J. Math. Anal. Appl. 372 (2010) 8–18

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On single-matrix graph-directed iterated function systems ☆

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ARTICLE INFO

Article history:
Received 20 March 2009
Available online 21 July 2010
Submitted by C.E. Wayne

Keywords:
Graph IFS
Weak norm
Open set condition

ABSTRACT

We study graph-directed function systems where each contraction in the system has the form $f_e(x) = A^{-1}(x + d_e)$, where A is an expanding matrix. We show that a certain discreteness implies the open set condition, and the latter implies the strong open set condition. Hausdorff measures and dimensions (w.r.t. a weak norm) of the invariant sets are investigated. The stationary Markov measures of the system are proved to be translation invariant.

1. Introduction

1.1. Graph IFS. Let (V, Γ) be a directed graph with vertex set $V = \{1, \ldots, N\}$ and edge set Γ. We call $\{f_e;\ e \in \Gamma\}$, a collection of contractions $f_e : \mathbb{R}^d \mapsto \mathbb{R}^d$, a *graph-directed iterated function system* (graph IFS).

Let Γ_{ij} be the set of edges from vertex i to j, then there are unique non-empty compact sets $\{E_i\}_{i=1}^N$ satisfying [27]

$$E_i = \bigcup_{j=1}^{N} \bigcup_{e \in \Gamma_{ij}} f_e(E_j), \quad 1 \leqslant i \leqslant N. \tag{1.1}$$

We call $(E_1, \ldots, E_N)$ the *invariant sets* of the graph IFS.

The graph IFS is said to satisfy the *open set condition* (OSC), if there exist open sets $U_1, \ldots, U_N$ such that

$$\bigcup_{j=1}^{N} \bigcup_{e \in \Gamma_{ij}} f_e(U_j) \subset U_i, \quad 1 \leqslant i \leqslant N,$$

and the left-hand side are non-overlapping unions [14,27]. In addition, if $U_i \cap E_i \neq \emptyset$ for all $1 \leqslant i \leqslant N$, then we say the graph IFS satisfies the *strong open set condition* (SOSC) [31].

Let us define $M = (m_{ij})_{1 \leqslant i,j \leqslant N}$ to be the associated matrix of (V, Γ), that is, $m_{ij} = \#\Gamma_{ji}$ counts the number of edges from j to i. We say (V, Γ) is *primitive* if M is a primitive matrix, i.e., M^n is a positive matrix for large n. From now on, we will always assume the graph (V, Γ) in consideration is primitive.

☆ The authors are supported by CNSF 10631040.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.07.001

1.2. Single-matrix IFS. If a graph (V, Γ) contains only one vertex, then the graph IFS $\{f_e;\ e \in \Gamma\}$ simplifies to an *iterated function system* $\{f_j\}_{j=1}^N$ ([14]). If all f_j have the form

$$f_j(x) = A^{-1}(x + d_j), \quad 1 \leqslant j \leqslant N, \tag{1.2}$$

where A is a $d \times d$ expanding matrix and $d_j \in \mathbb{R}^d$, then we call $\{f_j\}_{j=1}^N$ a *single-matrix IFS*. (A matrix is *expanding* if all its eigenvalues have moduli larger than 1.) Let us denote by $q = |\det A|$.

Some special cases of system (1.2) define number systems. The study of such number systems goes back as early as 1970's [18,28,15]. See also a recent survey [6].

Another special case of (1.2) is the so-called *self-affine tiling system*, when $N = q := |\det A|$ and the OSC holds. The self-affine tiling system has been studied by many authors [4,16,17,11,22–24,3].

In the above studies, we generally concern the following questions:

(Q1) *When does the system satisfy OSC?*
(Q2) *If the OSC holds, how to compute the Hausdorff dimension and Hausdorff measure of E, the invariant set of* (1.2)*?*
(Q3) *Does OSC imply SOSC?*

• If A is a similitude, these questions have satisfactory answers.
Let $\mathcal{D} = \{d_1, \ldots, d_N\}$ be the set of translations in (1.2). Define

$$\mathcal{D}_n = A^{n-1}\mathcal{D} + \cdots + A\mathcal{D} + \mathcal{D}, \quad n \geqslant 1.$$

Let $\dim E$ denote the Hausdorff dimension of E, and let $\mathcal{H}^s(E)$ be the s-dimensional Hausdorff measure of E. A set G is said to be *r-uniformly discrete* if $|x - y| > r$ for any $x, y \in G$. Then

(i) *OSC holds if and only if* $\#\mathcal{D}_n = N^n$ *and* $\mathcal{D}_n$ *is r-uniformly discrete for some* $r > 0$ *independent of n.*
(ii) *If OSC holds, then* $s = \dim E = \frac{d \log N}{\log q}$ *and* $0 < \mathcal{H}^s(E) < +\infty$.
(iii) *OSC implies SOSC.*

Especially in the self-affine tiling system case, many deep results on (Q1) and (Q2) have been obtained by Fourier transformation method [16,11,24]. For (Q3), Schief [31] gives a positive answer for general self-similar IFS.

• In case of A is not a similitude and N does not equal to $|\det A|$, it is much more complicated. Actually, in this case the second assertion does not hold. McMullen's carpets provide counter-examples [26].

To overcome the difficulty that A is not similitude, Lemarié-Rieusset [20] introduce a weak norm ω of $\mathbb{R}^d$ such that $\omega(Ax) = q^{1/d}\omega(x)$. Under the weak norm, A is a 'similitude'. He and Lau [12] introduce Hausdorff dimension and Hausdorff measure w.r.t. the weak norm, which will be denoted by $\dim_\omega$ and $\mathcal{H}_w^s$ respectively. [12] proved that the above results still hold except the second assertion is replaced by

(ii′) *If OSC holds, then* $s = \dim_\omega E = \frac{d \log N}{\log q}$ *and* $0 < \mathcal{H}_\omega^s(E) < +\infty$.

1.3. Single-matrix graph IFS. In this paper, we investigate the graph IFS $\{f_e;\ e \in \Gamma\}$ with the form

$$f_e(x) = A^{-1}(x + d_e), \tag{1.3}$$

where A is a $d \times d$ expanding matrix and $d_e \in \mathbb{R}^d$. The paper is motivated by the questions posed by Professor S. Ito in a conference in Beijing in 2006. To state the questions, we need some notations.

