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Abstract Let $T := T(A, \mathcal{D})$ be a disk-like self-affine tile generated by an integral expanding matrix A and a consecutive collinear digit set $\mathcal{D}$, and let $f(x) = x^2 + px + q$ be the characteristic polynomial of A. In the paper, we identify the boundary ∂T with a sofic system by constructing a neighbor graph and derive equivalent conditions for the pair $(A, \mathcal{D})$ to be a number system. Moreover, by using the graph-directed construction and a device of pseudo-norm ω, we find the generalized Hausdorff dimension $\dim_H^{\omega}(\partial T) = 2\log\rho(M)/\log|q|$ where $\rho(M)$ is the spectral radius of certain contact matrix M. Especially, when A is a similarity, we obtain the standard Hausdorff dimension $\dim_H(\partial T) = 2\log\rho/\log|q|$ where ρ is the largest positive zero of the cubic polynomial $x^3 - (|p| - 1)x^2 - (|q| - |p|)x - |q|$, which is simpler than the known result.

Keywords Boundary · Self-affine tile · Sofic system · Number system · Neighbor graph · Contact matrix · Graph-directed set · Hausdorff dimension

1 Introduction

Let $M_n(\mathbb{Z})$ denote the set of $n \times n$ matrices with entries in $\mathbb{Z}$ and let $A \in M_n(\mathbb{Z})$ be expanding (i.e., all eigenvalues of A have moduli >1). Assume $|\det(A)| = |q|$, and $\mathcal{D} = \{0, d_1, \ldots, d_{|q|-1}\} \subset \mathbb{Z}^n$ with $|q|$ distinct vectors. We call $\mathcal{D}$ a *digit set*

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and $(A, \mathcal{D})$ a *self-affine pair*. It is well known that there exists a unique self-affine set $T := T(A, \mathcal{D})$ [20] satisfying

$$T = A^{-1}(T + \mathcal{D}) = \Big\{ \sum_{i=1}^{\infty} A^{-i} d_{j_i} : d_{j_i} \in \mathcal{D} \Big\}.$$

If T has non-void interior, then there exists a subset $\mathcal{J} \subset \mathbb{Z}^n$ such that

$$T + \mathcal{J} = \mathbb{R}^n \quad \text{and} \quad (T + t)^\circ \cap (T + t')^\circ = \emptyset, \quad t \neq t', \ t, t' \in \mathcal{J},$$

thus T is called a *self-affine tile* and $\mathcal{J}$ a tiling set. $T + \mathcal{J}$ is called a tiling of $\mathbb{R}^n$, and a lattice tiling if $\mathcal{J}$ is a lattice [22].

The topological properties of self-affine tiles and their boundaries, such as connectedness, local connectedness, or disk-likeness (i.e., homeomorphic to the closed unit disk), have attracted a lot of interest. A systematical study on the connectedness of self-affine tiles was due to Kirat and Lau [19], they mainly concerned a class of tiles $T(A, \mathcal{D})$ generated by the *consecutive collinear* (CC) digit sets $\mathcal{D} := \mathcal{D}(v, |q|) = \{0, 1, \ldots, |q| - 1\}v$, $v \in \mathbb{Z}^n \setminus \{0\}$ via the algebraic property of the characteristic polynomial of the matrix A. More general cases on non-consecutive collinear or non-linear digit sets were considered by [6,18,25].

The question of disk-likeness was first investigated by Bandt and Gelbrich [3] for self-affine tiles in $\mathbb{R}^2$ with $|\det(A)| = 2$ or 3. They observed that the characteristic polynomial of $A \in M_2(\mathbb{Z})$ is of the form:

$$f(x) = x^2 + px + q, \ \text{with } |p| \le q, \ \text{if } q \ge 2; \quad |p| \le |q + 2|, \ \text{if } q \le -2.$$

By studying the neighborhood structure of T, Bandt and Wang [5] proved that a tile T with no more than six neighbors is disk-like if and only if T is connected. A translation of the tile $T + \ell$, $\ell \in \mathcal{J}$ is called a *neighbor* of T if $T \cap (T + \ell) \neq \emptyset$. Making use of this criterion, Leung and Lau [24] then gave a complete characterization of the disk-likeness of self-affine tiles with CC digit sets. Gmainer and Thuswaldner [14] considered the disk-likeness of tiles with non-collinear digit sets arising from polyominoes, and Kirat [18] proposed necessary and sufficient conditions for such tiles to be disk-like in general. By using the neighbor map technique, Bandt and Mesing [4] constructed a kind of finite type self-affine tiles and discussed their disk-likeness as well.

Theorem 1.1 ([24]) *Let $A \in M_2(\mathbb{Z})$ be an expanding matrix with characteristic polynomial $f(x) = x^2 + px + q$. Then for any CC digit set $\mathcal{D}(v, |q|)$ in $\mathbb{Z}^2$ such that v, Av are linearly independent, T is a disk-like tile if and only if $2|p| \le |q + 2|$.*

Moreover, when $p = 0$, T is a square tile; when $p \neq 0$, T is a hexagonal tile.

The boundary of a self-affine tile has more complicated geometric structure than the tile itself, hence it is also of considerable interest. The dimension of the boundary of a self-similar tile (where the expanding matrix A is a similarity) has been studied extensively in the literature. Strichartz and Wang [31] described the boundary set as a graph-directed set and gave an algorithm for finding the dimension of the boundary, various other methods can be founded in [8,17,23,32].

Recently, Akiyama and Loridant [1,2] provided a new method to parameterize the boundary set and reproved Theorem 1.1 by showing that the boundary of T is a simple closed curve. In the present paper, we go further to explore the structure of the boundary of the T defined in Theorem 1.1. For convenience, we call such T a *CC tile*. If it is also disk-like, we call it a *disk-like CC tile*.

First we establish a neighbor graph of T such that the boundary ∂T is identified as the union of all one-sided infinite paths of this graph. Hence ∂T determines a sofic system [11]. The neighbor graph technique is classical in the study of tiling theory [3,4]. However, it will be shown that we use the technique here from a different aspect. As self-affine tiles can be studied in the context of number systems [29], it is worth studying the conditions for the self-affine pair $(A, \mathcal{D})$ to be a number system. We give the answer when $T(A, \mathcal{D})$ is disk-like.

Theorem 1.2 *Let $T = T(A, \mathcal{D})$ be a disk-like CC tile. Then the following are equivalent:*

(i) $(A, \mathcal{D})$ *is a number system.*
(ii) $0 \in T^{\circ}$.
(iii) $f(x) = x^2 + px + q$ *with* $-1 \le p$ *and* $q \ge 2$.
(iv) *For all neighbors* $T + \ell$, $\ell = \sum_{i=0}^{k} a_i A^i v \in \mathcal{D}_{A,k+1}$ *for some* $k \in \mathbb{Z}$ *with* $a_k = 1$ *and* $a_i \in D$ *where* $0 \le i < k$.

In [31], Strichartz and Wang applied the graph-directed iterated function system (GIFS) to represent the boundary of a self-affine tile, but they were not sure whether the GIFS satisfies the open set condition or not. Our second aim is to give a positive answer for the disk-like CC tile and estimate the generalized Hausdorff dimension ($\dim_H^{\omega}$) of the boundary by using a pseudo-norm ω [16,26] instead of Euclidean norm.

Theorem 1.3 *The generalized Hausdorff dimension of the boundary of disk-like CC tile T is*

$$\dim_H^{\omega}(\partial T) = \frac{2\log \rho(M)}{\log |q|},$$

where $\rho(M)$ denotes the spectral radius of certain contact matrix M, and the corresponding measure is positive and finite.

When A is a similarity, we can improve the well-known Hausdorff dimension formula of the boundary in the following simpler way.

Theorem 1.4 *Let $A \in M_2(\mathbb{Z})$ be an expanding similarity with characteristic polynomial $f(x) = x^2 + px + q$ and $T = T(A, \mathcal{D})$ be a disk-like CC tile. Then*

$$\dim_H(\partial T) = \frac{2\log \rho}{\log |q|},$$

where ρ is the largest positive zero of the cubic polynomial $x^3 - (|p| - 1)x^2 - (|q| - |p|)x - |q|$.

The rest of the paper is organized as follows: In Sect. 2, we identify ∂T with a sofic system by constructing a neighbor graph and prove Theorem 1.2. In Sect. 3, we consider ∂T as a graph-directed set and prove Theorems 1.3 and 1.4. Finally all neighbor graphs, graph-directed sets and contact matrices corresponding to different characteristic polynomials $f(x)$ are listed in Appendices 1–3 for easy reference.

