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On the connectedness of planar self-affine sets ☆

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ABSTRACT

In this paper, we consider the connectedness of planar self-affine set $T(A,\mathcal{D})$ arising from an integral expanding matrix A with characteristic polynomial $f(x)=x^2+bx+c$ and a consecutive collinear digit set $\mathcal{D}=\{0,1,\ldots,m\}v$. The necessary and sufficient conditions only depending on b,c,m are given for the $T(A,\mathcal{D})$ to be connected. Moreover, we also consider the case that $\mathcal{D}$ is non-consecutively collinear.

1. Introduction

Let $A\in M_n(\mathbb{Z})$ be an expanding $n\times n$ integral matrix, i.e., all eigenvalues of A have moduli strictly greater than 1. Let $\mathcal{D}=\{d_1,\ldots,d_m\}$ be a finite set of m distinct vectors on $\mathbb{R}^n$. We call $\mathcal{D}$ a *digit set*. Then the maps

$$S_i(x)=A^{-1}(x+d_i),\quad 1\leqslant i\leqslant m$$

are contractive under a suitable norm in $\mathbb{R}^n$ [18], and it is well-known that there exists a unique non-empty compact set $T:=T(A,\mathcal{D})$ satisfying the set-valued functional equation

$$T=\bigcup_{i=1}^{m}S_i(T). \tag{1.1}$$

Usually T can also be written as the set of radix expansions

$$T=A^{-1}(T+\mathcal{D})=\left\{\sum_{i=1}^{\infty}A^{-1}d_{j_i}:d_{j_i}\in\mathcal{D}\right\}.$$

The T is called the *self-affine set (or attractor)* of the iterated function system (IFS) $\{S_i\}_{i=1}^m$. We call T a self-affine tile if it has positive Lebesgue measure and the union in 1.1 is essentially disjoint, i.e., the intersection $(T+d_i)\cap(T+d_j)$ has zero Lebesgue measure for $i\neq j$. In this situation, $T^\circ\neq\emptyset$ and $c:=|\det(A)|=m$.

There have been a lot of interests on the fundamental properties of self-affine tiles on $\mathbb{R}^n$ in the literature (see e.g. [7,18–20]). One of the very interesting aspects is the connectedness, in particular the disk-likeness (i.e., homeomorphic to a closed disk in the case $n=2$). The connected self-affine tiles have important applications to wavelet theory and number systems (see e.g. [2,4,7] or the survey papers [1,3]). Gröchenig and Haas [7] as well as Hacon et al. [8] first discussed a few special connected self-affine tiles. Subsequently Lau and his coworkers ([10,12–14]) studied a large class of connected self-affine tiles generated by the *consecutive collinear digit set* $\mathcal{D}=\{0,1,\ldots,c-1\}v$, and their disk-likeness in the plane by introducing an algebraic approach. Akiyama and Thuswaldner [3] investigated the connectedness of families of self-affine tiles associated to quadratic number systems and results on their fundamental group. On the other hand, Bandt and Wang [5] and Leung and Luo [17] also concerned the

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disk-like self-affine tiles or the boundary structure by using a technique of neighbor graphs.

Recently, on $\mathbb{R}^2$, Kirat [11] and Deng and Lau [6] found out the connected self-affine tiles $T(A,\mathcal{D})$ among classes of data $(A,\mathcal{D})$ with non-collinear digit sets $\mathcal{D}$ and characterized the disk-like ones. Leung and Luo ([15,16]) were also interested in the collinear digit set $\{0,1,m\}v$ and the non-collinear digit set $\{0,v,mAv\}$ with the restriction of $|\det A|=3$.

In this paper, we study more general self-affine sets $T(A,\mathcal{D})$ on $\mathbb{R}^2$ arising from an integral expanding matrix A with characteristic polynomial $f(x)=x^2+bx+c$ and the consecutive collinear digit set $\mathcal{D}=\{0,1,\dots,m\}v$. We obtain the following main results.

Theorem 1.1. *Let the characteristic polynomial of A be $f(x)=x^2+bx+c$ and a digit set $\mathcal{D}=\{0,1,\dots,m\}v$ where $m\geqslant 1$ and $v\in\mathbb{R}^2$ such that $\{v,Av\}$ are linearly independent. If $\Delta=b^2-4c\geqslant 0$ and the eigenvalues of A have moduli $\geqslant 2$, then*

(i) *if $c=4$, then $T(A,\mathcal{D})$ is connected if and only if $m\geqslant 2$;*
(ii) *otherwise $c\neq 4$, then $T(A,\mathcal{D})$ is connected if and only if*

$$m\geqslant\begin{cases}\max\{c-|b|+1,|b|-1\} & c>0,\\ |c|-|b|-1 & c<0.\end{cases}$$

If $\Delta=b^2-4c<0$, the eigenvalues of A are complex numbers, the self-affine set $T(A,\mathcal{D})$ becomes very complicated. However, under certain situations, we still obtain some interesting results.

Theorem 1.2. *Let the characteristic polynomial of A be $f(x)=x^2+bx+c$ and a digit set $\mathcal{D}=\{0,1,\dots,m\}v$ where $m\geqslant 1$ and $v\in\mathbb{R}^2$ such that $\{v,Av\}$ are linearly independent. If $\Delta=b^2-4c<0$, then $T(A,\mathcal{D})$ is connected if and only if*

$$m\geqslant\begin{cases}\max\{c-|b|+1,|b|-1\} & b^2=3c,\\ c-|b|+1 & b^2=2c,\ b^2=c,\\ c-1 & b=0.\end{cases}$$

On the other hand, when the characteristic polynomial of A is of the special form $f(x)=x^2-(p+q)x+pq$ where $|p|,|q|\geqslant 2$ are integers, and the digit set $\mathcal{D}$ may be non-consecutively collinear. By letting $f_1(x)=x^2\pm 4x+4$ and $f_2(x)=x^2\pm 7x+12$, we can characterize the connectedness of the associated self-affine tile $T(A,\mathcal{D})$ through the following theorem, which is also a generalization of [15].

Theorem 1.3. *Let the characteristic polynomial of A be $f(x)=x^2-(p+q)x+pq$ and a digit set $\mathcal{D}=\{0,1,\dots,|pq|-2,|pq|-1+s\}v$ where $s\geqslant 0,|p|,|q|\geqslant 2$ are integers and $v\in\mathbb{R}^2$ such that $\{v,Av\}$ are linearly independent. Then*

(i) *if $f\neq f_1,f_2$, then $T(A,\mathcal{D})$ is connected if and only if $s=0$;*
(ii) *if $f=f_1$ or f_2, then $T(A,\mathcal{D})$ is connected if and only if $s=0$ or 1.*

As in the papers previously cited, a lot of calculations are needed in the proofs. But the main methods are algebraic and make full use of the properties of the matrix A. We also provide many figures to illustrate our results.

The paper is organized as follows: In Section 2, we recall several well-known results on the connectedness of self-affine sets and prove a basic lemma; Theorems 1.1 and 1.2 are proved in Section 3, and conclude with an open problem; Theorem 1.3 is proved in Section 4.

