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ON THE CLASSIFICATION OF FRACTAL SQUARES

JUN JASON LUO*
College of Mathematics and Statistics
Chongqing University, Chongqing 401331, P. R. China
jun.luo@cqu.edu.cn

JING-CHENG LIU
Key Laboratory of High Performance Computing and Stochastic
Information Processing (Ministry of Education of China)
College of Mathematics and Computer Science, Hunan Normal University
Changsha 410081, Hunan, P. R. China
liujingcheng11@126.com

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Abstract

In the previous paper [K. S. Lau, J. J. Luo and H. Rao, Topological structure of fractal squares, *Math. Proc. Camb. Phil. Soc.* **155** (2013) 73–86], Lau, Luo and Rao completely classified the topological structure of so called *fractal square* F defined by $F = (F + \mathcal{D})/n$, where $\mathcal{D} \subsetneq \{0, 1, \ldots, n-1\}^2, n \geq 2$. In this paper, we further provide simple criteria for the F to be totally disconnected, then we discuss the Lipschitz classification of F in the case $n = 3$, which is an attempt to consider non-totally disconnected sets.

Keywords: Fractal Square; Totally Disconnected; Congruence; Lipschitz Equivalence.

*Corresponding author.

1. INTRODUCTION

For $n \geq 2$, let $\mathcal{D} = \{d_1, \ldots, d_m\} \subsetneq \{0, 1, \ldots, n-1\}^2$ be a digit set with cardinality $\#\mathcal{D} = m$, and let $\{S_i\}_{i=1}^m$ be an iterated function system (IFS) on $\mathbb{R}^2$, where $S_i(x) = \frac{1}{n}(x + d_i)$ where $d_i \in \mathcal{D}$. Then there exists a unique self-similar set $F \subset \mathbb{R}^2$ satisfying the set equation[1]:

$$F = \bigcup_{i=1}^m S_i(F) = \frac{1}{n}(F + \mathcal{D}) \tag{1.1}$$

which is called a *fractal square*.[2] The geometric construction of a fractal square seems like that of middle third Cantor set: First we divide a unit square into n^2 small equal squares of which m small squares are kept and the rest discarded, the positions of the m chosen squares depend on $\mathcal{D}$; Secondly, repeat the first step on every chosen square and continue in this way, we then obtain a fractal square by taking limits.

Lau *et al.* gave a detailed study on the topological structure of F, they completely classified the topology of F by three types: (i) F is totally disconnected; (ii) F contains a non-trivial component which is not a line segment; and (iii) All non-trivial components of F are parallel line segments.[2]

Let $\mathcal{F}_{n,m}$ denote the collection of all fractal squares satisfying (1.1). It is easy to see that the fractal squares in $\mathcal{F}_{n,m}$ have the common Hausdorff dimension ($\log m/\log n$) but distinct topological structures. In the above three types, the fractal squares of type (i) are called Cantor-type sets which play an important role in fractal geometry and dynamical systems, so we will give a further study on this case. Especially, we provide simple criteria for the existence of type (i) in $\mathcal{F}_{n,m}$.

Two sets E and F on $\mathbb{R}^d$ are said to be *Lipschitz equivalent*, and denoted by $E \simeq F$, if there is a bi-Lipschitz map g from E onto F, i.e. g is a bijection and there is a constant $C > 0$ such that

$$C^{-1}|x - y| \leq |g(x) - g(y)| \leq C|x - y|, \quad \forall\ x, y \in E.$$

It is well-known that if $E \simeq F$ then they have the same Hausdorff dimension, but the converse is not true in general. Lipschitz classification of sets has attracted a lot of interests in the literature. In fractal geometry, the fundamental works were due to Cooper and Pignataro[3] and Falconer and Marsh[4] on Cantor sets. Recently, many generalizations on totally disconnected self-similar sets (Cantor-type sets) have been extensively studied.[5–13] But there are few results on non-totally disconnected cases.[14] Motivated by that, our aim of the paper is to make an attempt in this direction.

For $\mathcal{F}_{n,m}$, the Lipschitz equivalence class is denoted by $\mathcal{F}_{n,m}/\simeq$. When $n = 3, m = 2, 3, 4, 5$, we have

Theorem 1.1. $\#(\mathcal{F}_{3,2}/\simeq) = 1$; $\#(\mathcal{F}_{3,3}/\simeq) = \#(\mathcal{F}_{3,4}/\simeq) = 2$; *and* $\#(\mathcal{F}_{3,5}/\simeq) \leq 10$.

The first three classes are simple, while $\mathcal{F}_{3,5}$ is complicated, as it contains all the three types of fractal squares. The complete classification seems very difficult, but we conjecture that $\#(\mathcal{F}_{3,5}/\simeq) = 10$ (see remarks in Sec. 4).

The paper is organized as follows: In Sec. 2, we discuss several criteria for a fractal square to be totally disconnected. We prove Theorem 1.1 by using various methods (see Theorems 3.3, 3.4, 3.6, and 3.10) in Sec. 3, and give some remarks on other cases in Sec. 4. Finally, we include all figures of fractal squares in $\mathcal{F}_{3,5}, \mathcal{F}_{3,6}, \mathcal{F}_{3,7}$ and $\mathcal{F}_{3,8}$ in an appendix.

2. CRITERIA FOR TOTAL DISCONNECTEDNESS

For fractal square F as in (1.1), we define a set on $\mathcal{D}$ by

$$\mathcal{E} = \{(d_i, d_j) : (F + d_i) \cap (F + d_j) \neq \emptyset, \quad d_i, d_j \in \mathcal{D}\}.$$

We say that d_i, d_j are *$\mathcal{E}$-connected* if there exists a finite sequence $\{d_{j_1}, \ldots, d_{j_k}\} \subset \mathcal{D}$ such that $d_i = d_{j_1}, d_j = d_{j_k}$ and $(d_{j_l}, d_{j_{l+1}}) \in \mathcal{E}, 1 \leq l \leq k-1$. The following criterion for connectedness was first proved by Hata[15] and rediscovered by Kirat and Lau.[16]

Lemma 2.1. *A fractal square F with a digit set $\mathcal{D}$ is connected if and only if any two $d_i, d_j \in \mathcal{D}$ are $\mathcal{E}$-connected.*

Let $B = [0, 1]^2$ be the unit square, $\Sigma = \{1, \ldots, m\}$. Let $\mathcal{D}_1 = \mathcal{D}$ and $\mathcal{D}_{k+1} = \mathcal{D} + n\mathcal{D}_k$, then

$$\mathcal{D}_k = \{d_{\mathbf{u}} := d_{j_k} + nd_{j_{k-1}} + \cdots + n^{k-1}d_{j_1} : \mathbf{u} = j_1 \cdots j_k \in \Sigma^k\}, \quad k \geq 1. \tag{2.1}$$

Denote by $S_{\mathbf{u}}(B) = S_{j_1} \circ \cdots \circ S_{j_k}(B) = n^{-k}(B + d_{\mathbf{u}})$, we call such $S_{\mathbf{u}}(B)$ (or any translation of n^{-k}

scaling of B) a *k-square*. Obviously, we have

$$F = \bigcap_{k=1}^{\infty} \bigcup_{\mathbf{u}\in\Sigma^k} S_{\mathbf{u}}(B). \tag{2.2}$$

By letting $F^{(k)} = \bigcup_{\mathbf{u}\in\Sigma^k} S_{\mathbf{u}}(B)$, we call $F^{(k)}$ a *kth approximation* of the fractal square F.

