On the connectedness of planar self-affine sets

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\textbf{A B S T R A C T}

In this paper, we consider the connectedness of planar self-affine set $T(A, D)$ arising from an integral expanding matrix $A$ with characteristic polynomial $f(x) = x^2 + bx + c$ and a consecutive collinear digit set $D = \{0, 1, \ldots, m\}$. The necessary and sufficient conditions only depending on $b, c, m$ are given for the $T(A, D)$ to be connected. Moreover, we also consider the case that $D$ is non-consecutively collinear.

The $T$ is called the\textit{ self-affine set (or attractor)} of the iterated function system (IFS) $\{S_i\}_{i=1}^n$. We call $T$ a self-affine tile if it has positive Lebesgue measure and the union in 1.1 is essentially disjoint, i.e., the intersection $(T + d_i) \cap (T + d_j)$ has zero Lebesgue measure for $i \neq j$. In this situation, $T \neq \emptyset$ and $c := |\det(A)| = m$.

There have been a lot of interests on the fundamental properties of self-affine tiles on $\mathbb{R}^n$ in the literature (see e.g. [7, 18–20]). One of the very interesting aspects is the connectedness, in particular the disk-likeness (i.e., homeomorphic to a closed disk in the case $n = 2$). The connected self-affine tiles have important applications to wavelet theory and number systems (see e.g. [2, 4, 7] or the survey papers [1, 3]). Gröchenig and Haas [7] as well as Hacon et al. [8] first discussed a few special connected self-affine tiles. Subsequently Lau and his coworkers ([10, 12–14]) studied a large class of connected self-affine tiles generated by the consecutive collinear digit set $D = \{0, 1, \ldots, c - 1\}$, and their disk-likeness in the plane by introducing an algebraic approach. Akiyama and Thuswaldner [3] investigated the connectedness of families of self-affine tiles associated to quadratic number systems and results on their fundamental group. On the other hand, Bandt and Wang [5] and Leung and Luo [17] also concerned the

\footnotesize{$T = A^{-1}(T + D) = \left\{ \sum_{i=1}^{m} A^{-1} d_i : d_i \in D \right\}$.}
disk-like self-affine tiles or the boundary structure by using a technique of neighbor graphs.

Recently, on $\mathbb{R}^2$, Kirat [11] and Deng and Lau [6] found out the connected self-affine tiles $T(A, D)$ among classes of data $(A, D)$ with non-collinear digit sets $D$ and characterized the disk-like ones. Leung and Luo ([15,16]) were also interested in the collinear digit set $\{0, 1, m\} \nu$ and the non-collinear digit set $\{0, \nu, m\} \nu$ with the restriction of $|\det A| = 3$.

In this paper, we study more general self-affine sets $T(A, D)$ on $\mathbb{R}^d$ arising from an integral expanding matrix $A$ with characteristic polynomial $f(x) = x^2 + bx + c$ and the consecutive collinear digit set $D = \{0, 1, \ldots, m\} \nu$. We obtain the following main results.

**Theorem 1.1.** Let the characteristic polynomial of $A$ be $f(x) = x^2 + bx + c$ and a digit set $D = \{0, 1, \ldots, m\} \nu$ where $m \geq 1$ and $\nu \in \mathbb{R}^d$ such that $(\nu, A\nu)$ are linearly independent. If $\Delta = b^2 - 4c > 0$ and the eigenvalues of $A$ have moduli $\geq 2$, then

\[
\begin{align*}
\text{(i) if } c = 4, \text{ then } T(A, D) \text{ is connected if and only if } m \geq 2; \\
\text{(ii) otherwise } c \neq 4, \text{ then } T(A, D) \text{ is connected if and only if } m \geq \begin{cases} 
\max(c - |b| + 1, |b| - 1) & c > 0, \\
|c| - |b| - 1 & c < 0.
\end{cases}
\end{align*}
\]

If $\Delta = b^2 - 4c < 0$, the eigenvalues of $A$ are complex numbers, the self-affine set $T(A, D)$ becomes very complicated. However, under certain situations, we still obtain some interesting results.

**Theorem 1.2.** Let the characteristic polynomial of $A$ be $f(x) = x^2 + bx + c$ and a digit set $D = \{0, 1, \ldots, m\} \nu$ where $m \geq 1$ and $\nu \in \mathbb{R}^2$ such that $(\nu, A\nu)$ are linearly independent. If $\Delta = b^2 - 4c < 0$, then $T(A, D)$ is connected if and only if

\[
m \geq \begin{cases} 
\max(c - |b| + 1, |b| - 1) & b^2 = 3c, \\
\max(c - |b| + 1, |b| - 1) & b^2 = 2c, b^2 = c, \\
c - 1 & b = 0.
\end{cases}
\]

On the other hand, when the characteristic polynomial of $A$ is of the special form $f(x) = x^2 - (p + q)x + pq$ where $|p|, |q| \geq 2$ are integers, and the digit set $D$ may be non-consecutively collinear. By letting $f_1(x) = x^2 \pm 4x + 4$ and $f_2(x) = x^2 \pm 7x + 12$, we can characterize the connectedness of the associated self-affine tile $T(A, D)$ through the following theorem, which is also a generalization of [15].