2. Preliminaries

In the section, we provide several elementary results on self-affine sets $T(A,\mathcal{D})$. We call the digit set $\mathcal{D}$ *collinear* if $\mathcal{D}=\{d_1,\dots,d_m\}v$ for some non-zero vector $v\in\mathbb{R}^n$ and $d_1<d_2<\cdots<d_m,\ d_i\in\mathbb{R}$; If $d_{i+1}-d_i=1$, then $\mathcal{D}$ is called a *consecutive collinear digit set*. Let $D=\{d_1,\dots,d_m\}$, $\Delta D=D-D=\{d=d_i-d_j:\ d_i,d_j\in D\}$. Then $\mathcal{D}=Dv$ and $\Delta\mathcal{D}=\Delta Dv$. It is easy to see that the connectedness of $T(A,\mathcal{D})$ is invariant under a translation of the digit set, hence we always assume that $d_1=0$ for simplicity. The following criterion for connectedness of $T(A,\mathcal{D})$ was due to [9] or [12].

Lemma 2.1. *A self-affine set $T(A,\mathcal{D})$ with a consecutive collinear digit set $\mathcal{D}=\{0,1,\dots,m\}v$ is connected if and only if $v\in T-T$.*

Let $\mathbb{Z}[x]$ be the set of polynomials with integer coefficients. A polynomial $f(x)\in\mathbb{Z}[x]$ is said to be *expanding* if all its roots have moduli strictly bigger than 1. Note that a matrix $A\in M_n(\mathbb{Z})$ is expanding if and only if its characteristic polynomial is expanding. We say that a monic polynomial $f(x)\in\mathbb{Z}[x]$ with $|f(0)|=c$ has the *Height Reducing Property* (HRP) if there exists $g(x)\in\mathbb{Z}[x]$ such that

$$g(x)f(x)=x^k+a_{k-1}x^{k-1}+\cdots+a_1x\pm c,$$

where $|a_i|\leqslant c-1,\ i=1,\dots,k-1$.

This property was introduced by Kirat and Lau [12] to study the connectedness of self-affine tiles with consecutive collinear digit sets. It was proved that:

Proposition 2.2. *Let $A\in M_n(\mathbb{Z})$ with $|\det(A)|=c$ be expanding and $\mathcal{D}=\{0,1,2,\dots,(c-1)\}v$. If the characteristic polynomial of A has the Height Reducing Property, then $T(A,\mathcal{D})$ is connected.*

In [13], Kirat et al. conjectured that all expanding integer monic polynomials have HRP. Akiyama and Gjini [1,2] confirmed it up to $n=4$. But it is still unclear for the higher dimensions. Recently, He et al. [10] developed an algorithm of polynomials about HRP. It may be a good attempt on this problem.

Denote the characteristic polynomial of A by $f(x)=x^2+bx+c$, where $b,c\in\mathbb{Z}$. We can regard A as the companion matrix of $f(x)$, i.e.,

$$A=\begin{bmatrix}0 & -c\\ 1 & -b\end{bmatrix}.$$

Let $\Delta=b^2-4c$ be the discriminant. Define α_i,β_i by

$$A^{-i}v=\alpha_iv+\beta_iAv,\quad i=1,2,\dots.$$

According to the Hamilton–Cayley theorem $f(A)=A^2+bA+cI=0$ where I is a 2×2 identity matrix, the following consequence is well-known (please refer to [15,16]).

Lemma 2.3. *Let α_i, β_i be defined as the above. Then $c\alpha_{i+2} + b\alpha_{i+1} + \alpha_i = 0$ and $c\beta_{i+2} + b\beta_{i+1} + \beta_i = 0$, i.e.,*

$$\begin{bmatrix}\alpha_{i+1}\\ \alpha_{i+2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1/c & -b/c\end{bmatrix}^i \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}; \quad \begin{bmatrix}\beta_{i+1}\\ \beta_{i+2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1/c & -b/c\end{bmatrix}^i \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}$$

and $\alpha_1 = -b/c, \alpha_2 = (b^2 - c)/c^2; \beta_1 = -1/b, \beta_2 = b/c^2$. Moreover for $\Delta \neq 0$, we have

$$\alpha_i = \frac{c(r_1^{i+1} - r_2^{i+1})}{\Delta^{1/2}} \quad \text{and} \quad \beta_i = \frac{-(r_1^i - r_2^i)}{\Delta^{1/2}},$$

where $r_1 = \frac{-b+\Delta^{1/2}}{2c}$ and $r_2 = \frac{-b-\Delta^{1/2}}{2c}$ are the two roots of $cx^2 + bx + 1 = 0$.

Set

$$\tilde{\alpha} := \sum_{i=1}^{\infty} |\alpha_i|, \quad \tilde{\beta} := \sum_{i=1}^{\infty} |\beta_i|.$$

Then $\tilde{\alpha}$ and $\tilde{\beta}$ are finite numbers as r_1, r_2 have moduli strictly less than 1.

Write $L := \{\gamma v + \delta A v : \gamma, \delta \in \mathbb{Z}\}$, then L is a *lattice* generated by $\{v, Av\}$. For $l \in L \setminus \{0\}$, we call $T + l$ a *neighbor* of T if $T \cap (T + l) \neq \emptyset$. It is clear that $T + l$ is a neighbor of T if and only if $l \in T - T$, hence l can be expressed as

$$l = \sum_{i=1}^{\infty} b_i A^{-i} v \in T - T, \quad \text{where } b_i \in \Delta D.$$

If $T + l$ is a neighbor of T where $l = \sum_{i=1}^{\infty} b_i A^{-i} v := \gamma v + \delta A v$, then

$$|\gamma| \leqslant \max_i |b_i| \tilde{\alpha} \quad \text{and} \quad |\delta| \leqslant \max_i |b_i| \tilde{\beta}. \tag{2.1}$$

By multiplying A on both sides of the expression of l and by using $f(A) = 0$, it follows that $T - (c\delta + b_1)v + (\gamma - b\delta)Av$ is also a neighbor of T. Repeatedly applying this neighbor-generating algorithm, we then can construct a sequence of neighbors: $\{T + l_n\}_{n=0}^{\infty}$, where $l_0 = l, l_n = \gamma_n v + \delta_n A v, n \geqslant 1$ and

$$\begin{bmatrix}\gamma_n\\ \delta_n\end{bmatrix} = A^n \begin{bmatrix}\gamma\\ \delta\end{bmatrix} - \sum_{i=1}^{n} A^{i-1} \begin{bmatrix}b_{n+1-i}\\ 0\end{bmatrix}. \tag{2.2}$$

Moreover, $|\gamma_n| \leqslant \max_i |b_i| \tilde{\alpha}$ and $|\delta_n| \leqslant \max_i |b_i| \tilde{\beta}$ hold for any $n \geqslant 0$.

Lemma 2.4. *If the characteristic polynomial of the expanding matrix A is $x^2 + bx + c$ and that of B is $x^2 - bx + c$, then the self-affine set $T(A, \mathcal{D})$ is connected if and only if $T(B, \mathcal{D})$ is connected where $\mathcal{D}$ is a consecutive collinear digit set.*

Proof. Let $B = -A$ and $T_1 = T(A, \mathcal{D}), T_2 = T(-A, \mathcal{D})$. If $l \in T_1 - T_1$, then

$$l = \sum_{i=1}^{\infty} b_i A^{-i} v = \sum_{i=1}^{\infty} b_{2i} (-A)^{-2i} v + \sum_{i=1}^{\infty} (-b_{2i-1})(-A)^{-2i+1} v.$$

Thus $l \in T_2 - T_2$, and vice versa. □

To get the necessary conditions of Theorems 1.1–1.3, we need the exact values of $\tilde{\alpha}$ and $\tilde{\beta}$.