Denote by Γ_{ij}^n the paths from vertex i to vertex j with length n. For $I = e_1 \cdots e_n \in \Gamma_{ij}^n$, set $f_I(x) := f_{e_1} \circ f_{e_2} \circ \cdots \circ f_{e_n}(x)$ and define

$$d_I := A^{n-1}d_{e_1} + A^{n-2}d_{e_2} + \cdots + Ad_{e_{n-1}} + d_{e_n},$$

then $f_I(x)$ has the form: $f_I(x) = A^{-n}(x + d_I)$. Set

$$\mathcal{D}_{ij}^n := \{d_I;\ I \in \Gamma_{ij}^n\}. \tag{1.4}$$

(P1) *Does r-uniformly discreteness of* $\mathcal{D}_{ij}^n$ *imply OSC?*
(P2) *Does OSC imply SOSC?*
(P3) *Let* μ_i *be the stationary Markov measures on* E_i*, are* μ_i *'translation invariant' on* E_i*?*

Stationary Markov measures will be introduced in Section 2. For a measure μ supported by a set E, we say μ is *translation invariant* on E if for any $B_1, B_2 \subset E$ and $B_1 = B_2 + x$, it holds that $\mu(B_1) = \mu(B_2)$.

1.4. Main results. In this paper, we generalize the results of [12] to single-matrix graph IFS. These results are worth to be documented, since graph IFS are frequently encountered in practice, for example, in the study of IFS with overlap structures

[30,7,13], in self-similar tiling theory [33,34,5], in Rauzy geometry [29,2,32,19], etc. Our results are in great general form and contain many previous results as special case. Also, they give satisfactory answers to the questions of Professor Ito.

Theorem 1.1. *For graph IFS* (1.3), *the following are equivalent*:

(i) *OSC.*
(ii) $\#\mathcal{D}_{ij}^n = \#\Gamma_{ij}^n$ *and there is an* $r > 0$ *such that* $\mathcal{D}_{ij}^n$ *is* r*-uniformly discrete for all* $1 \leqslant i, j \leqslant N$ *and* $n \geqslant 1$.
(iii) *SOSC.*

In Section 6, we show by examples that OSC does not imply SOSC if the system is not a single-matrix system.

Theorem 1.2. *For graph IFS* (1.3), *let* λ *be the maximal eigenvalue of* M, *the associate matrix of* (V, Γ). *If OSC holds, then for any* $1 \leqslant i \leqslant N$,

(i) $s = \dim_\omega E_i = d \log \lambda / \log q$.
(ii) $0 < \mathcal{H}_\omega^s(E_i) < +\infty$.
(iii) *The right-hand side of* (1.1) *is a disjoint union in sense of the measure* $\mathcal{H}_w^s$.

Remark 1.3. For Theorem 1.1 and Theorem 1.2, the case that A is a similitude has been studied by Li [21]. The case that (1.2) is a tiling system has been settled by Lagarias and Wang [25].

The next theorem answers question (P3). The translation invariance of a measure on a fractal set has never been considered before. Theorem 1.4 seems to be the first result of this type. Moreover, although the result seems very nature, it is hard to prove without using the weak norm technique.

Theorem 1.4. *For* $1 \leqslant i \leqslant N$, *the stationary Markov measure* μ_i *is equal to* $a_i^{-1}\mathcal{H}_\omega^s|_{E_i}$, *where* $a_i = \mathcal{H}_\omega^s(E_i)$. *Consequently,* μ_i *is translation invariant.*

1.5. Applications. Recently, Furukado, Ito and Rao [9] apply the above results to the study of atomic surfaces of hyperbolic substitutions and obtain some interesting results. According to a substitution σ, [9] constructs a single-matrix graph IFS. They define a *fractal domain-exchange transformation* Φ on $E = \bigcup_{i=1}^N E_i$, the union of the invariant sets. Φ preserve the stationary measure μ by our results. [9] shows that (E, Φ, μ) is (measure theoretically) isomorphic to the *substitution dynamical system* defined by σ, i.e., there exists a measure-preserving bijection between two systems except a measure zero set.

Akiyama and Loridant [1] apply our results to study the parametrization of boundaries of self-affine tiles.

The paper is organized as follows: In Section 2, we recall some known results on Markov measures. In Section 3, we give a brief introduction to weak norm. Theorem 1.1 is proved in Section 4, Theorem 1.2 and Theorem 1.4 are proved in Section 5. In Section 6 we give some remarks on SOSC.

2. SOSC and Markov measures

In this section, we consider general graph IFS (1.1).

2.1. Markov measures. Let $\mathcal{M}(\mathbb{R}^d)$ denote the collection of probability Borel measures on $\mathbb{R}^d$ with bounded support.

Let $p : \Gamma \mapsto (0, 1]$ be a function satisfying

$$\sum_{j=1}^N \sum_{e \in \Gamma_{ij}} p_e = 1, \quad 1 \leqslant i \leqslant N.$$

We shall call p a *probability weight* on graph (V, Γ).

It is well known that there is a unique vector $(\mu_1, \ldots, \mu_N) \in (\mathcal{M}(\mathbb{R}^d))^N$ satisfying the equations

$$\mu_i = \sum_{j=1}^N \sum_{e \in \Gamma_{ij}} p_e \cdot \mu_j \circ f_e^{-1}, \quad 1 \leqslant i \leqslant N, \tag{2.1}$$

we call $\mu_1, \ldots, \mu_N$ the *Markov measures* determined by the weights $\{p_e;\ e \in \Gamma\}$.

Remark 2.1. For an IFS, Eq. (2.1) simplifies to $\mu = \sum_{i=1}^N p_i \cdot \mu \circ f_i^{-1}$. The measure μ is called a *self-similar measure* when the mappings f_i are similitudes [14].

2.2. Measures on symbolic space and their projections. Let $\Sigma_i^* = \bigcup_{j=1}^N \bigcup_{k\geqslant 1} \Gamma_{ij}^k$ denote the collection of all finite paths with initial state (or vertex) i and $\Sigma^* = \bigcup_{i=1}^N \Sigma_i^*$. Denote $\Sigma_i^{\mathbb{N}}$ to be the collection of infinite paths with initial state i, denote $\Sigma = \bigcup_{i=1}^N \Sigma_i^{\mathbb{N}}$ be the set of all infinite paths.