**Theorem 1.3.** Let the characteristic polynomial of $A$ be $f(x) = x^2 - (p + q)x + pq$ and a digit set $D = \{0, 1, \ldots, |pq| - 2, |pq| - 1 + s\} \nu$ where $s \geq 0, |p|, |q| \geq 2$ are integers and $\nu \in \mathbb{R}^2$ such that $(\nu, A\nu)$ are linearly independent. Then

\[
\begin{align*}
\text{(i) if } f \neq f_1, f_2, \text{ then } T(A, D) \text{ is connected if and only if } s = 0; \\
\text{(ii) if } f = f_1 \text{ or } f_2, \text{ then } T(A, D) \text{ is connected if and only if } s = 0 \text{ or } 1.
\end{align*}
\]

As in the papers previously cited, a lot of calculations are needed in the proofs. But the main methods are algebraic and make full use of the properties of the matrix $A$. We also provide many figures to illustrate our results.

The paper is organized as follows: In Section 2, we recall several well-known results on the connectedness of self-affine sets and prove a basic lemma; Theorems 1.1 and 1.2 are proved in Section 3, and conclude with an open problem; Theorem 1.3 is proved in Section 4.

2. Preliminaries

In the section, we provide several elementary results on self-affine sets $T(A, D)$. We call the digit set $D$ collinear if $\nu = \{d_1, \ldots, d_m\} \nu$ for some non-zero vector $\nu \in \mathbb{R}^d$ and $d_1 < d_2 < \cdots < d_m$, $d_i \in \mathbb{R}$; if $d_1 - d_i = 1$, then $D$ is called a consecutive collinear digit set. Let $D = \{d_1, \ldots, d_m\}$, $D_D = D - D = \{d = d_i - d_j : d_i, d_j \in D\}$. Then $D = D_D$ and $\Delta D = \Delta D_D$. It is easy to see that the connectedness of $T(A, D)$ is invariant under a translation of the digit set, hence we always assume that $d_1 = 0$ for simplicity. The following criterion for connectedness of $T(A, D)$ was due to [9] or [12].

**Lemma 2.1.** A self-affine set $T(A, D)$ with a consecutive collinear digit set $D = \{0, 1, \ldots, m\} \nu$ is connected if and only if $\nu \in T - T$.

Let $Z[\nu]$ be the set of polynomials with integer coefficients. A polynomial $f(x) \in Z[\nu]$ is said to be expanding if all its roots have moduli strictly bigger than 1. Note that a matrix $A \in M_n(Z)$ is expanding if and only if its characteristic polynomial is expanding. We say that a monic polynomial $f(x) \in Z[x]$ with $|f(0)| = c$ has the Height Reducing Property (HRP) if there exists $g(x) \in Z[x]$ such that $g(x)f(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1x + c$, where $|a_i| \leq c - 1, i = 1, \ldots, k - 1$.

This property was introduced by Kirat and Lau [12] to study the connectedness of self-affine tiles with consecutive collinear digit sets. It was proved that:

**Proposition 2.2.** Let $A \in M_n(Z)$ with $|\det A| = c$ be expanding and $D = \{0, 1, 2, \ldots, (c - 1)\} \nu$. If the characteristic polynomial of $A$ has the Height Reducing Property, then $T(A, D)$ is connected.

In [13], Kirat et al. conjectured that all expanding integer monic polynomials have HRP. Akiyama and Gjimi [12] confirmed it up to $n = 4$. But it is still unclear for the higher dimensions. Recently, He et al. [10] developed an algorithm of polynomials about HRP. It may be a good attempt on this problem.

Denote the characteristic polynomial of $A$ by $f(x) = x^2 + bx + c$, where $b, c \in Z$. We can regard $A$ as the companion matrix of $f(x)$, i.e.,

$$
A = \begin{bmatrix}
0 & -c \\
1 & -b
\end{bmatrix}.
$$

Let $\Delta = b^2 - 4c$ be the discriminant. Define $\alpha, \beta$ by

$$
A^{-1}\nu = \alpha \nu + \beta A\nu, \quad i = 1, 2, \ldots,
$$

According to the Hamilton–Cayley theorem $f(A) = A^2 + bA + cI = 0$ where $I$ is a $2 \times 2$ identity matrix, the following consequence is well-known (please refer to [15,16]).
Lemma 2.3. Let \( x_i, \beta_i \) be defined as the above. Then
cx_{i+2} + bx_{i+1} + x_i = 0 and c\beta_{i+2} + b\beta_{i+1} + \beta_i = 0, \text{i.e.,}
\[
\begin{bmatrix}
\alpha_{i+1} \\
\alpha_{i+2}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-1/c & -b/c
\end{bmatrix}
\begin{bmatrix}
\alpha_i \\
\alpha_{i+1}
\end{bmatrix},
\]
and
\[
\alpha_i = -b/c, \alpha_2 = (b^2 - c)/c^2; \beta_i = -1/b, \beta_2 = b/c^2.
\]
Moreover for \( \Delta \neq 0 \), we have
\[
\alpha_i = \frac{c(t_i^1 - r_i^{1/2})}{\Delta^{1/2}} \quad \text{and} \quad \beta_i = \frac{-(r_i^1 - t_i^1)}{\Delta^{1/2}}.
\]
where \( r_1 = \frac{-b + \sqrt{b^2 - 4c}}{2c} \) and \( r_2 = \frac{-b - \sqrt{b^2 - 4c}}{2c} \) are the two roots of
\( cx^2 + bx + 1 = 0 \).

