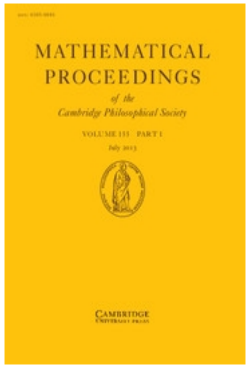

Topological structure of fractal squares

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Topological structure of fractal squares

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Abstract

Given an integer $n \geqslant 2$ and a digit set $\mathcal{D} \subsetneq \{0, 1, \dots, n-1\}^2$, there is a self-similar set $F \subset \mathbb{R}^2$ satisfying the set equation: $F = (F + \mathcal{D})/n$. We call such F a fractal square. By studying a periodic extension $H = F + \mathbb{Z}^2$, we classify F into three types according to their topological properties. We also provide some simple criteria for such classification.

1. *Introduction*

For $n \geqslant 2$, let $\mathcal{D} \subset \{0, 1, \dots, n-1\}^2$ and call it a *digit set*. We assume that $1 < \#\mathcal{D} < n^2$ to exclude the trivial case. Let $F \subset \mathbb{R}^2$ be the unique non-empty compact set satisfying the set equation (cf. [**5**])

$$F = (F + \mathcal{D})/n. \tag{1·1}$$

We shall call F a *fractal square*. A familiar example of a fractal square is the Sierpinski carpet. Let $I = [0, 1]^2$ be the unit square. We define $F_1 = (I + \mathcal{D})/n$, and recurrently, $F_{k+1} = (F_k + \mathcal{D})/n$ for $k \geqslant 1$. Then F_k is a union of squares of size $1/n^k$ (called them the *k-cells*). Clearly $F_{k+1} \subset F_k$ and $F = \bigcap_{k=1}^{\infty} F_k$.

The topological structure of self-similar/self-affine sets, including connectedness, local connectedness, disk-likeness, is an important topic in fractal geometry. In [**6**], Hata first gave a criterion for connectedness of self-similar sets. Subsequently, there are many works devoted to study such topological properties (cf. [**1**, **2**, **3**, **7**, **9**, **10**, **11**, **13**, **14**, **15**]). Recently, Taylor et al [**17**, **18**] considered the connectedness properties of the Sierpinski relatives with rotations and reflections. Xi and Xiong [**19**] showed that for a fractal square, it is totally disconnected if and only if *the number of cells in the connected components in each iteration* *is uniformly bounded*. Moreover, Roinestad [**16**] proved that a fractal square F is totally disconnected if and only if *for some* $k \geqslant 1$, $I \setminus F_k$ *contains a path that can reach the opposite side of the square*.

Our aim in the paper is to provide a complete characterization on the topological structure of the fractal square through the connected components. We introduce periodic extensions of F and F_k by defining

$$H = F + \mathbb{Z}^2 \quad \text{and} \quad H_k = F_k + \mathbb{Z}^2$$

and denote their complements by H^c and H_k^c, respectively. Then we can classify F into three classes topologically. We summarize the results in the following theorem. For brevity, we will use *component* to mean *connected component*, and a *non-trivial component* means it has more than one point.

THEOREM 1·1. *Let F be a fractal square as in* (1·1). *Then F satisfies either*:

(i) *H^c has a bounded component, which is also equivalent to: F contains a non-trivial component that is not a line segment*; *or*

(ii) *H^c has an unbounded component, then F is either totally disconnected or all non-trivial components of F are parallel line segments.*

It is easy to see that the Sierpinski carpet is of type (i). The two cases in type (ii) are not so obvious. The reader can refer to Section 5 for the examples and the figures. The above theorem is proved in Section 2 (Theorems 2·2, 2·5 and Corollary 2·6).

The three classes of fractal squares can be determined in finite steps. Indeed, we show that if F contains a line segment, then it can be detected in F_1 (Theorem 3·3). Also to show whether H^c has unbounded components, we make use of certain class of paths in H_k^c, which can be constructed inductively and is easy to check (Theorems 4·4, 4·6). These are proved in Sections 3 and 4. The implementation of these criteria and some examples are given in Section 5.

We remark that it is not straightforward to generalize the present classification to higher dimensions, as the technique here depends very much on the two dimensional topology. On the other hand it is possible to extend this consideration to disk-like self-affine tiles (cf. [**2**, **3**, **9**]) by replacing the square I here. Indeed for the totally disconnected case, this approach (and for more general self-similar sets) has been taken up by the authors [**8**] to study the Lipschitz equivalence problem. It may be a useful setting to study the classification problem introduced here.

2. *Classification of F by connected components*

For $H_k = F_k + \mathbb{Z}^2$, $H = F + \mathbb{Z}^2$, it is clear that $q + H = H$ for $q \in \mathbb{Z}^2$, and $H_{k+1} \subset H_k$. Moreover we have

$$H_{k+1} \subset H_k/n, \quad H \subset H/n \tag{2·1}$$

(as $H_{k+1} = F_{k+1} + \mathbb{Z}^2 = (F_k + \mathcal{D})/n + \mathbb{Z}^2 \subset (F_k + \mathbb{Z}^2)/n = H_k/n$). For the complement, we have $H^c = \bigcup_{k \geqslant 1} H_k^c$, $H_k^c \subset H_{k+1}^c$ and

$$H_k^c/n \subset H_{k+1}^c, \quad H^c/n \subset H^c. \tag{2·2}$$

LEMMA 2·1. *If there is a component in H^c that is bounded, then every component of H^c is bounded.*

Proof. Let U be a bounded component of H^c as in the assumption. Then U is an open set since H^c is open. Let I denote the unit square and let k be an integer so that $a + I \subset n^k U$ for some $a \in \mathbb{Z}^2$. For an arbitrary component V of H^c, choose a point $b \in \mathbb{Z}^2$ such that $V \cap (b + I) \neq \emptyset$. Then $(V - (b - a)) \cap (a + I) \neq \emptyset$, so that

$$\frac{1}{n^k}(V - (b - a)) \cap U \neq \emptyset.$$

Since U is a component and $\dfrac{1}{n^k}(V - (b - a))$ is connected, we conclude that

$$\frac{1}{n^k}(V - (b - a)) \subset U.$$

Therefore $V \subset b - a + n^k U$ and it is bounded.

THEOREM 2·2. *If H^c contains a bounded component, then the diameter of every component is uniformly bounded, say by $\sqrt{2}(n^2 + 1)^2/n$.*

Proof. We will prove the following claim: *If there is a curve $\gamma : [0, 1] \to H^c$ with*

$$\mathrm{diam}(\gamma) > \sqrt{2}(n^2 + 1)^2/n, \tag{2·3}$$

then there is a curve $\gamma' \subset H^c$ such that $\gamma'(1) - \gamma'(0) = q \in \mathbb{Z}^2 \setminus \{0\}$. For such γ', it has the property that $\gamma' + \mathbb{L} \subset H^c$ is a continuous curve with translational period q (here $\mathbb{L} := \{mq : m \in \mathbb{Z}\}$). Note that $\gamma^* := \gamma' + \mathbb{L}$ behaves asymptotically like a straight line through $\gamma'(0)$ with slope q. Therefore the component of H^c containing γ^* is unbounded, and by Lemma 2·1, every component of H^c is unbounded. The theorem is a contrapositive statement of this.