Definition 2.2. In B, a vertical path is a curve starting at point $(x,0)$ and ending at point $(x,1)$ for some $x \in [0,1]$; a horizontal path is a curve starting at point $(0,y)$ and ending at point $(1,y)$ for some $y \in [0,1]$; a cross path is the union of one vertical path and one horizontal path; a λ-path is the union $\gamma_1 \cup \gamma_2 \cup \gamma_3$ where γ_i are three arcs connecting an interior point of B and three corners of B, respectively. (see Fig. 1.)

Obviously, a vertical path and a horizontal path meet each other, so a cross path is connected and reaches four points of the four sides of B, respectively. A λ-path is also connected. Intuitively, the shape of the λ-path looks like the letter "λ" or its rotations. The simplest λ-path may be the union of a diagonal and half of the other in B. From (2.2), it can be seen that $B\backslash F$ contains a vertical path if and only if there exist an integer $k \geq 1$ and a chain of edge-adjacent k-squares outside $F^{(k)}$ which begins with $[\frac{j}{n^k}, \frac{j+1}{n^k}] \times [0, \frac{1}{n^k}]$ and ends with $[\frac{j}{n^k}, \frac{j+1}{n^k}] \times [1-\frac{1}{n^k}, 1]$ for some $j \in \{0, 1, \ldots, n^k - 1\}$. Similarly for the cross path and the λ-path. (see Fig. 2.)

The main use of the above four paths is to verify the total disconnectedness of F.

Proposition 2.3 (Ref. 11). *A fractal square F is totally disconnected if and only if $B\backslash F$ has a cross path.*

The following criterion is more convenient for many cases in our consideration. Note that F contains a vertical (horizontal) line segment if and only if $F^{(1)}$ does.

Theorem 2.4. *A fractal square F is totally disconnected if and only if F contains no vertical line segments and $B\backslash F$ contains a vertical path.*

Proof. If F is totally disconnected, then the necessity is obvious since $B\backslash F$ is open and pathwise connected. For the converse part, let C be a component of F, and $\mathrm{Proj}_x C$ denote the orthogonal

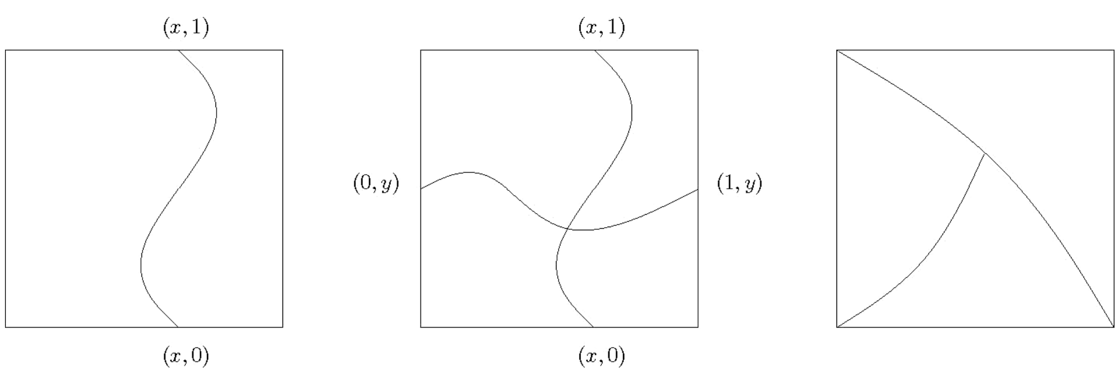


Fig. 1 From left to right: A vertical path, a cross path and a λ-path.

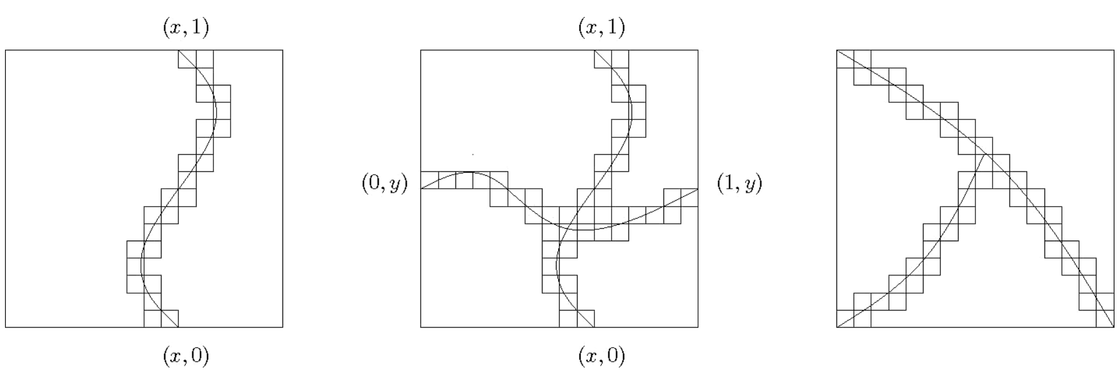


Fig. 2 Paths covered by squares.

projection of C on the x-axis, then $\mathrm{Proj}_x C$ is also pathwise connected. We claim $|\mathrm{Proj}_x C| = 0$. Indeed, if otherwise, $|\mathrm{Proj}_x C| > 0$. Choose an integer k large enough such that

$$C \cap \left[\frac{i}{n^k}, \frac{i+1}{n^k}\right] \times [0,1] \neq \emptyset,$$

$$C \cap \left[\frac{i+1}{n^k}, \frac{i+2}{n^k}\right] \times [0,1] \neq \emptyset,$$

$$C \cap \left[\frac{i+2}{n^k}, \frac{i+3}{n^k}\right] \times [0,1] \neq \emptyset$$

hold for some $i \in \{0, 1, \ldots, n^k - 3\}$. Let $I_j = [\frac{i+1}{n^k}, \frac{i+2}{n^k}] \times [\frac{j}{n^k}, \frac{j+1}{n^k}], j = 0, 1, \ldots, n^k - 1$ be the k-squares in the rectangle $[\frac{i+1}{n^k}, \frac{i+2}{n^k}] \times [0,1]$. Suppose α is a vertical path of $B \backslash F$. If I_j belongs to the kth approximation of F, we denote it by $S_{\mathbf{u}}(B)$ for some $\mathbf{u} \in \Sigma^k$. Then $S_{\mathbf{u}}(B \backslash F)$ contains a path $S_{\mathbf{u}}(\alpha) := \alpha_j$; if not, then $I_j \subset B \backslash F$. We can take a vertical line β_j in I_j with the same horizontal coordinate as the α_j. Hence we construct a vertical path in $B \backslash F$ by joining the paths α_j, β_j, which separate the component C. Thus, C must lie in one vertical line. By the assumption, C cannot be a vertical line segment, which implies C is just a singleton. Therefore, F is totally disconnected. □