Set
\[
\tilde{x} := \sum_{i=1}^{\infty} |\alpha_i|, \quad \tilde{\beta} := \sum_{i=1}^{\infty} |\beta_i|.
\]

Then \( \tilde{x} \) and \( \tilde{\beta} \) are finite numbers as \( r_1, r_2 \) have moduli strictly less than 1.

Write \( L := \left( \gamma \nu + \delta A \nu : \gamma, \delta \in \mathbb{Z} \right) \), then \( L \) is a lattice generated by \( \{ \nu, A \nu \} \). For \( l \in \mathbb{Z} \setminus \{ 0 \} \), we call \( T + l \) a neighbor of \( T \) if \( T \cap (T + l) \neq \emptyset \). It is clear that \( T + l \) is a neighbor of \( T \) if and only if \( l \in T - T \), hence \( l \) can be expressed as
\[
l = \sum_{i=1}^{\infty} b_i A^{-i} \nu \in T - T, \quad \text{where} \ b_i \in \Delta D.
\]

If \( T + l \) is a neighbor of \( T \) where \( l = \sum_{i=1}^{\infty} b_i A^{-i} \nu := \gamma \nu + \delta A \nu \), then
\[
|\gamma| \leq \max_{i} |b_i| \tilde{x} \quad \text{and} \quad |\delta| \leq \max_{i} |b_i| \tilde{\beta}.
\]

By multiplying \( A \) on both sides of the expression of \( l \) and by using \( f(A) = 0 \), it follows that \( T - (c\delta + b_1) \nu + (\gamma - b_\delta)A \nu \) is also a neighbor of \( T \). Repeatedly applying this neighbor-generating algorithm, we then can construct a sequence of neighbors: \( \{T + l_n\}_{n=0}^{\infty} \), where \( l_0 = 0, L_1 = \gamma \nu + \delta A \nu, n \geq 1 \) and
\[
\begin{bmatrix}
\gamma_n \\
\delta_n
\end{bmatrix} = A^n \begin{bmatrix}
\gamma \\
\delta
\end{bmatrix} - \sum_{i=1}^{n} A^{n-i} \begin{bmatrix}
b_i \\
0
\end{bmatrix}.
\]

Moreover, \( |\gamma_n| \leq \max_{i} |b_i| \tilde{x} \) and \( |\delta_n| \leq \max_{i} |b_i| \tilde{\beta} \) hold for any \( n \geq 0 \).

Lemma 2.4. If the characteristic polynomial of the expanding matrix \( A \) is \( x^2 + bx + c \) and that of \( B \) is \( x^2 - bx + c \), then the self-affine set \( T(A, D) \) is connected if and only if \( T(B, D) \) is connected where \( D \) is a consecutive collinear digit set.

Proof. Let \( B = -A \) and \( T_1 = T(A, D), T_2 = T(-A, D) \). If \( l \in T_1 - T_2 \), then
\[
l = \sum_{i=1}^{\infty} b_i A^{-i} \nu = \sum_{i=1}^{\infty} b_{2i} (-A)^{-2i} \nu + \sum_{i=1}^{\infty} (-b_{2i-1}) (-A)^{-2i-1} \nu.
\]

Thus \( l \in T_2 - T_2 \), and vice versa. \( \square \)

To get the necessary conditions of Theorems 1.1–1.3, we need the exact values of \( \tilde{x} \) and \( \tilde{\beta} \).

Lemma 2.5. Let the characteristic polynomial of the expanding matrix \( A \) be \( f(x) = x^2 + bx + c \), where \( b, c \) are integers and \( \Delta = b^2 - 4c > 0 \). Then
\[
\tilde{x} = \begin{cases}
\frac{1}{c - b^2 + 1} & \text{if } c > 0, \\
\frac{1}{c - b^2 - 1} & \text{if } c < 0.
\end{cases}
\]
\[
\tilde{\beta} = \begin{cases}
\frac{1}{c - b^2 - 1} & \text{if } c > 0, \\
\frac{1}{c - b^2 + 1} & \text{if } c < 0.
\end{cases}
\]

Proof. Let \( x_1, x_2 \) denote the roots of \( x^2 + bx + c = 0 \).