For $I = e_1 e_2 \cdots e_k \in \Gamma_{ij}^k$, we define $b(I) = i$ be the *initial state* of I, and $t(I) = j$ be the *terminate state* of I. Denote $E_I := f_I(E_j)$ where $j = t(I)$.

Denote $[I] := \{e_1 e_2 \cdots e_k \cdots \in \Sigma : e_1 e_2 \cdots e_k = I\}$, and call it a cylinder of Σ. Given $I, J \in \Sigma^*$ with $t(I) = b(J)$, denote by IJ the concatenation of I and J.

Let S be the shift operator on Σ where $S(e_1 e_2 e_3 \cdots) = e_2 e_3 \cdots$.

Let $\mathbb{P}_i$ be the probability measure on $\Sigma_i^{\mathbb{N}}$ satisfying the relations

$$\mathbb{P}_i\big([e_1 \cdots e_n]\big) = p_{e_1} \cdots p_{e_n}, \quad e_1 \cdots e_n \in \Sigma_i^*. \tag{2.2}$$

According to formula (1.1), it is seen that $\{f_{e_1 \cdots e_n}(E_{t(e_n)})\}_{n\geqslant 1}$ is a decreasing sequence of compact sets and their intersection is a single point in $E_{b(e_1)}$. Define a projection $\pi : (\Sigma_1^{\mathbb{N}}, \ldots, \Sigma_N^{\mathbb{N}}) \mapsto (\mathbb{R}^d, \ldots, \mathbb{R}^d)$, where $\pi_i : \Sigma_i^{\mathbb{N}} \mapsto \mathbb{R}^d$ is defined by

$$\big\{\pi_i(e_1 e_2 \cdots e_n \cdots)\big\} = \bigcap_{n\geqslant 1} f_{e_1 \cdots e_n}(E_{t(e_n)}).$$

Then for $e \in \Gamma_{ij}$ and $J \in \Sigma_j^*$,

$$f_e^{-1} \circ \pi_i(eJ) = \pi_j \circ S(eJ). \tag{2.3}$$

Set $\mu_i = \mathbb{P}_i \circ \pi_i^{-1}$, then μ_i is a probability measure supported by E_i.

Proposition 2.2. *The projection measures $(\mu_1, \ldots, \mu_N)$ satisfy Eq.* (2.1) *and thus are Markov measures on E_i.*

Proof. Pick any Borel set A in $\mathbb{R}^d$. For any edge $e \in \Gamma_{ij}$, set

$$H_e = \pi_j^{-1} \circ f_e^{-1}(A).$$

Then $H_e \subset \Sigma_j^{\mathbb{N}}$ and thus $eJ \in \Sigma_i^{\mathbb{N}}$ for any $J \in H_e$. Denote $eH_e := \{eJ\colon\ J \in H_e\}$, then $\mathbb{P}_i(eH_e) = p_e \mathbb{P}_j(H_e)$ by the definition of $\mathbb{P}_i$'s.

First we show that

$$\pi_i^{-1}(A) = \bigcup_{j=1}^N \bigcup_{e\in\Gamma_{ij}} eH_e. \tag{2.4}$$

Suppose $I \in \pi_i^{-1}(A)$ and the initial edge of I is $e \in \Gamma_{ij}$, then

$$\begin{aligned} I \in \pi_i^{-1}(A) &\iff f_e^{-1} \circ \pi_i(I) \in f_e^{-1}(A) \\ &\iff \pi_j \circ S(I) \in f_e^{-1}(A) \quad \big(\text{by formula (2.3)}\big) \\ &\iff S(I) \in \pi_j^{-1} \circ f_e^{-1}(A) = H_e \\ &\iff I \in eH_e. \end{aligned}$$

So formula (2.4) holds. Applying P_i to both sides of (2.4), we obtain

$$\begin{aligned} \mu_i(A) &= \sum_{j=1}^N \sum_{e\in\Gamma_{ij}} \mathbb{P}_i(eH_e) = \sum_{j=1}^N \sum_{e\in\Gamma_{ij}} p_e \circ \mathbb{P}_j(H_e) \\ &= \sum_{j=1}^N \sum_{e\in\Gamma_{ij}} p_e \circ \mathbb{P}_j \circ \pi_j^{-1} \circ f_e^{-1}(A) \\ &= \sum_{j=1}^N \sum_{e\in\Gamma_{ij}} p_e \circ \mu_j \circ f_e^{-1}(A). \qquad \square \end{aligned}$$

Two paths are said to be *comparable* if one of them is a prefix of the other one; otherwise, they are incomparable. The following measure separation property, proved by Fan and Lau [10, Theorem 2.2], illustrates the importance of SOSC.

Proposition 2.3. *Suppose a graph IFS $\{f_e;\ e\in\Gamma\}$ satisfies the strong open set condition with open sets $\{U_i\}_{1\leqslant i\leqslant N}$. Let μ_i be Markov measures on E_i, $1\leqslant i\leqslant N$. Then*

(i) $\mu_i(U_i)=1$,
(ii) $\mu_i(E_I\cap E_J)=0$ *for any incomparable* $I, J\in\Sigma_i^*$.

Question. *Does the inverse of Proposition 2.3 hold? That is, does the measure separation property imply SOSC?*

2.3. Stationary Markov measures. Recall that $M=(m_{ij})_{1\leqslant i,j\leqslant N}$ is the associated matrix of graph (V,Γ). Then M is a non-negative primitive matrix. Let $(c_1,\dots,c_N)$ be the left eigenvector of M corresponding to the maximal eigenvalue λ. By Perron–Frobenius Theorem, $\lambda>0$ and $(c_1,\dots,c_N)$ is a positive vector. Let us assume that $c_1+\dots+c_N=1$.

