prod_{v \in V(\Gamma)} B_{v} \prod_{e \in E(\Gamma)} A_{e}
$$

where

$$
B_{v}= \begin{cases}A_{v} \cdot \prod_{e \in E_{v}} \frac{1}{w_{(e, v)}-\psi_{(e, v)}}, & v \in V^{S} ; \\ \frac{A_{v}}{w_{(e, v)}+w_{\left(e^{\prime}, v\right)}}, & v \in V^{0,2}, E_{v}=\left\{e, e^{\prime}\right\} ; \\ A_{v}, & v \in V^{1,1} ; \\ A_{v} \cdot w_{(e, v)}, & v \in V^{0,1}, E_{v}=\{e\} .\end{cases}
$$

The formulae for $A_{v}, A_{e}$ and $w_{(e, v)}$ are provided below.
Lemma 2.1. Let $v$ and $v^{\prime}$ be the two vertices of an edge $e$.
(1) When $v \in V_{0}$, then $w_{(e, v)}=\frac{h_{e}+\mathfrak{t}}{d_{e}}$ and $w_{\left(e, v^{\prime}\right)}=-\frac{h_{e}+\mathfrak{t}}{d_{e}}$.
(2) When $v \in V_{\infty} \backslash V_{\infty}^{0,1}$, then $w_{(e, v)}=\frac{t}{r_{e} d_{e}}$ and $w_{\left(e, v^{\prime}\right)}=-\frac{\mathfrak{t}}{d_{e}}$.

Lemma 2.2 (Contribution from edges). For the two cases (1) $e \in E_{0}$, (2) $e \in E_{\infty},(e, v) \in F, v \in$ $V_{\infty} \backslash V_{\infty}^{0,1}$, we have respectively

$$
A_{e}=\frac{\prod_{j=1}^{5 d_{e}-1}\left(-5 h_{e}+\frac{j\left(h_{e}+\mathfrak{t}\right)}{d_{e}}\right)}{\prod_{j=1}^{d_{e}}\left(h_{e}-\frac{j\left(h_{e}+\mathfrak{t}\right)}{d_{e}}\right)^{5} \cdot \prod_{j=1}^{d_{e}} \frac{j\left(h_{e}+\mathfrak{t}\right)}{d_{e}}}, \quad \frac{\prod_{j=1}^{\left\lceil-d_{e}\right\rceil-1}\left(-\mathfrak{t}-\frac{j \mathfrak{t}}{d_{e}}\right)^{5}}{\prod_{j=1}^{-5 d_{e}}\left(-\frac{j \mathfrak{t}}{d_{e}}\right) \prod_{j=1}^{\left\lfloor-d_{e}\right\rfloor}\left(\frac{j \mathfrak{t}}{d_{e}}\right)} .
$$

Lemma 2.3 (Contribution from unstable vertices). If $v \in V^{0,2}$, then

$$
A_{v}= \begin{cases}h_{e}+\mathfrak{t}=h_{e^{\prime}}+\mathfrak{t}, & v \in V_{0}^{0,2} \text { and } E_{v}=\left\{e, e^{\prime}\right\} \\ -5 \mathfrak{t}^{6}, & v \in V_{1}^{0,2}\end{cases}
$$

If $v \in V_{1}^{1,1}, S_{v} \subset \Sigma_{(1, \rho)}^{\mathcal{C}}$, then we have $A_{v}=5 \mathfrak{t}$. If $v \in V_{0}^{1,1}$, we have $A_{v}=1$.
Let's introduce some notations.

- Given a stable vertex $v \in V^{S}$, let $\pi_{v}: \mathcal{C}_{v} \rightarrow \mathcal{W}_{v}$ be the universal curve, $\mathcal{L}_{v}$ be the universal line bundle over $\mathcal{C}_{v}$.
- Given $v \in V_{1}^{S}$, let $\mathbb{E}_{v}:=\pi_{*} \omega_{\pi_{v}}$ be the Hodge bundle, where $\omega_{\pi_{v}} \rightarrow \mathcal{C}_{v}$ is the relative dualizing sheaf. Then $\mathbb{E}_{v}^{V}=R^{1} \pi_{v *} \mathcal{O}_{\mathcal{C}_{v}}$.
- Let $\mathbf{L}_{k}$ be the one-dimensional weight $k T$-representation.

Lemma 2.4 (Contribution from all stable vertices). For $v \in V^{S}$, define

$$
A_{v}:= \begin{cases}\frac{1}{e_{T}\left(R \pi_{v *} \phi_{v}^{*} \mathcal{O}_{\mathbb{P}^{4}}(1) \otimes \mathbf{L}_{1}\right)} \prod_{e \in E_{v}}\left(h_{e}+\mathfrak{t}\right), & v \in V_{0}^{S}  \tag{4}\\ \left(\frac{e_{T}\left(\mathbb{E}_{v}^{\vee} \otimes \mathbf{L}_{-1}\right)}{-\frac{\mathfrak{t}}{}}\right)^{5} \cdot \frac{5 \mathfrak{t}}{e_{T}\left(\mathbb{E}_{v} \otimes \mathbf{L}_{5}\right)} \cdot\left(-\mathfrak{t}^{5}\right)^{\left|E_{v}\right|} \cdot\left(\frac{-\mathfrak{t}^{4}}{5}\right)^{\left|S_{v}^{(1, \varphi)}\right|}, & v \in V_{1}^{S} \\ \frac{1}{e_{T}\left(R \pi_{v *} \mathcal{L}_{v}^{\vee} \otimes \mathbf{L}_{-1}\right)}, & v \in V_{\infty}^{S}\end{cases}
$$

2.2. Types $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$. When the genus is one, we can divide the type of regular graphs associated to $T$-fixed points of $\mathcal{W}$ into four types $\mathrm{A}, \mathrm{B}, \mathrm{C}$ and D . They are given as follows.

Type A: This graph has one stable vertex $v \in V_{0}$ with genus $g_{v}=1$.
Type B: This graph has a loop with $\mathcal{C}_{0}$ consisting of possibly many rational curves and/or points and $\mathcal{C}_{1}$ being rational curves and/or points. $\mathcal{C}_{\infty}$ is empty. Here $\mathcal{C}_{0}=\mathcal{C} \cap\left(\nu_{1}=0\right)$ and similarly for $\mathcal{C}_{1}$ and $\mathcal{C}_{\infty}$.

Type C: This graph has one stable vertex $v \in V_{1}$ with genus $g_{v}=1$.
Type D: This graph has one stable vertex $v \in V_{\infty}$ with genus $g_{v}=1$. All edges $e$ in $E_{\infty}$ has $\operatorname{deg}\left(\mathcal{L} \mid \mathcal{C}_{e}\right)=-\frac{1}{5}$.


Each circle represents all possible graphs (with one marking) for genus zero and $d_{\infty}=0 \mathrm{MSP}$ moduli, where the marking lies on $V_{0}$ or $V_{1}$ accordingly.

## 3. Equivariant Mirror Theorem

3.1. Equivariant Mirror Theorem. Let $G=\mathbb{C}^{*}$ act on $\mathbb{P}^{5}$ as follows: $t \cdot\left[x_{0}, \ldots, x_{5}\right]=\left[x_{0}, \ldots, t x_{5}\right]$. Over

$$
H_{G}^{*}\left(\mathbb{P}^{5} ; \mathbb{Q}\right)=\mathbb{Q}[p, \mathfrak{t}] /\left\langle p^{5}(p+\mathfrak{t})\right\rangle
$$

there is a $G$-equivariant pairing given by $\left(p^{i}, p^{j}\right)=(-\mathfrak{t})^{i+j-5}$ if $i+j \geq 5$ and 0 otherwise. The ordered basis

$$
\left(\mathbb{T}^{0}, \mathbb{T}^{1}, \mathbb{T}^{2}, \mathbb{T}^{3}, \mathbb{T}^{4}, \mathbb{T}^{5}\right):=\left(p^{4}(p+\mathfrak{t}), p^{3}(p+\mathfrak{t}), p^{2}(p+\mathfrak{t}), p(p+\mathfrak{t}),(p+\mathfrak{t}), 1\right)
$$

is dual to the ordered basis

$$
\left(\mathbb{T}_{0}, \mathbb{T}_{1}, \mathbb{T}_{2}, \mathbb{T}_{3}, \mathbb{T}_{4}, \mathbb{T}_{5}\right):=\left(1, p, p^{2}, p^{3}, p^{4}, p^{5}\right)
$$

with respect to the $G$-equivariant Poincaré pairing, namely $\left(\mathbb{T}^{i}, \mathbb{T}_{j}\right)=\delta_{i, j}$.
Let $G^{\prime}=\mathbb{C}^{*}$ acts on $\mathbb{P}^{5}$ trivially. Consider the $G \times G^{\prime}$ equivariant bundle over $\mathbb{P}^{5}$

$$
\begin{equation*}
V=L_{H}^{\otimes 5} \oplus\left(L_{H} \otimes L_{\mathfrak{t}} \otimes L_{\mathfrak{t}^{\prime}}\right) \tag{5}
\end{equation*}
$$

where $\mathfrak{t}^{\prime}=c_{1}\left(\mathcal{O}_{B G^{\prime}}(-1)\right)$ (with $B G^{\prime} \cong \mathbb{C} \mathbb{P}^{\infty}$ ) denotes the first Chern class of $L_{\mathfrak{t}^{\prime}}$ and similarly for $\mathfrak{t}$ with the group $G^{\prime}$ replaced by $G$. Let $\left(\pi_{d}, \mathfrak{f}_{d}\right): \mathcal{C}_{d} \rightarrow \bar{M}_{0,1}\left(\mathbb{P}^{5}, d\right) \times \mathbb{P}^{5}$ be the universal family and $E_{d}=\pi_{d *} f_{d}^{*} V$.

Recall the conventions $q=e^{t}, f(q)=\sum_{d=0}^{\infty} \frac{(5 d)!}{(d!)^{5}} q^{d}=I_{0}(t), g_{l}(q)=\sum_{d=1}^{\infty} q^{d} \frac{(5 d)!}{(d!)^{5}}\left(\sum_{m=1}^{l d} \frac{1}{m}\right)$,

$$
I_{1}(t):=5\left[g_{5}-g_{1}\right](q)+t I_{0}(t), \quad T(t):=I_{1}(t) / I_{0}(t)
$$

We set $T_{0}=\hbar \log f+\mathfrak{t}^{\prime} \frac{g_{1}}{f}$. Then the equivariant mirror theorem Gi2] states that

$$
I(q)=q^{p / \hbar} 5 p\left(p+\mathfrak{t}+\mathfrak{t}^{\prime}\right) \sum_{d=0}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(5 p+m \hbar) \prod_{m=1}^{d}\left(p+\mathfrak{t}+\mathfrak{t}^{\prime}+m \hbar\right)}{\prod_{m=1}^{d}(p+m \hbar)^{5}(p+\mathfrak{t}+m \hbar)}
$$

is equal to, under the mirror map $\mathcal{Q}(q)=e^{I_{1}(t) / I_{0}(t)}=e^{T}$,

$$
J(\mathbb{Q})=e^{\left(T_{0}+p \log \mathfrak{Q}\right) / \hbar}\left(5 p\left(p+\mathfrak{t}+\mathfrak{t}^{\prime}\right)+\sum_{d=1}^{\infty} \mathbb{Q}^{d} \sum_{k=0}^{5} \mathbb{T}_{k} \int_{\overline{\mathcal{M}}_{0,1}\left(\mathbb{P}^{5}, d\right)} \mathrm{ev}^{*} \mathbb{T}^{k} \frac{e\left(E_{d}\right)}{\hbar(\hbar-\psi)}\right)
$$

3.2. Application to the MSP set-up. Recall $\mathcal{W}_{d}:=\mathcal{W}_{g=0,(1, \rho),(d, 0)}=\bar{M}_{0,1}\left(\mathbb{P}^{5}, d\right)$. Let

$$
F_{d}=\pi_{d *} f_{d}^{*} L_{H}^{\otimes 5}, \quad R_{d}=\pi_{d *} f_{d}^{*}\left(L_{H} \otimes L_{\mathfrak{t}}\right)
$$

Then rank $R_{d}=d+1$ and $E_{d}=F_{d} \oplus\left(R_{d} \otimes L_{\mathfrak{t}^{\prime}}\right)$. Thus

$$
e\left(E_{d}\right)=e\left(F_{d}\right) e\left(R_{d} \otimes L_{\mathfrak{t}^{\prime}}\right)=e\left(F_{d}\right) \sum_{k=0}^{\infty} c_{\mathrm{rank} R_{d}-k}\left(R_{d}\right)\left(\mathfrak{t}^{\prime}\right)^{k}
$$

With $\left[\overline{\mathcal{M}}_{0,1}\left(\mathbb{P}^{5}, d\right)\right] \cap e\left(F_{d}\right)=(-1)^{d+1}\left[\mathcal{W}_{g=0,(1, \rho),(d, 0)}\right]^{\text {vir }}$, the mirror theorem is rephrased as follows.
Corollary 3.1. The series $e^{-\left(T_{0}+p \log Q\right) / \hbar} I(q)$ is equal to

$$
5 p\left(p+\mathfrak{t}+\mathfrak{t}^{\prime}\right)-\sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \int_{\left[\mathcal{W}_{g=0,(1, \rho),(d, 0)}\right]^{\mathrm{vir}}}\left(\sum_{k=0}^{4} p^{k} \frac{e\left(R_{d} \otimes L_{\mathfrak{t}^{\prime}}\right)}{\hbar(\hbar-\psi)} \cup \operatorname{ev}_{1}^{*}\left[p^{4-k}(p+\mathfrak{t})\right]+p^{5} \frac{e\left(R_{d} \otimes L_{\mathfrak{t}^{\prime}}\right)}{\hbar(\hbar-\psi)}\right)
$$

For each $j \geq 0$, define

$$
U_{j}:=-\sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \int_{\left[\mathcal{W}_{g=0,(1, \rho),(d, 0)}\right]^{\text {vir }}} \frac{e\left(R_{d} \otimes L_{\mathfrak{t}^{\prime}}\right)}{\hbar(\hbar-\psi)} \cup \mathrm{ev}_{1}^{*} p^{j}
$$

Then $Y:=Y(p, q):=e^{-\left(T_{0}+p \log Q\right) / \hbar} I(q)$ in Corollary 3.1 equals

$$
5 p\left(p+\mathfrak{t}+\mathfrak{t}^{\prime}\right)+p^{0}\left(U_{5}+\mathfrak{t} U_{4}\right)+p^{1}\left(U_{4}+\mathfrak{t} U_{3}\right)+p^{2}\left(U_{3}+\mathfrak{t} U_{2}\right)+p^{3}\left(U_{2}+\mathfrak{t} U_{1}\right)+p^{4}\left(U_{1}+\mathfrak{t} U_{0}\right)+p^{5} U_{0}
$$

We also write $Y(p)=\sum_{j=0}^{5}\left(\operatorname{Coe}_{p^{j}} Y\right) p^{j}$. We define

$$
\begin{aligned}
\tilde{z}_{1,5}:= & -\sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\Gamma) \in \Xi_{d}^{1}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \frac{e\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma}\right)\right)\right)}{e\left(N_{\mathcal{W}_{(\Gamma)} / \mathcal{W}}^{v i r}\right)} \\
& =-\frac{1}{\mathfrak{t}^{\prime}} \sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\Gamma) \in \Xi_{d}^{1}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \frac{e\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\right)\right)}{e\left(N_{\mathcal{W}_{(\Gamma)} / \mathcal{W}}^{v i r}\right)}
\end{aligned}
$$

where $\Xi_{n}^{1}$ denotes localization graphs with the first marking mapped to $O$ and $\mathcal{L}_{\Gamma}$ is the universal line bundle. Then, as $\left.p^{4}\right|_{P_{i}}=0$ for $0 \leq i \leq 4$ and $\left.p\right|_{O}=-\mathfrak{t}$, by the localization formula, one has $U_{j}=(-\mathfrak{t})^{j} \cdot \mathfrak{t}^{\prime} \cdot \tilde{z}_{1,5}$ whenever $j \geq 4$. Thus $\operatorname{Coe}_{p^{0}} Y=0$ and

$$
\begin{aligned}
\mathfrak{t}^{4} \mathfrak{t}^{\prime} \tilde{z}_{1,5} & =U_{4}=-5\left(\mathfrak{t}+\mathfrak{t}^{\prime}\right)+\operatorname{Coe}_{p^{1}} Y-\mathfrak{t} U_{3}=-5\left(\mathfrak{t}+\mathfrak{t}^{\prime}\right)+\operatorname{Coe}_{p^{1}} Y-\mathfrak{t}\left(\operatorname{Coe}_{p^{2}} Y-\mathfrak{t} U_{2}-5\right) \\
& =-5 \mathfrak{t}^{\prime}+\operatorname{Coe}_{p^{1}} Y-\mathfrak{t} \operatorname{Coe}_{p^{2}} Y+\mathfrak{t}^{2} \operatorname{Coe}_{p^{3}} Y-\mathfrak{t}^{3} \operatorname{Coe}_{p^{4}} Y+\mathfrak{t}^{4} \operatorname{Coe}_{p^{5}} Y
\end{aligned}
$$

implies $-\mathfrak{t}^{5} \mathfrak{t}^{\prime} \tilde{z}_{1,5}=5 \mathfrak{t t ^ { \prime }}+\left.Y\right|_{p \mapsto-\mathfrak{t}}$. Thus we have
Theorem 3.2. For MSP moduli $\mathcal{W}_{g=0,(1, \rho),(d, 0)}$,

$$
\sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\Gamma) \in \Xi_{d}^{1}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\hbar\left(\hbar-\psi_{1}\right)} \frac{e\left(\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \otimes \mathcal{L}_{\mathrm{t}} \otimes \mathcal{L}_{\mathrm{t}^{\prime}}\left(-D_{\Gamma}\right)\right)\right)}{e\left(N_{\mathcal{W}_{\Gamma} / \mathcal{W}}^{v i r}\right)}
$$

(here $-D_{\Gamma} \subset \mathcal{C}_{\mathcal{W}_{\Gamma}}$ is from the marking) is equal to

$$
\frac{5}{\mathfrak{t}^{4}}+\frac{-5}{\mathfrak{t}^{4}} \frac{1}{I_{0}} \exp \left(\frac{\mathfrak{t}(T-\mathfrak{t})}{\hbar}-\frac{\mathfrak{t}^{\prime} g_{1}}{I_{0} \hbar}\right) \cdot \sum_{d=0}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(-5 \mathfrak{t}+m \hbar) \prod_{m=1}^{d}\left(\mathfrak{t}^{\prime}+m \hbar\right)}{d!\hbar^{d} \prod_{m=1}^{d}(-\mathfrak{t}+m \hbar)^{5}}
$$

We then have the following package for $c_{i}$-capped invariants.
Theorem 3.3. For MSP moduli $\mathcal{W}_{g=0,(1, \rho),(d, 0)}, \forall m \in \mathbb{N}_{\geq 1}$

$$
\sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\Gamma) \in \Xi_{d}^{1}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\mathrm{vir}}} \frac{1}{\hbar\left(\hbar-\psi_{1}\right)} \frac{c_{\mathrm{top}_{\Gamma}-m}\left(\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \otimes \mathcal{L}_{\mathrm{t}}\right)\right)}{e\left(N_{\mathcal{W}_{\Gamma} / \mathcal{W}}^{v i r}\right)}
$$

is the $\left(\mathfrak{t}^{\prime}\right)^{m-1}$ coefficient of

$$
\frac{5}{\mathfrak{t}^{4}}+\frac{-5}{\mathfrak{t}^{4}} \frac{1}{I_{0}} \exp \left(\frac{\mathfrak{t}(T-\mathfrak{t})}{\hbar}-\frac{\mathfrak{t}^{\prime} g_{1}}{I_{0} \hbar}\right) \cdot \sum_{d=0}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(-5 \mathfrak{t}+m \hbar) \prod_{m=1}^{d}\left(\mathfrak{t}^{\prime}+m \hbar\right)}{d!\hbar^{d} \prod_{m=1}^{d}(-\mathfrak{t}+m \hbar)^{5}}
$$

In the special case $\mathfrak{t}^{\prime}=-\mathfrak{t}$ of Corollary 3.1, one has the following.
Corollary 3.4. Write $\left.\mathfrak{A}_{d}=\pi_{d *}\right\}_{d}^{*} L_{H}$. It is a rank $d+1$ bundle over $\mathcal{W}_{0,(1, \rho),(d, 0)}$. The series

$$
5 p^{2}-\sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \sum_{k=0}^{4} \int_{\left[\mathcal{W}_{g=0,(1, \rho),(d, 0)}\right]_{\mathrm{vir}}}\left(p^{k} \frac{e\left(\mathfrak{A}_{d}\right)}{\hbar(\hbar-\psi)} \cup \operatorname{ev}_{1}^{*}\left[p^{4-k}(p+\mathfrak{t})\right]+p^{5} \frac{e\left(\mathfrak{A}_{d}\right)}{\hbar(\hbar-\psi)}\right)
$$

is equal to

$$
\begin{equation*}
\frac{5 p^{2}}{I_{0}} e^{\frac{1}{\hbar}\left(\frac{g_{1}}{T_{0}} \mathfrak{t}+p(t-T)\right)} \sum_{d=0}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(5 p+m \hbar)}{\prod_{m=1}^{d}(p+m \hbar)^{4}(p+\mathfrak{t}+m \hbar)} \tag{6}
\end{equation*}
$$

Taking coefficients of $\mathfrak{t}^{\prime}$ of the formula in Corollary 3.1, we obtain the following identity.
Corollary 3.5. The series

$$
5 p-\sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \sum_{k=0}^{4} \int_{\left[\mathcal{W}_{g=0,(1, \rho),(d, 0)}\right]^{\text {vir }}}\left(p^{k} \frac{c_{d}\left(R_{d}\right)}{\hbar(\hbar-\psi)} \cup \operatorname{ev}_{1}^{*}\left[p^{4-k}(p+\mathfrak{t})\right]+p^{5} \frac{c_{d}\left(R_{d}\right)}{\hbar(\hbar-\psi)}\right)
$$

is equal to the coefficient of $\mathfrak{t}^{\prime}$ in $e^{-\left(T_{0}+p \log Q\right) / \hbar} I(0, q)$, which is

$$
\frac{5 p(p+\mathfrak{t})}{I_{0}} e^{\frac{p}{\hbar}(t-T)}\left[\left(\frac{1}{p+\mathfrak{t}}-\frac{g_{1}}{\hbar I_{0}}\right) \sum_{d=0}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(5 p+m \hbar)}{\prod_{m=1}^{d}(p+m \hbar)^{5}}+\sum_{d=1}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(5 p+m \hbar)\left(\sum_{m=1}^{d} \frac{1}{p+\mathfrak{t}+m \hbar}\right)}{\prod_{m=1}^{d}(p+m \hbar)^{5}}\right] .
$$

3.3. From MSP to Givental I-functions. Consider

$$
\begin{gather*}
B_{d}(h):=\frac{\prod_{k=1}^{5 d}\left(-5 h+\frac{k(h+\mathfrak{t})}{d}\right)}{\prod_{k=1}^{d}\left(h-\frac{k(h+\mathfrak{t})}{d}\right)^{5}}=b_{0}+b_{1} \frac{h}{\mathfrak{t}}+b_{2} \frac{h^{2}}{\mathfrak{t}^{2}}+b_{3} \frac{h^{3}}{\mathfrak{t}^{3}} \in \mathbb{Q}\left[\mathfrak{t}, \mathfrak{t}^{-1}\right][h] /\left(h^{4}\right)  \tag{7}\\
A_{d}(h):=\frac{\prod_{k=1}^{5 d}(5 h+k)}{\prod_{k=1}^{d}(h+k)^{5}}=a_{0}+a_{1} h+a_{2} h^{2}+a_{3} h^{3} \in \mathbb{Q}[h] /\left(h^{4}\right)
\end{gather*}
$$

where both $a_{i}=a_{i}(d), b_{i}=b_{i}(d)$ are rational numbers. Apply the map $\theta: \mathbb{Q}\left[\mathfrak{t}, \mathfrak{t}^{-1}\right][h] /\left(h^{4}\right) \rightarrow$ $\mathbb{Q}[h] /\left(h^{4}\right)$ which sends $\mathfrak{t}$ to $(-1-h)$. Since $(-1-h)$ is invertible in $\mathbb{Q}[h] /\left(h^{4}\right)$, this defines a ring homomorphism. Then $\theta\left(B_{d}(h)\right)=(-1)^{5 d} A_{d}(d h)$ implies

$$
b_{0}=(-1)^{d} a_{0}, \quad b_{1}=(-1)^{d+1} a_{1} d, \quad b_{2}=(-1)^{d}\left[a_{2} d^{2}+a_{1} d\right], \quad b_{3}=(-1)^{d+1}\left[a_{3} d^{3}+2 a_{2} d^{2}+a_{1} d\right]
$$

Recall the definition of Givental's I function

$$
\begin{equation*}
I_{0}(t)+I_{1}(t) h+I_{2} h^{2}+\cdots+I_{3} h^{3}=e^{h t} \sum_{d=0}^{\infty} e^{d t} \frac{\prod_{r=1}^{5 d}(5 h+r)}{\prod_{r=1}^{d}(h+r)^{5}}=e^{h t} \sum_{d=0}^{\infty} e^{d t} A_{d}(h)\left(\bmod h^{4}\right) \tag{8}
\end{equation*}
$$

Recall $D_{t}:=q \frac{\partial}{\partial q}=\frac{\partial}{\partial t}$, and denote $D_{t} I_{k}$ as $I_{k}^{\prime}$. The relations in (8) imply
(9) $1+\sum_{d=1}^{\infty} b_{0}(-q)^{d}=1+\sum_{d=1}^{\infty} a_{0}(d) q^{d}=I_{0}$,

$$
\sum_{d=1}^{\infty} b_{1}(-q)^{d}=-\sum_{d=1}^{\infty} d a_{1}(d) q^{d}=-D_{t} I_{1}+D_{t}\left(t I_{0}\right)
$$

$$
\sum_{d=1}^{\infty} b_{2}(-q)^{d}=\sum_{d=1}^{\infty}\left[a_{2} d^{2}+a_{1} d\right] q^{d}=D_{t}^{2} I_{2}-D_{t}\left(t I_{1}^{\prime}\right)+D_{t}\left(\frac{t^{2}}{2} I_{0}^{\prime}\right)
$$

$$
\sum_{d=1}^{\infty} b_{3}(-q)^{d}=-\sum_{d=1}^{\infty}\left[a_{3} d^{3}+2 a_{2} d^{2}+a_{1} d\right] q^{d}=-D_{t}^{3} I_{3}+D_{t}\left(t I_{2}^{\prime \prime}\right)-D_{t}\left(\frac{t^{2}}{2} I_{1}^{\prime \prime}\right)+D_{t}\left(\frac{t^{3}}{6} I_{0}^{\prime \prime}\right)
$$

## 4. $\mathcal{K}$-series and $\mathcal{P}$-series

4.1. Definitions of $\mathcal{K}$-series and $\mathcal{P}$-series. Consider the localization of $\mathcal{W}_{g=0,1_{\rho},(n, 0)}$. Let $\Xi_{n}$ be the set of flat $g=0$ graphs with one leg decorated by $1_{\rho}$ and $\left(d_{0}, d_{\infty}\right)=(n, 0)$, and let $\Xi_{n}^{i} \subset \Xi_{n}, i=0$ or 1 , be the subset where the marking lies on vertices in $V_{i}$.

For $c>0$ and $k=0, \cdots, 5$, we set a formal power series in $\mathbb{Q}(\mathfrak{t})[[Q]]$

$$
\begin{equation*}
\mathcal{K}_{c, k}(\mathfrak{t}):=\sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \sum_{(\Gamma) \in \hat{\Xi}_{d}^{0}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\mathrm{vir}}} \frac{1}{\psi^{c}} \operatorname{ev}^{*} \mathbb{T}^{k} \frac{c_{\mathrm{top}_{\Gamma}-1}\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \otimes \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma}\right)\right)\right)}{e\left(N_{\mathcal{W}_{(\Gamma)}^{v i r} / \mathcal{W}}^{v i}\right)} \in \mathbb{Q}[[\mathcal{Q}]]^{2-k-c} \tag{10}
\end{equation*}
$$

where $\hat{\Xi}_{d}^{0}$ denotes the subset of graphs in $\Xi_{d}^{0}$ whose vertices contain an unstable $1_{\rho}$-leg, $D_{\Gamma} \subset \mathcal{C}_{\mathcal{W}_{\Gamma}}$ is the divisor given by the marking, and top ${ }_{\Gamma}$ is the rank of the bundle $R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \otimes \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma}\right)\right)$.

For convenience, we also consider formal power series in $\mathbb{Q}(\mathfrak{t})\left[\left[\mathfrak{t}^{\prime}\right]\right][[Q]]$

$$
\begin{aligned}
\tilde{\mathcal{K}}_{c, k}\left(\mathfrak{t}, \mathfrak{t}^{\prime}\right) & :=\sum_{d=1}^{\infty}(-\mathbb{Q})^{d} \sum_{(\Gamma) \in \hat{\Xi}_{d}^{0}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\psi^{c}} e v^{*} \mathbb{T}^{k} \frac{e\left(\pi_{\Gamma *} \mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma}\right)\right)}{e\left(N_{\mathcal{W}_{(\Gamma)}^{v i r} / \mathcal{W}}^{v i r}\right.} \\
\hat{\mathcal{K}}_{k}(w) & :=\sum_{c=1}^{\infty} \tilde{\mathcal{K}}_{c, k}(-w)^{c-1}
\end{aligned}
$$

$$
=\sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\Gamma) \in \hat{\Xi}_{d}^{0}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]} \frac{1}{\text { vir }} \frac{\lim ^{*}+T^{k}}{\psi^{k}+\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \otimes \mathcal{L}_{\mathfrak{t}} \otimes \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma}\right)\right)\right)} \underset{e\left(N_{\mathcal{W}_{(\Gamma)}}^{v i r} / \mathcal{W}\right)}{ }
$$

which is an expansion near $w=0$. Note that

$$
e\left(\pi_{\Gamma *} \mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma}\right)\right)=\sum_{\ell=0}^{\infty} c_{\mathrm{top}_{\Gamma}-\ell}\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \otimes \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma}\right)\right)\right)\left(\mathfrak{t}^{\prime}\right)^{\ell}
$$

By definition, $\mathcal{K}_{c, k}$ is the coefficient of $\left(\mathfrak{t}^{\prime}\right)^{1}$ in $\tilde{\mathcal{K}}_{c, k}$. Observe that the constant term of $\mathfrak{t}^{\prime}$-expansion of $\tilde{\mathcal{K}}_{c, k}$ vanishes because $\nu_{1}$ makes $e\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma}\right)\right)\right)$ vanishing. We call this cap geometry, and write the expansion

$$
\begin{equation*}
\tilde{\mathcal{K}}_{c, k}=\mathcal{K}_{c, k} \mathfrak{t}^{\prime}+\mathcal{K}_{c, k, 1}\left(\mathfrak{t}^{\prime}\right)^{2}+\cdots+\mathcal{K}_{c, k, \ell}\left(\mathfrak{t}^{\prime}\right)^{\ell+1}+\cdots \tag{11}
\end{equation*}
$$

We define $\mathcal{P}$-series as follows. For $c>0$, we set a formal power series in $\mathbb{Q}(\mathfrak{t})[[Q]]$

$$
\begin{gathered}
\mathcal{P}_{c}(\mathfrak{t}):=\sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \sum_{(\Gamma) \in \hat{\Xi}_{d}^{1}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\psi^{c}} \frac{c_{\mathrm{top}_{\Gamma}}\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \otimes \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma}\right)\right)\right)}{e\left(N_{\left.\mathcal{W}_{(\Gamma)}\right) / \mathcal{W}}^{v i r}\right)}, \\
\tilde{\mathcal{P}}_{c}\left(\mathfrak{t} . \mathfrak{t}^{\prime}\right):=\sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\Gamma) \in \hat{\Xi}_{d}^{1}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\mathrm{vir}}} \frac{1}{\psi^{c}} \frac{e\left(\pi_{\Gamma *} \mathcal{L}_{\Gamma} \mathcal{L}_{L^{\prime}} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma}\right)\right)}{e\left(N_{\mathcal{W}_{(\Gamma)}^{v} / \mathcal{W}}^{v i r}\right)}
\end{gathered}
$$

Let $\hat{\Xi}_{d}^{1}$ be the subset of graphs in $\Xi_{d}^{1}$ whose vertices contain the unstable $1_{\rho}$-leg. We also define

$$
\begin{gather*}
\hat{\mathcal{P}}(w):=\sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\Gamma) \in \hat{\Xi}_{d}^{1}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{w+\psi_{\Gamma}} \frac{e\left(\pi_{\Gamma *} \mathcal{L}_{\Gamma} \mathcal{L}_{t} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma}\right)\right)}{e\left(N_{\mathcal{W}_{(\Gamma)}^{v i r}}^{v i \gamma}\right)}=\tilde{\mathcal{P}}_{1}-w \tilde{\mathcal{P}}_{2}+w^{2} \tilde{\mathcal{P}}_{3}-\cdots,  \tag{12}\\
\tilde{\mathcal{Z}}_{j, k}:=-\sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\Gamma) \in \Xi_{d}^{j}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} \mathbb{T}^{k} \frac{e\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{t} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma}\right)\right)\right)}{e\left(N_{\mathcal{W}_{(\Gamma)}^{v i r} / \mathcal{W}}^{v i r}\right)}, \quad \text { for } j=0,1 \\
z_{j, k}:=-\sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\Gamma) \in \Xi_{d}^{j}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} \mathbb{T}^{k} \frac{c_{\text {top }-1}\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma}\right)\right)\right)}{e\left(N_{\mathcal{W}_{(\Gamma)}^{v i r} / \mathcal{W}}^{v i r}\right)}, \quad \text { for } j=0,1 \tag{13}
\end{gather*}
$$

Lemma 4.1. For $c \geq 1$, we have

$$
\tilde{\mathcal{K}}_{c, k}=\operatorname{Res}_{\hbar=0} \frac{\tilde{z}_{0, k}}{\hbar^{c-1}}, \quad \mathcal{K}_{c, k}=\operatorname{Res}_{\hbar=0} \frac{z_{0, k}}{\hbar^{c-1}}, \quad \tilde{\mathcal{P}}_{c}=\operatorname{Res}_{\hbar=0} \frac{\tilde{z}_{1,5}}{\hbar^{c-1}}, \quad \mathcal{P}_{c}=\operatorname{Res}_{\hbar=0} \frac{\mathcal{Z}_{1,5}}{\hbar^{c-1}} .
$$

Proof. We prove the first two identities. In the case $(\Gamma) \in \hat{\Xi}_{d}^{0}$, its $\psi$ is invertible and the expansion near $\hbar=0$ gives $\frac{1}{\hbar-\psi}=\frac{-1}{\psi}\left(1+\frac{\hbar}{\psi}+\frac{\hbar^{2}}{\psi^{2}}+\cdots\right)$. And in the case $(\Gamma) \in \Xi_{d}^{0}-\hat{\Xi}_{d}^{0}$, its $\psi^{r}$ will vanish on $\left[\mathcal{W}_{\Gamma}\right]^{\text {vir }}$ for some $r$. Thus we can write $\frac{1}{\hbar-\psi}=\frac{1}{\hbar}\left(1+\frac{\psi}{\hbar}+\cdots+\frac{\psi^{r}}{\hbar^{r}}\right), \forall \hbar \neq 0$. Therefore such $\Gamma$ doesn't contribute to the residue.
4.2. Calculations of $\tilde{z}_{1, k}$. For $k<5$, one has $\mathbb{T}^{k}=p^{4-k}(p+\mathfrak{t})$ and thus $\left.\mathbb{T}^{k}\right|_{O}=0$ implies $\tilde{\mathcal{Z}}_{1, k}=0$. Thus

$$
-\sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\Gamma) \in \Xi_{d}^{1}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \frac{e\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma}\right)\right)\right)}{e\left(N_{\left.\mathcal{W}_{(\Gamma)}\right) / \mathcal{W}}^{v i r}\right)} \mathrm{ev}^{*} p^{k}=(-\mathfrak{t})^{k} \cdot \tilde{z}_{1,5}
$$