For the $\gamma \subset H^c$, we consider the intersection of γ and H_1^c. Let $a_1 + I/n, \ldots, a_m + I/n$ be the squares intersecting γ and satisfying $a_i \in \mathcal{D}^c/n + \mathbb{Z}^2$.

Case 1. If $m = 0$, i.e., there are no such squares, then for any $b \in \mathcal{D}^c/n + \mathbb{Z}^2$, $\gamma \cap (b + F/n) \subset \gamma \cap (b + I/n) = \emptyset$. Also for any $b \in \mathcal{D}/n + \mathbb{Z}^2$, $\gamma \cap (b + F/n) \subset \gamma \cap H = \emptyset$. It follows that

$$(\gamma + b) \subset H^c/n \subset H^c, \qquad \forall\, b \in \mathbb{Z}^2/n. \tag{2·4}$$

As $\mathrm{diam}(\gamma) > n^2 \cdot \sqrt{2}/n$ (by (2·3)), γ intersects $n^2 + 1$ subsquares of size $1/n$. Hence the pigeon hole principle implies that there exist $b_1,\ b_2 \in \mathbb{Z}^2/n$ such that $b_2 - b_1 = q \in \mathbb{Z}^2 \setminus \{0\}$. Pick any $v \in \mathcal{D}^c/n$, then $\gamma - b_1 + v$ is a curve in H^c joining $v + I/n$ and $v + q + I/n$. We let $\gamma' \subset H^c$ be the sub-curve with a slight adjustment to start at $b_1 + v$ and to end at $b_2 + v$ (this is possible by the openness of H^c) and re-parameterize it to be on $[0, 1]$. Then $\gamma'(1) - \gamma'(0) = q$ and the claim follows.

Case 2. $0 < m \leqslant n^2$. We can assume without loss of generality that

$$\|\gamma(1) - \gamma(0)\| = \mathrm{diam}(\gamma)\ (> \sqrt{2}(n^2 + 1)^2/n)) \tag{2·5}$$

(otherwise we can pick $c, c' \in \gamma[0, 1]$ that attain the diameter and restrict γ to start and end at c, c'). Clearly, there exist sub-curves $\gamma_0, \ldots, \gamma_k \subset \gamma$ and $\{b_1, \ldots, b_k\} \subset \{a_1, \ldots, a_m\}$ such that:

(i) γ_0 joins $\gamma(0)$ and $b_1 + I/n$; γ_j joins $b_j + I/n$ and $b_{j+1} + I/n$ for $1 \leqslant j \leqslant k - 1$; γ_k joins $b_k + I/n$ and $\gamma(1)$;

(ii) each γ_j can intersect $\bigcup_{i=1}^m (a_i + I/n)$ only at its end points.

We observe that it is impossible to have $\mathrm{diam}(\gamma_j) \leqslant n^2 \cdot \sqrt{2}/n$ for all $0 \leqslant j \leqslant k(\leqslant m)$. Indeed in such case, by (2·5), we have

$$\mathrm{diam}(\gamma) \leqslant \mathrm{diam}(\textstyle\bigcup_{j=0}^{k}\gamma_j) + k\sqrt{2}/n \leqslant \sqrt{2}(n^2+1)^2/n.$$

This contradicts (2·3). Hence one of the γ_j satisfies $\mathrm{diam}(\gamma_j) > n^2 \cdot \sqrt{2}/n$, then the proof of Case 1 will imply that the γ' in the claim exists.

Case 3. If $m > n^2$, then by the pigeon hole principle again, there exist a_i and a_j such that $a_i - a_j = q \in \mathbb{Z}^2 \setminus \{0\}$. We modify the sub-arc of γ to obtain a $\gamma' \subset H^c$ that starts at the center of $a_i + I/n$ and ends at the center of $a_j + I/n$, which satisfies the claim.

LEMMA 2·3. *If F contains a line segment, then H contains a straight line with the same slope (disregarding whether the components of H^c are bounded or not).*

Proof. Let L_0 be a line segment in F. Then for $k \geqslant 1$, $n^k L_0 \subset n^k H \subset H = F + \mathbb{Z}^2$. Let u_k be the mid-point of $n^k L_0$, and let v_k be a point in $\mathbb{Z}^2$ such that $u_k - v_k \in F$. Set $L_k = n^k L_0 - v_k$ and $a_k = u_k - v_k$. Then $\{a_k\}_k$ is a sequence in F. Since F is compact, there is a convergent subsequence in $\{a_k\}_k$. For simplicity, we assume $\{a_k\}_k$ itself converges to $a \in F$.

Let L be the straight line passing through a and parallel to L_0. We assert that L must lie in H. Indeed let $b \in L$, then as the vector $b - a$ and L_k have the same slope as L_0, and each L_k has length $n^k|L_0|$, it follows that there exists k_0 such that $a_k + (b - a) \subset L_k \subset H$ holds for any $k \geqslant k_0$. Since $a_k + (b - a)$ converges to $a + (b - a) = b$ and H is a closed set, we get $b \in H$. Thus $L \subset H$.

COROLLARY 2·4. *If F contains two non-parallel line segments, then there is a non-trivial component of F which is not a line segment.*

Proof. Let L_1, L_2 be the two non-parallel line segments in F. By Lemma 2·3, there exist two straight lines L_1', L_2' in H and they are parallel to L_1, L_2, respectively, and that $L_1' \bigcap L_2'$ is in F. The corollary follows.

According to the above results, if F possesses non-trivial components, then either all the components are parallel line segments (see Figure 2) or one of them is not a line segment (see Figure 1).

THEOREM 2·5. *F contains a non-trivial component which is not a line segment if and only if every component of H^c is bounded.*

Proof. We first show the necessity, let $C \subset F$ be a non-trivial component which is not a line segment, then there are three distinct points $a, b, c \in C$ not in a line. Suppose a component of H^c is unbounded. By the proof of Theorem 2·2, there exists a curve $\gamma \subset H^c$ such that $\gamma(1) - \gamma(0) = q \in \mathbb{Z}^2 \setminus \{0\}$, and $\gamma^* := \gamma + \{mq : m \in \mathbb{Z}\}$ is a curve in H^c behaves like a straight line asymptotically. Hence H is separated by γ^*. Assume the line segment $[a, b]$ is not parallel to $[0, q]$ (otherwise we take $[a, c]$ instead). We can take a large k and a suitable $z \in \mathbb{Z}^2$ such that $n^k C - z \subset H$ and $n^k a - z$ and $n^k b - z$ are separated by γ^*, which contradicts the connectedness of $n^k C - z$.

