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Lipschitz equivalence of self-similar sets and hyperbolic boundaries

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Abstract

Kaimanovich (2003) [9] introduced the concept of *augmented tree* on the symbolic space of a self-similar set. It is hyperbolic in the sense of Gromov, and it was shown by Lau and Wang (2009) [12] that under the open set condition, a self-similar set can be identified with the hyperbolic boundary of the tree. In the paper, we investigate in detail a class of *simple* augmented trees and the Lipschitz equivalence of such trees. The main purpose is to use this to study the Lipschitz equivalence problem of the totally disconnected self-similar sets which has been undergoing some extensive development recently.

Keywords: Augmented tree; Hyperbolic boundary; Incidence matrix; Lipschitz equivalence; OSC; Primitive; Rearrangeable; Self-similar set; Self-affine set

1. Introduction

Two compact metric spaces (X, d_X) and (Y, d_Y) are said to be *Lipschitz equivalent*, and denote by $X \simeq Y$, if there is a bi-Lipschitz map σ from X onto Y, i.e., σ is a bijection and there is a constant $C > 0$ such that

$$C^{-1} d_X(x, y) \le d_Y(\sigma(x), \sigma(y)) \le C d_X(x, y) \quad \forall\, x, y \in X.$$

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It is easy to see that if $X \simeq Y$, then $\dim_H X = \dim_H Y = s$, where $\dim_H$ denotes the Hausdorff dimension. A more intensive study of this was due to Cooper and Pignartaro [2] in the late 80s, they showed that for certain Cantor sets X, Y on $\mathbb{R}$, the Lipschitz equivalence implies that there exists a bi-Lipschitz map σ and a $\lambda > 0$ such that $\mathcal{H}^s(\sigma(E)) = \lambda\mathcal{H}^s(E)$. In another consideration, Falconer and Marsh [5] proved that for quasi-self-similar circles, they are Lipschitz equivalent if and only if they have the same Hausdorff dimension.

The recent interest of the Lipschitz equivalence is due to the path breaking study of Rao, Ruan and Xi [18] on a question of David and Semmes [3] on certain special self-similar set on $\mathbb{R}$ to be dust-like (see Example 5.2). They observed a graph directed relationship in the underlying iterated function system (IFS), and made use of this to construct the needed bi-Lipschitz map. There is a number of generalizations on $\mathbb{R}$ and $\mathbb{R}^d$ for the totally disconnected self-similar sets with uniform contraction ratio or with logarithmically commensurable contraction ratios ([4,19,23–25]). For certain Cantor-type sets on $\mathbb{R}^d$ with non-equal contraction ratios, Rao, Ruan and Wang [17] had an elegant algebraic criterion for them to be Lipschitz equivalent, which improved a condition of Falconer and Marsh in [6]. Other considerations can be found in [15,16,22].

So far the investigation of the Lipschitz equivalence is very much restricted on a few special self-similar sets. In this paper, we will provide a broader framework to study the problem through the concept of *augmented (rooted) tree*. For an IFS $\{S_i\}_{i=1}^m$ of contractive similitudes on $\mathbb{R}^d$ (assume equal contraction ratio in the present situation) and the associated self-similar set K, we use $X = \bigcup_{n=0}^\infty \Sigma^n$, $\Sigma = \{1, \ldots, m\}$ to denote the symbolic space. Then X has a natural graph structure, and we denote the edge set by $\mathcal{E}_v$ (v for vertical). We define a horizontal edge for a pair $(\mathbf{u}, \mathbf{v})$ in $X \times X$ if $\mathbf{u}, \mathbf{v} \in \Sigma^n$ and $S_{\mathbf{u}}(K) \cap S_{\mathbf{v}}(K) \neq \emptyset$, and denote this set of edges by $\mathcal{E}_h$. The *augmented tree* is defined as the graph $(X, \mathcal{E})$ where $\mathcal{E} = \mathcal{E}_v \cup \mathcal{E}_h$.

Such augmented tree was first introduced by Kaimanovich [9] on the Sierpinski gasket in order to incorporate the intersection of the cells to the symbolic space, and was developed by Lau and Wang [12] to general self-similar sets. It was proved that if an IFS satisfies the open set condition (OSC), then the augmented tree is *hyperbolic* in the sense of Gromov. There is a hyperbolic metric ρ on X, which induces a *hyperbolic boundary* $(\partial X, \rho)$. The hyperbolic boundary is shown to be homeomorphic to K; moreover under certain mild condition, the homeomorphism is actually a Hölder equivalent map. This setup is used to study the random walks on $(X, \mathcal{E})$ and their Martin boundaries [13].

Based on this, our approach to the Lipschitz equivalence of the self-similar sets is to lift the consideration to the augmented trees $(X, \mathcal{E})$. We define a *horizontal connected component* of X to be the maximal connected horizontal subgraph T in some level Σ^n. Let $\mathscr{C}$ be the set of all horizontal connected components of X. For $T \in \mathscr{C}$, we use $T\Sigma$ to denote the set of offsprings of T, and consider $T \cup T\Sigma$ as a subgraph in X. We say that $T, T' \in \mathscr{C}$ are equivalent if they are graph isomorphic. We call X *simple* if there are finitely many equivalence classes. Under this condition, we can define an incidence matrix

$$A = [a_{ij}]$$

for the equivalence classes as follows: choose any component T belonging to the class $\mathscr{T}_i$, and let $Z_{i1}, \ldots, Z_{i\ell}$ be the connected components of the descendants of T. The entry a_{ij} denotes the number of Z_{ik} that belonging to the class $\mathscr{T}_j$.

It is shown that a simple augmented tree X is always hyperbolic, and the relationship of the hyperbolic boundary and the self-similar set is analogous to the case with OSC (Propositions 3.4 and 3.5). Our basic theorem, to put it into a simple statement, is (assuming the IFS has equal contraction ratio):

Theorem 1.1. *Suppose the augmented tree $(X, \mathcal{E})$ is simple and the corresponding incidence matrix A is primitive (i.e., $A^n > 0$ for some $n > 0$), then $\partial(X, \mathcal{E}) \simeq \partial(X, \mathcal{E}_v)$.*

We call a self-similar set K *dust-like* if it satisfies $S_i(K) \cap S_j(K) = \emptyset$ for $i \neq j$. By reducing the Lipschitz equivalence on the trees in Theorem 1.1 to the self-similar sets (Proposition 3.5, Theorem 3.10), we have

Theorem 1.2. (i) *If in addition to the condition in Theorem 1.1, the IFS satisfies some mild condition (H) (see Section 2), then K is Lipschitz equivalent to a dust-like self-similar set.*

(ii) *If K and K' are as in* (i) *and the two IFS's have the same number of similitudes and the same contraction ratio, then they are Lipschitz equivalent.*

The proof of Theorem 1.1 depends on constructing a *near-isometry* between the augmented tree $(X, \mathcal{E})$ and $(X, \mathcal{E}_v)$. The crux of the construction is to use a technique of *rearrangement* of edges (Section 4), which is based on an idea of Deng and He in [4]. Actually we prove a less restrictive form of Theorem 1.1 (Theorem 3.7) in terms of *rearrangeable matrices*. Theorem 1.1 follows from another theorem (Theorem 3.8) that the primitive property implies rearrangability.

We will provide an easy way to check an augmented tree being simple (Lemma 5.1), which is more efficient to apply to various examples than the graph directed systems that were used in the previous studies [4,18,24]. Theorem 1.2 essentially covers all the known cases so far, it also covers some new classes of IFS's that have overlaps and rotations (see Section 5). Moreover the theory can be extended from the self-similar IFS to the self-affine IFS: it is easy to see that the notion of augmented tree and the results on such tree remain unchanged. For self-affine sets on $\mathbb{R}^d$, we can still establish the Lipschitz equivalence, making use of a device in [8] which replaces the Euclidean distance by an ultra-metric adapted to the underlying self-affine system (see Theorem 3.14).

The paper is organized as follows. In Section 2, we recall some well-known results on hyperbolic graphs and set up the augmented tree. In Section 3, we introduce the notion of *simple* augmented tree, and derive its basic properties. Theorem 1.1 is stated there, and Theorem 1.2 together with other consequences is proved. The proof of Theorem 1.1 and the involved concept of rearrangement are given in Section 4. In Section 5, we provide several new examples to illustrate our results; some concluding remarks and open questions are given in Section 6.

2. Preliminaries

Let X be a countably infinite set, we say that X is a *graph* if it is associated with a symmetric subset $\mathcal{E}$ of $(X \times X) \setminus \{(x, x) : x \in X\}$; we call $x \in X$ a *vertex*, $(x, y) \in \mathcal{E}$ an *edge*, which is more conveniently denoted by $x \sim y$ (intuitively, x, y are neighborhoods to each other). By a *path* in X from x to y, we mean a finite sequence $x = x_0, x_1, \ldots, x_n = y$ such that $x_i \sim x_{i+1}$, $i = 0, \ldots, n-1$. We always assume that the graph X is connected, i.e., there is a path joining any two vertices $x, y \in X$. We call X a *tree* if the path between any two points is unique. We equip a graph X with an integer-valued metric $d(x, y)$, which is the minimum among the lengths of the paths from x to y; the corresponding geodesic path is denoted by $\pi(x, y)$ and its length by $|\pi(x, y)|(=d(x, y))$. Let $o \in X$ be a fixed point in X and call it the *root* of the graph. We use $|x|$ to denote $d(o, x)$, and say that x belongs to the n-th level of the graph if $d(o, x) = n$.

The notion of hyperbolic graph was introduced by Gromov [7,21]. First we define the Gromov product of any two points $x, y \in X$ by

$$|x \wedge y| = \frac{1}{2}(|x| + |y| - d(x, y)).$$

We say that X is *hyperbolic* (with respect to o) if there is $\delta > 0$ such that

$$|x \wedge y| \geq \min\{|x \wedge z|, |z \wedge y|\} - \delta \quad \forall\, x, y, z \in X.$$

Note that this is equivalent to a more geometric characterization: there exists a δ' such that for any three points in X, the geodesic triangle is δ'-*thin*: any point on one side of the triangle has distance less than δ' to some point on one of the other two sides.

For a fixed $a > 0$ with $a' = \exp(\delta a) - 1 < \sqrt{2} - 1$, we define an *ultra-metric* $\rho_a(\cdot, \cdot)$ on X by

$$\rho_a(x, y) = \begin{cases} \exp(-a|x \wedge y|) & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Then

$$\rho_a(x, y) \leq (1 + a') \max\{\rho_a(x, z), \rho_a(z, y)\} \quad \forall\, x, y, z \in X,$$

which is equivalent to the path metric

$$\theta_a(x, y) = \inf\left\{\sum_{i=1}^{n} \rho_a(x_{i-1}, x_i) : n \geq 1,\ x = x_0, x_1, \ldots, x_n = y,\ x_i \in X\right\}.$$

Since θ_a and ρ_a determine the same topology as long as $a' < \sqrt{2} - 1$, we will use ρ_a to replace θ_a for simplicity.

Definition 2.1. The hyperbolic boundary of X is defined as $\partial X = \hat{X} \setminus X$ where $\hat{X}$ is the completion of X under ρ_a.