When $k=5$, by Theorem 3.2 we have

$$
\begin{align*}
\tilde{z}_{1,5} & =-\sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \sum_{(\Gamma) \in \Xi_{d}^{1}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \frac{e\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma}\right)\right)\right)}{e\left(N_{\mathcal{W}_{(\Gamma)}^{v i r} / \mathcal{W}}^{v i r}\right)} \\
& =-\frac{5}{\mathfrak{t}^{4}}+\frac{5}{\mathfrak{t}^{4} I_{0}} \exp \left(\frac{\mathfrak{t}(T-t)}{\hbar}-\frac{\mathfrak{t}^{\prime} g_{1}}{I_{0} \hbar}\right) \cdot \sum_{d=0}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(-5 \mathfrak{t}+m \hbar) \prod_{m=1}^{d}\left(\mathfrak{t}^{\prime}+m \hbar\right)}{d!\hbar^{d} \prod_{m=1}^{d}(-\mathfrak{t}+m \hbar)^{5}} . \tag{14}
\end{align*}
$$

We write

$$
w:=-\frac{\mathfrak{t}}{\hbar}, \mathcal{R}(w, t):=\sum_{d=0}^{\infty} q^{d} \frac{\prod_{k=1}^{5 d}(5 w+k)}{\prod_{k=1}^{d}(w+k)^{5}} \in \mathbb{Q}[[q, w]], \Lambda:=\frac{1}{I_{0}} e^{(t-T) w} \mathcal{R}(w, t)
$$

## Corollary 4.1.

$$
\begin{gather*}
z_{1,5}=\operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{0}} \tilde{z}_{1,5}=-\frac{5}{\mathfrak{t}^{4}}+\frac{5}{\mathfrak{t}^{4}} \Lambda  \tag{15}\\
\sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\Gamma) \in \Xi_{d}^{1}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \frac{c_{d}\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma}\right)\right)\right)}{e\left(N_{\mathcal{W}_{(\Gamma)}^{v i r} / \mathcal{W}}^{v i r}\right)} e v^{*} p^{k}=5(-\mathfrak{t})^{k-4}(1-\Lambda)
\end{gather*}
$$

4.3. Calculation of $\mathcal{Z}_{0, k}$. Assume $k<5$, then $\mathbb{T}^{k}=p^{4-k}(p+\mathfrak{t})$. We have for each $(\Gamma) \in \Xi_{d}^{0}$,

$$
\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\right)=\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma}\right)\right)+\pi_{\Gamma *}\left(\left.\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\right|_{D_{\Gamma}}\right)=\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma}\right)\right)+\mathrm{ev}^{*} L_{H} L_{\mathfrak{t}},
$$

as classes in $K$-groups, and thus

$$
\begin{align*}
& c_{d}\left(\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\right)\right)=e\left(\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma}\right)\right)\right)+c_{d-1}\left(\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma}\right)\right)\right) \cdot c_{1}\left(\operatorname{ev}^{*} L_{H} L_{\mathfrak{t}}\right), \\
& \mathcal{Z}_{0, k}=-\sum_{d=1}^{\infty}(-\mathbb{Q})^{d} \sum_{(\Gamma) \in \Xi_{d}^{0}} \int_{\left[\mathcal{W}_{(\Gamma)}\right] \text { vir }} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} p^{4-k}(p+\mathfrak{t}) \frac{c_{d-1}\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma}\right)\right)\right)}{e_{G}\left(N_{\mathcal{W}_{(\Gamma)}^{v i r} / \mathcal{W}}^{v}\right)} \\
& =-\sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\Gamma) \in \Xi_{d}^{0}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} p^{4-k} \frac{c_{d}\left(\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\right)\right)}{e_{G}\left(N_{\mathcal{W}_{(\Gamma)} / \mathcal{W}}^{v i r}\right)}  \tag{16}\\
& +\sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \sum_{(\Gamma) \in \Xi_{d}^{0}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} p^{4-k} \frac{e\left(\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma}\right)\right)\right)}{e_{G}\left(N_{\mathcal{W}_{(\Gamma)} / \mathcal{W}}^{v i r}\right)}
\end{align*}
$$

The two integrals in (16) can be separately reduced as follows.
[i]. The first integral in (16) can be calculated as

$$
\begin{align*}
& -\sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \sum_{(\Gamma) \in \Xi_{d}^{0}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} e v^{*} p^{4-k} \frac{c_{d}\left(\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\right)\right)}{e_{G}\left(N_{\mathcal{W}_{(\Gamma)}^{v i r} / \mathcal{W}}\right)} \\
= & -\sum_{d=1}^{\infty}(-Q)^{d} \operatorname{Coe}_{\mathfrak{t}^{\prime}} \int_{\left[\mathcal{W}_{0,1_{\rho},(d, 0)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} p^{4-k} \cdot e\left(\pi_{\mathcal{W}_{*}}\left(\mathcal{L}_{\mathcal{W}} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\right)\right)  \tag{17}\\
+ & \sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\Gamma) \in \Xi_{d}^{1}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} p^{4-k} \frac{c_{d}\left(\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\right)\right)}{e_{G}\left(N_{\mathcal{W}_{(\Gamma)}^{v i r} / \mathcal{W}}^{v i}\right)}
\end{align*}
$$

where $\pi_{\mathcal{W}}$ is the projection from the universal family $\mathcal{C}_{\mathcal{W}}$ to $\mathcal{W}$ and $\mathcal{L}_{\mathcal{W}}$ is the pull-back of $\mathcal{O}_{\mathbb{P}^{5}}(1)$ via the universal family map $\mathcal{C}_{\mathcal{W}} \rightarrow \mathbb{P}^{5}$. Denote the first term in (17) as

$$
\mathfrak{B}_{k}:=-\operatorname{Coe}_{\mathfrak{t}^{\prime}} \sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \int_{\left[\mathcal{W}_{0,1_{\rho},(d, 0)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} p^{4-k} \cdot e\left(\pi_{\mathcal{W} *}\left(\mathcal{L}_{\mathcal{W}} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\right)\right)
$$

The second term in (17) is, by Corollary 4.1,

$$
\begin{aligned}
& \sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\Gamma) \in \Xi_{d}^{1}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} p^{4-k} \frac{c_{d}\left(\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\right)\right)}{e_{G}\left(N_{\mathcal{W}_{(\Gamma)}^{v i r} / \mathcal{W}}\right)} \\
= & 5(-\mathfrak{t})^{-k}\left(1-\frac{1}{I_{0}} e^{\frac{t}{\hbar} \cdot(T-t)} \sum_{d=0}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(-5 \mathfrak{t}+m \hbar)}{\prod_{m=1}^{d}(-\mathfrak{t}+m \hbar)^{5}}\right) .
\end{aligned}
$$

[ii]. The second integral in (16), denoted by $\mathfrak{R}_{k}$, is only contributed by those $\Gamma \in \Xi_{d}^{0}$ which have only one vertex $v$ in $V_{0}\left(\right.$ denote such graph as $\left.\Gamma_{v}\right)$, otherwise, $\nu_{1}$ makes $e\left(\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma}\right)\right)\right)$ vanish. More precisely

$$
\begin{aligned}
\mathfrak{R}_{k} & =\sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \int_{(-1)^{d+1}\left[\bar{M}_{0,1}(Q, d)\right]^{\text {vir }}} \frac{1}{\hbar^{2}}\left(1+\frac{\psi}{\hbar}\right) \cdot \mathrm{ev}^{*} h^{4-k} \cdot \frac{1}{e\left(\pi_{\Gamma *}\left(\left.\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\right|_{D_{\Gamma}}\right)\right)} \\
& =-\sum_{d=1}^{\infty} \mathbb{Q}^{d} \int_{\left[\bar{M}_{0,1}(Q, d)\right] \text { vir }} \frac{1}{\hbar^{2}}\left(1+\frac{\psi}{\hbar}\right) \cdot \mathrm{ev}^{*} h^{4-k} \cdot \mathrm{ev}^{*}\left(1-\frac{h}{\mathfrak{t}}\right) \frac{1}{\mathfrak{t}}
\end{aligned}
$$

where $h$ is the hyperplane class in $\mathbb{P}^{4}$. We calculate $\mathfrak{R}_{k}$ for different $k$ 's.

$$
\begin{aligned}
\mathfrak{R}_{4} & =-\sum_{d=1}^{\infty} \mathbb{Q}^{d} \int_{\left[\bar{M}_{0,1}(Q, d)\right]^{\text {vir }}} \frac{1}{\hbar^{2}} \cdot \operatorname{ev}^{*}\left(-\frac{h}{\mathfrak{t}}\right) \frac{1}{\mathfrak{t}}-\sum_{d=1}^{\infty} \mathbb{Q}^{d} \int_{\left[\bar{M}_{0,1}(Q, d)\right]^{\text {vir }}} \frac{1}{\hbar^{2}} \frac{\psi}{\hbar} \cdot \frac{1}{\mathfrak{t}} \\
& =\sum_{d=1}^{\infty} \mathbb{Q}^{d}\left(\frac{1}{\hbar^{2} \mathfrak{t}^{2}} d N_{0, d}-\frac{1}{\hbar^{3} \mathfrak{t}}(-2) N_{0, d}\right)
\end{aligned}
$$

Let $k<4$. The integral $\Re_{k}$ becomes

$$
-\sum_{d=1}^{\infty} Q^{d} \int_{\left[\bar{M}_{0,1}(Q, d)\right]^{\mathrm{vir}}} \frac{1}{\hbar^{2}} \mathrm{ev}^{*} h^{4-k} \cdot \frac{1}{\mathfrak{t}}= \begin{cases}-\sum_{d=1}^{\infty} Q^{d} \frac{1}{\hbar^{2} \mathfrak{t}} d N_{0, d}, & k=3 \\ 0 & k=0,1,2\end{cases}
$$

Therefore we proved the following.
Lemma 4.2. For $k=0,1, \cdots, 4$, we have

$$
z_{0, k}=\mathfrak{B}_{k}+5(-\mathfrak{t})^{-k}(1-\Lambda)+\mathfrak{R}_{k},
$$

where $\mathfrak{R}_{k}=0$ for $k=0,1,2$ and

$$
\mathfrak{R}_{3}=-\sum_{d=1}^{\infty} \mathbb{Q}^{d} \frac{1}{\hbar^{2} \mathfrak{t}} d N_{0, d}, \quad \mathfrak{R}_{4}=\sum_{d=1}^{\infty} \mathbb{Q}^{d}\left(\frac{1}{\hbar^{2} \mathfrak{t}^{2}} d N_{0, d}-\frac{1}{\hbar^{3} \mathfrak{t}}(-2) N_{0, d}\right)
$$

We now calculate $\mathfrak{B}_{k}$ for $k=0,1,2$.

$$
\begin{aligned}
\mathfrak{B}_{0} & =-\frac{1}{\mathfrak{t}} \operatorname{Coe}_{\mathfrak{t}^{\prime}} \sum_{d=1}^{\infty}(-\mathbb{Q})^{d} \int_{\left[\mathcal{W}_{0,1 \rho,(d, 0)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} p^{4}(p+\mathfrak{t}) \cdot e\left(\pi_{\mathcal{W}_{*}}\left(\mathcal{L}_{\mathcal{W}} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\right)\right) \\
& +\frac{1}{\mathfrak{t}} \operatorname{Coe}_{\mathfrak{t}^{\prime}} \sum_{d=1}^{\infty}(-\mathbb{Q})^{d} \int_{\left[\mathcal{W}_{0,1 \rho,(d, 0)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} p^{5} \cdot e\left(\pi_{\mathcal{W} *}\left(\mathcal{L}_{\mathcal{W}} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\right)\right) \\
& \left.=\frac{-\mathfrak{t}^{5}}{\mathfrak{t}} \sum_{d=1}^{\infty}(-\mathbb{Q})^{d} \sum_{(\Gamma) \in \Xi_{d}^{1}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \cdot \frac{c_{\text {top }-1}\left(\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}}\right)\right)}{e\left(N_{\mathcal{W}}^{\text {vir }} / \mathcal{W}\right.}\right) \\
& =-\mathfrak{t}^{4} \cdot\left(\frac{5}{\mathfrak{t}^{4}}+\frac{-5}{\mathfrak{t}^{4}} \frac{1}{I_{0}} \exp \left(\frac{\mathfrak{t}(T-t)}{\hbar}\right) \cdot \sum_{d=0}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(-5 \mathfrak{t}+m \hbar)}{\prod_{m=1}^{d}(-\mathfrak{t}+m \hbar)^{5}}\right)=5 \Lambda(w, t)-5
\end{aligned}
$$

where the third identity is by $G$-localization and the last line comes from Theorem 3.3 ,
Taking $\mathrm{Coe}_{p}$ of the Corollary 3.5, one has

$$
\begin{aligned}
\mathfrak{B}_{1} & \left.=-\frac{1}{\mathfrak{t}} \sum_{d=1}^{\infty}(-\mathbb{Q})^{d} \int_{\left[\mathcal{W}_{0,1},(d, 0)\right.}\right]^{\mathrm{vir}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} p^{3}(p+\mathfrak{t}) \cdot c_{\mathrm{top}-1}\left(\pi_{\mathcal{W} *}\left(\mathcal{L}_{\mathcal{W}} \mathcal{L}_{\mathfrak{t}}\right)\right) \\
& +\frac{1}{\mathfrak{t}} \sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \int_{\left[\mathcal{W}_{0,1_{\rho},(d, 0)}\right]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} p^{4} \cdot c_{\mathrm{top}-1}\left(\pi_{\mathcal{W} *}\left(\mathcal{L}_{\mathcal{W}} \mathcal{L}_{\mathfrak{t}}\right)\right)
\end{aligned}
$$

$$
=-\frac{5}{\mathfrak{t}} \Lambda(w, t)+\frac{-5 w}{\mathfrak{t} I_{0}}\left[\left(-\frac{I_{0}}{w}-g_{1}\right)+\sum_{d=1}^{\infty} q^{d} \frac{(5 d)!}{(d!)^{5}}\left(\sum_{m=1}^{d} \frac{1}{m-w}\right)\right]
$$

Taking Coe $_{p^{2}}$ of Corollary 3.5 we get

$$
\begin{aligned}
\mathfrak{B}_{2}= & \left.-\frac{1}{\mathfrak{t}} \sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \int_{\left[\mathcal{W}_{0,1},(d, 0)\right.}\right]^{\text {vir }} \\
& +\frac{1}{\mathfrak{t}} \sum_{d=1}^{\infty}(-Q)^{d} \int_{\left[\mathcal{W}_{0,1_{\rho},(d, 0)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} \operatorname{ev}^{*} p^{2}(p+\mathfrak{t}) \cdot c_{\mathrm{top}-1}\left(\pi_{\mathcal{W} *}\left(\mathcal{L}_{\mathcal{W}} \mathcal{L}_{\mathfrak{t}}\right)\right) \\
= & -\frac{1}{\mathfrak{t}} \mathfrak{B}_{1}-\frac{5}{I_{0}} \cdot \frac{g_{1}}{\hbar \mathfrak{t}} \\
& +\frac{5}{I_{0}} \sum_{d=1}^{\infty} q^{d} \frac{(5 d)!}{(d!)^{5}}\left(\left(\frac{1}{\mathfrak{t}}-\frac{T-t}{\hbar}+\sum_{\mathcal{W}_{*}}\left(\mathcal{L}_{\mathcal{W}} \mathcal{L}_{\mathfrak{t}}\right)\right)\right. \\
& \left.\left.\frac{5 d}{m \hbar}\right) \sum_{m=1}^{d} \frac{1}{\mathfrak{t}+m \hbar}-\sum_{m=1}^{d} \frac{1}{(\mathfrak{t}+m \hbar)^{2}}\right)
\end{aligned}
$$

Here we omit detailed calculations obtaining the second equality above.
4.4. Formula of $\mathcal{K}_{1,2}$ and $\mathcal{K}_{2,1}$. For $k=1,2$, we can now apply $\mathcal{K}_{c, k}=\operatorname{Res}_{\hbar=0} \frac{z_{0, k}}{\hbar^{c-1}}$.

Now we look at the case $k=1$. The residue at $\hbar=0$ of $5 \mathfrak{t}^{-1}(1-\Lambda)+\mathfrak{R}_{1}$ divided by $\hbar$ is

$$
\frac{5}{\mathfrak{t}}\left(1-\operatorname{Res}_{\hbar=0} \frac{\Lambda}{\hbar} d \hbar\right)=\frac{5}{\mathfrak{t}}\left(1-\operatorname{Res}_{w=\infty} \Lambda \cdot(-\mathfrak{t}) \cdot \frac{-d w}{w^{2}} \cdot \frac{-w}{\mathfrak{t}}\right)=\frac{5}{\mathfrak{t}}\left(1+\operatorname{Res}_{w=\infty} \Lambda \frac{d w}{w}\right) .
$$

From

$$
\mathfrak{B}_{1}=-\frac{5}{\mathfrak{t}} \Lambda(w, t)+\frac{-5 w}{\mathfrak{t} I_{0}}\left[-g_{1}(q)+\sum_{d=1}^{\infty} q^{d} \frac{(5 d)!}{(d!)^{5}}\left(\sum_{m=1}^{d} \frac{1}{m-w}\right)\right]+\frac{5}{\mathfrak{t}},
$$

one calculates $\operatorname{Res}_{w=\infty} \frac{d w}{m-w}=1$ for all $m \in \mathbb{Z}_{\geq 0}$ and thus

$$
\operatorname{Res}_{w=\infty} \mathfrak{B}_{1} \frac{d w}{w}=-\frac{5}{\mathfrak{t}}\left(\operatorname{Res}_{w=\infty} \Lambda(w, t) \frac{d w}{w}+\frac{1}{I_{0}} \frac{d I_{0}}{d t}\right)-\frac{5}{\mathfrak{t}}
$$

Thus we have

$$
\mathcal{K}_{2,1}=\operatorname{Res}_{\hbar=0} \frac{\mathfrak{B}_{1}}{\hbar}-\frac{5}{\mathfrak{t}}\left(1+\operatorname{Res}_{w=\infty} \Lambda \frac{d w}{w}\right)=\frac{5}{\mathfrak{t}} \frac{1}{I_{0}} \frac{d I_{0}}{d t} .
$$

For later purpose, we also use $\operatorname{Res}_{w=\infty} \frac{1}{m-w} \frac{d w}{w}=0$ for all $m \in \mathbb{Z}_{\geq 0}$ to calculate

$$
\begin{aligned}
& \operatorname{Res}_{\hbar=0} \mathfrak{B}_{1}=\operatorname{Res}_{\hbar=0} \mathfrak{B}_{1} d \hbar=\operatorname{tRes}_{w=\infty} \mathfrak{B}_{1} \frac{d w}{w^{2}} \\
= & -5 \operatorname{Res}_{w=\infty} \Lambda \frac{d w}{w^{2}}-\frac{5 g_{1}}{I_{0}}-\frac{5}{I_{0}} \sum_{d=0}^{\infty} q^{d} \frac{(5 d)!}{(d!)^{5}}\left(\sum_{m=0}^{d} \operatorname{Res}_{w=\infty} \frac{1}{m-w} \frac{d w}{w}\right) \\
= & -5 \operatorname{Res}_{w=\infty} \Lambda \frac{d w}{w^{2}}-\frac{5 g_{1}}{I_{0}} .
\end{aligned}
$$

We now calculate $\mathcal{K}_{1,2}$. Using $d \hbar=\mathfrak{t} \frac{d w}{w^{2}}$,

$$
\begin{aligned}
& \operatorname{Res}_{\hbar=0} \mathfrak{B}_{2} \\
= & \frac{1}{\mathfrak{t}}\left(-5 \operatorname{Res}_{w=\infty} \Lambda \frac{d w}{w^{2}}+\frac{5 g_{1}}{I_{0}}\right)-\frac{5}{I_{0}} \cdot \frac{g_{1}}{\mathfrak{t}}+\frac{5}{I_{0}} \sum_{d=1}^{\infty} q^{d} \frac{(5 d)!}{(d!)^{5}}\left(\frac{(t-T) d}{\mathfrak{t}}+\sum_{m=d+1}^{5 d} \frac{5 d}{m \mathfrak{t}}\right) \\
= & \frac{5}{\mathfrak{t}} \operatorname{Res}_{w=\infty} \Lambda \frac{d w}{w^{2}}+\frac{5}{I_{0} t} \sum_{d=1}^{\infty} d q^{d} \frac{(5 d)!}{(d!)^{5}}\left(-(T-t)+\sum_{m=d+1}^{5 d} \frac{5}{m}\right) .
\end{aligned}
$$

By Lemma 4.2, $z_{0,2}=\mathfrak{B}_{2}+5 \mathfrak{t}^{-2}(1-\Lambda)$. Thus $\mathcal{K}_{1,2}=\operatorname{Res}_{\hbar=0} \mathcal{Z}_{0,2}$ gives

$$
\begin{align*}
\mathcal{K}_{1,2} & =\operatorname{Res}_{\hbar=0} \mathfrak{B}_{2}+\frac{-5}{\mathfrak{t}}\left(\operatorname{Res}_{w=\infty} \Lambda \cdot \frac{d w}{w^{2}}\right)=\frac{5}{\mathfrak{t} I_{0}} \sum_{d=1}^{\infty} d q^{d} \frac{(5 d)!}{(d!)^{5}}\left(t-T+\sum_{m=d+1}^{5 d} \frac{5}{m}\right) \\
& =\frac{5}{\mathfrak{t} I_{0}}\left[q \frac{d}{d q}\left(I_{1}-t I_{0}\right)-(T-t) \cdot q \frac{d}{d q} I_{0}\right] . \tag{18}
\end{align*}
$$

Lemma 4.3.

$$
\begin{equation*}
\mathcal{K}_{1,2}=\frac{5}{\mathfrak{t}} q \frac{d}{d q}(T-t)=\frac{5}{\mathfrak{t}} \frac{d}{d t}(T-t) \quad \text { and } \quad \mathcal{K}_{2,1}=\frac{5}{\mathfrak{t}} \frac{1}{I_{0}} \frac{d I_{0}}{d t} . \tag{19}
\end{equation*}
$$

4.5. Relations obtained by substituting $\mathfrak{t}^{\prime}$ by $-\mathfrak{t}$. The trick used in previous calulation of $\mathcal{z}_{0, k}$ may be used to calculate $\tilde{z}_{0, k}(\mathfrak{t},-\mathfrak{t})$. For each $(\Gamma) \in \Xi_{d}^{0}$, we have

$$
\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma}\right)=\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma}\left(-D_{\Gamma}\right)\right)+\pi_{\Gamma *}\left(\left.\mathcal{L}_{\Gamma}\right|_{D_{\Gamma}}\right)=\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma}\left(-D_{\Gamma}\right)\right)+\mathrm{ev}^{*} L_{H}
$$

and thus

$$
e\left(\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma}\right)\right)=e\left(\pi_{\Gamma *}\left(\mathcal{L}_{\Gamma}\left(-D_{\Gamma}\right)\right)\right) \cdot \operatorname{ev}^{*} H
$$

Suppose $k<4$ and thus $\mathbb{T}^{k}=p^{4-k}(p+\mathfrak{t})$. Set

$$
\begin{aligned}
\tilde{\mathcal{Z}}_{0, k}(\mathfrak{t},-\mathfrak{t}) & :=-\sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \sum_{(\Gamma) \in \Xi_{d}^{0}} \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{\mathrm{vir}}} \frac{1}{\hbar(\hbar-\psi)} e v^{*} \mathbb{T}^{k} \frac{e\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma}\left(-D_{\Gamma}\right)\right)\right)}{e\left(N_{\mathcal{W}_{(\Gamma)} / \mathcal{W}}^{v i r}\right)} \\
& =-\sum_{d=1}^{\infty}(-\mathbb{Q})^{d} \sum_{(\Gamma) \in \Xi_{d}^{0} \cup \Xi_{d}^{1}} \int_{\left[\mathcal{W}_{(\Gamma)}\right)} \frac{1}{\operatorname{vir}} \frac{1}{\hbar(\hbar-\psi)} e v^{*} \mathbb{T}^{k+1} \frac{e\left(R \pi_{\Gamma *} \mathcal{L}_{\Gamma}\right)}{e\left(N_{\mathcal{W}_{(\Gamma)}^{v i r} / \mathcal{W}}^{v i r}\right)} \\
& =-\sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \int_{\left[\mathcal{W}_{0,1_{\rho},(d, 0)}\right]^{\text {vir }}} \frac{1}{\hbar(\hbar-\psi)} e v^{*} \mathbb{T}^{k+1} e\left(R \pi_{\Gamma *} \mathcal{L}_{\Gamma}\right) .
\end{aligned}
$$

In the case $k=1, \mathbb{T}^{2}=p^{2}(p+\mathfrak{t})$. Taking coefficient of $p^{2}$ in (6), Corollary 3.4 implies

$$
\begin{equation*}
\tilde{z}_{0,1}(\mathfrak{t},-\mathfrak{t})=-5+\left[\frac{5}{I_{0}} e^{\frac{t}{\hbar}\left(\frac{g_{1}}{I_{0}}\right)} \sum_{d=0}^{\infty} q^{d} \frac{(5 d)!\hbar^{d}}{(d!)^{4} \prod_{m=1}^{d}(\mathfrak{t}+m \hbar)}\right] \tag{20}
\end{equation*}
$$

## 5. Graph Type A

In genus one cap-MSP, we apply localization formula to

$$
\begin{equation*}
0=\int_{\left[\mathcal{W}_{g=1, \emptyset,(n, 0)}\right]^{\mathrm{vir}}} \mathfrak{t} c_{n-1}\left(R \pi_{\mathcal{W} *} \mathcal{L}_{\mathcal{W}} \mathcal{N}_{\mathcal{W}} \mathcal{L}_{\mathfrak{t}}\right) \tag{21}
\end{equation*}
$$

for all $n$. We say a graph $\Gamma$ is of type $A$ if for some $v \in V_{0}(\Gamma)$ we have $g_{v}=1$.
5.1. Packaging graph $A_{+}$and $A_{\text {single }}$ using $\mathcal{K}$ series. If $\Gamma$ is of type $A$ with $d_{v}>0$, one can assume $\Gamma$ has a single vertex at $V_{0}$ without edges, or $\Gamma$ is as Figure 1 below. All other type of graphs have zero contribution to (21). This is because if $\Gamma$ has more than one vertices at $V_{1}$, one removes $v$ from $\Gamma$ to obtain connected graphs $\Gamma_{i}$ 's for $i=1, \cdots, \ell$, with $\ell>1$. This expresses $\mathcal{C}_{\mathcal{W}_{\Gamma}}$ as a union of $\mathcal{C}_{\mathcal{W}_{v}}$ and a family of rational curves $\mathcal{C}_{\mathcal{W}_{\Gamma_{i}}}$. Then $\left\{\left.\nu_{1}\right|_{\mathcal{C}_{\Gamma_{i}}}\right\}_{i=1}^{\ell}$ are nowhere zero sections of the locally free sheaves $\left\{\left.R \pi_{\mathcal{W}_{\Gamma_{i}} *}\left(\mathcal{L}_{\mathcal{W}} \mathcal{N}_{\mathcal{W}} \mathcal{L}_{\mathfrak{t}}\right)\right|_{\mathcal{C}_{\mathcal{W}_{i}}}\right\}_{i=1}^{\ell}$. Since

$$
\left.c\left(R \pi_{\mathcal{W}_{v} *}\left(\mathcal{L}_{\mathcal{W}} \mathcal{N}_{\mathcal{W}} \mathcal{L}_{\mathfrak{t}}\right)\right)\right|_{\mathcal{C}_{\mathcal{W}_{v}}} \cap\left[\mathcal{W}_{v}\right]^{\mathrm{vir}} \cong c\left(\mathbb{C}^{d_{v}} \mathfrak{t}\right) \cap\left[\mathcal{W}_{v}\right]^{\mathrm{vir}}
$$

are chern classes of a bundle, the fact $n=\operatorname{rank} R \pi_{\mathcal{W} *} \mathcal{L}_{\mathcal{W}} \mathcal{N}_{\mathcal{W}} \mathcal{L}_{\mathrm{t}}$ and the condition $\ell>1$ imply $c_{n-1}\left(R \pi_{\mathcal{W} *} \mathcal{L}_{\mathcal{W}} \mathcal{N}_{\mathcal{W}} \mathcal{L}_{\mathfrak{t}}\right)$ vanishes on $\mathcal{W}_{\Gamma}$.

1

0


Figure 1. The labelled graph $\Gamma_{A}\left(d_{v}, d_{e}, d_{1}, \ldots, d_{k}\right): g_{v}=1 ; g_{v_{0}}=0$; for $i=1, \ldots, k$, $d_{i}:=d_{e_{i}} ; v_{i} \in V_{0}, g_{v_{i}}=0$ and $d_{v_{i}} \in \mathbb{Z}_{\geq 0}$.

In this subsection we calculate contributions of such graphs with $d_{v}>0$. Let $A_{\text {single }}$ be the graph type that no edges are attached to $v$, then Contri $\left(A_{\text {single }}\right)$ is

$$
\begin{align*}
& \sum_{d_{v}=1}^{\infty} Q^{d_{v}} \int_{\left[\bar{M}_{1}\left(Q, d_{v}\right)\right]^{\mathrm{vir}}} \frac{\mathfrak{t} c_{d_{v}-1}\left(R \pi_{v *} \mathcal{L}_{v} \mathcal{L}_{\mathfrak{t}}\right)}{e_{T}\left(R \pi_{v *} \mathcal{L}_{v} \mathcal{L}_{\mathfrak{t}}\right)}=\sum_{d_{v}=1}^{\infty} \mathbb{Q}^{d_{v}} \int_{\left[\bar{M}_{1}\left(Q, d_{v}\right)\right]^{\mathrm{vir}}} \frac{\mathfrak{t}\left(d_{v} \mathfrak{t}^{d_{v}-1}\right)}{\mathfrak{t}^{d_{v}}} \\
= & \sum_{d_{v}=1}^{\infty} N_{1, d_{v}} d_{v} Q^{d_{v}}=\frac{d}{d T} \tilde{F}_{1}(Q) \tag{22}
\end{align*}
$$

Let $A_{+}$be the graph type with $d_{v}>0$ and $\sum_{j}\left(d_{e_{j}}+d_{v_{j}}\right)>0$. For each such $\Gamma$ we replace $v$ by a $(1, \rho)$-marking to obtain $\Gamma^{\prime} \in \hat{\Xi}_{m}^{0}$ (see the sentences below (10)) for $m=d_{\Gamma}-d_{v}$. Using the Poincaré dual of the quintic diagonal $P D:=5^{-1} \sum_{j=0}^{3} h^{j} \otimes h^{3-j}$ and $\operatorname{deg}\left(\psi_{(e, v)} \cap \bar{M}_{1,1}\left(Q, d_{v}\right)\right)=0$, the contribution we want is

$$
\begin{aligned}
& \sum_{d_{v}, m=1}^{\infty}(-\mathbb{Q})^{d_{v}+m} \sum_{\left(\Gamma^{\prime}\right) \in \hat{\Xi}_{m}^{0}} \int_{\left[\bar{M}_{1,1}\left(\mathbb{P}^{4}, d_{v}\right)^{p}\right]^{v} \times\left[\mathcal{W}_{\left(\Gamma^{\prime}\right)}\right]^{v}} \frac{\mathfrak{t} \cdot e_{T}\left(R \pi_{v *} \mathcal{L}_{v} \otimes \mathcal{L}_{\mathfrak{t}}\right)}{e_{T}\left(R \pi_{v *} \mathcal{L}_{v} \otimes \mathcal{L}_{\mathfrak{t}}\right)} \frac{(-1) P D\left(h_{e}+\mathfrak{t}\right)}{w_{(e, v)}-\psi_{(e, v)}} \\
& \cdot \frac{c_{\text {top }-1}\left(\pi_{\Gamma^{\prime} *} \mathcal{L}_{\Gamma^{\prime}} \mathcal{L}_{\mathrm{t}}\left(-D_{\Gamma^{\prime}}\right)\right)}{e\left(N_{\mathcal{W}_{\Gamma^{\prime}} / \mathcal{W}}^{\mathrm{vir}}\right)} \\
& =\frac{1}{5} \sum_{j=0}^{3} \sum_{d_{v}=1}^{\infty}(-Q)^{d_{v}} \int_{\left[\bar{M}_{1,1}\left(\mathbb{P}^{4}, d_{v}\right)^{p}\right]^{v}} \operatorname{tev}^{*} h^{j} \\
& \cdot \sum_{m=1}^{\infty}(-\mathcal{Q})^{m} \sum_{\left(\Gamma^{\prime}\right) \in \hat{\Xi}_{m}^{0}} \int_{\left[\mathcal{W}_{\left(\Gamma^{\prime}\right)}\right]^{\text {vir }}} \frac{-\mathrm{ev}^{*} h^{3-j}(h+\mathfrak{t})}{w_{(e, v)}} \frac{c_{\text {top }-1}\left(\pi_{\Gamma^{\prime} *} \mathcal{L}_{\Gamma^{\prime}} \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma^{\prime}}\right)\right)}{e\left(N_{\mathcal{W}_{\Gamma^{\prime}} / \mathcal{W}}^{\text {vir }}\right)} \\
& =\frac{1}{5}\left(\sum_{d_{v}=1}^{\infty} d_{v} N_{1, d_{v}} \mathfrak{Q}^{d_{v}}\right) \sum_{m=1}^{\infty}(-\mathcal{Q})^{m} \sum_{\left(\Gamma^{\prime}\right) \in \hat{\Xi}_{m}^{0}} \int_{\left[\mathcal{W}_{\left(\Gamma^{\prime}\right)}\right]^{\mathrm{vir}}} \frac{\mathfrak{t}}{\psi_{\Gamma}^{\prime}} \operatorname{ev}^{*}\left(h^{2}(h+\mathfrak{t})\right) \frac{c_{\mathrm{top}-1}\left(\pi_{\Gamma^{\prime} *} \mathcal{L}_{\Gamma^{\prime}} \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma^{\prime}}\right)\right)}{e\left(N_{\mathcal{W}_{\Gamma^{\prime}} / \mathcal{W}}^{\mathrm{vir}}\right)} \\
& =\frac{1}{5}\left(\sum_{d_{v}=1}^{\infty} d_{v} N_{1, d_{v}} Q^{d_{v}}\right) \cdot \mathfrak{t} \mathcal{K}_{1,2}=\frac{\mathfrak{t}}{5} \cdot \mathcal{K}_{1,2} \cdot \frac{d}{d T} \mathcal{F}_{1} .
\end{aligned}
$$

where $\mathcal{F}_{1}:=\sum_{d=1}^{\infty} N_{1, d} Q^{d}$ with $\mathcal{Q}=e^{T}$.
5.2. Contribution from type $A$ graphs $\Gamma$ with $d_{v}=0, v \in V_{0}(\Gamma), g_{v}=1$. Let $A_{+}^{0}$ be the graph type of $d_{v}=0, v \in V_{0}(\Gamma), g_{v}=1$. Such graphs are of the following shape.


In the above graph, $\Gamma$ is decomposed uniquely to (i) a $\Gamma_{0}=\{v\} \in \Xi_{0}$ with $d_{v}=0$, and (ii) a sequence $\Gamma_{1}, \cdots, \Gamma_{k} \in \hat{\Xi}^{0}$ where each edge containing the marking is denoted by $e_{i}$ (for $i=1, \cdots, k$ ).