Proposition 2.5. *Let F be a fractal square. If $B \backslash F$ contains a λ-path, then F is totally disconnected. Conversely, if F is totally disconnected and at most one corner of B is in F, then there exists a λ-path in $B \backslash F$.*

Proof. The proof is essentially the same as above. We mention that if F is totally disconnected and at most one corner of B is in F, then $B \backslash F$ contains at least three corners of B. Hence we can construct a λ-path in $B \backslash F$ by using the pathwise connectedness of $B \backslash F$. □

Theorem 2.6. *If $m \leq n^2 - n - [\frac{n}{2}]$ then $\mathcal{F}_{n,m}$ contains a totally disconnected fractal square.*

Proof. Let $\mathcal{D}_1 = \{(i,i) : i = 0, 1, \ldots, [\frac{n}{2}]\} \cup \{(j, n-j-1) : j = 0, 1, \ldots, n-1\}$, and a digit set $\mathcal{D} = \{0, 1, \ldots, n-1\}^2 \backslash \mathcal{D}_1$. Then $\#\mathcal{D} = n^2 - n - [\frac{n}{2}] := m$, and $F = \frac{1}{n}(F + \mathcal{D})$ belongs to $\mathcal{F}_{n,m}$. Since the set $\mathcal{D}_1$ determines a λ-path in $B \backslash F^{(2)}$, so in $B \backslash F$, it implies that F is totally disconnected by Proposition 2.5. □

3. CLASSIFICATION OF FRACTAL SQUARES WHEN $n = 3$

Lemma 3.1 (Refs. 6 and 12). *Let $F, F' \in \mathcal{F}_{n,m}$ be two fractal squares. If F, F' are totally disconnected then $F \simeq F'$.*

However, if two fractal squares are not totally disconnected, there are few results about their Lipschitz equivalence. In this section, we make an attempt on some special cases, such as connected fractal squares or fractal squares containing parallel line segments. We try to classify the Lipschitz equivalence classes of $\mathcal{F}_{n,m}$ for $n = 3, m = 2, 3, 4, 5$. For convenience, we use an $n \times n$ matrix $M = (m_{ij})_{1 \leq i,j \leq n}$ to represent a fractal square F where

$$m_{ij} = \begin{cases} 1 & \text{if } (j-1, n-i) \in \mathcal{D} \\ 0 & \text{otherwise.} \end{cases}$$

We call M the *label matrix* of F. It is easy to see that there is a one-to-one correspondence between F and M. So we prefer to use the label matrix to depict the fractal square for simplicity.

Geometrically, two sets are called *congruent* if one can be transformed into the other by some rigid motions. From (2.2), it is seen that two fractal squares are congruent if their first approximations are congruent, which can be immediately observed from the label matrices.

Lemma 3.2. *Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be a linear transformation defined by $g(x) = Ax + v$ where A is a $d \times d$ invertible matrix and $v \in \mathbb{R}^d$. Then g is a bi-Lipschitz map.*

Proof. Since A is invertible, for any $x, y \in \mathbb{R}^d$, we have

$$|A(x-y)| \leq \|A\| |x-y|$$

and

$$|x-y| = |A^{-1}A(x-y)| \leq \|A^{-1}\| |A(x-y)|,$$

where $\|A\|$ denotes the norm of matrix A. Hence

$$\|A^{-1}\|^{-1} |x-y| \leq |g(x) - g(y)| \leq \|A\| |x-y|$$

proving that g is a bi-Lipschitz map. □

Theorem 3.3. $\#(\mathcal{F}_{3,2}/\simeq) = 1$; $\#(\mathcal{F}_{3,3}/\simeq) = \#(\mathcal{F}_{3,4}/\simeq) = 2$.

Proof. Since $\log 2 / \log 3 < 1$, all the fractal squares in $\mathcal{F}_{3,2}$ are totally disconnected.[1] Hence $\#(\mathcal{F}_{3,2}/\simeq) = 1$ by Lemma 3.1.

In $\mathcal{F}_{3,3}$, every fractal square is either totally disconnected or connected (a line segment). Hence $\#(\mathcal{F}_{3,3}/\simeq) = 2$. In $\mathcal{F}_{3,4}$, the totally disconnected fractal squares form one Lipschitz equivalence class by Lemma 3.1. Moreover, it can be easily checked that, up to congruence, there are only 6 different non-totally disconnected fractal squares, denoted by $F_i = \frac{1}{3}(F_i + \mathcal{D}_i)$ where $i = 1, \ldots, 6$. The corresponding label matrices are listed as follows:

$$\begin{bmatrix}1&0&0\\0&0&0\\1&1&1\end{bmatrix}, \quad \begin{bmatrix}0&1&0\\0&0&0\\1&1&1\end{bmatrix}, \quad \begin{bmatrix}0&0&0\\1&0&0\\1&1&1\end{bmatrix},$$
$$\begin{bmatrix}0&0&0\\0&1&0\\1&1&1\end{bmatrix}, \quad \begin{bmatrix}0&0&1\\0&1&1\\1&0&0\end{bmatrix}, \quad \begin{bmatrix}0&0&1\\0&1&0\\1&0&1\end{bmatrix}.$$

We define a linear transformation $g : \mathbb{R}^2 \to \mathbb{R}^2$ by $g(x) = Ax$ where $A = \left[\begin{smallmatrix}1&1/2\\0&1\end{smallmatrix}\right]$. Then $\mathcal{D}_2 = A\mathcal{D}_1$. Hence $AF_1 = \frac{1}{3}(AF_1 + A\mathcal{D}_1) = \frac{1}{3}(AF_1 + \mathcal{D}_2)$, implying $F_2 = g(F_1)$ by the uniqueness of attractor. So we get $F_1 \simeq F_2$ as g is a bi-Lipschitz map by Lemma 3.2.

Similarly, it is easy to verify that $\mathcal{D}_3 = A_1\mathcal{D}_1$, $\mathcal{D}_4 = A_1\mathcal{D}_2$, $\mathcal{D}_6 = A_2\mathcal{D}_4$ and $\mathcal{D}_5 = A_3\mathcal{D}_4$, where

$$A_1 = \begin{bmatrix}1&0\\0&1/2\end{bmatrix}, \quad A_2 = \begin{bmatrix}1&1\\1&-1\end{bmatrix},$$
$$A_3 = \begin{bmatrix}1&1\\1&0\end{bmatrix}.$$

Therefore, $F_1 \simeq F_2 \simeq \cdots \simeq F_6$, proving $\#(\mathcal{F}_{3,4}/\simeq) = 2$. $\square$

In $\mathcal{F}_{3,5}$, the total number of fractal squares is C_9^5. Up to congruence, there are 21 distinct fractal squares among them. However, in the rest of this section, we will show that there are at most 10 Lipschitz equivalence classes. First we use $\{F_i\}_{i=1}^{21}$ to denote 21 fractal squares, and each F_i takes the following M_i as its label matrix:

Type (i): totally disconnected fractal squares:

$$M_1 = \begin{bmatrix}0&1&0\\1&0&1\\1&0&1\end{bmatrix}, \quad M_2 = \begin{bmatrix}1&0&0\\0&1&1\\1&1&0\end{bmatrix},$$
$$M_3 = \begin{bmatrix}1&1&0\\0&0&1\\1&0&1\end{bmatrix}, \quad M_4 = \begin{bmatrix}1&1&0\\0&0&1\\0&1&1\end{bmatrix},$$
$$M_5 = \begin{bmatrix}1&1&0\\1&0&1\\0&1&0\end{bmatrix}.$$