(1) \( c > 0 \). If \( |x_1| > |x_2| \), then \( |x_1| = \frac{c}{|x_2|} \) \( \left( \frac{1}{|x_2|} - \frac{1}{|x_1|} \right) \). Hence
\[
\tilde{x} = \sum_{i=1}^{\infty} |\alpha_i| = \sum_{i=1}^{\infty} c \left( \frac{1}{|x_2|^i} - \frac{1}{|x_1|^i} \right) = \left( \frac{|x_1| + |x_2| - 1}{|x_2|} \right) = \frac{1}{|x_1|} - \frac{1}{|x_2|} = \frac{|b|}{c - |b| + 1}.
\]

(2) \( c < 0 \). Without loss of generality, we can assume \( |x_1| > |x_2| \), then
\[
\tilde{x} = \sum_{i=1}^{\infty} |\alpha_i| = \sum_{i=1}^{\infty} \left( |x_1| - |x_2| \right) \left( \frac{1}{|x_2|^{2i+1}} + \frac{1}{|x_1|^{2i+1}} \right)
\]
\[
= \sum_{i=1}^{\infty} \left( \frac{1}{|x_2|^{2i+1}} - \frac{1}{|x_1|^{2i+1}} \right)
\]
\[
= \sum_{i=1}^{\infty} \left( \frac{1}{|x_1|^{2i+1}} - \frac{1}{|x_2|^{2i+1}} \right)
\]
\[
= \sum_{i=1}^{\infty} \left( \frac{1}{|x_1|^{2i+1}} - \frac{1}{|x_2|^{2i+1}} \right)
\]
\[
= \frac{1}{|x_1|} - \frac{1}{|x_2|} = \frac{|b| + 1}{|x_1| - |x_2|} = \frac{1}{|x_1| - |x_2|}.
\]
\[
\bar{\beta} = \sum_{i=1}^{n} |\beta_i| = \frac{1}{|x_1| + |x_2|} \left( \sum_{i=1}^{n} \left( \frac{1}{|x_1|} + \frac{1}{|x_2|} \right) \right) + \sum_{i=1}^{n} \left( \frac{1}{|x_1|} - \frac{1}{|x_2|} \right) = \frac{1}{|x_1| + |x_2|} \left( \frac{|x_1|}{|x_1|^2 - 1} + \frac{|x_2|}{|x_2|^2 - 1} + \frac{1}{|x_1|^2 - 1} - \frac{1}{|x_2|^2 - 1} \right) = \frac{1}{|x_1| + |x_2|} \left( \frac{1}{|x_2| - 1} + \frac{1}{|x_1| + 1} \right) = \frac{1}{|x_1| - |x_2| - 1}.
\]

3. Consecutive collinear digit set

In the section, we characterize the connectedness of the self-affine sets \( T(A, D) \) associated with digit sets \( D = \{0, 1, \ldots, m\} \). The necessary and sufficient conditions are given for \( T(A, D) \) to be connected.

**Theorem 3.1.** Let the characteristic polynomial of the expanding integer matrix \( A \) be \( f(x) = x^2 + bx + c \) and a digit set \( D = \{0, 1, \ldots, m\} \) where \( m > 1 \) is integral and \( v \in \mathbb{R}^2 \) such that \( \langle v, Av \rangle \) are linearly independent. If \( \Delta - b^2 - 4c > 0 \) and the eigenvalues of \( A \) have moduli \( \geq 2 \), then

(i) If \( c = 4 \), then \( T(A, D) \) is connected if and only if \( m \geq 2 \);
(ii) Otherwise \( c \neq 4 \), then \( T(A, D) \) is connected if and only if

\[
m \geq \left\{ \begin{array}{ll}
\max\{c - |b| + 1, |b| - 1\} & c > 0, \\
|c| - |b| - 1 & c < 0.
\end{array} \right.
\]

**Proof.** Let \( T + I \) be a neighbor of \( T \), then \( I = \gamma v + \Delta A v = \sum_{i=1}^{n} b_i A_i \ v, b_i \in \Delta D = \{0, \pm 1, \pm 2, \ldots, \pm m\} \). By (2.1), we have

\[|\gamma| \leq m \bar{\gamma}; \quad |\delta| \leq m \bar{\delta}. \tag{3.1}\]

Suppose \( (T + I) \cap (T + \epsilon_2 v) \neq \emptyset \) for \( 0 \leq \epsilon_1 < \epsilon_2 \leq m \), then \( (\epsilon_2 - \epsilon_1) v = \sum_{i=1}^{m} b_i A_i \ v, b_i \in \Delta D \). By (2.2), we obtain

\[
I = -((\epsilon_2 - \epsilon_1) c + 2b) v - (b_2 (\epsilon_2 - \epsilon_1) + b_1 A v).
\]

(i) If \( c = 4 \), then \( |\delta| = 4 \), by (3.1) and Lemma 2.5, the connectedness of \( T(A, D) \) can imply that

\[
(\epsilon_2 - \epsilon_1)|b| - m \leq |(\epsilon_2 - \epsilon_1) b + b_1| \leq \frac{m}{|c - |b| + 1}|c - |b| - 1| = m,
\]

which further implies that \( m \geq 2(\epsilon_2 - \epsilon_1) \geq 2 \).

Conversely, if \( b = -4 \), then \( f(x) = x^2 - 4x + 4 \). By using \( f(A) = 0 \) and \( A f(A) = 0 \), we have \( A^3 - 3A^2 + 4I = 0 \) and \( A^2 - 2A = 2A(A = I) + 2(A = I) - 2I \).\)

which yields

\[
(A - I) = 2A^{-1}(A = I) + 2A^{-2}(A = I) - 2A^{-2}
\]

and

\[
I = 2A^{-1} + 2A^{-2} - 2\sum_{i=3}^{\infty} A^{-i}.
\]

Then

\[
v = 2A^{-1} v + 2A^{-2} v - 2\sum_{i=3}^{\infty} A^{-i} v \in T - T.
\]

Therefore \( T \) is connected by Lemma 2.1. (see Fig. 1). If \( b = 4 \), then \( f(x) = x^2 - 4x + 4 \). By Lemma 2.4, the connectedness of \( T(A, D) \) is the same as that of \( T(-A, D) \) in which the characteristic polynomial of \(-A\) is \( f(x) = x^2 - 4x + 4 \).