Denote $\psi_{\left(e_{i}, v\right)}=\psi_{i}$ and $w_{\left(e_{i}, v\right)}:=-\psi_{\Gamma_{i}}$, where $\psi_{\Gamma_{i}}$ is the $\psi$-class of $\mathcal{C}_{e_{i}}$. Then the contribution we want is

$$
\begin{aligned}
& \mathfrak{t} \sum_{m, k=1}^{\infty} \frac{(-Q)^{m}}{k!} \sum_{\substack{\left(\Gamma_{i}\right) \in \hat{\Xi}_{m_{i}}^{0} \\
\sum m_{i}=m}} \sum_{\substack{a+1=\sum_{a} a_{i} \\
a, a_{i} \geq 0}} \int_{\left[\mathcal{W}_{\Gamma_{0}}\right]^{v} \times \Pi\left[\mathcal{W}_{\left(\Gamma_{i}\right)}\right]^{v}} \frac{c_{a}\left(R \pi_{\Gamma_{0} *} \mathcal{L}_{\Gamma_{0}} \mathcal{L}_{\mathfrak{t}}\right)}{e_{T}\left(R \pi_{\Gamma_{0} *} \mathcal{L}_{\Gamma_{0}} \otimes \mathcal{L}_{\mathfrak{t}}\right)} \\
& \cdot \prod_{i=1}^{k}\left(-P D_{i}\right) \frac{h_{e_{i}}+\mathfrak{t}}{w_{\left(e_{i}, v\right)}-\psi_{i}} \frac{c_{m_{i}-a_{i}}\left(\pi_{\Gamma_{i} *} \mathcal{L}_{\Gamma_{i}} \mathcal{L}_{\mathfrak{t}}\left(-D_{\Gamma_{i}}\right)\right)}{e\left(N_{\mathcal{W}_{\Gamma_{i}} / \mathcal{W}}^{\mathrm{vir}}\right)} \\
& =\mathfrak{t} \sum_{m, k=1}^{\infty} \frac{(-Q)^{m}}{k!} \sum_{\substack{\left(\Gamma_{i}\right) \in \hat{\Xi}_{m_{i}}^{0} \\
\sum m_{i}=m}} \sum_{a=0}^{\infty} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{a+1}} \int_{\left[\mathcal{W}_{\Gamma_{0}}\right]^{v} \times \Pi\left[\mathcal{W}_{\left(\Gamma_{i}\right)}\right]^{v}} \frac{c_{a}\left(R \pi_{\Gamma_{0} *} \mathcal{L}_{\Gamma_{0}} \mathcal{L}_{\mathfrak{t}}\right)}{e_{T}\left(R \pi_{\Gamma_{0} *} \mathcal{L}_{\Gamma_{0}} \otimes \mathcal{L}_{\mathfrak{t}}\right)} \\
& \cdot \prod_{i=1}^{k} P D_{i} \frac{h_{e_{i}}+\mathfrak{t}}{\psi_{\Gamma_{i}}+\psi_{i}} \frac{e\left(\pi_{\Gamma_{i} *} \mathcal{L}_{\Gamma_{i}} \mathcal{L}_{\mathrm{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma_{i}}\right)\right)}{e\left(N_{\mathcal{W}_{\Gamma_{i}} / \mathcal{W}}^{\text {vir }}\right)},
\end{aligned}
$$

where $P D_{i}=\frac{1}{5} \sum_{j=0}^{3} h^{j} \otimes h_{e_{i}}^{3-j}$. Using $\lambda^{2}=h^{2}=\lambda h=0$ over $\left[\mathcal{W}_{\Gamma_{0}}\right]^{\text {vir }}=\left(\bar{M}_{1, k} \times Q\right) \cap\left[-40 h^{3}-10 \lambda h^{2}\right]$, $\frac{c\left(R \pi_{\Gamma_{0} *} \mathcal{L}_{\Gamma_{0}} \mathcal{L}_{\mathfrak{t}}\right)}{e\left(R \pi_{\Gamma_{0} *} \mathcal{L}_{\Gamma_{0}} \mathcal{L}_{\mathfrak{t}}\right)}=\frac{1+h+\mathfrak{t}}{h+\mathfrak{t}} \frac{\mathfrak{t}+h-\lambda}{1+\mathfrak{t}+h-\lambda}=1-\frac{\lambda}{\mathfrak{t}(1+\mathfrak{t})}=1-\frac{\lambda}{\mathfrak{t}}+\lambda-\lambda \mathfrak{t}+\lambda \mathfrak{t}^{2}-\lambda \mathfrak{t}^{3}+\cdots+(-1)^{\ell} \lambda \mathfrak{t}^{\ell}+\cdots$.
Hence we get

$$
\frac{c_{a}\left(R \pi_{\Gamma_{0} *} \mathcal{L}_{\Gamma_{0}} \mathcal{L}_{\mathfrak{t}}\right)}{e\left(R \pi_{\Gamma_{0} *} \mathcal{L}_{\Gamma_{0}} \mathcal{L}_{\mathfrak{t}}\right)}= \begin{cases}(-1)^{a-1} \lambda \mathfrak{t}^{a-1} & \text { for } a \in \mathbb{N} \\ 1-\lambda \mathfrak{t}^{-1} & \text { for } a=0\end{cases}
$$

We express the contribution as the sum over $a=0,1,2,3, \cdots$ as follows.
Case $a \in \mathbb{N}$ : As $\left[-40 h^{3}-10 \lambda h^{2}\right]\left((-1)^{a-1} \lambda \mathfrak{t}^{a-1}\right)=(-1)^{a} 40 h^{3} \lambda \mathfrak{t}^{a-1}$, the contribution is

$$
\begin{aligned}
& \mathfrak{t} \sum_{k, m_{i}=1}^{\infty} \frac{(-Q)^{m_{1}+\cdots m_{k}}}{k!} \sum_{\left(\Gamma_{i}\right) \in \hat{\Xi}_{m_{i}}^{0}} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{a+1}} \int_{\bar{M}_{1, k} \times Q \times \Pi\left[\mathcal{W}_{\left(\Gamma_{i}\right)}\right]^{v}}(-1)^{a} 40 h^{3} \lambda \mathfrak{t}^{a-1} \\
& \cdot \prod_{i=1}^{k} \frac{h_{e_{i}}^{3}}{5} \frac{h_{e_{i}}+\mathfrak{t}}{\psi_{\Gamma_{i}}+\psi_{i}} \frac{e\left(\pi_{\Gamma_{i} *} \mathcal{L}_{\Gamma_{i}} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma_{i}}\right)\right)}{e\left(N_{\mathcal{W}_{\Gamma_{i}} / \mathcal{W}}^{v i r}\right)} \\
= & \mathfrak{t} \sum_{k=1}^{\infty} \frac{1}{k!} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{a+1}} \int_{\bar{M}_{1, k} \times Q} \frac{(-1)^{a}}{5^{k}} 40 h^{3} \lambda \mathfrak{t}^{a-1} \hat{\mathcal{K}}_{1}\left(\psi_{1}\right) \hat{\mathcal{K}}_{1}\left(\psi_{2}\right) \cdots \hat{\mathcal{K}}_{1}\left(\psi_{k}\right) \\
= & (-1)^{a} \mathfrak{t}^{a} \sum_{k=1}^{\infty} \frac{200}{5^{k} k!} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{a+1}} \int_{\bar{M}_{1, k}} \lambda \hat{\mathcal{K}}_{1}\left(\psi_{1}\right) \hat{\mathcal{K}}_{1}\left(\psi_{2}\right) \cdots \hat{\mathcal{K}}_{1}\left(\psi_{k}\right) .
\end{aligned}
$$

Case $a=0$ : As $\left[-40 h^{3}-10 \lambda h^{2}\right]\left(1-\frac{\lambda}{\mathfrak{t}}\right)=-40 h^{3}+40 h^{3} \frac{\lambda}{\mathfrak{t}}-10 \lambda h^{2}$, the contribution is the sum of:

$$
\begin{aligned}
& \text { (i) } \mathfrak{t} \sum_{k, m_{i}=1}^{\infty} \frac{(-Q)^{m_{1}+\cdots m_{k}}}{k!} \sum_{\left(\Gamma_{i}\right) \in \hat{\Xi}_{m_{i}}^{0}} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{1}} \int_{\bar{M}_{1, k} \times Q \times \Pi\left[\mathcal{W}_{\left(\Gamma_{i}\right)}\right]^{v}} 40\left(h^{3} \frac{\lambda}{\mathfrak{t}}-h^{3}\right) \\
& \cdot \prod_{i=1}^{k} P D_{i} \frac{h_{e_{i}}+\mathfrak{t}}{\psi_{\Gamma_{i}}+\psi_{i}} \frac{e\left(\pi_{\Gamma_{i} *} \mathcal{L}_{\Gamma_{i}} \mathcal{L}_{\mathrm{t}} \mathcal{L}_{\mathrm{t}^{\prime}}\left(-D_{\Gamma_{i}}\right)\right)}{e\left(N_{\mathcal{W}_{\Gamma_{i}} / \mathcal{W}}^{\text {vir }}\right)} \\
& =\mathfrak{t} \sum_{k=1}^{\infty} \frac{1}{5^{k} k!} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{1}} \int_{\bar{M}_{1, k} \times Q}\left(-40 h^{3}+40 h^{3} \frac{\lambda}{\mathfrak{t}}\right) \hat{\mathcal{K}}_{1}\left(\psi_{1}\right) \cdots \tilde{\mathcal{K}}_{1}\left(\psi_{k}\right) \\
& =\mathfrak{t} \sum_{k=1}^{\infty} \frac{1}{5^{k} k!} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{1}} \int_{\bar{M}_{1, k}} 200\left(\frac{\lambda}{\mathfrak{t}}-1\right) \hat{\mathcal{K}}_{1}\left(\psi_{1}\right) \cdots \tilde{\mathcal{K}}_{1}\left(\psi_{k}\right) \text {, }
\end{aligned}
$$

and (ii), recalling $P D_{i}=\frac{1}{5} \sum_{j=0}^{3} h^{j} \otimes h_{e_{i}}^{3-j}$,

$$
\begin{aligned}
& \mathfrak{t} \sum_{k, m_{i}=1}^{\infty} \frac{(-Q)^{m_{1}+\cdots m_{k}}}{k!} \sum_{\left(\Gamma_{i}\right) \in \hat{\Xi}_{m_{i}}^{0}} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{1}} \int_{\bar{M}_{1, k} \times Q \times \Pi\left[\mathcal{W}_{\left(\Gamma_{i}\right)}\right]^{v}}\left(-10 \lambda h^{2}\right) \\
& \cdot \prod_{i=1}^{k} P D_{i} \cdot \frac{h_{e_{i}}+\mathfrak{t}}{\psi_{\Gamma_{i}}+\psi_{i}} \frac{e\left(\pi_{\Gamma_{i} *} \mathcal{L}_{\Gamma_{i}} \mathcal{L}_{t} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma_{i}}\right)\right)}{e\left(N_{\mathcal{W}_{\Gamma_{i}} / \mathcal{W}}^{\operatorname{vir}}\right)} \\
= & \mathfrak{t} \sum_{k, m_{i}=1}^{\infty} \frac{(-Q)^{m_{1}+\cdots m_{k}}}{5^{k} k!} \sum_{\left(\Gamma_{i}\right) \in \hat{\Xi}_{m_{i}}^{0}} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{1}} \int_{\bar{M}_{1, k} \times Q \times \Pi\left[\mathcal{W}_{\left(\Gamma_{i}\right)}\right]^{v}}\left(-10 \lambda h^{2}\right), \\
& \left(\sum_{j=1}^{k}\left(h^{0} \otimes h_{e_{1}}^{3}\right) \cdots\left(h^{0} \otimes h_{e_{j-1}}^{3}\right)\left(h^{1} \otimes h_{e_{j}}^{2}\right)\left(h^{0} \otimes h_{e_{j+1}}^{3}\right) \cdots\left(h^{0} \otimes h_{e_{k}}^{3}\right)\right) \\
& \cdot\left(\prod_{i=1}^{k} \frac{h_{e_{i}}+\mathfrak{t}}{\psi_{\Gamma_{i}}+\psi_{i}} \frac{e\left(\pi_{\Gamma_{i} *} \mathcal{L}_{\Gamma_{i}} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma_{i}}\right)\right)}{e\left(N_{\mathcal{W}_{\Gamma_{i}} / \mathcal{W}^{v}}^{v i r}\right)}\right) \\
= & \mathfrak{t} \sum_{k=1}^{\infty} \frac{1}{5^{k} k!} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{1}} \int_{\bar{M}_{1, k}}(-50) \lambda \sum_{j=1}^{k} \hat{\mathcal{K}}_{1}\left(\psi_{1}\right) \hat{\mathcal{K}}_{1}\left(\psi_{2}\right) \cdots \hat{\mathcal{K}}_{1}\left(\psi_{j-1}\right) \hat{\mathcal{K}}_{2}\left(\psi_{j}\right) \hat{\mathcal{K}}_{1}\left(\psi_{j+1}\right) \cdots \hat{\mathcal{K}}_{1}\left(\psi_{k}\right) .
\end{aligned}
$$

Therefore the contribution of graph type $A_{+}^{0}$ is

$$
\begin{aligned}
& \mathfrak{t} \sum_{k=1}^{\infty} \frac{(-50)}{5^{k} k!} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{1}} \int_{\bar{M}_{1, k}} \lambda \sum_{j=1}^{k} \hat{\mathcal{K}}_{1}\left(\psi_{1}\right) \hat{\mathcal{K}}_{1}\left(\psi_{2}\right) \cdots \hat{\mathcal{K}}_{1}\left(\psi_{j-1}\right) \hat{\mathcal{K}}_{2}\left(\psi_{j}\right) \hat{\mathcal{K}}_{1}\left(\psi_{j+1}\right) \cdots \hat{\mathcal{K}}_{1}\left(\psi_{k}\right) \\
& -\mathfrak{t} \sum_{k=1}^{\infty} \frac{200}{5^{k} k!} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{1}} \int_{\bar{M}_{1, k}} \hat{\mathcal{K}}_{1}\left(\psi_{1}\right) \cdots \hat{\mathcal{K}}_{1}\left(\psi_{k}\right) \\
& +\sum_{a=0}^{\infty}(-1)^{a} \mathfrak{t}^{a} \sum_{k=1}^{\infty} \frac{200}{5^{k} k!} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{a+1}} \int_{\bar{M}_{1, k}} \lambda \hat{\mathcal{K}}_{1}\left(\psi_{1}\right) \hat{\mathcal{K}}_{1}\left(\psi_{2}\right) \cdots \hat{\mathcal{K}}_{1}\left(\psi_{k}\right) \\
= & \frac{(-50)}{5} \mathfrak{t C o e}\left(\mathfrak{t}^{\prime}\right)^{1} \int_{\bar{M}_{1,1}} \lambda \hat{\mathcal{K}}_{2}\left(\psi_{1}\right)-\frac{200}{5} \mathfrak{t C o e}_{\left(\mathfrak{t}^{\prime}\right)^{1}} \int_{\bar{M}_{1,1}} \hat{\mathcal{K}}_{1}\left(\psi_{1}\right) \\
= & -\frac{1}{\mathfrak{t}} \sum_{k=1}^{\infty} \frac{200}{5^{k} k!} \sum_{a=0}^{\infty}(-\mathfrak{t})^{a+1} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{a+1}} \int_{\bar{M}_{1, k}} \lambda \hat{\mathcal{K}}_{1}\left(\psi_{1}\right) \hat{\mathcal{K}}_{1}\left(\psi_{2}\right) \cdots \hat{\mathcal{K}}_{1}\left(\psi_{k}\right) \\
= & \frac{40}{24} \mathfrak{t}_{2,1}-\left.\left.\left.\frac{1}{\mathfrak{t}} \sum_{k=1}^{\infty} \frac{200}{5^{k} k!} \int_{\bar{M}_{1, k}} \lambda \hat{\mathcal{K}}_{1}\right|_{\mathfrak{t}^{\prime}=-\mathfrak{t}}\left(\psi_{1}\right) \hat{\mathcal{K}}_{1}\right|_{\mathfrak{t}^{\prime}=-\mathfrak{t}}\left(\psi_{2}\right) \cdots \hat{\mathcal{K}}_{1}\right|_{\mathfrak{t}^{\prime}=-\mathfrak{t}}\left(\psi_{k}\right),
\end{aligned}
$$

where the first equality comes from the fact that both $\tilde{\mathcal{K}}_{c, k}$ and $\hat{\mathcal{K}}_{k}$ for any $c, k$ are formal power series of $\mathfrak{t}^{\prime}$ with coefficients of $\left(\mathfrak{t}^{\prime}\right)^{0}$ vanishing.

Using (11) and Lemma 4.1 the last term in the above formula equals

$$
\begin{equation*}
-\frac{200}{\mathfrak{t}} \cdot\left(\int_{\bar{M}_{1,1}} \lambda\right) \cdot \sum_{k=1}^{\infty} \sum_{\substack{i_{1}+\ldots+i_{k}=k-1 \\ i_{1}, \ldots, i_{k} \geq 0}} \frac{1}{k} \cdot \prod_{j=1}^{k}\left(\frac{(-1)^{i_{j}}}{i_{j}!} \operatorname{Res}_{\hbar=0}\left\{\hbar^{-i_{j}} \frac{\tilde{z}_{0,1}(\mathfrak{t},-\mathfrak{t})}{5}\right\}\right), \tag{23}
\end{equation*}
$$

where we used $\int_{\bar{M}_{1, k}} \lambda \psi_{1}^{i_{1}} \cdots \psi_{k}^{i_{k}}=\frac{(k-1)!}{i_{1}!\cdots i_{k}!} \int_{\bar{M}_{1,1}} \lambda$ whenever $i_{1}+\ldots+i_{k}=k-1$. By (34) in the Appendix, we have (23) equals $-\frac{200}{24} \mathfrak{t}^{-1} \eta=-\frac{25}{3} \frac{g_{1}}{I_{0}}$. Thus the contribution of graph type $A_{+}^{0}$ equals

$$
\frac{40 \mathrm{t}}{24} \cdot \mathcal{K}_{2,1}-\frac{10 \mathrm{t}}{24} \cdot \mathcal{K}_{1,2}-\frac{25}{3} \cdot \frac{g_{1}}{I_{0}}=\frac{25}{3 I_{0}} \frac{\mathrm{~d} I_{0}}{\mathrm{~d} t}-\frac{25}{12} \cdot \frac{d}{d t}(T-t)-\frac{25}{3} \cdot \frac{g_{1}}{I_{0}}
$$

Thus type A contribution is, by (19),

$$
\begin{aligned}
& \left(1+\frac{\mathfrak{t}}{5} \mathcal{K}_{1,2}\right) \frac{d \mathcal{F}_{1}}{d T}+\frac{25}{3 I_{0}} \frac{\mathrm{~d} I_{0}}{\mathrm{~d} t}-\frac{25}{12} \cdot \frac{d(T-t)}{d t}-\frac{25}{3} \cdot \frac{g_{1}}{I_{0}} \\
= & \frac{d}{d t} \mathcal{F}_{1}-\frac{25}{12} \cdot\left(\frac{d T}{d t}-1\right)+\frac{40}{24} \cdot 5 \frac{1}{I_{0}} \frac{d I_{0}}{d t}-\frac{25}{3} \cdot \frac{g_{1}}{I_{0}} .
\end{aligned}
$$

## 6. Graph Type B

The contribution from Graph B is the most involved. It splits into two parts. One part is in this section, another is in the Appendix C
6.1. Notations for $g=0$ two-point functions. We define in MSP weight a series

$$
\tilde{z}_{66}^{*}:=\frac{1}{2 \hbar_{1} \hbar_{2}} \sum_{d=1}^{\infty} \mathbb{Q}^{d} \int_{\bar{M}_{0,2}\left(\mathbb{P}^{5}, d\right)} \frac{e\left(\tilde{E}_{d}\right)}{\left(\hbar_{1}-\psi_{1}\right)\left(\hbar_{2}-\psi_{2}\right)} e v_{1}^{*} H^{5} e v_{2}^{*} H^{5}
$$

where $\tilde{E}_{d}:=\tilde{\pi}_{*} \tilde{f}^{*}\left(L_{H}^{\otimes 5} \oplus\left(L_{H} L_{\mathfrak{t}} L_{\mathfrak{t}^{\prime}}\right)\right)$ is a rank $6 d+2$ bundle over $\bar{M}_{0,2}\left(\mathbb{P}^{5}, d\right)$, with $\tilde{\pi}: \tilde{\mathcal{C}}_{d} \rightarrow \bar{M}_{0,2}\left(\mathbb{P}^{5}, d\right)$ and $\tilde{f}: \mathcal{C}_{d} \rightarrow \bar{M}_{0,2}\left(\mathbb{P}^{5}, d\right)$ the universal family. Then

$$
\begin{aligned}
& \tilde{\mathcal{Z}}_{66}^{*}=\frac{-1}{2 \hbar_{1} \hbar_{2}} \sum_{d=1}^{\infty}(-Q)^{d} \int_{\left[\mathcal{W}_{0,(1, \rho)^{2},(d, 0)}\right]^{\text {vir }}} \frac{e\left(\tilde{\pi}_{*} \tilde{f}^{*}\left(L_{H} L_{\mathfrak{t}} L_{\mathfrak{t}^{\prime}}\right)\right)}{\left(\hbar_{1}-\psi_{1}\right)\left(\hbar_{2}-\psi_{2}\right)} e v_{1}^{*} H^{5} e v_{2}^{*} H^{5} \\
& =\frac{-1}{2 \hbar_{1} \hbar_{2}} \sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\tilde{\Gamma}) \in \tilde{\Delta}_{d}^{11}} \int_{\left[\mathcal{W}_{(\tilde{\Gamma})}\right]^{\mathrm{vir}}} \frac{e\left(\tilde{\pi}_{*} \tilde{f}^{*}\left(L_{\tilde{\Gamma}} L_{\mathfrak{t}} L_{\mathfrak{t}^{\prime}}\right)\right)}{e\left(N_{\mathcal{W}_{(\tilde{\Gamma})}^{\mathrm{vir}} / \mathcal{W}}\right)} \frac{\mathfrak{t}^{10}}{\left(\hbar_{1}-\psi_{1}\right)\left(\hbar_{2}-\psi_{2}\right)},
\end{aligned}
$$

where $\tilde{\Delta}_{d}^{i j}$ denotes the set of all (regular) graphs appearing in localizing $\mathbb{C}^{*}$-action on $\mathcal{W}_{0,(1, \rho)^{2},(d, 0)}=$ $\bar{M}_{0,2}\left(\mathbb{P}^{5}, d\right)$ which have the first marking mapped to $V_{i}$ and the second marking mapped to $V_{j}$, for $i, j \in\{0,1\}$.

Let $\tilde{\Delta}_{d, S e p}^{11}$ denote the set of all graphs in $\tilde{\Delta}_{d}^{11}$ with two markings not on the same vertex.

$$
\tilde{Z}_{66, S e p}^{*}:=\frac{-1}{2 \hbar_{1} \hbar_{2}} \sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\tilde{\Gamma}) \in \tilde{\Delta}_{d, S e p}^{11}} \int_{\left[\mathcal{W}_{(\tilde{\Gamma})}\right]^{\mathrm{vir}}} \frac{e\left(\tilde{\pi}_{*} \tilde{f}^{*}\left(L_{\tilde{\Gamma}} L_{\mathrm{t}} L_{\mathbf{t}^{\prime}}\right)\right)}{e\left(N_{\mathcal{W}_{(\tilde{\Gamma})}^{\mathrm{vir}} / \mathcal{W}}\right)} \frac{\mathfrak{t}^{10}}{\left(\hbar_{1}-\psi_{1}\right)\left(\hbar_{2}-\psi_{2}\right)}
$$

Let $\mathfrak{D}_{1}, \mathfrak{D}_{2} \subset \mathcal{C}_{\tilde{\Gamma}} \rightarrow \mathcal{W}_{(\Gamma)}$ be the universal family and divisors given by the two markings. We also have
$\tilde{\mathcal{Z}}_{66, S e p}^{*}=\frac{-1}{2 \hbar_{1} \hbar_{2}} \sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \sum_{(\tilde{\Gamma}) \in \tilde{\Delta}_{d, S e p}^{11}} \int_{\left[\mathcal{W}_{(\tilde{\Gamma})}\right]^{\text {vir }}}\left(\mathfrak{t}^{\prime}\right)^{2} \cdot \frac{e\left(\tilde{\pi}_{*} \tilde{f}^{*}\left(L_{\tilde{\Gamma}} L_{\mathfrak{t}} L_{\mathfrak{t}^{\prime}}\left(-\mathfrak{D}_{1}-\mathfrak{D}_{2}\right)\right)\right)}{e\left(N_{\mathcal{W}_{(\tilde{\Gamma})}^{\mathrm{vir}} / \mathcal{W}}\right)} \frac{\mathfrak{t}^{10}}{\left(\hbar_{1}-\psi_{1}\right)\left(\hbar_{2}-\psi_{2}\right)}$.

$$
\begin{align*}
& \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{2}} \tilde{z}_{66, S e p}^{*} \\
= & \frac{-1}{2 \hbar_{1} \hbar_{2}} \sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\tilde{\Gamma}) \in \tilde{\Delta}_{d, S e p}^{11}} \int_{\left[\mathcal{W}_{(\tilde{\Gamma})}\right]^{\mathrm{vir}}} \frac{e\left(\tilde{\pi}_{*} \tilde{f}^{*}\left(L_{\tilde{\Gamma}} L_{\mathfrak{t}}\left(-\mathfrak{D}_{1}-\mathfrak{D}_{2}\right)\right)\right)}{e\left(N_{\mathcal{W}_{(\tilde{\Gamma})}^{\mathrm{vir}} / \mathcal{W}}\right)} \frac{\mathfrak{t}^{10}}{\left(\hbar_{1}-\psi_{1}\right)\left(\hbar_{2}-\psi_{2}\right)} . \tag{24}
\end{align*}
$$

6.2. Two types of contributions. Any graph $\Gamma$ without positive genus vertex (called loop graph) that gives nonzero contribution to (21) necessarily has $\left|V_{1}(\Gamma)\right|=1$, because of the integrand in (21). We call it the geometry of cap. We say such a graph is of type $B$, as associated to the left graph below. Let $\tilde{\Gamma}$ be the subgraph of $\Gamma$ shown below on the right.


Here all vertices are of genus zero. Let $\Gamma_{0}$ be the graph of single vertex $\tilde{v} \in V_{1}, g_{\tilde{v}}=0$, no edges, and with $k+2$ many $(1, \rho)$ markings. The tail of $\tilde{v}$ that contains $e_{i}$ is denoted by $\Gamma_{i}$. Let the total degree of $\mathcal{L}$ over objects in $\Gamma_{i}$ be $d_{i}$. Then $d=d_{\tilde{\Gamma}}+d_{1}+d_{2}+\cdots+d_{k}$ is the degree $\mathcal{L}$. We divide into two cases.
(1) the first case is when $k=0, \tilde{v}$ unstable. We leave its $C_{o n t}$ to next few subsections.
(2) the other case is when $k>0$ and $\tilde{v}$ stable whose contribution is denoted by $\operatorname{Cont}_{I I}$.

Let's consider the second case. By [LLL2],

$$
\left[\mathcal{W}_{(\Gamma)}\right]^{\mathrm{vir}}=|A u t \tilde{\Gamma}| \cdot \frac{\prod\left|A u t \Gamma_{i}\right|}{|A u t \Gamma|}\left[\mathcal{W}_{\left(\Gamma_{0}\right)}\right]^{\mathrm{vir}} \times\left[\mathcal{W}_{(\tilde{\Gamma})}\right]^{\mathrm{vir}} \times \prod\left[\mathcal{W}_{\left(\Gamma_{i}\right)}\right]^{\mathrm{vir}}
$$

Thus the contribution of $\Gamma$ is

$$
\begin{aligned}
& \operatorname{Contri}_{(\Gamma)}:=\int_{\left[\mathcal{W}_{(\Gamma)}\right]^{v}} \frac{\left.{\mathfrak{t} c_{d-1} R \pi_{\mathcal{W}_{\Gamma} *}\left(\mathcal{L}_{\mathcal{W}_{\Gamma}} \mathcal{N}_{\mathcal{W}_{\Gamma}} \mathcal{L}_{\mathfrak{t}}\right)}_{e\left(N_{\mathcal{W}_{(\Gamma)}}^{\operatorname{vir}}\right)}{ }^{2}\right)}{} \\
& =\frac{\mathfrak{t}|A u t \tilde{\Gamma}| \prod\left|A u t \Gamma_{i}\right|}{|A u t \Gamma|} \sum_{\ell+\ell_{0}+\sum \ell_{i}=d-1} \int_{\left[\mathcal{W}_{\left(\Gamma_{0}\right)}\right]^{v}} \frac{c_{\ell}\left(R \pi_{\mathcal{W}_{\Gamma_{0}} *}\left(\mathcal{L}_{\mathcal{W}_{\Gamma_{0}}} \mathcal{L}_{\mathfrak{t}}\right)\right)}{e\left(N_{\mathcal{W}_{\left(\Gamma_{0}\right)} \mathrm{vir}}^{v i}\right)} \\
& \cdot \int_{\left[\mathcal{W}_{(\tilde{\Gamma})}\right]^{v}} \frac{\mathfrak{t}^{8} c_{\ell_{0}}\left(R \pi_{\mathcal{W}_{\tilde{\Gamma}}^{*}}\left(\mathcal{L}_{\mathcal{W}_{\tilde{\Gamma}}} \mathcal{L}_{\mathfrak{t}}\left(-\mathfrak{D}_{1}-\mathfrak{D}_{2}\right)\right)\right)}{5^{2}\left(\psi_{1}+\psi_{1}^{\prime}\right)\left(\psi_{2}+\psi_{2}^{\prime}\right) e\left(N_{\mathcal{W}_{(\tilde{\Gamma})}^{\mathrm{vi}}}^{\mathrm{vi}}\right)} \prod \int_{\left[\mathcal{W}_{\left(\Gamma_{i}\right)}\right]^{v}} \frac{-\mathfrak{t}^{4} c_{\ell_{i}}\left(R \pi_{\mathcal{W}_{\Gamma_{i}} *}\left(\mathcal{L}_{\mathcal{W}_{\Gamma_{i}}} \mathcal{L}_{\mathfrak{t}}\left(-D_{i}\right)\right)\right)}{5\left(-\psi_{\Gamma_{i}}-\hat{\psi}_{i}\right) e\left(N_{\mathcal{W}_{\left(\Gamma_{i}\right)}^{\mathrm{vir}}}\right)} .
\end{aligned}
$$

From the shape of the graph, one has $R \pi_{\mathcal{W}_{\Gamma_{0}} *}\left(\mathcal{L}_{\mathcal{W}_{\Gamma_{0}}} \mathcal{L}_{\mathfrak{t}}\right) \cong \mathcal{O}_{\mathcal{W}_{\Gamma_{0}}}$. Both $R \pi_{\mathcal{W}_{\Gamma}{ }^{*}}\left(\mathcal{L}_{\mathcal{W}_{\tilde{\Gamma}}} \mathcal{L}_{\mathfrak{t}}\left(-\mathfrak{D}_{1}-\mathfrak{D}_{2}\right)\right)$ and $\oplus_{i} R \pi_{\mathcal{W}_{\Gamma_{i}} *}\left(\mathcal{L}_{\mathcal{W}_{\Gamma_{i}}} \mathcal{L}_{\mathfrak{t}}\left(-D_{i}\right)\right)$ are bundles, and the sum of their ranks is $d-1$. We see the only possible contribution to $\operatorname{Contri}_{(\Gamma)}$ is from $\ell=0$ and all $c_{\ell_{i}}$ are top Chern classes. Therefore, with $\frac{1}{e\left(N_{\mathcal{W}}^{\left(\Gamma_{0}\right)}\right)}=\frac{5 \mathfrak{t}}{-\mathfrak{t}^{5}}$, Contri $_{(\Gamma)}$ equals

$$
\begin{aligned}
& \frac{-|A u t \tilde{\Gamma}| \prod\left|A u t \Gamma_{i}\right|}{5 \mathfrak{t}^{5}|A u t \Gamma|} \sum_{a, b, b_{1}, \cdots, b_{k}=0}^{\infty} \int_{\bar{M}_{0,2+k}}\left(-\psi_{1}^{\prime}\right)^{a}\left(-\psi_{2}^{\prime}\right)^{b} \prod_{i}\left(-\hat{\psi}_{i}\right)^{b_{i}}\left(\int_{\left[\mathcal{W}_{(\tilde{\Gamma})}\right]^{\text {vir }}} \frac{\mathfrak{t}^{10}}{\left(\psi_{1}\right)^{a+1}\left(\psi_{2}\right)^{b+1}}\right. \\
& \left.\cdot \frac{e\left(\pi_{\mathcal{W}_{\tilde{\Gamma}^{*}}}\left(\mathcal{L}_{\mathcal{W}_{\tilde{\Gamma}}} \mathcal{L}_{\mathfrak{t}}\left(-\mathfrak{D}_{1}-\mathfrak{D}_{2}\right)\right)\right)}{e\left(N_{\mathcal{W}_{(\widetilde{\Gamma})}}^{\text {vir }}\right)}\right) \prod \int_{\left[\mathcal{W}_{\left(\Gamma_{i}\right)}\right]^{\text {vir }}} \frac{\mathfrak{t}^{4}}{5} \frac{1}{\left(\psi_{\Gamma_{i}}\right)^{b_{i}+1}} \frac{e\left(\pi_{\mathcal{W}_{\Gamma_{i}} *}\left(\mathcal{L}_{\mathcal{W}_{\Gamma_{i}}} \mathcal{L}_{\mathfrak{t}}\left(-D_{i}\right)\right)\right)}{e\left(N_{\mathcal{W}_{\left(\Gamma_{i}\right)}}^{\text {vir }}\right)} \\
& =\frac{-|A u t \tilde{\Gamma}| \prod\left|A u t \Gamma_{i}\right|}{5 \mathfrak{t}^{5}|A u t \Gamma|}(k-1)!\sum_{a+b+b_{1}+\cdots b_{k}=k-1}^{\infty} \frac{(-1)^{a+b}}{a!b!}\left(\int_{\left[\mathcal{W}_{(\tilde{\Gamma})}\right]} \frac{\mathfrak{t}^{\text {vir }}}{} \frac{{ }^{10}}{\left(\psi_{1}\right)^{a+1}\left(\psi_{2}\right)^{b+1}}\right. \\
& \left.\cdot \frac{e\left(\pi_{\mathcal{W}_{\tilde{\Gamma}} *}\left(\mathcal{L}_{\mathcal{W}_{\tilde{\Gamma}}} \mathcal{L}_{\mathfrak{t}}\left(-\mathfrak{D}_{1}-\mathfrak{D}_{2}\right)\right)\right)}{e\left(N_{\mathcal{W}_{(\tilde{\Gamma})}^{\text {vir }}}\right)}\right) \prod_{i=1}^{k} \frac{(-1)^{b_{i}}}{b_{i}!} \frac{\mathfrak{t}^{4}}{5} \int_{\left[\mathcal{W}_{\left(\Gamma_{i}\right)}\right]^{\text {vir }}} \frac{1}{\left(\psi_{\Gamma_{i}}\right)^{b_{i}+1}} \frac{e\left(\pi_{\mathcal{W}_{\Gamma_{i}} *}\left(\mathcal{L}_{\mathcal{W}_{\Gamma_{i}}} \mathcal{L}_{\mathfrak{t}}\left(-D_{i}\right)\right)\right)}{e\left(N_{\mathcal{W}_{\left(\Gamma_{i}\right)} \mathrm{vi}}\right)} .
\end{aligned}
$$