For $e\in\Gamma_{ij}$, set

$$p_e=\frac{c_j}{c_i}\lambda^{-1}.$$

It is easy to see that $\sum_{b(e)=i}p_e=1$ for each i. Therefore $\{p_e;\ e\in\Gamma\}$ is a probability weight. Let $\mathbb{P}_1,\dots,\mathbb{P}_N$ be the measures on $\Sigma_1^{\mathbb{N}},\dots,\Sigma_N^{\mathbb{N}}$ defined by formula (2.2), and let $\mu_1,\dots,\mu_N$ be the Markov measures on $E_1,\dots,E_N$ defined by the weight $\{p_e;\ e\in\Gamma\}$. We shall call $\mu_1,\dots,\mu_N$ the *stationary Markov measures* of the graph IFS $\{f_e;\ e\in\Gamma\}$.

Lemma 2.4. *Suppose the SOSC holds for a graph IFS $\{f_e;\ e\in\Gamma\}$. Then for any $I\in\Gamma_{ij}^k$, we have*

$$\mu_i\big(f_I(E_j)\big)=\frac{c_j}{c_i}\lambda^{-k}.$$

Proof. It is obvious that $\mu_i(E_I)\geqslant\mathbb{P}_i([I])$.

The strong open set condition implies that $\mu_i(E_I\cap E_J)=0$ for any incomparable $I, J\in\Sigma_i^*$ (Proposition 2.3). Hence

$$\mu_i(E_I)=\mu_i\big(E_I\setminus\cup\big\{E_J;\ |J|=|I|,\ J\neq I,\ J\in\Sigma_i^*\big\}\big)\leqslant\mathbb{P}_i\big([I]\big).$$

Therefore

$$\mu_i\big(f_I(E_j)\big)=\mu_i(E_I)=\mathbb{P}_i\big([I]\big)=p_{e_1}\cdots p_{e_k}=\frac{c_j}{c_i}\lambda^{-k}.\qquad\square$$

3. Pseudo-norm

Let A be a $d\times d$ real expanding matrix with $|\det A|=q$. With respect to A, [20] defines a pseudo norm ω on $\mathbb{R}^d$ as follows (see also [12]).

Denote $B(x,r)$ the open ball with center x and radius r. Then $V=A(B(0,1))\setminus B(0,1)$ is an annular region. Choose any $0<\delta<\frac{1}{2}$ and any C^∞ function $\phi_\delta(x)$ with support in $B(0,\delta)$ such that $\phi_\delta(x)=\phi_\delta(-x)$ and $\int\phi_\delta(x)\,dx=1$, define a pseudo norm $\omega(x)$ in $\mathbb{R}^d$ by

$$\omega(x)=\sum_{n\in\mathbb{Z}}q^{-n/d}h\big(A^nx\big),\tag{3.1}$$

where $h(x)=\chi_V*\phi_\delta(x)$ is the convolution of the characteristic function χ_V and $\phi_\delta(x)$.

We list some basic properties of $\omega(x)$.

Proposition 3.1. *(See [12].) The $\omega(x)$ is a C^∞ function on $\mathbb{R}^d$ and satisfies*

(i) $\omega(x)\geqslant 0$; $\omega(x)=0$ *if and only if* $x=0$.
(ii) $\omega(x)=\omega(-x)$.
(iii) $\omega(Ax)=q^{1/d}\omega(x)\geqslant\omega(x)$ *for all* $x\in\mathbb{R}^d$.
(iv) *There exists β such that $\omega(x+y)\leqslant\beta\max\{\omega(x),\omega(y)\}$ for any $x, y\in\mathbb{R}^d$.*

Item (iii) says that matrix A is a similitude in this weak norm, a fact plays a central role in [12] as well as in our paper.

The next proposition shows that the pseudo-norm $\omega(x)$ is comparable with the Euclidean norm $|x|$ through $\lambda_{\max}$ and $\lambda_{\min}$, the maximal and minimal modulus of the eigenvalues of A.

Proposition 3.2. *(See [12].) For any $0<\varepsilon<\lambda_{\min}-1$, there exists $C>0$ (depends on ε) such that*

$$C^{-1}|x|^{\ln q/d\ln(\lambda_{\max}+\varepsilon)}\leqslant\omega(x)\leqslant C|x|^{\ln q/d\ln(\lambda_{\min}-\varepsilon)},\quad \textit{if } |x|>1,$$
$$C^{-1}|x|^{\ln q/d\ln(\lambda_{\min}-\varepsilon)}\leqslant\omega(x)\leqslant C|x|^{\ln q/d\ln(\lambda_{\max}+\varepsilon)},\quad \textit{if } |x|\leqslant 1.$$

Let E be a subset of $\mathbb{R}^d$. Define $\operatorname{diam}_\omega E = \sup\{\omega(x-y)\colon x, y \in E\}$ be the ω-diameter of E. Now we can define a Hausdorff measure of E with respect to the pseudo norm $\omega(x)$.

$$\mathcal{H}^s_{\omega,\delta}(E) = \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam}_\omega E_i)^s\colon E \subset \bigcup_i E_i,\ \operatorname{diam}_\omega E_i \leqslant \delta\right\}.$$

Since $\mathcal{H}^s_{\omega,\delta}(E)$ is increasing when δ tends to 0, we can define

$$\mathcal{H}^s_{\omega}(E) = \lim_{\delta\to 0}\mathcal{H}^s_{\omega,\delta}(E).$$

It is shown that $\mathcal{H}^s_\omega(E)$ is an outer measure and is a regular measure on the family of Borel subsets on $\mathbb{R}^d$. It is translation invariant and has the scaling property; precisely,

$$\mathcal{H}^s_\omega(E+x) = \mathcal{H}^s_\omega(E) \quad \text{and} \quad \mathcal{H}^s_\omega\big(A^{-1}E\big) = q^{-s/d}\mathcal{H}^s_\omega(E). \tag{3.2}$$

A Hausdorff dimension with respect to the pseudo norm thus can be defined to be

$$\dim_\omega E = \inf\big\{s;\ \mathcal{H}^s_\omega(E) = 0\big\} = \sup\big\{s;\ \mathcal{H}^s_\omega(E) = \infty\big\}.$$

The relation of $\dim_\omega E$ and the classical Hausdorff dimension $\dim_H(E)$ has been studied in [12].