Type (ii): connected fractal squares:

$$M_6 = \begin{bmatrix}0&1&0\\0&1&0\\1&1&1\end{bmatrix}, \quad M_7 = \begin{bmatrix}1&0&0\\1&0&0\\1&1&1\end{bmatrix},$$
$$M_8 = \begin{bmatrix}0&0&1\\0&1&0\\1&1&1\end{bmatrix}, \quad M_9 = \begin{bmatrix}1&0&1\\0&1&0\\1&0&1\end{bmatrix},$$
$$M_{10} = \begin{bmatrix}0&1&0\\1&1&1\\0&1&0\end{bmatrix}, \quad M_{11} = \begin{bmatrix}0&1&1\\0&1&0\\1&1&0\end{bmatrix}.$$

Type (iii): fractal squares containing parallel line segments:

$$M_{12} = \begin{bmatrix}1&0&1\\0&0&0\\1&1&1\end{bmatrix}, \quad M_{13} = \begin{bmatrix}0&0&0\\1&0&1\\1&1&1\end{bmatrix},$$
$$M_{14} = \begin{bmatrix}0&0&0\\1&1&0\\1&1&1\end{bmatrix}, \quad M_{15} = \begin{bmatrix}1&1&0\\0&0&0\\1&1&1\end{bmatrix},$$
$$M_{16} = \begin{bmatrix}0&0&1\\0&1&1\\1&1&0\end{bmatrix}, \quad M_{17} = \begin{bmatrix}0&1&0\\0&0&1\\1&1&1\end{bmatrix},$$
$$M_{18} = \begin{bmatrix}1&0&0\\0&0&1\\1&1&1\end{bmatrix}, \quad M_{19} = \begin{bmatrix}0&0&1\\1&1&0\\1&1&0\end{bmatrix},$$
$$M_{20} = \begin{bmatrix}0&1&0\\0&1&1\\1&1&0\end{bmatrix}, \quad M_{21} = \begin{bmatrix}1&0&1\\0&1&0\\1&1&0\end{bmatrix}.$$

By the criteria in the last section, especially Theorem 2.4, fractal squares of type (i) are indeed totally disconnected. Hence $F_i, 1 \le i \le 5$ are Lipschitz equivalent by Lemma 3.1. For types (ii) and (iii), we have

Theorem 3.4. *$F_7 \simeq F_8$; $F_9 \simeq F_{10} \simeq F_{11}$; $F_{12} \simeq F_{13}$; $F_{14} \simeq F_{15} \simeq F_{16}$; $F_{19} \simeq F_{20}$.*

Proof. Let $F_i = \frac{1}{3}(F_i + \mathcal{D}_i), i = 1, \ldots, 21$, and let

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then $\mathcal{D}_8 = A_1\mathcal{D}_7, \mathcal{D}_{11} = A_2\mathcal{D}_9, \mathcal{D}_{13} = A_3\mathcal{D}_{12}, A_3\mathcal{D}_{15} = \mathcal{D}_{14} = A_4^{-1}\mathcal{D}_{16}$. By defining linear transformations $g_i(x) = A_i x$ for $i = 1, 2, 3, 4$, we can obtain $g_1(F_7) = F_8, g_2(F_9) = F_{11}, g_3(F_{15}) = F_{14}$ and $g_4(F_{14}) = F_{16}$, where g_i are bi-Lipschitz maps.

Moreover, let $g_5(x) = A_5 x + v$, $g_6(x) = A_6 x + \frac{v'}{2}$ where $v = \frac{1}{2}\left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]$, $v' = \frac{1}{2}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. Then $\mathcal{D}_9 = A_5\mathcal{D}_{10} + 2v$ and $\mathcal{D}_{20} = A_6\mathcal{D}_{19} + v'$. Hence

$$\begin{aligned} g_5(F_{10}) &= g_5\left(\frac{1}{3}(F_{10} + \mathcal{D}_{10})\right) \\ &= \frac{1}{3}(A_5F_{10} + A_5\mathcal{D}_{10} + 3v) \\ &= \frac{1}{3}(g_5(F_{10}) + \mathcal{D}_9) \end{aligned}$$

and

$$\begin{aligned} g_6(F_{19}) &= g_6\left(\frac{1}{3}(F_{19} + \mathcal{D}_{19})\right) \\ &= \frac{1}{3}\left(A_6F_{19} + A_6\mathcal{D}_{19} + \frac{3}{2}v'\right) \\ &= \frac{1}{3}(g_6(F_{19}) + \mathcal{D}_{20}). \end{aligned}$$

That implies $g_5(F_{10}) = F_9$ and $g_6(F_{19}) = F_{20}$, finishing the proof. □

Lemma 3.5. *F_7 is a connected set which equals the closure of a union of infinitely countable circles, and so does F_8.*

Proof. The connectedness can be obtained easily by Lemma 2.1. Let $C = [0,1] \times \{0,1\} \cup \{0,1\} \times [0,1]$, then C is a circle, and $\frac{1}{3}C \subset F_7, \frac{1}{3}(\frac{1}{3}C + \mathcal{D}) \subset F_7$ (see Fig. 3). By induction, we can get for any $k \geq 1$,

$$\frac{C}{3^k} + \frac{\mathcal{D}_{k-1}}{3^{k-1}} = \frac{C}{3^k} + \frac{\mathcal{D}}{3^{k-1}} + \cdots + \frac{\mathcal{D}}{3} \subset F_7.$$

Hence

$$\overline{\bigcup_{k=1}^{\infty}\left(\frac{C}{3^k} + \frac{\mathcal{D}_{k-1}}{3^{k-1}}\right)} \subset F_7.$$

On the other hand, for any $x \in F_7$, there exists a sequence $\{d_{j_i}\}_i$ with $d_{j_i} \in \mathcal{D}$ such that $x = \sum_{i=1}^{\infty} 3^{-i} d_{j_i}$. By (2.1), for all $k \geq 1$, we have $\sum_{i=1}^{k} 3^{-i} d_{j_i} \in 3^{-k}\mathcal{D}_k \subset \bigcup_{k=1}^{\infty}(\frac{C}{3^k} + \frac{\mathcal{D}_{k-1}}{3^{k-1}})$, then $x \in \overline{\bigcup_{k=1}^{\infty}(\frac{C}{3^k} + \frac{\mathcal{D}_{k-1}}{3^{k-1}})}$. Hence

$$F_7 \subset \overline{\bigcup_{k=1}^{\infty}\left(\frac{C}{3^k} + \frac{\mathcal{D}_{k-1}}{3^{k-1}}\right)}.$$

We omit the proof for F_8 as it is the same as above. □

A nonempty compact set $T \subset \mathbb{R}^2$ is called a *tree-like set* if for any two distinct points $x, y \in T$, there is a unique path (or curve) in T connecting them.