2 Sofic System and Number System

We first introduce some terminology of symbolic dynamics from [27]. Let $\mathcal{G} = \mathcal{G}(\mathcal{V}, \mathcal{E})$ be a directed graph where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ the set of edges. Let $\mathcal{A}$ be a finite set (called *alphabet*). If there exists a mapping (called *labeling*) $\mathcal{L}: \mathcal{E} \to \mathcal{A}$, then the ordered pair $\mathbf{G} = (\mathcal{G}, \mathcal{L})$ is called a *labeled directed graph*. All the infinite paths $\xi = e_1 e_2 e_3 \ldots$ on $\mathcal{G}$ constitute the so-called *edge shift* $\mathbf{X}_{\mathcal{G}}$. Define the *label of the path* ξ by

$$\mathcal{L}_\infty(\xi) := \mathcal{L}(e_1)\mathcal{L}(e_2)\mathcal{L}(e_3)\ldots \in \mathcal{A}^{\mathbb{N}}.$$

Here $\mathcal{A}^{\mathbb{N}}$ is called the *full shift* of $\mathcal{A}$. The set of all such labels is denoted by

$$\mathbf{X}_{\mathbf{G}} = \left\{x \in \mathcal{A}^{\mathbb{N}} : x = \mathcal{L}_\infty(\xi) \text{ for some } \xi \in \mathbf{X}_{\mathcal{G}}\right\}.$$

Any subset of $\mathcal{A}^{\mathbb{N}}$ which can be defined by a labeled directed graph as above, is called a *sofic shift* or *sofic system* [11,27]. Weiss [33] coined the term *sofic* which is derived from the Hebrew word for *finite* [27].

Let $D = \{0, 1, \ldots, |q| - 1\}$ and the difference set $\Delta D := D - D$, then the CC digit set $\mathcal{D} = Dv$ and $\Delta\mathcal{D} := \mathcal{D} - \mathcal{D} = \Delta D v$. Without loss of generality, we assume the digit set $\mathcal{D}$ is primitive, i.e., the lattice $\mathcal{J}$ generated by $\mathcal{D}$ and $A\mathcal{D}$ in $\mathbb{Z}^2$ is equal to $\mathbb{Z}^2$. For otherwise, there exists an invertible $B \in M_2(\mathbb{Z})$ such that $\tilde{\mathcal{D}} = B^{-1}\mathcal{D} \subset \mathbb{Z}^2$ is primitive and $T(A, \mathcal{D}) = BT(\tilde{A}, \tilde{\mathcal{D}})$ where $\tilde{A} = B^{-1}AB \in M_2(\mathbb{Z})$ [21] and we can consider $\tilde{A}, \tilde{\mathcal{D}}$ instead. Hence we set $\mathbb{Z}^2 = \{\gamma v + \delta A v : \gamma, \delta \in \mathbb{Z}\}$. It is easy to see that $T + \ell$ where $\ell \in \mathbb{Z}^2$ is a neighbor of T if and only if $\ell \in T - T$. More precisely, ℓ can be expressed as

$$\ell = \sum_{i=1}^{\infty} b_i A^{-i} v \in T - T, \quad b_i \in \Delta D.$$

The following is a neighbor-generating formula which plays a key role in constructing the labeled directed graph for the boundary.

Lemma 2.1 ([24]) *Suppose $T + \ell$ is a neighbor of T with $\ell = \gamma v + \delta A v = \sum_{i=1}^{\infty} b_i A^{-i} v$, then we get another neighbor $T + \ell'$ satisfying $\ell' = A\ell - b_1 v = \gamma' v + \delta' A v$ with $\gamma' = -(q\delta + b_1)$ and $\delta' = \gamma - p\delta$.*

Inductively, we can construct a sequence of neighbors: $\{T + \ell_n\}_{n=0}^{\infty}$ where $\ell_0 = \ell$ and $\ell_{n+1} = A\ell_n - b_{n+1} v$.

Let T be a disk-like CC tile and $T_\ell = T \cap (T+\ell)$ for any $\ell \in \mathbb{Z}^2$. Let $\mathcal{V} = \{\ell \in \mathbb{Z}^2 : \ell \neq 0 \text{ and } T \cap T_\ell \neq \emptyset\}$. Then the boundary of T can be written as

$$\partial T = \bigcup_{\ell \in \mathcal{V}} T_\ell. \tag{2.1}$$

Define an edge set $\mathcal{E} := \{e = (\ell, \ell') : \ell, \ell' \in \mathcal{V} \text{ and } \ell' = A\ell - b_1 v \text{ for some } b_1 \in \Delta D\}$ and a labeling $\mathcal{L} : \mathcal{E} \to \mathcal{A}$ by $\mathcal{L}(e) = b_1$ where $\mathcal{A} = \Delta D$. Then by the definition above, $\mathbf{G} = (\mathcal{G}, \mathcal{L})$ is a labeled directed graph and it determines a sofic shift. We call $\mathbf{G}$ the *neighbor graph* of T.

Proposition 2.2 *Let* $\mathbf{G}$ *be the neighbor graph of a CC disk-like tile* T. *If* $x = \sum_{i=1}^{\infty} a_i A^{-i} v = \ell + \sum_{i=1}^{\infty} a_i' A^{-i} v \in T_\ell$ *where* $a_i, a_i' \in D$, *then* $\{b_i := a_i - a_i'\}_{i=1}^{\infty}$ *is the sequence of labeling of the edges of an infinite path starting at* ℓ *(or simply called a label sequence starting at* ℓ*). Conversely, any label sequence* $\{b_i\}_{i=1}^{\infty}$ *(with* $b_i \in \Delta D$*) starting at* ℓ *defines a set*

$$\left\{x : x = \sum_{i=1}^{\infty} a_i A^{-i} v = \ell + \sum_{i=1}^{\infty} a_i' A^{-i} v,\ a_i - a_i' = b_i,\ a_i, a_i' \in D \text{ for } i = 1, 2, \ldots\right\}$$

of boundary points of T.

Proof Since $\ell = \sum_{i=1}^{\infty} b_i A^{-i} v$ with $b_i = a_i - a_i'$, by Lemma 2.1, we have a sequence of neighbors $\{T + \ell_n\}_{n=0}^{\infty}$ where $\ell_0 = \ell$ and $\ell_{n+1} = A\ell_n - b_{n+1} v$, hence $\{b_i\}_{i=1}^{\infty}$ is a label sequence starting at ℓ by the definition.

Conversely, if $\ell = \sum_{i=1}^{\infty} b_i A^{-i} v$ where $b_i \in \Delta D$, then $b_i = a_i - a_i'$ for $a_i, a_i' \in D$ and $\ell = \sum_{i=1}^{\infty} (a_i - a_i') A^{-i} v$. It follows that

$$x = \sum_{i=1}^{\infty} a_i A^{-i} v = \ell + \sum_{i=1}^{\infty} a_i' A^{-i} v \in T \cap (T + \ell) = T_\ell. \tag{2.2}$$

□

We can verify whether the origin 0 is a boundary point of T in the following way.

Corollary 2.3 $0 \in \partial T$ *if and only if there exists an infinite path in* $\mathbf{G}$ *with all edge labels either non-positive or non-negative.*

Proof Suppose $0 \in T \cap (T + \ell)$ for some neighbor $T + \ell$. Putting $a_i = 0$ for all i into (2.2), we have

$$\ell = \sum_{i=1}^{\infty} (-a_i') A^{-i} v.$$

Since $a_i' \in D$, the label sequence $\{b_i = -a_i'\}_{i=1}^{\infty}$ starting at ℓ has all labels non-positive. Similarly $\{b_i' = a_i\}_{i=1}^{\infty}$ is a sequence starting at $-\ell$ with all labels non-negative. By reversing the argument, we can prove the converse. □

Table 1 Relation among all neighbors of T associated with $f(x)=x^2+px+q,\ p,q\geq 2,\ 2p\leq q+2$ (excluding $p=q=2$)

ℓ	b_1	ℓ'
v	$-(p-1)$	$Av+(p-1)v$
	$-p$	$Av+pv$
$Av+(p-1)v$	$-(q-p)$	$-Av-pv$
	$-(q-p+1)$	$-Av-(p-1)v$
$Av+pv$	$-(q-1)$	$-v$
$-v$	$p-1$	$-Av-(p+1)v$
	p	$-Av-pv$
$-Av-(p-1)v$	$q-p$	$Av+pv$
	$q-p+1$	$Av+(p-1)v$
$-Av-pv$	$q-1$	v

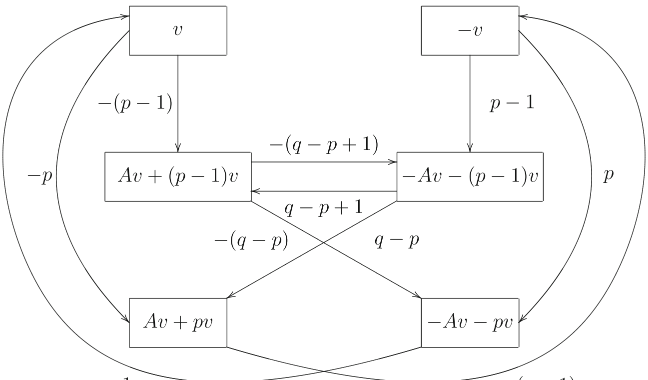

Fig. 1 The neighbor graph of T associated with $f(x)=x^2+px+q,\ p,q\geq 2,\ 2p\leq q+2$ (excluding $q=p=2$)

In fact, we can determine the neighbor graph **G** for any disk-like CC tile T. Let us take the case of $f(x)=x^2+px+q,\ p,q\geq 2$ (excluding $p=q=2$) as an example. By Theorem 1.1, T is a hexagonal tile with six neighbors [24] and

$$\mathcal{V}=\{\pm v,\ \pm\big(Av+(p-1)v\big),\ \pm(Av+pv)\}. \tag{2.3}$$

In view of the definition of $\mathcal{E}$, if $\ell=v$ we take $b_1=-p$ and $\ell'=Av+pv$ or $b_1=-(p-1)$ and $\ell'=Av+(p-1)v$; if $\ell=Av+pv$, using $f(A)v=0$, we have $b_1=-(q-1)$ and $\ell'=-v$. Proceeding similarly with all ℓ, we obtain Table 1. Then we establish the neighbor graph (Fig. 1). The neighbor graphs corresponding to other $f(x)$ are given in Appendix 1.

