Journal of Functional Analysis 272 (2017) 599–612

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Spectral property of self-affine measures on $\mathbb{R}^n$ ☆

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ARTICLE INFO

Article history:
Received 5 February 2016
Accepted 13 October 2016
Available online 17 October 2016
Communicated by L. Gross

MSC:
primary 42C05
secondary 28A80

Keywords:
Self-affine measure
Spectral measure
Hadamard triple
Fourier transform

ABSTRACT

We study spectral properties of the self-affine measure $\mu_{M,\mathcal{D}}$ generated by an expanding integer matrix $M \in M_n(\mathbb{Z})$ and a consecutive collinear digit set $\mathcal{D} = \{0, 1, \dots, q-1\}v$ where $v \in \mathbb{Z}^n \setminus \{0\}$ and $q \geq 2$ is an integer. Some sufficient conditions for $\mu_{M,\mathcal{D}}$ to be a spectral measure or to have infinitely many orthogonal exponentials are given. Moreover, for some special cases, we can obtain a necessary and sufficient condition on the spectrality of $\mu_{M,\mathcal{D}}$. Our study generalizes the one dimensional results proved by Dai, et al. [4,5].

1. Introduction

Let μ be a Borel probability measure with compact support on $\mathbb{R}^n$. We call μ a *spectral measure* if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that the set of complex exponentials

☆ The research is supported in part by the NNSF of China (No. 11171100, No. 11301175, No. 11301322, No. 11571104), the program for excellent talents in Hunan Normal University (No. ET14101), Specialized Research Fund for the Doctoral Program of Higher Education of China (20134402120007), the Fundamental and Frontier Research Project of Chongqing (No. cstc2015jcyjA00035).
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http://dx.doi.org/10.1016/j.jfa.2016.10.011
0022-1236/

$E(\Lambda) := \{e^{2\pi i<\lambda,x>} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. The set Λ is called a *spectrum* for μ. The interest of spectral measures was initialed by Fuglede [9] and his famous conjecture: $\chi_\Omega dx$ is a spectral measure on $\mathbb{R}^n$ if and only if Ω is a translational tile. Later it was proved that the conjecture is false in both directions on $\mathbb{R}^n$ for $n \geq 3$ [25,13,14]; but it is still open for $n = 1, 2$. Moreover, the problem of spectral measures is also very attractive when we replace the Lebesgue measure μ by fractal measures such as self-similar/affine measures.

Let $M \in M_n(\mathbb{Z})$ be an expanding matrix (i.e., all the moduli of eigenvalues of M are strictly larger than one) and $\mathcal{D} \subset \mathbb{Z}^n$ be a finite digit set. Then the maps

$$\phi_d(x) = M^{-1}(x + d), \quad d \in \mathcal{D}$$

are contractive with respect to a suitable norm in $\mathbb{R}^n$ [16] and it is well known that there is a unique nonempty compact set $T := T(M, \mathcal{D})$ satisfying the set-valued equation [11]:

$$T = \bigcup_{d\in\mathcal{D}} \phi_d(T).$$

Such T is called the *self-affine set* (or *attractor*) of the *iterated function system (IFS)* $\{\phi_d\}_{d\in\mathcal{D}}$. A self-affine set can be equipped with a unique invariant probability measure $\mu := \mu_{M,\mathcal{D}}$ defined by

$$\mu = \frac{1}{|\mathcal{D}|} \sum_{d\in\mathcal{D}} \mu \circ \phi_d^{-1},$$

and $\mu_{M,\mathcal{D}}$ is supported on T. We call $\mu_{M,\mathcal{D}}$ a *self-affine measure* [11]. In particular, if M is a multiple of an orthonormal matrix, T and $\mu_{M,\mathcal{D}}$ are often called *self-similar set* and *self-similar measure*, respectively.