Theorem 3.6. *F_6 and F_7 are not homeomorphic, hence are not Lipschitz equivalent.*

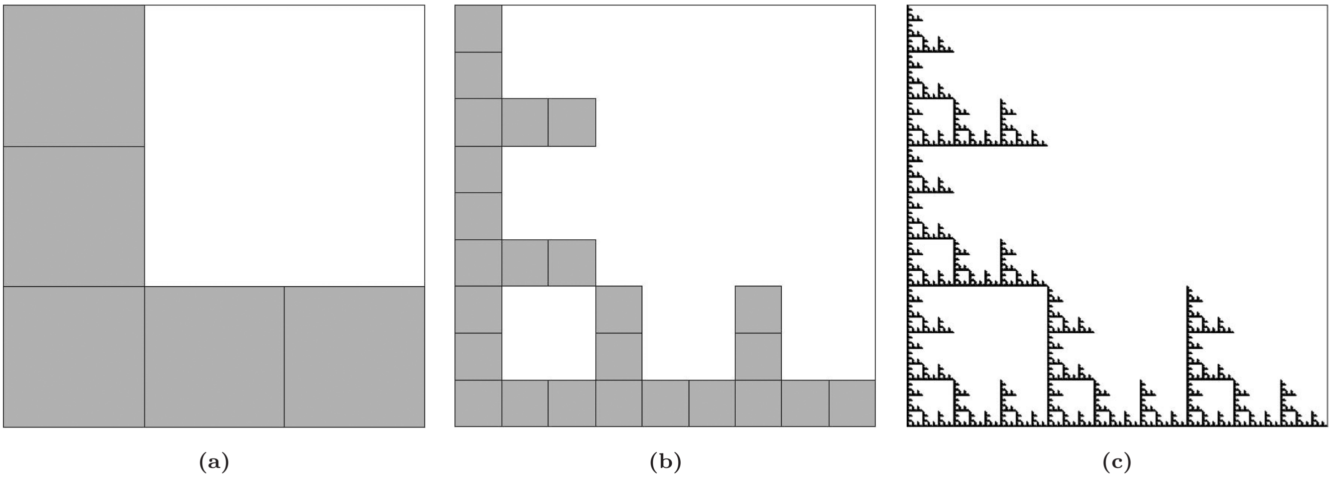


Fig. 3 Fractal square F_7.

Proof. Lemma 3.5 implies F_7 is not a tree-like set, so it suffices to show that F_6 is a tree-like set. By Lemma 2.1, F_6 is connected, hence is pathwise connected.[15] Thus, for any two distinct points $x, y \in F_6$, there is a path $\pi(x, y)$ in F_6 connecting them. Next we show F_6 is a tree-like set by proving the uniqueness of the path $\pi(x, y)$.

Following notation of (2.2), let $\mathbf{u}, \mathbf{v} \in \Sigma^k$. It is known that if $S_{\mathbf{u}}(B) \cap S_{\mathbf{v}}(B)$ is singleton then $S_{\mathbf{u}i}(B) \cap S_{\mathbf{v}j}(B) = \emptyset$ for any $i, j \in \Sigma$; if $S_{\mathbf{u}}(B) \cap S_{\mathbf{v}}(B)$ is a line segment, say L_k, then there exists a unique pair $(i, j) \in \Sigma \times \Sigma$ such that $S_{\mathbf{u}i}(B) \cap S_{\mathbf{v}j}(B)$ $(:= L_{k+1})$ is also a line segment with length

$$|L_{k+1}| = \frac{|L_k|}{3} = \frac{1}{3^{k+1}}$$

(see Fig. 4). Define

$$\mathcal{E}_k = \left\{(d_{\mathbf{u}}, d_{\mathbf{v}}) : |S_{\mathbf{u}}(B) \cap S_{\mathbf{v}}(B)| = \frac{1}{3^k},\right.$$
$$\left. d_{\mathbf{u}}, d_{\mathbf{v}} \in \mathcal{D}_k\right\}$$

to be the set of edges for $\mathcal{D}_k$. Then $(\mathcal{D}_k, \mathcal{E}_k)$ forms a tree by the argument above for any $k \geq 1$.

Assume $\pi'(x, y)$ is a path different from $\pi(x, y)$. Then there exists a point $z_0 \in \pi'(x, y) \backslash \{x, y\}$ such that

$$\epsilon_0 := \inf\{|z - z_0| : z \in \pi(x, y) \backslash \{x, y\}\} > 0.$$

Since $x, y \in F_6 \subset \bigcup_{\mathbf{u} \in \Sigma^k} S_{\mathbf{u}}(B)$ and $x \neq y$, there is a large enough $k_0 \geq \log_3 \frac{\sqrt{2}}{\epsilon_0} + 1$ such that, for any $k \geq k_0$, there exist $\mathbf{u}, \mathbf{v} \in \Sigma^k$ such that $x \in S_{\mathbf{u}}(B), y \in S_{\mathbf{v}}(B)$ and $S_{\mathbf{u}}(B) \cap S_{\mathbf{v}}(B) = \emptyset$. By the tree structure of $(\mathcal{D}_k, \mathcal{E}_k)$, we can find a unique finite sequence $\mathbf{u}_1, \ldots, \mathbf{u}_\ell$ satisfying $\mathbf{u} = \mathbf{u}_1$, $\mathbf{v} = \mathbf{u}_\ell$ and $(\mathbf{u}_i, \mathbf{u}_{i+1}) \in \mathcal{E}_k$ for $i = 1, \ldots, \ell - 1$. Thus $\pi(x, y), \pi'(x, y) \subset \bigcup_{i=1}^{\ell} S_{\mathbf{u}_i}(B)$. Suppose $z_0 \in S_{\mathbf{u}_{i_0}}(B)$, then let $z_1 \in \pi(x, y) \cap S_{\mathbf{u}_{i_0}}(B)$, we get

$$|z_0 - z_1| \leq \operatorname{diam}(S_{\mathbf{u}_{i_0}}(B)) = \frac{\sqrt{2}}{3^k} < \epsilon_0.$$

That contradicts $|z_0 - z_1| \geq \epsilon_0$. □

Let $E \subset \mathbb{R}^2$ be a nonempty connected set. We say a point $a \in E$ is a *k-branch point* if $E \backslash \{a\}$ consists of k components. It is known that k-branch points are topological invariants. A 1-branch point of E is often called a *top* of E. The following lemma is obvious.