The metric ρ_a can be extended onto ∂X, and under which ∂X is a compact set. It is often useful to identify $\xi \in \partial X$ with a *geodesic ray* in X that converge to ξ, i.e., an infinite path $\pi[x_1, x_2, \ldots]$ such that $x_i \sim x_{i+1}$ and any finite segment of the path is a geodesic. It is known that two geodesic rays $\pi[x_1, x_2, \ldots], \pi[y_1, y_2, \ldots]$ represent the same $\xi \in \partial X$ if and only if $|x_n \wedge y_n| \to \infty$ as $n \to \infty$.

Our interest is on the following tree structure introduced by Kaimanovich which is used to study the self-similar sets ([9,12]). For a tree X with a root o, we use $\mathcal{E}_v$ to denote the set of edges (v for vertical). We introduce additional edges on each level $\{x : d(o, x) = n\}$, $n \in \mathbb{N}$ as follows. Let x^{-k} denote the k-th ancestor of x, the unique point in the $(n-k)$-th level that is joined by a unique path.

Definition 2.2. Let X be a tree with a root o. Let $\mathcal{E}_h \subset (X \times X) \setminus \{(x, x) : x \in X\}$ such that it is symmetric and satisfies:

$$(x, y) \in \mathcal{E}_h \Rightarrow |x| = |y| \quad \text{and either} \quad x^{-1} = y^{-1} \text{ or } (x^{-1}, y^{-1}) \in \mathcal{E}_h.$$

We call elements in $\mathcal{E}_h$ *horizontal edges*, and for $\mathcal{E} = \mathcal{E}_v \cup \mathcal{E}_h$, $(X, \mathcal{E})$ is called an *augmented tree*.

Following [12], we say that a path $\pi(x, y)$ is a *horizontal geodesic* if it is a geodesic and it consists of horizontal edges only. It is called a *canonical geodesic* if there exist $u, v \in \pi(x, y)$ such that:

(i) $\pi(x, y) = \pi(x, u) \cup \pi(u, v) \cup \pi(v, y)$ with $\pi(u, v)$ a horizontal path and $\pi(x, u), \pi(v, y)$ vertical paths;

(ii) for any geodesic path $\pi'(x, y)$, $\mathrm{dist}(o, \pi(x, y)) \leq \mathrm{dist}(o, \pi'(x, y))$.

Note that condition (ii) is to require the horizontal part of the canonical geodesic to be on the highest level. The following basic theorem was proved in [12]:

Theorem 2.3. *Let X be an augmented tree. Then*

(i) *Let $\pi(x, y)$ be a canonical geodesic, then $|x \wedge y| = l - h/2$, where l and h are the level and the length of the horizontal part of the geodesic.*

(ii) *X is hyperbolic if and only if there exists a constant $k > 0$ such that any horizontal part of a geodesic is bounded by k.*

The premier application of the augmented trees is to use their hyperbolic boundaries to study the self-similar sets. Throughout the paper, we assume a self-similar set K is generated by an iterated function system (IFS) $\{S_i\}_{i=1}^m$ on $\mathbb{R}^d$ where

$$S_i(x) = rR_i x + d_i, \quad i = 1, \ldots, m \tag{2.2}$$

with $0 < r < 1$, R_i is an orthogonal matrix, and $d_i \in \mathbb{R}^d$. It is well-known that K satisfies $K = \bigcup_{i=1}^m S_i(K)$. Let $\Sigma = \{1, \ldots, m\}$ and let $X = \bigcup_{n=0}^\infty \Sigma^n$ be the symbolic space representing the IFS (by convention, $\Sigma^0 = \emptyset$, and we still denote it by o). For $\mathbf{u} = i_1 \cdots i_n$, we use $S_{\mathbf{u}}$ to denote the composition $S_{i_1} \circ \cdots \circ S_{i_n}$.

Let $\mathcal{E}_v$ be the set of vertical edges corresponding to the nature tree structure on X with o as a root. In [12], a set of horizontal edges $\mathcal{E}_h$ is defined as

$$\mathcal{E}_h = \{(\mathbf{u}, \mathbf{v}) : |\mathbf{u}| = |\mathbf{v}|, \mathbf{u} \neq \mathbf{v} \text{ and } K_{\mathbf{u}} \cap K_{\mathbf{v}} \neq \emptyset\},$$

where $K_{\mathbf{u}} = S_{\mathbf{u}}(K)$. Let $\mathcal{E} = \mathcal{E}_v \cup \mathcal{E}_h$, then $(X, \mathcal{E})$ is an augmented tree induced by the self-similar set.

If the IFS is strongly separated (i.e., $S_i(K) \cap S_j(K) = \emptyset$ for $i \neq j$), then K is called *dust-like*. It is clear that in this case, $\mathcal{E}_h = \emptyset$, and ρ_a coincides with the natural metric on the symbolic space:

$$\varrho(x, y) = \exp(-a \max\{k : x_i = y_i,\ i \leq k\}).$$

In [12], it was proved that under the *open set condition* (OSC), Theorem 2.3(ii) implies that the above augmented tree is hyperbolic, and the nature map $\Phi : \partial X \to K$ is a homeomorphism. Moreover if in addition, the IFS satisfies

condition (H): there exists a constant $c > 0$ such that for any integer $n \geq 1$ and words $\mathbf{u}, \mathbf{v} \in \Sigma^n$,

$$K_{\mathbf{u}} \cap K_{\mathbf{v}} = \emptyset \Rightarrow \operatorname{dist}(K_{\mathbf{u}}, K_{\mathbf{v}}) \geq cr^n. \tag{2.3}$$

Then for $\alpha = -\log r/a$, Φ satisfies the following Hölder equivalent property:

$$C^{-1}|\Phi(\xi) - \Phi(\eta)| \leq \rho_a(\xi, \eta)^\alpha \leq C|\Phi(\xi) - \Phi(\eta)| \quad \forall\, \xi, \eta \in \partial X.$$

Condition (H) is satisfied by the standard self-similar sets, for example, the generating IFS has the OSC and all the parameters of the similitudes are integers. However there are also examples that condition (H) is not satisfied (see [12] for an example such that the similitudes involve irrational translations).

From Definition 2.2, we see that the choice of the horizontal edges for the augmented tree can be quite flexible, for example we can use $K_{\mathbf{u}} \cap K_{\mathbf{v}}$ to have positive dimension to define $\mathcal{E}_h$. In this paper, we will use another setting by replacing K with a bounded closed invariant set J (i.e., $S_i(J) \subset J$ for each i), namely

$$\mathcal{E}_h = \{(\mathbf{u}, \mathbf{v}) : |\mathbf{u}| = |\mathbf{v}|, \mathbf{u} \neq \mathbf{v} \text{ and } J_{\mathbf{u}} \cap J_{\mathbf{v}} \neq \emptyset\}. \tag{2.4}$$

We can take $J = K$ as before or in many situations, take $J = \overline{U}$ for the U in the OSC (see the examples in Section 5). The above statements on the hyperbolicity of the augmented tree and the homeomorphism of the boundary still valid by adopting the same proof; for the Hölder equivalence, we use the following modification of condition (H) for J, which will be used again in proving Proposition 3.5.

Lemma 2.4. *Suppose the IFS in (2.2) satisfies condition (H), then for any bounded closed invariant set J, there exists $c' > 0$ and $k \geq 0$ such that for any $n \geq 0$ and $\mathbf{u}, \mathbf{v} \in \Sigma^n$,*

$$J_{\mathbf{u}} \cap J_{\mathbf{v}} = \emptyset \Rightarrow dist(J_{\mathbf{ui}}, J_{\mathbf{vj}}) \geq c' r^n \quad \forall\, \mathbf{i},\ \mathbf{j} \in \Sigma^k.$$

Proof. Let c be the constant in the definition of condition (H). For the bounded closed invariant set J, we have $K \subseteq J$ and the Hausdorff distance $d_H(K_{\mathbf{i}}, J_{\mathbf{i}}) \leq c_1 r^k$ for all $\mathbf{i} \in \Sigma^k$. In particular we choose k such that $c_1 r^k < c/3$.

Now if $\mathbf{u}, \mathbf{v} \in \Sigma^n$, $J_{\mathbf{u}} \cap J_{\mathbf{v}} = \emptyset$, then $K_{\mathbf{u}} \cap K_{\mathbf{v}} = \emptyset$, it follows from condition (H) that $\mathrm{dist}(K_{\mathbf{u}}, K_{\mathbf{v}}) \geq cr^n$ for $\mathbf{u}, \mathbf{v} \in \Sigma^n$. Applying this and the above to $n + k$, we have

$$\begin{aligned} \mathrm{dist}(J_{\mathbf{ui}}, J_{\mathbf{vj}}) &\geq \mathrm{dist}(K_{\mathbf{ui}}, K_{\mathbf{vj}}) - d_H(K_{\mathbf{ui}}, J_{\mathbf{ui}}) - d_H(K_{\mathbf{vj}}, J_{\mathbf{vj}}) \\ &\geq cr^n - (2c/3)\, r^n \geq (c/3) r^n \quad \forall\, \mathbf{i},\ \mathbf{j} \in \Sigma^k. \end{aligned}$$

The lemma follows by taking $c' = c/3$. $\square$

We remark that the augmented tree $(X, \mathcal{E})$ depends on the choice of the bounded invariant set J. But under the OSC, the hyperbolic boundary is the same as they can be identified with the underlying self-similar set.

We conclude this section with the following simple relationship of the totally disconnected self-similar set and the structure of the augmented tree. The more explicit study of their Lipschitz equivalence will be carried out in detail in the rest of the paper. By a *horizontal connected component* of an augmented tree, we mean a maximal connected horizontal subgraph on some level Σ^n of X.

Proposition 2.5. *Suppose the cardinality of any horizontal component in the augmented tree induced by the IFS in (2.2) is uniformly bounded, then the associated self-similar set K is totally disconnected.*

The converse is also true if the about IFS is on $\mathbb{R}^1$ and satisfies the OSC.

Proof. Suppose K is not totally disconnected, then there is a connected component $C \subset K$ contains more that one point. Note that for any $n > 0$, $K = \bigcup_{\mathbf{i} \in \Sigma^n} K_{\mathbf{i}}$. Let $K_{\mathbf{i}_1} \cap C \neq \emptyset$. If $C \setminus K_{\mathbf{i}_1} \neq \emptyset$, then it is a relatively open set in C, and as C is connected,

$$\partial_C(C \setminus K_{\mathbf{i}_1}) \cap \partial_C(K_{\mathbf{i}_1} \cap C) \neq \emptyset.$$

($\partial_C(E)$ means the relative boundary of E in C). Let x be in the intersection, there exists $\mathbf{i}_2 \in \Sigma^n$ such that $x \in K_{\mathbf{i}_1} \cap K_{\mathbf{i}_2}$ and $K_{\mathbf{i}_2} \cap (C \setminus K_{\mathbf{i}_1}) \neq \emptyset$.

Inductively, if $\bigcup_{j=1}^k K_{\mathbf{i}_j}$ does not cover C, then we can repeat the same procedure to find $\mathbf{i}_{k+1} \in \Sigma^n$ such that

$$K_{\mathbf{i}_{k+1}} \cap \left(\bigcup_{j=1}^k K_{\mathbf{i}_j} \right) \neq \emptyset \quad \text{and} \quad K_{\mathbf{i}_{k+1}} \cap \left(C \setminus \bigcup_{j=1}^k K_{\mathbf{i}_j} \right) \neq \emptyset.$$

Since $K = \bigcup_{\mathbf{i}\in\Sigma^n} K_{\mathbf{i}}$, this process must end at some step, say ℓ, and in this case $C \subset \bigcup_{j=1}^{\ell} K_{\mathbf{i}_j}$. Since the diameter $|K_{\mathbf{i}_j}| = r^n|K| \to 0$ as $n \to \infty$, ℓ must tend to infinity, which contradicts the uniform boundedness of the horizontal connected components Σ^n.