For the sufficiency, suppose U is a bounded component of H^c. Let V be the unbound component of $\mathbb{R}^2 \setminus \overline{U}$, then V is a simply connected domain. Hence the boundary ∂V is connected, $\partial V \subset H$, and it is not a line segment. It follows that the non-trivial components of F can not be parallel line segments. Hence F contains a non-trivial component which is not a line segment by Corollary 2·4.

COROLLARY 2·6. *If the components of H^c are unbounded, then either F is totally disconnected, or all non-trivial components of F are parallel line segments.*

In particular, in the second case, there are infinitely many unbounded components in H^c.

Proof. Since H^c contains unbounded components, by Theorem 2·5, it follows that F is either totally disconnected, or the non-trivial components of F are parallel line segments.

To prove the last statement, let L be a line in H, let $u \in \mathbb{Z}^2$ be a vector such that the line segment $[0, u]$ is not parallel to L. Then $L + mu$ are parallel lines for $m \in \mathbb{Z}$. Let U be the region bounded by $L + mu$ and $L + (m+1)u$, then $U \setminus H$ is an open set and it is not empty since $\dim_H F < 2$. Hence $U \setminus H$ contains at least one component, and each component in $U \setminus H$ is unbounded by Lemma 2·1.

3. *F containing line segments*

It is clear that F contains a vertical line segment (or horizontal line segment) if and only if F_1 does. Hence we will not include these two special cases in the following consideration. It follows from Lemma 2·3 that F contains a line segment if and only if $H = F + \mathbb{Z}^2$ contains a line. Suppose L is a line in H, then $\widetilde{L} = L/\mathbb{Z}^2$ can be regarded as a helix in the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Since the closure of $\widetilde{L}$ is contained in $(F + \mathbb{Z}^2)/\mathbb{Z}^2 (= F)$, which is a proper subset of $\mathbb{T}^2$. The helix $\widetilde{L}$ is not dense in $\mathbb{T}^2$, the slope of L must be a rational number. The same conclusion holds for a line in H_k. Let us denote the slope of L by $\tau = r/s$, where r and s are co-prime integers and $s \geqslant 1$. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection along the line L, that is, $\pi(x, y) = x - \tau y$, and let

$$\Omega = \{\omega \in \mathbb{R} : L_\omega \subset H\}, \qquad \Omega_k = \{\omega \in \mathbb{R} : L_\omega \subset H_k\},$$

where L_ω denotes the line with slope r/s and the x-intercept ω.

LEMMA 3·1. *With the above notation, then*:

(i) $\Omega + 1/s = \Omega$, $\Omega_k + 1/s = \Omega_k$; *and*

(ii) $\pi(H^c) = \mathbb{R} \setminus \Omega$, $\pi(H_k^c) = \mathbb{R} \setminus \Omega_k$.

Proof. (i) Note that $H + (1, 0) = H$ and $H + (0, 1) = H$, the projection π yields $\Omega + 1 = \Omega$ and $\Omega - r/s = \Omega$ respectively. Since r, s are co-prime, there exist integers k_1, k_2 such that $k_1 r + k_2 s = 1$. Hence

$$\Omega = \Omega + (k_2 + k_1 r/s) = \Omega + 1/s.$$

The same proof holds for Ω_k. Part (ii) is clear from the definition of the projection π.

Let $Tx = nx \pmod 1$ be a transformation on $[0, 1)$, and let $\widetilde{\Omega} = \Omega \bigcap [0, 1)$, $\widetilde{\Omega}_k = \Omega_k \bigcap [0, 1)$. The following lemma is crucial.

LEMMA 3·2. $\alpha \in \widetilde{\Omega}$ *if and only if the orbit* $\{T^k \alpha : k \geqslant 0\} \subset \widetilde{\Omega}_1$.

Proof. Suppose $\alpha \in \widetilde{\Omega}$, then $L_\alpha \subset H \subset H/n$ (by (2·1)). It follows that $nL_\alpha \subset H$, which implies $T\alpha = n\alpha \pmod 1 \in \widetilde{\Omega} \subset \widetilde{\Omega}_1$ and the necessity follows.

For the sufficiency, we claim that if $n\beta \in \Omega_k$, then $\beta \in \Omega_{k+1}$. Indeed we let A be the set of lattice points $d \in \mathbb{Z}^2/n$ such that $(d + I/n) \subset H_1$ and $L_\beta \bigcap (d + I^\circ/n) \neq \emptyset$. That $nL_\beta = L_{n\beta} \subset H_k$ (since $n\beta \in \Omega_k$) implies that

$$nL_\beta \bigcap (nd + I^\circ) \subset (nd + F_k)\,, \quad d \in A.$$

Taking the union of both sides for all $d \in A$ and the closure, we obtain

$$nL_\beta \subset \bigcup\nolimits_{d\in A}(nd + F_k).$$

Therefore $L_\beta \subset \bigcup_{d\in A}(d + F_k/n)) \subset H_{k+1}$, and the claim is proved.

For $k \geqslant 0$, from $T^k\alpha \in \widetilde{\Omega}_1$, we infer that $nT^{k-1}\alpha = T^k\alpha + m \in \widetilde{\Omega}_1 + m \subset \Omega_1$, where m is an integer. It follows from the claim above that $T^{k-1}\alpha \in \Omega_2$, and indeed $T^{k-1}\alpha \in \widetilde{\Omega}_2$. By repeating this argument, we obtain that $\alpha \in \widetilde{\Omega}_{k+1}$. Hence $\alpha \in \widetilde{\Omega}$.

The following theorem provides a simple way to determine whether the fractal square F contains a line segment.

THEOREM 3·3. *H contains a line if and only if $\widetilde{\Omega}_1$ contains either an interval or the T-orbit of one point in $\mathbb{Z}/ns$ (degenerate interval).*

Proof. As $H_1^c = (I \setminus F_1) + \mathbb{Z}^2$, it is easy to see that $\pi(H_1^c)$ is a union of open intervals of length $(1 + |r|/s)/n$, and with end points in $\mathbb{Z}/ns$. Hence $\widetilde{\Omega}_1$ (if nonempty) contains closed intervals with end points in $\mathbb{Z}/ns$, or $\widetilde{\Omega}_1 \subset \mathbb{Z}/ns$. In the later case, $\widetilde{\Omega}_1$ contains a T-orbit by Lemma 3·2. The necessity is proved.