(ii) Necessity: If \( c > 0 \), then by (3.1) and Lemma 2.5, we have

\[
(\epsilon_2 - \epsilon_1)c - m \leq |(\epsilon_2 - \epsilon_1)c + b_2| \leq \frac{m}{|c - |b| + 1}|c - |b| + 1|; \tag{3.2}
\]

\[
(\epsilon_2 - \epsilon_1)|b| - m \leq |(\epsilon_2 - \epsilon_1)b + b_1| \leq \frac{m}{|c - |b| + 1}|c - |b| + 1|. \tag{3.3}
\]

(3.2) implies that \( m \geq (\epsilon_2 - \epsilon_1)(c - |b| + 1) \geq c - |b| + 1 \). Now we prove \( m \geq |b| - 1 \). Let \( x_1, x_2 \) denote the roots of \( x^2 + bx + c = 0 \), \( x < |b| - 1 \), then \(|x| \leq |b| - 2 \leq |x_1| - 1 + |x_2| - 1 \). We have

\[
\frac{m}{(|x_1| - 1)(|x_2| - 1)} \leq \frac{1}{|x_1| - 1} + \frac{1}{|x_2| - 1} < 2. \quad \tag{3.4}
\]

The last strict inequality holds due to the fact that \( c > 4 \) and \(|x_1|, |x_2| \geq 2 \). From (3.3) and (3.4), it follows that

\[
2 \leq |b| - m \leq (\epsilon_2 - \epsilon_1)|b| - m \leq \frac{m}{|c - |b| + 1}|c - |b| + 1| - \frac{m}{|x_1| - 1} - \frac{m}{|x_2| - 1} < 2,
\]

which is a contradiction. Hence \( m \geq |b| - 1 \).

If \( c < 0 \), then by (3.1) and Lemma 2.5, we have

\[
(\epsilon_2 - \epsilon_1)|c| - m \leq |(\epsilon_2 - \epsilon_1)c + b_2| \leq \frac{m}{|c - |b| + 1}|c - |b| - 1| \tag{3.6}
\]

implying that \( m \geq (\epsilon_2 - \epsilon_1)(|c| - |b| - 1) \geq |c| - |b| - 1 \).

Sufficiency: If \( c > 0 \), and \( m \geq \max\{c - |b| + 1, |b| - 1\} \), it suffices to show that \( v \in T - T \) by Lemma 2.1.

When \( b < 0 \), by using \( f(A) = A^2 + bA + cI = 0 \), we have \( A^2 + bA = (b+1)I = -(c+b+1)I, i.e., (A-I)(A+(b+1)I) = -(c+b+1)I \). It follows that

\[
I + (b+1)A^{-1} = -(c+b+1)A^{-1}\sum_{i=1}^{\infty} A^{-i}.
\]

Hence

\[
I = (b+1)A^{-1} + \sum_{i=2}^{\infty} -(c+b+1)A^{-i}.
\]

Then \( v = (|b| - 1)A^{-1} v + \sum_{i=2}^{\infty} -(|c| - |b| + 1)A^{-i} v \in T - T, \) and \( T \) is connected. When \( b > 0 \), Lemma 2.4 and the above argument also yield that \( T \) is connected. (see Fig. 2).

If \( c < 0 \), suppose \( m \geq |c| - |b| - 1 \), then \( m \geq |b| + 1 \) (Indeed, if \( x_1, x_2 \) are the roots of \( x^2 + bx + c = 0 \), without loss of generality, we let \( |x_1| > |x_2| \), then \(|c| - |b| - 1 = |x_1| + 1 > |x_1| - |x_2| + 1 = |x_1| - |x_2| + |x_2| - 1 = (|x_1| - 1)(|x_2| - 1) = (|x_1| + 1)(|x_2| - 1) \geq |x_1| + 1 > |x_1| - |x_2| + 1 = |b| + 1 \).
When $b < 0$, by using $f(A) = A^2 + bA + cI = 0$, we have $A^2 + A = (-b + 1)(A + I) + (-c + b - 1)I$. Then

$$I = (-b + 1)A^{-1} + (-c + b - 1)\sum_{i=2}^{\infty}(-A)^{-i}$$

$$= (|b| + 1)A^{-1} + (|c| - |b| - 1)\sum_{i=2}^{\infty}(-A)^{-i}$$

$$= (|b| + 1)A^{-1} + (|c| - |b| - 1)\sum_{k=1}^{\infty}A^{-2k} - (|c| - |b| - 1)\sum_{k=1}^{\infty}A^{-2k-1}.$$ 

Hence $\nu \in T - T$ and $T$ is connected by Lemma 2.1. When $b > 0$, Lemma 2.4 and the above argument also yield that $T$ is connected. (see Fig. 3). \( \Box \)

In the proof above, the condition that eigenvalues of $A$ have moduli $\geq 2$ is essential. If otherwise, in the case that the moduli of the eigenvalues $< 2$, e.g., the moduli are close to 1, we have no idea about the conditions for $T(A, D)$ to be connected by estimating $\bar{\alpha}$ or $\bar{\beta}$.