For the converse, we assume that the IFS is defined on $\mathbb{R}$. Note that if $K \subset \mathbb{R}$ is totally disconnected, then $\dim_H K = s < 1$ [20]. Let μ denote the restriction of the s-Hausdorff measure on K. Without loss of generality, we assume $\mu(K) = 1$. Then it is well-known that for any point $x \in K$ and any $0 < t < |K|$ (where $|K|$ denotes the diameter of K),

$$C_1 < \frac{\mu(B(x,t))}{t^s} < C_2,$$

where C_1, C_2 are constants independent of x and t.

Suppose $\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_k$ is a finite sequence of distinct words in Σ^n and is in a horizontal connected component (we take $J = K$ for convenience), i.e., $K_{\mathbf{i}_j} \cap K_{\mathbf{i}_{j+1}} \neq \emptyset$ for $1 \le j \le k-1$. Let G be the smallest interval containing $K_{\mathbf{i}_1}, \ldots, K_{\mathbf{i}_k}$. Let $x \in G \cap K$, and let $t = |G|$. Then the Hausdorff measure μ implies

$$\mu(B(x,t)) \ge kr^{ns}.$$

This together with $t = |G| \le kr^n|K|$ implies

$$\frac{k^{1-s}}{|K|^s} = \frac{kr^{ns}}{(kr^n|K|)^s} \le \frac{\mu(B(x,t))}{t^s} \le C_2.$$

Hence k is uniformly bounded. $\square$

3. Lipschitz equivalence and the main theorems

In this rest of the paper, unless otherwise stated, we will assume the augmented tree $(X, \mathcal{E})$ is associated with the symbolic space of the IFS $\{S_i\}_{i=1}^m$ in (2.2), and $\mathcal{E} = \mathcal{E}_h \cup \mathcal{E}_v$ where $\mathcal{E}_h$ is defined by a fixed bounded closed invariant set J as in (2.4). We introduce a class of mappings between two hyperbolic graphs which plays a key role in the Lipschitz equivalence.

Definition 3.1. Let X and Y be two hyperbolic graphs and let $\sigma : X \to Y$ be a bijective map. We say that σ is a *near-isometry* if there exists $c > 0$ such that

$$\big||\pi(\sigma(x), \sigma(y))| - |\pi(x,y)|\big| \le c \quad \forall\, x, y \in X.$$

Remark. By checking $\pi(o,x)$, it is easy to show that the above definition implies $\big||\sigma(x)| - |x|\big| \le c + 3|\sigma(o)| + k$ where k is the bound of the horizontal geodesic in Theorem 2.3(ii). Hence the above definition is equivalent to

$$\big||\sigma(x)| - |x|\big| < c, \qquad \big||\pi(\sigma(x), \sigma(y))| - |\pi(x,y)|\big| \le c \quad \forall\, x, y \in X$$

(with different constant c).

Proposition 3.2. *Let X, Y be two hyperbolic augmented trees that are equipped with the hyperbolic metrics with the same parameter a (as in (2.1)). Suppose there exists a near-isometry $\sigma : X \to Y$, then $\partial X \simeq \partial Y$.*

Proof. With the notation as in Theorem 2.3(i), it follows that for $x \neq y \in X$,

$$|\pi(x,y)| = |x| + |y| - 2l + h, \qquad |\pi(\sigma(x), \sigma(y))| = |\sigma(x)| + |\sigma(y)| - 2l' + h'.$$

From the definition of σ (and the remark), we have

$$\big||\sigma(x)| - |x|\big|, \qquad \big||\sigma(y)| - |y|\big| \le c, \qquad |l - l'| \le 3c/2 + k/2, \quad \text{and} \quad |h - h'| \le k$$

for some $k > 0$ (where k is the hyperbolic constant as in Theorem 2.3(ii)). By Theorem 2.3(i),

$$|x \wedge y| = l - h/2 \quad \text{and} \quad |\sigma(x) \wedge \sigma(y)| = l' - h'/2.$$

It follows that

$$\big||\sigma(x) \wedge \sigma(y)| - |x \wedge y|\big| = |l' - h'/2 - l + h/2| \le 3c/2 + k := k'.$$

Let $\lambda = e^{ak'}$, together with the definition of the ultra-metric $\rho_a(x, y) = \exp(-a|x \wedge y|)$ in (2.1), we conclude that

$$\lambda^{-1}\rho_a(x, y) \le \rho_a(\sigma(x), \sigma(y)) \le \lambda\rho_a(x, y) \quad \forall\, x, y \in X.$$

By extending the metrics to the boundaries $\partial X,\ \partial Y$, the above implies σ is a bi-Lipschitz map from ∂X onto ∂Y. $\square$

Let $\mathscr{C}$ be the set of all horizontal connected components of X. For $T \in \mathscr{C}$, we let $T\Sigma = \{\mathbf{u}i : \mathbf{u} \in T, i \in \Sigma\}$ be the set of offsprings of T. Note that if two distinct components $T, T' \in \mathscr{C}$ lie in the same level, then $T\Sigma$ is not connected to $T'\Sigma$, equivalently,

$$\left(\bigcup_{\mathbf{i}\in T\Sigma} J_{\mathbf{i}}\right) \cap \left(\bigcup_{\mathbf{j}\in T'\Sigma} J_{\mathbf{j}}\right) = \emptyset. \tag{3.1}$$

This follows easily from $S_{\mathbf{u}i}(J) \cap S_{\mathbf{v}j}(J) \subset S_{\mathbf{u}}(J) \cap S_{\mathbf{v}}(J) = \emptyset$ for all $\mathbf{u} \in T,\ \mathbf{v} \in T',\ i, j \in \Sigma$.

By regarding $T \cup T\Sigma$ as a subgraph in X. We say that $T, T' \in \mathscr{C}$ are *equivalent*, denoted by $T \sim T'$, if there exists a graph isomorphism

$$g : \ T \cup T\Sigma \ \to \ T' \cup T'\Sigma,$$

that is, g is a bijection such that g and g^{-1} preserve the vertical and horizontal edges. It is easy to check that $\sim$ is indeed an equivalence relation. We use $[T]$ to denote the equivalence class and call it a *connected class* determined by T. Obviously, $\{o\}$ is the connected class determined by the root o.

Definition 3.3. An augmented tree X is called *simple* (with respect to the defining bounded closed invariant set J) if there are finitely many connected classes, i.e., $\mathscr{C}/\sim$ is finite.

Proposition 3.4. *A simple augmented tree is always hyperbolic.*

Proof. Note that for each geodesic $\pi(x, y)$ in X, the horizontal part must be contained in a horizontal component of the augmented tree. Since there are finitely many connected classes $[T]$, and each T contains finitely many vertices, it follows that the horizontal part of $\pi(x, y)$ is uniformly bounded, and hence hyperbolic by Theorem 2.3(ii). $\square$

In the following we show that the hyperbolic boundary of a simple augmented tree is Hölder equivalent to the self-similar set, which is similar to the case in [12] for the OSC.

Proposition 3.5. *Let $\{S_i\}_{i=1}^m$ be an IFS satisfies condition (H) in (2.3), and assume that the corresponding augmented tree $(X, \mathcal{E})$ is simple. Then there exists a bijection $\Phi : \partial X \to K$*

satisfying the Hölder equivalent property:

$$C^{-1}|\Phi(\xi) - \Phi(\eta)| \le \rho_a(\xi, \eta)^\alpha \le C|\Phi(\xi) - \Phi(\eta)|, \tag{3.2}$$

where $\alpha = -\log r/a$ *and* $C > 0$ *is a constant.*

Proof. The proof is essentially the same as in [12] by replacing K with the invariant set J in $\mathcal{E}_h$. We sketch the main idea of proof here. For any geodesic ray $\xi = \pi[\mathbf{u}_1, \mathbf{u}_2, \ldots]$, we define

$$\Phi(\xi) = \lim_{n\to\infty} S_{\mathbf{u}_n}(x_0)$$

for some $x_0 \in J$. Then the mapping is well-defined and is a bijection (see Lemma 4.1, Theorem 4.3 in [12]).

To show that Φ satisfies (3.2), let $\xi = \pi[\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots]$, $\eta = \pi[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots]$ be any two non-equivalent geodesic rays in X. Then there is a canonical bilateral geodesic γ joining ξ and η:

$$\gamma = \pi[\ldots, \mathbf{u}_{n+1}, \mathbf{u}_n, \mathbf{t}_1, \ldots, \mathbf{t}_\ell, \mathbf{v}_n, \mathbf{v}_{n+1}, \ldots]$$

with $\mathbf{u}_n, \mathbf{t}_1, \ldots, \mathbf{t}_\ell, \mathbf{v}_n \in \Sigma^n$. It follows that

$$|S_{\mathbf{u}_n}(x_0) - S_{\mathbf{v}_n}(x_0)| \le (\ell + 2)r^n|J|.$$

Since X is simple, ℓ is uniformly bounded (by Proposition 3.4). Note that $\Phi(\xi) \in J_{\mathbf{u}_k}$ and $\Phi(\eta) \in J_{\mathbf{v}_k}$ for all $k \ge 0$, hence

$$|\Phi(\xi) - S_{\mathbf{u}_n}(x_0)|, \qquad |\Phi(\eta) - S_{\mathbf{v}_n}(x_0)| \le r^n|J|.$$

Therefore

$$\begin{aligned}|\Phi(\xi) - \Phi(\eta)| &\le |\Phi(\xi) - S_{\mathbf{u}_n}(x_0)| + |S_{\mathbf{u}_n}(x_0) - S_{\mathbf{v}_n}(x_0)| + |\Phi(\eta) - S_{\mathbf{v}_n}(x_0)| \\ &\le C_1 r^n.\end{aligned}$$

Since γ is a bilateral canonical geodesic, we have $|\xi \wedge \eta| = n - (\ell + 1)/2$ and ℓ is uniformly bounded. By using $\rho_a(\xi, \eta) = \exp(-a|\xi \wedge \eta|)$, we see that

$$|\Phi(\xi) - \Phi(\eta)| \le C\rho_a(\xi, \eta)^\alpha.$$

On the other hand, assume that $\xi \ne \eta$. Since γ is a geodesic, it follows that $(\mathbf{u}_{n+1}, \mathbf{v}_{n+1}) \notin \mathcal{E}_h$, and hence $J_{\mathbf{u}_{n+1}} \cap J_{\mathbf{v}_{n+1}} = \emptyset$. By Lemma 2.4, there is k (independent of n) such that

$$J_{\mathbf{u}} \cap J_{\mathbf{v}} = \emptyset \Rightarrow \mathrm{dist}(J_{\mathbf{ui}}, J_{\mathbf{vj}}) \ge c' r^n \quad \forall\, \mathbf{i},\ \mathbf{j} \in \Sigma^k.$$

Referring to $\gamma = \pi[\ldots, \mathbf{u}_{n+1}, \mathbf{u}_n, \mathbf{t}_1, \ldots, \mathbf{t}_\ell, \mathbf{v}_n, \mathbf{v}_{n+1}, \ldots]$, we have $\Phi(\xi) \in J_{\mathbf{u}_{n+k+1}}$, $\Phi(\xi) \in J_{\mathbf{v}_{n+k+1}}$. It follows that

$$|\Phi(\xi) - \Phi(\eta)| \ge \mathrm{dist}(J_{\mathbf{u}_{n+k+1}}, J_{\mathbf{v}_{n+k+1}}) \ge c' r^n,$$

and $|\Phi(\xi) - \Phi(\eta)| \ge c''\rho_a(\xi, \eta)^\alpha$ follows by the definition of ρ_a as the above. $\square$

For a simple augmented tree X, we label the connected classes as $\{\mathscr{T}_1, \ldots, \mathscr{T}_r\}$ and introduce an $r \times r$ incidence matrix

$$A = [a_{ij}]_{r\times r} \tag{3.3}$$

for the connected classes. The entries a_{ij} are defined as follows. For any $1 \le i \le r$, take a horizontal connected component T in X such that $[T] = \mathscr{T}_i$. Let $Z_{i1}, \ldots, Z_{i\ell}$ be the horizontal connected components consisting of offsprings generated by T, i.e., $T\Sigma = \bigcup_{k=1}^{\ell} Z_{ik}$, and define

$$a_{ij} = \#\{k : 1 \le k \le \ell, [Z_{ik}] = \mathscr{T}_j\}.$$

Observe that a_{ij} is independent of the choice of the components in the equivalence classes. It is clear that for $T, T' \in \mathscr{C}$, $[T] = [T']$ implies $\#T = \#T'$. But the converse is not true. As a direct consequence of the definition, we have

Proposition 3.6. *Let* $\mathbf{b} = [b_1, \ldots, b_r]^t$ *where* $b_i = \#T$ *where* $[T] = \mathscr{T}_i$, *then* $A\mathbf{b} = m\mathbf{b}$.