Following [20], we let $\mathcal{D}_{A,k}=\{\sum_{i=0}^{k-1}a_iA^iv : a_i\in D\},\ \Delta\mathcal{D}_{A,k}=\mathcal{D}_{A,k}-\mathcal{D}_{A,k}=\{\sum_{i=0}^{k-1}b_iA^iv : b_i\in\Delta D\}$ and $\mathcal{D}_{A,\infty}=\bigcup_{k=1}^{\infty}\mathcal{D}_{A,k}$.

Proposition 2.4 *Let T be a disk-like CC tile and $T+\ell$ a neighbor. Then* $\ell = \sum_{i=0}^{k} b_i A^i v \in \Delta\mathcal{D}_{A,k+1}$ *for some* $k \in \mathbb{Z}$ *with* $b_k \in \{-1, 1\}$, $b_i \in \Delta D$ *for* $0 \le i < k$. *When* $f(x) = x^2 \pm 2x + 2$, $k = 3$; *and* $k = 1$ *otherwise.*

Proof It follows from (2.3) that $\ell \in \Delta\mathcal{D}_{A,2}$ excluding the case of $f(x) = x^2 \pm 2x + 2$. For $f(x) = x^2 \pm 2x + 2$, we have $Av \pm 2v = A^3 v \pm A^2 v + Av \in \Delta\mathcal{D}_{A,4}$ by using $(A \mp I) f(A) = 0$.

A more desirable property is that any $\ell \in \mathbb{Z}^2$ can be expressed as $\ell = \sum_{i=0}^{k} a_i A^i v \in \mathcal{D}_{A,k+1}$ (instead of $\Delta\mathcal{D}_{A,k+1}$) for some $k \in \mathbb{Z}$ with $a_k = 1$ and $a_i \in D$ where $0 \le i < k$. But this is not always the case. This property is closely related to a *number system* defined below (see also [29]).

Definition 2.5 Let $A \in M_2(\mathbb{Z})$ be expanding and $\mathcal{D}$ be a CC digit set. The self-affine pair $(A, \mathcal{D})$ is said to be a number system if for any $\ell \in \mathbb{Z}^2$, it has a unique representation $\ell = \sum_{i=0}^{k} A^i v_i'$ with $v_i' \in \mathcal{D}$.

For convenience, we sometimes write a point of the form $x = \sum_{i=1}^{\infty} a_i A^{-i} v \in \mathbb{R}^2$ as *radix expansion*: $0.a_1 a_2 a_3 \ldots$ An overbar denotes repeating digits as in $0.12\overline{301} = 0.12301301301\ldots$ Likewise, $a_{-2}a_{-1}a_0.a_1a_2a_3\ldots$ represents a point $a_{-2}A^2 v + a_{-1}Av + a_0 v + \sum_{i=1}^{\infty} a_i A^{-i} v$. Note that shifting a radix place to the left means multiplying A to x. When x is on the boundary of T, the radix expansion of x is not unique. Now we give some equivalent conditions for the self-affine pair $(A, \mathcal{D})$ to be a number system.

Theorem 2.6 *Let $T = T(A, \mathcal{D})$ be a disk-like CC tile. Then the following are equivalent:*

(i) $(A, \mathcal{D})$ *is a number system.*
(ii) $0 \in T^\circ$.
(iii) $f(x) = x^2 + px + q$ *with* $-1 \le p$ *and* $q \ge 2$.
(iv) *For all neighbors* $T + \ell$, $\ell = \sum_{i=0}^{k} a_i A^i v \in \mathcal{D}_{A,k+1}$ *for some* $k \in \mathbb{Z}$ *with* $a_k = 1$ *and* $a_i \in D$ *where* $0 \le i < k$.

Proof (i) $\Rightarrow$ (ii) Suppose $0 \notin T^\circ$. Then $0 \in T \cap (T + \ell)$ for some $\ell \in \mathbb{Z}^2 \setminus \{0\}$. Since $(A, \mathcal{D})$ is a number system, $\ell = \sum_{i=-k}^{0} a_i A^{-i} v$ with $a_i \in D$ and $a_{-k} > 0$. Hence $0 = a_{-k}a_{-(k-1)}\ldots a_{-1}a_0.a_1a_2a_3\ldots$. Shifting the radix point k places to the left, we get $0 = a_{-k}.a_{-(k-1)}\ldots a_{-1}a_0a_1a_2a_3\ldots$. That means $T + a_{-k}v$ is a neighbor of T. By Proposition 2.4, $a_{-k} = 1$. Hence 0 corresponds to an infinite path starting at v with non-positive labels $b_i = -a_i$. But by checking all the neighbor graphs in Appendix 1, we find no such path.

(ii) $\Rightarrow$ (i) It suffices to show that $\mathbb{Z}^2 \subset \mathcal{D}_{A,\infty}$. By the lattice tiling property, 0 is the only lattice point in T, i.e., $\mathbb{Z}^2 \cap T = \{0\}$. It follows that $\mathbb{Z}^2 \cap A^n T = \sum_{i=0}^{n-1} A^i \mathcal{D} = \mathcal{D}_{A,n}$ for $n \ge 1$. If $\ell \in \mathbb{Z}^2$, there exists a large integer n such that $\ell \in A^n T$ as $0 \in T^\circ$, then $\ell \in \mathcal{D}_{A,n} \subset \mathcal{D}_{A,\infty}$.

(ii) $\Leftrightarrow$ (iii) By inspecting all neighbor graphs in Appendix 1, we find that in each graph corresponding to $f(x) = x^2 + px + q$ with $-1 \le p$ and $q \ge 2$, there exists no infinite path with edge labels either all non-positive or all non-negative, hence $0 \in T^\circ$

Table 2 Infinite paths representing a boundary point 0

$f(x)$	Neighbor	Path
x^2-2x+2	$Av-v$	$\overline{(-1)}$
	$-Av+v$	$\overline{1}$
x^2-px-q	v	$\overline{p(q-1)}$
	$-v$	$\overline{(-p)[-(q-1)]}$
x^2+px-q	$Av+(p+1)v$	$\overline{(q-p-1)}$
	$-Av-(p+1)v$	$\overline{[-(q-p-1)]}$
x^2-q	v	$\overline{0(q-1)}$
	$-v$	$\overline{0[-(q-1)]}$
	$Av+v$	$\overline{(q-1)}$
	$-Av-v$	$\overline{[-(q-1)]}$
x^2-px+q	$Av-(p-1)v$	$\overline{[-(q-p+1)]}$
	$-Av+(p-1)v$	$\overline{(q-p+1)}$

by Corollary 2.3. In every other case, there always exists such a path. All these paths are listed in Table 2.

(iii) ⇒ (iv) Let $f(x)$ be one of the cases: x^2+q, x^2+x+q, x^2+px+q ($p\geq 2$, excluding $p=q=2$), x^2+2x+2, x^2-x+q, where $p\geq 0$ and $q\geq 2$. In each case, we can rewrite their neighbors as the desired form in (iv). By using $0=f(A)v$, $0=(A-I)f(A)v$, $0=(A+I)f(A)v$, we have

Case (1) $f(x)=x^2+q$. $Av-v=A^2v+Av+(q-1)v$, $-v=A^2v+(q-1)v$, $-Av=A^3v+(q-1)Av$, $-Av+v=A^3v+(q-1)Av+v$, $-Av-v=A^3v+A^2v+(q-1)Av+(q-1)v$.

Case (2) $f(x)=x^2+x+q$. $-v=A^2v+Av+(q-1)v$, $-Av=A^3v+A^2v+(q-1)Av$, $-Av-v=A^2v+(q-1)v$.

Case (3) $f(x)=x^2+px+q$ ($p\geq 2$). $-v=A^2v+pAv+(q-1)v$, $-Av-(p-1)v=A^2v+(p-1)Av+(q-p+1)v$, $-Av-pv=A^2v+(p-1)Av+(q-p)v$.

Case (4) $f(x)=x^2+2x+2$. $Av+2v=A^3v+A^2v+Av$, $-v=A^4v+A^3v+A^2v+v$, $-Av-v=A^2v+Av+v$, $-Av-2v=A^2v+Av$.

Case (5) $f(x)=x^2-x+q$. $Av-v=A^2v+(q-1)v$, $-v=A^3v+(q-1)Av+(q-1)v$, $-Av=A^4v+(q-1)A^2v+(q-1)Av$, $-Av+v=A^4v+(q-1)A^2v+(q-1)Av+v$.

(iv) ⇒ (ii) Suppose $0\notin T^\circ$. By the same argument as in the proof of (i)⇒(ii) above, there should be an infinite path in the neighbor graph starting at v with edge labels all non-positive. But we find no such path by inspecting all the neighbor graphs in Appendix 1. □

Remark 2.7 Gilbert [13] obtained some related results in the context of quadratic number fields. We conjecture that Theorem 2.6 can be extended to non-disk-like tiles.