4. Uniform discreteness and OSC

We prove Theorem 1.1 in this section. Our proof is an analogue of [12], where the basic idea belongs to [31].

Proof of Theorem 1.1. (i) ⇒ (ii). Suppose the open set condition holds and $U_1, \ldots, U_N$ are open sets such that

$$\bigcup_{j=1}^{N}\bigcup_{e\in\Gamma_{ij}} f_e(U_j) \subset U_i, \quad 1 \leqslant i \leqslant N, \tag{4.1}$$

and the left-hand side of (4.1) is a disjoint union.

First we note that $f_I(U_j) \subset U_i$ holds for $I \in \Gamma^n_{ij}$.

Let $I = e_1 \cdots e_n$ and $I' = e'_1 \cdots e'_n$ be two elements of Γ^n_{ij}. Using (4.1) repeatedly, one can show that $f_I(U_j)$ and $f_{I'}(U_j)$ belong to U_i and they are disjoint. Hence $A^{-n}(U_j + d_I) \cap A^{-n}(U_j + d_{I'}) = \emptyset$ and so that $(U_j + d_I) \cap (U_j + d_{I'}) = \emptyset$. It follows that $d_I - d_{I'} \neq 0$ and thus $\#\mathcal{D}^n_{ij} = \#\Gamma^n_{ij}$.

Set $\eta_j = \inf_{x\in\mathbb{R}^d}\{|x|\colon (U_j + x) \cap U_j = \emptyset\}$, then $\eta_j > 0$ since U_j is an open set. Put $r = \min\{\eta_j\colon 1 \leqslant j \leqslant N\}$. Since $(U_j + d_I) \cap (U_j + d_{I'}) = \emptyset$, we conclude that $|d_I - d_{I'}| \geqslant r$ for $I, I' \in \mathcal{D}^n_{ij}$.

(ii) ⇒ (iii). This is the difficult part of this theorem.

We define the δ-parallel body of a set E with respect to the weak norm ω as follows: $[E]^\delta = \{x \in \mathbb{R}^d\colon \omega(x-y) < \delta$ for some $y \in E\}$. (Usually $[E]_\delta$ stands for the δ-parallel body of E. Here we use an unusual notation to avoid confusion.)

Pick any $\delta > 0$ and fix it. For any $J \in \Gamma^n_{ij}$, define

$$G_J = f_J\big([E_j]^\delta\big),$$

then $G_J = [E_J]^{q^{-n/d}\delta}$ by Proposition 3.1(iii). We define $V(J)$, neighbors of J, to be a collection of paths giving by

$$V(J) = \big\{I \in \Sigma^*_i\colon |I| = n \text{ and } E_I \cap G_J \neq \emptyset\big\}.$$

Note that $E_I \cap G_J \neq \emptyset$ if and only if $(E_{t(I)} + d_I) \cap ([E_j]^\delta + d_J) \neq \emptyset$.

We shall show that $\#V(J)$ have a uniform bound, namely,

$$\gamma = \sup\big\{\#V(J)\colon J \in \Sigma^*\big\} < +\infty.$$

Let R be a real number such that $\bigcup_{1\leqslant j\leqslant N} E_j \subset B(0,R)$, and let η be a positive number such that $\bigcup_{j=1}^N [E_j]^\delta \subset B(0,\eta)$. Then for $J \in \Gamma^n_{ij}$,

$$\begin{aligned}\#V(J) &= \#\big\{I \in \Sigma^*_i\colon |I| = n,\ (E_{t(I)} + d_I) \cap \big([E_j]^\delta + d_J\big) \neq \emptyset\big\}\\ &\leqslant \#\big\{I \in \Sigma^*_i\colon |I| = n,\ (E_{t(I)} + d_I) \cap \big(B(0,\eta) + d_J\big) \neq \emptyset\big\}\\ &\leqslant \sup_{x\in\mathbb{R}^d}\#\big\{I \in \Sigma^*_i\colon |I| = n,\ (E_{t(I)} + d_I) \cap B(x,\eta) \neq \emptyset\big\}\end{aligned}$$

$$\leqslant \sum_{j=1}^{N} \sup_{x\in\mathbb{R}^d} \#\{d_I:\ d_I \in B(x,\eta+R)\cap D_{ij}^n\}$$
$$<\infty,$$

by the uniform discreteness of D_{ij}^n. Therefore $\gamma < +\infty$.

Let $J \in \Sigma^*$ be a path such that $\#V(J)=\gamma$ attains the maximum. We claim that if $IJ \in \Sigma^*$, then

$$V(IJ) = IV(J) := \{IL:\ L \in V(J)\}.$$

For any $L \in V(J)$, we have $E_L \cap G_J \neq \emptyset$, hence $E_{IL}\cap G_{IJ} = f_I(E_L)\cap f_I(G_J)\neq\emptyset$. It follows that $IL \in V(IJ)$ and so that $IV(J)\subset V(IJ)$. By the maximality of $\#V(J)$, we obtain $IV(J)=V(IJ)$. The claim is proved.

Let us assume that $J \in \Gamma_{hj}^n$. Let $\delta' = \frac{\delta}{\beta}$, where the constant β is as in Proposition 3.1(iv). Define

$$U_i = \bigcup_{n=1}^{\infty}\bigcup_{I\in\Gamma_{ih}^n} f_{IJ}\big([E_j]^{\delta'}\big), \quad 1\leqslant i\leqslant N. \tag{4.2}$$

Clearly $U_i \cap E_i \neq \emptyset$. We shall show these open sets satisfy SOSC.

For $e \in \Gamma_{ik}$, we have

$$f_e(U_k) = \bigcup_{n=1}^{\infty}\bigcup_{I\in\Gamma_{kh}^n} f_{eIJ}\big([E_j]^{\delta'}\big) \subset \bigcup_{n=2}^{\infty}\bigcup_{I'\in\Gamma_{ih}^n} f_{I'J}\big([E_j]^{\delta'}\big)\subset U_i.$$

So

$$\bigcup_{k=1}^{N}\bigcup_{e\in\Gamma_{ik}} f_e(U_k)\subset U_i. \tag{4.3}$$

It remains to show that the left-hand side of (4.3) is a disjoint union.