The following theorem is for Lipschitz equivalence on the hyperbolic boundaries, it is the crucial step to establish the equivalence for the self-similar sets.

Theorem 3.7. *Suppose the augmented tree* $(X, \mathcal{E})$ *is simple, and suppose the corresponding incidence matrix* A *is* $(m, \mathbf{b})$*-rearrangeable (where the* m *and* $\mathbf{b}$ *are defined as in Proposition 3.6). Then there is a near-isometry between* $(X, \mathcal{E})$ *and* $(X, \mathcal{E}_v)$, *so that* $\partial(X, \mathcal{E}) \simeq \partial(X, \mathcal{E}_v)$.

The notion of *rearrangeable matrix* is an important tool to construct the near-isometry. Since the concept is a little complicated, we will introduce this in more detail, and prove Theorem 3.7 together with the following theorem in the next section.

Theorem 3.8. *If the incidence matrix* A *is primitive, then* A^k *is* $(m^k, \mathbf{b})$*-rearrangeable for some* $k > 0$. *Consequently* $\partial(X, \mathcal{E}) \simeq \partial(X, \mathcal{E}_v)$.

As a direct consequence, we have

Corollary 3.9. *Under the assumption on Theorem 3.7 (or Theorem 3.8), then* $(\partial(X, \mathcal{E}), \rho_a)$ *is totally disconnected.*

By Theorem 3.7 we obtain the following Lipschitz equivalence on the self-similar sets.

Theorem 3.10. *Let* K *and* K' *be self-similar sets that are generated by two IFS's as in (2.2) with the same number of similitudes and the same contraction ratio, and satisfy condition (H) in (2.3). Assume the associated augmented trees are simple and the incidence matrices are* $(m, \mathbf{b})$*-rearrangeable (in particular, primitive). Then* K *and* K' *are Lipschitz equivalent, and are also Lipschitz equivalent to a dust-like self-similar set.*

Proof. It follows from Theorem 3.7 that

$$\partial(X, \mathcal{E}) \simeq \partial(X, \mathcal{E}_v) = \partial(Y, \mathcal{E}_v) \simeq \partial(Y, \mathcal{E}) \tag{3.4}$$

(for the respective metrics ρ_a). Let $\varphi : \partial(X, \mathcal{E}) \to \partial(Y, \mathcal{E})$ be the bi-Lipschitz map. With no confusion, we just denote these two boundaries by ∂X, ∂Y as before.

By Proposition 3.5, there exist two bijections $\Phi_1 : \partial X \to K$ and $\Phi_2 : \partial Y \to K'$ satisfying (3.2) with constants C_1, C_2, respectively. Define $\tau : K \to K'$ as

$$\tau = \Phi_2 \circ \varphi \circ \Phi_1^{-1}.$$

Then

$$\begin{aligned} |\tau(x) - \tau(y)| &\le C_2\, \rho_a(\varphi \circ \Phi_1^{-1}(x), \varphi \circ \Phi_1^{-1}(y))^\alpha \\ &\le C_2 C_0^\alpha\, \rho_a(\Phi_1^{-1}(x), \Phi_1^{-1}(y))^\alpha \\ &\le C_2 C_0^\alpha C_1\, |x - y|. \end{aligned}$$

Let $C' = C_2 C_0^{\alpha} C_1$, then

$$|\tau(x) - \tau(y)| \leq C'|x - y|.$$

Similarly, we have $C'^{-1}|x - y| \leq |\tau(x) - \tau(y)|$. Therefore $\tau : K \to K'$ is a bi-Lipschitz map.

For the last statement, we can regard $(X, \mathcal{E}_v)$ as the augmented tree of an IFS that is strongly separated, and then apply the above conclusion. □

Corollary 3.11. *The IFS in Theorem 3.10 satisfies the OSC.*

Proof. We make use of the following well-known result of Schief [20] on a self-similar set K: let s be the similarity dimension of K, then the IFS satisfies the OSC if and only if $0 < \mathcal{H}^s(K) < \infty$.

Let K be the self-similar set as in Theorem 3.10, then it is Lipschitz equivalent to a dust-like set K''. It follows that $0 < \mathcal{H}^s(K'') < \infty$, so is K by the Lipschitz equivalence. Hence by Schief's criterion, the IFS for K satisfies the OSC. □

In Section 5, we will provide some interesting examples for the Lipschitz equivalence of the totally disconnected self-similar sets in Theorem 3.10. We also remark that in Theorem 3.10 the condition on the augmented tree can be weaken, and the proof still yields a very useful result.

Proposition 3.12. *Let K and K' be self-similar sets that are generated by two IFS's as in (2.2) that have the same number of similitudes, same contraction ratio, and satisfy condition (H) in (2.3). Suppose the two IFS's satisfy either* (i) *the OSC, or* (ii) *the augmented trees are simple. Then*

$$K \simeq K' \Leftrightarrow \partial X \simeq \partial Y. \tag{3.5}$$

Proof. The sufficiency of (3.5) is always satisfied, as we can replace (3.4) by the given condition $\partial X \simeq \partial Y$, then follows from the same proof of Theorem 3.10. The necessity follows by making use of the Hölder equivalence (3.2) which is satisfied for cases (i) and (ii), and proceeds with a similar estimation for $\varphi = \Phi_2^{-1} \circ \tau \circ \Phi_1$. □

We remark that the above theory of Lipschitz equivalence can also be applied to study the self-affine systems. Let B be a $d \times d$ expanding matrix (i.e., all the eigenvalues have moduli > 1) and let $\{S_i\}_{i=1}^m$ with $S_i(x) = B^{-1}(x + d_i)$, $d_i \in \mathbb{R}^d$ be the IFS. For the part of simple augmented tree, it is clear that the notion can be defined and the hyperbolicity in Proposition 3.4 follows by the same way. Moreover we have

Proposition 3.13. *For the IFS $\{S_i\}_{i=1}^m$ of self-affine maps as the above, Theorems 3.7 and 3.8 remain valid.*

For the part involves the self-affine set on $\mathbb{R}^d$, we need to use a device in [8] by replacing the Euclidean norm with an "ultra-norm" adapted to the matrix B. By renorming, we can assume without loss of generality that $\|x\| \leq \|Bx\|$. For $0 < \delta < 1/2$, let $\varphi \geq 0$ be a C^∞ function supported in the open ball U_δ centering at 0 with $\varphi(x) = \varphi(-x)$ and $\int_{\mathbb{R}^d} \varphi = 1$. Let $V = BU_1 \setminus U_1$, and let $h = \chi_V * \varphi$ be the convolution of the indicator function χ_V and φ. Let $q = |\det(B)|$ and define

$$w(x) = \sum_{n=-\infty}^{\infty} q^{-n/q} h(B^n x) \quad x \in \mathbb{R}^d.$$

Then $w(x)$ satisfies (i) $w(x) = w(-x)$ and $w(x) = 0$ if and only if $x = 0$, (ii) $w(Bx) = q^{1/d}w(x)$, and (iii) there exists $\beta > 1$ such that $w(x+y) \le \beta \max\{w(x), w(y)\}$. This w is used as a distance (ultra-metric) to replace the Euclidean distance to define the generalized Hausdorff measure $\mathcal{H}^\alpha_w$, Hausdorff dimension $\dim^w_H$, box dimension $\dim^w_B$. Under this setting, most of the basic properties for the self-similar sets (including Schief's basic result on OSC) can be carried to the self-affine sets [8]. To apply to here, we need to adjust condition (H) (2.3) to

$$K_{\mathbf{u}} \cap K_{\mathbf{v}} = \emptyset \Rightarrow \mathrm{dist}_w(K_{\mathbf{u}}, K_{\mathbf{v}}) \ge cq^{-n/d}$$

and to replace the r^n in the proofs of Lemma 2.4 and Proposition 3.5 by $q^{-n/d}$. Then we have

Theorem 3.14. *With K and K' self-affine sets satisfying the conditions in Theorem 3.10. The K and K' are Lipschitz equivalent under the ultra-metric defined by w, and they are Lipschitz equivalent to a dust-like self-affine set.*

4. Rearrangeable matrix and proofs of the main theorems

The proof of the Lipschitz equivalence of the simple augmented tree to the original tree in Theorem 3.7 is to construct a near-isometry between them, which is based on a device of "rearrangement" of graphs. The idea of rearrangement was introduced by Deng and He [4]. A similar technique of "equal decomposition" was also used to consider the Lipschitz equivalence in [18] (see also [24,25]). First we give a detail discussion of the concept of rearrangement.

Definition 4.1. Given $m, r \in \mathbb{N}$. Suppose $\mathbf{a} = [a_1, \ldots, a_r] \in \mathbb{Z}^r_+$ is a row vector, and $\mathbf{b} = [b_1, \ldots, b_r]^t \in \mathbb{N}^r$ is a column vector. We say that $\mathbf{a}$ is $(m, \mathbf{b})$-rearrangeable if there exists an integer $p > 0$, and a nonnegative integral matrix $C = [c_{ij}]_{p\times r}$ such that

$$\mathbf{a} = [1, \ldots, 1]C \quad \text{and} \quad C\mathbf{b} = [m, \ldots, m]^t.$$

(Note that in this case $\mathbf{ab} = pm$ for some $p \in \mathbb{N}$.)

A matrix A is called $(m, \mathbf{b})$-rearrangeable if each row of A is $(m, \mathbf{b})$-rearrangeable.

Remarks. (1) The intuitive explanation of the definition is as follows. Let a_i be the number of balls with the same weight b_i. That $\mathbf{a}$ is $(m, \mathbf{b})$-rearrangeable means we can rearrange these balls into p groups (the p-rows in C) such that in each group the number of balls with weight b_j is c_{ij} and the total weight is exactly m. It is clear that the total weight of all balls is $pm = \mathbf{ab} = \mathbf{1}C\mathbf{b}$.