3 Dimension of the Boundary of *T*

For a directed graph $\mathcal{G} = \mathcal{G}(\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{v_1, v_2, \ldots, v_m\}$, we write $\mathcal{E}_{i,j}$ for the set of edges from vertex v_i to vertex v_j, and we add a contraction mapping $F_e : \mathbb{R}^2 \to \mathbb{R}^2$ for each edge $e \in \mathcal{E}$. Then the family of contractions $\{F_e : e \in \mathcal{E}\}$ is called a *graph-directed iterated function system* (GIFS) and there exists a unique family of non-empty compact subsets $E_1, \ldots, E_m$ of $\mathbb{R}^2$ [9,28] such that

$$E_i = \bigcup_{j=1}^{m} \bigcup_{e \in \mathcal{E}_{i,j}} F_e(E_j). \tag{3.1}$$

We call $E := \bigcup_{i=1}^{m} E_i$ a *graph-directed set*. Define $M = (M_{ij})_{1 \le i,j \le m}$ as the *contact matrix* [15] of $\mathcal{G}$ with $M_{ij} = \#\mathcal{E}_{i,j}$ counting the number of edges from v_i to v_j.

The GIFS $\{F_e : e \in \mathcal{E}\}$ is said to satisfy the *open set condition* (OSC) if there exists a family of open sets $\{O_1, \ldots, O_m\}$ such that

$$O_i \supset \bigcup_{j=1}^{m} \bigcup_{e \in \mathcal{E}_{i,j}} F_e(O_j) \quad \text{for } i = 1, 2, \ldots, m \tag{3.2}$$

with disjoint unions, i.e., $F_e(O_j) \cap F_{e'}(O_{j'}) = \emptyset$ whenever $(e, j) \neq (e', j')$. With this OSC, we then can compute the dimension of the graph-directed set.

In this section, we first identify the boundary of T with a graph-directed set by making use of the well-known method [17,31], then calculate its dimension in the self-affine case and the self-similar case, respectively.

Proposition 3.1 *Let* $\ell = \gamma v + \delta A v$, $\ell' = \gamma' v + \delta' A v \in \mathcal{V}$ *such that* $\ell' = A\ell - b_1 v$ *for some* $b_1 \in \Delta D$, *then*

$$A^{-1}(T_{\ell'} + jv) \subset T_\ell \quad \textit{for all} \quad j \in I_{b_1} := \begin{cases} \{b_1, b_1 + 1, \ldots, q - 1\} & \textit{if} \quad b_1 \ge 0; \\ \{0, 1, \ldots, q - 1 + b_1\} & \textit{if} \quad b_1 < 0. \end{cases}$$

Moreover,

$$T_\ell = \bigcup_{\ell' \in B_\ell} \bigcup_{j \in I_{b_1}} A^{-1}(T_{\ell'} + jv)$$

where $B_\ell := \{\ell'' \in \mathcal{V} : \ell'' = A\ell - b_1' v$ *for some* $b_1' \in \Delta D\}$. *Hence the boundary* $\partial T = \bigcup_{\ell \in \mathcal{V}} T_\ell$ *is a graph-directed set.*

Proof When $b_1 \ge 0$, if $x \in T_{\ell'}$ then the radix expansion is

$$x = 0.c_1 c_2 c_3 \ldots = \delta' \gamma' . c_1' c_2' c_3' \ldots.$$

It follows from Lemma 2.1 and $0 = A^{-1} f(A) v$ that

$$A^{-1}x + (b_1 + k)A^{-1}v = 0.(b_1 + k)c_1 c_2 c_3 \ldots = \delta\gamma.k c_1' c_2' c_3' \ldots \in T_\ell$$

for $k = 0, 1, \ldots, q - 1 - b_1$. The case when $b_1 < 0$ can be proved similarly.

For the second part, we only need to show

$$T_\ell \subset \bigcup_{\ell' \in B_\ell} \bigcup_{j \in I_{b_1}} A^{-1}(T_{\ell'} + jv).$$

Let $y = 0.a_1a_2a_3\ldots = \delta\gamma.a_1'a_2'a_3'\ldots \in T_\ell$. It follows that $Ay - a_1v = 0.a_2a_3a_4\ldots = \delta\gamma(a_1' - a_1).a_2'a_3'\ldots \in T_{\ell'}$, where $\ell' = A\ell - (a_1 - a_1')v$. This implies $y \in A^{-1}(T_{\ell'} + a_1v)$. By definition, we see that $a_1 \in I_{b_1}$ for $b_1 = a_1 - a_1'$. □

It should be mentioned that the graph for the GIFS comes from the neighbor graph by adding more edges, or equivalently the neighbor graph is a reduced graph for the GIFS. The following example about Fig. 1 can illustrate their relationship. All the other cases are given in Appendix 2.

Example 3.2 Consider the case $f(x)=x^2+px+q$ $(p, q \geq 2$, excluding $p=q=2)$. When $\ell=v$, from Table 1 we have $B_\ell=B_v=\{Av+pv, Av+(p-1)v\}$. When $\ell'=Av+pv$, $b_1=-p$ and $I_{-p}=\{0,1,2,\ldots,q-1-p\}$; when $\ell'=Av+(p-1)v$, $b_1=-(p-1)$ and $I_{-(p-1)}=\{0,1,2,\ldots,q-p\}$. Thus by Proposition 3.1, the first set equation comes out. Similarly the other five can be deduced. For simplicity we let $u_1=v$, $u_2=Av+(p-1)v$, $u_3=Av+pv$. Then the sets $T_{\pm u_1}$, $T_{\pm u_2}$, $T_{\pm u_3}$, representing ∂T satisfy

$$\begin{aligned}
AT_{u_1} &= \bigcup_{j=0}^{q-p}(T_{u_2}+jv) \cup \bigcup_{j=0}^{q-p-1}(T_{u_3}+jv),\\
AT_{u_2} &= \bigcup_{j=0}^{p-2}(T_{-u_2}+jv) \cup \bigcup_{j=0}^{p-1}(T_{-u_3}+jv),\\
AT_{u_3} &= T_{-u_1},\\
AT_{-u_1} &= \bigcup_{j=p-1}^{q-1}(T_{-u_2}+jv) \cup \bigcup_{j=p}^{q-1}(T_{-u_3}+jv),\\
AT_{-u_2} &= \bigcup_{j=q-p+1}^{q-1}(T_{u_2}+jv) \cup \bigcup_{j=q-p}^{q-1}(T_{u_3}+jv),\\
AT_{-u_3} &= T_{u_1}+(q-1)v.
\end{aligned}$$

The Hausdorff dimension ($\dim_H$) (see e.g., [9,10]) is the most common and important dimension in fractal geometry. The case of self-similar sets has been studied extensively with or without separation conditions. However the case of self-affine sets is still hard to handle. Recently, He and Lau [16] defined the generalized Hausdorff dimension ($\dim_H^\omega$) and Hausdorff measure ($\mathcal{H}_\omega^s$) for self-affine fractals by replacing the Euclidean norm with a pseudo-norm ω for which the expanding matrix A becomes a similarity:

$$\omega(Ax) = |\det A|^{1/2}\omega(x).$$

Under this setting, most of the basic properties for the self-similar sets can be carried to the self-affine sets. Moreover, Luo and Yang [26] extended this technique to the self-affine GIFS and obtained a dimension formula of the graph-directed set we need.

Proposition 3.3 ([26]) *For the GIFS as in* (3.1) *with the affine mappings* $F_e(x) = A^{-1}(x + d_e)$ *where* A *is an expanding matrix and* $|\det A| = |q|$, *let* $\rho(M)$ *be the spectral radius of the contact matrix* M. *If the OSC holds, then* $s = \dim_H^{\omega} E = 2\log\rho(M)/\log|q|$ *and* $0 < \mathcal{H}_{\omega}^{s}(E) < \infty$.

By using this, we can establish our first-dimensional result about the boundary of T as follows.

Theorem 3.4 *The generalized Hausdorff dimension of the boundary of disk-like CC tile* T *is*

$$\dim_H^{\omega}(\partial T) = 2\log\rho(M)/\log|q|$$

and the corresponding measure is positive and finite.

Proof From Propositions 3.1 and 3.3, it suffices to show the GIFS representing the boundary of T satisfies the OSC. Replacing T_ℓ by $(T+\ell)^\circ$, we can check the OSC holds case by case. We illustrate the idea by proving the case $f(x) = x^2+px+q$ ($p \geq 2, q \geq 2,$ excluding $p = q = 2$). In view of Example 3.2, we need to show

$$\begin{aligned}
A(T+u_1)^\circ &\supset \bigcup_{j=0}^{q-p}((T+u_2)^\circ + jv) \cup \bigcup_{j=0}^{q-p-1}((T+u_3)^\circ + jv),\\
A(T+u_2)^\circ &\supset \bigcup_{j=0}^{p-2}((T-u_2)^\circ + jv) \cup \bigcup_{j=0}^{p-1}((T-u_3)^\circ + jv),\\
A(T+u_3)^\circ &\supset (T-u_1)^\circ,\\
A(T-u_1)^\circ &\supset \bigcup_{j=p-1}^{q-1}((T-u_2)^\circ + jv) \cup \bigcup_{j=p}^{q-1}((T-u_3)^\circ + jv),\\
A(T-u_2)^\circ &\supset \bigcup_{j=q-p+1}^{q-1}((T+u_2)^\circ + jv) \cup \bigcup_{j=q-p}^{q-1}((T+u_3)^\circ + jv),\\
A(T-u_3)^\circ &\supset (T+u_1)^\circ + (q-1)v,
\end{aligned}$$

with disjoint unions. Since T is a CC tile, it follows that

$$AT^\circ \supset \bigcup_{j=0}^{q-1}(T+jv)^\circ = \bigcup_{j=0}^{q-1}(T^\circ + jv) \tag{3.3}$$

with disjoint union. By using (3.3) and $0 = f(A)v = A^2v + pAv + qv$ extensively, we prove the first two set inequalities in the following. The remaining four can be verified similarly.