Lemma 2.5. *Let the characteristic polynomial of the expanding matrix A be $f(x) = x^2 + bx + c$, where b, c are integers and $\Delta = b^2 - 4c \geqslant 0$. Then*

$$\tilde{\alpha} = \begin{cases} \frac{|b|-1}{c-|b|+1} & c > 0, \\ \frac{|b|+1}{|c|-|b|-1} & c < 0; \end{cases} \quad \tilde{\beta} = \begin{cases} \frac{1}{c-|b|+1} & c > 0, \\ \frac{1}{|c|-|b|-1} & c < 0. \end{cases}$$

Proof. Let x_1, x_2 denote the roots of $x^2 + bx + c = 0$.

(1) $c > 0$. If $|x_1| > |x_2|$, then $|\alpha_i| = \frac{c}{|x_1 - x_2|}\left(\frac{1}{|x_2|^{i+1}} - \frac{1}{|x_1|^{i+1}}\right)$, $|\beta_i| = \frac{1}{|x_1 - x_2|}\left(\frac{1}{|x_2|^{i}} - \frac{1}{|x_1|^{i}}\right)$. Hence

$$\tilde{\alpha} = \sum_{i=1}^{\infty} |\alpha_i| = \sum_{i=1}^{\infty} \frac{c}{|x_1 - x_2|}\left(\frac{1}{|x_2|^{i+1}} - \frac{1}{|x_1|^{i+1}}\right) = \frac{|x_1 + x_2| - 1}{(|x_1| - 1)(|x_2| - 1)} = \frac{|b| - 1}{c - |b| + 1};$$

$$\tilde{\beta} = \sum_{i=1}^{\infty} |\beta_i| = \sum_{i=1}^{\infty} \frac{1}{|x_1 - x_2|}\left(\frac{1}{|x_2|^{i}} - \frac{1}{|x_1|^{i}}\right) = \frac{1}{(|x_1| - 1)(|x_2| - 1)} = \frac{1}{c - |b| + 1}.$$

Similarly for $|x_2| > |x_1|$.
If $|x_1| = |x_2| = |b|/2$, by Lemma 2.3 and a simple calculation, it follows that $|\alpha_i| = \frac{i+1}{|x_1|^i}$ and $|\beta_i| = \frac{i}{|x_1|^{i+1}}$. Thus

$$\left(1 - \frac{1}{|x_1|}\right)\tilde{\alpha} = \sum_{i=1}^{\infty} \frac{i+1}{|x_1|^i} - \sum_{j=1}^{\infty} \frac{j+1}{|x_1|^{j+1}} = \frac{2}{|x_1|} + \sum_{\ell=1}^{\infty} \frac{1}{|x_1|^{\ell+1}} = \frac{2}{|x_1|} + \frac{1}{|x_1|(|x_1| - 1)} = \frac{2|x_1| - 1}{|x_1|(|x_1| - 1)};$$

$$(|x_1| - 1)\tilde{\beta} = \sum_{i=1}^{\infty} \frac{i}{|x_1|^i} - \sum_{j=1}^{\infty} \frac{j}{|x_1|^{j+1}} = \frac{1}{|x_1|} + \sum_{\ell=1}^{\infty} \frac{1}{|x_1|^{\ell+1}} = \frac{1}{|x_1|} + \frac{1}{|x_1|(|x_1| - 1)} = \frac{1}{(|x_1| - 1)},$$

which implies $\tilde{\alpha} = \frac{2|x_1| - 1}{(|x_1| - 1)^2} = \frac{|b| - 1}{c - |b| + 1}$ and $\tilde{\beta} = \frac{1}{(|x_1| - 1)^2} = \frac{1}{c - |b| + 1}$.

(2) $c < 0$. Without loss of generality, we can assume $|x_1| \geqslant |x_2|$, then

$$\begin{aligned}\tilde{\alpha} &= \sum_{i=1}^{\infty} |\alpha_i| = \frac{|c|}{|x_1| + |x_2|}\left(\sum_{i=1}^{\infty}\left(\frac{1}{|x_1|^{2i+1}} + \frac{1}{|x_2|^{2i+1}}\right) + \sum_{i=1}^{\infty}\left(\frac{1}{|x_2|^{2i}} - \frac{1}{|x_1|^{2i}}\right)\right)\\ &= \frac{|c|}{|x_1| + |x_2|}\left(\frac{1}{|x_1|(|x_1|^2 - 1)} + \frac{1}{|x_2|(|x_2|^2 - 1)} + \frac{1}{|x_2|^2 - 1} - \frac{1}{|x_1|^2 - 1}\right)\\ &= \frac{|c|}{|x_1| + |x_2|}\left(\frac{1}{|x_2|(|x_2| - 1)} - \frac{1}{|x_1|(|x_1| + 1)}\right)\\ &= \frac{|x_1| - |x_2| + 1}{(|x_1| + 1)(|x_2| - 1)} = \frac{|b| + 1}{|c| - |b| - 1};\end{aligned}$$

$$\begin{aligned}\tilde{\beta} &= \sum_{i=1}^{\infty}|\beta_i| = \frac{1}{|x_1|+|x_2|}\left(\sum_{i=1}^{\infty}\left(\frac{1}{|x_1|^{2i-1}}+\frac{1}{|x_2|^{2i-1}}\right)\right.\\ &\quad \left.+\sum_{i=1}^{\infty}\left(\frac{1}{|x_2|^{2i}}-\frac{1}{|x_1|^{2i}}\right)\right)\\ &= \frac{1}{|x_1|+|x_2|}\left(\frac{|x_1|}{|x_1|^2-1}+\frac{|x_2|}{|x_2|^2-1}+\frac{1}{|x_2|^2-1}-\frac{1}{|x_1|^2-1}\right)\\ &= \frac{1}{|x_1|+|x_2|}\left(\frac{1}{|x_2|-1}+\frac{1}{|x_1|+1}\right)\\ &= \frac{1}{|c|-|b|-1}. \quad \square\end{aligned}$$

3. Consecutive collinear digit set

In the section, we characterize the connectedness of the self-affine sets $T(A,\mathcal{D})$ associated with digit sets $\mathcal{D}=\{0,1,\ldots,m\}v$. The necessary and sufficient conditions are given for $T(A,\mathcal{D})$ to be connected.

Theorem 3.1. *Let the characteristic polynomial of the expanding integer matrix A be $f(x)=x^2+bx+c$ and a digit set $\mathcal{D}=\{0,1,\ldots,m\}v$ where $m\geqslant 1$ is integral and $v\in\mathbb{R}^2$ such that $\{v,Av\}$ are linearly independent. If $\Delta=b^2-4c\geqslant 0$ and the eigenvalues of A have moduli $\geqslant 2$, then*

(i) *if $c=4$, then $T(A,\mathcal{D})$ is connected if and only if $m\geqslant 2$;*
(ii) *otherwise $c\neq 4$, then $T(A,\mathcal{D})$ is connected if and only if*

$$m \geqslant \begin{cases}\max\{c-|b|+1,|b|-1\} & c>0,\\ |c|-|b|-1 & c<0.\end{cases}$$

Proof. Let $T+l$ be a neighbor of T, then $l=\gamma v+\delta Av=\sum_{i=1}^{\infty}b_iA^{-i}v, b_i\in\Delta D=\{0,\pm1,\pm2,\ldots,\pm m\}$. By (2.1), we have

$$|\gamma|\leqslant m\tilde{\alpha};\quad |\delta|\leqslant m\tilde{\beta}. \tag{3.1}$$

Suppose $(T+\ell_1v)\cap(T+\ell_2v)\neq\emptyset$ for $0\leqslant\ell_1<\ell_2\leqslant m$, then $(\ell_2-\ell_1)v=\sum_{i=1}^{\infty}b_iA^{-i}v,\ b_i\in\Delta D$. By (2.2), we obtain $l^*=-((\ell_2-\ell_1)c+b_2)v-(b(\ell_2-\ell_1)+b_1)Av$.