On the other hand, if $\Delta = b^2 - 4c < 0$, it is also difficult to compute the exact values of $\bar{\alpha}$ and $\bar{\beta}$ in general. However, under certain special situations, the exact values of $\bar{\alpha}$ and $\bar{\beta}$ can still be calculated as well.

**Theorem 3.2.** Let the characteristic polynomial of $A$ be $f(x) = x^2 + bx + c$ and a digit set $D = \{0, 1, \ldots, m\} \nu$ where $m \geq 1$ and $\nu \in \mathbb{R}^2$ such that $(\nu, A\nu)$ are linearly independent. If $\Delta = b^2 - 4c < 0$, then $T(A, D)$ is connected if and only if

$$m \geq \begin{cases} \max\{c - |b| + 1, |b| - 1\} & b^2 = 3c, \\
 c - |b| + 1 & b^2 = 2c, b^2 = c, \\
 c - 1 & b = 0. \end{cases}$$

(3.7)

**Proof.** From Lemma 2.4, we can suppose $b < 0$. Let $r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$ be the complex roots of $cx^2 + bx + 1 = 0$ as in Lemma 2.3 and let $r := |r_1| = |r_2| = \frac{1}{r}$. Then $r_1 = re^{i\theta}$ and $r_2 = re^{-i\theta}$ where $\theta$ is the argument of $r_1$. We show the necessity first by assuming $T(A, D)$ is connected.
Fig. 3. (a) is disconnected and (b) is connected where $A = [6, 0, -1, -4]$, $v = (1, 0)^t$.

Fig. 4. (a) is disconnected and (b) is connected where $A = [0, -12, 1, -6]$, $v = (1, 0)^t$.

(i) If $b^3 = 3c$, then $\theta = \frac{\pi}{6}$ and

$$|x| = \frac{2c r^{i+1} |\sin((i+1) \frac{\pi}{6})|}{\sqrt{c}};$$

$$|y| = \frac{2c r^{i+1} |\sin((i+1) \frac{\pi}{6})|}{\sqrt{c}}.$$  

Hence

$$\ddot{x} = \sum_{i=1}^{\infty} |x| = \sqrt{c} \left( (\sqrt{3}r^2 + 2r^3 + \sqrt{3}r^4 + r^7) \sum_{j=0}^{\infty} r^{3j} \right)$$

$$= \frac{3|b|^5 + 6|b|^4 + 9|b|^3 + 9|b|^2 + 27}{b^6 - 27};$$

Analogous to (3.2), we have

$$c - m \leq (\ell_2 - \ell_1)c - m \leq |(\ell_2 - \ell_1)c + b_2|$$

$$\leq m \frac{3|b|^5 + 6|b|^4 + 9|b|^3 + 9|b|^2 + 27}{b^6 - 27}$$

which implies that

$$m \geq \frac{b^6 - 27}{3(b^6 + 3|b|^3 + 6b^2 + 9|b| + 9)}$$

$$= \frac{b^6 - |b| + 3b^4 + 9|b|^3 + 18b^2 + 27|b| - 27}{3b^6 + 9|b|^3 + 18b^2 + 27|b| + 27}$$

$$= c - |b| + \frac{3b^4 + 9|b|^3 + 18b^2 + 27|b| - 27}{3b^4 + 9|b|^3 + 18b^2 + 27|b| + 27}.$$  

Thus $m \geq c - |b| + 1$ as $0 < \frac{3b^4 + 9|b|^3 + 18b^2 + 27|b| - 27}{3b^4 + 9|b|^3 + 18b^2 + 27|b| + 27} < 1$ and $m$ is integral. (see Fig. 4).

If $|b| > 3$, then $c - |b| + 1 > |b| - 1$ is always true; if $|b| = 3$ then $c = 3$, and

$$\ddot{\mu} = \sum_{i=1}^{\infty} |\mu| = \frac{1}{\sqrt{c}} \left( (r + \sqrt{3}r^2 + 2r^3 + \sqrt{3}r^4 + r^7) \sum_{j=0}^{\infty} r^{3j} \right) = \frac{14}{13}$$

Analogous to (3.3), we have

$$3 - m \leq (\ell_2 - \ell_1)3 - m \leq |(\ell_2 - \ell_1)b + b_1| \leq m \frac{14}{13}.$$
and $m \geq \frac{39}{27}$. Therefore $m \geq 2 = |b| - 1$.