For $j = 0, 1, \dots, q - p$,

$$(T + u_2)^\circ + jv = T^\circ + (p - 1 + j)v + Av \subset A(T + u_1)^\circ.$$

For $j = 0, 1, \dots, q - p - 1$,

$$(T + u_3)^\circ + jv = T^\circ + (p + j)v + Av \subset A(T + u_1)^\circ.$$

For $j = 0, 1, \dots, p - 2$,

$$\begin{aligned}(T - u_2)^\circ + jv &= T^\circ + (j - p + 1)v - Av \\ &= T^\circ + (q + j - p + 1)v + A^2v + (p - 1)Av \\ &\subset A(T + u_2)^\circ.\end{aligned}$$

For $j = 0, 1, \dots, p - 1$,

$$\begin{aligned}(T - u_3)^\circ + jv &= T^\circ + (j - p)v - Av \\ &= T^\circ + (q + j - p)v + A^2v + (p - 1)Av \\ &\subset A(T + u_2)^\circ.\end{aligned}$$

By the same way, all the other cases follow and hence the theorem is proved. □

In the rest of this section, we will find the exact value of Hausdorff dimension $\dim_H(\partial T)$ for certain particular cases that A is a similarity. We state the simplest one first.

Proposition 3.5 *Let $A \in M_2(\mathbb{Z})$ be expanding with characteristic polynomial $f(x) = x^2 + q$ ($|q| \geq 2$) and $T(A, \mathcal{D})$ a disk-like CC tile. Then* $\dim_H(\partial T) = 1$.

Proof By Theorem 1.1, T is a square tile (parallelogram). Hence $\dim_H(\partial T) = 1$. □

Geometrically, a similarity is a multiple of either a reflection or a rotation. We call the former a *scaled reflection* and the latter a *scaled rotation*; algebraically, a similarity is a multiple of an orthogonal matrix. The case that A is a scaled reflection is solved already as its characteristic polynomial is of the form $f(x) = x^2 - q$ ($q > 0$). So we focus our attention on those A that are scaled rotations.

Lemma 3.6 *Let A be a scaled rotation. Then its characteristic polynomial has positive constant term and A has either two distinct non-real eigenvalues or two equal real eigenvalues.*

Proof Let $A = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix}$. The characteristic polynomial is given by $x^2 - 2r\cos\theta\, x + r^2$. It has two equal real zeros when $\theta = 0$ or π and two distinct non-real zeros otherwise. □

Table 3 The contact matrix of T associated with $f(x) = x^2 + px + q,\ p, q \geq 2, 2p \leq q + 2$ (excluding $p = q = 2$)

	v	$Av + (p-1)v$	$Av + pv$	$-v$	$-Av - (p-1)v$	$-Av - pv$
v	0	$q - p + 1$	$q - p$	0	0	0
$Av + (p-1)v$	0	0	0	0	$p - 1$	p
$Av + pv$	0	0	0	1	0	0
$-v$	0	0	0	0	$q - p + 1$	$q - p$
$-Av - (p-1)v$	0	$p - 1$	p	0	0	0
$-Av - pv$	1	0	0	0	0	0

The following dimension formula on the boundaries of self-similar tiles has been investigated in the literature by various methods (see [8,31,32,17,23]). We shall apply this formula to obtain our second dimensional result which is simpler than the known one.

Proposition 3.7 *If A is a similarity with $|\det(A)| = |q| \geq 2$, then the Hausdorff dimension of ∂T is given by*

$$\dim_H(\partial T) = \log \rho(M)/\log r = 2\log \rho(M)/\log |q|, \tag{3.4}$$

where $\rho(M)$ denotes the spectral radius of the contact matrix M and $r = |q|^{1/2}$ is the expansion ratio of A.

Let ℓ, ℓ', b_1 and B_ℓ be defined as in Proposition 3.1. We first find the contact matrix M. Since $\mathcal{D}$ is a CC digit set, we have the entry $M_{\ell\ell'} = \#I_{b_1} = q - |b_1|$ where I_{b_1} is as in Proposition 3.1. Recall that b_1 is the label of the edge from ℓ to ℓ'. Hence we obtain the contact matrix M of T from its neighbor graph with different edge labels (i.e., replace b_1 by $q - |b_1|$).

Moreover, it is easy to see that there is a one-to-one correspondence between the contact matrix and the neighbor graph. For example, the contact matrix for the case $f(x) = x^2 + px + q$ ($p, q \geq 2,\ 2p \leq q + 2$ excluding $p = q = 2$) can be found in Table 3, and the related neighbor graph is shown by Fig. 1. The contact matrices for the other cases are given in Appendix 3.

If M is irreducible (i.e., for each entry M_{ij}, there exists an integer $n \geq 0$ such that $(M^n)_{ij} > 0$), then the spectral radius $\rho(M) = \lambda_M$ where λ_M is the *Perron–Frobenius eigenvalue* of M as stated in the following simplified version of the Perron–Frobenius Theorem.

Theorem 3.8 ([12,30]) *Let M be an irreducible non-negative matrix. Then there exists a positive eigenvalue λ_M such that $\lambda_M \geq |\mu|$ for all eigenvalues μ of M. Moreover, λ_M is a simple zero of the characteristic polynomial of M.*

It is known that a contact matrix is irreducible if and only if the neighbor graph it represents is strongly connected. A directed graph is called *strongly connected* if for any two vertices v_i, v_j there exists a path starting at v_i and ending at v_j.

Theorem 3.9 *Let $A \in M_2(\mathbb{Z})$ be an expanding similarity with characteristic polynomial $f(x) = x^2 + px + q$ and $T(A, \mathcal{D})$ be a disk-like CC tile. Then $\rho(M)$ is the largest positive zero of the cubic polynomial*

$$x^3 - (|p| - 1)x^2 - (|q| - |p|)x - |q|.$$

Hence $\dim_H(\partial T) = 2 \log \rho(M)/ \log |q|$.

Proof Since $|\det(A)| = |q|$, it is more convenient to work with $f(x) = x^2 \pm px \pm q$ ($p \geq 0$, $q \geq 2$). Also we ignore those $f(x)$ of the form $f(x) = x^2 \pm px - q$ ($p > 0$, $q \geq 2$) as they cannot be characteristic polynomials of similarities (Lemma 3.6). We can see from Appendix 1 or 3 that the contact matrix is irreducible if and only if $f(x) = x^2 \pm px + q$ where $p > 0$.

Case (1) $f(x) = x^2 + px + q$. The characteristic polynomial of the corresponding M is $(x - 1)(x^2 + px + q)[x^3 - (p - 1)x^2 - (q - p)x - q]$. Notice that $\rho(M) \neq 1$. Indeed, if $\rho(M) = 1$, then $\dim_H(\partial T) = 0$, which implies ∂T is totally disconnected (Proposition 2.5, [10]). This is not possible for the boundary of a topological disk. The zeros of $x^2 + px + q$ are either both negative or both non-real. Hence $\rho(M)$ is the largest positive real zero of $x^3 - (p - 1)x^2 - (q - p)x - q$.

Case (2) $f(x) = x^2 - px + q$. The characteristic polynomial of the corresponding M is $(x + 1)(x^2 - px + q)[x^3 - (p - 1)x^2 - (q - p)x - q]$. Since $f(x)$ cannot have unequal real zeros (Lemma 3.6), we have $p^2 - 4q \leq 0$. When $p^2 - 4q < 0$, the zeros of $x^2 - px + q$ are non-real. Then $\rho(M)$ is the largest positive real zero of $x^3-(p-1)x^2-(q-p)x-q$. When $p^2-4q = 0$, the two zeros of x^2-px+q are equal. But the Perron–Frobenius eigenvalue should be a simple zero of the characteristic polynomial of M (Theorem 3.8), so $\rho(M)$ is also the largest positive real zero of $x^3 - (p - 1)x^2 - (q - p)x - q$.

Case (3) $f(x) = x^2 + q$. The contact matrix M is reducible. Its characteristic polynomial is $(x^2 - q)(x^2 + q)(x - 1)(x + 1)(x^2 + 1)$. We see that $\rho(M) = q^{1/2}$, which is the largest positive zero of $x^3 + x^2 - qx - q = (x^2 - q)(x + 1)$.