It is natural to ask whether $\mu_{M,\mathcal{D}}$ is a spectral measure. Jorgenson and Pederson [12] first studied the spectral property of certain Cantor measures. It was found that the 1/4-Cantor measure $\mu_{1/4}$ on $\mathbb{R}$ is a spectral measure while the 1/3-Cantor measure $\mu_{1/3}$ is not. Hu and Lau [10] further investigated the spectrality of Bernoulli convolutions μ_ρ, and observed that $L^2(\mu_\rho)$ contains an infinite orthonormal set of exponential functions if and only if the contraction ratio ρ is the n-th root of a fraction p/q where p is odd and q is even. Recently, Dai [3] completely settled the problem that the only spectral Bernoulli convolutions are of the contraction ratio $1/2k$. As a generalization of the Bernoulli convolution, Dai, He and Lai [4] studied the self-similar measure generated by a positive integer b and consecutive digits $\mathcal{D} = \{0, 1, \ldots, q-1\}$, it was proved that $L^2(\mu_{b,\mathcal{D}})$ has infinitely many orthogonal exponentials if and only if $\gcd(q, b) > 1$, and $\mu_{b,\mathcal{D}}$ is a spectral measure provided $q|b$. After that, replacing the integer b by any real number $b' > 1$, Dai, He and Lau [5] showed that $\mu_{b',\mathcal{D}}$ is a spectral measure if and only if $b' \in \mathbb{N}$ and $q|b'$. For more general cases such as Moran measures, we refer to [1,2].

However, there are few results on the spectrality of self-affine measures [6–8,17–22], among which Li [17–22] mainly considered the spectrality or non-spectrality of some self-affine measures in low dimensions. For example, Li and Wen [22] studied the measure $\mu_{M,\mathcal{D}}$ generated by $M \in M_2(\mathbb{Z})$ and $\mathcal{D} = \{0, 1\}v$, where $v \in \mathbb{Z}^2 \setminus \{0\}$. They obtained that $\mu_{M,\mathcal{D}}$ admits an infinite orthonormal set if $\det(M)$ is even; if v is the eigenvector of M with eigenvalue ℓ, then $\mu_{M,\mathcal{D}}$ is a spectral measure if and only if ℓ is even.

Motivated by the above results, in this paper, we study the spectrality of self-affine measures on $\mathbb{R}^n$ with consecutive collinear digits. First we need a decomposition of integer matrices, which may be known but we can not find the reference, so we provide a proof in Section 3 (see Lemma 3.5).

Let $M \in M_n(\mathbb{Z}), v \in \mathbb{Z}^n \setminus \{0\}$, and let r be the rank of the set of vectors $\{v, Mv, \ldots, M^{n-1}v\}$. Then there exists a unimodular matrix $B \in M_n(\mathbb{Z})$ such that $Bv = (x_1, \ldots x_r, 0, \ldots, 0)^t$ and

$$BMB^{-1} = \begin{bmatrix} M_1 & C \\ 0 & M_2 \end{bmatrix} \tag{1.1}$$

where $M_1 \in M_r(\mathbb{Z})$, $M_2 \in M_{n-r}(\mathbb{Z})$ and $C \in M_{r,n-r}(\mathbb{Z})$. By making use of this matrix decomposition, we have the main theorem of the paper.

Theorem 1.1. *Let $M \in M_n(\mathbb{Z})$ be an expanding matrix, $\mathcal{D} = \{0, 1, \ldots, q-1\}v$ be a digit set where $q \geq 2$ is an integer and $v \in \mathbb{Z}^n \setminus \{0\}$. If the set of vectors $\{v, Mv, \ldots, M^{n-1}v\}$ has rank r and M, M_1 are expressed as in (1.1), then*

(i) $\mu_{M,\mathcal{D}}$ has infinitely many orthogonal exponentials if $\gcd(q, \det(M_1)) > 1$;
(ii) $\mu_{M,\mathcal{D}}$ is a spectral measure if $q | \det(M_1)$.

In particular, if the characteristic polynomial of M_1 is of the simple form $f(x) = x^r + c$, then we have the following necessary and sufficient condition on the spectrality of $\mu_{M,\mathcal{D}}$.

Theorem 1.2. *Under the same assumption of Theorem 1.1, if the characteristic polynomial of M_1 is $f(x) = x^r + c$, then*

(i) $\mu_{M,\mathcal{D}}$ has infinitely many orthogonal exponentials if and only if $\gcd(q, \det(M_1)) > 1$;
(ii) $\mu_{M,\mathcal{D}}$ is a spectral measure if and only if $q | \det(M_1)$.