(ii) If $b^2 = 2c$, then $\theta = \frac{\pi}{4}$ and

$$|\lambda| = \left| \frac{c(r_1^1 - r_2^1)}{\Delta^{1/2}} \right| = \frac{2c^{1/2} \sin((i + 1) \frac{\pi}{2})}{|b|}.$$ 

Then

$$\tilde{x} = \sum_{i=1}^{\infty} |\lambda_i| = |b| \left( \sum_{j=0}^{\infty} r_j^3 + \frac{\sqrt{2}}{2} \sum_{k=0}^{\infty} r_k^2 \right)$$

$$= \frac{2|b|^3 + 2b^2 + 4}{b^4 - 4}.$$

Analogous to (3.2), we have

$$c - m \leq (\ell_2 - \ell_1)c - m \leq |(\ell_2 - \ell_1)c + b_2|$$

$$\leq \frac{m}{b^4 - 4} \frac{2|b|^3 + 2b^2 + 4}{b^4 - 4},$$

implying that

$$m \geq \frac{b^4 - 4}{2(b^2 + 2|b| + 2)}$$

$$= \frac{b^2}{2} - |b| + \frac{2b^2 + 4|b| - 4}{2b^2 + 4|b| + 4}$$

$$= c - |b| + \frac{2b^2 + 4|b| - 4}{2b^2 + 4|b| + 4}.$$

Hence $m \geq c - |b| + 1$ as $0 < \frac{2b^2 + 4|b| - 4}{2b^2 + 4|b| + 4} < 1$ and $m$ is integral.

(iii) If $b^2 = c$, then $\theta = \frac{\pi}{4}$. By the similar discussion of (i) above, it follows that

$$\tilde{x} = \sum_{i=1}^{\infty} |\lambda_i| = \frac{b^2 + 1}{|b|^3 - 1}$$

and

$$m \geq \frac{|b|^3 - 1}{|b| + 1} = b^2 - |b| + \frac{|b| - 1}{|b| + 1} = c - |b| + \frac{|b| - 1}{|b| + 1}.$$

Fig. 5. (a) is disconnected and (b) is connected where $A = [0, -8; 1, -4], \, \nu = (1, 0)^t$.

Fig. 6. (a) is disconnected and (b) is connected where $A = [0, -9; 1, -3], \, \nu = (1, 0)^t$. 

Hence $m \geq c - |b| + 1$ as $0 < \frac{c-1}{c} < 1$. (see Fig. 6)

(iv) If $b = 0$, then $\theta = \frac{\pi}{2}$. Similarly, we have

$$\tilde{\alpha} = \sum_{i=1}^{\infty} |\alpha_i| = \frac{c}{\sqrt{c}} \sum_{i=1}^{\infty} \sqrt{c+1} = \frac{1}{c-1}$$

and then $m \geq c - 1$.

On the contrary, for the sufficiency, if $m$ satisfies (3.7), then $m \geq |b| - 1$ for cases (ii) and (iii). With the similar proof as in Theorem 3.1, we can conclude that $\nu \in T - T$ and $T(A, D')$ is connected. The connectedness of case (iv) comes from the Proposition 2.2 directly.

For other unsolved cases, by observing computer graphs, we conclude with the following conjecture.

**Conjecture 3.3.** Let the characteristic polynomial of $A$ be $f(x) = x^2 + bx + c$ and a digit set $D = \{0, 1, \ldots, m\} \nu$ where $m \geq 1$ and $\nu \in \mathbb{R}^2$ such that $\langle \nu, Av \rangle$ are linearly independent. Then

(i) if $c = |b| = 4$, then $T(A, D')$ is connected if and only if $m \geq 2$,

(ii) otherwise, $T(A, D')$ is connected if and only if

$$m \geq \begin{cases} \max\{c - |b| + 1, |b| - 1\} & \text{if } c > 0, |b| \geq 2, \\ c - 1 & \text{if } c > 0, |b| \leq 1, \\ \max\{|c| - |b| - 1, |b| + 1\} & \text{if } c < 0. \end{cases}$$

4. Non-consecutive collinear digit set

By previous Section 2, we know that if $A \in M_2(\mathbb{Z})$ is an expanding matrix, then its characteristic polynomial has HRP. Let $D' = \{0, 1, \ldots, (|\det(A)| - 1)\} \nu$ be a consecutive collinear digit set with $\#D' = |\det(A)|$. By Proposition 2.2, then the associated self-affine tile $T(A, D')$ is always connected. However there are few results on the non-consecutive collinear digit sets. In [15], Leung and Luo first study this case, by checking 10 eligible characteristic polynomials of the $A$ with $|\det A| = 3$ case by case, they obtained a complete characterization for connectedness of $T(A, D')$. 

Fig. 7. (a) is connected and (b) is disconnected where $A = [6, 0, -1, 4], \nu = (1, 0)'$.

Fig. 8. (a) is connected and (b) is disconnected where $A = [2, 0, -1, 2], \nu = (1, 0)'$. 

with \( D = \{0, 1, m\} \nu \). In the section, we further study this kind of digit sets in more general situations. Suppose the characteristic polynomial of \( A \) is of the form \( f(x) = x^2 - (p + q)x + pq \) where \( |p|, |q| \geq 2 \) are integers, and \( D = \{0, 1, \ldots, |pq| - 2, |pq| - 1 + s\} \nu \). By letting \( f_1(x) = x^2 + 4x + 4 \) and \( f_2(x) = x^2 + 7x + 12 \), we have the following criterion for the connectedness.