(i) If $c=4$, then $|b|=4$, by (3.1) and Lemma 2.5, the connectedness of $T(A,\mathcal{D})$ can imply that

$$(\ell_2-\ell_1)|b|-m\leqslant|(\ell_2-\ell_1)b+b_1|\leqslant\frac{m}{c-|b|+1}=m,$$

which further implies that $m\geqslant 2(\ell_2-\ell_1)\geqslant 2$.
Conversely, if $b=-4$, then $f(x)=x^2-4x+4$. By using $f(A)=0$ and $Af(A)=0$, we have $A^3-3A^2+4I=0$ and $A^3-A^2=2A(A-I)+2(A-I)-2I$ which yields

$$(A-I)=2A^{-1}(A-I)+2A^{-2}(A-I)-2A^{-2}$$

and

$$I=2A^{-1}+2A^{-2}-2\sum_{i=3}^{\infty}A^{-i}.$$

Then

$$v=2A^{-1}v+2A^{-2}v-2\sum_{i=3}^{\infty}A^{-i}v\in T-T.$$

Therefore T is connected by Lemma 2.1. (see Fig. 1).
If $b=4$, then $f(x)=x^2+4x+4$. By Lemma 2.4, the connectedness of $T(A,\mathcal{D})$ is the same as that of $T(-A,\mathcal{D})$ in which the characteristic polynomial of $-A$ is $f(x)=x^2-4x+4$.

(ii) Necessity: If $c>0$, then by (3.1) and Lemma 2.5, we have

$$\begin{aligned}(\ell_2-\ell_1)c-m&\leqslant|(\ell_2-\ell_1)c+b_2|\\ &\leqslant m\frac{|b|-1}{c-|b|+1};\end{aligned} \tag{3.2}$$

$$\begin{aligned}(\ell_2-\ell_1)|b|-m&\leqslant|(\ell_2-\ell_1)b+b_1|\\ &\leqslant\frac{m}{c-|b|+1}.\end{aligned} \tag{3.3}$$

(3.2) implies that $m\geqslant(\ell_2-\ell_1)(c-|b|+1)\geqslant c-|b|+1$.
Now we prove $m\geqslant|b|-1$. Let x_1,x_2 denote the roots of $x^2+bx+c=0$, if $m<|b|-1$, then $m\leqslant|b|-2\leqslant|x_1|-1+|x_2|-1$. We have

$$\frac{m}{(|x_1|-1)(|x_2|-1)}\leqslant\frac{1}{|x_1|-1}+\frac{1}{|x_2|-1}<2. \tag{3.4}$$

The last strict inequality holds due to the fact that $c>4$ and $|x_1|,|x_2|\geqslant 2$. From (3.3) and (3.4), it follows that

$$\begin{aligned}2&\leqslant|b|-m\leqslant(\ell_2-\ell_1)|b|-m\leqslant\frac{m}{c-|b|+1}\\ &=\frac{m}{(|x_1|-1)(|x_2|-1)}<2,\end{aligned} \tag{3.5}$$

which is a contradiction. Hence $m\geqslant|b|-1$.
If $c<0$, then by (3.1) and Lemma 2.5, we have

$$\begin{aligned}(\ell_2-\ell_1)|c|-m&\leqslant|(\ell_2-\ell_1)c+b_2|\\ &\leqslant m\frac{|b|+1}{|c|-|b|-1}\end{aligned} \tag{3.6}$$

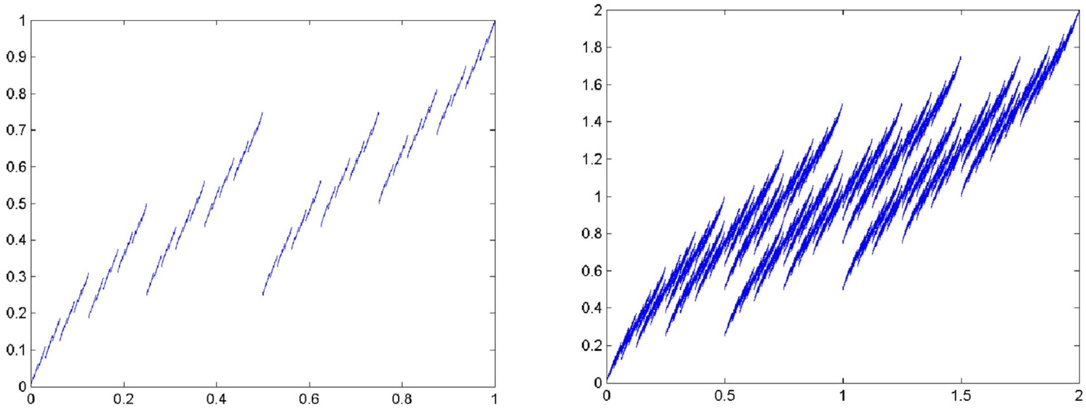
implying that $m\geqslant(\ell_2-\ell_1)(|c|-|b|-1)\geqslant|c|-|b|-1$.
Sufficiency: If $c>0$, and $m\geqslant\max\{c-|b|+1,|b|-1\}$, it suffices to show that $v\in T-T$ by Lemma 2.1.
When $b<0$, by using $f(A)=A^2+bA+cI=0$, we have $A^2+bA-(b+1)I=-(c+b+1)I$, i.e., $(A-I)(A+(b+1)I)=-(c+b+1)I$. It follows that

$$I+(b+1)A^{-1}=-(c+b+1)A^{-1}\sum_{i=1}^{\infty}A^{-i}.$$

Hence

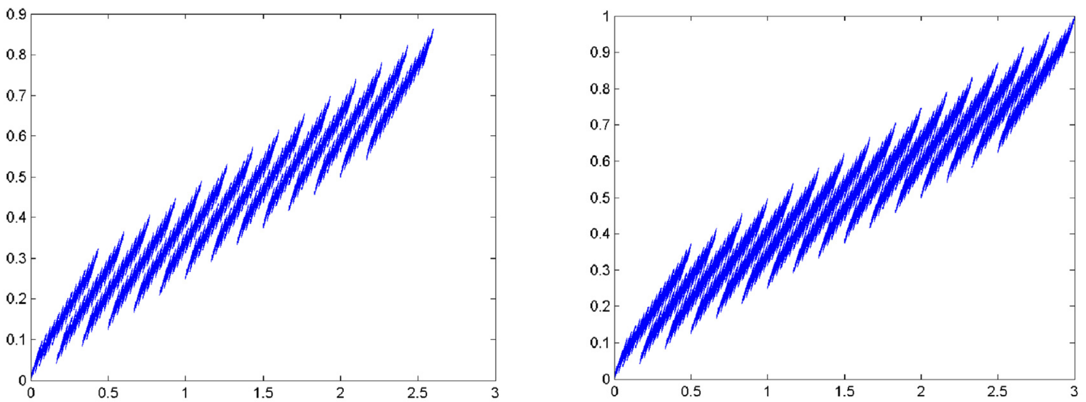
$$\begin{aligned}I&=(-b-1)A^{-1}+\sum_{i=2}^{\infty}-(c+b+1)A^{-i}\\ &=(|b|-1)A^{-1}+\sum_{i=2}^{\infty}-(c-|b|+1)A^{-i}.\end{aligned}$$