(2) It follows easily from the definition that if $\mathbf{a}$ is $(m, \mathbf{b})$-rearrangeable, then $\max\{b_i : a_i b_i \neq 0\} \le m$.

(3) If we write an $r \times r$ matrix A as $\begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_r \end{bmatrix}$. Then A is $(m, \mathbf{b})$-rearrangeable is equivalent to the existence of $\mathbf{p} = [p_1, \ldots, p_r]^t \in \mathbb{N}^r$ such that $A\mathbf{b} = m\mathbf{p}$ and a sequence of nonnegative integral matrices $\{C_i\}_{i=1}^r$ such that $\mathbf{a}_i = \mathbf{1}C_i$ and $C_i\mathbf{b} = [m, \ldots, m]^t \in \mathbb{N}^{p_i}$ for all i.

We will prove some sufficient conditions for $(m, \mathbf{b})$-rearrangeable in the sequel (Lemma 4.7, Proposition 4.8). In the following we use two examples to illustrate more on such notion.

Example 4.2. Let $\mathbf{a} \in \mathbb{Z}^r_+$, and let $\mathbf{b} = \mathbf{1}^t$. Suppose $\sum_i a_i = m$, then trivially, $\mathbf{a}$ is $(m, \mathbf{b})$-rearrangeable (with $p = 1$, $C = \mathbf{a}$, i.e., intuitively we put everything in one group). Consequently, for any nonnegative integral matrix A with m as an eigenvalue and $\mathbf{b} = \mathbf{1}^t$ as the eigenvector, it is $(m, \mathbf{b})$-rearrangeable.

Example 4.3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

and $\mathbf{b} = [1, 2, 3]^t$. Then $A\mathbf{b} = 3\mathbf{b}$, and A is $(3, \mathbf{b})$-rearrangeable, so is A^2.

Proof. To show that A is $(3, \mathbf{b})$-rearrangeable, it suffices to check on each row of A is $(3, \mathbf{b})$-rearrangeable:

for $\mathbf{a} = [1, 1, 0]$, then $\mathbf{ab} = 3$, we take $C = \mathbf{a}$;

for $\mathbf{a} = [1, 1, 1]$, then $\mathbf{ab} = 2 \times 3$, we take $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

for $\mathbf{a} = [1, 1, 2]$, then $\mathbf{ab} = 3 \times 3$, we take $C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

For $A^2 = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 3 & 3 \\ 4 & 4 & 5 \end{bmatrix}$, we can proceed in the same way to show that A^2 is also $(3, \mathbf{b})$-rearrangeable. The corresponding matrices C for the three rows of A^2 are the transposes of the following matrices: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$. $\square$

As a matter of fact, the above example is typical by the following proposition.

Proposition 4.4. *Let $\mathbf{b} = [b_1, \ldots, b_r]^t \in \mathbb{N}^r$. If a matrix $A = [a_{ij}]_{r\times r}$ is $(m, \mathbf{b})$-rearrangeable, then A^n is $(m, \mathbf{b})$-rearrangeable for $n \geq 1$.*

If in addition, the eigen-relation $A\mathbf{b} = m\mathbf{b}$ is satisfied, then A^n is also $(m^n, \mathbf{b})$-rearrangeable.

Proof. We use induction to prove the first part. Let $\mathbf{a}_i$, $1 \leq i \leq r$ be the row vectors of A. Since A is $(m, \mathbf{b})$-rearrangeable, there exist $\mathbf{p} = [p_1, \ldots, p_r]^t \in \mathbb{N}^r$ such that $A\mathbf{b} = m\mathbf{p}$ and a sequence of matrices $\{C_i\}_{i=1}^r$ such that $\mathbf{a}_i = \mathbf{1}C_i$ and $C_i\mathbf{b} = [m, \ldots, m]^t \in \mathbb{N}^{p_i}$ for all i.

Assume that A^{n-1} is $(m, \mathbf{b})$-rearrangeable, then $A^{n-1}\mathbf{b} = m\widetilde{\mathbf{p}}$ for some positive integer vector $\widetilde{\mathbf{p}}$. Consider A^n, let $\boldsymbol{\alpha}_i$ be the i-th row of A^n. Since $A^n\mathbf{b} = mA\widetilde{\mathbf{p}}$, we have $\boldsymbol{\alpha}_i\mathbf{b} = m\mathbf{a}_i\widetilde{\mathbf{p}} := mq_i$. Write $A^{n-1} = [\widetilde{a}_{ij}]$, it follows that

$$\boldsymbol{\alpha}_i = \sum_{j=1}^{r} \widetilde{a}_{ij}\mathbf{a}_j = \sum_{j=1}^{r} \widetilde{a}_{ij}\mathbf{1}C_j.$$

Let

$$C^{(i)} = [\underbrace{C_1, \ldots, C_1}_{\widetilde{a}_{i1}}, \ldots, \underbrace{C_r, \ldots, C_r}_{\widetilde{a}_{ir}}]^t$$

where the transpose means transposing the row of matrices into a column of matrices (without transposing the C_j itself). Then

$$\boldsymbol{\alpha}_i = \mathbf{1}C^{(i)} \quad \text{and} \quad C^{(i)}\mathbf{b} = [m, \ldots, m]^t \in \mathbb{N}^{q_i}.$$

Hence $\boldsymbol{\alpha}_i$ is $(m, \mathbf{b})$-rearrangeable, and A^n is $(m, \mathbf{b})$-rearrangeable.

For the second part, since $A\mathbf{b} = m\mathbf{b}$ and $A^n\mathbf{b} = m^n\mathbf{b}$, we can replace the previous integral vector $\mathbf{p}$ by $\mathbf{b}$. Then $\boldsymbol{\alpha}_i\mathbf{b} = mq_i = m^n p_i$. We then replace the $q_i \times r$ matrix $C^{(i)}$ in the above by the $p_i \times r$ matrix $D^{(i)}$ which is obtained by summing every consecutive m^{n-1} row vectors of $C^{(i)}$ (note that $q_i = m^{n-1}p_i$). Hence

$$\boldsymbol{\alpha}_i = \mathbf{1}D^{(i)} \quad \text{and} \quad D^{(i)}\mathbf{b} = [m^n, \ldots, m^n]^t \in \mathbb{N}^{p_i}$$

so that $\boldsymbol{\alpha}_i$ is $(m^n, \mathbf{b})$-rearrangeable. Consequently, A^n is $(m^n, \mathbf{b})$-rearrangeable. $\square$

Proof of Theorem 3.7. Recall that $A\mathbf{b} = m\mathbf{b}$ where $\mathbf{b} = [b_1, \ldots, b_r]^t$ with $b_i = \#T$ where $[T] = \mathscr{T}_i$ (Proposition 3.6). First we claim that we can assume without loss of generality that $\max_i b_i \leq m$. For otherwise, let k be sufficiently large such that $\max_i b_i \leq m^k$, by Proposition 4.4, A^k is $(m^k, \mathbf{b})$-rearrangeable. The IFS of the k-th iteration of $\{S_i\}_{i=1}^m$ has symbolic space $X' = \bigcup_{n=0}^{\infty} \Sigma^{kn}$ and the augmented tree has incidence matrix A^k; moreover the two hyperbolic boundaries $\partial X'$ and ∂X are identical. Hence we can consider A^k instead if $\max_i b_i \leq m$ is not satisfied.

Let $X_1 = (X, \mathcal{E}),\ X_2 = (X, \mathcal{E}_v)$. In view of Proposition 3.2, it suffices to show that there exists a near-isometry σ between X_1 and X_2, and hence $\partial(X, \mathcal{E}) \simeq \partial(X, \mathcal{E}_v)$. We define this σ to be a one-to-one mapping from Σ^n (in X_1) to Σ^n (in X_2) inductively as follows: Let

$$\sigma(o) = o \quad \text{and} \quad \sigma(i) = i, \quad i \in \Sigma.$$

Suppose σ is defined on the level n such that for every horizontal connected component T, $\sigma(T)$ has the same parent (see Fig. 1), i.e.,

$$\sigma(x)^{-1} = \sigma(y)^{-1} \quad \forall\, x, y \in T \subset \Sigma^n. \tag{4.1}$$

To define the map σ on Σ^{n+1}, we note that T in Σ^n gives rise to horizontal connected components in Σ^{n+1}, which are accounted by the incidence matrix A. We can write

$$T\Sigma = \bigcup_{k=1}^{\ell} Z_k$$

where Z_k are horizontal connected components consisting of offsprings of T. If T belongs to the connected class $\mathscr{T}_i$, then $\#T = b_i$. By the definition of the incidence matrix A and $A\mathbf{b} = m\mathbf{b}$, we have

$$b_i m = \sum_{k=1}^{\ell} \#Z_k = \sum_{j=1}^{r} a_{ij} b_j.$$

Since A is $(m, \mathbf{b})$-rearrangeable, for the $\mathbf{a}_i$, there exists a nonnegative integral matrix $C = [c_{sj}]_{b_i \times r}$ (depends on i) such that

$$\mathbf{a}_i = \mathbf{1}C \quad \text{and} \quad C\mathbf{b} = [m, \ldots, m]^t.$$

We decompose $\mathbf{a}_i$ into b_i groups according to C as follows. Note that a_{ij} denotes the number of Z_k that belongs to $\mathscr{T}_j$. For each $1 \leq s \leq b_i$ and $1 \leq j \leq r$, we choose c_{sj} of those Z_k, and denote by Λ_s the set for all the chosen k with $1 \leq j \leq r$. Then we can write the index set $\{1, 2, \ldots, \ell\}$ as a disjoint union:

$$\{1, 2, \ldots, \ell\} = \bigcup_{s=1}^{b_i} \Lambda_s.$$

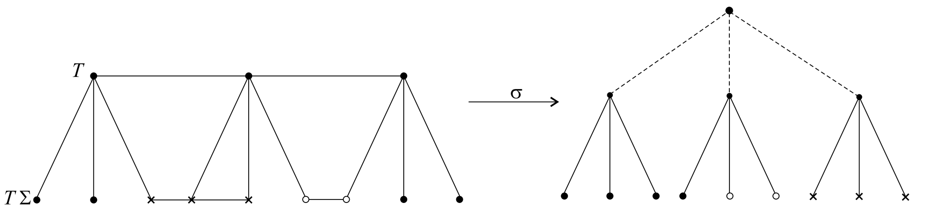


Fig. 1. An illustration of the rearrangement by σ, the $\bullet$, $\circ$ and $\times$ on the left denote the types of connected components.

Hence $\bigcup_{k=1}^{\ell} Z_k$ can be rearranged as b_i groups so that the total size of every group is equal to m, namely,

$$\bigcup_{k=1}^{\ell} Z_k = \bigcup_{k\in\Lambda_1} Z_k \cup \cdots \cup \bigcup_{k\in\Lambda_{b_i}} Z_k. \tag{4.2}$$

Note that each set on the right has m elements.

For the connected component $T = \{\mathbf{i}_1, \ldots, \mathbf{i}_{b_i}\} \subset \Sigma^n$, we have defined σ on Σ^n and $\sigma(T) = \{\mathbf{j}_1 = \sigma(\mathbf{i}_1), \ldots, \mathbf{j}_{b_i} = \sigma(\mathbf{i}_{b_i})\}$. In view of (4.2), we define σ on $T\Sigma = \bigcup_{k=1}^{\ell} Z_k$ by assigning each $\bigcup_{k\in\Lambda_s} Z_k$ (it has m elements) the m descendants of $\mathbf{j}_s$ (see Fig. 1). It is clear that σ is well-defined on $T\Sigma$ and satisfies (4.1) for $x, y \in T\Sigma$. We apply the same construction of σ on the offsprings of every horizontal connected component in Σ^n. It follows that σ is well-defined and satisfies (4.1) on Σ^{n+1}. Inductively, σ can be defined from X_1 to X_2 and is bijective.