Lemma 3.7. *Suppose that x is a k-branch point in $E \subset \mathbb{R}^2$. Then for any $U \subset \mathbb{R}^2, E \backslash U$ has at least k components provided that U contains x and the diameter U is small enough.*

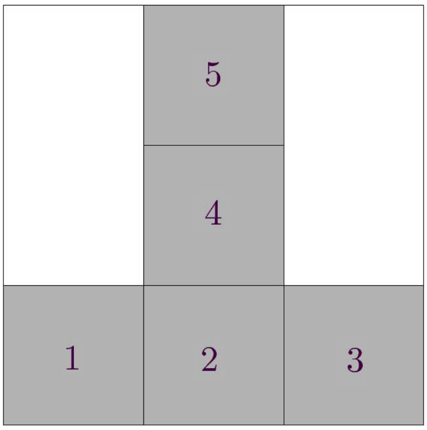
Proof. Let $E_1, E_2, \ldots, E_k$ be the components of $E \backslash \{x\}$. Let δ be the minimum of the diameters $\operatorname{diam}(E_j), 1 \leq j \leq k$. If $\operatorname{diam}(U) < \delta/2$, then $(E \backslash U) \cap E_j$ is not empty and contributes at least one component to $E \backslash U$. □

Let $F \in \mathcal{F}_{n,m}$ and $\Sigma = \{1, \ldots, m\}$. For any point $x \in F$, there exists an infinite word $i_1 i_2 \cdots$ such that

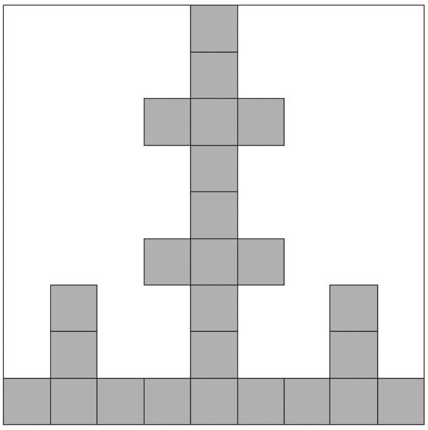
$$\{x\} = \bigcap_{k=1}^{\infty} S_{i_1 \cdots i_k}(F),$$

where $i_j \in \Sigma$ and $S_{i_1 \cdots i_k} = S_{i_1} \circ \cdots \circ S_{i_k}$. We call $i_1 i_2 \cdots$ a *coding* of x, and $(F)_{i_1 \cdots i_k} := S_{i_1 \cdots i_k}(F)$ a *cylinder* of F.

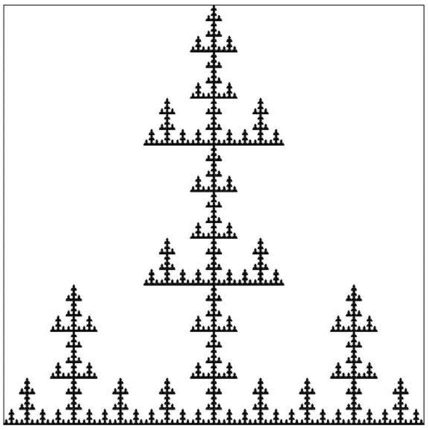
Lemma 3.8. *Let $\{S_i\}_{i=1}^5$ be the IFS of F_6, which is depicted by Fig. 4a. Suppose x belongs to F_6.*



(a)

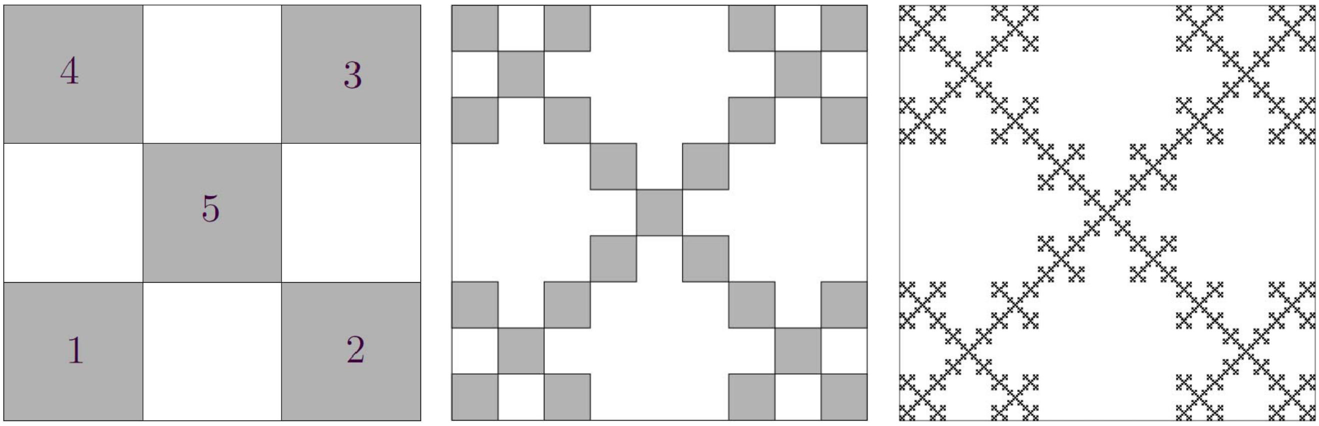


(b)



(c)

Fig. 4 Fractal square F_6.



(a) (b) (c)

Fig. 5 Fractal square F_9.

Then

(i) *If the coding of x is unique and contains finitely many symbols $2, 4$, then x is a 1-branch point.*
(ii) *Suppose the coding of x is unique and contains infinitely many symbols $2, 4$. If the coding is not eventually 2, then x is a 2-branch point; otherwise x is a 3-branch point.*
(iii) *If x has more than one coding, then x is either a 2-branch or a 4-branch point.*

Proof. (i) Clearly if the coding of x does not contain the symbols $2, 4$, then x is a 1-branch point, namely, x is a top of F_6 (see Fig. 4c). Indeed, we can show by induction that if we delete the cylinder $(F_6)_{i_1 \cdots i_k}$ from F_6, then the resulting set is still connected. Hence, x is a 1-branch point by Lemma 3.7.

Now suppose that $i_1 i_2 \cdots$ contains symbols $2, 4$, say i_k is the last symbol in $2, 4$ and i_j belongs to $\{1, 3, 5\}$ for all $j > k$. Then x is a top of the cylinder $(F_6)_{i_1 \cdots i_k}$. If x is not a top of F_6, then x must belong to another cylinder, which means x has more than one coding.

(ii) Suppose $i_1 i_2 \cdots$ contains infinitely many symbols $2, 4$, and it is not eventually 2. This means 4 will appear infinitely many times. Suppose $i_k = 4$. Let $U = (F_6)_{i_1 \cdots i_k} \backslash \{(F_6)_{i_1 \cdots i_k 5^\infty}\}$. Since $(F_6)_{i_1 \cdots i_{k-1}} \backslash U$ consists of two components and U does not intersect other cylinders of F_6, we conclude that $F_6 \backslash U$ has only two components. Therefore, x is a 2-branch point.

Now suppose that $i_1 i_2 \cdots$ is eventually 2. Suppose $i_k = 2$ for all $k \geq k_0$. Delete $(F_6)_{i_1 \cdots i_{k_0}}$ but keep the three intersecting points with other cylinders, the resulting set consists of three components. Hence, x is a 3-branch point.

(iii) Now suppose x has more than one coding. If x has no coding of eventually 2, then x must be the common top of two cylinders and it is a 2-branch point. If x has a coding of eventually 2, say $i_1 \cdots i_k 2^\infty$. If we delete x, then $(F_6)_{i_1 \cdots i_k}$ is partitioned into three pieces. The other part of F_6 either connects to the top of $(F_6)_{i_1 \cdots i_k}$ or connects to x. Hence, x is a 4-branch point. □

Lemma 3.9. *Let $\{S_i\}_{i=1}^5$ be the IFS of F_9, which is depicted by Fig. 5a. Suppose x belongs to F_9. Then*

(i) *If the coding of x is unique and contains finitely many symbols 5, then x is a 1-branch point.*
(ii) *If the coding of x is unique and contains infinitely many symbols 5, then x is a 4-branch point.*
(iii) *If x has more than one coding, then x is a 2-branch point.*

Proof. (i) Clearly if the coding of x does not contain the symbol 5, then x is a 1-branch point, namely, x is a top corner of F_9 (see Fig. 5c). Indeed, we can show by induction that if we delete the cylinder $(F_9)_{i_1 \cdots i_k}$ from F_9, then the resulting set is still connected. Hence x is a 1-branch point by Lemma 3.7.