Then $v=(|b|-1)A^{-1}v+\sum_{i=2}^{\infty}-(|c|-|b|+1)A^{-i}v\in T-T$, and T is connected. When $b>0$, Lemma 2.4 and the above argument also yield that T is connected. (see Fig. 2).
If $c<0$, suppose $m\geqslant|c|-|b|-1$, then $m\geqslant|b|+1$ (Indeed, if x_1,x_2 is the roots of $x^2+bx+c=0$, without loss of generality, we let $|x_1|\geqslant|x_2|$, then $|c|-|b|-1=|x_1x_2|-(|x_1|-|x_2|)-1=(|x_1|+1)(|x_2|-1)\geqslant|x_1|+1>|x_1|-|x_2|+1=|b|+1$).



(a) m=1 (b) m=2

Fig. 1. (a) is disconnected and (b) is connected where $A = [2, 0; -1, 2]$, $v = (1, 0)^t$.



(a) m=13 (b) m=15

Fig. 2. (a) is disconnected and (b) is connected where $A = [6, 0; -1, 4]$, $v = (1, 0)^t$.

When $b < 0$, by using $f(A) = A^2 + bA + cI = 0$, we have $A^2 + A = (-b+1)(A+I) + (-c+b-1)I$. Then

$$\begin{aligned} I &= (-b+1)A^{-1} + (-c+b-1)\sum_{i=2}^{\infty}(-A)^{-i} \\ &= (|b|+1)A^{-1} + (|c|-|b|-1)\sum_{i=2}^{\infty}(-A)^{-i} \\ &= (|b|+1)A^{-1} + (|c|-|b|-1)\sum_{k=1}^{\infty}A^{-2k} \\ &\quad - (|c|-|b|-1)\sum_{k=1}^{\infty}A^{-2k-1}. \end{aligned}$$

Hence $v \in T - T$ and T is connected by Lemma 2.1. When $b > 0$, Lemma 2.4 and the above argument also yield that T is connected. (see Fig. 3). □

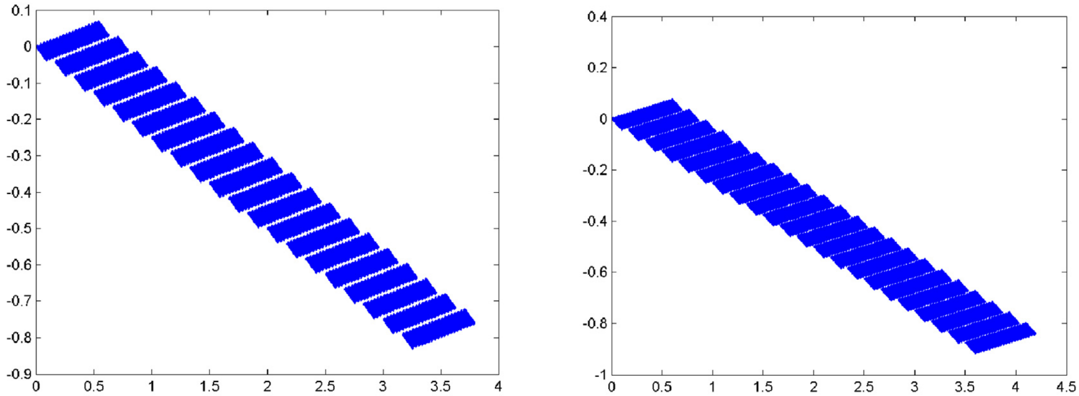
In the proof above, the condition that eigenvalues of A have moduli $\geqslant 2$ is essential. If otherwise, in the case that the moduli of the eigenvalues < 2, e.g., the moduli are close to 1, we have no idea about the conditions for $T(A, \mathcal{D})$ to be connected by estimating $\tilde{\alpha}$ or $\tilde{\beta}$.

On the other hand, if $\Delta = b^2 - 4c < 0$, it is also difficult to compute the exact values of $\tilde{\alpha}$ and $\tilde{\beta}$ in general. However, under certain special situations, the exact values of $\tilde{\alpha}$ and $\tilde{\beta}$ can still be calculated as well.

Theorem 3.2. *Let the characteristic polynomial of A be $f(x) = x^2 + bx + c$ and a digit set $\mathcal{D} = \{0, 1, \ldots, m\}v$ where $m \geqslant 1$ and $v \in \mathbb{R}^2$ such that $\{v, Av\}$ are linearly independent. If $\Delta = b^2 - 4c < 0$, then $T(A, \mathcal{D})$ is connected if and only if*

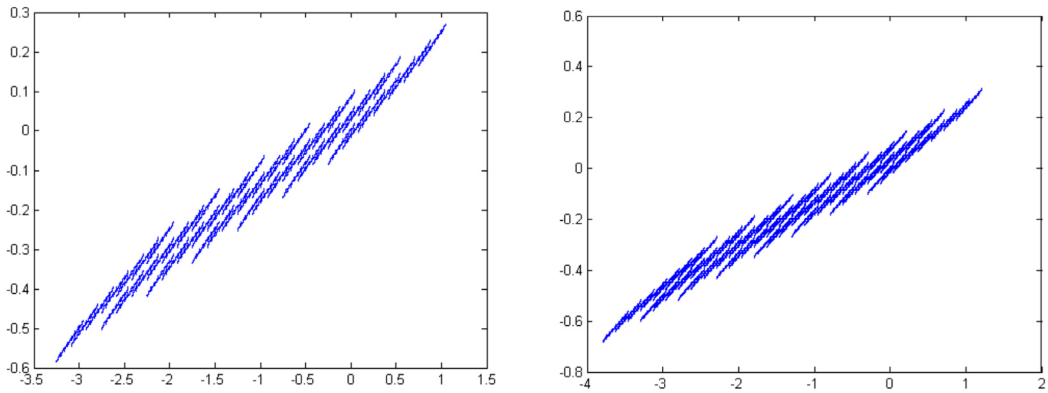
$$m \geqslant \begin{cases} \max\{c - |b| + 1, |b| - 1\} & b^2 = 3c, \\ c - |b| + 1 & b^2 = 2c,\ b^2 = c, \\ c - 1 & b = 0. \end{cases} \tag{3.7}$$

Proof. From Lemma 2.4, we can suppose $b < 0$. Let $r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2c}$ be the complex roots of $cx^2 + bx + 1 = 0$ as in Lemma 2.3 and let $r := |r_1| = |r_2| = \frac{1}{\sqrt{c}}$. Then $r_1 = re^{i\theta}$ and $r_2 = re^{-i\theta}$ where θ is the argument of r_1. We show the necessity first by assuming $T(A, \mathcal{D})$ is connected.