Suppose this is false. Then $f_e(U_k)\cap f_{e'}(U_{k'})\neq\emptyset$ for some $e\in\Gamma_{ik}$, $e'\in\Gamma_{ik'}, e\neq e'$ (it may happen that $k=k'$). By the construction of the open sets, there exist two paths I_1 from k to h and I_2 from k' to h such that

$$f_{eI_1J}\big([E_j]^{\delta'}\big)\cap f_{e'I_2J}\big([E_j]^{\delta'}\big)\neq\emptyset.$$

Suppose y is a point in this intersection, then

$$y = f_{eI_1J}(y_1) = f_{e'I_2J}(y_2)$$

for some $y_1, y_2 \in [E_j]^{\delta'}$. Choose $z_1, z_2 \in E_j$ such that

$$\omega(y_1 - z_1) < \delta', \qquad \omega(y_2 - z_2) < \delta'.$$

Without loss of generality, we assume that $|I_1| \leqslant |I_2|$.

On one hand, as $f_I(x) = A^{-|I|}(x+d_I)$, we have

$$\begin{aligned}\omega\big(f_{eI_1J}(z_1) - f_{e'I_2J}(z_2)\big) &= \omega\big(f_{eI_1J}(z_1) - y + y - f_{e'I_2J}(z_2)\big)\\ &= \omega\big(A^{-|eI_1J|}(z_1-y_1) + A^{-|e'I_2J|}(y_2-z_2)\big)\\ &< \beta\max\big\{\delta' q^{-\frac{|eI_1J|}{d}}, \delta' q^{-\frac{|e'I_2J|}{d}}\big\} \quad \big(\text{by Proposition 3.1(iv)}\big)\\ &= \beta\delta' q^{-\frac{|eI_1J|}{d}}\\ &= q^{-\frac{|eI_1J|}{d}}\delta.\end{aligned}$$

On the other hand, for any $L \in \Sigma_{k'}^*$ with length $|L| = |I_1J|$, clearly $e'L \notin V(eI_1J) = eI_1V(J)$; so we have

$$f_{e'L}(E_{t(L)}) \cap f_{eI_1J}\big([E_j]^{\delta}\big) = \emptyset. \tag{4.4}$$

As $|e'L| = |eI_1J| \leqslant |e'I_2J|$, we have

$$f_{e'I_2J}(E_j) \subset \bigcup_L f_{e'L}(E_{t(L)}),$$

where L runs over the paths in $\Sigma_{k'}^*$ with length $|L| = |I_1J|$. Therefore, the point $f_{e'I_2J}(z_2)\in f_{e'I_2J}(E_j)$ must belong to some cylinder of the form $f_{e'L}(E_{t(L)})$; as a result, it does not belong to $f_{eI_1J}([E_j]^{\delta})$ by (4.4). It follows that

$$\omega\big(f_{eI_1J}(z_1) - f_{e'I_2J}(z_2)\big) \geqslant q^{-\frac{|eI_1J|}{d}}\delta,$$

which is a contradiction.

(iii) ⇒ (i) is trivial. □

5. Hausdorff measure in weak norm

Mass distribution principle is a powerful method to estimate the lower bound of Hausdorff dimension (cf. Falconer [8]). It also works for Hausdorff dimension w.r.t. a weak norm.

Let ω be a weak norm and μ be a measure of E. If there exist constants $c > 0$ and $\delta > 0$ such that $\mu(B) \leqslant c(\operatorname{diam}_\omega B)^s$ for any set B with $\operatorname{diam}_\omega B < \delta$ and $B \cap E \neq \emptyset$, then

$$\mathcal{H}^s_\omega(E) \geqslant c^{-1}\mu(E).$$

The proof is analogues to the classical case.

Theorem 1.2. *For graph IFS* (1.3), *let λ be the maximal eigenvalue of M, the associate matrix of (V, Γ). If OSC holds, then for any $1 \leqslant i \leqslant N$,*

(i) $s = \dim_\omega E_i = d\log\lambda/\log q$.
(ii) $0 < \mathcal{H}^s_\omega(E_i) < +\infty$.
(iii) *The right-hand side of* (1.1) *is a disjoint union in sense of the measure $\mathcal{H}^s_w$.*

Proof. Recall that M is the associated matrix of the graph (V, Γ), λ be the Perron–Frobenius eigenvalue of M, and $q = |\det A|$. Let $s = d\log\lambda/\log q$.

(i) and (ii). We first show that $\mathcal{H}^s_\omega(E_i) < \infty$, $1 \leqslant i \leqslant N$. Let $\alpha = \max_{1\leqslant j\leqslant N}\operatorname{diam}_\omega(E_j)$. Then $\bigcup_{j=1}^N\{f_I(E_j);\ I \in \Gamma^k_{ij}\}$ provides a δ_k-covering of E_i with $\delta_k = q^{-\frac{k}{d}}\alpha$ and consequently,

$$\mathcal{H}^s_{\omega,\delta_k}(E_i) \leqslant \sum_{j=1}^N \sum_{I\in\Gamma^k_{ij}} \big(q^{-\frac{k}{d}}\alpha\big)^s = \lambda^{-k}\alpha^s\sum_{j=1}^N \#\Gamma^k_{ij} \leqslant C\alpha^s$$

by Perron–Frobenius Theorem. Since $\{\delta_k\}_{k\geqslant 1}$ decreases to 0, we obtain $\mathcal{H}^s_\omega(E_i) < \infty$.

To prove $\mathcal{H}^s_\omega(E_i) > 0$, we use the mass-distribution principle.

Pick any set F with $\operatorname{diam}_\omega(F) = \delta < 1$. Let k be the integer such that $q^{-\frac{k}{d}} \leqslant \delta < q^{-\frac{k-1}{d}}$, then $1 \leqslant \operatorname{diam}_\omega(A^kF) < q^{1/d}$. By Proposition 3.2, we have that $|A^kF| \leqslant C_1$, where C_1 is a constants independent of F.