To prove the sufficiency, we can identify the interval [0, 1) with $\mathbb{T}$ for the convenience to use the map $T(x) = nx \pmod 1$. For the degenerate case, the theorem follows immediately by Lemma 3·2. For the non-degenerate case, by assumption, we let $J_0 = [m/ns, m + 1/ns]$ be an interval in $\widetilde{\Omega}_1$. Then $T(J_0) = [m/s, m + 1/s]$ is an interval in $\mathbb{T}$ with length $1/s$. Since $\widetilde{\Omega}_1 + 1/s = \widetilde{\Omega}_1$, $T(J_0) \bigcap \widetilde{\Omega}_1$ contains a translation of J_0, which we denote by J_1. By the same argument, there is J_2, a translation of J_1, contained in $T(J_1) \bigcap \widetilde{\Omega}_1$. Therefore, we can find intervals $J_1, J_2, \dots$ such that they are translations of J_0, all of them are subsets of $\widetilde{\Omega}_1$, and $J_j \subset T(J_{j-1})$ for $j \geqslant 1$. Since there are only n different translations of J_0, we conclude that the sequence must be eventually periodic and hence $J_{k_0} = J_{k_0+p}$ holds for some $k_0 \geqslant 0$, $p \geqslant 1$. Hence $J_{k_0} \subset T^p(J_{k_0})$, and it follows that there is a p-periodic point α of T in J_{k_0} and $T^k\alpha \in J_{k_0+k}$ (see Sarkovskii's Theorem in [**4**]). Moreover, the orbit of α is in $\widetilde{\Omega}_1$. By Lemma 3·2, $\alpha \in \widetilde{\Omega}$ and H contains a line.

We remark that if a line with slope τ is contained in H, then $\tau = r/s$ with $1 \leqslant |r| + s \leqslant n$, $r, s \in \mathbb{Z}$, $s \geqslant 1$. For otherwise, since $\pi(H_1^c)$ is a union of open intervals of length $(1+|r|/s)/n$ and $\pi(H_1^c)+1/s = \pi(H_1^c)$, $(1+|r|/s)/n > 1/s$ implies that $\pi(H_1^c) = \mathbb{R}$ and thus $\Omega_1 = \emptyset$. Hence there are at most n^2 choices of τ. That the components of F are line segments can be checked directly on Ω_1.

4. *H^c and its components*

In this section we will study in more detail on the set H_k^c, and provide a criterion to determine the boundedness of the components of H^c. For $q \in \mathbb{Z}^2$, we define

$$H_k(0, q) = F_k + \left(\mathbb{Z}^2 \setminus \{0, q\}\right),$$

and denote by $H_k^c(0, q)$ its complement. Clearly $H_k^c(0, q)$ is an open set and contains I° and $q + I^\circ$, and $H_k^c \subset H_k^c(0, q)$.

Definition 4·1. A vector $q \in \mathbb{Z}^2$ is said to be an admissible vector of order $k \geqslant 1$ if $H_k^c(0, q)$ has a component containing the open squares I° and $q + I^\circ$. We denote by Q_k the set of admissible vectors of order k. By convention we let $Q_0 = \{0, \pm e_1, \pm e_2\}$. (Here $I = [0, 1]^2$, $e_1 = (1, 0)$, $e_2 = (0, 1)$.)

Remarks. (1) It follows that $q \in Q_k$ if and only if there exists a curve $\gamma \subset H_k^c(0, q)$ that starts from I° and ends in $q + I^\circ$. Roughly speaking, the attachment of these two auxiliary unit squares to a curve in H_k^c is for the sake of normalization and for convenience. For the curve γ, we can choose one that passes through a chain of non-repeated squares of size $1/n^\ell$ (or $1/n^\ell$-squares) where $0 < \ell \leqslant k$ in H_k^c, and γ is composed of line segments connecting the centers of these squares.

(2) Clearly if H_k^c has an unbounded component, then Q_k is an infinite set.

LEMMA 4·2. *With the above notation, then*:

(i) $Q_k \subset Q_{k+1}$;

(ii) *the components of H^c are bounded if and only if $\{Q_k\}_k$ is uniformly bounded. In this case there exists k_0 such that $Q_{k+1} = Q_k$ for all $k \geqslant k_0$.*

Proof. Part (i) follows from the fact $F_{k+1} \subset F_k$. For part (ii), if $\{Q_k\}_k$ is unbounded, then there exists $q_k \in Q_k$ such that $\|q_k\| \to \infty$ as $k \to \infty$, it follows that the corresponding sub-curve $\gamma^* \subset H_k^c$ is unbounded, and hence the components of H^c are unbounded. Also the above implications are reversible.

In the following, we give a detail consideration on the structure of Q_k. Recall that $\mathcal{D}$ is the digit set of F, and let $\mathcal{D}^c = \{0, 1, \ldots, n-1\}^2 \setminus \mathcal{D}$. We define a set of vertices by

$$\mathcal{V} = \mathcal{D}^c/n := \{v_1, \ldots, v_\ell\}.$$

Let Q be a subset of $\mathbb{Z}^2$ and assume that $0 \in Q$, we define a graph $\mathcal{G}_Q$ as follows: let $b \in \mathbb{Z}^2$, and $u, v \in \mathcal{V}$, by an *edge b from u to v*, we mean

$$n(v + b - u) \in Q \tag{4·1}$$

and denote this edge by $(u, v; b)$. If Q is symmetric (i.e., $Q = -Q$) and if there is an edge $(u, v; b)$, then there is an edge $(v, u; -b)$. By a *path* of $\mathcal{G}_Q$, we mean a finite sequence $\{(u_i, u_{i+1}; b_i)\}_{i=1}^m \subset \mathcal{G}_Q$; in addition, if $\sum_{i=1}^m b_i = 0$, we call it a 0-*path*. This is useful for sorting the vertices into equivalence classes (see Section 5). A path is a *loop* if $u_1 = u_{m+1} (= u)$. In the case that $\sum_{i=1}^m b_i \neq 0$, we refer this as a *non-zero* loop; otherwise we call it 0-loop. The edge $(u, u; 0) \in \mathcal{G}_Q$ for any $u \in \mathcal{V}$, and we sometimes call it a trivial loop. Note that $(u, u; b) \in \mathcal{G}_Q$ with $b \neq 0$ is a non-zero loop.

We remark that a 0-path is not necessary a loop. The reader can refer to Figure 2(a) in Example 2 for an illustration. In the example, $Q = Q_0 = \{0, \pm e_1, \pm e_2\}$, then

$$(v_1, v_1; 0), \quad (v_1, v_2; 0), \quad (v_2, v_3; 0) \in \mathcal{G}_Q.$$

They are edges associated with 0 and are 0-paths, the first one is a trivial loop, but the last two are not loops.

We are interested in the graphs $\mathcal{G}_{Q_k}, k \geqslant 1$. We remind the reader that in the sequel, a "path" is reserved for a sequence of edges in the graph $\mathcal{G}_{Q_k}$, and a "curve" is referred to a path in $H_k^c \subset \mathbb{R}^2$. The main motivation of this notion of graph is due to the following simple proposition.