(a) m=19 (b) m=21

Fig. 3. (a) is disconnected and (b) is connected where $A = [6, 0; -1, -4]$, $v = (1, 0)^t$



(a) m=6 (b) m=7

Fig. 4. (a) is disconnected and (b) is connected where $A = [0, -12; 1, -6]$, $v = (1, 0)^t$.

(i) If $b^2 = 3c$, then $\theta = \frac{\pi}{6}$ and

$$|\alpha_i| = \left|\frac{c(r_1^{i+1} - r_2^{i+1})}{\Delta^{1/2}}\right| = \frac{2cr^{i+1}|\sin((i+1)\frac{\pi}{6})|}{\sqrt{c}};$$

$$|\beta_i| = \left|\frac{(r_1^{i+1} - r_2^{i+1})}{\Delta^{1/2}}\right| = \frac{2r^{i+1}|\sin((i+1)\frac{\pi}{6})|}{\sqrt{c}}.$$

Hence

$$\tilde{\alpha} = \sum_{i=1}^{\infty}|\alpha_i| = \sqrt{c}\left((\sqrt{3}r^2 + 2r^3 + \sqrt{3}r^4 + r^5 + r^7)\sum_{j=0}^{\infty} r^{6j}\right) = \frac{3|b|^5 + 6b^4 + 9|b|^3 + 9b^2 + 27}{b^6 - 27};$$

Analogous to (3.2), we have

$$c - m \leqslant (\ell_2 - \ell_1)c - m \leqslant |(\ell_2 - \ell_1)c + b_2| \leqslant m\frac{3|b|^5 + 6b^4 + 9|b|^3 + 9b^2 + 27}{b^6 - 27}$$

which implies that

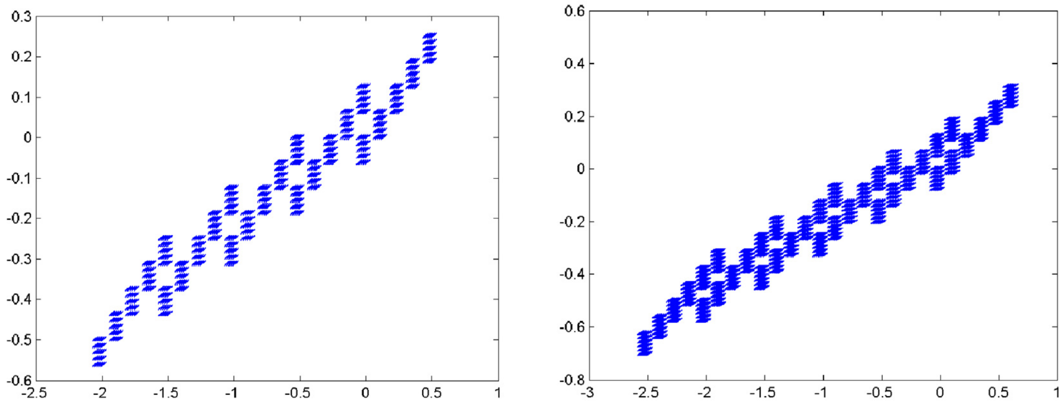
$$m \geqslant \frac{b^6 - 27}{3(b^4 + 3|b|^3 + 6b^2 + 9|b| + 9)} = \frac{b^2}{3} - |b| + \frac{3b^4 + 9|b|^3 + 18b^2 + 27|b| - 27}{3b^4 + 9|b|^3 + 18b^2 + 27|b| + 27} = c - |b| + \frac{3b^4 + 9|b|^3 + 18b^2 + 27|b| - 27}{3b^4 + 9|b|^3 + 18b^2 + 27|b| + 27}.$$

Thus $m \geqslant c - |b| + 1$ as $0 < \frac{3b^4 + 9|b|^3 + 18b^2 + 27|b| - 27}{3b^4 + 9|b|^3 + 18b^2 + 27|b| + 27} < 1$ and m is integral. (see Fig. 4).
If $|b| > 3$, then $c - |b| + 1 > |b| - 1$ is always true; if $|b| = 3$ then $c = 3$, and

$$\tilde{\beta} = \sum_{i=1}^{\infty}|\beta_i| = \frac{1}{\sqrt{c}}\left((r + \sqrt{3}r^2 + 2r^3 + \sqrt{3}r^4 + r^5)\sum_{j=0}^{\infty} r^{6j}\right) = \frac{14}{13}.$$

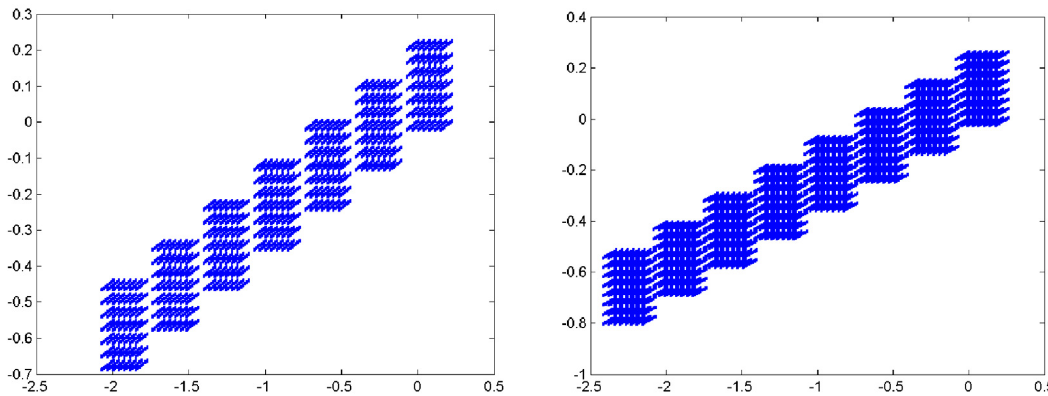
Analogous to (3.3), we have

$$3 - m \leqslant (\ell_2 - \ell_1)3 - m \leqslant |(\ell_2 - \ell_1)b + b_1| \leqslant m\frac{14}{13}$$



(a) m=4 (b) m=5

Fig. 5. (a) is disconnected and (b) is connected where $A = [0, -8; 1, -4]$, $v = (1, 0)^t$.



(a) m=6 (b) m=7

Fig. 6. (a) is disconnected and (b) is connected where $A = [0, -9; 1, -3]$, $v = (1, 0)^t$.

and $m \geqslant \frac{39}{27}$. Therefore $m \geqslant 2 = |b| - 1$.