Case (4) $f(x) = x^2 - q$. The contact matrix M is also reducible and its characteristic polynomial is found to be $(x^2 - q)^2(x + 1)(x - 1)^3$. As in the previous case, $\rho(M) = q^{1/2}$, which is also the largest positive zero of $x^3 + x^2 - qx - q$. □

Remark 3.10 It is interesting to see that the signs of p and q do not matter in the calculation of $\dim_H(\partial T)$ when A is a similarity. Notice also for the last two cases, $f(x) = x^2 + q$ ($|q| \geq 2$), we have $\rho(M) = |q|^{1/2}$. It follows that $\dim_H(\partial T) = 1$, as expected for the boundary of a parallelogram (Proposition 3.5).

We observe that $\dim_H(\partial T)$ is independent of the choice of the vector v in the following sense.

Corollary 3.11 *Let $A \in M_2(\mathbb{Z})$ be an expanding similarity with characteristic polynomial $f(x) = x^2 + px + q$ ($|q| \geq 2$). Let $\mathcal{D} = \mathcal{D}(v, |q|)$ and $\mathcal{D}' = \mathcal{D}(v', |q|)$ be two CC digit sets such that each of $\{v, Av\}$ and $\{v', Av'\}$ is an independent set. If $2|p| \leq |q + 2|$, then*

$$\dim_H(\partial T(A, \mathcal{D})) = \dim_H(\partial T(A, \mathcal{D}')).$$

Proof As $2|p| \leq |q+2|$, both $T(A, \mathcal{D})$ and $T(A, \mathcal{D}')$ are disk-like CC tiles (Theorem 1.1). Hence the corollary follows from Theorem 3.9. ◻

Remark 3.12 We conjecture that Theorem 3.9 and Corollary 3.11 are also valid when $2|p| > |q+2|$, i.e., T is non-disk-like. The major difficulty in justifying these conjectures is that, in general, there is no upper bound on the number of neighbors of a non-disk-like CC tile [7].

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Appendix 1: Neighbor Graphs

Let $f(x) = x^2 \pm px \pm q$ ($p \geq 0$, $q \geq 2$). The neighbor graphs of disk-like tiles are classified by $f(x)$ and listed in Figs. 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11.

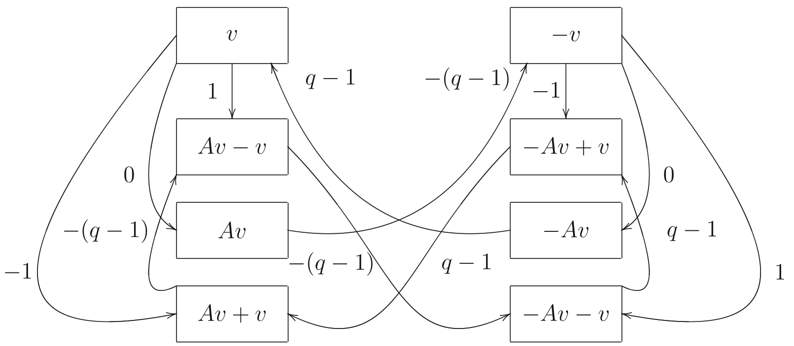
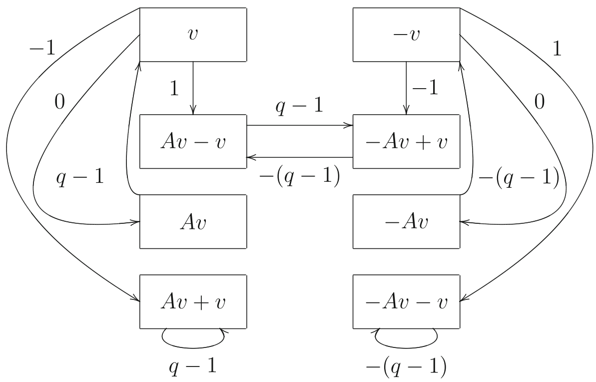


Fig. 2 The neighbor graph of T associated with $f(x) = x^2 + q$

Fig. 3 The neighbor graph of T associated with $f(x) = x^2 - q$



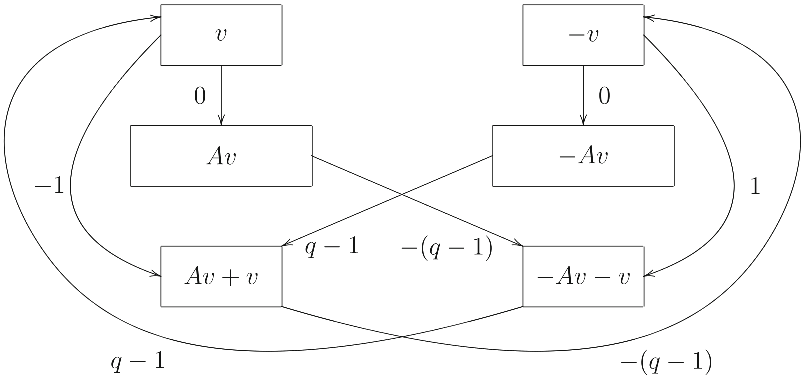


Fig. 4 The neighbor graph of T associated with $f(x) = x^2 + x + q$

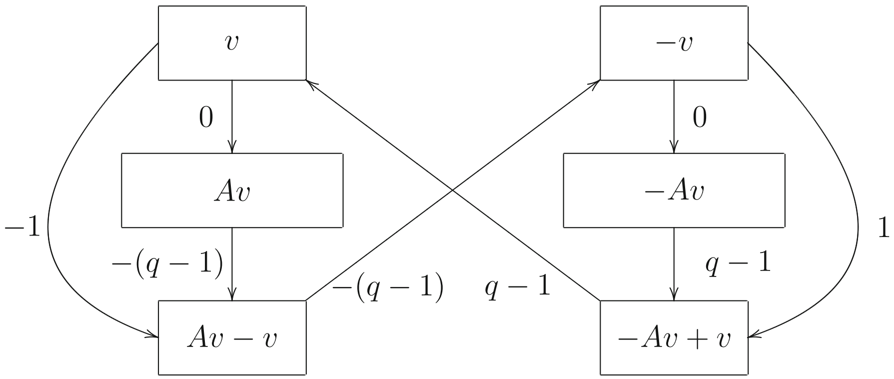


Fig. 5 The neighbor graph of T associated with $f(x) = x^2 - x + q$

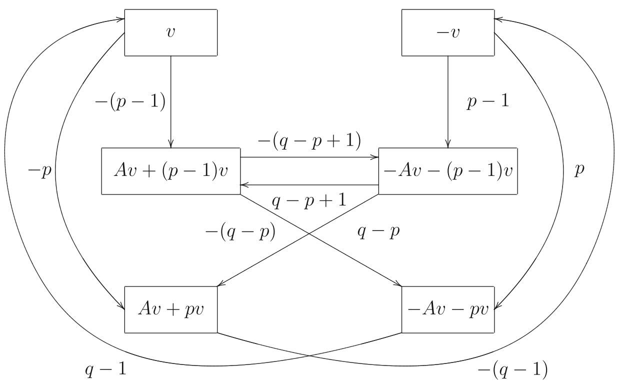


Fig. 6 The neighbor graph of T associated with $f(x) = x^2 + px + q$, $p \geq 2, 2p \leq q + 2$ (excluding $q = p = 2$)

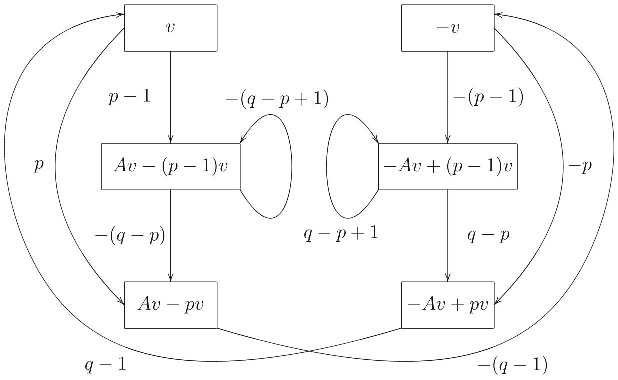


Fig. 7 The neighbor graph of T associated with $f(x) = x^2 - px + q$, $p \geq 2, 2p \leq q + 2$ (excluding $q = p = 2$)

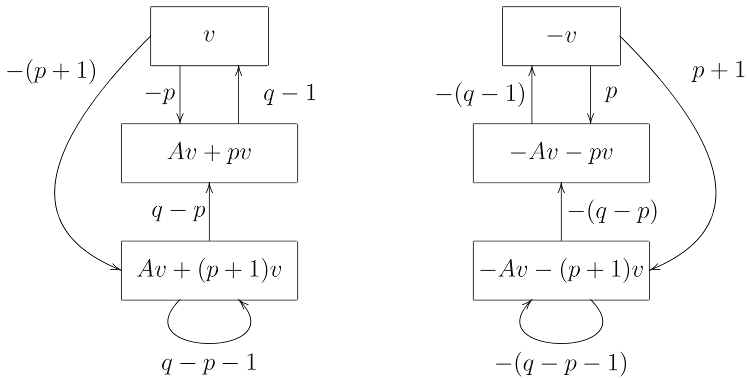


Fig. 8 The neighbor graph of T associated with $f(x) = x^2 + px - q$, $p \geq 1, 2p \leq q - 2$