LEMMA 4·3. *Let* $u, v \in \mathcal{V}$, $b \in \mathbb{Z}^2$.

(i) *If* $(u, v; b) \in \mathcal{G}_{Q_{k-1}}$, *then there is a curve* $\gamma \subset H_k^c$ *joining* $u + I^\circ/n$ *and* $b + v + I^\circ/n$.

(ii) *Conversely, if there is a curve* $\gamma \subset H_k^c$ *joining* $u + I^\circ/n$ *and* $b + v + I^\circ/n$, *and the curve does not intersect the closure of any other* $1/n$*-squares in* H_k^c, *then* $(u, v; b) \in \mathcal{G}_{Q_{k-1}}$.

Proof. (i) Note that $(u, v; b) \in \mathcal{G}_{Q_{k-1}}$ means $q = n(v+b-u) \in Q_{k-1}$, which means there is a curve γ' connecting I° and $q + I^\circ$ in $H_{k-1}^c(0, q)$. Then $\gamma' + nu$ is a curve connecting $nu + I^\circ$ and $q + nu + I^\circ$ in $H_{k-1}^c(0, q) + nu$. We claim that

$$\frac{1}{n} H_{k-1}^c(0, q) + u \subset H_k^c.$$

This will imply $\gamma = \gamma'/n + u$ is a curve joining $u + I^\circ/n$ and $v + b + I^\circ/n$ in H_k^c and (i) follows.

To prove the claim we first observe that for $u \in \mathcal{V}$, then $nu \in \mathcal{D}^c$, and it is easy to check that $\mathcal{D} \subset (\mathbb{Z}^2 \setminus \{0, q\}) + nu$. Hence

$$nH_k = (F_{k-1} + \mathcal{D}) + n\mathbb{Z}^2 \subset F_{k-1} + (\mathbb{Z}^2 \setminus \{0, q\}) + nu = H_k(0, q) + nu.$$

The claim follows by taking the complement of the above.

(ii) Suppose γ is a curve in H_k^c as in the lemma. Let $\gamma^* = \gamma(t_1, t_2)$ be an open sub-arc of γ such that γ^* does not intersect the two squares $u + I/n$ and $v + b + I/n$. By the same argument as Case 1 in Theorem 2·2, we have that $\gamma^* \subset H_{k-1}^c/n$ as in (2·4), which implies that $n\gamma^* - nu \subset H_{k-1}^c$. It is seen that $n\gamma^* - nu$ can be extended to I° and $n(v+b-u) + I^\circ$, hence $n(v + b - u) \in Q_{k-1}$, which implies $(u, v; b) \in \mathcal{G}_{Q_{k-1}}$.

By using (4·1), we introduce several auxiliary classes of edge sets. Let $\widetilde{\mathcal{V}} = \{h/n : h \in \{0, 1, \ldots, n-1\}^2\}$ and define

$$\widetilde{\mathcal{G}}_Q = \{(u, v; b) : u,\ v \in \widetilde{\mathcal{V}}\}.$$

Then for an edge $(u, v; b) \in \widetilde{\mathcal{G}}_{Q_{k-1}}$, it has the same property as in Lemma 4·3 except by replacing H_k^c with $H_k^c \cup (u + I^\circ/n) \cup (v + b + I^\circ/n)$ (notice that $u + I^\circ/n$ and $(v + b + I^\circ/n)$ are subsets of H_k^c when $u, v \in \mathcal{V}$).

Especially, we set

$$\mathcal{G}'_Q = \{(u, v; b) : u \in \widetilde{\mathcal{V}},\ v \in \mathcal{V}\} \quad \text{and} \quad \mathcal{G}''_Q = \{(u, v; b) : u \in \mathcal{V},\ v \in \widetilde{\mathcal{V}}\}.$$

Now we can give the inductive relationship of Q_k.

THEOREM 4·4. *For any* $k \geqslant 1$, Q_k *equals the set of* $q = b_0 + \cdots + b_m$ *from the path* $\{(u_i, u_{i+1}; b_i)\}_{i=0}^m$ *with* $m = 0, 1$ *or*

$$(u_0, u_1; b_0) \in \mathcal{G}'_{Q_{k-1}},\ \{(u_i, u_{i+1}; b_i)\}_{i=1}^{m-1} \subset \mathcal{G}_{Q_{k-1}},\ (u_m, u_{m+1}; b_m) \in \mathcal{G}''_{Q_{k-1}}. \tag{4·2}$$

Proof. If $m = 0$, then $(u_0, u_1; b)$ is an edge in $\widetilde{\mathcal{G}}_{Q_{k-1}}$. By the remark above, $u_0 + I^\circ/n$ and $u_1 + b + I^\circ/n$ are in the same component of $H_k^c \cup (u_0 + I^\circ/n) \cup (u_1 + b + I^\circ/n) \subset H_k^c(0, b)$ and hence $b \in Q_k$. Similarly for $m = 1$.

Assume $m \geqslant 2$ and $\{(u_i, u_{i+1}; b_i)\}_{i=1}^{m-1} \subset \mathcal{G}_{Q_{k-1}}$, we have by Lemma 4·3 that there is a component in H_k^c containing

$$u_1 + I^\circ/n,\ \ b_1 + u_2 + I^\circ/n,\ \ldots,\ \ (b_1 + \cdots + b_{m-1}) + u_m + I^\circ/n.$$

So there is a curve γ joining $u_1 + I^\circ/n$ and $(b_1 + \cdots + b_{m-1}) + u_m + I^\circ/n$. We add an initial curve γ' and a final curve γ'' corresponding to $(u_0, u_1; b_0)$ and $(u_m, u_{m+1}; b_m)$ respectively, and $\gamma' \cup \gamma \cup \gamma''$ is a new curve joining the following squares (replacing $u_0 + I^\circ/n$ and $u_{m+1} + I^\circ/n$ at the two ends by I°, since they are subsets of I°):

$$I^\circ,\ b_0 + u_1 + I^\circ/n,\ \ldots,\ (b_0 + \cdots + b_{m-1}) + u_m + I^\circ/n,\ (b_0 + \cdots + b_m) + I^\circ.$$

It is in $H_k^c(0, q)$, and this implies that $q = b_0 + \cdots + b_m$ is in Q_k.