(ii) If $b^2 = 2c$, then $\theta = \frac{\pi}{4}$ and

$$|\alpha_i| = \left|\frac{c(r_1^{i+1} - r_2^{i+1})}{\Delta^{1/2}}\right| = \frac{2cr^{i+1}|\sin((i+1)\frac{\pi}{4})|}{|b|}.$$

Then

$$\begin{aligned}\tilde{\alpha} = \sum_{i=1}^{\infty}|\alpha_i| &= |b|\left(r^2\sum_{j=0}^{\infty}r^{4j} + \frac{\sqrt{2}}{2}r^3\sum_{k=0}^{\infty}r^{2k}\right)\\ &= \frac{2|b|^3 + 2b^2 + 4}{b^4 - 4}.\end{aligned}$$

Analogous to (3.2), we have

$$\begin{aligned}c - m \leqslant (\ell_2 - \ell_1)c - m &\leqslant |(\ell_2 - \ell_1)c + b_2|\\ &\leqslant m\frac{2|b|^3 + 2b^2 + 4}{b^4 - 4}\end{aligned}$$

implying that

$$\begin{aligned}m &\geqslant \frac{b^4 - 4}{2(b^2 + 2|b| + 2)}\\ &= \frac{b^2}{2} - |b| + \frac{2b^2 + 4|b| - 4}{2b^2 + 4|b| + 4}\\ &= c - |b| + \frac{2b^2 + 4|b| - 4}{2b^2 + 4|b| + 4}.\end{aligned}$$

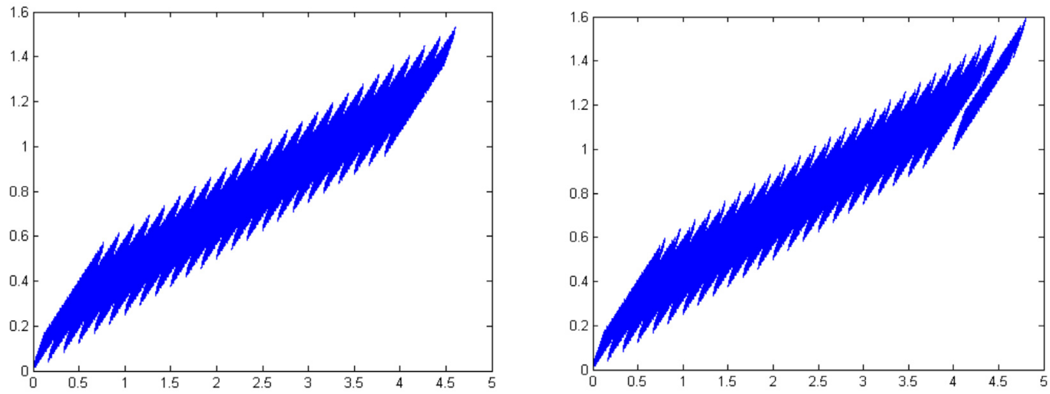
Hence $m \geqslant c - |b| + 1$ as $0 < \frac{2b^2+4|b|-4}{2b^2+4|b|+4} < 1$ and m is integral. (see Fig. 5)

(iii) If $b^2 = c$, then $\theta = \frac{\pi}{3}$. By the similar discussion of (i) above, it follows that

$$\tilde{\alpha} = \sum_{i=1}^{\infty}|\alpha_i| = \frac{b^2 + 1}{|b|^3 - 1}$$

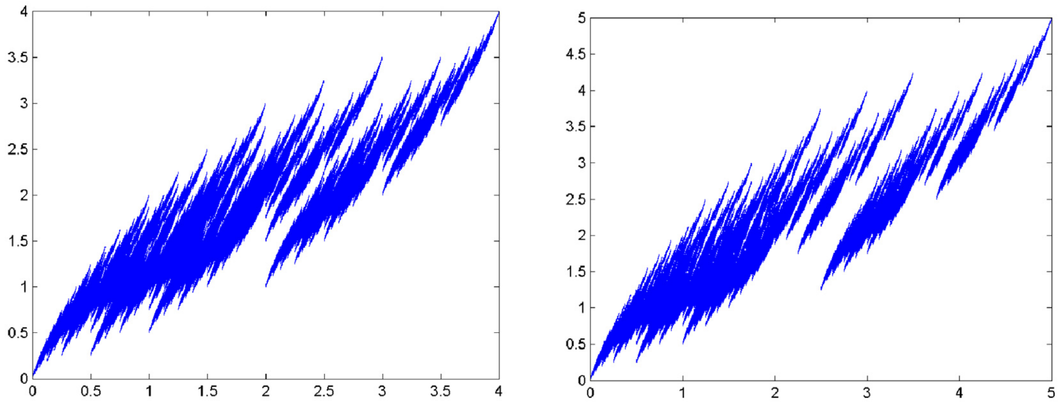
and

$$m \geqslant \frac{|b|^3 - 1}{|b| + 1} = b^2 - |b| + \frac{|b| - 1}{|b| + 1} = c - |b| + \frac{|b| - 1}{|b| + 1}.$$



(a) s=0 (b) s=1

Fig. 7. (a) is connected and (b) is disconnected where $A = [6, 0; -1, 4]$, $v = (1, 0)^t$.



(a) s=1 (b) s=2

Fig. 8. (a) is connected and (b) is disconnected where $A = [2, 0; -1, 2]$, $v = (1, 0)^t$.

Hence $m \geqslant c - |b| + 1$ as $0 < \frac{|b|-1}{|b|+1} < 1$. (see Fig. 6)

(iv) If $b = 0$, then $\theta = \frac{\pi}{2}$. Similarly, we have

$$\tilde{\alpha} = \sum_{i=1}^{\infty} |\alpha_i| = \frac{c}{\sqrt{c}} \sum_{i=1}^{\infty} r^{2i+1} = \frac{1}{c-1}$$

and then $m \geqslant c - 1$.

On the contrary, for the sufficiency, if m satisfies (3.7), then $m \geqslant |b| - 1$ for cases (ii) and (iii). With the similar proof as in Theorem 3.1, we can conclude that $v \in T - T$ and $T(A, \mathcal{D})$ is connected. The connectedness of case (iv) comes from the Proposition 2.2 directly. □

For other unsolved cases, by observing computer graphs, we conclude with the following conjecture.

Conjecture 3.3. Let the characteristic polynomial of A be $f(x) = x^2 + bx + c$ and a digit set $\mathcal{D} = \{0, 1, \ldots, m\}v$ where $m \geqslant 1$ and $v \in \mathbb{R}^2$ such that $\{v, Av\}$ are linearly independent. Then

(i) if $c = |b| = 4$, then $T(A, \mathcal{D})$ is connected if and only if $m \geqslant 2$;

(ii) otherwise, $T(A, \mathcal{D})$ is connected if and only if

$$m \geqslant \begin{cases} \max\{c - |b| + 1, |b| - 1\} & c > 0, |b| \geqslant 2, \\ c - 1 & c > 0,\ |b| \leqslant 1, \\ \max\{|c| - |b| - 1, |b| + 1\} & c < 0. \end{cases}$$

4. Non-consecutive collinear digit set

By previous Section 2, we know that if $A \in M_2(\mathbb{Z})$ is an expanding matrix, then its characteristic polynomial has HRP. Let $\mathcal{D}' = \{0, 1, \ldots, (|\det(A)| - 1)\}v$ be a consecutive collinear digit set with $\#\mathcal{D}' = |\det(A)|$. By Proposition 2.2, then the associated self-affine tile $T(A, \mathcal{D}')$ is always connected. However there are few results on the non-consecutive collinear digit sets. In [15], Leung and Luo first study this case, by checking 10 eligible characteristic polynomials of the A with $|\det A| = 3$ case by case, they obtained a complete characterization for connectedness of $T(A, \mathcal{D})$

with $\mathcal{D} = \{0, 1, m\}v$. In the section, we further study this kind of digit sets in more general situations. Suppose the characteristic polynomial of A is of the form $f(x) = x^2 - (p+q)x + pq$ where $|p|, |q| \geqslant 2$ are integers, and $\mathcal{D} = \{0, 1, \ldots, |pq| - 2, |pq| - 1 + s\}v$. By letting

$$f_1(x) = x^2 \pm 4x + 4 \quad \text{and} \quad f_2(x) = x^2 \pm 7x + 12,$$

we have the following criterion for the connectedness.