Let $\tilde{\Delta}_{d}^{11, \circ}$ denote the graphs in $\tilde{\Delta}_{d}^{11}$ which has only one vertex at $V_{0}$ and whose both markings are unstable. Using (24) and the geometry of cap, the sum of all above contributions is

$$
\begin{aligned}
& \operatorname{Cont}_{I I}=\sum_{k=1}^{\infty} \frac{-(k-1)!}{5 \mathfrak{t}^{5} \cdot 2 \cdot k!} \sum_{\substack{a+b+b_{1}+\cdots b_{k}=k-1 \\
a, b, b_{i} \geq 0}}^{\infty} \frac{(-1)^{a+b}}{a!b!} \sum_{(\tilde{\Gamma}) \in \tilde{\Delta}_{d}^{11, o}}(-Q)^{d_{\tilde{\Gamma}}}\left(\int_{\left[\mathcal{W}_{(\tilde{\Gamma})}\right]^{\text {vir }}} \frac{\mathfrak{t}^{10}}{\left(\psi_{1}\right)^{a+1}\left(\psi_{2}\right)^{b+1}}\right. \\
&=\left.\frac{e\left(\pi_{\mathcal{W}_{\tilde{\Gamma}^{*}}}\left(\mathcal{L}_{\mathcal{W}_{\tilde{\Gamma}}} \mathcal{L}_{\mathfrak{t}}\left(-\mathfrak{D}_{1}-\mathfrak{D}_{2}\right)\right)\right)}{e\left(N_{\mathcal{W}_{\tilde{\Gamma}}}^{\text {vir }}\right)}\right) \prod_{i=1}^{k} \frac{(-1)^{b_{i}}}{b_{i}!} \frac{\mathfrak{t}^{4}}{5} \sum_{\left(\Gamma_{i}\right) \in \hat{\Xi}^{1}}(-Q)^{d_{\Gamma_{i}}} \int_{\left[\mathcal{W}_{\left(\Gamma_{i}\right)}\right]^{\text {vir }}} \frac{e\left(\pi_{\Gamma_{i} *} \mathcal{L}_{\Gamma_{i}} \mathcal{L}_{\mathfrak{t}}\left(-D_{i}\right)\right)}{\left(\psi_{\Gamma_{i}}\right)^{b_{i}+1} e\left(N_{\left.\mathcal{W}_{\left(\Gamma_{i}\right)} / \mathcal{W}\right)}^{v i r}\right)} \\
&=\sum_{k=1}^{\infty} \frac{1}{5 \mathfrak{t}^{5} k} \sum_{\substack{a+b+b_{1}+\cdots b_{k}=k-1 \\
a, b, b_{i} \geq 0}}^{\infty} \frac{(-1)^{a+b}}{a!b!} \Re_{\hbar_{2}=0} \Re_{\hbar_{1}=0}\left\{\hbar_{1}^{-a} \hbar_{2}^{-b} C o e_{\left(\mathfrak{t}^{\prime}\right)^{2}} \tilde{z}_{66, S e p}^{*}\right\} \prod_{i=1}^{k} \frac{(-1)^{b_{i}}}{b_{i}!} \frac{\mathfrak{t}^{4}}{5} \Re_{\hbar=0} \frac{\mathcal{Z}_{1,5}}{\hbar^{b_{i}}},
\end{aligned}
$$

where we used $\mathcal{P}_{c}=\operatorname{Res}_{\hbar=0} \frac{z_{1,5}}{\hbar^{c-1}}$ in Lemma 4.1. By (13), $\mathcal{Z}_{1,5}=\left.\tilde{z}_{1,5}\right|_{\mathfrak{t}^{\prime}=0}$ where $\tilde{z}_{1,5}$ is regular in $\mathfrak{t}^{\prime}$. Applying (34), the above becomes $\left(\mathcal{Z}_{6}^{*}:=\frac{\mathfrak{t}^{4}}{5} \tilde{z}_{1,5}\right)$

$$
\begin{aligned}
& \operatorname{Cont}_{I I}=\frac{1}{5 \mathfrak{t}^{5}} \sum_{a, b \geq 0}^{\infty} \frac{(-1)^{a+b}}{a!b!} \Re_{\hbar_{2}=0} \Re_{\hbar_{1}=0}\left\{\hbar_{1}^{-a} \hbar_{2}^{-b} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{2}} \tilde{z}_{66, S e p}^{*}\right\} \\
& \left.\cdot \sum_{m=1}^{\infty} \frac{1}{m}\left(\sum_{a+b+\sum_{b_{i} \geq 0} b_{\ell}=m-1}^{\infty} \prod_{\ell=1}^{m} \frac{(-1)^{b_{\ell}}}{b_{\ell}!} \Re_{\hbar=0} \frac{\mathcal{Z}_{6}^{*}}{\hbar^{b_{\ell}}}\right)\right|_{\mathfrak{t}^{\prime}=0} \\
= & \frac{1}{5 \mathfrak{t}^{5}} \sum_{a, b \geq 0}^{\infty} \frac{(-1)^{a+b}}{a!b!} \Re_{\hbar_{2}=0} \Re_{\hbar_{1}=0}\left\{\hbar_{1}^{-a} \hbar_{2}^{-b} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{2}} \tilde{z}_{66, S e p}^{*}\right\} \frac{\eta_{0}^{a+b+1}}{a+b+1},
\end{aligned}
$$

where we used $\sum_{a, b \geq 0} \frac{y^{a+b+1}}{(a+b+1) a!b!\hbar_{1}^{a} \hbar_{2}^{b}}=\frac{\hbar_{1} \hbar_{2}}{\hbar_{1}+\hbar_{2}}\left(e^{y\left(\frac{1}{\hbar_{1}}+\frac{1}{\hbar_{2}}\right)}-1\right)$, and we denote $\eta_{0}:=\left.\eta\right|_{\mathfrak{t}^{\prime}=0}$.
For Cont $_{I}$, using $\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}=\frac{1}{2}\left[\mathcal{W}_{(\tilde{\Gamma})}\right]^{\text {vir }}$, MSP II gives us

$$
\left.\left.\operatorname{Cont}_{I}=\frac{1}{2} \sum_{d=1}^{\infty}(-\mathbb{Q})^{d} \sum_{(\tilde{\Gamma}) \in \tilde{\Delta}_{d}^{11, \mathrm{o}}} \int_{\left[\mathcal{W}_{(\tilde{\Gamma})}\right]} \frac{e\left(\tilde{\pi}_{*} \tilde{f}^{\text {vir }}\right.}{} \tilde{f}_{\tilde{\Gamma}} L_{\mathfrak{t}}\left(-\mathfrak{D}_{1}-\mathfrak{D}_{2}\right)\right)\right), \frac{\mathfrak{t}}{e\left(N_{\mathcal{W}_{(\tilde{\Gamma})}^{\text {vir }} / \mathcal{W}}\right)} \frac{-5 \mathfrak{t}^{6}}{(5 \mathfrak{t})^{2}}-\psi_{1}-\psi_{2} .
$$

6.3. Computations of Contri II. Let $\tilde{\Delta}_{d}^{11,0}$ denote the graphs in $\tilde{\Delta}_{d}^{11}$ which has only one vertex at $V_{0}$ and whose both markings are unstable. Using (24) and the geometry of cap, we can argue to obtain

$$
\begin{align*}
\mathfrak{X} & :=\mathfrak{R}_{\hbar_{2}=0} \mathfrak{R}_{\hbar_{1}=0}\left\{\hbar_{1}^{-a} \hbar_{2}^{-b} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{2}} \tilde{\mathcal{Z}}_{66, S e p}^{*}\right\} \\
& =\frac{-1}{2} \sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\tilde{\Gamma}) \in \tilde{\Delta}_{d}^{11, o}} \int_{\left[\mathcal{W}_{(\tilde{\Gamma})}\right]} \frac{e\left(\tilde{\pi}_{*} \tilde{f}^{*}\left(L_{\tilde{\Gamma}} L_{\mathfrak{t}}\left(-\mathfrak{D}_{1}-\mathfrak{D}_{2}\right)\right)\right)}{e\left(N_{\mathcal{W}_{(\tilde{\Gamma})}^{\text {vir }} / \mathcal{W}}\right)} \cdot \mathfrak{t}^{10} \frac{-1}{\psi_{1}^{a+1}} \cdot \frac{-1}{\psi_{2}^{b+1}} . \tag{25}
\end{align*}
$$

For each $\tilde{\Gamma} \in \tilde{\Delta}_{d}^{11, \circ}$, let $\bar{\psi}_{i}$ be the $\psi$-class of the marking (seperated from the node) on $\mathcal{C}_{v}$. For convenience, we introduce $W_{\mathfrak{c}_{i}}:=Q$ with $\left[W_{\mathfrak{c}_{i}}\right]^{\text {vir }}=-Q / d_{\mathfrak{c}_{i}}$, and we let $d_{i}=d_{\mathfrak{e}_{i}}$, and $h_{\mathfrak{e}_{i}}, h_{i}$ be the classes on $W_{\mathfrak{e}_{i}}, \bar{M}_{0,2}\left(\mathbb{P}^{4}, d_{v}\right)^{p}$ respectively induced by the hyperplane of $\mathbb{P}^{4}$. Also let $D_{1}, D_{2}$ be divisors of $\mathcal{C}_{e_{1}}, \mathcal{C}_{e_{2}}$ given by the nodes. There are two cases, depending on $d_{v}>0$ or $d_{v}=0$, which we denote by $C_{o n t}^{+}$and Cont $t_{I I}^{0}$ respectively. Then Cont $_{I I}=$ Cont $_{I I}^{+}+$Cont $_{I I}^{0}$.
(1) Case one: $d_{v}>0$. We have an exact sequence of sheaves

$$
\left.\left.\left.0 \rightarrow L_{\tilde{\Gamma}} L_{\mathfrak{t}}\left(-\mathfrak{D}_{1}\right)\right|_{\mathcal{C}_{\mathfrak{c}_{1}}}\left(-D_{1}\right) \oplus L_{\tilde{\Gamma}} L_{\mathfrak{t}}\left(-\mathfrak{D}_{2}\right)\right|_{\mathcal{C}_{\mathfrak{c}_{2}}}\left(-D_{2}\right) \rightarrow L_{\tilde{\Gamma}} L_{\mathfrak{t}}\left(-\mathfrak{D}_{1}-\mathfrak{D}_{2}\right) \rightarrow L_{\tilde{\Gamma}} L_{\mathfrak{t}}\right|_{\mathcal{C}_{v}} \rightarrow 0
$$

Thus, up to capping with $\left[\mathcal{W}_{\tilde{\Gamma}}\right]^{\text {vir }}$, one has

$$
\begin{equation*}
e\left(\tilde{\pi}_{*} \tilde{f}^{*}\left(L_{\tilde{\Gamma}} L_{\mathfrak{t}}\left(-\mathfrak{D}_{1}-\mathfrak{D}_{2}\right)\right)\right)=\prod_{a=1}^{2} e\left(\tilde{\pi}_{*} \tilde{f}^{*}\left(L_{\mathcal{C}_{\mathfrak{e}_{a}}} L_{\mathfrak{t}}\left(-\mathfrak{D}_{a}-D_{a}\right)\right)\right) \cdot e\left(\left.\tilde{\pi}_{*} \tilde{f}^{*} L_{\tilde{\Gamma}} L_{\mathfrak{t}}\right|_{\mathcal{C}_{v}}\right) \tag{26}
\end{equation*}
$$

$$
=\prod_{j=1}^{d_{1}-1} \frac{j\left(h_{\mathfrak{e}_{1}}+\mathfrak{t}\right)}{d_{1}} \cdot \prod_{j=1}^{d_{2}-1} \frac{j\left(h_{\mathfrak{e}_{2}}+\mathfrak{t}\right)}{d_{2}} \cdot \mathfrak{t}^{d_{v}+1} .
$$

Recall $\frac{1}{5} \sum_{j=0}^{3} h^{j} \otimes h_{\mathfrak{e}_{i}}^{3-j}$ is the Poincaré dual of the diagonal in $Q \times Q$. By CLLL2], $e_{T}\left(R \pi_{v *} \mathcal{L}_{v} \otimes L_{\mathfrak{t}}\right)=$ $\mathfrak{t}^{d_{v}+1}$,

$$
\begin{aligned}
{\left[\mathcal{W}_{(\tilde{\Gamma})}\right]^{\mathrm{vir}}=} & {\left[\prod_{i=1}^{2}\left(\frac{1}{5} \sum_{j=0}^{3} h_{i}^{j} \otimes h_{\mathfrak{c}_{i}}^{3-j}\right)\right] \cap\left(\left[\bar{M}_{0,2}\left(\mathbb{P}^{4}, d_{v}\right)^{p}\right]^{\mathrm{vir}} \times\left(\left[W_{\mathfrak{c}_{1}}\right]^{\mathrm{vir}} \times\left[W_{\mathfrak{c}_{2}}\right]^{\mathrm{vir}}\right)\right), } \\
& \frac{1}{e\left(N_{\left.\mathcal{W}_{(\overline{\mathrm{r}})}^{\mathrm{vir}} / \mathcal{W}\right)}\right)}=\frac{(5 \mathfrak{t})^{2}\left(h_{\mathfrak{c}_{1}}+\mathfrak{t}\right)\left(h_{\mathfrak{c}_{2}}+\mathfrak{t}\right)}{e_{T}\left(R \pi_{v *} \mathcal{L}_{v} \otimes L_{\mathfrak{t}}\right)} \frac{A_{\mathfrak{c}_{1}} \cdot A_{\mathfrak{c}_{2}}}{\left(\psi_{1}-\bar{\psi}_{1}\right)\left(\psi_{2}-\bar{\psi}_{2}\right)} .
\end{aligned}
$$

And for $i=1,2, \psi_{i}=\frac{h_{i}+\mathfrak{t}}{d_{i}}$ and $\bar{\psi}_{i}{ }^{3}=0$ give $\frac{1}{\psi_{i}-\psi_{i}}=\sum_{\ell_{i}=0}^{2} \frac{\bar{\psi}_{i}{ }_{i}}{\psi_{i}^{\epsilon_{i}+1}}$. The above relation and (26) gives

$$
\begin{equation*}
\frac{e\left(\tilde{\pi}_{*} \tilde{f}^{*}\left(L_{\tilde{\Gamma}} L_{\mathfrak{t}}\left(-\mathfrak{D}_{1}-\mathfrak{D}_{2}\right)\right)\right)}{e\left(N_{\mathcal{W}_{(\tilde{\Gamma})}^{\text {vir }} / \mathcal{W}}\right)}=\frac{B_{d_{1}}\left(h_{\mathfrak{e}_{1}}\right) B_{d_{2}}\left(h_{\mathfrak{e}_{2}}\right)}{\left(\psi_{1}-\bar{\psi}_{1}\right)\left(\psi_{2}-\bar{\psi}_{2}\right)} \tag{27}
\end{equation*}
$$

Thus contribution of case $d_{v}>0$ to (25) is

$$
\begin{aligned}
\mathfrak{X}^{+}= & \frac{-1}{2} \sum_{j_{1}, j_{2}=0}^{3} \sum_{\ell_{1}, \ell_{2}=0}^{2}\left(\sum_{d_{v}=1}^{\infty}(-Q)^{d_{v}} \int_{\left[\bar{M}_{0,2}\left(\mathbb{P}^{4}, d_{v}\right)^{p}\right]_{\text {vir }}} \bar{\psi}_{1}^{\ell_{1}} e v_{1}^{*} h_{1}^{j_{1}} \bar{\psi}_{2}^{\ell_{2}} e v_{2}^{*} h_{2}^{j_{2}} \frac{1}{\mathfrak{t}^{d_{v}+1}} \mathfrak{t}^{d_{v}+1}\right) \\
& \cdot\left(\frac{5 \mathfrak{t}}{5} \sum_{d_{1}=1}^{\infty}(-Q)^{d_{1}} \int_{\left[W_{\mathfrak{e}_{1}}\right]^{\text {vir }}} h_{\mathfrak{e}_{1}}^{3-j_{1}} \frac{h_{\mathfrak{e}_{1}}+\mathfrak{t}}{\psi_{1}^{\ell_{1}+1}} A_{\mathfrak{e}_{1}}\left(\prod_{j=1}^{d_{1}-1} \frac{j\left(h_{\mathfrak{e}_{1}}+\mathfrak{t}\right)}{d_{1}}\right) \frac{\mathfrak{t}^{5}}{\psi_{1}^{a+1}}\right) \\
& \cdot\left(\frac{5 \mathfrak{t}}{5} \sum_{d_{2}=1}^{\infty}(-Q)^{d_{2}} \int_{\left[W_{\mathfrak{c}_{2}}\right] \text { vir }} h_{\mathfrak{e}_{2}}^{3-j_{2}} \frac{h_{\mathfrak{e}_{2}+\mathfrak{t}}}{\psi_{2}^{\ell_{2}+1}} A_{\mathfrak{e}_{2}}\left(\prod_{j=1}^{d_{2}-1} \frac{j\left(h_{\mathfrak{e}_{2}}+\mathfrak{t}\right)}{d_{2}}\right) \frac{\mathfrak{t}^{5}}{\psi_{2}^{b+1}}\right) \\
= & \frac{1}{2} \sum_{j_{1}, j_{2}=0}^{1} \sum_{\ell_{1}, \ell_{2}=0}^{2}\left(\sum_{d_{v}=1}^{\infty} Q^{d_{v}}<\tau_{\ell_{1}}\left(h^{j_{1}}\right) \tau_{\ell_{2}}\left(h^{j_{2}}\right)>_{g=0, d_{v}}\right) \\
& \cdot\left(\frac{\mathfrak{t}^{5}}{5} \sum_{d_{1}=1}^{\infty}(-Q)^{d_{1}} \int_{\left[W_{\mathfrak{e}_{1}}\right] \text { vir }} \frac{h_{\mathfrak{e}_{1}}^{3-j_{1}}}{\psi_{1}^{\ell_{1}+2+a}} B_{d_{1}}\left(h_{\mathfrak{e}_{1}}\right)\right) \frac{\mathfrak{t}^{5}}{5} \sum_{d 2=1}^{\infty}(-Q)^{d_{2}} \int_{\left[W_{\mathfrak{e}_{2}}\right]^{\text {vir }}} \frac{h_{\mathfrak{e}_{2}}^{3-j_{2}}}{\psi_{2}^{\ell_{2}+2+b}} B_{d_{2}}\left(h_{\mathfrak{e}_{2}}\right) .
\end{aligned}
$$

Thus the contribution from Case one is

$$
\begin{aligned}
& \text { Cont }_{I I}^{+}=\frac{1}{5 \mathfrak{t}^{5}} \sum_{a, b \geq 0}^{\infty} \frac{(-1)^{a+b}}{a!b!} \mathfrak{X}^{+} \frac{\eta_{0}^{a+b+1}}{a+b+1}=\frac{1}{2} \frac{\mathfrak{t}^{5}}{5} \sum_{j_{1}, j_{2}=0}^{1} \sum_{\ell_{1}, \ell_{2}=0}^{2} \sum_{d_{v}=1}^{\infty} \mathbb{Q}^{d_{v}}<\tau_{\ell_{1}}\left(h^{j_{1}}\right) \tau_{\ell_{2}}\left(h^{j_{2}}\right)>_{g=0, d_{v}} \\
& \cdot\left(\frac{1}{5^{2}} \sum_{d_{1}, d_{2}=1}^{\infty}(-Q)^{d_{1}+d_{2}} \int_{\left[W_{\mathfrak{e}_{1}}\right]^{\mathrm{vir}} \times\left[W_{\mathfrak{e}_{2}}\right]^{\mathrm{vir}}} \frac{h_{\mathfrak{e}_{1}}^{3-j_{1}}}{\psi_{1}^{\ell_{1}+1}} \frac{h_{\mathfrak{e}_{2}}^{3-j_{2}}}{\psi_{2}^{\ell_{2}+1}} B_{d_{1}}\left(h_{\mathfrak{e}_{1}}\right) B_{d_{2}}\left(h_{\mathfrak{e}_{2}}\right) \frac{e^{\left(-\eta_{0}\right)\left(\frac{1}{\psi_{1}}+\frac{1}{\psi_{2}}\right)}-1}{\psi_{1}+\psi_{2}}\right) .
\end{aligned}
$$

(2) Case two: $d_{v}=0$. Then $v$ is unstable. Similarly by [CLLL2], $\left[\mathcal{W}_{(\tilde{\Gamma})}\right]^{\mathrm{vir}}=\frac{[-Q]}{d_{1} d_{2}}, \psi_{i}=\frac{h_{e_{i}+\mathfrak{t}}}{d_{i}}$,

$$
\begin{aligned}
& \frac{1}{e\left(N_{\mathcal{W}_{(\tilde{\Gamma})} / \mathcal{W}}^{\text {vir }}\right)}=(5 \mathfrak{t})^{2} \frac{(h+\mathfrak{t}) A_{\mathfrak{e}_{1}} A_{\mathfrak{e}_{2}}}{\psi_{1}+\psi_{2}}, e\left(\tilde{\pi}_{*} \tilde{f}^{*}\left(L_{\tilde{\Gamma}} L_{\mathfrak{t}}\left(-\mathfrak{D}_{1}-\mathfrak{D}_{2}\right)\right)\right) \\
= & (h+\mathfrak{t}) \prod_{j=1}^{d_{1}-1} \frac{j\left(h_{\mathfrak{e}_{1}}+\mathfrak{t}\right)}{d_{1}} \prod_{j=1}^{d_{2}-1} \frac{j\left(h_{\mathfrak{e}_{2}}+\mathfrak{t}\right)}{d_{2}} .
\end{aligned}
$$

Thus contribution of Case two to (25) is,

$$
\mathfrak{X}^{0}:=\frac{-1}{2} \sum_{d_{1}, d_{2}=1}^{\infty} \frac{(-Q)^{d_{1}+d_{2}}}{d_{1} d_{2}}\left(\int_{[-Q]}(5 \mathfrak{t})^{2} \frac{h+\mathfrak{t}}{\psi_{1}+\psi_{2}} A_{\mathfrak{e}_{1}} A_{\mathfrak{e}_{2}}\right.
$$

$$
\begin{aligned}
& \left.\cdot\left(\prod_{j=1}^{d_{1}-1} \frac{j\left(h_{\mathfrak{e}_{1}}+\mathfrak{t}\right)}{d_{1}}\right)\left(\prod_{j=1}^{d_{2}-1} \frac{j\left(h_{\mathfrak{e}_{2}}+\mathfrak{t}\right)}{d_{2}}\right)(h+\mathfrak{t}) \mathfrak{t}^{10} \frac{1}{\psi_{1}^{a+1}} \frac{1}{\psi_{2}^{b+1}}\right) \\
= & \frac{1}{2} \sum_{d_{1}, d_{2}=1}^{\infty} \frac{(-Q)^{d_{1}+d_{2}}}{d_{1}+d_{2}} \int_{Q}(5 \mathfrak{t})^{2} A_{\mathfrak{e}_{1}} A_{\mathfrak{e}_{2}}\left(\prod_{j=1}^{d_{1}-1} \frac{j\left(h_{\mathfrak{e}_{1}}+\mathfrak{t}\right)}{d_{1}}\right)\left(\prod_{j=1}^{d_{2}-1} \frac{j\left(h_{\mathfrak{e}_{2}}+\mathfrak{t}\right)}{d_{2}}\right)(h+\mathfrak{t}) \mathfrak{t}^{10} \frac{1}{\psi_{1}^{a+1}} \frac{1}{\psi_{2}^{b+1}} .
\end{aligned}
$$

We see that the contribution of Case 2 to Contri ${ }_{I I}$ is

$$
\text { Cont }_{I I}^{0}=\frac{1}{5 \mathfrak{t}^{5}} \sum_{a, b=0}^{\infty} \frac{(-1)^{a+b}}{a!b!} \mathfrak{X}^{0} \frac{\eta_{0}^{a+b+1}}{a+b+1}
$$

6.4. Computations of Contri I. The graph of the contribution type $I$ is a type $B$ graph $\Gamma$ with the unique vertex $\tilde{v} \in V_{1}(\Gamma)$ unstable. Let $\tilde{\Gamma}$ be the graph obtained by resolving the node at $V_{1}$ of $\Gamma$ shown in the Figure Graph $\tilde{\Gamma}$. Then $\tilde{\Gamma}$ is a graph in $\tilde{\Delta}_{d}^{11, \circ}$. We have $\left[\mathcal{W}_{(\Gamma)}\right]^{\text {vir }}=\frac{1}{2}\left[\mathcal{W}_{(\tilde{\Gamma})}\right]^{\text {vir }}$. The contribution of $\Gamma$ is

$$
\frac{1}{2} \sum_{d=1}^{\infty}(-Q)^{d} \sum_{(\tilde{\Gamma}) \in \tilde{\Delta}_{d}^{11, \mathrm{o}}} \int_{\left[\mathcal{W}_{(\tilde{\Gamma})}\right]^{\mathrm{vir}}} \frac{e\left(\tilde{\pi}_{*} \tilde{f}^{*}\left(L_{\tilde{\Gamma}} L_{\mathfrak{t}}\left(-\mathfrak{D}_{1}-\mathfrak{D}_{2}\right)\right)\right)}{e\left(N_{\mathcal{W}_{(\tilde{\Gamma})}^{\text {vir }} / \mathcal{W}}\right)} \cdot \frac{\mathfrak{t}}{(5 \mathfrak{t})^{2}} \frac{-5 \mathfrak{t}^{6}}{-\psi_{1}-\psi_{2}}
$$

There are two cases, depending on $d_{v}>0$ or $d_{v}=0$.
(1) Case one: $d_{v}>0$. The factors are the same as those in Cont $_{I I}^{+}$. By (27), one has

$$
\begin{aligned}
\text { Cont }_{I}^{+}= & \frac{-1}{2} \frac{\mathfrak{t}^{5}}{5} \sum_{j_{1}, j_{2}=0}^{1} \sum_{\ell_{1}, \ell_{2}=0}^{2}\left(\sum_{d_{v}=1}^{\infty} Q^{d_{v}}<\tau_{\ell_{1}}\left(h^{j_{1}}\right) \tau_{\ell_{2}}\left(h^{j_{2}}\right)>_{g=0, d_{v}}\right) \\
& \left(\frac{1}{5^{2}} \sum_{d_{1}, d_{2}=1}^{\infty}(-Q)^{d_{1}+d_{2}} \int_{\left[W_{\mathfrak{e}_{1}}\right]^{\mathrm{vir}} \times\left[W_{\mathfrak{e}_{2}}\right]^{\mathrm{vir}}} \frac{h_{\mathfrak{e}_{1}}^{3-j_{1}}}{\psi_{1}^{\ell_{1}+1}} B_{d_{1}}\left(h_{\mathfrak{e}_{1}}\right) \frac{h_{\mathfrak{e}_{2}}^{3-j_{2}}}{\psi_{2}^{\ell_{2}+1}} B_{d_{2}}\left(h_{\mathfrak{e}_{2}}\right) \frac{1}{\psi_{1}+\psi_{2}}\right) .
\end{aligned}
$$

(2) Case two: $d_{v}=0$. The factors are the same as those in $C_{o n t}^{0}$. Thus contribution of this case is, noting $h_{\mathfrak{e}_{1}}=h_{\mathfrak{e}_{2}}=h$ over the diagonal $Q \subset Q \times Q$,

$$
\begin{aligned}
\text { Cont }_{I}^{0} & =\frac{1}{2} \sum_{d_{1}, d_{2}=1}^{\infty} \frac{(-Q)^{d_{1}+d_{2}}}{d_{1} d_{2}} \int_{[-Q]}(5 \mathfrak{t})^{2} \frac{(h+\mathfrak{t}) A_{\mathfrak{e}_{1}} A_{\mathfrak{e}_{2}}}{\psi_{1}+\psi_{2}}\left(\prod_{j=1}^{d_{1}-1} \frac{j\left(h_{\mathfrak{e}_{1}}+\mathfrak{t}\right)}{d_{1}}\right)\left(\prod_{j=1}^{d_{2}-1} \frac{j\left(h_{\mathfrak{e}_{2}}+\mathfrak{t}\right)}{d_{2}}\right) \frac{\mathfrak{t}(h+\mathfrak{t})}{\psi_{1}+\psi_{2}} \frac{\mathfrak{t}^{4}}{5} \\
& =-\frac{1}{2} \frac{\mathfrak{t}^{5}}{5} \sum_{d_{1}, d_{2}=1}^{\infty} \frac{(-Q)^{d_{1}+d_{2}}}{d_{1}+d_{2}} \int_{Q}(5 \mathfrak{t})^{2} A_{\mathfrak{e}_{1}} A_{\mathfrak{c}_{2}}\left(\prod_{j=1}^{d_{1}-1} \frac{j\left(h_{\mathfrak{e}_{1}}+\mathfrak{t}\right)}{d_{1}}\right)\left(\prod_{j=1}^{d_{2}-1} \frac{j\left(h_{\mathfrak{e}_{2}}+\mathfrak{t}\right)}{d_{2}}\right) \frac{(h+\mathfrak{t})}{\psi_{1}+\psi_{2}}
\end{aligned}
$$

6.5. Summations $\operatorname{Cont}_{I I}^{+}+\operatorname{Cont}_{I}^{+}$and $\operatorname{Cont}_{I I}^{0}+\operatorname{Cont}_{I}^{0}$. Using

$$
\sum_{a, b=0}^{\infty} \frac{(-1)^{a+b}}{a!b!} \frac{1}{\psi_{1}^{a+1}} \frac{1}{\psi_{2}^{b+1}} \frac{\eta_{0}^{a+b+1}}{a+b+1}=\frac{-1}{\psi_{1}+\psi_{2}}\left(e^{\left(-\eta_{0}\right)\left(\frac{1}{\psi_{1}}+\frac{1}{\psi_{2}}\right)}-1\right)
$$

we see the sum $\operatorname{Cont}_{I I}^{+}+$Cont $_{I}^{+}$equals

$$
\begin{align*}
& \text { Cont }^{+}:=\text {Cont }_{\text {II }}^{+}+\text {Cont }_{I}^{+}=\frac{1}{2} \frac{\mathfrak{t}^{5}}{5} \sum_{j_{1}, j_{2}=0}^{1} \sum_{\ell_{1}, \ell_{2}=0}^{2}\left(\sum_{d_{v}=1}^{\infty} Q^{d_{v}}<\tau_{\ell_{1}}\left(h^{j_{1}}\right) \tau_{\ell_{2}}\left(h^{j_{2}}\right)>_{g=0, d_{v}}\right) \\
& \cdot\left(-\frac{1}{5^{2}} \sum_{d_{1}, d_{2}=1}^{\infty}(-Q)^{d_{1}+d_{2}} \int_{\left[W_{\mathfrak{e}_{1}}\right]^{\mathrm{vir}} \times\left[W_{\mathfrak{e}_{2}}\right]^{\mathrm{vir}}} \frac{h_{\mathfrak{e}_{1}}^{3-j_{1}}}{\psi_{1}^{\ell_{1}+1}} \frac{h_{\mathfrak{e}_{2}}^{3-j_{2}}}{\psi_{2}^{\ell_{2}+1}} B_{d_{1}}\left(h_{\mathfrak{e}_{1}}\right) B_{d_{2}}\left(h_{\mathfrak{e}_{2}} \frac{e^{\left(-\eta_{0}\right)\left(\frac{1}{\psi_{1}}+\frac{1}{\psi_{2}}\right)}}{\psi_{1}+\psi_{2}}\right) .\right. \tag{28}
\end{align*}
$$

A similar combination holds for $\operatorname{Cont}_{I I}^{0}+\operatorname{Cont}_{I}^{0}$. Recall $h_{\mathfrak{e}_{1}}=h=h_{\mathfrak{e}_{2}}$ over diagonal $Q \subset Q \times Q$. We have

$$
\begin{align*}
& \text { Cont }^{0}:=\text { Cont }_{I I}^{0}+\text { Cont }_{I}^{0}=-\frac{1}{2} \frac{\mathfrak{t}^{5}}{5} \sum_{d_{1}, d_{2}=1}^{\infty} \frac{(-\mathcal{Q})^{d_{1}+d_{2}}}{d_{1}+d_{2}}\left(\int_{Q}(5 \mathfrak{t})^{2} A_{\mathfrak{e}_{1}} A_{\mathfrak{e}_{2}}\right.  \tag{29}\\
& \left.\cdot\left(\prod_{j=1}^{d_{1}} \frac{j\left(h_{\mathfrak{e}_{1}}+\mathfrak{t}\right)}{d_{1}}\right)\left(\prod_{j=1}^{d_{2}} \frac{j\left(h_{\mathfrak{e}_{2}}+\mathfrak{t}\right)}{d_{2}}\right) \frac{1}{d_{1} d_{2}} \frac{1}{\psi_{1}} \frac{1}{\psi_{2}} \frac{h+\mathfrak{t}}{\psi_{1}+\psi_{2}} e^{\left(-\eta_{0}\right)\left(\frac{1}{\psi_{1}}+\frac{1}{\psi_{2}}\right)}\right) .
\end{align*}
$$

6.6. $S$-series and $I$-functions. In this subsection we introduce a $S$-series which is a generating function of MSP's $A_{e}$ for all edges $e$ connecting $V_{0}$ with $V_{1}$. We express $S$-series using Givental's $I$ -unctions. This will enable us to express the two series Cont $I_{I I}^{+}+$Cont $_{I I}^{0}$ and $C_{o n t}^{+}+C o n t_{I}^{0}$ in previous subsections using Givental's $I$-functions.

For $j \in \mathbb{Z}_{\leq 3}, \psi=\frac{h_{\mathfrak{c}}+\mathfrak{t}}{d_{\mathfrak{e}}}$, and for arbitrary $m \in \mathbb{Z}$, we formulate

$$
T_{j, m}:=\frac{\mathfrak{t}^{5}}{5} \sum_{d=1}^{\infty}(-\mathcal{Q})^{d} \int_{\left[W_{\mathrm{e}}\right] \mathrm{vir}} \frac{h^{3-j}}{\psi^{m}} B_{d}(h):=S_{j, m-2}
$$

Note that this implies $S_{j, m}=0$ whenever $j<0$. Also, by definition, we have

$$
\int_{\left[W_{\mathrm{c}}\right]^{\mathrm{vir}}} d \frac{h^{3-j}}{\psi^{m+2}} B_{d}(h)=\int_{\left[W_{\mathfrak{e}}\right]_{\mathrm{vir}}}(h+\mathfrak{t}) \frac{h^{3-j}}{\psi^{m+3}} B_{d}(h)=\int_{\left[W_{\mathrm{e}}\right]^{\mathrm{vir}}} \frac{h^{3-(j-1)}}{\psi^{m+3}} B_{d}(h)+\mathfrak{t} \int_{\left[W_{\mathfrak{c}}\right]^{\mathrm{vir}}} \frac{h^{3-j}}{\psi^{m+3}} B_{d}(h)
$$

This implies directly

$$
\partial_{T} S_{j, m}=S_{j-1, m+1}+\mathfrak{t} S_{j, m+1}, \quad \forall j \in \mathbb{Z}_{\leq 3}, m \in \mathbb{Z}
$$

We also have

$$
\left.S_{j, m}\right|_{T=-\infty}=0,\left.\quad T^{k} S_{j, m}\right|_{T=-\infty}=0, \forall k \in \mathbb{R}
$$

where $-\infty$ refers to the limit of the point on the real line with the real part approaching the negative infinity.