Conversely, let $q \in Q_k$, then there exists a simple curve $\gamma \subset H_k^c(0, q)$ connecting I° and $q + I^\circ$ (as in Remark (1) of Definition 4·1). Let γ^* be the part of the curve by deleting the parts in I° and $q + I^\circ$. Let $\{a_i + I/n\}_{i=1}^m$ be the $1/n$-squares that intersect γ^* with $a_i \in D^c/n + \mathbb{Z}^2$ (if exist; otherwise, reduce to $m = 0$). Without loss of generality, we may assume that γ passes each square $a_i + I^\circ/n$ only once and that $\{a_i + I^\circ/n\}_{i=1}^m$ are arranged in the order according to the advance of γ. Then we can write them as

$$a_i = c_i + u_i, \qquad u_i \in \mathcal{V},\ \ c_i \in \mathbb{Z}^2.$$

We add in two more $1/n$-squares $a_0 + I^\circ/n$ and $a_{m+1} + I^\circ/n$ as follows: since the curve γ^* has an extension into I° and is contained in $H_k^c(0, q)$, we use $a_0 + I^\circ/n = u_0 + I^\circ/n$ to denote the $1/n$-square in I° that contains the extension where $a_0 \in \widetilde{\mathcal{V}}$. Similarly we choose $a_{m+1} + I^\circ/n = q + u_{m+1} + I^\circ/n$ in $q + I^\circ$ where $u_{m+1} \in \widetilde{\mathcal{V}}$. Let

$$b_0 = c_1,\ \ b_i = c_{i+1} - c_i,\ \ 1 \leqslant i \leqslant m-1 \quad \text{and} \quad b_m = q - c_m.$$

It follows that (by Lemma 4·3 (ii)) the sequence $\{(u_i, u_{i+1}; b_i)\}_{i=0}^m$ satisfies (4·2), since the curve between $a_i + I^\circ/n$ and $a_{i+1} + I^\circ/n$ does not intersect any other $1/n$-squares.

COROLLARY 4·5. *If $Q_k = Q_{k+1}$ for some $k \geqslant 1$, then $Q_k = Q_{k+p}$ for all $p \geqslant 1$.*

We remark that $\mathcal{V} \subset \widetilde{\mathcal{V}}$, hence a path $\{(u_i, u_{i+1}; b_i)\}_{i=0}^m$ in $\mathcal{G}_{Q_{k-1}}$ by itself satisfies (4·2) by treating u_1, u_m as u_0, u_{m+1}. For brevity, we write

$$\mathcal{G}_k := \mathcal{G}_{Q_k} \quad \text{and} \quad \widetilde{\mathcal{G}}_k := \widetilde{\mathcal{G}}_{Q_k}.$$

The key role of the graph $\widetilde{\mathcal{G}}_k$ is to illustrate the relation between Q_k and Q_{k+1} as in Theorem 4·4. However, to determine the boundedness of the components of H^c, only the information of the graph $\mathcal{G}_k$ is needed.

THEOREM 4·6. *The components of H^c are unbounded if and only if there is a non-zero loop in some $\mathcal{G}_k$.*

Proof. For the sufficiency, by Lemma 4·3, the assumption implies that there is a curve γ in H_{k+1}^c satisfying $\gamma(1) - \gamma(0) = b \in \mathbb{Z}^2 \setminus \{0\}$. This implies $H_{k+1}^c \subset H^c$ has an unbounded component, and by Lemma 2·1, all the components of H^c are unbounded.

For the necessity, if the components of H^c are unbounded, then there exists a curve $\gamma \subset H^c$ such that $\gamma(1) - \gamma(0) = q \in \mathbb{Z}^2 \setminus \{0\}$. Let $\gamma^* = \gamma + \{mq;\ m \geqslant 0\}$.

Case 1. If γ^* intersects a square $a + I/n$ with $a \in D^c/n + \mathbb{Z}^2$, then γ^* also intersects $a + q + I/n$. Let $\gamma' \subset \gamma^*$ be a sub-arc joining $a + I/n$ and $a + q + I/n$, a similar argument as the second part of the proof of Theorem 4·4 implies there is a non-zero loop in $\mathcal{G}_k$.

Case 2. If γ^* does not intersect any square $a + I/n$ with $a \in D^c/n + \mathbb{Z}^2$, then $\gamma^* \subset H^c/n$. Hence $\gamma^* + b \subset H^c/n \subset H^c$ for any $b \in \mathbb{Z}^2/n$ (by (2·4)). Pick any $u \in D^c/n$, we can choose b so that $\gamma^* + b$ passes $u + I/n$ and the result follows by Case 1.

As a direct consequence, if there is no non-zero loop in some $\mathcal{G}_k$ and $\widetilde{\mathcal{G}}_k = \widetilde{\mathcal{G}}_{k+1}$, then there is no non-zero loop in all $\mathcal{G}_k$, hence all the components of H^c are bounded. Another simple observation is, if $b \in Q_k$ and $b \in n\mathbb{Z}^2 \setminus \{0\}$, then in $\mathcal{G}_k$ we have a non-zero loop $(v, v; b/n)$ for any $v \in \mathcal{V}$, hence the components of H^c are unbounded. Moreover, Lemma 4·2 implies that if $\mathcal{G}_k$ has infinitely many edges (equivalently, Q_{k+1} is unbounded), then the components of H^c are unbounded as well.

These criteria provide a convenient way to classify the topology of the fractal square F, which will be explained by using several instructive examples in the next section.

5. *Algorithm and examples*

In Section 2, we have shown that the fractal square F can be classified into three types according to their topological structure: (i) F is totally disconnected; (ii) the non-trivial components of F are parallel line segments; and (iii) F contains a non-trivial component that is not a line segment. For some of the simple cases, it is easy to inspect these types directly. However, in general, it is difficult to see the topology of F in an obvious manner. By making use of the construction in Section 4, it is possible to devise an algorithm to obtain the classification. The basic idea of the algorithm is as follows

$$Q_0 \xrightarrow{(4\cdot 1)} \widetilde{\mathcal{G}}_0 \xrightarrow{Theorem\ 4\cdot 4} Q_1 \xrightarrow{(4\cdot 1)} \widetilde{\mathcal{G}}_1 \cdots \tag{5·1}$$

Then we can use Theorem 4·6 to determine whether the components of H^c are bounded, which distinguishes type (iii) from types (i) and (ii). By Theorem 3·3, we can separate types (i) and (ii). The following proposition justifies the finiteness of the algorithm described in (5·1).