Theorem 4.1. *Let the characteristic polynomial of A be $f(x) = x^2 - (p+q)x + pq$ and a digit set $\mathcal{D} = \{0, 1, \ldots, |pq| - 2, |pq| - 1 + s\}v$ where $s \geqslant 0, |p|, |q| \geqslant 2$ are integers and $v \in \mathbb{R}^2$ such that $\{v, Av\}$ are linearly independent. Then*

(i) *if $f \neq f_1, f_2$, then $T(A, \mathcal{D})$ is connected if and only if* $s = 0$;
(ii) *if $f = f_1$ or f_2, then $T(A, \mathcal{D})$ is connected if and only if* $s = 0$ or 1.

Proof. If $s = 0$, then $T(A, \mathcal{D})$ is always connected by Proposition 2.2.

(i) Suppose $T(A, \mathcal{D})$ is connected, then $(T + (|pq| - 1 + s)v) \cap (T + iv) \neq \emptyset$ for some $0 \leqslant i \leqslant |pq| - 2$. Let $r = |pq| - 1 + s - i$, then $rv \in T - T$, i.e., $rv = \sum_{i=1}^{\infty} b_i A^{-i} v$ where $b_i \in \Delta D$. By (2.2), we obtain a new neighbor $T + l^*$ where $l^* = -(rpq + b_2)v + (r(p+q) - b_1)Av$. If $pq > 0$, then by (2.1) and Lemma 2.5, we have

$$\begin{aligned}(1+s)|pq| - (|pq| - 1 + s) &\leqslant r|pq| \\ - (|pq| - 1 + s) &\leqslant |rpq + b_2| \\ &\leqslant (|pq| - 1 + s)\frac{|p| + |q| - 1}{(|p| - 1)(|q| - 1)};\end{aligned} \tag{4.1}$$

$$\begin{aligned}&(1+s)|p+q| - (|pq| - 1 + s) \leqslant |r(p+q) - b_1| \\ &\leqslant \frac{|pq| - 1 + s}{(|p| - 1)(|q| - 1)}.\end{aligned} \tag{4.2}$$

It follows from (4.1) that

$$s \leqslant \frac{|p| + |q| - 2}{|pq| - |p| - |q|}. \tag{4.3}$$

Let

$$\begin{aligned} t &= \frac{|p| + |q| - 1}{(|p| - 1)(|q| - 1)} \\ &= \frac{1}{|p| - 1} + \frac{1}{|q| - 1} + \frac{1}{(|p| - 1)(|q| - 1)}. \end{aligned}$$

It is easy to see that $t < 1$ if $|p|, |q| \geqslant 4$ or one of $|p|, |q|$ is equal to 3 and the other one is larger than 5. Therefore $|p| + |q| - 2 < |pq| - |p| - |q|$, and $s < 1$, i.e., $s = 0$. (see Fig. 7)
If one of $|p|, |q|$ is equal to 2 and the other one is larger than 3, without loss of generality, suppose $|p| = 2$ and $|q| \geqslant 3$. From (4.2) we get

$$s \leqslant \frac{|q|^2 - 2|q| + 2}{|q|^2 - 2} = 1 - \frac{2|q| - 4}{|q|^2 - 2} < 1. \tag{4.4}$$

Hence $s = 0$.
If $pq < 0$, analogous to the (4.1), then

$$\begin{aligned}&(1+s)|pq| - (|pq| - 1 + s) \\ &\quad \leqslant (|pq| - 1 + s)\frac{|p+q| + 1}{|pq| - |p+q| - 1}.\end{aligned} \tag{4.5}$$

We have

$$\begin{aligned} s &\leqslant \frac{|p+q|}{|pq| - |p+q| - 2} \\ &= 1 - \frac{|pq| - 2|p+q| - 2}{|pq| - |p+q| - 2}. \end{aligned} \tag{4.6}$$

Since $pq < 0$, without loss of generality, we let $|p| \geqslant |q|$, it follows that $|pq| - |p+q| - 1 = |pq| - (|p| - |q|) \quad -1 = (|p| + 1)(|q| - 1) \geqslant |p| + 1 > |p| - |q| + 1 = |p+q| + 1$. Thus $|pq| - 2|p+q| - 2 > 0$, and $s = 0$.

(ii) If $f = f_1$, then (4.4) implies that $s \leqslant 1$ (see Fig. 8); if $f = f_2$, then (4.3) implies that $s \leqslant 1$ (see Fig. 9). Conversely, for $s = 1$, let $\Delta D_1 = \{0, \pm 1, \pm 2, \pm 3, \pm 4\}$ and $\Delta D_2 = \{0, \pm 1, \ldots, \pm 12\}$ and let A_1 and A_2 denote the matrices of f_1 and f_2 respectively. We only need to

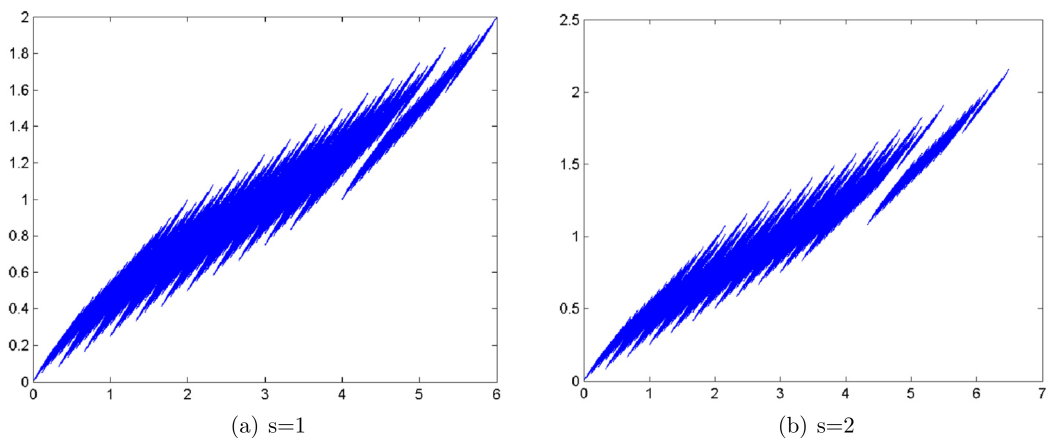


Fig. 9. (a) is connected and (b) is disconnected where $A = [3, 0; -1, 4]$, $v = (1, 0)^t$.

show that $v,\ 2v \in T - T$ (see [12] or [15]). Let $\Delta D_1' = \{0, \pm 1, \pm 2\}$ and $\Delta D_2' = \{0, \pm 1, \ldots, \pm 6\}$. By Theorem 3.1, there exist sequences $\{b_{1i}\}_{i=1}^{\infty}$ where $b_{1i} \in \Delta D_1'$ and $\{b_{2i}\}_{i=1}^{\infty}$ where $b_{2i} \in \Delta D_2'$ such that $v = \sum_{i=1}^{\infty} b_{1i} A_1^{-i} v \in T - T$ and $v = \sum_{i=1}^{\infty} b_{2i} A_2^{-i} v \in T - T$. Moreover, $2b_{1i} \in \Delta D_1$ and $2b_{2i} \in \Delta D_2$. Hence $2v \in T - T$ as well. □

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