Finally we show that σ is indeed a near-isometry and complete the proof. Since $\sigma : X_1 \to X_2$ preserves the levels, without loss of generality, it suffices to prove the near-isometry for $\mathbf{x}, \mathbf{y}$ belong to the same level. Let $\pi(\mathbf{x}, \mathbf{y})$ be the canonical geodesic connecting them, which can be written as

$$\pi(\mathbf{x}, \mathbf{y}) = [\mathbf{x}, \mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{t}_1, \ldots, \mathbf{t}_k, \mathbf{v}_n, \ldots, \mathbf{v}_1, \mathbf{y}]$$

where $[\mathbf{t}_1, \ldots, \mathbf{t}_k]$ is the horizontal part and $[\mathbf{x}, \mathbf{u}_1, \ldots, \mathbf{u}_n, \mathbf{t}_1]$, $[\mathbf{t}_k, \mathbf{v}_n, \ldots, \mathbf{v}_1, \mathbf{y}]$ are vertical parts. Clearly, $\{\mathbf{t}_1, \ldots, \mathbf{t}_k\}$ must be included in one horizontal connected component of X_1, we denote it by T_j. With the notation as in Theorem 2.3(i), it follows that for $\mathbf{x} \neq \mathbf{y} \in X_1$,

$$|\pi(\mathbf{x}, \mathbf{y})| = |\mathbf{x}| + |\mathbf{y}| - 2l + h, \qquad |\pi(\sigma(\mathbf{x}), \sigma(\mathbf{y}))| = |\sigma(\mathbf{x})| + |\sigma(\mathbf{y})| - 2l' + h'.$$

We have

$$\big||\pi(\sigma(\mathbf{x}), \sigma(\mathbf{y}))| - |\pi(\mathbf{x}, \mathbf{y})|\big| \le |h - h'| + 2|l' - l| \le k + 2|l' - l|$$

where k is a hyperbolic constant as in Theorem 2.3(ii). If T_j is a singleton, then

$$|l' - l| = 0.$$

If T_j contains more than one point, then the elements of $\sigma(T_j)$ share the same parent. Then the confluence of $\sigma(\mathbf{x})$ and $\sigma(\mathbf{y})$ (as a tree) is $\sigma(\mathbf{x})^{-1}$ $(=\sigma(\mathbf{y})^{-1})$. Hence

$$|l' - l| = 1.$$

Consequently

$$\big||\pi(\sigma(\mathbf{x}), \sigma(\mathbf{y}))| - |\pi(\mathbf{x}, \mathbf{y})|\big| \le k + 2.$$

This completes the proof that σ is a near-isometry and the theorem is established. $\square$

Corollary 4.5. *Under the same assumption of Theorem 3.7. If there exists $k \in \mathbb{N}$ such that A^k is $(m^k, \mathbf{b})$-rearrangeable, then $\partial(X, \mathcal{E}) \simeq \partial(X, \mathcal{E}_v)$.*

To prove of Theorem 3.8 that primitive implies $(m, \mathbf{b})$-rearrangeable, we need a combinatorial lemma due to Xi and Xiong [24]. We include their proof for completeness.

Lemma 4.6. *Let p, m and $\{n_i\}_{i\in\Lambda}$ be positive integers with $\sum_{i\in\Lambda} n_i = pm$. Suppose there exists an integer l with $n_i \le l < m$ for all $i \in \Lambda$, and $\#\{i \in \Lambda : n_i = 1\} \ge pl$. Then there is a decomposition $\Lambda = \bigcup_{s=1}^{p} \Lambda_s$ satisfying $\sum_{i\in\Lambda_s} n_i = m$ for $1 \le s \le p$.*

Proof. The lemma is trivially true for $p = 1$, suppose it is true for p. For $p + 1$, let $\Omega \subset \{i : n_i = 1\}$ with $\#\Omega = (p + 1)l$, and select a maximal subset Δ_1 of $\Lambda \setminus \Omega$ such that $\sum_{i\in\Delta_1} n_i < m$. We claim that

$$\sum_{i\in\Delta_1} n_i \ge m - l.$$

For otherwise, take $i_0 \in \Lambda \setminus (\Delta_1 \cup \Omega)$, then $\sum_{i\in\Delta_1\cup\{i_0\}} n_i < (m - l) + l = m$, which contradicts the maximality of Δ_1 and the claim follows. Choose a subset Ω_1 from Ω such that $\#\Omega_1 = m - \sum_{i\in\Delta_1} n_i$ $(\le l)$ and set $\Lambda_1 = \Delta_1 \cup \Omega_1$, then $\sum_{i\in\Lambda_1} n_i = m$.

Note that for $\Lambda' = \Lambda \setminus \Lambda_1$, $\#\{i \in \Lambda' : n_i = 1\} \ge pl$. Applying the inductive hypothesis on p, we get a decomposition $\Lambda' = \bigcup_{s=2}^{p+1} \Lambda_s$ with $\sum_{i\in\Lambda_s} n_i = m$ for $s \ge 2$. Therefore, the assertion for $p + 1$ holds, and the lemma is proved. □

Lemma 4.6 yields the following rearrangement lemma we need (see also [4]).

Lemma 4.7. *Let $\mathbf{b} = [b_1, \dots, b_r]^t \in \mathbb{N}^r$ with $b_1 = 1$. Let $\ell = \max_j b_j$ and $\mathbf{a} = [a_1, \dots, a_r] \in \mathbb{Z}_+^r$ such that $a_1 \ge \ell^2$. Suppose there exists $m > \ell$ such that $\mathbf{ab} = pm$ for some integer $0 < p \le \ell$, then $\mathbf{a}$ is $(m, \mathbf{b})$-rearrangeable.*

Proof. We first assume that all $a_i > 0$. By the assumption, we have $\ell < m$ and $a_1 \ge \ell^2 \ge p\ell$. Define a sequence $\{n_j\}_{j=1}^r$ by

$$n_j = \begin{cases} b_1 & j = 1, \dots, a_1; \\ b_2 & j = a_1 + 1, \ \dots, \ a_1 + a_2; \\ \vdots & \\ b_r & j = \sum_{j=1}^{r-1} a_j + 1, \dots, \sum_{j=1}^{r} a_j. \end{cases}$$

Note that $n_j \le \ell$. Let $\Lambda = \{1, 2, \ \dots, \ \sum_{j=1}^r a_j\}$ be the index set. Then the assumption $\mathbf{ab} = pm$ is equivalent to:

$$\sum_{j\in\Lambda} n_j = pm.$$

By Lemma 4.6, there is a decomposition $\Lambda = \bigcup_{s=1}^{p} \Lambda_s$ satisfying $\sum_{j\in\Lambda_s} n_j = m$ for $1 \le s \le p$. Counting the number of b_j's in each group s,

$$c_{sj} = \#\{k \in \Lambda_s : n_k = b_j\}, \quad 1 \le s \le p, \ 1 \le j \le r.$$

The lemma follows by letting the matrix $C = [c_{sj}]_{p\times r}$.

If some of the a_i equals zero. Without loss generality we assume that $a_r = 0$ and $a_i > 0, \ 1 \le i \le r-1$. Let $\mathbf{a}' = [a_1, \ldots, a_{r-1}]$ and $\mathbf{b}' = [b_1, \ldots, b_{r-1}]$, then $\mathbf{a}'\mathbf{b}' = pm$ and by the above conclusion, $\mathbf{a}'$ is $(m, \mathbf{b})$-rearrangeable by a $p \times (r-1)$ matrix C'. Let C be the $p \times r$ matrix obtained by adding a last column $\mathbf{0}$ to C'. Then it is easy to see that $\mathbf{a}$ is $(m, \mathbf{b})$-rearrangeable. □

Proposition 4.8. *Let A be an $r \times r$ nonnegative integral matrix and is primitive (i.e., $A^n > 0$ for some $n > 0$). Let $\mathbf{b} = [b_1, \ldots, b_r]^t \in \mathbb{N}^r$ with $b_1 = 1$ and $A\mathbf{b} = m\mathbf{b}$ for some $m \in \mathbb{N}$. Then A^k is $(m^k, \mathbf{b})$-rearrangeable for some $k > 0$.*

Proof. Let $\ell = \max_j b_j$. Observe that A is primitive, there exists a large k such that $\ell < m^k$, and $A^k > 0$ with every entry greater than ℓ^2. Hence Lemma 4.7 implies A^k is $(m^k, \mathbf{b})$-rearrangeable. □

Proof of Theorem 3.8. Let A be the incidence matrix, and $A\mathbf{b} = m\mathbf{b}$ where $\mathbf{b} = [b_1, \ldots, b_r]^t$ with $b_i = \#T$ where $[T] = \mathscr{T}_i$ (Proposition 3.6). If we let the root o to be $\mathscr{T}_1$, then it is clear from the definitions of A, $\mathbf{b}$ and $A\mathbf{b} = m\mathbf{b}$ that $b_1 = 1$. Hence Proposition 4.8 implies that A^k is $(m^k, \mathbf{b})$-rearrangeable for some $k > 0$.

To prove the last part, we can assume without loss of generality that A *is* $(m, \mathbf{b})$-*rearrangeable and* $\max_i b_i \le m$. For otherwise, by the primitive assumption on A and by the same reasoning as in the proof of Theorem 3.7, we can consider the augmented tree as the IFS defined by the k-th iteration of $\{S_i\}_{i=1}^m$, and the corresponding A^k is $(m^k, \mathbf{b})$-rearrangeable. Hence we can apply Theorem 3.7 and complete the proof of Theorem 3.8. □

5. Examples

In this section, we provide several examples to illustrate our theorems of simple augmented tree and Lipschitz equivalence. Note that all the IFS's considered here satisfy condition (H); also the bounded closed invariant sets J we use are connected tiles and they have finite number of neighbors. By using the following lemma, we see that the process of finding the connected components will end in finitely many steps.

We assume the IFS consists of contractive similitudes $S_i(x) = B^{-1}(x + d_i), i = 1, \ldots, m$, where $B^{-1} = rR$ is the scaled orthogonal matrix. Let J be a closed subset such that $S_i(J) \subset J$ for each i. Then for $\mathbf{i} = i_1 \cdots i_k \in \Sigma^k$, we have $J_{\mathbf{i}} = S_{\mathbf{i}}(J) = B^{-k}(J + d_{\mathbf{i}})$ where $d_{\mathbf{i}} = d_{i_k} + Bd_{i_{k-1}} + \cdots + B^{k-1}d_{i_1}$.

Lemma 5.1. *For two horizontal connected components $T_1 \subset \Sigma^{k_1}$, $T_2 \subset \Sigma^{k_2}$ with $\#T_1 = \#T_2 = \ell$. If there exist similitudes $\phi_i(x) = B^{k_i}x + c_i$, $i = 1, 2$, where $c_i \in \mathbb{R}^d$ such that*

$$\phi_1(\cup_{\mathbf{i}\in T_1} J_{\mathbf{i}}) = \phi_2(\cup_{\mathbf{j}\in T_2} J_{\mathbf{j}}) = J \cup (J + v_1) \cup \cdots \cup (J + v_{\ell-1})$$

for some vectors $v_1, \ldots, v_{\ell-1} \in \mathbb{R}^d$, then $T_1 \sim T_2$.