Let μ_i be the stationary Markov measure on E_i. Let

$$\mathcal{N} = \bigcup_{j=1}^N\big\{I \in \Gamma^k_{ij};\ E_I \cap F \neq \emptyset\big\}$$

be the set of cylinders intersecting F. Clearly

$$\mu_i(F) \leqslant \sum_{I\in\mathcal{N}}\mu_i(E_I). \tag{1.1}$$

Notice that $E_I \cap F \neq \emptyset$ if and only if $(E_j + d_I) \cap A^kF \neq \emptyset$. Set $R = C_1 + \max_{1\leqslant j\leqslant N}|E_j|$ and pick any $x_0 \in A^kF$, then $I \in \mathcal{N}$ implies that $d_I \in B(x_0, R)$. By the uniform discreteness of d_I, there exists a constant C_2 such that $\#\mathcal{N} < C_2$.

Set $C_3 = \max\{c_j/c_i;\ 1 \leqslant i, j \leqslant N\}$, where $(c_1, \ldots, c_N)$ is the left maximal eigenvector of M. Then by Lemma 2.4 we have $\mu_i(E_I) \leqslant C_3\lambda^{-k}$. So by (1.1), we have

$$\mu_i(F) \leqslant (\#\mathcal{N})C_3\lambda^{-k} < C_2C_3\lambda^{-k} = C_2C_3q^{(-ks)/d} \leqslant C_2C_3(\operatorname{diam}_\omega F)^s.$$

Now the mass distribution principle implies that $\mathcal{H}^s_\omega(E_i) \geqslant \frac{\mu_i(E_i)}{C_2C_3} > 0$. Therefore, $\dim_\omega(E_i) = s = d\ln\lambda/\ln q$ and $0 < \mathcal{H}^s_\omega(E_i) < \infty$.

(iii) By formula (3.2), for $e \in \Gamma_{ij}$, $\mathcal{H}^s_\omega(f_e(E_j) = \lambda^{-1}\mathcal{H}^s_\omega(E_j)$. Therefore, by $E_i = \bigcup_{j=1}^N\bigcup_{e\in\Gamma_{ij}} f_e(E_j)$ and the subadditivity of $\mathcal{H}^s_\omega$, one has

$$\mathcal{H}^s_\omega(E_i) \leqslant \frac{1}{\lambda}\sum_{j=1}^N m_{ji}\mathcal{H}^s_\omega(E_j).$$

So

$$\lambda(a_1, \ldots, a_N) \leqslant (a_1, \ldots, a_N)M$$

where $a_i = \mathcal{H}^s_\omega(E_i)$. Since $(a_1, \dots, a_N)$ is a positive vector, by Perron–Frobenius Theorem, the inequality above is actually an equality. Therefore $(a_1, \dots, a_N)$ is an eigenvector of M and

$$\mathcal{H}^s_\omega(E_i) = \sum_{j=1}^{N} \sum_{e \in \Gamma_{ij}} \mathcal{H}^s_\omega\big(f_e(E_j)\big). \qquad \square$$

We denote by $\mathcal{H}^s_\omega|_B$ the restriction measure

$$\mathcal{H}^s_\omega|_B(F) = \mathcal{H}^s_\omega(F \cap B).$$

Theorem 1.4. *For $1 \leqslant i \leqslant N$, the stationary Markov measure μ_i is equal to $a_i^{-1}\mathcal{H}^s_\omega|_{E_i}$, where $a_i = \mathcal{H}^s_\omega(E_i)$. Consequently, μ_i are translation invariant.*

Proof. For any $I \in \Gamma^k_{ij}$, $\mu_i(E_I) = \frac{c_j}{c_i}\lambda^{-k}$ by Lemma 2.4, $\mathcal{H}^s_\omega(E_I) = a_j\lambda^{-k}$ by the scaling property of $\mathcal{H}^s_\omega$. Since both $(c_1, \dots, c_N)$ and $(a_1, \dots, a_N)$ are Perron–Frobenius eigenvectors of M, we have

$$(c_1, \dots, c_N) = \kappa(a_1, \dots, a_N)$$

for some $\kappa > 0$. Hence $\mu_i(E_I) = a_i^{-1}\mathcal{H}^s_\omega(E_I)$.

To show $\mu_i = a_i^{-1}\mathcal{H}^s_\omega|_{E_i}$, it suffices to show that

$$\mu_i(G \cap E_i) = a_i^{-1}\mathcal{H}^s_\omega(G \cap E_i) \tag{5.2}$$

for any open set $G \subset \mathbb{R}^d$. For $I = e_1 \cdots e_k$, let us denote by $I^* := e_1 \cdots e_{k-1}$ the ancestor of I. Clearly $G \cap E_i$ can be written as

$$G \cap E_i = \bigcup \big\{E_I;\ I \in \Sigma^*_i \text{ with } E_I \subset G \text{ and } E_{I^*} \not\subset G\big\}.$$

Moreover, if E_I and E_J belong to the union above, then I and J are incomparable. Thus, by Theorem 1.2(iii), we have

$$\mathcal{H}^s_\omega(G \cap E_i) = \sum \big\{\mathcal{H}^s_\omega(E_I);\ I \in \Sigma^*_i,\ E_I \subset G,\ E_{I^*} \not\subset G\big\};$$

on the other hand, by Proposition 2.3,

$$\mu_i(G \cap E_i) = \sum \big\{\mu_i(E_I);\ I \in \Sigma^*_i,\ E_I \subset G,\ E_{I^*} \not\subset G\big\}.$$

Since $\mu_i(E_I) = a_i^{-1}\mathcal{H}^s_\omega(E_I)$ holds for arbitrary $I \in \Sigma^*_i$, (5.2) follows from the above two equations.

μ_i is translation invariant since $\mathcal{H}^s_\omega|_{E_i}$ is translation invariant. $\square$

6. Two remarks on SOSC

In Section 4, we show that OSC implies SOSC for single-matrix systems. The following examples show that OSC does not imply SOSC in general.

Let K be the unit square $[0, 1]^2$. Let A, B be two 2×2 non-singular matrices such that

$$AK \subset K, \qquad BK \subset K, \qquad AK \cap BK = \{0\}.$$

(See Fig. 1(a).) Then the IFS $\{f_1(x) = Ax, f_2(x) = Bx\}$ satisfies OSC but does not satisfy SOSC. Clearly the interior of K is an open set for the OSC. Since the invariant set $E = \{0\}$ is a single point, if U is a strong open set, then U contains a neighborhood of 0; consequently $AU \cap BU \neq \emptyset$.