PROPOSITION 5·1. *There exists k ($\leqslant 38n^{10}$) such that the process in* (5·1) *ends; at such k, either*:

(i) *$\mathcal{G}_k$ contains a non-zero loop*; *or*

(ii) *there is no non-zero loop in $\mathcal{G}_k$ and $\widetilde{\mathcal{G}}_k = \widetilde{\mathcal{G}}_{k-1}$.*

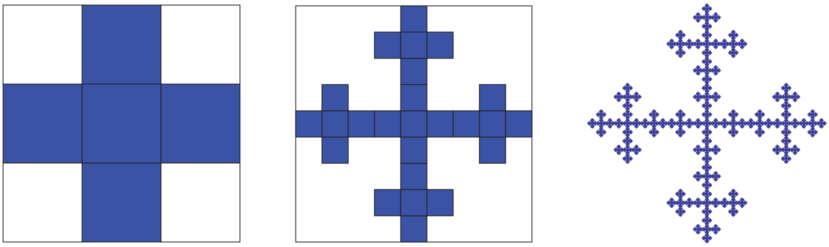
Proof. Let k be the maximal integer such that there is no non-zero loop in $\mathcal{G}_k$ and $\widetilde{\mathcal{G}}_k \neq \widetilde{\mathcal{G}}_{k-1}$. Then $\widetilde{\mathcal{G}}_0 \subsetneq \widetilde{\mathcal{G}}_1 \subsetneq \cdots \subsetneq \widetilde{\mathcal{G}}_k$, and $\#\widetilde{\mathcal{G}}_{k-1} \geqslant k - 1 + n^2$ since $\widetilde{\mathcal{G}}_0$ contains at least n^2 trivial edges.

As there is no non-zero loop in $\mathcal{G}_k$, the components of H_k^c are bounded (otherwise there exists $b \in Q_k$ such that $b \in n\mathbb{Z}^2 \setminus \{0\}$). For edges $(u_0, u; b) \in \widetilde{\mathcal{G}}_{k-1}$, there is a curve $\gamma \subset H_k^c$ joining $u_0 + I$ and $u + b + I$. It follows by Theorem 2·2 that $\mathrm{diam}(\gamma) \leqslant \sqrt{2}(n^2+1)^2/n$. Hence

$$\|b\| \leqslant \mathrm{diam}(\gamma) + 2\sqrt{2} \leqslant \sqrt{2}(n^3 + 2n + 2 + 1/n) \leqslant 29\sqrt{2}n^3/16.$$

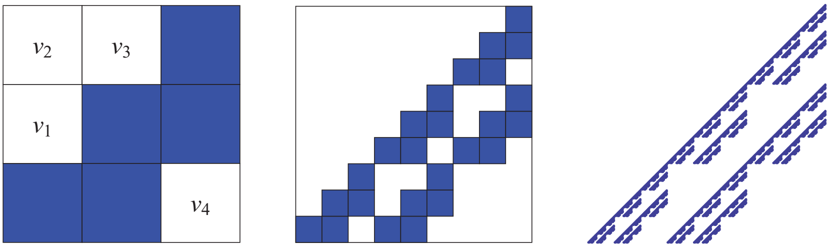
That implies $\#\widetilde{\mathcal{G}}_{k-1} \leqslant (2\|b\| + 1)^2 \cdot n^4 < 38n^{10}$, and $k < 38n^{10} - n^2 + 1 < 38n^{10}$. The proposition follows.

If $\mathcal{G}_k$ has no non-zero loops, then the number of non-zero paths as in Theorem 4·4 is uniformly bounded. Therefore, to produce Q_{k+1}, we only need to check finitely many steps. We also point out that the estimate of the steps in Proposition 5·1 is very rough. In practice, the number of steps really needed is far less, as is seen in the following examples.



(a) (b) (c)

Fig. 1. (Colour online) Vicsek fractal.



(a) (b) (c)

Fig. 2. (Colour online) F composed of parallel line segments.

Example 1. The fractal square in Figure 1 is the well-known Vicsek fractal. It is clear that F contains dendrite curves. It is also easy to see that H_k contains horizontal and vertical lines which divide H_k^c into bounded components for any $k \geqslant 1$.

Example 2. Consider the fractal square in Figure 2, the vertex set $\mathcal{V} = \mathcal{D}^c/n = \{v_1, v_2, v_3, v_4\}$ is depicted in Figure 2(a). It is easy to see that H_1 contains the line $x = y$, and $\widetilde{\Omega}_1 = \{0\}$, hence H contains the line by Theorem 3·3. Moreover, this line is also a component of H. It follows from Corollary 2·6 that the non-trivial components of F are parallel line segments.

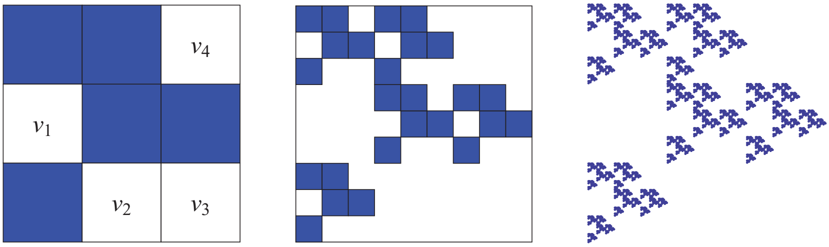
In the sequel, we use two examples to demonstrate the inductive method for the classification derived from (5·1). Before we do that, we simplify the graph $\widetilde{\mathcal{G}}_k$ by identifying some of the vertices as follows.

1. *Identifying vertices in* $\widetilde{\mathcal{V}}$. We introduce an abstract vertex and denote it by ε_0. Set $\mathcal{V}^0 = \{\varepsilon_0\} \bigcup \mathcal{V}$. Then we define a graph $\mathcal{G}_Q^0$ to be an extension of $\mathcal{G}_Q$ by adding the following edges: for $u \in \mathcal{V}$, $(\varepsilon_0, u; b) \in \mathcal{G}_Q^0$ if and only if $(u_0, u; b) \in \widetilde{\mathcal{G}}_Q$ for some $u_0 \in \widetilde{\mathcal{V}}$, and $(u, \varepsilon_0; b) \in \mathcal{G}_Q^0$ is defined similarly; moreover, $(\varepsilon_0, \varepsilon_0; b) \in \mathcal{G}_Q^0$ if and only if $(u_1, u_2; b) \in \widetilde{\mathcal{G}}_Q$ for some $u_1, u_2 \in \widetilde{\mathcal{V}}$. Write $\mathcal{G}_k^0 = \mathcal{G}_{Q_k}^0$, and note that

(i) The restriction of $\mathcal{G}_k^0$ to $\mathcal{V}$ is $\mathcal{G}_k$;

(ii) $Q_{k+1} = \left\{\sum_{i=0}^m b_i \;:\; \{(u_i, u_{i+1}; b_i)\}_{i=0}^m \text{ is a loop containing } \varepsilon_0 \text{ in } \mathcal{G}_k^0\right\}$.