Proof. By the definition and (3.1), it suffices to prove $T_1\Sigma$ and $T_2\Sigma$ have the same connectedness structure, equivalently, to prove this for $\bigcup_{\mathbf{i}\in T_1} \bigcup_{i=1}^m J_{\mathbf{i}i}$ and $\bigcup_{\mathbf{j}\in T_2} \bigcup_{j=1}^m J_{\mathbf{j}j}$. Letting $v_0 = 0$, we have

$$\begin{aligned}\phi_1\left(\bigcup_{\mathbf{i}\in T_1}\bigcup_{i=1}^{m} J_{\mathbf{i}i}\right) &= B^{k_1}\left(\bigcup_{\mathbf{i}\in T_1}\bigcup_{i=1}^{m} J_{\mathbf{i}i}\right)+c_1\\ &= B^{k_1}\left(\bigcup_{\mathbf{i}\in T_1}\bigcup_{i=1}^{m} B^{-k_1-1}(J+d_i+Bd_{\mathbf{i}})\right)+c_1\\ &= \bigcup_{\mathbf{i}\in T_1}\bigcup_{i=1}^{m}(J_i+d_{\mathbf{i}})+c_1=\bigcup_{j=0}^{\ell-1}\bigcup_{i=1}^{m}(J_i+v_j).\end{aligned}$$

Similarly we have $\phi_2\big(\bigcup_{\mathbf{j}\in T_2}\bigcup_{j=1}^{m} J_{\mathbf{j}j}\big)=\bigcup_{j=0}^{\ell-1}\bigcup_{i=1}^{m}(J_i+v_j)$. Since ϕ_1,ϕ_2 are similarity maps which preserve the connectedness, the lemma follows. □

We remark that it is a lot simpler to use the approach here than the method of constructing the graph directed system as in [4,18,24]. In fact Example 5.4 shows that it may be difficult to use the later to prove the Lipschitz equivalence. The same example also shows that the converse of the above lemma does not hold.

Example 5.2. Let $\mathcal{D}_1=\{0,2,4\}$ and $\mathcal{D}_2=\{0,3,4\}$ be two digit sets, and let K_1 and K_2 be the two self-similar sets defined by:

$$K_1=\frac{1}{5}(K_1+\mathcal{D}_1),\qquad K_2=\frac{1}{5}(K_2+\mathcal{D}_2).$$

Then K_1 is clearly dust-like and $K_1\simeq K_2$ (see Fig. 2).

This is a well-known example of Lipschitz equivalence of self-similar sets called the $\{1,3,5\}$–$\{1,4,5\}$ *problem* [3,18], which was proved by showing that they have the same graph directed system. In our approach, the associated augmented tree for K_1 is just the standard rooted tree with no horizontal edges, and the incidence matrix is [3]. For K_2, by letting $S_i(x)=(x+d_i)/3,\ d_i\in\mathcal{D}_2$, and use $J=[0,1]$ in the definition of the augmented tree, we see that in total, there are three connected classes (after three iterations): $\mathscr{T}_1=[o]$, $\mathscr{T}_2=[\{2,3\}]$ and $\mathscr{T}_3=[\{22,23,31\}]$. Hence X is simple and the incidence matrix is

$$A=\begin{bmatrix}1&1&0\\1&1&1\\1&1&2\end{bmatrix}.$$

Clearly A is primitive, and $K_2\simeq K_1$ by Theorem 3.10. □

The following proposition is a generalization of Example 5.2 on $\mathbb{R}$ [18].

Proposition 5.3. *Let $\mathcal{D}$ be a proper subset of $\{0,1,\ldots,n-1\}$, and let K the self-similar set on $\mathbb{R}$ satisfying*

$$K=\frac{1}{n}(K+\mathcal{D}).$$

Then the corresponding augmented tree is simple and the incidence matrix is primitive. Consequently K is Lipschitz equivalent to a dust-like set.

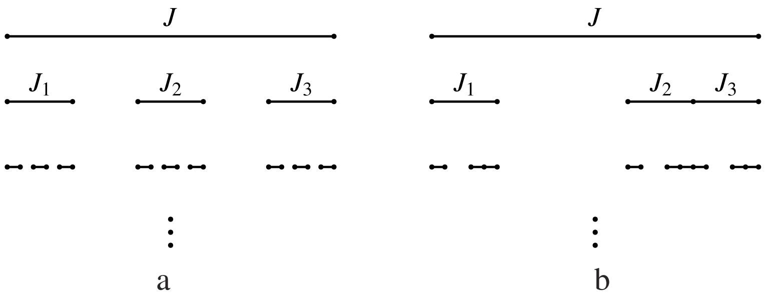


Fig. 2.

Proof. Let $S_j(x) = \frac{1}{n}(x + d_j),\ d_j \in \mathcal{D}$ be the IFS for K with $\#\mathcal{D} = m(< n)$. We will ignore the trivial case that the IFS is strongly separated. We use $J = [0, 1]$ to define the associated augmented tree X, and it is easy to see that it is simple. Our main task is to show that the incidence matrix is primitive. Let $F_k = \bigcup_{\mathbf{i} \in \Sigma^k} S_{\mathbf{i}}(J)$ be the k-th iteration. Let $L_i, 1 \le i \le \ell$ denote the disjoint closed sub-intervals of F_1 arranged from left to right.

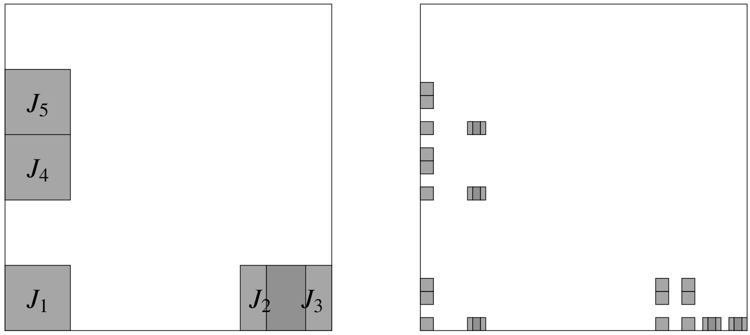
We first consider the case that $\mathcal{D}$ does not contain both 0 and $n-1$. In this case, the connected components of the augmented tree is determined by the root and the components in the first level. Let o be the root which represents the connected class $\mathcal{T}_1$. For the others, let $b_i\ (= n|L_i|)$ be the number of vertices in the connected component where $i = 1, \ldots, r$ for some $r \le \ell$. For convenience, we assume $b_1 = 1$ and b_r corresponds the rightmost sub-interval of F_1. Note that $b_i > 1$ for $2 \le i \le r$. Let a_i denote the number of such components. Then it is easy to see that the incidence matrix is given by

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_r \\ a_1 b_2 & a_2 b_2 & \cdots & a_r b_2 \\ \vdots & \vdots & & \vdots \\ a_1 b_r & a_2 b_r & \cdots & a_r b_r \end{bmatrix}.$$

Hence A is primitive and the proposition follows.

For the case that $\{0, n-1\} \subset \mathcal{D}$, we note that for an interval L_i in F_1 with $b_i > 1$, if we inspect the sub-intervals of F_2 contained in L_i, we find that there is a group of new sub-intervals that come from union of two intervals: one with right end points $S_{im}(1)$ and one with left end-point the $S_{i+1,1}(0)$ (see Fig. 2(b), the fourth interval on the third level). There are $b_i - 1$ of them and the length is $1 + b_r$. If there is already a component of this length, say the second one, then the incidence matrix is an $r \times r$ matrix

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_{r-1} & a_r \\ a_1 b_2 - (b_2 - 1) & a_2 b_2 + (b_2 - 1) & \cdots & a_{r-1} b_2 & a_r b_2 - (b_2 - 1) \\ \vdots & \vdots & & \vdots & \vdots \\ a_1 b_r - (b_r - 1) & a_2 b_r + (b_r - 1) & \cdots & a_{r-1} b_r & a_r b_r - (b_r - 1) \end{bmatrix}.$$



a b

Fig. 3. The first two iterations for K.

If this is a new component, then the incidence matrix is an $(r+1)\times(r+1)$ matrix with

$$A=\begin{bmatrix} a_1 & a_2 & \cdots & a_{r-1} & a_r & 0\\ a_1b_2-(b_2-1) & a_2b_2 & \cdots & a_{r-1}b_2 & a_rb_2-(b_2-1) & b_2-1\\ \vdots & \vdots & & \vdots & \vdots & \vdots\\ a_1b_r-(b_r-1) & a_2b_r & \cdots & a_{r-1}b_r & a_rb_r-(b_r-1) & b_r-1\\ a_1(1+b_r)-b_r & a_2(1+b_r) & \cdots & a_{r-1}(1+b_r) & a_r(1+b_r)-b_r & b_r \end{bmatrix}.$$

Again, A is primitive that implies the proposition. $\square$

The above proposition also has a higher dimensional extension to the totally disconnected *fractal cubes* of the form $K=\frac{1}{n}(K+\mathcal{D})$ where $\mathcal{D}\subset\{0,\ldots,n-1\}^d$ by Xi and Xiong [24] via constructing a more complicated graph directed system. Their result can also be put into the framework of Theorem 3.8, we will omit the detail. In the following we will consider other totally disconnected self-similar sets of more general forms.

The following shows that the $S_j(J)$'s can have large overlaps (though the OSC is still satisfied by Corollary 3.11). Moreover it shows that the converse of Lemma 5.1 does not hold.

Example 5.4. On $\mathbb{R}^2$, let K be the self-similar set defined by the IFS

$$S_i(x)=\frac{1}{5}(x+d_i)\quad\text{where } d_i\in\left\{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}18/5\\0\end{bmatrix},\begin{bmatrix}4\\0\end{bmatrix},\begin{bmatrix}0\\2\end{bmatrix},\begin{bmatrix}0\\3\end{bmatrix}\right\}.$$

We use $J=[0,1]^2$ as the invariant set to define an augmented tree X for K. It can be shown that, in the graph X, the two components $T_1=\{2,3\}$ and $T_2=\{4,5\}$ are equivalent in the augmented tree, as the $T_1\cup T_1\Sigma$ and $T_2\cup T_2\Sigma$ are graph isomorphic (see Fig. 3), but there do not exist similarity maps ϕ_1,ϕ_2 as in Lemma 5.1 such that $\phi_1(J_2\cup J_3)=\phi_2(J_4\cup J_5)$.

On the other hand, if we continue the process by iterating, we can see that X is simple with two connected classes $\mathscr{T}_1=[o]$ and $\mathscr{T}_2=[T_1]$, and the incidence matrix is $\begin{bmatrix}1&2\\2&4\end{bmatrix}$ which is primitive. Hence K is Lipschitz equivalent to a dust-like self-similar set by Theorem 3.10.