The above example seems to be the only known example for OSC without SOSC. It is trivial in the sense its invariant set is a singleton. In the following, we give a second example with this property.

6.1. An IFS satisfying OSC but SOSC fails. We consider the IFS $\{f_1, f_2\}$ on $\mathbb{R}^2$, where

$$f_1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \qquad f_2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

Let U be the interior of the triangle with vertices $(0, 0), (\frac{1}{2}, 1), (1, 0)$, then $f_1(U)$ and $f_2(U)$ are indicated by Fig. 1(b). Clearly the IFS satisfies OSC and U is an open set. The invariant set is the segment $E = \{(x, 0);\ 0 \leqslant x \leqslant 1\}$.

Suppose V is a strong open set for the IFS. Then there is a $\bar{x} \in [0, 1]$ such that $B((\bar{x}, 0), r) \subset V$. So we can find $x_0 \in (0, 1)$, $y_0 < 0$ such that the segment from $G_0 = (x_0, 0)$ to $H_0 = (x_0, y_0)$ is contained in V. Consequently $f_I(\overline{G_0H_0}) \subset V$ for any finite word I over $\{1, 2\}$.

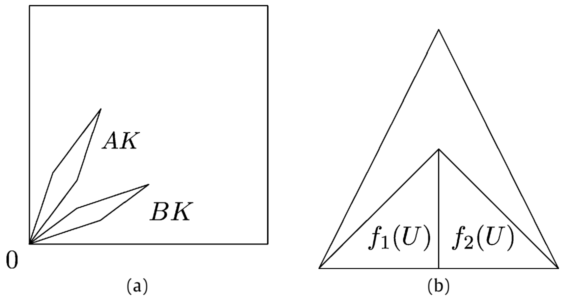

Fig. 1.

Let us denote $H_n = f_1^n(H_0) = (x_n, y_n)$ and define $s_n = x_n/y_n$. Then

$$s_{n+1} = \frac{x_{n+1}}{y_{n+1}} = \frac{\frac{1}{2}x_n + \frac{1}{4}y_n}{\frac{1}{2}y_n} = s_n + \frac{1}{2}.$$

Hence sooner or later, s_n will be positive and then tends to $+\infty$. Hence we can choose n large so that $\frac{1}{2}x_n - \frac{1}{4}y_n < 0$.

Let us consider the set $f_2 f_1^n(\overline{G_0 H_0})$, which is a line segment from $f_2 f_1^n(G_0)$ to $f_2 f_1^n(H_0)$. Since

$$f_2 f_1^n(G_0) = \left(\frac{x_0}{2^{n+1}} + \frac{1}{2}, 0\right), \qquad f_2 f_1^n(H_0) = \left(\frac{x_n}{2} - \frac{y_n}{4} + \frac{1}{2}, \frac{y_0}{2^{n+1}}\right),$$

clearly $f_2 f_1^n(G_0)$ is a point on the x-axis and on the right side of $L = \{(x, y);\ x = 1/2\}$, and $f_2 f_1^n(H_0)$ is on the left side of L.

A similar argument shows that $f_1 f_2^n(G_0)$ is on the x-axis and on the left side of L, $f_1 f_2^n(H_0)$ is on the right side of L and with second coordinate $y_0/2^{n+1}$.

Therefore, $f_2 f_1^n(\overline{G_0 H_0}) \cap f_1 f_2^n(\overline{G_0 H_0}) \neq \emptyset$. It follows that $f_2 f_1^n(V) \cap f_1 f_2^n(V) \neq \emptyset$, which is a contradiction.

Remark 6.1. The disadvantage of our example is that the invariant set is contained in a subspace of $\mathbb{R}^2$, so it is still degenerated in this sense. It is interesting to find a non-degenerated example.

6.2. A method to construct strong open set. Let $\{f_j\}_{1\leqslant j\leqslant N}$ be an IFS on $\mathbb{R}^d$, namely, $f_i : \mathbb{R}^d \mapsto \mathbb{R}^d$ are contractive and injective.

Clearly f_i are continues. We show that f_i are open mappings. Let $U \subset \mathbb{R}^d$ be a bounded open set. Choose $K = \overline{B}(0, R)$ be a closed ball such that $\overline{U} \subset K^\circ$. Since f_i is contractive and it is injection, the compactness implies that f_i is a homeomorphism form K to $f_i(K)$. So $f_i(U)$ is a relative open set in $f_i(K)$, and thus an open set of $\mathbb{R}^d$.

If U is an open set satisfying the SOSC, we will say that U is a *strong open set* in short.

Proposition 6.2. *Let $\{f_j\}_{1\leqslant j\leqslant N}$ be an IFS satisfying OSC with open set U, then $V = (\overline{U})^\circ$, the interior of the closure of U, is still an open set satisfying OSC.*

For example, let C be the Middle-third Cantor set, then $U = [0, 1] \setminus C$ is an open set for OSC but it is not a strong open set. But $V = (\overline{U})^\circ = (0, 1)$ is clearly a strong open set for OSC.

Proof. First we show that $f_i(V) \subset V$ for $1 \leqslant i \leqslant N$. Since $V \subset \overline{U}$, we have that $f_i(V) \subset f_i(\overline{U}) \subset \overline{f_i(U)}$. But $f_i(V)$ is an open set, so we have $f_i(V) \subset (\overline{f_i(U)})^\circ \subset (\overline{U})^\circ = V$.

Secondly, we show that $f_i(V) \cap f_j(V) = \emptyset$ for $i \neq j$. Let A, B be two disjoint open sets, then $\overline{A} \cap B = \emptyset$, and hence $(\overline{A})^\circ \cap B = \emptyset$. Therefore $(\overline{A})^\circ \cap \overline{B} = \emptyset$ and finally $(\overline{A})^\circ \cap (\overline{B})^\circ = \emptyset$.

Since we have proved that $f_i(V) \subset (\overline{f_i(U)})^\circ$, from $f_i(U) \cap f_j(U) = \emptyset$, we deduce that $f_i(V) \cap f_j(V) = \emptyset$. □

Acknowledgments

The authors thank the referee for many useful suggestions and comments.

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