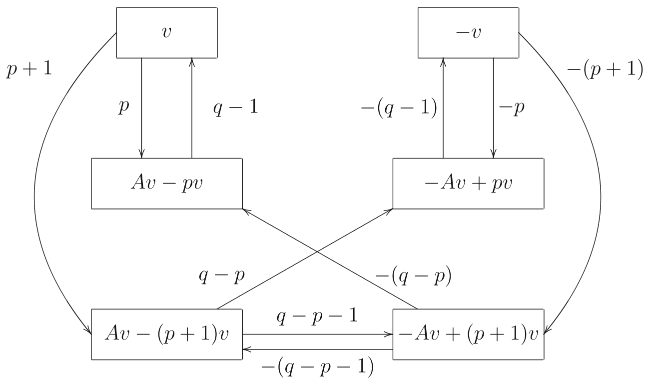


Fig. 9 The neighbor graph of T associated with $f(x) = x^2 - px - q$, $p \geq 1, 2p \leq q - 2$

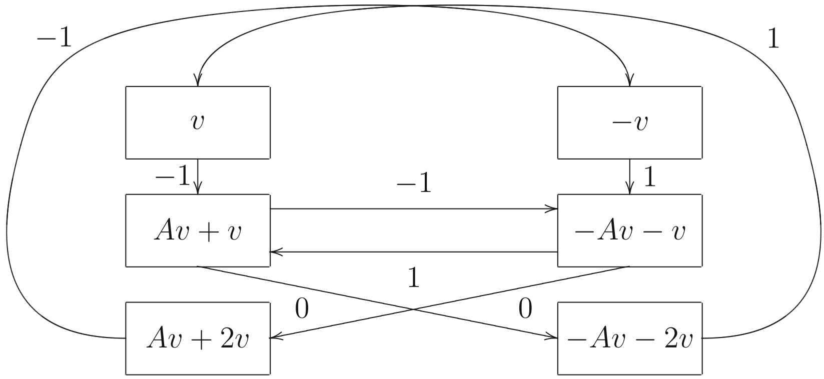
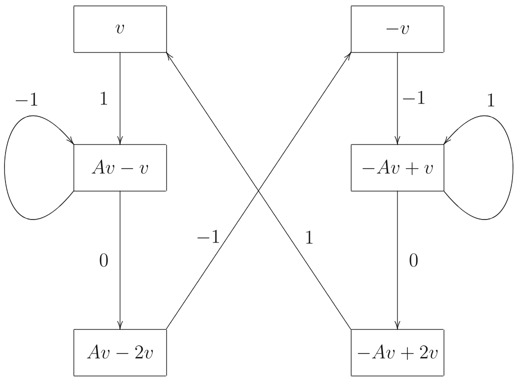


Fig. 10 The neighbor graph of T associated with $f(x) = x^2 + 2x + 2$

Fig. 11 The neighbor graph of T associated with $f(x) = x^2 - 2x + 2$



Appendix 2: Graph-Directed Sets

Let $f(x) = x^2 \pm px \pm q$ $(p \geq 0,\ q \geq 2)$. The graph-directed sets representing the boundary ∂T are classified by $f(x)$ and listed below.

(1) $f(x) = x^2 + q$. Convention: $u_1 = v, u_2 = Av - v, u_3 = Av, u_4 = Av + v$.

$$\begin{aligned}
AT_{u_1} &= \bigcup_{j=1}^{q-1}(T_{u_2} + jv) \cup \bigcup_{j=0}^{q-1}(T_{u_3} + jv) \cup \bigcup_{j=0}^{q-2}(T_{u_4} + jv),\\
AT_{u_2} &= T_{-u_4},\\
AT_{u_3} &= T_{-u_1},\\
AT_{u_4} &= T_{u_2},\\
AT_{-u_1} &= \bigcup_{j=0}^{q-2}(T_{-u_2} + jv) \cup \bigcup_{j=0}^{q-1}(T_{-u_3} + jv) \cup \bigcup_{j=1}^{q-1}(T_{-u_4} + jv),\\
AT_{-u_2} &= T_{u_4} + (q-1)v,
\end{aligned}$$

$$
\begin{aligned}
AT_{-u_3} &= T_{u_1} + (q-1)v,\\
AT_{-u_4} &= T_{-u_2} + (q-1)v.
\end{aligned}
$$

(2) $f(x) = x^2 - q$. Convention: $u_1 = v, u_2 = Av - v, u_3 = Av, u_4 = Av + v$.

$$
\begin{aligned}
AT_{u_1} &= \bigcup_{j=1}^{q-1}(T_{u_2} + jv) \cup \bigcup_{j=0}^{q-1}(T_{u_3} + jv) \cup \bigcup_{j=0}^{q-2}(T_{u_4} + jv),\\
AT_{u_2} &= T_{-u_2} + (q-1)v,\\
AT_{u_3} &= T_{u_1} + (q-1)v,\\
AT_{u_4} &= T_{u_4} + (q-1)v,\\
AT_{-u_1} &= \bigcup_{j=0}^{q-2}(T_{-u_2} + jv) \cup \bigcup_{j=0}^{q-1}(T_{-u_3} + jv) \cup \bigcup_{j=1}^{q-1}(T_{-u_4} + jv),\\
AT_{-u_2} &= T_{u_2},\\
AT_{-u_3} &= T_{-u_1},\\
AT_{-u_4} &= T_{-u_4}.
\end{aligned}
$$

(3) $f(x) = x^2 + x + q$. Convention: $u_1 = v, u_2 = Av, u_3 = Av + v$.

$$
\begin{aligned}
AT_{u_1} &= \bigcup_{j=0}^{q-1}(T_{u_2} + jv) \cup \bigcup_{j=0}^{q-2}(T_{u_3} + jv),\\
AT_{u_2} &= T_{-u_3},\\
AT_{u_3} &= T_{-u_1},\\
AT_{-u_1} &= \bigcup_{j=0}^{q-1}(T_{-u_2} + jv) \cup \bigcup_{j=1}^{q-1}(T_{-u_3} + jv),\\
AT_{-u_2} &= T_{u_3} + (q-1)v,\\
AT_{-u_3} &= T_{u_1} + (q-1)v.
\end{aligned}
$$

(4) $f(x) = x^2 - x + q$. Convention: $u_1 = v, u_2 = Av, u_3 = Av - v$.

$$
\begin{aligned}
AT_{u_1} &= \bigcup_{j=0}^{q-1}(T_{u_2} + jv) \cup \bigcup_{j=0}^{q-2}(T_{u_3} + jv),\\
AT_{u_2} &= T_{u_3},\\
AT_{u_3} &= T_{-u_1},\\
AT_{-u_1} &= \bigcup_{j=0}^{q-1}(T_{-u_2} + jv) \cup \bigcup_{j=1}^{q-1}(T_{-u_3} + jv),\\
AT_{-u_2} &= T_{-u_3} + (q-1)v,\\
AT_{-u_3} &= T_{u_1} + (q-1)v.
\end{aligned}
$$

(5) $f(x) = x^2 + px + q,\ p \geq 2, 2p \leq q + 2$ (excluding $p = q = 2$). Convention: $u_1 = v, u_2 = Av + (p-1)v, u_3 = Av + pv$.

$$
\begin{aligned}
AT_{u_1} &= \bigcup_{j=0}^{q-p} (T_{u_2} + jv) \cup \bigcup_{j=0}^{q-p-1} (T_{u_3} + jv),\\
AT_{u_2} &= \bigcup_{j=0}^{p-2} (T_{-u_2} + jv) \cup \bigcup_{j=0}^{p-1} (T_{-u_3} + jv),\\
AT_{u_3} &= T_{-u_1},\\
AT_{-u_1} &= \bigcup_{j=p-1}^{q-1} (T_{-u_2} + jv) \cup \bigcup_{j=p}^{q-1} (T_{-u_3} + jv),\\
AT_{-u_2} &= \bigcup_{j=q-p+1}^{q-1} (T_{u_2} + jv) \cup \bigcup_{j=q-p}^{q-1} (T_{u_3} + jv),\\
AT_{-u_3} &= T_{u_1} + (q-1)v.
\end{aligned}
$$

(6) $f(x) = x^2 - px + q,\ p \geq 2, 2p \leq q + 2$ (excluding $p = q = 2$). Convention: $u_1 = v, u_2 = Av - (p-1)v, u_3 = Av - pv$.

$$
\begin{aligned}
AT_{u_1} &= \bigcup_{j=p-1}^{q-1} (T_{u_2} + jv) \cup \bigcup_{j=p}^{q-1} (T_{u_3} + jv),\\
AT_{u_2} &= \bigcup_{j=0}^{p} (T_{u_2} + jv) \cup \bigcup_{j=0}^{p-1} (T_{u_3} + jv),\\
AT_{u_3} &= T_{-u_1},\\
AT_{-u_1} &= \bigcup_{j=0}^{q-p} (T_{-u_2} + jv) \cup \bigcup_{j=0}^{q-p-1} (T_{-u_3} + jv),\\
AT_{-u_2} &= \bigcup_{j=q-p+1}^{q-1} (T_{-u_2} + jv) \cup \bigcup_{j=q-p}^{q-1} (T_{-u_3} + jv),\\
AT_{-u_3} &= T_{u_1} + (q-1)v.
\end{aligned}
$$

(7) $f(x) = x^2 + px - q,\ p \geq 1, 2p \leq q - 2$. Convention: $u_1 = v, u_2 = Av + pv, u_3 = Av + (p+1)v$.