Now suppose that $i_1 i_2 \cdots$ contains symbol 5, say i_k is the last 5 and $i_j \in \{1, 2, 3, 4\}$ for all $j > k$. Then x is a top of the cylinder $(F_9)_{i_1 \cdots i_k}$. If x is not a top of F_9, then x must belong to another cylinder, which means x has more than one coding.

(ii) If $i_1 i_2 \cdots$ contains infinitely many 5, suppose $i_k = 5$. Let $U = (F_9)_{i_1 \cdots i_k}$. Since $(F_9)_{i_1 \cdots i_{k-1}} \backslash U$ consists of four components and U does not intersect other cylinders of F_9, we conclude that $F_9 \backslash U$ has only four components. Therefore, x is a 4-branch point.

(iii) If x has more than one coding, then x must be the common top of two cylinders, and it is a 2-branch point. □

Theorem 3.10. *F_6 and F_9 are not homeomorphic, hence are not Lipschitz equivalent.*

Proof. By Lemmas 3.8 and 3.9, we know that F_6 contains 3-branch points, while F_9 contains no 3-branch points. Therefore, they are not homeomorphic. □

4. REMARKS

Because of irregularity, it is difficult to study the remaining F_{13}, F_{14}, F_{17}, F_{18}, F_{20}, F_{21} of $\mathcal{F}_{3,5}$. We conjecture that they are not Lipschitz equivalent at all, and $\#(\mathcal{F}_{3,5}/\simeq) = 10$.

For the cases of $\mathcal{F}_{3,6}, \mathcal{F}_{3,7}$ and $\mathcal{F}_{3,8}$, we summarize their topological classifications as follows: up to congruence, $\mathcal{F}_{3,6}$ only contains 16 fractal squares of which 6 are disconnected and 10 are connected; $\mathcal{F}_{3,7}$ only contains 8 connected fractal squares; and $\mathcal{F}_{3,8}$ only contains 3 connected fractal squares (please see their figures in the next Appendix section). Recently, Ruan and Wang[17] proved that $\#(\mathcal{F}_{3,7}/\simeq) = 8$ and $\#(\mathcal{F}_{3,8}/\simeq) = 3$ by making use of an old result called Whyburn's theorem. However, it is still hopeless to handle the other cases completely.

For the general $\mathcal{F}_{n,m}$, we can make a further discussion to get similar results as Sec. 3, but the process will become more complicated.

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REFERENCES

1. K. J. Falconer, *Fractal Geometry: Mathematical Foundation and Applications* (New York: John Wiley & Sons, 1990).
2. K. S. Lau, J. J. Luo and H. Rao, Topological structure of fractal squares, *Math. Proc. Camb. Philos. Soc.* **155** (2013) 73–86.
3. D. Cooper and T. Pignataro, On the shape of Cantor sets, *J. Diff. Geom.* **28** (1988) 203–221.
4. K. J. Falconer and D. T. Marsh, On the Lipschitz equivalence of Cantor sets, *Mathematika* **39** (1992) 223–233.
5. G. T. Deng, K. S. Lau and J. J. Luo, Lipschitz equivalence of self-similar sets and hyperbolic boundaries II, *J. Fractal Geom.* **2** (2015) 53–79.
6. J. J. Luo and K. S. Lau, Lipschitz equivalence of self-similar sets and hyperbolic boundaries, *Adv. Math.* **235** (2013) 555–579.
7. H. Rao, H. J. Ruan and L. F. Xi, Lipschitz equivalence of self-similar sets, *CR Acad. Sci. Paris, Ser. I* **342** (2006) 191–196.
8. H. Rao, H. J. Ruan and Y. Wang, Lipschitz equivalence of Cantor sets and algebraic properties of contraction ratios, *Trans. Am. Math. Soc.* **364** (2012) 1109–1126.
9. H. J. Ruan, Y. Wang and L. Xi, Lipschitz equivalence of self-similar sets with touching structures, *Nonlinearity* **27**(6) (2014) 1299–1321.
10. H. Rao and Y. Zhang, The higher dimensional Frobenius problem and Lipschitz equivalence of Cantor sets, *J. Math. Pure App.* **104** (2015) 868–881.
11. K. A. Roinestad, Geometry of fractal squares, Ph.D. Thesis, The Virginia Polytechnic Institute and State University (2010).
12. L. F. Xi and Y. Xiong, Self-similar sets with initial cubic patterns, *CR Acad. Sci. Paris, Ser. I* **348** (2010) 15–20.
13. L. F. Xi and Y. Xiong, Lipschitz equivalence class, ideal class and the Gauss class number problem, arXiv:1304.0103.
14. Z. Wen, Z. Zhu and G. Deng, Lipschitz equivalence of a class of general Sierpinski carpets, *J. Math. Anal. Appl.* **385** (2012) 16–23.
15. M. Hata, On the structure of self-similar sets, *Jpn. J. Appl. Math.* **2** (1985) 381–414.
16. I. Kirat and K. S Lau, On the connectedness of self-affine tiles, *J. London Math. Soc.* **62** (2000) 291–304.
17. H. J. Ruan and Y. Wang, Topological invariants and Lipschitz equivalence of fractal sets, preprint.

APPENDIX A. FIGURES OF FRACTAL SQUARES

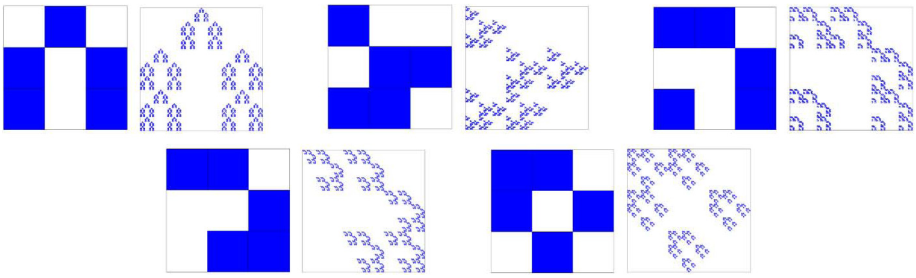

Fig. A.1 Five totally disconnected fractal squares in $\mathcal{F}_{3,5}$.

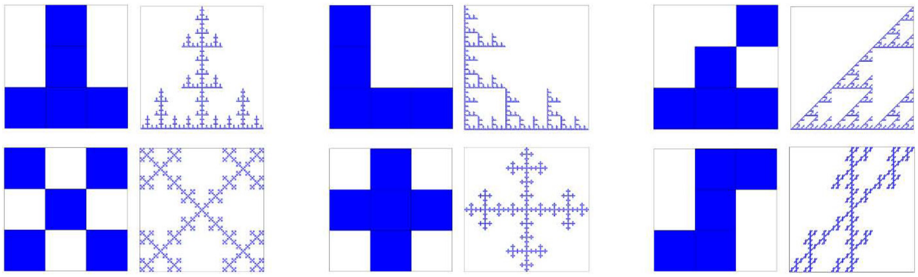

Fig. A.2 Six connected fractal squares in $\mathcal{F}_{3,5}$.