On the other hand, denote $I_{\ell}=0=b_{\ell}(d)$ whenever $\ell<0$. By (7), for $j=0, \cdots, 3$, and $m \in \mathbb{Z}$, we have

$$
T_{j, m}=\mathfrak{t}^{5-j-m} \sum_{d=1}^{\infty} d^{m-1}(-Q)^{d}\left[-b_{j}(d)+m b_{j-1}(d)-\binom{m+1}{2} b_{j-2}(d)+\binom{m+2}{3} b_{j-3}(d)\right] .
$$

We denote $I_{j}(y)=\left.I_{j}(t)\right|_{t \mapsto y}$, and $I_{j}^{(k)}=\partial_{y}^{k} I_{j}(y)$. As an example, we have

$$
\begin{gathered}
S_{0,-1}=T_{0,1}=-\mathfrak{t}^{5} \sum_{d=1}^{\infty}(-Q)^{d} b_{0}(d) \frac{1}{\mathfrak{t}}=-\left.\mathfrak{t}^{4}\left(I_{0}(y)-1\right)\right|_{y \mapsto T} \\
S_{1,-1}=T_{1,1}=\left.\mathfrak{t}^{3} I_{1}^{\prime}\right|_{t \mapsto T}-\left.\mathfrak{t}^{3} T I_{0}^{\prime}(y)\right|_{y \mapsto T}-\mathfrak{t}^{3}
\end{gathered}
$$

Applying the formulae in (9), one directly computes
Lemma 6.1. Denote $I_{\ell}=0$ whenever $\ell<0$. Then for $m>0$ and $j=0, \cdots, 3$, we have

$$
\begin{aligned}
T_{j, m}= & \left.(-1)^{j+1} \mathfrak{t}^{5-j-m}\left(I_{j}^{(m+j-1)}-t I_{j-1}^{(m+j-1)}+\frac{t^{2}}{2} I_{j-2}^{(m+j-1)}-\frac{t^{3}}{6} I_{j-3}^{(m+j-1)}\right)\right|_{t \mapsto T} \\
& +(-1)^{j} \mathfrak{t}^{5-j-m}\binom{m+j-1}{j} \delta_{m, 1}
\end{aligned}
$$

Lemma 6.2. For arbitrary $\ell \in \mathbb{Z}_{\geq-1}, j=0, \cdots, 3$, we have

$$
\begin{aligned}
\sum_{a=0}^{\infty} \frac{(-1)^{a}}{a!} S_{j, a+\ell}(-\mathfrak{t}(y-T))^{a}= & (-1)^{j+1} \mathfrak{t}^{3-j-\ell}\left(I_{j}^{(\ell+j+1)}(y)-T I_{\ell+j}^{(\ell+j+1)}(y)+\frac{T^{2}}{2} I_{\ell+j-1}^{(\ell+j+1)}(y)\right. \\
& \left.-\frac{T^{3}}{6} I_{\ell+j-2}^{(\ell+j+1)}(y)\right)+(-1)^{j} \mathfrak{t}^{3-j-\ell}\binom{\ell+j+1}{j} \delta_{\ell,-1} .
\end{aligned}
$$

6.7. Expression of Cont $^{+}$by $I$-functions. Let us define a function, for $j_{1}, j_{2} \in \mathbb{Z}_{\leq 3}, \ell_{1}, \ell_{2} \in \mathbb{Z}$,

$$
\begin{aligned}
& M_{j_{1}, j_{2}, \ell_{1}, \ell_{2}}(x, T) \\
& :=-\frac{1}{2} \frac{\mathfrak{t}^{6}}{5} \sum_{d_{1}, d_{2}=1}^{\infty} \frac{\left(-e^{T}\right)^{d_{1}+d_{2}}}{d_{1}+d_{2}} \int_{\left[W_{\mathfrak{c}_{1}}\right]^{\mathrm{vir}} \times\left[W_{\mathfrak{c}_{2}}\right]^{\mathrm{vir}}} \frac{1}{5} \frac{h_{\mathfrak{e}_{1}}^{3-j_{1}}}{\psi_{1}^{\ell_{1}+1}} \frac{h_{\mathfrak{e}_{2}}^{3-j_{2}}}{\psi_{2}^{\ell_{2}+1}} B_{d_{1}}\left(h_{\mathfrak{e}_{1}}\right) B_{d_{2}}\left(h_{\mathfrak{e}_{2}}\right) \frac{e^{(\mathrm{t} x)\left(\frac{1}{\psi_{1}}+\frac{1}{\psi_{2}}\right)}}{\psi_{1}+\psi_{2}} .
\end{aligned}
$$

Then

$$
\begin{aligned}
& \partial_{T} M_{j_{1}, j_{2}, \ell_{1}, \ell_{2}}(x, T) \\
= & -\frac{1}{2} \frac{\mathfrak{t}^{6}}{5} \sum_{d_{1}, d_{2}=1}^{\infty}\left(-e^{T}\right)^{d_{1}+d_{2}} \int_{\left[W_{\mathfrak{e}_{1}}\right]^{\operatorname{vir}} \times\left[W_{\mathfrak{e}_{2}}\right]^{\operatorname{vir}}} \frac{1}{5} \frac{h_{\mathfrak{e}_{1}}^{3-j_{1}}}{\psi_{1}^{\ell_{1}+1}} \frac{h_{\mathfrak{e}_{2}}^{3-j_{2}}}{\psi_{2}^{\ell_{2}+1}} B_{d_{1}}\left(h_{\mathfrak{e}_{1}}\right) B_{d_{2}}\left(h_{\mathfrak{e}_{2}}\right) \frac{e^{(\mathfrak{t} x)\left(\frac{1}{\psi_{1}}+\frac{1}{\psi_{2}}\right)}}{\psi_{1}+\psi_{2}} .
\end{aligned}
$$

We have, when $x$ or $T$ approaches real $-\infty$,

$$
\left.M_{j_{1}, j_{2}, \ell_{1}, \ell_{2}}(x, T)\right|_{x \rightarrow-\infty}=0,\left.\quad \partial_{T} M_{j_{1}, j_{2}, \ell_{1}, \ell_{2}}(x, T)\right|_{x \rightarrow-\infty}=0,\left.\quad M_{j_{1}, j_{2}, \ell_{1}, \ell_{2}}(x, T)\right|_{T \rightarrow-\infty}=0
$$

This implies

$$
\begin{gathered}
\partial_{T} M_{j_{1}, j_{2}, \ell_{1}, \ell_{2}}(z, T)=\int_{-\infty}^{z} \partial_{x} \partial_{T} M_{j_{1}, j_{2}, \ell_{1}, \ell_{2}}(x, T) d x \\
M_{j_{1}, j_{2}, \ell_{1}, \ell_{2}}(x, u)=\int_{-\infty}^{u} \partial_{w} M_{j_{1}, j_{2}, \ell_{1}, \ell_{2}}(x, w) d w=\int_{-\infty}^{u} \int_{-\infty}^{x} \partial_{z} \partial_{w} M_{j_{1}, j_{2}, \ell_{1} \ell_{2}}(z, w) d z d w .
\end{gathered}
$$

We have a closed formula

$$
\partial_{x} \partial_{T} M_{j_{1}, j_{2}, \ell_{1}, \ell_{2}}(x, T)=-\frac{1}{2} \frac{1}{\mathfrak{t}^{3}} \sum_{a, b=0}^{\infty} S_{j_{1}, \ell_{1}+a} S_{j_{2}, \ell_{2}+b} \frac{(\mathfrak{t} x)^{a}}{a!} \frac{(\mathfrak{t} x)^{b}}{b!}
$$

The definition of $M$ enables us to simplify, by (28), (29) and the above formula of $\partial_{x} \partial_{T} M$,

$$
\begin{aligned}
& \text { Cont }^{+}:=\text {Cont }_{I I}^{+}+\text {Cont }_{I}^{+} \\
= & \left.\frac{1}{5 \mathfrak{t}} \sum_{j_{1}, j_{2}=0}^{1} \sum_{\ell_{1}, \ell_{2}=0}^{2}\left(\sum_{d_{v}=1}^{\infty} \mathbb{Q}^{d_{v}}<\tau_{\ell_{1}}\left(h^{j_{1}}\right) \tau_{\ell_{2}}\left(h^{j_{2}}\right)>_{g=0, d_{v}}\right)\left(\partial_{T} M_{j_{1}, j_{2}, \ell_{1}, \ell_{2}}(x, T)\right)\right|_{x=t-T} \\
= & \left.\frac{1}{5 \mathfrak{t}} \partial_{T}^{2} F_{0}\left(\partial_{T} M_{1,1,0,0}(x, T)\right)\right|_{x=t-T} \\
& +\frac{1}{5 \mathfrak{t}} 2 \partial_{T} F_{0}\left(\left.\left(\partial_{T} M_{1,0,1,0}(x, T)\right)\right|_{x=t-T}-\left.\left(\partial_{T} M_{1,0,0,1}(x, T)\right)\right|_{x=t-T}\right) \\
& +\frac{1}{5 \mathfrak{t}} 2 F_{0}\left(\left.\left(\partial_{T} M_{0,0,1,1}(x, T)\right)\right|_{x=t-T}-\left.2\left(\partial_{T} M_{0,0,2,0}(x, T)\right)\right|_{x=t-T}\right) \\
= & -\frac{1}{10} \partial_{T}^{2} F_{0} \int_{-\infty}^{t}\left(I_{1}^{(2)}-T I_{0}^{(2)}\right)^{2}(y) d y \\
& -\frac{1}{10} 2 \partial_{T} F_{0}\left(-\int_{-\infty}^{t}\left(I_{1}^{(3)}-T I_{0}^{(3)}\right)(y) I_{0}^{(1)}(y) d y+\int_{-\infty}^{t}\left(I_{1}^{(2)}-T I_{0}^{(2)}\right)(y) I_{0}^{(2)}(y)\right) \\
& -\frac{1}{10} 2 F_{0}\left(\int_{-\infty}^{t}\left(I_{0}^{(2)}\right)^{2}(y)-2 \int_{-\infty}^{t} I_{0}^{(3)}(y) I_{0}^{(1)}(y) d y\right),
\end{aligned}
$$

where $I_{a}^{(k)}(y):=\frac{d^{k}}{d y^{k}}\left(\left.I_{a}(t)\right|_{t \mapsto y}\right)$, and in last identity we used Lemma 6.2 Furthermore, using

$$
F_{0}=\frac{5}{2}\left(J_{1} J_{2}-J_{3}\right)-\frac{5}{6} T^{3}, \quad \partial_{T} F_{0}=5 J_{2}-5 \cdot \frac{T^{2}}{2}, \quad \partial_{T}^{2} F_{0}=5 J_{2}^{\prime}-5 T
$$

one easily calculates

$$
\begin{aligned}
& \text { Cont }^{+}=-\frac{1}{2} \partial_{T} J_{2} \int_{-\infty}^{t}\left(I_{1}^{(2)}-T I_{0}^{(2)}\right)^{2} d y-\frac{1}{2} 2 J_{2}\left(\int_{-\infty}^{t}\left(I_{1}^{(2)}-T I_{0}^{(2)}\right) I_{0}^{(2)}-\left(I_{1}^{(3)}-T I_{0}^{(3)}\right) I_{0}^{\prime} d y\right) \\
& -\frac{1}{2} \partial_{T}^{-1}\left(2 J_{2}\right) \int_{-\infty}^{t}\left(I_{0}^{(2)}\right)^{2}-2 I_{0}^{(3)} I_{0}^{(1)} d y+\frac{1}{2} T \int_{-\infty}^{t}\left(I_{1}^{(2)}\right)^{2} d y-\frac{1}{2} T^{2}\left(I_{1}^{\prime \prime}(t)-\frac{T}{3} I_{0}^{\prime \prime}(t)\right) I_{0}^{\prime}(t)
\end{aligned}
$$

6.8. Expression of $C o n t^{0}$ by $I$-functions. In this subsection we use $I_{r}$ to mean Givental's $I$ functions $I_{r}(t)$, which is a power series in $q=e^{t}$ as usual. The difference of the $I_{r}$ notation from before is just the variable $y$ changed to variable $t$ here.

$$
\begin{aligned}
& \text { Cont }_{I I}^{0}+\text { Cont }_{I}^{0}=-\frac{1}{2} \frac{\mathfrak{t}^{5}}{5} \sum_{d_{1}, d_{2}=1}^{\infty} \frac{(-1)^{d_{1}+d_{2}}}{\left(d_{1}+d_{2}\right)^{2}} \int_{Q} B_{d_{1}}(h) \cdot B_{d_{2}}(h) \frac{1}{\psi_{1}} \frac{1}{\psi_{2}} q^{d_{1}+d_{2}} \\
& \cdot\left(1+\left(d_{1}+d_{2}\right) \frac{(T-t) h}{h+\mathfrak{t}}+\frac{1}{2}\left(d_{1}+d_{2}\right)^{2}(T-t)^{2} \frac{h^{2}}{\mathfrak{t}^{2}}\right)+\frac{1}{2}(T-t)^{2}\left(I_{0}^{\prime}\right)^{2}-\frac{1}{12}(T-t)^{3} \partial_{t}\left(I_{0}^{\prime} \cdot I_{0}^{\prime}\right)
\end{aligned}
$$

Let us calculate the three terms appeared above respectively.

$$
\begin{aligned}
& -\frac{1}{2} \frac{\mathfrak{t}^{5}}{5} \sum_{d_{1}, d_{2}=1}^{\infty} \frac{(-1)^{d_{1}+d_{2}}}{\left(d_{1}+d_{2}\right)^{2}} \int_{Q} B_{d_{1}}(h) \cdot B_{d_{2}}(h) \frac{1}{\psi_{1}} \frac{1}{\psi_{2}} q^{d_{1}+d_{2}} \cdot\left(\frac{1}{2}\left(d_{1}+d_{2}\right)^{2}(T-t)^{2} \frac{h^{2}}{\mathfrak{t}^{2}}\right) \frac{(h+\mathfrak{t})^{2}}{(h+\mathfrak{t})^{2}} \\
= & -\frac{\mathfrak{t}^{5}}{20}(T-t)^{2} \sum_{d_{1}, d_{2}=1}^{\infty}(-1)^{d_{1}+d_{2}} \int_{Q} \frac{B_{d_{1}}(h) \cdot B_{d_{2}}(h)}{\psi_{1} \psi_{2}} \frac{1}{(h+\mathfrak{t})^{2}} q^{d_{1}+d_{2}} \cdot h^{2} \\
- & \frac{\mathfrak{t}^{5}}{10}(T-t)^{2} \sum_{d_{1}, d_{2}=1}^{\infty}(-1)^{d_{1}+d_{2}} \int_{Q} \frac{B_{d_{1}}(h) \cdot B_{d_{2}}(h)}{\psi_{1} \psi_{2}} \frac{1}{(h+\mathfrak{t})^{2}} q^{d_{1}+d_{2}} \cdot \frac{h^{3}}{\mathfrak{t}} \\
= & -\frac{\mathfrak{t}^{5}}{100}(T-t)^{2} \sum_{j=0}^{1} \sum_{d_{1}, d_{2}=1}^{\infty}(-1)^{d_{1}+d_{2}} \int_{\left[W_{\left.\mathfrak{c}_{1}\right]}\right] \mathrm{vir}^{\mathrm{vi}}\left[W_{\mathfrak{c}_{2}}\right] \mathrm{vir}} h_{1}^{j+2} h_{2}^{3-j} \frac{B_{d_{1}}\left(h_{1}\right) \cdot B_{d_{2}}\left(h_{2}\right)}{\psi_{1}^{2} \psi_{2}^{2}} q^{d_{1}+d_{2}} \\
- & \frac{1}{2}(T-t)^{2} \sum_{d_{1}, d_{2}=1}^{\infty}(-1)^{d_{1}+d_{2}} d_{1} d_{2} b_{0}\left(d_{1}\right) b_{0}\left(d_{2}\right) q^{d_{1}+d_{2}} \\
= & -\left.\frac{\mathfrak{t}^{5}}{100}(T-t)^{2} \sum_{j=0}^{1} \frac{5^{2}}{\mathfrak{t}^{10}}\left(S_{1-j, 0} S_{j, 0}\right)\right|_{T \mapsto t}-\frac{1}{2}(T-t)^{2}\left(I_{0}^{\prime}\right)^{2}=\frac{1}{2}(T-t)^{2}\left(\left(I_{1}^{\prime \prime}-t I_{0}^{\prime \prime}\right) I_{0}^{\prime}-\left(I_{0}^{\prime}\right)^{2}\right) .
\end{aligned}
$$

Similar calculations give

$$
\begin{aligned}
- & \frac{1}{2} \frac{\mathfrak{t}^{5}}{5} \sum_{d_{1}, d_{2}=1}^{\infty} \frac{(-1)^{d_{1}+d_{2}}}{\left(d_{1}+d_{2}\right)^{2}} \int_{Q} B_{d_{1}}(h) \cdot B_{d_{2}}(h) \frac{1}{\psi_{1}} \frac{1}{\psi_{2}} q^{d_{1}+d_{2}} \cdot\left(\left(d_{1}+d_{2}\right) \frac{(T-t) h}{h+\mathfrak{t}}\right) \\
= & -\frac{1}{2}(T-t) \partial_{t}^{-1}\left(\left(I_{1}^{\prime \prime}(t)-t I_{0}^{\prime \prime}(t)\right)^{2}+2\left(I_{2}^{\prime \prime \prime}(t)-t I_{1}^{\prime \prime \prime}(t)+\frac{t^{2}}{2} I_{0}^{\prime \prime \prime}(t)\right) I_{0}^{\prime}(t)\right) \\
& +(T-t) \partial_{t}^{-1}\left(I_{1}^{\prime \prime}(t)-t I_{0}^{\prime \prime}(t)\right) I_{0}^{\prime}(t), \\
& -\frac{1}{2} \frac{\mathfrak{t}^{5}}{5} \sum_{d_{1}, d_{2}=1}^{\infty} \frac{(-1)^{d_{1}+d_{2}}}{\left(d_{1}+d_{2}\right)^{2}} \int_{Q} B_{d_{1}}(h) \cdot B_{d_{2}}(h) \frac{1}{\psi_{1}} \frac{1}{\psi_{2}} q^{d_{1}+d_{2}} . \\
= & \partial_{t}^{-2}\left(\left(I_{1}^{\prime \prime}-t I_{0}^{\prime \prime}\right) I_{0}^{\prime}\right)-\partial_{t}^{-2}\left(\left(I_{1}^{\prime \prime}-t I_{0}^{\prime \prime}\right)^{2}+2\left(I_{2}^{\prime \prime \prime}-t I_{1}^{\prime \prime \prime}+\frac{t^{2}}{2} I_{0}^{\prime \prime \prime}\right) I_{0}^{\prime}\right) \\
& +\partial_{t}^{-2}\left(\left(I_{3}^{\prime \prime \prime \prime}-t I_{2}^{\prime \prime \prime \prime}+\frac{t^{2}}{2} I_{1}^{\prime \prime \prime \prime}-\frac{t^{3}}{6} I_{0}^{\prime \prime \prime \prime}\right) I_{0}^{\prime}+\left(I_{2}^{\prime \prime \prime}-t I_{1}^{\prime \prime \prime}+\frac{t^{2}}{2} I_{0}^{\prime \prime \prime}\right)\left(I_{1}^{\prime \prime}-t I_{0}^{\prime \prime}\right)\right),
\end{aligned}
$$

where $I_{k}, I_{k}^{\prime}, I_{k}^{\prime \prime}, \cdots$ denotes $I_{k}(t), I_{k}^{\prime}(t), I_{k}^{\prime \prime}(t), \cdots$. We then conclude

$$
\begin{aligned}
& \text { Cont }_{I I}^{0}+\text { Cont }_{I}^{0}=-\frac{1}{12}(T-t)^{3} \partial_{t}\left(I_{0}^{\prime} \cdot I_{0}^{\prime}\right)+\frac{1}{2}(T-t)^{2}\left(I_{1}^{\prime \prime}-t I_{0}^{\prime \prime}\right) I_{0}^{\prime} \\
& -\frac{1}{2}(T-t) \partial_{t}^{-1}\left(\left(I_{1}^{\prime \prime}-t I_{0}^{\prime \prime}\right)^{2}+2\left(I_{2}^{\prime \prime \prime}-t I_{1}^{\prime \prime \prime}+\frac{t^{2}}{2} I_{0}^{\prime \prime \prime}\right) I_{0}^{\prime}\right)+(T-t) \partial_{t}^{-1}\left(\left(I_{1}^{\prime \prime}-t I_{0}^{\prime \prime}\right) I_{0}^{\prime}\right) \\
& +\partial_{t}^{-2}\left(\left(I_{1}^{\prime \prime}-t I_{0}^{\prime \prime}\right) I_{0}^{\prime}\right)-\partial_{t}^{-2}\left(\left(I_{1}^{\prime \prime}-t I_{0}^{\prime \prime}\right)^{2}+2\left(I_{2}^{\prime \prime \prime}-t I_{1}^{\prime \prime \prime}+\frac{t^{2}}{2} I_{0}^{\prime \prime \prime}\right) I_{0}^{\prime}\right) \\
& +\partial_{t}^{-2}\left(\left(I_{3}^{\prime \prime \prime \prime}-t I_{2}^{\prime \prime \prime \prime}+\frac{t^{2}}{2} I_{1}^{\prime \prime \prime \prime}-\frac{t^{3}}{6} I_{0}^{\prime \prime \prime \prime}\right) I_{0}^{\prime}+\left(I_{2}^{\prime \prime \prime}-t I_{1}^{\prime \prime \prime}+\frac{t^{2}}{2} I_{0}^{\prime \prime \prime}\right)\left(I_{1}^{\prime \prime}-t I_{0}^{\prime \prime}\right)\right)
\end{aligned}
$$

One can combine a few terms to write in terms of total derivatives,

$$
\begin{aligned}
\operatorname{Cont}^{0}:=\operatorname{Cont}_{I I}^{0}+\operatorname{Cont}_{I}^{0} & =-\frac{1}{12}(T-t)^{3} \partial_{t}\left(I_{0}^{\prime} \cdot I_{0}^{\prime}\right)+\frac{1}{2}(T-t)^{2}\left(I_{1}^{\prime \prime}-t I_{0}^{\prime \prime}\right) I_{0}^{\prime} \\
& -\frac{1}{2}(T-t) \partial_{t}^{-1}\left(\left(I_{1}^{\prime \prime}-t I_{0}^{\prime \prime}\right)^{2}+2\left(I_{2}^{\prime \prime}-t I_{1}^{\prime \prime}+\frac{t^{2}}{2} I_{0}^{\prime \prime}\right)^{\prime} I_{0}^{\prime}\right) \\
& +\partial_{t}^{-2}\left(\left(I_{3}^{\prime \prime \prime}-t I_{2}^{\prime \prime \prime}+\frac{t^{2}}{2} I_{1}^{\prime \prime \prime}-\frac{t^{3}}{6} I_{0}^{\prime \prime \prime}\right)^{\prime} I_{0}^{\prime}+\left(I_{2}^{\prime \prime}-t I_{1}^{\prime \prime}+\frac{t^{2}}{2} I_{0}^{\prime \prime}\right)^{\prime}\left(I_{1}^{\prime}-t I_{0}^{\prime}\right)^{\prime}\right)
\end{aligned}
$$

After simplifications one has

$$
\begin{aligned}
\text { Cont }^{0}= & \left(\frac{T^{2}}{2} I_{1}^{\prime \prime}-\frac{T^{3}}{6} I_{0}^{\prime \prime}\right) I_{0}^{\prime}(t)-\frac{1}{2} T \partial_{t}^{-1}\left(I_{1}^{\prime \prime}\right)^{2}-(T-t) \partial_{t}^{-1}\left(I_{2}^{\prime \prime \prime} I_{0}^{\prime}\right) \\
& +\partial_{t}^{-2}\left(\left(I_{3}^{\prime \prime \prime}-t I_{2}^{\prime \prime \prime}\right)^{\prime} I_{0}^{\prime}+I_{2}^{\prime \prime \prime}\left(I_{1}^{\prime}-t I_{0}^{\prime}\right)^{\prime}\right)
\end{aligned}
$$

Combining previous subsection and using Proposition C.4 the type $B$ contribution is

$$
\begin{align*}
& \text { Cont }^{+}+\text {Cont }^{0}  \tag{30}\\
= & -\frac{1}{2} \partial_{T} J_{2} \int_{-\infty}^{t}\left(I_{1}^{(2)}-T I_{0}^{(2)}\right)^{2} d y-J_{2}\left(\int_{-\infty}^{t}\left(I_{1}^{(2)}-T I_{0}^{(2)}\right) I_{0}^{(2)}-\left(I_{1}^{(3)}-T I_{0}^{(3)}\right) I_{0}^{(1)} d y\right) \\
& -\partial_{T}^{-1}\left(J_{2}\right) \int_{-\infty}^{t}\left(I_{0}^{(2)}\right)^{2}-2 I_{0}^{(3)} I_{0}^{(1)} d y+\partial_{t}^{-2}\left(\left(I_{3}^{\prime \prime \prime}-t I_{2}^{\prime \prime \prime}\right)^{\prime} I_{0}^{\prime}+I_{2}^{\prime \prime \prime}\left(I_{1}^{\prime}-t I_{0}^{\prime}\right)^{\prime}\right)-(T-t) \partial_{t}^{-1}\left(I_{2}^{\prime \prime \prime} I_{0}^{\prime}\right) \\
= & \frac{1}{2} \frac{T^{\prime \prime}}{T^{\prime}}+2 \frac{I_{0}^{\prime}}{I_{0}}+\frac{\ln \left(1-5^{5} e^{t}\right)}{5} .
\end{align*}
$$

## 7. Graph Type C

A type $C$ graph $\Gamma$ is of the following form:


Separating all black nodes, $\Gamma$ is decomposed into $v$ and rational tails $\left\{\Gamma_{i}\right\}_{i=1}^{k}$. Let $d_{i}=\operatorname{deg} \Gamma_{i}$. Then $d_{1}+d_{2}+\cdots+d_{k}=n$. Thus the contribution of $\Gamma$ to (11) is

$$
\frac{\mathfrak{t} \prod\left|A u t \Gamma_{i}\right|}{|A u t \Gamma|} \sum_{a+\sum a_{i}=n-1} \int_{\left[\mathcal{W}_{v}\right]^{\mathrm{vir}}} \frac{c_{a}\left(R \pi_{\mathcal{W}_{v} *}\left(\mathcal{L}_{\mathcal{W}_{v}} \mathcal{L}_{\mathfrak{t}}\right)\right)}{e\left(N_{\mathcal{W}_{v}}^{\mathrm{vir}}\right)} \prod_{i=1}^{k} \int_{\left[\mathcal{W}_{\left(\Gamma_{i}\right)}\right]^{\mathrm{vir}}} \frac{-\mathfrak{t}^{4} c_{a_{i}}\left(\pi_{\mathcal{W}_{\Gamma_{i}} *}\left(\mathcal{L}_{\mathcal{W}_{\Gamma_{i}}} \mathcal{L}_{\mathfrak{t}}\left(-D_{i}\right)\right)\right)}{5\left(-\psi_{\Gamma_{i}}-\psi_{i}\right) e\left(N_{\mathcal{W}_{\left(\Gamma_{i}\right)} \mathrm{vi}}\right)}
$$

$$
\begin{aligned}
= & \frac{\prod\left|\operatorname{Aut} \Gamma_{i}\right|}{|\operatorname{Aut} \Gamma|} \frac{\mathfrak{t}^{4 k+1}}{5^{k}} \sum_{a=0}^{n-1} \int_{\left[\mathcal{W}_{v}\right]^{\text {vir }}} \frac{c_{a}\left(\pi_{\mathcal{W}_{v} *}\left(\mathcal{L}_{\mathcal{W}_{v}} \mathcal{L}_{\mathrm{t}}\right)\right)}{e\left(N_{\mathcal{W}_{v}}^{\text {vir }}\right)} \\
& \cdot \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{a+1}} \prod \int_{\left[\mathcal{W}_{\left(\Gamma_{i}\right)}\right]^{\text {vir }}} \frac{e\left(\pi_{\mathcal{W}_{\Gamma_{i}} *}\left(\mathcal{L}_{\mathcal{W}_{\Gamma_{i}}} \mathcal{L}_{\mathrm{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{i}\right)\right)\right)}{\left(\psi_{\Gamma_{i}}+\psi_{i}\right) e\left(N_{\mathcal{W}_{\left(\Gamma_{i}\right)}^{\text {vir }}}\right)} .
\end{aligned}
$$

Since $\mathcal{W}_{v}=\bar{M}_{1, k}$ and, over its universal curve, the section $\nu_{1}$ trivializes $\mathcal{L}_{\mathcal{W}_{v}} \mathcal{L}_{\mathrm{t}}, R \pi_{\mathcal{W}_{v} *}\left(\mathcal{L}_{\mathcal{W}_{v}} \mathcal{L}_{\mathrm{t}}\right)=$ $R \pi_{\mathcal{W}_{v} *} \mathcal{O}_{\mathcal{W}_{v}}$ and $c\left(R \pi_{\mathcal{W}_{v} *}\left(\mathcal{L}_{\mathcal{W}_{v}} \mathcal{L}_{\mathfrak{t}}\right)\right)=\frac{1}{1-\lambda}=1+\lambda$. Also by (4), over $\left[\mathcal{W}_{v}\right]^{\text {vir }}$ one has

$$
\frac{1}{e\left(N_{\mathcal{W}_{(v)}}^{\mathrm{vir}}\right)}=\left(\frac{\lambda+\mathfrak{t}}{\mathfrak{t}}\right)^{5} \cdot\left(\frac{5 \mathfrak{t}}{\lambda+5 \mathfrak{t}}\right)=\left(1+\frac{5 \lambda}{\mathfrak{t}}\right)\left(1-\frac{\lambda}{5 \mathfrak{t}}\right)=1+\frac{24}{5} \frac{\lambda}{\mathfrak{t}}
$$

Let $\Delta_{C}$ be the subset of type $C$ graphs with degree $(d, 0)$ for $d=1,2, \cdots$.
Then the total contribution of type $C$ graphs is, using (12),

$$
\begin{aligned}
& \operatorname{Contri}(C)=\sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{\mathfrak{t}^{4}}{5}\right)^{k} \sum_{a=0}^{1} \int_{\bar{M}_{1, k}} c_{a}\left(R \pi_{\mathcal{W}_{v} *}\left(\mathcal{L}_{\mathcal{W}_{v}} \mathcal{L}_{\mathfrak{t}}\right)\right)\left(\mathfrak{t}+\frac{24}{5} \lambda\right) \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{a+1}} \prod_{i=1}^{k} \hat{\mathcal{P}}\left(\psi_{i}\right) \\
&= \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{\mathfrak{t}^{4}}{5}\right)^{k} \int_{\bar{M}_{1, k}}\left[\frac{24}{5} \lambda \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{1}} \prod_{i=1}^{k} \hat{\mathcal{P}}\left(\psi_{i}\right)+\lambda \mathfrak{t} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{2}} \prod_{i=1}^{k} \hat{\mathcal{P}}\left(\psi_{i}\right)+\mathfrak{t C o e}\left(\mathfrak{t}^{\prime}\right)^{1}\right. \\
&\left.\prod_{i=1}^{k} \hat{\mathcal{P}}\left(\psi_{i}\right)\right] .
\end{aligned}
$$

Since $\mathcal{Z}_{6}^{*}:=\frac{\mathfrak{t}^{4}}{5} \tilde{z}_{1,5}(\hbar, Q)$ is regularizable(c.f. Lemma A.2), by (12) we have

$$
\begin{aligned}
& \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{\mathfrak{t}^{4}}{5}\right)^{k} \int_{\bar{M}_{1, k}} \lambda \prod_{i=1}^{k} \hat{\mathcal{P}}\left(\psi_{i}\right)=\frac{1}{24} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!}\left(\frac{\mathfrak{t}^{4}}{5}\right)^{k} \sum_{\substack{\sum_{\begin{subarray}{c}{k=1 \\
a_{\ell}=k-1 \\
a_{\ell} \geq 0} }}^{k}}\end{subarray}}\binom{k-1}{a_{1}, \cdots, a_{k}} \prod_{\ell=1}^{k} \tilde{\mathcal{P}}_{1+a_{\ell}} \\
= & \frac{1}{24} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{\sum_{\begin{subarray}{c}{ \\
\ell=1 \\
a_{\ell} \geq 0} }}^{k} a_{\ell=k-1}}\end{subarray}} \prod_{\ell=1}^{k} \frac{(-1)^{a_{\ell}}}{a_{\ell}!} \operatorname{Res}_{\hbar=0}\left\{\hbar^{-a_{\ell}} \mathcal{Z}_{6}^{*}(\hbar, \mathcal{Q})\right\}=\frac{1}{24} \operatorname{Res}_{\hbar=0} \ln \left(1+Z_{6}^{*}(\hbar, \mathcal{Q})\right) .
\end{aligned}
$$

Using (36) in Appendix A the last term in (31) becomes

$$
\left.\sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{\mathfrak{t}^{4}}{5}\right)^{k} \int_{\bar{M}_{1, k}} \mathfrak{t C o e}_{\mathfrak{t}^{\prime}} \hat{\mathcal{P}}\left(\psi_{1}\right) \hat{\mathcal{P}}\left(\psi_{2}\right) \cdots \hat{\mathcal{P}}\left(\psi_{k}\right)=-\frac{\mathfrak{t}}{24} \operatorname{Coe}_{\mathfrak{t}^{\prime}} \operatorname{Res}_{\hbar=0}\left(\ln \left(1+\bar{z}_{6}^{*}\right)\right) / \hbar\right)
$$

Thus, the type $C$ contribution is
$\left.\operatorname{Contri}(C)=\left(\frac{24}{5} \operatorname{Coe}_{\mathfrak{t}^{\prime}}+\mathfrak{t C o e}_{\left(\mathfrak{t}^{\prime}\right)^{2}}\right) \frac{1}{24} \operatorname{Res}_{\hbar=0} \ln \left(1+z_{6}^{*}(\hbar)\right)-\frac{\mathfrak{t}}{24} \operatorname{Coe}_{\mathfrak{t}^{\prime}} \operatorname{Res}_{\hbar=0}\left(\hbar^{-1} \ln \left(1+Z_{6}^{*}\right)\right)\right)$.
Using (37), the type $C$ contribution is

$$
\begin{aligned}
& \left.\left(\frac{24}{5} \operatorname{Coe}_{\mathfrak{t}^{\prime}}+\mathfrak{t C o e}_{\left(\mathfrak{t}^{\prime}\right)^{2}}\right) \frac{\eta}{24}-\frac{\mathfrak{t}}{24} \operatorname{Coe}_{\mathfrak{t}^{\prime}} \operatorname{Res}_{\hbar=0}\left(\hbar^{-1} \log \left(1+z_{6}^{*}\right)\right)\right) \\
= & -\frac{1}{5} \frac{g_{1}}{I_{0}}-\frac{1}{5} \ln \left(1-5^{5} q\right)-\frac{1}{12} \cdot \frac{5^{5} q}{1-5^{5} q}
\end{aligned}
$$

## 8. Graph Type D


$\begin{array}{rr}\Gamma & \Gamma_{\ell} \\ d_{1} & d_{\ell}\end{array}$

Let $d_{i}=\operatorname{deg} \Gamma_{i}$ and $d=d_{1}+\ldots+d_{\ell}$. Each $\Gamma_{i}$ is in $\Xi_{d_{i}}^{1}$. The $\tilde{\psi}_{i}$ denotes the $\psi$-class of the marking on $\mathfrak{C}_{\Gamma_{i}}$ and $\psi_{i}$ is the $\psi$-class of the $i$-th marking on the curve $\mathcal{C}_{v}$. The contribution from type D graphs is
(31) $\sum_{\ell=1} \sum_{d_{i}>0,1 \leq i \leq \ell} \sum_{\left(\Gamma_{i}\right) \in \hat{\Xi}_{d_{i}}^{1}} \frac{(-Q)^{d_{1}+\ldots+d_{\ell}}}{\left|\operatorname{Aut}\left(\left\{\Gamma_{1}, \ldots, \Gamma_{\ell}\right\}\right)\right|} \int_{\bar{M}_{1, \ell}^{1 / 5,5 p} \times\left[\mathcal{W}_{\left(\Gamma_{1}\right)}\right]^{v i r} \times \cdots \times\left[\mathcal{W}_{\left(\Gamma_{\ell}\right)}\right]^{v i r}} \mathfrak{t} \cdot \prod_{i=1}^{\ell} \frac{1}{-\mathfrak{t}-\psi_{i}}$

$$
\frac{1}{e_{T}\left(R \pi_{v}\left(\mathcal{L}_{v}^{\vee} \mathcal{L}_{\mathfrak{t}}\right)\right)} \cdot \frac{\left(-\mathfrak{t}^{5}\right)^{\ell}}{(5 \mathfrak{t})^{\ell}} \cdot \prod_{i=1}^{\ell}\left(\frac{1}{5 \mathfrak{t}-\tilde{\psi}_{i}} \cdot \frac{1}{e_{T}\left(N_{\mathcal{W}_{\Gamma_{1}} / \mathcal{W}}^{v i r}\right)}\right) \cdot\left(\frac{1}{1-\lambda} c\left(R \pi_{\mathcal{W}_{\Gamma} *}\left(\mathcal{L}_{\mathcal{W}_{\Gamma}} \mathcal{N}_{\mathcal{W}_{\Gamma}} \mathcal{L}_{\mathfrak{t}}\right)\right)\right)_{d-1}
$$

where $\bar{M}_{1, \ell}^{1 / 5,5 p}$ is the moduli of 5 -spin curves of genus 1 with five $P$-fields and $(\cdots)_{d-1}$ is the degree $d-1$ component.