$$
\begin{aligned}
AT_{u_1} &= \bigcup_{j=0}^{q-p-1} (T_{u_2} + jv) \cup \bigcup_{j=0}^{q-p-2} (T_{u_3} + jv),\\
AT_{u_2} &= T_{u_1} + (q-1)v,
\end{aligned}
$$

$$
\begin{aligned}
AT_{u_3} &= \bigcup_{j=q-p}^{q-1} (T_{u_2} + jv) \cup \bigcup_{j=q-p-1}^{q-1} (T_{u_3} + jv),\\
AT_{-u_1} &= \bigcup_{j=p}^{q-1} (T_{-u_2} + jv) \cup \bigcup_{j=p+1}^{q-1} (T_{-u_3} + jv),\\
AT_{-u_2} &= T_{u_1},\\
AT_{-u_3} &= \bigcup_{j=0}^{p-1} (T_{-u_2} + jv) \cup \bigcup_{j=0}^{p} (T_{-u_3} + jv).
\end{aligned}
$$

(8) $f(x) = x^2 - px - q,\ p \geq 1, 2p \leq q - 2$. Convention: $u_1 = v, u_2 = Av - pv, u_3 = Av - (p+1)v$.

$$
\begin{aligned}
AT_{u_1} &= \bigcup_{j=p}^{q-1} (T_{u_2} + jv) \cup \bigcup_{j=p+1}^{q-1} (T_{u_3} + jv),\\
AT_{u_2} &= T_{u_1} + (q-1)v,\\
AT_{u_3} &= \bigcup_{j=q-p}^{q-1} (T_{-u_2} + jv) \cup \bigcup_{j=q-p-1}^{q-1} (T_{-u_3} + jv),\\
AT_{-u_1} &= \bigcup_{j=0}^{q-p-1} (T_{-u_2} + jv) \cup \bigcup_{j=0}^{q-p-2} (T_{-u_3} + jv),\\
AT_{-u_2} &= T_{-u_1},\\
AT_{-u_3} &= \bigcup_{j=0}^{p-1} (T_{u_2} + jv) \cup \bigcup_{j=0}^{p-2} (T_{u_3} + jv).
\end{aligned}
$$

(9) $f(x) = x^2 + 2x + 2$. Convention: $u_1 = v, u_2 = Av + v, u_3 = Av + 2v$.

$$
\begin{aligned}
AT_{u_1} &= T_{u_2},\\
AT_{u_2} &= T_{-u_2} \cup T_{-u_3} \cup (T_{-u_3} + v),\\
AT_{u_3} &= T_{-u_1},\\
AT_{-u_1} &= T_{-u_2} + v,\\
AT_{-u_2} &= (T_{u_2} + v) \cup T_{u_3} \cup (T_{u_3} + v),\\
AT_{-u_3} &= T_{u_1} + v.
\end{aligned}
$$

(10) $f(x) = x^2 - 2x + 2$. Convention: $u_1 = v, u_2 = Av - v, u_3 = Av - 2v$.

$$
\begin{aligned}
AT_{u_1} &= T_{u_2} + v,\\
AT_{u_2} &= T_{u_2} \cup T_{u_3} \cup (T_{u_3} + v),\\
AT_{u_3} &= T_{-u_1},
\end{aligned}
$$

$$\begin{aligned} AT_{-u_1} &= T_{-u_2}, \\ AT_{-u_2} &= (T_{-u_2} + v) \cup T_{-u_3} \cup (T_{-u_3} + v), \\ AT_{-u_3} &= T_{u_1} + v. \end{aligned}$$

Appendix 3: Contact Matrices

Let $f(x) = x^2 \pm px \pm q$ ($p \geq 0$, $q \geq 2$). The contact matrices (in table form) of disk-like tiles are classified by $f(x)$ and listed in Tables 4, 5, 6, 7, 8, 9, 10, 11, 12, and 13.

Table 4 $f(x) = x^2 + q$

	v	Av	$-v$	$-Av$	$Av - v$	$-Av - v$	$-Av + v$	$Av + v$
v	0	q	0	0	$q - 1$	0	0	$q - 1$
Av	0	0	1	0	0	0	0	0
$-v$	0	0	0	q	0	$q - 1$	$q - 1$	0
$-Av$	1	0	0	0	0	0	0	0
$Av - v$	0	0	0	0	0	1	0	0
$-Av - v$	0	0	0	0	0	0	1	0
$-Av + v$	0	0	0	0	0	0	0	1
$Av + v$	0	0	0	0	1	0	0	0

Table 5 $f(x) = x^2 - q$

	v	Av	$-v$	$-Av$	$Av - v$	$-Av + v$	$Av + v$	$-Av - v$
v	0	q	0	0	$q - 1$	0	$q - 1$	0
Av	1	0	0	0	0	0	0	0
$-v$	0	0	0	q	0	$q - 1$	0	$q - 1$
$-Av$	0	0	1	0	0	0	0	0
$Av - v$	0	0	0	0	0	1	0	0
$-Av + v$	0	0	0	0	1	0	0	0
$Av + v$	0	0	0	0	0	0	1	0
$-Av - v$	0	0	0	0	0	0	0	1

Table 6 $f(x) = x^2 + x + q$

	v	Av	$Av + v$	$-v$	$-Av$	$-Av - v$
v	0	q	$q - 1$	0	0	0
Av	0	0	0	0	0	1
$Av + v$	0	0	0	1	0	0
$-v$	0	0	0	0	q	$q - 1$
$-Av$	0	0	1	0	0	0
$-Av - v$	1	0	0	0	0	0

Table 7 $f(x) = x^2 - x + q$

	v	Av	$Av - v$	$-v$	$-Av$	$-Av + v$
v	0	q	$q - 1$	0	0	0
Av	0	0	1	0	0	0
$Av - v$	0	0	0	1	0	0
$-v$	0	0	0	0	q	$q - 1$
$-Av$	0	0	0	0	0	1
$-Av + v$	1	0	0	0	0	0

Table 8 $f(x) = x^2 + px + q,\ p \geq 2,\ 2p \leq q + 2$ (excluding $p = q = 2$)

	v	$Av + (p-1)v$	$Av + pv$	$-v$	$-Av - (p-1)v$	$-Av - pv$
v	0	$q - p + 1$	$q - p$	0	0	0
$Av + (p-1)v$	0	0	0	0	$p - 1$	p
$Av + pv$	0	0	0	1	0	0
$-v$	0	0	0	0	$q - p + 1$	$q - p$
$-Av - (p-1)v$	0	$p - 1$	p	0	0	0
$-Av - pv$	1	0	0	0	0	0

Table 9 $f(x) = x^2 - px + q,\ p \geq 2,\ 2p \leq q + 2$ (excluding $p = q = 2$)

	v	$Av - (p-1)v$	$Av - pv$	$-v$	$-Av + (p-1)v$	$-Av + pv$
v	0	$q - p + 1$	$q - p$	0	0	0
$Av - (p-1)v$	0	$p - 1$	p	0	0	0
$Av - pv$	0	0	0	1	0	0
$-v$	0	0	0	0	$q - p + 1$	$q - p$
$-Av + (p-1)v$	0	0	0	0	$p - 1$	p
$-Av + pv$	1	0	0	0	0	0

Table 10 $f(x) = x^2 + px - q,\ p \geq 1,\ 2p \leq q - 2$

	v	$Av + pv$	$Av + (p+1)v$	$-v$	$-Av - pv$	$-Av - (p+1)v$
v	0	$q - p$	$q - p - 1$	0	0	0
$Av + pv$	1	0	0	0	0	0
$Av + (p+1)v$	0	p	$p + 1$	0	0	0
$-v$	0	0	0	0	$q - p$	$q - p - 1$
$-Av - pv$	0	0	0	1	0	0
$-Av - (p+1)v$	0	0	0	0	p	$p + 1$

Table 11 $f(x) = x^2 - px - q,\ p \geq 1,\ 2p \leq q - 2$

	v	$Av - pv$	$Av - (p+1)v$	$-v$	$-Av + pv$	$-Av + (p+1)v$
v	0	$q - p$	$q - p - 1$	0	0	0
$Av - pv$	1	0	0	0	0	0
$Av - (p+1)v$	0	0	0	0	p	$p + 1$
$-v$	0	0	0	0	$q - p$	$q - p - 1$
$-Av + pv$	0	0	0	1	0	0
$-Av + (p+1)v$	0	p	$p + 1$	0	0	0

Table 12 $f(x) = x^2 + 2x + 2$

	v	$Av + v$	$Av + 2v$	$-v$	$-Av - v$	$-Av - 2v$
v	0	1	0	0	0	0
$Av + v$	0	0	0	0	1	2
$Av + 2v$	0	0	0	1	0	0
$-v$	0	0	0	0	1	0
$-Av - v$	0	1	2	0	0	0
$-Av - 2v$	1	0	0	0	0	0

Table 13 $f(x) = x^2 - 2x + 2$

	v	$Av - v$	$Av - 2v$	$-v$	$-Av + v$	$-Av + 2v$
v	0	1	0	0	0	0
$Av - v$	0	1	2	0	0	0
$Av - 2v$	0	0	0	1	0	0
$-v$	0	0	0	0	1	0
$-Av + v$	0	0	0	0	1	2
$-Av + 2v$	1	0	0	0	0	0

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