According to the dependence of the set of vectors $\{v, Mv, \ldots, M^{n-1}v\}$, we will prove the above theorems in two steps: $r = n$ (Theorems 3.2, 3.4) and $r < n$ (Theorem 3.6). The main idea is to use the techniques of linear algebra and matrix analysis.

If the rank $r = 1$, then v becomes an eigenvector of M and $M_1 = \ell$ with characteristic polynomial $f(x) = x - l$, by Theorem 1.2, the following corollary is immediate.

Corollary 1.3. *Under the same assumption of Theorem 1.1, if $r = 1$, in this case v is an eigenvector of M and $M_1 = \ell$ is the corresponding eigenvalue, then*

(i) $\mu_{M,\mathcal{D}}$ *has infinitely many orthogonal exponentials if and only if* $\gcd(q,\ell) > 1$*;*
(ii) $\mu_{M,\mathcal{D}}$ *is a spectral measure if and only if* $q|\ell$*.*

The organization of the paper is as follows: we recall several basic concepts and lemmas in Section 2 and prove the main results orderly in Section 3.

2. Preliminaries

In this section, we give some preliminary definitions and lemmas that we need in proving our main results.

Definition 2.1. Let $M \in M_n(\mathbb{Z})$ be an expanding matrix. Let $\mathcal{D}$ and S be finite subsets of $\mathbb{Z}^n$ with the same cardinality q. We say $(M^{-1}\mathcal{D}, S)$ is an integral compatible pair if the matrix

$$H = \frac{1}{\sqrt{q}} \left[e^{2\pi i < M^{-1}d, s>} \right]_{d\in\mathcal{D}, s\in S}$$

is unitary, i.e., $H^*H = I$, where H^* denotes the transposed conjugate of H. At this time, $(M, \mathcal{D}, S)$ is called Hadamard triple.

It is fairly easy to construct an infinite mutually orthogonal set of exponential functions using the Hadamard triple assumption. However, to check these exponentials form a Fourier basis for $L^2(\mu)$ is a much more difficult task. Jorgensen and Pedersen conjectured that all Hadamard triples will generate self-affine spectral measures. The conjecture was solved in dimension one by Laba and Wang [15]. Under various additional conditions, it was also valid in the high dimensions (see [8,23,24]). Until recently, Dutkay, Haussermann and Lai [7] have completely proved the conjecture.

Lemma 2.1. *([7]) Let* $(M, \mathcal{D}, S)$ *be a Hadamard triple. Then the self-affine measure* $\mu_{M,\mathcal{D}}$ *is a spectral measure.*

The Fourier transform of $\mu_{M,\mathcal{D}}$ plays a key role in studying the spectrality of the measure. We recall the definition by

$$\hat{\mu}_{M,\mathcal{D}}(\xi) := \int e^{2\pi i <x,\xi>} d\mu_{M,\mathcal{D}}(x) = \prod_{j=1}^{\infty} m_{\mathcal{D}}(M^{*-j}\xi), \quad \xi \in \mathbb{R}^n \tag{2.2}$$

where $m_{\mathcal{D}}(\cdot) = \frac{1}{|\mathcal{D}|}\sum_{d\in\mathcal{D}} e^{2\pi i\langle d,\cdot\rangle}$ is the mask polynomial of the digit set $\mathcal{D}$. It is easy to see that $m_{\mathcal{D}}$ is a $\mathbb{Z}^n$-periodic function for $\mathcal{D} \subset \mathbb{Z}^n$. Let $Z_{\mathcal{D}}^n := \{x \in [0,1)^n : m_{\mathcal{D}}(x) = 0\}$ be the zero set of $m_{\mathcal{D}}$ in $[0,1)^n$. Then we have the following useful criterion for the existence of infinitely many orthogonal systems.