Using the variable $\mathfrak{t}^{\prime}$, we can rewrite $\left(\frac{1}{1-\lambda} c\left(R \pi_{\mathcal{W}_{\Gamma} *}\left(\mathcal{L}_{\mathcal{W}_{\Gamma}} \mathcal{N}_{\mathcal{W}_{\Gamma}} \mathcal{L}_{\mathfrak{t}}\right)\right)\right)_{d-1}$ as

$$
\left(\operatorname{Coe}_{\mathfrak{t}^{\prime}}+\lambda \operatorname{Coe}_{\mathfrak{t}^{\prime 2}}\right) \prod_{i=1}^{\ell} e\left(R \pi_{\mathcal{W}_{\Gamma_{i}} *}\left(\mathcal{L}_{\mathcal{W}_{\Gamma_{i}}} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\right)\right)
$$

Let $h_{\ell}: \bar{M}_{1,1^{\ell}}^{1 / 5,5 p} \rightarrow \bar{M}_{1,1^{\ell-1}}^{1 / 5,5 p}$ be the contracting map forgetting the last marking point. Let $D_{i}$ be the boundary divisor of $\bar{M}_{1,1^{\ell}}^{1 / 5,5 p}$ which is the graph of the section of $h_{\ell}$ induced by $\ell$-th marking. For $i \leq \ell-1$, we have the identity $\psi_{i}^{k}=h_{\ell}^{*} \psi_{i}^{k}+\left(D_{i} / 5\right) h_{\ell}^{*} \psi_{i}^{k-1}$. This implies

$$
\frac{1}{\mathfrak{t}+\psi_{i}}=\frac{1}{\mathfrak{t}}\left(1-\frac{h_{\ell}^{*} \psi_{i}+D_{i} / 5}{\mathfrak{t}}+\frac{h_{\ell}^{*}\left(\psi_{i}\right)^{2}+h_{\ell}^{*} \psi_{i} \cdot D_{i} / 5}{\mathfrak{t}^{2}}+\ldots\right)=\frac{1-D_{i} / 5 \mathfrak{t}}{\mathfrak{t}+h_{\ell}^{*} \psi_{i}}
$$

We denote $e_{T}\left(R \pi_{v *} e v^{*} \mathcal{L}^{\vee} \mathcal{L}_{\mathfrak{t}}\right)^{-1}=\mathfrak{e}_{\ell}$. It is direct to check $\mathfrak{e}_{\ell}=\left(-\mathfrak{t}-\psi_{\ell}\right) h_{\ell}^{*} \mathfrak{e}_{\ell-1}$. Therefore

$$
\begin{aligned}
& \int_{\bar{M}_{1,1 \ell}^{1 / 5,5 p}} \frac{1}{-\mathfrak{t}-\psi_{1}} \cdots \frac{1}{-\mathfrak{t}-\psi_{\ell}} \mathfrak{e}_{\ell}=\int_{\bar{M}_{1,1}^{1 / \ell}} \frac{1-D_{1} / 5 \mathfrak{t}}{-\mathfrak{t}-h_{\ell}^{*} \psi_{1}} \cdots \frac{1-D_{\ell-1} / 5 \mathfrak{t}}{-\mathfrak{t}-h_{\ell}^{*} \psi_{\ell-1}} h_{\ell}^{*} \ell_{\ell-1} \\
= & -\frac{\ell-1}{5 \mathfrak{t}} \int_{\bar{M}_{1,1}^{1 / \ell-1,5 p}} \frac{1}{-\mathfrak{t}-\psi_{1}} \cdots \frac{1}{-\mathfrak{t}-\psi_{\ell-1}} \mathfrak{e}_{\ell-1}=(-1)^{\ell-1} \frac{(\ell-1)!}{(5 \mathfrak{t})^{\ell-1}} \int_{\bar{M}_{1,1}^{1 / 5,5 p}} \frac{1}{-\mathfrak{t}-\psi_{1}} \mathfrak{e}_{1} \\
= & (-1)^{\ell} \frac{(\ell-1)!}{(5 \mathfrak{t})^{\ell}} \cdot \frac{128}{3} .
\end{aligned}
$$

Here we used the fact that $D_{i} \cdot D_{j}=0$ if $i \neq j$ and a reduction Lemma about FJRW invariants.
Using $\lambda=5 \psi_{1}$ and a computation of a FJRW invariant, we obtain similarly

$$
\int_{\bar{M}_{1,1 \ell}^{1 / 5,5 p}} \frac{1}{-\mathfrak{t}-\psi_{1}} \cdots \frac{1}{-\mathfrak{t}-\psi_{\ell}} \mathfrak{e}_{\ell}=(-1)^{\ell-1} \frac{(\ell-1)!}{(5 \mathfrak{t})^{\ell-1}} \int_{\bar{M}_{1,1}^{1 / 5,5 p}} \frac{-1}{\mathfrak{t}}\left(1-\frac{\psi_{1}}{\mathfrak{t}}\right) 5 \psi_{1} \mathfrak{e}_{1}=(-1)^{\ell} \frac{(\ell-1)!}{(5 \mathfrak{t})^{\ell-1}} \cdot \frac{25}{3} .
$$

Using (37), the type D contribution (31) is

$$
\begin{aligned}
& \quad \sum_{\ell=1} \sum_{d_{i}>0,1 \leq i \leq \ell} \sum_{\left(\Gamma_{i}\right) \in \hat{\Xi}_{d_{i}}^{1}}(-Q)^{d_{1}+\ldots+d_{\ell}} \frac{\mathfrak{t}}{\left|\operatorname{Aut}\left(\left\{\Gamma_{1}, \ldots, \Gamma_{\ell}\right\}\right)\right|} \cdot \frac{\left(-\mathfrak{t}^{5}\right)^{\ell}}{(5 \mathfrak{t})^{\ell}}\left((-1)^{\ell} \frac{(\ell-1)!}{(5 \mathfrak{t})^{\ell}} \cdot \frac{128}{3} \operatorname{Coe}_{\mathfrak{t}^{\prime}}\right. \\
& \left.\quad+(-1)^{\ell} \frac{(\ell-1)!}{(5 \mathfrak{t})^{\ell-1}} \cdot \frac{25}{3} \operatorname{Coe}_{\mathfrak{t}^{\prime 2}}\right) \prod_{i=1}^{\ell}\left(\int_{\left[\mathcal{W}_{\left(\Gamma_{i}\right)}\right]^{v i r}} \frac{1}{5 \mathfrak{t}-\tilde{\psi}_{i}} \frac{e\left(R \pi_{\Gamma_{i} *} \mathcal{L}_{\Gamma_{i}}\left(-D_{\Gamma_{i}}\right) \otimes \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\right)}{e_{T}\left(N_{\mathcal{W}_{\Gamma_{i}}^{v i r}}^{v i}\right)}\right) \\
& =\left.\mathfrak{t} \cdot \frac{128}{3} \sum_{\ell=1}^{\infty} \frac{1}{\ell}\left(\sum_{d=1}^{\infty} \sum_{\Gamma \in \Sigma_{d}^{1}}(-Q)^{d} \cdot \frac{\mathfrak{t}^{4}}{5} \cdot \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{v i r}} \frac{1}{5 \mathfrak{t} \cdot(5 \mathfrak{t}-\tilde{\psi})} \frac{e\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma}\right)\right)\right)}{e_{T}\left(N_{\mathcal{W}_{\Gamma} / \mathcal{W}}^{v i r}\right)}\right)^{\ell}\right|_{\mathfrak{t}^{\prime}} \\
& \quad+\left.\frac{125}{3} \mathfrak{t}^{2} \cdot \sum_{\ell=1}^{\infty} \frac{1}{\ell}\left(\sum_{d=1}^{\infty} \sum_{\Gamma \in \Sigma_{d}^{1}}(-Q)^{d} \frac{\mathfrak{t}^{4}}{5} \cdot \int_{\left[\mathcal{W}_{(\Gamma)}\right]^{v i r}} \frac{1}{5 \mathfrak{t} \cdot(5 \mathfrak{t}-\tilde{\psi})} \frac{e\left(R \pi_{\Gamma *}\left(\mathcal{L}_{\Gamma} \mathcal{L}_{\mathfrak{t}} \mathcal{L}_{\mathfrak{t}^{\prime}}\left(-D_{\Gamma}\right)\right)\right)}{e_{T}\left(N_{\mathcal{W}_{\Gamma} / \mathcal{W}}^{v i r}\right)}\right)^{\ell}\right|_{\mathfrak{t}^{\prime 2}}
\end{aligned}
$$

$$
\begin{aligned}
& =-\mathfrak{t} \cdot \frac{128}{3} \operatorname{Coe}_{\mathfrak{t}^{\prime}} \ln \left(1+\frac{\mathfrak{t}^{4}}{5} \tilde{z}_{1,5}^{*}(5 \mathfrak{t})\right)-\frac{125}{3} \mathfrak{t}^{2} \operatorname{Coe}_{\mathfrak{t}^{\prime 2}} \ln \left(1+\frac{\mathfrak{t}^{4}}{5} \tilde{z}_{1,5}^{*}(5 \mathfrak{t})\right) \\
& =-\frac{128}{3} \mathfrak{t}\left(-\frac{g_{1}}{5 \mathfrak{t} I_{0}}\right)=\frac{128}{3} \cdot \frac{g_{1}}{5 I_{0}} .
\end{aligned}
$$

$$
\text { 9. } \mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}
$$

Let's recall the results from previous sections about the contributions of graph type A, B, C, and D.

$$
\begin{array}{ll}
\text { Graphs A: } & \mathcal{F}_{1}^{\prime}-\frac{25}{12} \cdot\left(T^{\prime}-1\right)+\frac{40}{24} \cdot 5 \frac{I_{0}^{\prime}}{I_{0}}-\frac{25}{3} \cdot \frac{g_{1}}{I_{0}} \\
\text { Graphs B: } & 2 \frac{I_{0}^{\prime}}{I_{0}}+\frac{\ln \left(1-5^{5} q\right)}{5}+\frac{1}{2} \frac{T^{\prime \prime}(t)}{T^{\prime}(t)} \\
\text { Graphs C: } & -\frac{1}{5} \frac{g_{1}}{I_{0}}-\frac{1}{5} \ln \left(1-5^{5} e^{t}\right)-\frac{1}{12} \cdot \frac{5^{5} e^{t}}{1-5^{5} e^{t}} \\
\text { Graphs D: } & \frac{128}{3} \frac{g_{1}}{5 I_{0}}
\end{array}
$$

Here $\frac{d}{d t}$ is replaced by ${ }^{\prime}$.
Taking the summation of the four contributions above, we get

$$
\mathcal{F}_{1}^{\prime}-\frac{25}{12} \cdot\left(T^{\prime}-1\right)+\frac{31}{3} \frac{I_{0}^{\prime}}{I_{0}}+\frac{1}{2} \frac{T^{\prime \prime}(t)}{T^{\prime}(t)}-\frac{1}{12} \cdot \frac{5^{5} e^{t}}{1-5^{5} e^{t}}=0
$$

This implies the formula for $\mathcal{F}_{1}=\sum_{d=1}^{\infty} N_{1, d} e^{T d}$ as follows:

$$
\begin{equation*}
\mathcal{F}_{1}=\frac{25}{12} \cdot(T-t)-\ln \left(I_{0}^{\frac{31}{3}} \cdot T^{\prime}(t)^{\frac{1}{2}} \cdot\left(1-5^{5} e^{t}\right)^{\frac{1}{12}}\right) \tag{32}
\end{equation*}
$$

since both side of (32) are power series in $q$ with no $q^{0}$ term. It is exactly what A. Zinger obtained in [Zi2].

## A. Appendix: Regularity and some identities

A.1. Regularity. We prove all the results involving regularities of series here.

Recall the definition of regularizable power series introduced by Zinger in [Zi2]. Let $\mathbb{Q}_{\mathfrak{t}}=\mathbb{Q}\left[\mathfrak{t}, \mathfrak{t}^{-1}, \mathfrak{t}^{\prime}\right]$.
Definition A.1. A power series $\mathcal{Z}^{*}=\mathcal{Z}^{*}(\hbar, q) \in \mathbb{Q}_{\mathfrak{t}}(\hbar)[[q]]$ is regularizable at $\hbar=0$ if there exist power series

$$
\eta=\eta(q) \in \mathbb{Q}_{\mathfrak{t}}[[q]] \quad \text { and } \quad \overline{\mathcal{Z}}^{*}=\overline{\mathcal{Z}}^{*}(\hbar, q) \in \mathbb{Q}_{\mathfrak{t}}(\hbar)[[q]]
$$

with no degree-zero term in $q$ such that $\overline{\mathcal{Z}}^{*}$ is regular at $\hbar=0$ and

$$
1+\mathcal{Z}^{*}(\hbar, q)=e^{\eta(q) / \hbar}\left(1+\overline{\mathcal{Z}}^{*}(\hbar, q)\right)
$$

If $\mathcal{Z}^{*}$ is regularizable at $\hbar=0$, then $\mathcal{Z}^{*}$ has no degree-zero term in $u$ and the regularizing pair $\left(\eta, \overline{\mathcal{Z}}^{*}\right)$ is unique. It is determined by $\eta(u)=\operatorname{Res}_{\hbar=0} \ln \left(1+\mathcal{Z}^{*}(\hbar, u)\right)$.
Lemma A.1. $\frac{\tilde{z}_{0,1}(\mathfrak{t},-\mathfrak{t})}{5} \in \mathbb{Q}_{\mathfrak{t}}(\hbar)[[q]]$ is regularizable.
Proof. From (20), we have

$$
1+\frac{\tilde{z}_{0,1}(\mathfrak{t},-\mathfrak{t})}{5}=e^{\frac{g_{1} \mathfrak{t}}{I_{0} \hbar}} \cdot \frac{1}{I_{0}} \cdot \sum_{d=0}^{\infty} q^{d} \frac{\hbar^{d}(5 d)!}{(d!)^{4} \prod_{m=1}^{d}(\mathfrak{t}+m \hbar)}:=e^{\frac{g_{1} \mathfrak{t}}{I_{0} \hbar}}\left(1+\bar{z}_{0,1}\right)
$$

Then $\frac{g_{1} t}{I_{0}}$ and $\bar{z}_{0,1}$ has no degree-zero terms in $q$, and $\bar{z}_{0,1}$ is regular at $\hbar=0$.

Lemma A.2. $\mathcal{Z}_{6}^{*}:=\frac{\mathfrak{t}^{4}}{5} \tilde{\mathcal{Z}}_{1,5}(\hbar, \mathcal{Q}) \in \mathbb{Q}\left(\mathfrak{t}, \mathfrak{t}^{\prime}\right)(\hbar)[[\mathbb{Q}]]$ is regularizable at $\hbar=0$
Proof. The argument in [Zi2, lemm 2.3] applies here.
Thus we can express

$$
\begin{equation*}
1+Z_{6}^{*}(\hbar)=e^{\eta / \hbar}\left(1+\bar{z}_{6}^{*}(\hbar)\right), \quad \eta=\operatorname{Res}_{\hbar=0} \ln \left(1+Z_{6}^{*}(\hbar)\right) \tag{33}
\end{equation*}
$$

where $\bar{Z}_{6}^{*} \in \mathbb{Q}\left(\mathfrak{t}, \mathrm{t}^{\prime}\right)(\hbar)[[\mathbb{Q}]] \mathbb{Q}$ is regular at $\hbar=0$.
Mimicking [Zi2, Lemm 2.2(i)(ii)], one can show
Lemma A.3. If $\left(\eta, \bar{Z}^{*}\right)$ is the regularizing pair for $Z^{*}$ at $\hbar=0$, then for every $a \geq 0$,

$$
\begin{equation*}
\sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{m \\
\sum_{\begin{subarray}{c}{ \\
a_{\ell}=m-1-a \\
a_{\ell} \geq 0} }}}\end{subarray}}\left(\prod_{\ell=1}^{m} \frac{(-1)^{a_{\ell}}}{a_{\ell}!} \operatorname{Res}_{\hbar=0}\left\{\hbar^{-a_{\ell}} Z^{*}(\hbar, q)\right\}\right)=\frac{\eta^{a+1}}{a+1} \tag{34}
\end{equation*}
$$

A.2. Some identities $I$. Using methods in Zinger's [Zi2, proof of (2.4)], we want to calculate

$$
\begin{aligned}
\mathbf{X} & :=\sum_{m=1}^{\infty} \frac{1}{m!}\left(\frac{\mathfrak{t}^{4}}{5}\right)^{m} \int_{\bar{M}_{1, m}} \hat{\mathcal{P}}\left(\psi_{1}\right) \hat{\mathcal{P}}\left(\psi_{2}\right) \cdots \hat{\mathcal{P}}\left(\psi_{m}\right) \\
& =\sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\vec{a}=\left(a_{1}, \cdots, a_{m}\right)} \vec{a}!\int_{\bar{M}_{1, m}} \psi_{1}^{a_{1}} \cdots \psi_{m}^{a_{m}} \prod_{\ell=1}^{m} \frac{(-1)^{a_{\ell}}}{a_{\ell}!} \operatorname{Res}_{\hbar=0}\left\{\hbar^{-a_{\ell}} Z_{6}^{*}(\hbar, q)\right\}
\end{aligned}
$$

Writing $\overline{\mathcal{Z}}_{6}^{*}=\sum_{m=0}^{\infty} C_{m} \hbar^{m}$, one has

$$
\operatorname{Res}_{\hbar=0} \hbar^{a} \mathcal{Z}_{6}^{*}=\sum_{\substack{p-m=1+a \\ p, m \geq 0}} \frac{\eta^{p}}{p!} C_{m}+ \begin{cases}\frac{\eta^{a+1}}{(a+1)!} & a \geq 0 \\ 0 & \text { otherwise }\end{cases}
$$

For fixed $k \geq 0$, let $\alpha=\left(\alpha_{1}, \cdots, \alpha_{k}\right) \in \mathbb{N}^{k}$, where $N=\alpha_{1}+\cdots+\alpha_{k}:=|\alpha|$. Pick a subset of $\mathbb{Z}_{\geq 0}$

$$
\begin{equation*}
m=\left\{m_{1}=\cdots=m_{\alpha_{1}}, m_{\alpha_{1}+1}=\cdots=m_{\alpha_{1}+\alpha_{2}}, \cdots, m_{\alpha_{1}+\cdots+\alpha_{k-1}+1}=\cdots=m_{N}\right\} \tag{35}
\end{equation*}
$$

be a sequence of N distinct numbers. We look at the coefficient of $C_{m}^{\alpha}:=C_{m_{1}}^{\alpha_{1}} C_{m_{\alpha_{1}+\alpha_{2}}}^{\alpha_{2}} \cdots C_{m_{N}}^{\alpha_{N}}$ in $\mathbf{X}$. For each $\beta:\{1, \cdots, N\} \rightarrow \mathbb{Z}_{\geq 0}$ and a choice of $\alpha$ above, the term $C_{m}^{\alpha}$ appears in the $n=|\beta|:=$ $\beta_{1}+\cdots+\beta_{N}$ summand in $\mathbf{X}$ with the coefficient

$$
\begin{aligned}
& \eta^{n-N} \prod_{\ell}\left(\beta_{\ell}!\right) \int_{\bar{M}_{1,|\beta|}}\left(\prod_{s=1}^{N} \psi_{s}^{\beta_{s}}\right) \prod_{\ell=1}^{N} \frac{(-1)^{\beta_{\ell}}}{\beta_{\ell}!} \frac{\eta^{m_{\ell}+1-\beta_{\ell}}}{\left(m_{\ell}+1-\beta_{\ell}\right)!} \\
= & \frac{\eta^{\alpha \cdot m}}{(m+1)!^{\alpha}} \prod_{\ell}\left(\beta_{\ell}!\right) \int_{\bar{M}_{1,|\beta|}}\left(\prod_{s=1}^{N} \psi_{s}^{\beta_{s}}\right) \prod_{\ell=1}^{N}(-1)^{\beta_{\ell}}\binom{m_{\ell}+1}{\beta_{\ell}},
\end{aligned}
$$

where $\alpha \cdot m:=\sum_{\ell} m_{\ell}$ and $(m+1)!^{\alpha}:=\prod_{\ell=1}^{N}\left(m_{\ell}+1\right)!$. The number of such choices of $\alpha$ is $\binom{|\beta|}{\alpha,|\beta|-\alpha}$.
Set $b=|\beta|$ and sum over it. We then have

$$
\begin{aligned}
\operatorname{Coe}_{C_{m}^{\alpha}} \mathbf{X} & =\frac{\eta^{\alpha \cdot m}}{(m+1)!^{\alpha}} \sum_{b=0}^{\infty} \frac{(-1)^{b}}{b!}\binom{b}{\alpha, b-N} \sum_{\substack{\beta_{1}+\cdots+\beta_{N}=b \\
\beta_{\ell} \geq 0}} \prod_{\ell}\left(\beta_{\ell}!\right) \int_{\bar{M}_{1, b}} \psi_{1}^{\beta_{1}} \cdots \psi_{N}^{\beta_{N}} \prod_{\ell=1}^{N}\binom{m_{\ell}+1}{\beta_{\ell}} \\
& =\frac{\eta^{\alpha \cdot m}}{(m+1)!^{\alpha}} \frac{N!}{\alpha!} \sum_{b=N}^{\infty} \frac{(-1)^{b}}{b!}\binom{b}{N} \Lambda_{b}\left(m_{1}+1, \cdots, m_{N}+1\right)
\end{aligned}
$$

where for $b \geq N, r_{1}, \cdots, r_{N} \geq 0$, we set

$$
\Lambda_{b}\left(r_{1}, \cdots, r_{N}\right):=\sum_{\substack{\beta_{1}+\cdots+\beta_{N}=b \\ r_{\ell} \geq \beta_{\ell} \geq 0}} \int_{\bar{M}_{1, b}} \psi_{1}^{\beta_{1}} \cdots \psi_{N}^{\beta_{N}} \prod_{\ell=1}^{N} \frac{r_{\ell}!}{\left(r_{\ell}-\beta_{\ell}\right)!}
$$

Lemma A.4. Set $\Lambda_{b}\left(r_{1}, \cdots, r_{N}\right):=0$ if some $r_{i}<0$. Then $\forall b \geq N$, the series $\Lambda_{b+1}\left(r_{1}, \cdots, r_{N}\right)$ is equal to

$$
r_{1} \Lambda_{b}\left(r_{1}-1, r_{2}, \cdots, r_{N}\right)+r_{2} \Lambda_{b}\left(r_{1}, r_{2}-1, r_{3}, \cdots, r_{N}\right)+\cdots+r_{N} \Lambda_{b}\left(r_{1}, r_{2}, \cdots, r_{N-1}, r_{N}-1\right)
$$

The identity is a simple consequence of the string equation. Using the Lemma, we obtain

$$
\begin{aligned}
& \frac{N!}{\alpha!} \sum_{b=N}^{\infty} \frac{(-1)^{b}}{b!}\binom{b}{N} \Lambda_{b}\left(r_{1}, \cdots, r_{N}\right)=\frac{1}{\alpha!} \sum_{b=N}^{\infty} \frac{(-1)^{b}}{(b-N)!} \Lambda_{b}\left(r_{1}, \cdots, r_{N}\right) \\
& =\frac{(-1)^{N}}{\alpha!} \sum_{b=0}^{\infty} \frac{(-1)^{b}}{b!} \Lambda_{b+N}\left(r_{1}, \cdots, r_{N}\right) \\
& =\frac{(-1)^{N}}{\alpha!} \sum_{0 \leq a_{i} \leq r_{i}} \frac{(-1)^{|r|-|a|}}{(|r|-|a|)!}\binom{|r|-|a|}{r_{1}-a_{1}, \cdots, r_{N}-a_{N}} \frac{r_{1}!}{a_{1}!} \cdots \frac{r_{N}!}{a_{N}!} \Lambda_{N}\left(a_{1}, \cdots, a_{N}\right) \\
& =\frac{(-1)^{N}}{\alpha!} \sum_{\substack{\beta_{1}+\cdots+\beta_{N}=N \\
r_{\ell} \geq \beta_{\ell} \geq 0}} \sum_{\beta_{i} \leq a_{i} \leq r_{i}}\left((-1)^{|r|-|a|} \frac{1}{\left(r_{1}-a_{1}\right)!\cdots\left(r_{N}-a_{N}\right)!}\right. \\
& \left.\cdot \frac{r_{1}!}{\left(a_{1}-\beta_{1}\right)!} \cdots \frac{r_{N}!}{\left(a_{N}-\beta_{N}\right)!} \cdot \int_{\bar{M}_{1, N}} \psi_{1}^{\beta_{1}} \cdots \psi_{N}^{\beta_{N}}\right) \\
& =\frac{(-1)^{N}}{\alpha!} \sum_{\substack{\beta_{1}+\cdots+\beta_{N}=N \\
r_{\ell} \geq \beta_{\ell} \geq 0}}(-1)^{|\beta|-|r|} \prod_{i=1}^{N} \frac{r_{i}!}{\left(r_{i}-\beta_{i}\right)!} \sum_{\beta_{i} \leq a_{i} \leq r_{i}}(-1)^{|a|-|\beta|} \prod_{i=1}^{N}\binom{r_{i}-\beta_{i}}{a_{i}-\beta_{i}} \int_{\bar{M}_{1, N}} \psi_{1}^{\beta_{1}} \cdots \psi_{N}^{\beta_{N}} \\
& = \begin{cases}0 & \text { if } r_{1}+r_{2}+\cdots+r_{N} \neq N, \\
\frac{(-1)^{N}}{\alpha!} r_{1}!\cdots r_{N}!\int_{\bar{M}_{1, N}} \psi_{1}^{r_{1}} \cdots \psi_{N}^{r_{N}} & \text { if } r_{1}+r_{2}+\cdots+r_{N}=N,\end{cases}
\end{aligned}
$$

where we used $\sum_{\beta_{i} \leq a_{i} \leq r_{i}}(-1)^{a_{i}-\beta_{i}}\binom{r_{i}-\beta_{i}}{a_{i}-\beta_{i}}=0$ because $\sum_{k=0}^{n}(-1)^{n}\binom{n}{k}=0$. The second case implies $r_{1}=r_{2}=\cdots=r_{N}=1, k=1$ and $N=\alpha_{1} \in \mathbb{N}$ by (35). Therefore we have

$$
\operatorname{Coe}_{C_{m}^{\alpha}} \mathbf{X}=\frac{(-1)^{N}}{N!} \int_{\bar{M}_{1, N}} \psi_{1} \cdots \psi_{N}=\frac{(-1)^{N}}{24 N}
$$

whenever $k=1\left(N=\alpha_{1}\right)$ and $|m|=0$, and $\operatorname{Coe}_{C_{m}^{\alpha}} \mathbf{X}=0$ otherwise. Taking Res ${ }_{\hbar=0} \ln$ to (33) we have

$$
\begin{equation*}
\left.\mathbf{X}=\sum_{N=1}^{\infty} \frac{(-1)^{N}}{24 N} C_{0}^{N}=-\frac{1}{24} \ln \left(1+C_{0}\right)=-\frac{1}{24} \ln \left(1+\left.\bar{z}_{6}^{*}\right|_{\hbar=0}\right)=-\frac{1}{24} \operatorname{Res}_{\hbar=0}\left(\ln \left(1+\bar{z}_{6}^{*}\right)\right) / \hbar\right) \tag{36}
\end{equation*}
$$

A.3. Some identities $I I$. We now determine $\mathrm{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{i}} \ln \left(1+z_{6}^{*}\right)$ for $i=1,2$. By (14), we have

$$
z_{6}^{*}=\frac{\mathfrak{t}^{4}}{5} \tilde{z}_{1,5}(\hbar)=-1+\frac{1}{I_{0}} \exp \left(-\frac{\mathfrak{t}}{\hbar}(t-T)-\frac{\mathfrak{t}^{\prime} g_{1}}{\hbar I_{0}}\right) \cdot \sum_{d=0}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(-5 \mathfrak{t}+m \hbar) \prod_{m=1}^{d}\left(m+\frac{\mathfrak{t}^{\prime}}{\hbar}\right)}{d!\prod_{m=1}^{d}(-\mathfrak{t}+m \hbar)^{5}}
$$

As $z_{6}^{*}$ has no $q$-degree zero term, so is $\ln \left(1+z_{6}^{*}\right)$. Together with the fact that $z_{6}^{*}$ is a formal power series in $\mathfrak{t}^{\prime}$, we know $\ln \left(1+z_{6}^{*}\right)$ is also a formal power series in $\mathfrak{t}^{\prime}$.

Lemma A.5. Let $R$ be a ring and $A_{j} \in R[[q]] q^{j}, j=0,1,2,3, \cdots$. Then

$$
\begin{aligned}
& \ln \left(1+A_{0}+A_{1} \mathfrak{t}^{\prime}+A_{2}\left(\mathfrak{t}^{\prime}\right)^{2}+\cdots\right) \\
= & \ln \left(1+A_{0}\right)+\frac{A_{1}}{1+A_{0}} \mathfrak{t}^{\prime}+\left(\frac{A_{2}}{1+A_{0}}-\frac{A_{1}^{2}}{2\left(1+A_{0}\right)^{2}}\right)\left(\mathfrak{t}^{\prime}\right)^{2}+\text { higher powers of } \mathfrak{t}^{\prime}
\end{aligned}
$$

$$
\text { Introduce } \mathcal{R}(\hbar, \mathfrak{t}):=\sum_{d=0}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(-5 \mathfrak{t}+m \hbar)}{\prod_{m=1}^{d}(-\mathfrak{t}+m \hbar)^{5}} \text { and observe first }
$$

$$
\sum_{d=1}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(-5 \mathfrak{t}+m \hbar) \prod_{m=1}^{d}\left(m+\frac{\mathfrak{t}^{\prime}}{\hbar}\right)}{d!\prod_{m=1}^{d}(-\mathfrak{t}+m \hbar)^{5}}=(\mathcal{R}-1)+C_{1} \frac{\mathfrak{t}^{\prime}}{\hbar}+C_{2} \frac{\left(\mathfrak{t}^{\prime}\right)^{2}}{\hbar^{2}}+\text { higher powers of } \mathfrak{t}^{\prime}
$$

where

$$
C_{1}=\sum_{d=1}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(-5 \mathfrak{t}+m \hbar) \sum_{m=1}^{d} \frac{1}{m}}{\prod_{m=1}^{d}(-\mathfrak{t}+m \hbar)^{5}} \quad \text { and } \quad C_{2}=\sum_{d=2}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(-5 \mathfrak{t}+m \hbar) \sum_{1 \leq a<b \leq d} \frac{1}{a b}}{\prod_{m=1}^{d}(-\mathfrak{t}+m \hbar)^{5}}
$$

Applying Lemma A.5 and above observations to the last term of

$$
\ln \left(1+z_{6}^{*}\right)=-\ln I_{0}-\frac{\mathfrak{t}}{\hbar}(t-T)-\frac{\mathfrak{t}^{\prime} g_{1}}{\hbar I_{0}}+\ln \left(1+\sum_{d=1}^{\infty} q^{d} \frac{\prod_{m=1}^{5 d}(-5 \mathfrak{t}+m \hbar) \prod_{m=1}^{d}\left(m+\frac{\mathfrak{t}^{\prime}}{\hbar}\right)}{d!\prod_{m=1}^{d}(-\mathfrak{t}+m \hbar)^{5}}\right)
$$

we obtain

$$
\mathrm{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{1}} \ln \left(1+\mathcal{Z}_{6}^{*}\right)=-\frac{g_{1}}{\hbar I_{0}}+\frac{C_{1}}{\hbar \mathcal{R}}, \quad \mathrm{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{2}} \ln \left(1+z_{6}^{*}\right)=\frac{1}{\hbar^{2}}\left(\frac{C_{2}}{\mathcal{R}}-\frac{C_{1}^{2}}{2 \mathcal{R}^{2}}\right)
$$

For simplicity, denote and find

$$
\begin{gathered}
A:=\sum_{d=1}^{\infty}\left(5^{5} q\right)^{d} \sum_{m=1}^{d} \frac{1}{m}, \quad \text { and } \quad B:=\sum_{d=1}^{\infty}\left(5^{5} q\right)^{d} \sum_{1 \leq a<b \leq d} \frac{1}{a b} . \\
A\left(1-5^{5} q\right)=-\ln \left(1-5^{5} q\right) \quad \text { and } \quad B\left(1-5^{5} q\right)=\sum_{d=2}^{\infty}\left(5^{5} q\right)^{d} \frac{1}{d} \sum_{a=1}^{d-1} \frac{1}{a} .
\end{gathered}
$$

Then

$$
C_{1}=A+\frac{\hbar}{\mathfrak{t}} \cdot 2 \frac{d}{d t} A+\text { higher powers of } \hbar, \quad \text { and } \quad C_{2}=B+\frac{\hbar}{\mathfrak{t}} \cdot 2 \frac{d}{d t} B+\text { higher powers of } \hbar
$$

$$
\mathcal{R}(\hbar, \mathfrak{t})=\sum_{d=0}^{\infty}\left(5^{5} q\right)^{d}+\frac{\hbar}{\mathfrak{t}} \sum_{d=0}^{\infty} 2 d\left(5^{5} q\right)^{d}+\text { higher powers of } \hbar=\frac{1}{1-5^{5} q}+\frac{\hbar}{\mathfrak{t}} \frac{d}{d t} \frac{2}{1-5^{5} q}\left(\bmod \hbar^{2}\right)
$$

where the expansion is near $\hbar=0$. We can now calculate the followings:

$$
\begin{align*}
& \left.\operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{0}} \operatorname{Res}_{\hbar=0} \ln \left(1+z_{6}^{*}\right)=\operatorname{Res}_{\hbar=0}\left(\operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{0}} \ln \left(1+z_{6}^{*}\right)\right)\right)=-\mathfrak{t}(t-T),  \tag{37}\\
& \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{1}} \operatorname{Res}_{\hbar=0} \frac{\ln \left(1+z_{6}^{*}\right)}{\hbar}=\operatorname{Res}_{\hbar=0} \frac{C_{1}}{\hbar^{2} \mathcal{R}}=\frac{2}{\mathfrak{t}} \frac{d}{d t}\left(A\left(1-5^{5} q\right)\right)=\frac{2}{\mathfrak{t}} \cdot \frac{5^{5} q}{1-5^{5} q}, \\
& \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{1}} \eta=\operatorname{Res}_{\hbar=0} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{1}} \ln \left(1+z_{6}^{*}\right)=\operatorname{Res}_{\hbar=0}\left(-\frac{g_{1}}{\hbar I_{0}}+\frac{C_{1}}{\hbar \mathcal{R}}\right)
\end{align*}
$$

$$
\begin{aligned}
= & -\frac{g_{1}}{I_{0}}+A\left(1-5^{5} q\right)=-\frac{g_{1}}{I_{0}}-\ln \left(1-5^{5} q\right) \\
& \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{2}} \eta=\operatorname{Res}_{\hbar=0} \operatorname{Coe}_{\left(\mathfrak{t}^{\prime}\right)^{2}} \ln \left(1+Z_{6}^{*}\right)=\operatorname{Res}_{\hbar=0}\left(\frac{1}{\hbar^{2}}\left(\frac{C_{2}}{\mathcal{R}}-\frac{C_{1}^{2}}{2 \mathcal{R}^{2}}\right)\right) \\
= & \frac{2}{\mathfrak{t}} \frac{d B\left(1-5^{5} q\right)}{d t}+\frac{1}{2} \frac{4}{\mathfrak{t}} A\left(1-5^{5} q\right) \frac{d A\left(1-5^{5} q\right)}{d t} \\
= & \frac{2}{\mathfrak{t}} \sum_{d=2}^{\infty}\left(5^{5} q\right)^{d} \sum_{a=1}^{d-1} \frac{1}{a}+\frac{2}{\mathfrak{t}} \frac{5^{5} q}{1-5^{5} q} \ln \left(1-5^{5} q\right)=0 .
\end{aligned}
$$

## B. Appendix: Differential Relations arising from geometry

We shall derive some differential relations among periods from geometry in such a way that we keep track of the arithmetic origins of the coefficients in the relations. These relations are needed to prove Proposition C. 4 for type B contribution studied in $g=1$ MSP localization. We separate the discussions into this appendix since the relations obtained seem to be of independent interest.

The discussion applies to any one-parameter family of Calabi-Yau threefolds, but the focus of this work is on the quintic/mirror quintic family. The corresponding differential operator is

$$
\mathcal{L}=\theta^{4}-\alpha \prod_{k=1}^{4}\left(\theta+\frac{k}{5}\right)
$$

where $\alpha=5^{5} q=5^{5} e^{t}$ and $\theta=\alpha \frac{\partial}{\partial \alpha}$. This can be regarded as the Picard-Fuchs operator for the mirror quintic family, or the quantum differential equation for the quintic family itself. The properties are about the differential operator itself, regardless of whether we are looking at the A-model or B-model. However, we shall use the language for the B-model, which is more convenient.