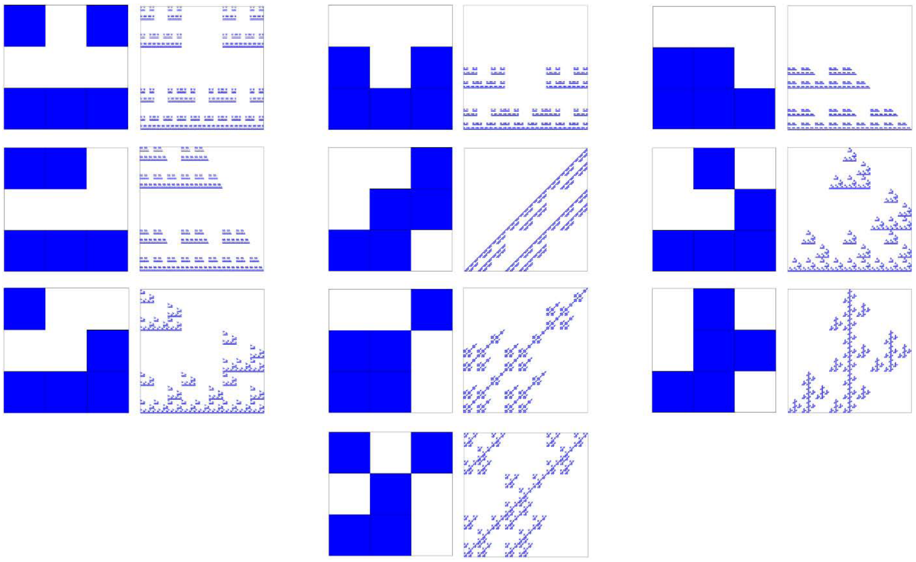

Fig. A.3 10 fractal squares containing parallel line segments in $\mathcal{F}_{3,5}$.

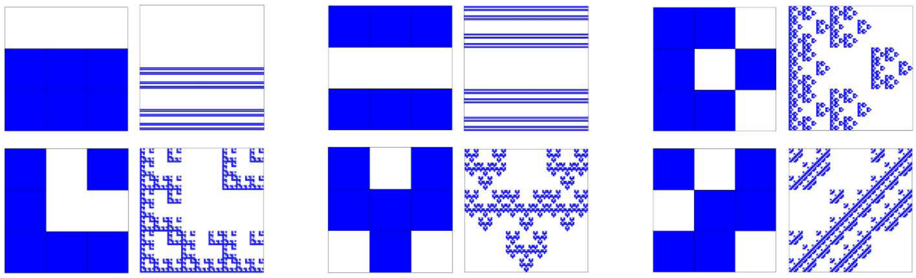

Fig. A.4 Six disconnected fractal squares in $\mathcal{F}_{3,6}$.

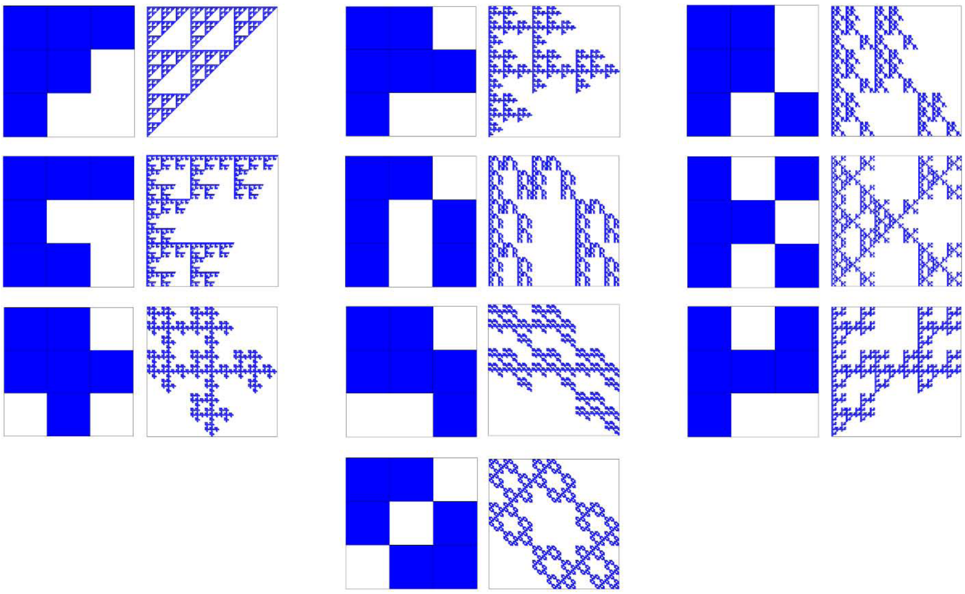

Fig. A.5 10 connected fractal squares in $\mathcal{F}_{3,6}$.

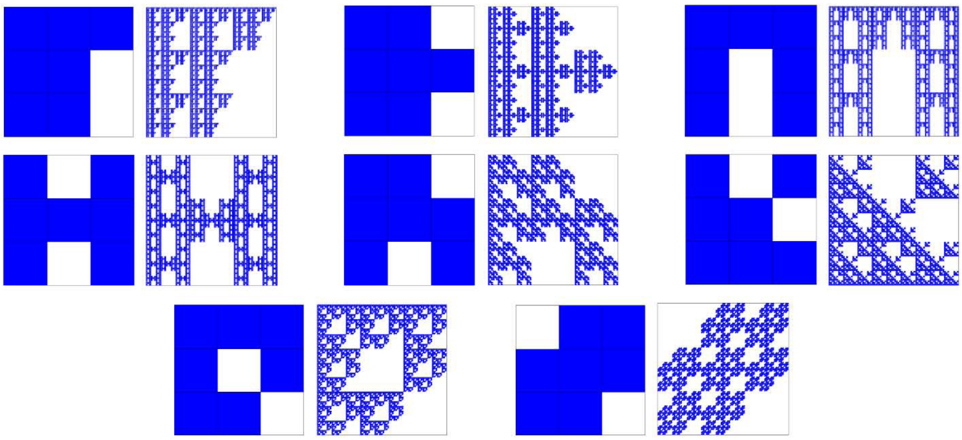

Fig. A.6 Eight fractal squares in $\mathcal{F}_{3,7}$.

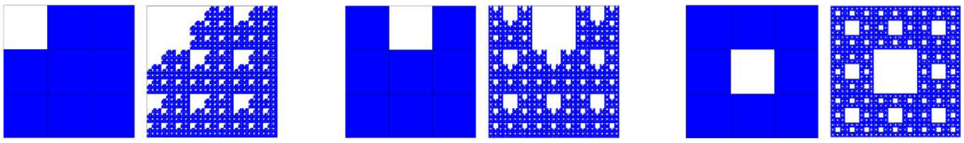

Fig. A.7 Three fractal squares in $\mathcal{F}_{3,8}$.