2. *Identifying vertices in* $\mathcal{V}$. We start with $\mathcal{G}_0$, two vertices $u, v \in \mathcal{V}$ are said to be *equivalent* in $\mathcal{G}_0$ if there is a 0-path joining u, v (i.e., there is a finite sequence $\{(u_i, u_{i+1}; b_i)\}_{i=1}^m \subset \mathcal{G}_0$ such that $u = u_1, v = u_{m+1}$ and $\sum_{i=1}^m b_i = 0$); note that in this case $u + I^\circ/n$ and $v + I^\circ/n$ are in I° and are connected in H_1^c. We use $[u]$ to denote the equivalence class containing u, and $\mathcal{V}_0^*$ the set of equivalence classes. We introduce a graph $\mathcal{G}_0^*$ on $\mathcal{V}_0^*$, call it a *reduced graph* of $\mathcal{G}_0$, by defining edges $([u], [v]; b) \in \mathcal{G}_0^*$ if there exist $u' \in [u]$ and $v' \in [v]$ such that $(u', v'; b) \in \mathcal{G}_0$.



(a) (b) (c)

Fig. 3. (Colour online) Totally disconnected F.

Similar to Part 1, we define a reduced graph $\mathcal{G}_0^{0*}$ on $\mathcal{V}_0^{0*} = \{\varepsilon_0\} \cup \mathcal{V}_0^*$. Inductively, we can perform the same reduction on each $\mathcal{G}_k$ (resp. $\mathcal{G}_k^{0*}$) and obtain a compatible sequence of vertex sets $\mathcal{V}_k^*$ (resp. $\mathcal{V}_k^{0*}$) and reduced graphs $\mathcal{G}_k^*$ (resp. $\mathcal{G}_k^{0*}$).

Example 3. Consider the fractal square in Figure 3, the vertex set $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ is given as in Figure 3(a).

Let $Q_0 = \{0, \pm e_1, \pm e_2\}$. Clearly v_2, v_3 are equivalent in $\mathcal{G}_0$, we denote the class by $[v_2]$. Then $\mathcal{V}_0^{0*} = \{\varepsilon_0, v_1, [v_2], v_4\}$, the non-trivial edges in the reduced graph $\mathcal{G}_0^{0*}$ are

$$(\varepsilon_0, [v_2]; -e_1); \quad (\varepsilon_0, [v_2]; e_2); \quad (\varepsilon_0, v_1; e_1);$$
$$(\varepsilon_0, v_4; -e_2); \quad (\varepsilon_0, v_4; -e_1); \quad ([v_2], v_4; -e_2).$$

The two non-zero paths satisfying (4·2) are

$$\{(\varepsilon_0, [v_2]; -e_1),\ ([v_2], \varepsilon_0; -e_2)\} \text{ and } \{(\varepsilon_0, v_4; -e_2),\ (v_4, \varepsilon_0; e_1)\}$$

which give $q = -e_1 - e_2 = -(1, 1)$ and $e_1 - e_2 = (1, -1)$. Therefore $Q_1 = Q_0 \cup \{\pm(1, 1),\ \pm(1, -1)\}$.

Next, for $\mathcal{G}_1 := \mathcal{G}_{Q_1}$, there are new edges $(v_1, v_2; 0)$, $(v_1, v_4; -e_1)$, and their reverse edges. Hence v_1, v_2, v_3 are equivalent in $\mathcal{G}_1$, we denote by $[v_1]$ the equivalence class. The vertex set of equivalence classes is $\mathcal{V}_1^{0*} = \{\varepsilon_0, [v_1], v_4\}$, and the reduced graph $\mathcal{G}_1^{0*}$ consists of edges

$$\{(\varepsilon_0, \varepsilon_0; b) : b \in Q_1\}; \quad ([v_1], v_4; -e_1); \quad ([v_1], v_4; -e_2).$$

This yields a non-zero loop $\{([v_1], v_4; -e_1),\ (v_4, [v_1]; e_2)\}$ in $\mathcal{G}_1^*$. Therefore $\mathcal{G}_1$ has a non-zero loop, and the components of H^c are unbounded by Theorem 4·6.

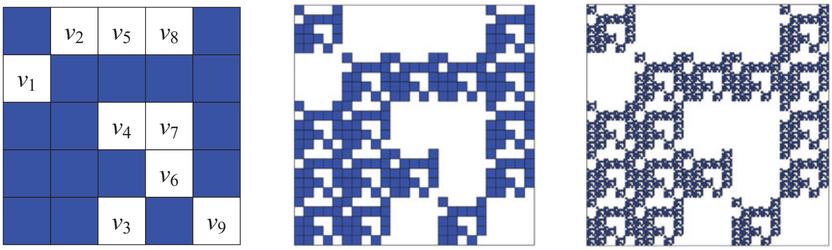
On the other hand, it is easy to observe that $\Omega_1 = \emptyset$ for any slope τ, hence there are no line segments in F by Theorem 3·3. Consequently, F is totally disconnected.

Finally, we consider one more example of which the classification is not so obvious by observation, and it relies on using the above technique to check the Q_k and $\mathcal{G}_k^{0*}$.

Example 4. Let F be the fractal square in Figure 4, and the vertex set $\mathcal{V} = \{v_1, \ldots, v_9\}$ is as in Figure 4(a). We only sketch the main steps and omit the straightforward but tedious verification. The details can be found in [**12**].

Clearly in $\mathcal{G}_0$, v_2, v_5, v_8 are in the same equivalence class, and v_4, v_6, v_7 are in another equivalence class. Let $\mathcal{V}_0^{0*} = \{\varepsilon_0, v_1, [v_2], v_3, [v_4], v_9\}$, and from the reduced graph $\mathcal{G}_0^{0*}$ we obtain $Q_1 = Q_0 \cup \{\pm(1, 1)\}$.

In $\mathcal{G}_1$, we check that $[v_1] = [v_2]$; $[v_3] = [v_4]$; and $[v_3] = [v_9]$. Then $\mathcal{V}_1^{0*} = \{\varepsilon_0, [v_1], [v_3]\}$, and from the reduced graph $\mathcal{G}_1^{0*}$, we show that $Q_2 = Q_1 \cup \{\pm(2, 1)\}$.



(a) (b) (c)

Fig. 4. (Colour online)

In $\mathcal{G}_2$, there is no new reduction on the equivalence class and we use the same vertex set $\mathcal{V}_2^{0*} = \mathcal{V}_1^{0*}$, and by checking the reduced graph $\mathcal{G}_2^{0*}$, we have $Q_3 = Q_2 \cup \{\pm(1,2)\}$.

Now in $\mathcal{G}_3$, we obtain $[v_1] = [v_3]$, so that $\mathcal{V}_3^{0*} = \{\varepsilon_0, [v_1]\}$. Also we have from the above, there is already an edge $([v_1], [v_3]; e_2) \in \mathcal{G}_1^*$. This leads to a non-zero loop $\{([v_1], [v_1]; e_2)\}$ in $\mathcal{G}_3^*$. Therefore $\mathcal{G}_3$ has a non-zero loop, and the components of H^c are unbounded. On the other hand, it is easy to see that $\Omega_1 = \emptyset$ for any slope τ, hence there are no line segments in F by Theorem 3·3. Consequently, F is totally disconnected.

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