**Theorem 4.1.** Let the characteristic polynomial of \( A \) be \( f(x) = x^2 - (p + q)x + pq \) and a digit set \( D = \{0, 1, \ldots, |pq| - 2, |pq| - 1 + s\} \nu \) where \( s \geq 0, |p|, |q| \geq 2 \) are integers and \( \nu \in \mathbb{R}^2 \) such that \( \{v, Av\} \) are linearly independent. Then

1. If \( f \neq f_1, f_2 \), then \( T(A, D) \) is connected if and only if \( s = 0 \);
2. If \( f = f_1 \) or \( f = f_2 \), then \( T(A, D) \) is connected if and only if \( s = 0 \) or 1.

**Proof.** If \( s = 0 \), then \( T(A, D) \) is always connected by Proposition 2.2.

(i) Suppose \( T(A, D) \) is connected, then \((T + (|pq| - 1 + s)\nu) \cap (T + i\nu) \neq \emptyset \) for some \( 0 \leq i \leq |pq| - 2 \). Let \( r = |pq| - 1 + s - i \), then \( r\nu \in T - T \), i.e., \( r\nu = \sum_{i=1}^{s-1} b_i A^i \nu \) where \( b_i \in \Delta D \). By (2.2), we obtain a new neighbor \( T + T' \) where \( T' = (rpq + b_2)\nu + (r(p + q) - b_1)Av \).

If \( pq > 0 \), then by (2.1) and Lemma 2.5, we have

\[
(1 + s)|pq| - (|pq| - 1 + s) \leq r|pq|
- (|pq| - 1 + s) \leq r|pq| + b_2|
\leq (|pq| - 1 + s)(|p| + |q| - 1)
\]

\[
(1 + s)|p + q| - (|pq| - 1 + s) \leq r(p + q) - b_1|
\leq (|pq| - 1 + s)(|p| + |q| - 1).
\]

It follows from (4.1) that

\[
s \leq \frac{|p| + |q| - 2}{|pq| - |p| - |q|}.
\]

Let

\[
t = \frac{|p| + |q| - 1}{|p| - 1 + (|p| - 1)(|q| - 1)}.
\]

It is easy to see that \( t < 1 \) if \( |p|, |q| \geq 4 \) or one of \( |p|, |q| \) is equal to 3 and the other one is larger than 5. Therefore \( |p| + |q| - 2 < |pq| - |p| - |q| \), and \( s < 1 \), i.e., \( s = 0 \). (see Fig. 7)

If one of \( |p|, |q| \) is equal to 2 and the other one is larger than 3, without loss of generality, suppose \( |p| = 2 \) and \( |q| \geq 3 \). From (4.2) we get

\[
s \leq \frac{|q|^2 - 2|q| + 2}{|q|^2 - 2} = 1 - \frac{2|q| - 4}{|q|^2 - 2} < 1.
\]

Hence \( s = 0 \).

If \( pq < 0 \), analogous to the (4.1), then

\[
(1 + s)|pq| - (|pq| - 1 + s)
\leq (|pq| - 1 + s)\frac{|p + q| + 1}{|p| - |p| + 1}.
\]

We have

\[
s \leq \frac{|p + q|}{|pq| - |p + q| - 2}
= \frac{1 - |pq| - 2|p + q| - 2}{|pq| - |p + q| - 2}.
\]

Since \( pq < 0 \), without loss of generality, we let \( |p| \geq |q| \), it follows that \( |pq| - |p + q| - 1 = |pq| - (|p| - |q|) = -1 = (|p| + 1)(|q| - 1) \geq |p| + 1 > |p| - |q| + 1 = |p + q| + 1 \). Thus \( |pq| - 2|p + q| - 2 > 0 \), and \( s = 0 \).

(ii) If \( f = f_1 \), then (4.4) implies that \( s \leq 1 \) (see Fig. 8); if \( f = f_2 \), then (4.3) implies that \( s \leq 1 \) (see Fig. 9). Conversely, for \( s = 1 \), let \( \Delta D = \{0, \pm 1, \pm 2, \pm 3, \pm 4\} \) and \( \Delta D = \{0, \pm 1, \ldots, \pm 12\} \) and let \( A_1 \) and \( A_2 \) denote the matrices of \( f_1 \) and \( f_2 \) respectively. We only need to
show that \( v, 2v \in T - T \) (see [12] or [15]). Let \( \Delta D' = \{0, \pm 1, \pm 2\} \) and \( \Delta D_2 = \{0, \pm 1, \ldots, \pm 6\} \). By Theorem 3.1, there exist sequences \( \{b_{1i}\}_{i=1}^{\infty} \) where \( b_{1i} \in \Delta D' \) and \( \{b_{2i}\}_{i=1}^{\infty} \) where \( b_{2i} \in \Delta D_2 \) such that 
\[
\begin{align*}
v = \sum_{i=1}^{\infty} b_{1i} A^{-1} v & \in T - T \\
v = \sum_{i=1}^{\infty} b_{2i} A_2^{-1} v & \in T - T.
\end{align*}
\]
Moreover, \( 2b_{1i} \in \Delta D_1 \) and \( 2b_{2i} \in \Delta D_2 \). Hence \( 2v \in T - T \) as well. □

References