We first expand the above operator as

$$
\mathcal{L}_{\mathrm{hyp}}=\theta^{4}-\alpha \sum_{k=0}^{3} \sigma_{4-k} \theta^{k}
$$

where $\sigma_{k}$ means the $k$ th fundamental symmetric polynomials of the numbers $1 / 5,2 / 5,3 / 5,4 / 5$. For the quintic/mirror quintic case in consideration, one has

$$
\begin{equation*}
\sigma_{1}=2, \quad \sigma_{2}=\frac{7}{5}, \quad \sigma_{3}=\frac{2}{5}, \quad \sigma_{4}=\frac{24}{5^{4}} \tag{38}
\end{equation*}
$$

Introducing the coordinate $t=\ln q$, then the above differential operator becomes

$$
\begin{equation*}
\mathcal{L}_{\mathrm{hyp}}=(1-\alpha) \partial_{t}^{4}-\alpha \sum_{k=0}^{3} \sigma_{4-k} \partial_{t}^{k} \tag{39}
\end{equation*}
$$

Introducing the notation (recall $\alpha=5^{5} q=5^{5} e^{t}$ )

$$
\begin{equation*}
C_{t t t}=\frac{1}{1-\alpha}=\frac{1}{1-5^{5} q}, \quad C=\partial_{t} \ln C_{t t t}=\frac{\alpha}{1-\alpha}=\frac{5^{5} q}{1-5^{5} q} \tag{40}
\end{equation*}
$$

then we can normalize the leading term of the differential operator to 1 and get

$$
\mathcal{L}:=\frac{1}{1-\alpha} \mathcal{L}_{\mathrm{hyp}}=\partial_{t}^{4}+\sum_{k=0}^{3} a_{k} \partial_{t}^{k}, \quad a_{k}=-\sigma_{4-k} C
$$

The 4 solutions of the corresponding differential equation obtained by the Frobenius method near the singular point $\alpha=0$ are denoted by $\left(I_{0}, I_{1}, I_{2}, I_{3}\right)$. In particular, one has

$$
\begin{equation*}
I_{0}={ }_{4} F_{3}\left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} ; 1,1,1 ; \alpha\right), \quad I_{1}=\log \left(\frac{\alpha}{5^{5}}\right) \cdot I_{0}+\cdots . \tag{41}
\end{equation*}
$$

Then one has

$$
T:=\frac{I_{1}}{I_{0}}=t+\cdots
$$

In the B-model, the operator $\mathcal{L}_{\text {hyp }}$ is the Picard-Fuchs operator, and the four solutions above are the periods of the mirror quintic. In the A-model, they are the first four coefficients in the $H$-expansion of the $I$-function $e^{H q} J$, where $H$ is the hyperplane class of $\mathbb{P}^{4}$ and $J$ is the $J$-function.
B.1. Special geometry relation. The set of periods has the special geometry structure $[S, W$ which implies that there exists a holomorphic function $F$ (called the prepotential) of $T=I_{1} / I_{0}$, such that the set of periods above has the following structure

$$
\begin{equation*}
\left(I_{0}, I_{1}, I_{2}, I_{3}\right)=I_{0}\left(1, T, F_{T}, T F_{T}-2 F\right) \tag{42}
\end{equation*}
$$

This fact can be easily proved by using Griffiths transversality. This structure in fact holds everywhere, but we shall only focus on a neighborhood near the point $\alpha=0$ near which we have

$$
\begin{equation*}
F(T)=\frac{1}{3!} T^{3}+\text { quadratic polynomial in } T+\mathcal{O}\left(e^{T}\right) \tag{43}
\end{equation*}
$$

It follows from this structure (called the special geometry structure) that the 4 solutions $I_{k}, k=0,1,2,3$ above, after normalization by the fundamental period $I_{0}$, are solutions to the following equation (see CDFLL, CDFLLR

$$
\partial_{T}^{2} F_{T T T}^{-1} \partial_{T}^{2}\left(\frac{I_{k}}{I_{0}}\right)=0, \quad k=0,1,2,3
$$

By comparing the leading terms, we can see that the Picard-Fuchs operator (39) must satisfy the relation

$$
\begin{equation*}
\mathcal{L}=\mathcal{D} \tag{44}
\end{equation*}
$$

where

$$
\mathcal{D}:=\left(\frac{1}{I_{0}} F_{T T T}^{-1}\left(\frac{\partial t}{\partial T}\right)^{4}\right)^{-1} \circ \partial_{T}^{2} F_{T T T}^{-1} \partial_{T}^{2} \circ \frac{1}{I_{0}}
$$

We now study the consequences of this identity by using the "computing twice" trick. We shall see that this identity (44) gives rise to some differential relations.

First, it is easy to see that we can rewrite $\mathcal{D}$ as
$\mathcal{D}=\left(\partial_{t}+\partial_{t} \log F_{T T T}^{-1}+\partial_{t} \log \left(\frac{\partial t}{\partial T}\right)^{3}+\partial_{t} \log \frac{1}{I_{0}}\right) \circ\left(\partial_{t}+\partial_{t} \log F_{T T T}^{-1}+\partial_{t} \log \left(\frac{\partial t}{\partial T}\right)^{2}+\partial_{t} \log \frac{1}{I_{0}}\right)$

$$
\begin{equation*}
\circ\left(\partial_{t}+\partial_{t} \log \left(\frac{\partial t}{\partial T}\right)+\partial_{t} \log \frac{1}{I_{0}}\right) \circ\left(\partial_{t}+\partial_{t} \log \frac{1}{I_{0}}\right) . \tag{45}
\end{equation*}
$$

For convenience, we introduce the following notations

$$
\begin{align*}
& c_{1}=\partial_{t} \log \frac{1}{I_{0}}, \quad c_{2}=\partial_{t} \log \left(\frac{\partial t}{\partial T}\right)+\partial_{t} \log \frac{1}{I_{0}} \\
& c_{3}=\partial_{t} \log F_{T T T}^{-1}+\partial_{t} \log \left(\frac{\partial t}{\partial T}\right)^{2}+\partial_{t} \log \frac{1}{I_{0}}, \quad c_{4}=\partial_{t} \log F_{T T T}^{-1}+\partial_{t} \log \left(\frac{\partial t}{\partial T}\right)^{3}+\partial_{t} \log \frac{1}{I_{0}} \tag{46}
\end{align*}
$$

They satisfy the obvious relation

$$
\begin{equation*}
c_{2}-c_{1}=c_{4}-c_{3} . \tag{47}
\end{equation*}
$$

Straightforward computations show that 1

$$
\mathcal{D}=\partial_{t}^{4}+b_{3} \partial_{t}^{3}+b_{2} \partial_{t}^{2}+b_{1} \partial_{t}+b_{0},
$$

[^1]where
\[

$$
\begin{align*}
b_{3}= & \sum_{k} c_{k}, \quad b_{2}=\sum_{k \neq l} c_{k} c_{l}+3 c_{1}^{\prime}+2 c_{2}^{\prime}+c_{3}^{\prime} \\
b_{1}= & \sum_{j \neq k \neq l} c_{j} c_{k} c_{l}+\left(2 c_{2}+2 c_{3}+2 c_{4}\right) c_{1}^{\prime}+\left(2 c_{1}+c_{3}+c_{4}\right) c_{2}^{\prime}+\left(c_{1}+c_{2}\right) c_{3}^{\prime}+3 c_{1}^{\prime \prime}+c_{2}^{\prime \prime}, \\
b_{0}= & c_{1} c_{2} c_{3} c_{4}+\left(c_{2} c_{3}+c_{2} c_{4}+c_{3} c_{4}\right) c_{1}^{\prime}+\left(c_{1} c_{3}+c_{1} c_{4}\right) c_{2}^{\prime}+2 c_{1}^{\prime} c_{2}^{\prime}+c_{1} c_{2} c_{3}^{\prime} \\
& +c_{1}^{\prime} c_{3}^{\prime}+c_{2} c_{1}^{\prime \prime}+c_{3} c_{1}^{\prime \prime}+c_{4} c_{1}^{\prime \prime}+c_{1} c_{2}^{\prime \prime}+c_{1}^{\prime \prime \prime} \tag{48}
\end{align*}
$$
\]

B.1.1. Matching degree three terms: Yukawa coupling $F_{T T T}$ in terms of period integrals. The matching of the coefficients of $\partial_{t}^{3}$ yields $b_{3}=a_{3}$ which according to (48) is

$$
c_{1}+c_{2}+c_{3}+c_{4}=a_{3}
$$

Plugging in the relation $a_{k}=-\sigma_{4-k} C$ and (46), we get the relation

$$
\begin{equation*}
\partial_{t} \log F_{T T T}+2 \partial_{t} \log I_{0}+3 \partial_{t} \log \frac{\partial T}{\partial t}=-\frac{1}{2} a_{3}=\sigma_{1} C \tag{49}
\end{equation*}
$$

This gives a relation between the transcendental series $I_{0}, \frac{\partial T}{\partial t}$ etc., and the rational function $C$.
For later use, we introduce furthermore

$$
\begin{equation*}
A=\partial_{t} \log \frac{\partial T}{\partial t}, \quad B=\partial_{t} \log I_{0}, \quad Y=\partial_{t} \log F_{T T T} \tag{50}
\end{equation*}
$$

The above relation is then (recall (40))

$$
Y+2 B+3 A=\frac{1}{2} \sigma_{1} C .
$$

That is

$$
F_{T T T}=\frac{1}{(1-\alpha)^{\frac{\sigma_{1}}{2}}} \frac{1}{I_{0}^{2}} \frac{1}{T^{\prime 3}} \cdot \text { const }
$$

for some constant const which can be fixed to be 1 by using the boundary conditions in (41) and the known asymptotic behaviour of the the Yukawa coupling $F_{T T T}$ following from (43).
B.1.2. Matching degree two terms: flatness of Gauss-Manin. For the next coefficients, we have $b_{2}=a_{2}$. By (48) this is

$$
\sum_{k \neq l} c_{k} c_{l}+3 c_{1}^{\prime}+2 c_{2}^{\prime}+c_{3}^{\prime}=-\sigma_{2} C
$$

Using the $b_{3}$-relation to eliminate $c_{3}$, we can see that the above relation becomes

$$
\begin{equation*}
-A^{\prime}-4 B^{\prime}-C^{\prime}-A^{2}-2 B^{2}+2 B C+C^{2}-2 A B+A C=-\sigma_{2} C \tag{51}
\end{equation*}
$$

This relation is discussed in $[\mathrm{LY}$ ] by using the Wronskian method. It is further studied in $\mathrm{YY}, \mathrm{H}, \mathrm{Z}$. In particular, as pointed out explicitly in YY, Z, this relation is equivalent to the flatness of the GaussManin connection on the Hodge bundle. Note that the above differential relation about $A, B, C$ is in fact an identity without referring to the prepotential, although in eliminating $c_{3}$ above we have used the $b_{3}$-relation whose definition does require the existence of prepotential. In the next section, we shall see that this relation can be naturally rephrased in terms of Wronskians, again without using the prepotential, which offers a more intrinsic way of explaining what this relation is.
B.1.3. Matching degree one terms: symplectic structure. We now match the degree one terms in (44). By straightforward calculation from (48), this is equivalent to

$$
\begin{equation*}
b_{1}=b_{2}^{\prime}+\frac{1}{2} b_{2} b_{3}-\frac{1}{8} b_{3}^{3}-\frac{1}{2} b_{3}^{\prime \prime}-\frac{3}{4} b_{3} b_{3}^{\prime} . \tag{52}
\end{equation*}
$$

This relation is studied in details in CDFLL, CDFLLR, LY, AZ. In particular, as pointed out in CDFLL, CDFLLR, this relation means that the Gauss-Manin connection is symplectic. Note that, in this relation, the $c_{k}$ 's are combined in such a way that they do not appear individually. Hence this relation does not give new differential relations among them.
B.1.4. Matching degree zero terms: Picard-Fuchs for fundamental period. Consider the degree zero terms in (44). By using the relation $b_{3}=a_{3}$ and (48), we get an ODE for $c_{1}$ :

$$
\begin{equation*}
b_{0}=c_{1}^{\prime \prime \prime}-c_{1}^{4}+6 c_{1}^{\prime} c_{1}^{2}-4 c_{1} c_{1}^{\prime \prime}-3 c_{1}^{\prime} c_{1}^{\prime}+b_{3}\left(c_{1}^{\prime \prime}-3 c_{1} c_{1}^{\prime}+c_{1}^{3}\right)+b_{2}\left(c_{1}^{\prime}-c_{1}^{2}\right)+b_{1} c_{1} . \tag{53}
\end{equation*}
$$

On the other hand, the Picard-Fuchs equation for the fundamental period $I_{0}$ reads (recall $B=\partial_{t} \log I_{0}$ )

$$
0=\frac{I_{0}^{\prime \prime \prime \prime}}{I_{0}}+a_{3} \frac{I_{0}^{\prime \prime \prime}}{I_{0}}+a_{2} \frac{I_{0}^{\prime \prime}}{I_{0}}+a_{1} \frac{I_{0}^{\prime}}{I_{0}}+a_{0}
$$

Recall that $c_{1}=-B$, we can see that the Picard-Fuchs equation for $I_{0}$ above is exactly the relation $b_{0}=a_{0}$.
B.1.5. A summary. We remark that the $b_{0}$-relation (53) is always true, independent of the geometry structure. The $b_{1}$-structure (52) is the symplectic structure, which can be further translated into relations on the periods, see the discussion in the next section below. It does not refer to the prepotential, as we have pointed out earlier that it is not a relation on the individual $c_{k}$ 's. The $b_{2}$-relation (51), the flatness condition, can also be phrased without referring to the prepotential, which can also be seen from the explicit expression in terms of Wronskians discussed in the next section. Finally, the $b_{3}$-relation (49) gives an identity that the prepotential satisfies, in case the special geometry structure is present.
B.2. Wronskian method. The Wronskian method and the exterior square structure are useful in simplifications and in thinking about what the quadratic differential polynomials in MSP calculation are. A computational evidence is that the derivative of a normalized period is naturally expressed in terms of Wronskians.

It is shown in AZ] that for a generic order 4 differential equation $\mathcal{L}$, the Wronskians

$$
M_{a b}=\operatorname{det}\left(\begin{array}{cc}
I_{a} & I_{b} \\
I_{a}^{\prime} & I_{b}^{\prime}
\end{array}\right)
$$

satisfies an order 6 differential equation. This is the exterior square of $\mathcal{L}$. We now review the results in AZ, which are important for the proof of the main result in this work. We denote

$$
\begin{aligned}
& u_{1}=M_{a b}, \quad u_{2}=M_{a b}^{\prime}=\operatorname{det}\left(\begin{array}{cc}
I_{a} & I_{b} \\
I_{a}^{\prime \prime} & I_{b}^{\prime \prime}
\end{array}\right) \\
& u_{3}=\operatorname{det}\left(\begin{array}{cc}
I_{a} & I_{b} \\
I_{a}^{\prime \prime \prime} & I_{b}^{\prime \prime \prime}
\end{array}\right), \quad u_{4}=\operatorname{det}\left(\begin{array}{cc}
I_{a}^{\prime} & I_{b}^{\prime} \\
I_{a}^{\prime \prime} & I_{b}^{\prime \prime}
\end{array}\right) \\
& u_{5}=\operatorname{det}\left(\begin{array}{cc}
I_{a}^{\prime} & I_{b}^{\prime} \\
I_{a}^{\prime \prime \prime} & I_{b}^{\prime \prime \prime}
\end{array}\right), \quad u_{6}=\operatorname{det}\left(\begin{array}{cc}
I_{a}^{\prime \prime} & I_{b}^{\prime \prime} \\
I_{a}^{\prime \prime \prime} & I_{b}^{\prime \prime \prime}
\end{array}\right) .
\end{aligned}
$$

Recall that the $I_{a}$ 's are annihilated by the order 4 differential operator

$$
\mathcal{L}=\partial^{4}+a_{3} \partial^{3}+a_{2} \partial^{2}+a_{1} \partial+a_{0}
$$

Then the set $\left\{u_{k}, k=1,2, \cdots 6\right\}$ satisfy the following relations

$$
\begin{align*}
& u_{1}^{\prime}=u_{2}, \quad u_{2}^{\prime}=u_{3}+u_{4}, \quad u_{3}^{\prime}=u_{5}-a_{1} u_{1}-a_{2} u_{2}-a_{3} u_{3} \\
& u_{4}^{\prime}=u_{5}, \quad u_{5}^{\prime}=u_{6}+a_{0} u_{1}-a_{2} u_{4}-a_{3} u_{5}, \quad u_{6}^{\prime}=a_{0} u_{2}+a_{1} u_{4}-a_{3} u_{6} \tag{54}
\end{align*}
$$

B.2.1. Symplectic structure. It is shown in AZ] that the above system gives rise to an order 6 differential equation for $u_{1}$. The solutions are $M_{a b}, a, b=1,2, \cdots 4$. When the symplectic structure (52)

$$
\begin{equation*}
a_{1}=\frac{1}{2} a_{2} a_{3}-\frac{1}{8} a_{3}^{3}+a_{2}^{\prime}-\frac{3}{4} a_{3} a_{3}^{\prime}-\frac{1}{2} a_{3}^{\prime \prime} \tag{55}
\end{equation*}
$$

is present, the order 6 differential equation becomes an order 5 differential equation. We now recall the computations therein. Define

$$
U=u_{1}^{\prime \prime \prime}+a_{3} u_{1}^{\prime \prime}+a_{2} u_{1}^{\prime}+a_{1} u_{1}
$$

then

$$
\begin{equation*}
U-a_{3} u_{4}=2 u_{5} \tag{56}
\end{equation*}
$$

Taking the derivative, one gets

$$
U^{\prime}-a_{3}^{\prime} u_{4}-a_{3} u_{5}=2 u_{6}+2 a_{0} u_{1}-2 a_{2} u_{4}-2 a_{3} u_{5}
$$

Eliminating $u_{5}$ using (56), we obtain

$$
\begin{equation*}
U^{\prime}+\frac{1}{2} a_{3} U=u_{4}\left(a_{3}^{\prime}+\frac{1}{2} a_{3}^{2}-2 a_{2}\right)+2 a_{0} u_{1}+2 u_{6}:=C_{1} u_{4}+2 a_{0} u_{1}+2 u_{6} \tag{57}
\end{equation*}
$$

Taking one more derivative and eliminating $u_{5}$ using (56), we obtain

$$
U^{\prime \prime}+\frac{1}{2} a_{3}^{\prime} U+\frac{1}{2} a_{3} U^{\prime}=C_{1}^{\prime} u_{4}+\frac{1}{2} C_{1}\left(U-a_{3} u_{4}\right)+2 a_{0}^{\prime} u_{1}+2 a_{0} u_{1}^{\prime}+2\left(a_{0} u_{1}^{\prime}+a_{1} u_{4}-a_{3} u_{6}\right)
$$

Using (57) to eliminate $u_{6}$, we obtain

$$
\begin{aligned}
& U^{\prime \prime}+\frac{1}{2} a_{3}^{\prime} U+\frac{1}{2} a_{3} U^{\prime} \\
= & C_{1}^{\prime} u_{4}+\frac{1}{2} C_{1}\left(U-a_{3} u_{4}\right)+2 a_{0}^{\prime} u_{1}+4 a_{0} u_{1}^{\prime}+2 a_{1} u_{4}-a_{3}\left(U^{\prime}+\frac{1}{2} a_{3} U-C_{1} u_{4}-2 a_{0} u_{1}\right) .
\end{aligned}
$$

Simplifying, one has

$$
\begin{aligned}
W & :=U^{\prime \prime}+\frac{1}{2} a_{3}^{\prime} U+\frac{3}{2} a_{3} U^{\prime}-\frac{1}{2} C_{1} U+\frac{1}{2} a_{3} U-2 a_{0}^{\prime} u_{1}-4 a_{0} u_{1}^{\prime}-2 a_{0} a_{3} u_{1} \\
& =C_{1}^{\prime} u_{4}+\frac{3}{2} C_{1} a_{3} u_{4}+2 a_{1} u_{4}:=C_{2} u_{4}
\end{aligned}
$$

If the condition $C_{2}=0$ is satisfied, then we are done. It turns out that this condition is exactly the symplectic structure relation (55). Otherwise, we get

$$
W^{\prime}=C_{2} u_{4}^{\prime}+C_{2}^{\prime} u_{4}=C_{2} u_{5}+C_{2}^{\prime} u_{4}=\frac{1}{2} C_{2}\left(U-a_{3} u_{4}\right)+C_{2}^{\prime} u_{4}=\frac{1}{2} C_{2} U+\left(C_{2}^{\prime}-\frac{1}{2} C_{2} a_{3}\right) u_{4}
$$

Hence

$$
\begin{equation*}
W^{\prime}=\frac{1}{2} C_{2} U+\left(C_{2}^{\prime}-\frac{1}{2} C_{2} a_{3}\right) \frac{W}{C_{2}} . \tag{58}
\end{equation*}
$$

This gives an order 6 ODE as desired.
B.2.2. Simplication of Wronskians in the presence of symplectic structure. As shown in [AZ], the symplectic structure condition (55) is equivalent to either of the following relations

$$
\begin{aligned}
\frac{\partial^{2}}{\partial T^{2}} \frac{I_{3}}{I_{0}} & =T \frac{\partial^{2}}{\partial T^{2}} \frac{I_{2}}{I_{0}} \\
\frac{\partial^{2}}{\partial T^{2}} \frac{I_{2}}{I_{0}} & =\frac{1}{I_{0}^{2}\left(\partial_{t} T\right)^{3}} \exp \left(-\frac{1}{2} \int a_{3}\right)
\end{aligned}
$$

This condition is evidently satisfied if the stronger special geometry structure (42) holds, as it should be. In this case, the second condition above condition is nothing but the $b_{3}$-relation in (49). Note that this should be thought of as a relation on $b_{1}$ instead of one on $b_{3}$, since in general there is no notion of prepotential but one can always talk about the condition above.

In terms of Wronskians, this is equivalent to

$$
W=U^{\prime \prime}+\frac{1}{2} a_{3}^{\prime} U+\frac{3}{2} a_{3} U^{\prime}-\frac{1}{2} C_{1} U+\frac{1}{2} a_{3} U-2 a_{0}^{\prime} u_{1}-4 a_{0} u_{1}^{\prime}-2 a_{0} a_{3} u_{1}=0
$$

where

$$
C_{1}=\left(a_{3}^{\prime}+\frac{1}{2} a_{3}^{2}-2 a_{2}\right), \quad U-a_{3} u_{4}=2 u_{5} .
$$

We now focus on the relation

$$
\frac{\partial^{2}}{\partial T^{2}} \frac{I_{3}}{I_{0}}=T \frac{\partial^{2}}{\partial T^{2}} \frac{I_{2}}{I_{0}}
$$

above. Integrating, one gets

$$
\frac{\partial}{\partial T} \frac{I_{3}}{I_{0}}=T \frac{\partial}{\partial T} \frac{I_{2}}{I_{0}}-\frac{I_{2}}{I_{0}}+\text { const }=T^{2} \frac{\partial}{\partial T}\left(\frac{1}{T} \frac{I_{2}}{I_{0}}\right)+\text { const }
$$

for some constant const. Using the boundary conditions given in (41), (42), (43) (in fact, only the $\log \alpha$ terms) for the periods $I_{2}, I_{3}$ one can see that const $=0$.

Now since

$$
\frac{\partial}{\partial T}=\frac{1}{T^{\prime}} \frac{\partial}{\partial t}
$$

the above is equivalent to

$$
\frac{\partial}{\partial t}\left(\frac{I_{3}}{I_{0}}\right)=T^{2} \frac{\partial}{\partial t}\left(\frac{1}{T} \frac{I_{2}}{I_{0}}\right)=T^{2} \frac{\partial}{\partial t}\left(\frac{I_{2}}{I_{1}}\right)
$$

This gives

$$
\begin{equation*}
I_{0} I_{3}^{\prime}-I_{0}^{\prime} I_{3}=I_{1} I_{2}^{\prime}-I_{1}^{\prime} I_{2} \tag{59}
\end{equation*}
$$

That is,

$$
\operatorname{det}\left(\begin{array}{ll}
I_{0} & I_{3} \\
I_{0}^{\prime} & I_{3}^{\prime}
\end{array}\right)=\operatorname{det}\left(\begin{array}{ll}
I_{1} & I_{2} \\
I_{1}^{\prime} & I_{2}^{\prime}
\end{array}\right)
$$

This is why the order 6 differential equation (58) for the Wronskians, which has 6 solutions $M_{a b}$, becomes an order 5 differential equation.
B.2.3. Flatness condition in terms of Wronskians in the present of symplectic structure. The exterior square structure above, which is the underlying geometry of Wronskians, provides a nice way to express the identities we found above by matching coefficients in (44).

We have done this for the symplectic structure relation in terms of

$$
\frac{\partial^{2}}{\partial T^{2}} \frac{I_{3}}{I_{0}}=T \frac{\partial^{2}}{\partial T^{2}} \frac{I_{2}}{I_{0}}
$$

or equivalently in terms of the Wronskians

$$
I_{0} I_{3}^{\prime}-I_{0}^{\prime} I_{3}=I_{1} I_{2}^{\prime}-I_{1}^{\prime} I_{2}
$$

We now do this for the flatness condition without assuming the symplectic structure. We specialize to $M_{a b}$ with $a=0, b=1$ and again use the following convention $u_{k}, k=1,2, \cdots 6$ for the Wronskians. One has the relations

$$
u_{2}^{\prime}=u_{3}+u_{4}
$$

and also

$$
\frac{u_{2}^{\prime}}{u_{1}}=\left(\frac{u_{2}}{u_{1}}\right)^{\prime}+\left(\frac{u_{2}}{u_{1}}\right)^{2} .
$$

The latter is due to $u_{2}=u_{1}^{\prime}$. We write them in terms of the coefficients $c_{k}$ in (46) and the Wronskians. From $c_{1}=-B, c_{2}=-A-B$, it is easy to see that

$$
\begin{gathered}
\frac{u_{2}}{u_{1}}=-\left(c_{1}+c_{2}\right), \quad\left(\frac{u_{2}}{u_{1}}\right)^{\prime}=-\left(c_{1}^{\prime}+c_{2}^{\prime}\right), \quad \frac{u_{3}}{u_{1}}=-2 c_{1}^{\prime}-c_{2}^{\prime}+3 c_{1} c_{2}+\left(c_{1}-c_{2}\right)^{2}, \quad \frac{u_{4}}{u_{1}}=c_{1}^{\prime}+c_{1} c_{2}, \\
\frac{u_{4}+u_{3}}{u_{1}}=\frac{u_{2}^{\prime}}{u_{1}}=\left(\frac{u_{2}}{u_{1}}\right)^{\prime}+\left(\frac{u_{2}}{u_{1}}\right)^{2}=-\left(c_{1}^{\prime}+c_{2}^{\prime}\right)+\left(c_{1}+c_{2}\right)^{2}, \quad \frac{u_{4}-u_{3}}{u_{1}}=3 c_{1}^{\prime}+c_{2}^{\prime}-\left(c_{1}^{2}+c_{2}^{2}\right) .
\end{gathered}
$$

Recall that the $b_{2}, b_{3}$-terms in (48) are

$$
b_{2}=\frac{1}{2}\left(\left(\sum c_{k}\right)^{2}-\sum c_{k}^{2}\right)+3 c_{1}^{\prime}+2 c_{2}^{\prime}+c_{3}^{\prime}, \quad b_{3}=\sum c_{k}
$$

Using the relation (47), we can solve for $c_{3}, c_{4}$ as follows

$$
c_{3}=\frac{1}{2} b_{3}-c_{2}, \quad c_{4}=\frac{1}{2} b_{3}-c_{1} .
$$

Plugging these into the $b_{2}$-coefficient, we have

$$
\begin{aligned}
b_{2} & =\frac{1}{2} b_{3}^{2}-\frac{1}{2}\left(c_{1}^{2}+c_{2}^{2}+c_{3}^{2}+c_{4}^{2}\right)+3 c_{1}^{\prime}+2 c_{2}^{\prime}+c_{3}^{\prime} \\
& =\frac{1}{4} b_{3}^{2}-\left(c_{1}^{2}+c_{2}^{2}\right)+\frac{1}{2} b_{3}\left(c_{1}+c_{2}\right)+\frac{1}{2} b_{3}^{\prime}+3 c_{1}^{\prime}+c_{2}^{\prime}
\end{aligned}
$$

In terms of Wronskians, we have

$$
c_{1}^{\prime}=\frac{u_{4}}{u_{1}}-c_{1} c_{2}, \quad 2 c_{1}^{\prime}+c_{2}^{\prime}=-\frac{u_{3}}{u_{1}}+3 c_{1} c_{2}+\left(c_{1}-c_{2}\right)^{2}
$$

Eliminating the derivatives from the $b_{2}$-expression, we get

$$
\begin{aligned}
b_{2} & =\frac{1}{4} b_{3}^{2}-\left(c_{1}^{2}+c_{2}^{2}\right)+\frac{1}{2} b_{3}\left(c_{1}+c_{2}\right)+\frac{1}{2} b_{3}^{\prime}+3 c_{1}^{\prime}+c_{2}^{\prime} \\
& =\frac{1}{4} b_{3}^{2}-\left(c_{1}^{2}+c_{2}^{2}\right)+\frac{1}{2} b_{3}\left(c_{1}+c_{2}\right)+\frac{1}{2} b_{3}^{\prime}+\frac{u_{4}}{u_{1}}-\frac{u_{3}}{u_{1}}+2 c_{1} c_{2}+\left(c_{1}-c_{2}\right)^{2} \\
& =\frac{1}{4} b_{3}^{2}+\frac{1}{2} b_{3}\left(c_{1}+c_{2}\right)+\frac{1}{2} b_{3}^{\prime}+\frac{u_{4}-u_{3}}{u_{1}} .
\end{aligned}
$$

Now using the relation $u_{2} / u_{1}=-\left(c_{1}+c_{2}\right)$ above, this gives

$$
-\frac{1}{2} b_{3} \frac{u_{2}}{u_{1}}+\frac{u_{4}-u_{3}}{u_{1}}=b_{2}-\frac{1}{2} b_{3}^{\prime}-\frac{1}{4} b_{3}^{2}
$$

We hence get the flatness condition in terms of Wronskians $u_{1}, u_{2}, u_{3}, u_{4}$ :

$$
\begin{equation*}
\left(u_{4}-u_{3}\right)=\left(b_{2}-\frac{1}{2} b_{3}^{\prime}-\frac{1}{4} b_{3}^{2}\right) u_{1}+\frac{1}{2} b_{3} u_{2} \tag{60}
\end{equation*}
$$

Substituting the relations $b_{3}=a_{3}=-\sigma_{1} C=-2 C$ and $b_{2}=a_{2}=-\sigma_{2} C$, recall that $C=\frac{\alpha}{1-\alpha}$ satisfies

$$
C^{\prime}=C^{2}+C
$$

we then get

$$
\left(u_{4}-u_{3}\right)=-C\left(\left(\sigma_{2}-1\right) u_{1}+\frac{1}{2} \sigma_{1} u_{2}\right) .
$$

That is,

$$
5(1-\alpha)\left(u_{4}-u_{3}\right)=-\alpha\left(2 u_{1}+5 u_{2}\right)
$$

as $\sigma_{2}=7 / 5$ from (38). It is easy to check that this equation is the same as (51), as it should be. Note that here we did not use the $b_{1}$-relation (52), hence no symplectic structure is used.

Remark B.1. The quantity $C=\partial_{t} \ln C_{t t t}$ is independent of the $\sigma_{k}$ 's. This is in contrast with the value of the Yukawa coupling $C_{t t t}:=\int \Omega \wedge \partial_{t}^{3} \Omega$ which is computationally solved from

$$
\partial_{t} \log C_{t t t}=-\frac{a_{3}}{2}
$$

and is given by

$$
C_{t t t}=c o n s t \cdot \exp \left(\int-\frac{a_{3}}{2}\right)=c o n s t \cdot \exp \left(\int \frac{\sigma_{1}}{2} C\right)
$$

for some constant const. Hence it depends on the specific value of $a_{3}$ or equivalently $\sigma_{1}$ for the present hypergeometric case.
C. Appendix: Antiderivatives of differential polynomials, and its application
C.1. Preparation: anti-derivatives. The structure in the relation (44) has other interesting consequences besides the differential relations we obtain in Appendix $B$. This structure is written as

$$
\begin{equation*}
\mathcal{L}=\mathcal{D}, \tag{61}
\end{equation*}
$$

where as in (45)

$$
\begin{equation*}
\mathcal{L}=\partial_{t}^{4}+\sum_{k=0}^{3}\left(-\sigma_{4-k} C\right) \partial_{t}^{k} \tag{62}
\end{equation*}
$$

and as in (46)

$$
\begin{equation*}
\mathcal{D}=\left(\partial_{t}+c_{4}\right)\left(\partial_{t}+c_{3}\right)\left(\partial_{t}+c_{2}\right)\left(\partial_{t}+c_{1}\right) . \tag{63}
\end{equation*}
$$

For later use, we shall denote

$$
\beta=1-\alpha, \quad \mathcal{D}_{k}=\left(\partial_{t}+c_{k}\right), \quad k=1,2,3,4
$$

C.1.1. Commuting differential operator actions on periods. Some first properties of the relation (44) are as follows.

Lemma C.1. For the four solutions $I_{k}, k=0,1,2,3$ to the Picard-Fuchs equation, one has

$$
\mathcal{L}_{\mathrm{hyp}} \circ \partial_{t} I_{k}=\partial_{t}^{4} I_{k}
$$

Proof. By definition,

$$
\mathcal{L}_{\text {hyp }} \circ \partial_{t} I_{k}=\left[\mathcal{L}_{\text {hyp }}, \partial_{t}\right] I_{k}+\partial_{t} \circ \mathcal{L}_{\text {hyp }} I_{k}=-\left[\partial_{t}, \mathcal{L}_{\text {hyp }}\right] I_{k}
$$

Now using (62) one has

$$
\left[\partial_{t}, \mathcal{L}_{\mathrm{hyp}}\right]=\left(\partial_{t} \beta\right) \partial_{t}^{4}+\sum_{k=0}^{3}\left(-\sigma_{4-k} \partial_{t} \alpha\right) \partial_{t}^{k}=(\beta-1) \partial_{t}^{4}+\sum_{k=0}^{3}\left(-\sigma_{4-k} \alpha\right) \partial_{t}^{k}=-\partial_{t}^{4}+\mathcal{L}_{\mathrm{hyp}}
$$

Therefore, one obtains

$$
\mathcal{L}_{\mathrm{hyp}} \circ \partial_{t} I_{k}=\partial_{t}^{4} I_{k}-\mathcal{L}_{\mathrm{hyp}} I_{k}=\partial_{t}^{4} I_{k}
$$

Lemma C.2. The following relation for the differential operators holds

$$
I_{0} \mathcal{L}_{\text {hyp }}=\partial_{t} \circ\left(I_{0} \beta\right) \circ \mathcal{D}_{3} \circ \mathcal{D}_{2} \circ \mathcal{D}_{1}
$$

Proof. The relation (61) reads that

$$
\mathcal{L}_{\text {hyp }}=\beta \mathcal{D}_{4} \circ \mathcal{D}_{3} \circ \mathcal{D}_{2} \circ \mathcal{D}_{1}, \quad \mathcal{D}_{4}=\partial_{t}+c_{4}=\partial_{t}-Y-B-3 A
$$

where the quantities $C, A, B, Y$ are as defined in (40) and (50). According to the $b_{3}$-relation in (49) and (40), we have

$$
-Y-B-3 A=B-C=\partial_{t} \ln I_{0}-\partial_{t} \ln C_{t t t}=\partial_{t} \ln \left(I_{0} \beta\right)
$$

Therefore, we obtain

$$
\mathcal{D}_{4}=\partial_{t}+\partial_{t} \log \left(I_{0} \beta\right)
$$

It follows that

$$
\left(I_{0} \beta\right) \circ \mathcal{D}_{4}=\partial_{t} \circ\left(I_{0} \beta\right)
$$

According to (61), we obtain

$$
I_{0} \mathcal{L}_{\text {hyp }}=\partial_{t} \circ\left(I_{0} \beta\right) \circ \mathcal{D}_{3} \circ \mathcal{D}_{2} \circ \mathcal{D}_{1}
$$

Proposition C.1. For any solution $I_{k}, k=0,1,2,3$ to the Picard-Fuchs equation, one has

$$
\int I_{0} \partial_{t}^{4} I_{k}=\left(I_{0} \beta\right) \circ \mathcal{D}_{3} \circ \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{k}+C_{k}
$$

for some constant $C_{k}$.
Proof. Combining Lemma C. 1 and Lemma C.2, we obtain

$$
I_{0} \partial_{t}^{4} I_{k}=I_{0} \mathcal{L} \circ \partial_{t} I_{k}=\partial_{t} \circ\left(I_{0} \beta\right) \circ \mathcal{D}_{3} \circ \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{k}
$$

The claim then follows.