In the following we consider the self-similar set selected from a self-similar tile (also the more general self-affine case) to replace the square and sub-squares in the previous case. Let B

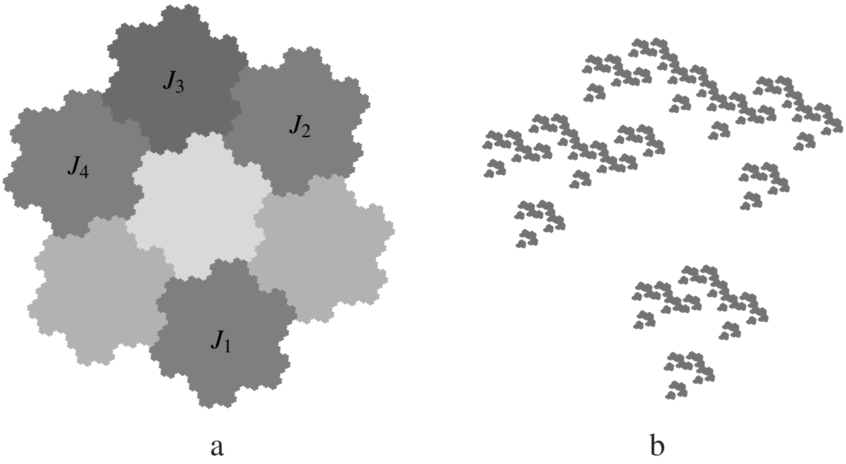


Fig. 4.

be a $d \times d$ expanding matrix (i.e., all the eigenvalues have moduli > 1) and $|\det(B)| = q$; let $\mathcal{D} = \{d_1, \ldots, d_q\} \subset \mathbb{R}^d$ be a digit set. The attractor J generated by

$$S_i(x) = B^{-1}(x + d_i), \quad d_i \in \mathcal{D}$$

satisfies $J = B^{-1}(J + \mathcal{D})$ (more conveniently, we use $BJ = J + \mathcal{D}$). It is called a *self-affine tile* if the interior of J is nonempty [10]. If in addition, B is a similar matrix, then J is a *self-similar tile*. It is well-known that if the tile J is disc-like (i.e., homeomorphic to the unit disc), then J has either six (hexagonal) or eight (square) neighbors [1]. This is needed to determine the connected classes of the augmented tree of the underlying self-similar set.

Example 5.5. $B = \begin{bmatrix} 5/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 5/2 \end{bmatrix}$ and $\mathcal{D} = \{0, \pm v_1, \pm v_2, \pm(v_1 + v_2)\}$. With $v_1 = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \end{bmatrix}$. If we let $S_i(x) = B^{-1}(x + d_i)$, $d_i \in \mathcal{D}$, then J is a self-similar tile on $\mathbb{R}^2$, and is called the *Gosper island* (Fig. 4(a)). If we let

$$\mathcal{D}' = \{v_2,\ v_1,\ -v_2,\ -(v_1 + v_2)\}$$

and let K be the corresponding self-similar set (see Fig. 4(b)). We use J to define the augmented tree. There are six neighbors of J, they are

$$J \pm v_1, \qquad J \pm v_2, \qquad J \pm (v_1 + v_2). \tag{5.1}$$

By making use of (5.1), we can show that the incidence matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 1 & 0 & 1 \\ 4 & 1 & 1 & 0 & 1 & 1 \\ 6 & 1 & 1 & 0 & 1 & 2 \end{bmatrix}$$

and it is primitive. Hence K is Lipschitz equivalent to a dust-like set. We omit the detail.

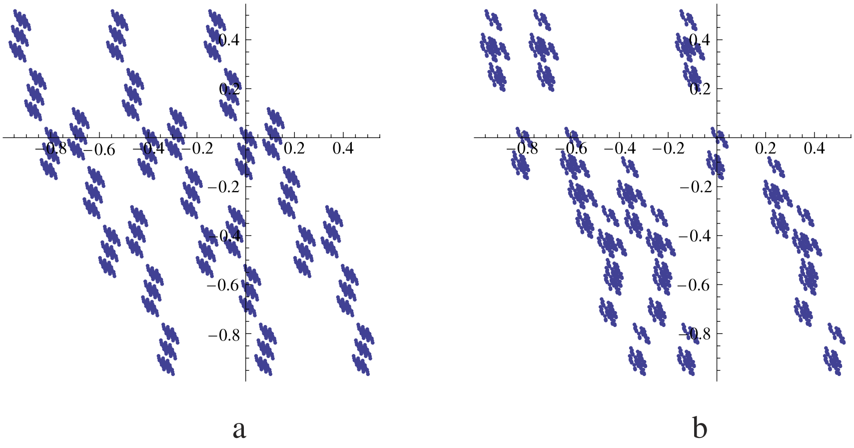


Fig. 5.

Finally we consider a case of self-affine sets which is a direct analog of Example 5.2. We make use the modified Lipschitz equivalence in Theorem 3.14 for self-affine sets.

Example 5.6. Let

$$B = \begin{bmatrix} 0 & 1 \\ -5 & -3 \end{bmatrix} \quad \text{and} \quad \mathcal{D} = \left\{ \begin{bmatrix} 0 \\ i \end{bmatrix} : i = 0, 1, 2, 3, 4 \right\}.$$

Then J is a self-affine tile; moreover it is known that J is homeomorphic to the unit disc, and it has six neighbors (Theorem 4.1 in [14]):

$$J \pm v, \qquad J \pm (Bv + 2v), \qquad J \pm (Bv + 3v) \quad \text{with } v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{5.2}$$

If we let

$$\mathcal{D}_1 = \left\{ \begin{bmatrix} 0 \\ i \end{bmatrix} : i = 0, 2, 4 \right\} \quad \text{and} \quad \mathcal{D}_2 = \left\{ \begin{bmatrix} 0 \\ i \end{bmatrix} : i = 0, 3, 4 \right\}$$

and let K_1 and K_2 be the corresponding self-affine sets. Then the augmented trees are simple, K_1 is dust-like and $K_1 \simeq K_2$ (see Fig. 5).

Indeed, it is easy to see that the IFS for K_1 is strongly separated. As the moduli of the eigenvalues of B are equal to $\sqrt{5}$, by Theorem 1.3 and Corollary 3.2 of [8], $\dim_H K_1 = \dim_H^w K_1 = 2\log 3/\log 5$.

For K_2, we let $S_j(x) = B^{-1}(x + d_j)$, $d_j \in \mathcal{D}_2$ be the IFS. Note that the tile J satisfies $BJ = J + \mathcal{D}$, we will use this together with the neighborhood relation of J to select the corresponding ones for $\mathcal{D}_2$. Let $\mathscr{T}_1 = \{o\}$, we observe that $(J + 3v) \cap (J + 4v) \neq \emptyset$ (equivalently, $J \cap (J + v) \neq \emptyset$), hence there is only one horizontal edge 2–3 in the first level of the augmented tree, let $\mathscr{T}_2 = [\{2, 3\}]$. To find the next connected class from $\mathscr{T}_2$, we note that the connectedness of $\{S_{2j}(J), S_{3j}(J)\}_{j=1}^3$ is the same as (with a B^2 multiple)

$$J, \ (J + 3v), \ (J + 4v) \quad \text{and} \quad (J + Bv), \ (J + Bv + 3v), \ (J + Bv + 4v)$$

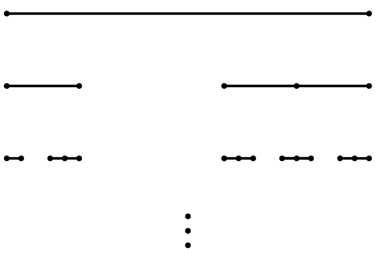


Fig. 6. The totally disconnected K defined by (6.1).

(which are the cells of $B(J \cup (J+v))$ corresponding to $\mathcal{D}_2$). By (5.2) (use the neighbors $J \pm v$ and $J \pm (Bv + 3v)$), we conclude that $\{2,3\}$ generates three components

$$\{21, 32, 33\},\ \{22, 23\},\ \{31\}.$$

The first one belongs to a new class, for the iteration, we selected from $B(J \cup (J + Bv + 3v) \cup (J + Bv + 4v))$ those cells corresponding to $\mathcal{D}_2$ and use Hamilton–Cayley theorem to obtain

$$J,\ (J+3v),\ (J+4v); \qquad (J-5v),\ (J-2v),\ (J-v);$$
$$(J+Bv-5v),\ (J+Bv-2v),\ (J+Bv-v).$$

Again the neighborhood relation in (5.2) determines the components

$$\{211\},\ \{212, 213\},\ \{321\},\ \{322, 323\},\ \{331\},\ \{332, 333\}.$$

There are no new ones if we continue the iteration, hence the three connected classes are: $\mathscr{T}_1 = [o]$, $\mathscr{T}_2 = [\{2,3\}]$ and $\mathscr{T}_3 = [\{21, 32, 33\}]$. The incidence matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & 3 & 0 \end{bmatrix}.$$

Clearly A is primitive, and $K_2 \simeq K_1$ by Theorem 3.14 under the ultra-metric defined by w. $\square$

6. Remarks and open questions

We note that not all incidence matrices of simple augmented trees are rearrangeable. For example, we let

$$K = \frac{1}{5}K \cup \frac{1}{5}(-K+4) \cup \frac{1}{5}(K+4). \tag{6.1}$$

(It is a modification of K_2 in Example 5.2 by putting in a reflection in the middle term.) Then the incidence matrix is $\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$. It is not rearrangeable, and it is easy to show that K is a countable union of dust-like sets, but we do not know if it is Lipschitz equivalent to a dust-like set (see Fig. 6).

Q1. Find conditions on the simple augmented tree that describe the above situation. Also in view of Theorem 3.8, is an irreducible incidence matrix A rearrangeable?

We have shown that if an augmented tree is simple, then the self-similar set is totally disconnected (Proposition 2.5); also the converse is true on $\mathbb{R}$ under the OSC. We ask

Q2. Suppose $K \subset \mathbb{R}^d$ is defined by (2.2), if K is totally disconnected, does it imply that augmented tree is simple, equivalently, the horizontal connected component of the augmented tree is uniformly bounded (with or without assuming the OSC)?

In recent literature, the investigations of the Lipschitz equivalence are mainly for the *fractal cubes* of the form $K = \frac{1}{n}(K + \mathcal{D})$ where $\mathcal{D} \subset \{0, \ldots, n-1\}^d$ [24,25]. In Examples 5.5 and 5.6, we bring in another classes of self-similar/self-affine sets K that are based on the disc-like tiles on $\mathbb{R}^2$. Note that such tiles have six or eight neighbors [1], and for such K, the augmented trees depend very much on this neighborhood relationship. For the self-affine tiles generated by consecutive collinear digit sets, the disc-like property has been completely characterized in [14] through the characteristic polynomials of the expanding matrices. This provides a wealth of source to study the topological property of such K. A general question is

Q3. For the classes of self-similar/self-affine sets K stated above, can we characterize the digit set $\mathcal{D}$ so that K is totally disconnected?

Also in [11], the authors have made a head start to investigate the classification of the non-totally disconnected fractal squares.

Q4. Can we put such classification into the framework of augmented trees?

In our consideration, we have restricted the IFS to have equal contraction ratio. On the other hand, there is also interesting study for different contraction ratios (e.g., [17,19,23]). For the augmented tree, the setup in [12] is actually for non-equal contraction ratios that the horizontal level Σ^n can be replaced by the standard level set: $\Lambda_n = \{\mathbf{i} = i_1 \cdots i_k : r_{i_1} \cdots r_{i_k} \le r^n < r_{i_1} \cdots r_{i_{k-1}}\}$ where r_i's are the contraction ratios. More generally, the authors have observed that the augmented trees can also be defined for Moran fractals and the hyperbolic property still holds. It is possible that the augmented trees can also be useful for the Lipschitz equivalence of fractal sets in those cases.

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