Proposition C.2. For any solution $I_{k}, k=0,1,2,3$ to the Picard-Fuchs equation, one has

$$
\int T^{\prime} \int I_{0} \partial_{t}^{4} I_{k}=\left(I_{0} \beta T^{\prime}\right) \circ \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{k}+C_{k} T+D_{k}
$$

for some constants $C_{k}, D_{k}$.
Proof. Since

$$
I_{0} \beta T^{\prime} \mathcal{D}_{3}=I_{0} \beta T^{\prime} \circ\left(\partial_{t}+\partial_{t} \log \left(\beta I_{0} T^{\prime}\right)\right)=\partial_{t} \circ\left(I_{0} \beta T^{\prime}\right)
$$

the claim then follows from Proposition C.1.
For simplicity in what follows we shall omit the integration constants, equipped with the known boundary conditions provided in (41),(42),(43).

By integration by parts, one has
Proposition C.3. One has

$$
\int I_{k} I_{0}^{\prime \prime \prime \prime}=\int J_{k} I_{0} I_{0}^{\prime \prime \prime \prime}=J_{k} \int I_{0} I_{0}^{\prime \prime \prime \prime}-\int J_{k}^{\prime} \int I_{0} I_{0}^{\prime \prime \prime \prime}, \quad J_{k}=\frac{I_{k}}{I_{0}}
$$

In particular, when $k=1$, this is Proposition C.2. When $k=2$, one has

$$
\begin{aligned}
\int J_{k}^{\prime} \int I_{0} I_{0}^{\prime \prime \prime \prime} & =\int F_{T T} \cdot T^{\prime} \int I_{0} I_{0}^{\prime \prime \prime \prime}=F_{T T} \cdot \int T^{\prime} \int I_{0} I_{0}^{\prime \prime \prime \prime}-\int F_{T T}^{\prime} \int T^{\prime} \int I_{0} I_{0}^{\prime \prime \prime \prime} \\
& =F_{T T} \cdot\left(T^{\prime} I_{0} \beta\right) \mathcal{D}_{2} \circ \mathcal{D}_{1} \partial_{t} I_{0}-\left(F_{T T T} T^{\prime 2} I_{0} \beta\right) \mathcal{D}_{1} \partial_{t} I_{0}
\end{aligned}
$$

C.1.2. Anti-derivatives of degree four differential polynomials of periods. One can now obtain the antiderivatives of degree four differential polynomials of periods. We now compute some which shall be used later. We have

$$
\begin{align*}
& \int I_{0}^{\prime \prime} I_{0}^{\prime \prime}=I_{0}^{\prime \prime} I_{0}^{\prime}-I_{0}^{\prime \prime \prime} I_{0}+\int I_{0}^{\prime \prime \prime \prime} I_{0}=I_{0}^{\prime \prime} I_{0}^{\prime}-I_{0}^{\prime \prime \prime} I_{0}+\left(I_{0} \beta\right) \mathcal{D}_{3} \circ \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{0}  \tag{64}\\
& \int I_{1}^{\prime \prime} I_{0}^{\prime \prime}=I_{0}^{\prime} I_{1}^{\prime \prime}-I_{0} I_{1}^{\prime \prime \prime}+\int I_{0} I_{1}^{\prime \prime \prime}=I_{0}^{\prime} I_{1}^{\prime \prime}-I_{0} I_{1}^{\prime \prime \prime}+\left(I_{0} \beta\right) \mathcal{D}_{3} \circ \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{1} \tag{65}
\end{align*}
$$

For the anti-derivative of $I_{1}^{\prime \prime} I_{1}^{\prime \prime}$, one has

$$
\begin{align*}
\int I_{1}^{\prime \prime} I_{1}^{\prime \prime} & =I_{1}^{\prime \prime} I_{1}^{\prime}-I_{1}^{\prime \prime \prime} I_{1}+\int T I_{1}^{\prime \prime \prime \prime} I_{0}=I_{1}^{\prime \prime} I_{1}^{\prime}-I_{1}^{\prime \prime \prime} I_{1}+T \int I_{0} I_{1}^{\prime \prime \prime \prime}-\int T^{\prime} \int I_{0} I_{1}^{\prime \prime \prime \prime} \\
& =I_{1}^{\prime \prime} I_{1}^{\prime}-I_{1}^{\prime \prime \prime} I_{1}+T\left(I_{0} \beta\right) \mathcal{D}_{3} \circ \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{1}-\left(T^{\prime} I_{0} \beta\right) \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{1} \tag{66}
\end{align*}
$$

C.1.3. Relations arising from integration by parts. There are sometimes multiple ways to compute the anti-derivatives. One can simpliy apply PropositionC.1 or alternatively combine PropositionC. 1 and Proposition C.3. For example, we have besides (65) the following

$$
\begin{aligned}
\int I_{1}^{\prime \prime} I_{0}^{\prime \prime} & =I_{0}^{\prime \prime} I_{1}^{\prime}-I_{0}^{\prime \prime \prime} I_{1}+\int I_{1} I_{0}^{\prime \prime \prime \prime}=I_{0}^{\prime \prime} I_{1}^{\prime}-I_{0}^{\prime \prime \prime} I_{1}+\int T I_{0} I_{0}^{\prime \prime \prime \prime} \\
& =I_{0}^{\prime \prime} I_{1}^{\prime}-I_{0}^{\prime \prime \prime} I_{1}+T \int I_{0} I_{0}^{\prime \prime \prime \prime}-\int T^{\prime} \int I_{0} I_{0}^{\prime \prime \prime} \\
& =I_{0}^{\prime \prime} I_{1}^{\prime}-I_{0}^{\prime \prime \prime} I_{1}+T\left(I_{0} \beta\right) \mathcal{D}_{3} \circ \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{0}-\left(T^{\prime} I_{0} \beta\right) \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{0}
\end{aligned}
$$

Taking the difference between these two equations, we have

$$
\begin{aligned}
& I_{0}^{\prime} I_{1}^{\prime \prime}-I_{0} I_{1}^{\prime \prime \prime}-\left(I_{0}^{\prime \prime} I_{1}^{\prime}-I_{0}^{\prime \prime \prime} I_{1}\right)=\int\left(I_{1} I_{0}^{\prime \prime \prime \prime}-I_{1}^{\prime \prime \prime \prime} I_{0}\right) \\
= & T\left(I_{0} \beta\right) \mathcal{D}_{3} \circ \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{0}-\left(T^{\prime} I_{0} \beta\right) \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{0}-\left(I_{0} \beta\right) \mathcal{D}_{3} \circ \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{1} .
\end{aligned}
$$

Similarly, in computing the anti-derivative of $I_{1}^{\prime \prime \prime} I_{0}^{\prime}$, one can get the same identity. While directly computing the right hand side in the above seems complicated, the left hand side is easy. That is, by integration by parts and the integration formulas in Proposition C. 1 and Proposition C.3, we can easily produce nontrivial relations.
C.2. Proof of an identity. The antiderivatives of $I_{j}^{(a)} I_{k}^{(b)}$ 's in previous section admits an application in packaging loop contribution in $g=1 \mathrm{MSP}$. We shall prove the following identity.

Proposition C.4. In the following, $I_{j}=I_{j}(y)$ in integrand and' means derivative with respect to $y$.

$$
\begin{aligned}
& -\frac{1}{2} \partial_{T} J_{2} \int_{-\infty}^{t}\left(I_{1}^{(2)}-T I_{0}^{(2)}\right)^{2} d y-J_{2}\left(\int_{-\infty}^{t}\left(I_{1}^{(2)}-T I_{0}^{(2)}\right) I_{0}^{(2)}-\left(I_{1}^{(3)}-T I_{0}^{(3)}\right) I_{0}^{(1)} d y\right) \\
& -\left(\partial_{T}^{-1} J_{2}\right) \int_{-\infty}^{t}\left(I_{0}^{(2)}\right)^{2}-2 I_{0}^{(3)} I_{0}^{(1)} d y+\partial_{t}^{-2}\left(\left(I_{3}^{\prime \prime \prime}-t I_{2}^{\prime \prime \prime}\right)^{\prime} I_{0}^{\prime}+I_{2}^{\prime \prime \prime}\left(I_{1}^{\prime}-t I_{0}^{\prime}\right)^{\prime}\right)-(T-t) \partial_{t}^{-1}\left(I_{2}^{\prime \prime \prime} I_{0}^{\prime}\right) \\
& =\frac{1}{2} \frac{T^{\prime \prime}}{T^{\prime}}+2 \frac{I_{0}^{\prime}}{I_{0}}+\frac{\ln \left(1-5^{5} e^{t}\right)}{5}
\end{aligned}
$$

By (42), we have $J_{2}=I_{2} / I_{0}=F_{T}$ and $\partial_{T} J_{2}=F_{T T}, \partial_{T}^{-1} J_{2}=F$. By (40) and (50), the R.H.S. of the above identity is $\frac{1}{2} A+2 B+\frac{1}{5} \ln C_{t t t}^{-1}$.

The rest of this section will be devoted to proving Proposition C. 4 In what follows, we denote the left and right hand sides of the identity in Proposition C. 4 by $L H S, R H S$ respectively.
C.2.1. First simplification: anomalous term. We first focus on the anomalous term on the left hand side:

$$
\begin{aligned}
& \iint\left(I_{3}^{\prime \prime \prime}(y)-t I_{2}^{\prime \prime \prime}(y)\right)^{\prime} I_{0}^{\prime}(y) d y+\iint I_{2}^{\prime \prime \prime}(y)\left(I_{1}^{\prime}(y)-t I_{0}^{\prime}(y)\right)^{\prime} d y-(T-t) \int^{t} I_{2}^{\prime \prime \prime}(y) I_{0}^{\prime}(y) d y \\
= & \iint\left(I_{3}^{\prime \prime \prime \prime}(y) I_{0}^{\prime}(y)+I_{2}^{\prime \prime \prime}(y) I_{1}^{\prime \prime}(y)\right) d y-T \int^{t} I_{2}^{\prime \prime \prime}(y) I_{0}^{\prime}(y) d y
\end{aligned}
$$

By integration by parts, one has

$$
\int I_{3}^{\prime \prime \prime \prime} I_{0}^{\prime}=I_{3}^{\prime \prime \prime} I_{0}^{\prime}-\int I_{3}^{\prime \prime \prime} I_{0}^{\prime \prime}
$$

This gives

$$
\begin{aligned}
& \iint\left(I_{3}^{\prime \prime \prime \prime}(y) I_{0}^{\prime}(y)+I_{2}^{\prime \prime \prime}(y) I_{1}^{\prime \prime}(y)\right) d y-T \int^{t} I_{2}^{\prime \prime \prime}(y) I_{0}^{\prime}(y) d y \\
= & \iint\left(-I_{3}^{\prime \prime \prime}(y) I_{0}^{\prime \prime}(y)+I_{2}^{\prime \prime \prime}(y) I_{1}^{\prime \prime}(y)\right) d y+\int^{t} I_{3}^{\prime \prime \prime}(y) I_{0}^{\prime}(y) d y-T \int^{t} I_{2}^{\prime \prime \prime}(y) I_{0}^{\prime}(y) d y
\end{aligned}
$$

Now applying integration by parts again, we get

$$
\int I_{3}^{\prime \prime \prime} I_{0}^{\prime \prime}=I_{3}^{\prime \prime} I_{0}^{\prime \prime}-\int I_{3}^{\prime \prime} I_{0}^{\prime \prime \prime}
$$

Hence

$$
\int I_{3}^{\prime \prime \prime} I_{0}^{\prime \prime}=\frac{1}{2} I_{3}^{\prime \prime} I_{0}^{\prime \prime}+\frac{1}{2} \int\left(I_{3}^{\prime \prime \prime} I_{0}^{\prime \prime}-I_{3}^{\prime \prime} I_{0}^{\prime \prime \prime}\right)
$$

Similarly,

$$
\int I_{2}^{\prime \prime \prime} I_{1}^{\prime \prime}=\frac{1}{2} I_{2}^{\prime \prime} I_{1}^{\prime \prime}+\frac{1}{2} \int\left(I_{2}^{\prime \prime \prime} I_{1}^{\prime \prime}-I_{2}^{\prime \prime} I_{1}^{\prime \prime \prime}\right)
$$

Therefore,

$$
\int\left(-I_{3}^{\prime \prime \prime} I_{0}^{\prime \prime}+I_{2}^{\prime \prime \prime} I_{1}^{\prime \prime}\right)=-\frac{1}{2} I_{3}^{\prime \prime} I_{0}^{\prime \prime}+\frac{1}{2} I_{2}^{\prime \prime} I_{1}^{\prime \prime}-\frac{1}{2} \int\left(I_{3}^{\prime \prime \prime} I_{0}^{\prime \prime}-I_{3}^{\prime \prime} I_{0}^{\prime \prime \prime}\right)+\frac{1}{2} \int\left(I_{2}^{\prime \prime \prime} I_{1}^{\prime \prime}-I_{2}^{\prime \prime} I_{1}^{\prime \prime \prime}\right)
$$

To compute

$$
-\frac{1}{2}\left(I_{3}^{\prime \prime \prime} I_{0}^{\prime \prime}-I_{3}^{\prime \prime} I_{0}^{\prime \prime \prime}\right)+\frac{1}{2}\left(I_{2}^{\prime \prime \prime} I_{1}^{\prime \prime}-I_{2}^{\prime \prime} I_{1}^{\prime \prime \prime}\right)=-\frac{1}{2} u_{6}^{(03)}+\frac{1}{2} u_{6}^{(12)}
$$

we use the Wronskian method discussed in Section B.2.1 above. Here $u_{6}^{(03)}$ means $u_{6}$ with $(a, b)=(0,3)$ and $u_{6}^{(12)}$ is similar. We recall that the two $u_{1}$ 's are equal, hence so are

$$
u_{2}=u_{1}^{\prime}, \quad u_{2}^{\prime}=u_{3}+u_{4}
$$

and

$$
U=a_{3} u_{4}+2 u_{5}, \quad C_{1} u_{4}+2 a_{0} u_{1}+2 u_{6}
$$

where

$$
C_{1}=a_{3}^{\prime}+\frac{1}{2} a_{3}^{2}-2 a_{2}=\frac{4}{5} C
$$

The difference

$$
\Delta u_{6}=u_{6}^{(03)}-u_{6}^{(12)}=M_{03}-M_{12}
$$

can be found by using the Wronskian method as follows. Observe that all of the equations involved are linear. To figure out $\Delta u_{6}$, we recall

$$
u_{6}^{\prime}=a_{0} u_{2}+a_{1} u_{4}-a_{3} u_{6}
$$

This gives

$$
\Delta\left(C_{1} u_{4}+2 a_{0} u_{1}+2 u_{6}\right)=C_{1} \Delta u_{4}+2 \Delta u_{6}=0
$$

and

$$
\Delta\left(u_{6}^{\prime}-a_{0} u_{2}-a_{1} u_{4}+a_{3} u_{6}\right)=\left(\Delta u_{6}\right)^{\prime}-a_{1}\left(\Delta u_{4}\right)+a_{3} \Delta u_{6}=0
$$

It follows that

$$
\left(\Delta u_{6}\right)^{\prime}+a_{1} \frac{2}{C_{1}} \Delta u_{6}+a_{3} \Delta u_{6}=0
$$

Plugging in $C_{1}=\frac{4}{5} C, a_{1}=-\frac{2}{5} C, a_{3}=-2 C$, we obtain

$$
\Delta u_{6}=k C^{\prime}=k C(C+1)
$$

for some constant $k$. This constant can be fixed by looking at the boundary conditions of $-\frac{1}{2}\left(I_{3}^{\prime \prime \prime} I_{0}^{\prime \prime}-\right.$ $\left.I_{3}^{\prime \prime} I_{0}^{\prime \prime \prime}\right)+\frac{1}{2}\left(I_{2}^{\prime \prime \prime} I_{1}^{\prime \prime}-I_{2}^{\prime \prime} I_{1}^{\prime \prime \prime}\right)$ obtained by the first few terms of the series obeying (42) and (43). This fixes $k=-\frac{1}{5}$.

Remark C.5. To minimize the computations, we can also prove the identity as follows. Note that the same reasoning as above tells us that

$$
\begin{gathered}
\Delta u_{3}+\Delta u_{4}=0 \\
\left(\Delta u_{3}\right)^{\prime}=\left(\Delta u_{4}\right)^{\prime}-a_{3} \Delta u_{3}=\left(-\Delta u_{3}\right)^{\prime}-a_{3} \Delta u_{3} .
\end{gathered}
$$

Solving the latter differential equation, we see that $\Delta u_{3}=a / \beta$ for some constant $a$. Then we get $\Delta u_{4}=-a / \beta$ and hence

$$
\Delta u_{6}=-\frac{C_{1}}{2} \Delta u_{4}=\frac{C_{1}}{2} \frac{a}{\beta}=\frac{2}{5} C \frac{a}{\beta}
$$

Hence the constant $k$ in the previous discussion is related to a by $k=\frac{2}{5}$ a. But now computing $\Delta u_{3}$ or $\Delta u_{4}$ is much easier: one has

$$
\begin{aligned}
\Delta u_{3} & =-\frac{1}{2}\left(I_{3}^{\prime \prime \prime} I_{0}-I_{3} I_{0}^{\prime \prime \prime}\right)+\frac{1}{2}\left(I_{2}^{\prime \prime \prime} I_{1}-I_{2} I_{1}^{\prime \prime \prime}\right) \\
\Delta u_{4} & =-\frac{1}{2}\left(I_{3}^{\prime \prime} I_{0}^{\prime}-I_{3}^{\prime} I_{0}^{\prime \prime}\right)+\frac{1}{2}\left(I_{2}^{\prime \prime} I_{1}^{\prime}-I_{2}^{\prime} I_{1}^{\prime \prime}\right)
\end{aligned}
$$

Due to the asymptotic behaviour of $1 / \beta$, this amounts to computing the constant term. We consider $\Delta u_{4}=-a / \beta=-a+\mathcal{O}(q)$ for simplicity. Now knowing that the first term $I_{0}$ is 1 from (41) is enough to fix $-a=1 / 2$. This gives $k=-1 / 5$.

Therefore, we obtain

$$
\int\left(-I_{3}^{\prime \prime \prime} I_{0}^{\prime \prime}+I_{2}^{\prime \prime \prime} I_{1}^{\prime \prime}\right)=-\frac{1}{2} I_{3}^{\prime \prime} I_{0}^{\prime \prime}+\frac{1}{2} I_{2}^{\prime \prime} I_{1}^{\prime \prime}-\frac{1}{5} C
$$

It follows then the anomalous term considered above is given by

$$
\int\left(-\frac{1}{2} I_{3}^{\prime \prime}(y) I_{0}^{\prime \prime}(y)+\frac{1}{2} I_{2}^{\prime \prime}(y) I_{1}^{\prime \prime}(y)\right) d y+\int^{t} I_{3}^{\prime \prime \prime}(y) I_{0}^{\prime}(y) d y-T \int^{t} I_{2}^{\prime \prime \prime}(y) I_{0}^{\prime}(y) d y
$$

Now the left hand side of the desired identity in Proposition C. 4 is simplified into

$$
\begin{aligned}
L H S= & -\frac{1}{2} F_{T T} \int^{t}\left(I_{1}^{\prime \prime}(y)-T(t) I_{0}^{\prime \prime}(y)\right)^{2} d y \\
& -F_{T} \int^{t}\left(I_{1}^{\prime \prime}(y)-T(t) I_{0}^{\prime \prime}(y)\right) I_{0}^{\prime \prime}(y) d y+F_{T} \int^{t}\left(I_{1}^{\prime \prime \prime}(y)-T(t) I_{0}^{\prime \prime \prime}(y)\right) I_{0}^{\prime}(y) d y \\
& -F \int^{t} I_{0}^{\prime \prime}(y)^{2} d y+2 F \int^{t} I_{0}^{\prime \prime \prime}(y) I_{0}^{\prime}(y) d y \\
& +\int\left(-\frac{1}{2} I_{3}^{\prime \prime} I_{0}^{\prime \prime}+\frac{1}{2} I_{2}^{\prime \prime} I_{1}^{\prime \prime}\right)^{\prime}+\int^{t} I_{3}^{\prime \prime \prime} I_{0}^{\prime}-T \int^{t} I_{2}^{\prime \prime \prime} I_{0}^{\prime}+\int-\frac{1}{5} C \\
= & -\frac{1}{2} F_{T T} \int^{t}\left(I_{1}^{\prime \prime}(y)-T(t) I_{0}^{\prime \prime}(y)\right)^{2} d y \\
& -2 F_{T} \int^{t}\left(I_{1}^{\prime \prime}(y)-T(t) I_{0}^{\prime \prime}(y)\right) I_{0}^{\prime \prime}(y) d y-3 F \int^{t} I_{0}^{\prime \prime}(y)^{2} d y \\
& +F_{T}\left(I_{1}^{\prime \prime} I_{0}^{\prime}-T I_{0}^{\prime \prime} I_{0}^{\prime}\right)+2 F I_{0}^{\prime \prime} I_{0}^{\prime} \\
& +\int\left(-\frac{1}{2} I_{3}^{\prime \prime} I_{0}^{\prime \prime}+\frac{1}{2} I_{2}^{\prime \prime} I_{1}^{\prime \prime}\right)+\int^{t} I_{3}^{\prime \prime \prime} I_{0}^{\prime}-T \int^{t} I_{2}^{\prime \prime \prime} I_{0}^{\prime}+\int-\frac{1}{5} C
\end{aligned}
$$

C.2.2. Taking the derivative. We can apply Proposition C.1. Proposition C. 3 and integral by parts to simply the single integrals involving $I_{0}$ in the quantity $L H S$. But the double integral seems to be out of reach. Both this and the $\log C_{t t t}^{-1}=\int C$ term on the right hand side of the expected identity in Proposition C. 4 suggest that one takes the derivative of the identity.

The derivative is then computed to be

$$
\partial_{t} L H S=-\frac{1}{2} F_{T T T} T^{\prime} \int^{t}\left(I_{1}^{\prime \prime}(y)-T(t) I_{0}^{\prime \prime}(y)\right)^{2} d y
$$

$$
\begin{aligned}
& -F_{T T} T^{\prime} \int^{t}\left(I_{1}^{\prime \prime}(y)-T(t) I_{0}^{\prime \prime}(y)\right) I_{0}^{\prime \prime}(y) d y-F_{T} T^{\prime} \int^{t} I_{0}^{\prime \prime}(y)^{2} d y \\
& -T^{\prime} \int I_{2}^{\prime \prime \prime} I_{0}^{\prime}-\frac{1}{2} F_{T T}\left(I_{1}^{\prime \prime}-T I_{0}^{\prime \prime}\right)^{2}-F_{T}\left(I_{1}^{\prime \prime}-T I_{0}^{\prime \prime}\right) I_{0}^{\prime \prime}-F I_{0}^{\prime \prime 2}+I_{0}^{\prime}\left(F_{T}\left(I_{1}^{\prime \prime} I_{0}^{\prime}-T I_{0}^{\prime \prime}\right)+2 F I_{0}^{\prime \prime}\right)^{\prime} \\
& +\left(-\frac{1}{2} I_{3}^{\prime \prime} I_{0}^{\prime \prime}+\frac{1}{2} I_{2}^{\prime \prime} I_{1}^{\prime \prime}\right)+I_{3}^{\prime \prime \prime} I_{0}^{\prime}-T I_{2}^{\prime \prime \prime} I_{0}^{\prime}-\frac{1}{5} C .
\end{aligned}
$$

After plugging in (64), (65), (66), the first two terms above reduce to expressions only involving $\mathcal{D}_{2}, \mathcal{D}_{1}$ thanks to Proposition C. 1 Proposition C.3

$$
\begin{aligned}
\int\left(I_{1}^{\prime \prime}(y)-T(t) I_{0}^{\prime \prime}(y)\right)^{2} d y & =T^{\prime} I_{0}\left(I_{1}^{\prime \prime}-T I_{0}^{\prime \prime}\right)-\left(T^{\prime} I_{0} \beta\right) \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{1}+T\left(T^{\prime} I_{0} \beta\right) \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{0} \\
\int\left(I_{1}^{\prime \prime}(y)-T(t) I_{0}^{\prime \prime}(y)\right) I_{0}^{\prime \prime}(y) d y & =I_{0} I_{0}^{\prime \prime} T^{\prime}-\left(T^{\prime} I_{0} \beta\right) \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{0}
\end{aligned}
$$

We also apply integration by parts, Proposition C. 1 and Proposition C. 3 to simplify the other integrals. This then gives a differential polynomial in the generators $A, B, C$ displayed in (50). Below are the details.

$$
\begin{aligned}
-F_{T} T^{\prime} \int^{t} I_{0}^{\prime \prime}(s)^{2} d s= & -F_{T} T^{\prime}\left(I_{0}^{\prime \prime} I_{0}^{\prime}-I_{0}^{\prime \prime \prime} I_{0}\right)-F_{T} T^{\prime}\left(I_{0} \beta\right) \mathcal{D}_{3} \circ \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{0} \\
-T^{\prime} \int I_{2}^{\prime \prime \prime} I_{0}^{\prime}= & T^{\prime}\left[-I_{2}^{\prime \prime} I_{0}^{\prime}+I_{2}^{\prime} I_{0}^{\prime \prime}-I_{2} I_{0}^{\prime \prime \prime}+F_{T}\left(I_{0} \beta\right) \mathcal{D}_{3} \circ \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{0}\right. \\
& \left.-F_{T T}\left(T^{\prime} I_{0} \beta\right) \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{0}+\left(F_{T T T} T^{\prime 2} I_{0} \beta\right) \mathcal{D}_{1} \circ \partial_{t} I_{0}\right]
\end{aligned}
$$

C.2.3. Completion of the proof of Proposition C.4. Plugging in all of the above expressions for the anti-derivatives, we obtain

$$
\begin{aligned}
\partial_{t} L H S= & -\frac{1}{2} F_{T T T} T^{\prime} T^{\prime} I_{0}\left(I_{1}^{\prime \prime}-T I_{0}^{\prime \prime}\right)-F_{T T} T^{\prime} I_{0} I_{0}^{\prime \prime} T^{\prime}-F_{T} T^{\prime}\left(I_{0}^{\prime \prime} I_{0}^{\prime}-I_{0}^{\prime \prime \prime} I_{0}\right) \\
& -\frac{1}{2} F_{T T}\left(I_{1}^{\prime \prime}-T I_{0}^{\prime \prime}\right)^{2}-F_{T}\left(I_{1}^{\prime \prime}-T I_{0}^{\prime \prime}\right) I_{0}^{\prime \prime}-F I_{0}^{\prime \prime 2} \\
& +I_{0}^{\prime} F_{T T} T^{\prime}\left(I_{1}^{\prime \prime}-T I_{0}^{\prime \prime}\right)+I_{0}^{\prime} F_{T}\left(I_{1}^{\prime \prime}-T I_{0}^{\prime \prime}\right)^{\prime}+2 F_{T} T^{\prime} I_{0}^{\prime} I_{0}^{\prime \prime}+2 F I_{0}^{\prime} I_{0}^{\prime \prime \prime} \\
& +T^{\prime}\left(I_{2}^{\prime \prime \prime} I_{0}-I_{2}^{\prime \prime} I_{0}^{\prime}+I_{2}^{\prime} I_{0}^{\prime \prime}-I_{2} I_{0}^{\prime \prime}\right)+\left(-\frac{1}{2} I_{3}^{\prime \prime} I_{0}^{\prime \prime}+\frac{1}{2} I_{2}^{\prime \prime} I_{1}^{\prime \prime}\right)+\left(I_{3}^{\prime \prime \prime} I_{0}^{\prime}-I_{2}^{\prime \prime \prime} I_{1}^{\prime}\right) \\
& +\frac{1}{2} F_{T T T} T^{\prime}\left(T^{\prime} I_{0} \beta\right) \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{1} \\
& -\frac{1}{2} F_{T T T} T^{\prime} T\left(T^{\prime} I_{0} \beta\right) \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{0}+T^{\prime}\left(F_{T T T} T^{2} I_{0} \beta\right) \mathcal{D}_{1} \circ \partial_{t} I_{0}-\frac{1}{5} C \\
= & 0+\frac{1}{2} F_{T T T} T^{\prime}\left(T^{\prime} I_{0} \beta\right) \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{1}-\frac{1}{2} F_{T T T} T^{\prime} T\left(T^{\prime} I_{0} \beta\right) \mathcal{D}_{2} \circ \mathcal{D}_{1} \circ \partial_{t} I_{0} \\
& +T^{\prime}\left(F_{T T T} T^{\prime 2} I_{0} \beta\right) \mathcal{D}_{1} \circ \partial_{t} I_{0}-\frac{1}{5} C .
\end{aligned}
$$

Keeping simplifying, we obtain

$$
\begin{aligned}
\partial_{t} L H S= & +\frac{1}{2} F_{T T T} T^{\prime}\left(T^{\prime} I_{0} \beta\right)\left[T^{\prime \prime \prime} I_{0}+3 T^{\prime \prime} I_{0}^{\prime}+3 T^{\prime} I_{0}^{\prime \prime}\right. \\
& \left.-(A+2 B)\left(T^{\prime \prime} I_{0}+2 T^{\prime} I_{0}^{\prime}\right)+\left(A B+B^{2}-B^{\prime}\right) T^{\prime} I_{0}\right] \\
& +T^{\prime}\left(F_{T T T} T^{\prime 2} \beta\right)\left(I_{0}^{\prime \prime} I_{0}-I_{0}^{\prime} I_{0}^{\prime}\right)-\frac{1}{5} C \\
= & \left(\frac{1}{2} A^{\prime}+B^{\prime}\right)+B^{\prime}-\frac{1}{5} C .
\end{aligned}
$$

This is identical to

$$
\partial_{t} R H S=\partial_{t}\left(\frac{1}{2} A+2 B+\frac{1}{5} \ln C_{t t t}^{-1}\right)=\frac{1}{2} A^{\prime}+2 B^{\prime}-\frac{1}{5} C .
$$

The identity $L H S=R H S$ then follows from the above equation $\partial_{t} L H S=\partial_{t} R H S$, and the fact that they have the same boundary conditions which follow from (41), (42) and (43).

## References

[AZ] G. Almkvist and W. Zudilin, Differential equations, mirror maps and zeta values, Mirror symmetry V, 481-515 (2006).
[BCOV] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Holomorphic Anomalies in Topological Field Theories, Nucl.Phys. B 405 279-304 (1993); Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitudes, Comm. Math. Phys. Volume 165, no. 2, 311-427 (1994).
[CDFLL] A. Ceresole, R. D'Auria, S. Ferrara, W. Lerche, and J. Louis, Picard-Fuchs equations and special geometry, Int.J.Mod.Phys. A8 79-114 (1993).
[CDFLLR] A. Ceresole, R. D'Auria, S. Ferrara, W. Lerche, J. Louis, and T. Regge, Picard-Fuchs equations, special geometry and target space duality, Mirror symmetry II, 281-353 (1997).
[CJ] A. Collino, M. Jinzenji, "On the structure of the small quantum cohomology rings of projective hypersurfaces," Comm. Math. Phys. 206, no. 1, 157-183 (1999).
[CK1] I. Ciocan-Fontanine and B. Kim, Moduli stacks of stable toric quasimaps, Advances in Math. 225, no. 6, 3022-3051 (2010).
[CK2] I. Ciocan-Fontanine and B. Kim, Quasimap Wall-crossings and Mirror Symmetry, arXiv:1611.05023.
[CL] H.-L. Chang and J. Li, Gromov-Witten invariants of stable maps with fields, Int. Math. Res. Not. 18, 4163-4217 (2012).
[CL1] H.-L. Chang and Jun Li, An algebraic proof of the hyperplane property of the genus one $G W$-invariants of quintics, J. Diff. Geom. 100, no. 2, p 251-299 (2015).
[CLL] H.-L. Chang, J. Li, W.-P. Li, Witten's top Chern classes via cosection localization, Invent. Math. 200, no 3, 1015-1063 (2015).
[CLLL1] H.-L. Chang, J. Li, W.-P. Li, C.-C. Melissa Liu, Mixed-Spin-P fields of Fermat quintic polynomials, math.AG. arXiv:1505.07532.
[CLLL2] H.-L. Chang, J. Li, W.-P. Li, C.-C. Melissa Liu, An effective theory of GW and FJRW invariants of quintics Calabi- Yau manifolds, arXiv:1603.06184 (2016).
[CLLL3] H.-L. Chang, J. Li, W.-P. Li, C.-C. Melissa Liu, A survey on mixed spin P-fields, Chin. Ann. Math. Ser. B 38, no. 4, 869-882 (2017).
[F] D. Freed, Special Kähler manifolds, Comm. Math. Phys. 203, no. 1, 31-52 (1999).
[FJR1] H.-J. Fan, T. J. Jarvis, Y.-B. Ruan, The Witten equation, mirror symmetry, and quantum singularity theory, Ann. of Math (2) 178, no. 1, 1-106 (2013).
[FJR2] H-J Fan, T. J. Jarvis and Y.-B. Ruan, The Witten equation and its virtual fundamental cycle, math.AG. arXiv:0712.4025.
[Gi1] A. Givental, The mirror formula for quintic threefolds. Northern California Symplectic Geometry Seminar, 49-62, Amer. Math. Soc. Transl. Ser. 2, 196, Amer. Math. Soc., Providence, RI, (1999).
[Gi2] A. Givental, "Equivariant Gromov-Witten invariants," math.berkeley.edu/ giventh/papers/eqv.pdf.
[GP] T. Graber, R. Pandharipande, Localization of virtual classes, Invent. Math. 135, no. 2, 487-518 (1999).
$[\mathrm{H}] \quad \mathrm{S}$. Hosono, BCOV ring and holomorphic anomaly equation, Adv.Stud.Pure Math. 59, 79-110 (2008).
[KL] Y.H. Kiem and J. Li, Localized virtual cycle by cosections, J. Amer. Math. Soc. 26, no. 4, 1025-1050 (2013).
[KLh] B. Kim and H. Lho, Mirror Theorem for Elliptic Quasimap Invariants, arXiv:1506.03196.
[LY] B. Lian and S.-T. Yau, Arithmetic properties of mirror map and quantum coupling, Commun.Math.Phys. 176, 163-192 (1996).
[LZ] J. Li and A. Zinger, On the Genus-One Gromov-Witten Invariants of Complete Intersections, J. Diff. Geom. 82, no. 3, 641-690 (2009).
[LLY] B. Lian, K.F. Liu and S.-T. Yau, Mirror principle. I Surveys in differential geometry: differential geometry inspired by string theory, 405-454, Surv. Differ. Geom. 5, Int. Press, Boston, MA, (1999).
[MP] D. Maulik, and R. Pandharipande, A topological view of Gromov-Witten theory, Topology 45, no. 5, 887-918 (2006).
[P] R. Pandharipande, "Rational curves on hypersurfaces [after A. Givental]," Séminaire Bourbaki 848, 50ème année, 1997-1998.
[PZ] A. Popa and A. Zinger, "Mirror symmetry for closed, open, and unoriented Gromov-Witten invariants," Adv. Math. 259, 448-510 (2014).
[S] A. Strominger, Special Geometry, Commun.Math.Phys. 133, 163-180 (1990) .
[VZ] R. Vakil and A. Zinger, A Desingularization of the Main Component of the Moduli Space of Genus-One Stable Maps into $\mathbb{P}^{n}$, Geom. Topol. 12, no. 1, 1-95 (2008).
[YY] S. Yamaguchi and S.-T. Yau, Topological string partition functions as polynomials, JHEP (2004) no.7, 047, 20pp.
[Z] J. Zhou, Differential Rings from Special Kähler Geometry, arXiv:1310.3555 [hep-th].
[Zi2] A. Zinger "The Reduced Genus One Gromov-Witten Invariants of Calabi-Yau Hypersurfaces", JAMS. 22, no.3, 691-737 (2009).

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[^1]:    ${ }^{1}$ Hereafter ${ }^{\prime}$ means $\partial_{t}$ and similar convention is used for ${ }^{\prime \prime}$ etc.