Lemma 2.2. *([19]) Let $M \in M_n(\mathbb{Z})$ be an expanding matrix and $\mathcal{D} \subset \mathbb{Z}^n$ be a finite digit set, if there exist $\alpha \in Z_{\mathcal{D}}^n$ and $\ell \in \mathbb{N}$ such that $M^{*\ell}\alpha \in \mathbb{Z}^n$. Then there are infinitely many orthogonal exponentials $E(\Lambda)$ in $L^2(\mu_{M,\mathcal{D}})$ with $\Lambda \subseteq \mathbb{Z}^n$.*

Let M, $\widetilde{M}$ be $n \times n$ integer matrices, and the finite sets $\mathcal{D}, S, \widetilde{\mathcal{D}}, \widetilde{S}$ be in $\mathbb{Z}^n$. We say that two triples $(M, \mathcal{D}, S)$ and $(\widetilde{M}, \widetilde{\mathcal{D}}, \widetilde{S})$ are *conjugate* (through the matrix B) if there exists an integer matrix B such that $\widetilde{M} = B^{-1}MB$, $\widetilde{\mathcal{D}} = B^{-1}\mathcal{D}$ and $\widetilde{S} = B^*S$. The following lemma is trivial.

Lemma 2.3. *Suppose that $(M, \mathcal{D}, S)$ and $(\widetilde{M}, \widetilde{\mathcal{D}}, \widetilde{S})$ are two conjugate triples, through the matrix B. Then*

(i) if $(M, \mathcal{D}, S)$ is a Hadamard triple then so is $(\widetilde{M}, \widetilde{\mathcal{D}}, \widetilde{S})$;

(ii) $\mu_{M,\mathcal{D}}$ is a spectral measure with spectrum Λ if and only if $\mu_{\widetilde{M},\widetilde{\mathcal{D}}}$ is a spectral measure with spectrum $B^*\Lambda$;

(iii) $\mu_{M,\mathcal{D}}$ has infinitely many orthogonal exponentials $E(\Lambda)$ in $L^2(\mu_{M,\mathcal{D}})$ if and only if $\mu_{\widetilde{M},\widetilde{\mathcal{D}}}$ has infinitely many orthogonal exponentials $E(B^*\Lambda)$ in $L^2(\mu_{\widetilde{M},\widetilde{\mathcal{D}}})$.

Proof. The proofs of (i) and (ii) can be found in [8] (or [7]). In fact, it is easy to see that $E(\Lambda)$ is an orthogonal set in $L^2(\mu_{M,\mathcal{D}})$ if and only if $(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}_{M,\mathcal{D}}) := \{\xi \in \mathbb{R}^n : \hat{\mu}_{M,\mathcal{D}}(\xi) = 0\}$. Let $\lambda_1, \lambda_2 \in \Lambda$, then

$$
\begin{aligned}
m_{\widetilde{\mathcal{D}}}(\widetilde{M}^{*-j}B^*(\lambda_1 - \lambda_2)) &= m_{\widetilde{\mathcal{D}}}(B^*M^{*-j}(\lambda_1 - \lambda_2)) \\
&= \frac{1}{|\mathcal{D}|} \sum_{d \in \mathcal{D}} e^{2\pi i \langle B^{-1}d, B^*M^{*-j}(\lambda_1 - \lambda_2)\rangle} = m_{\mathcal{D}}(M^{*-j}(\lambda_1 - \lambda_2)).
\end{aligned}
$$

Hence by (2.2), $\lambda_1 - \lambda_2 \in \mathcal{Z}(\hat{\mu}_{M,\mathcal{D}})$ if and only if $B^*\lambda_1 - B^*\lambda_2 \in \mathcal{Z}(\hat{\mu}_{\widetilde{M},\widetilde{\mathcal{D}}})$, proving (iii). □

3. Main results

In this section, we first consider the case that $\{v, Mv, \ldots, M^{n-1}v\}$ are linearly independent for expanding matrix $M \in M_n(\mathbb{Z})$ and $v \in \mathbb{Z}^n \setminus \{0\}$. We will get the sufficient conditions for $\mu_{M,\mathcal{D}}$ to have infinitely many orthogonal exponentials or to be a spectral measure that only depend on the value of $\det(M)$. The following simple lemma is an important tool in constructing a suitable conjugate Hadamard triple.

Lemma 3.1. *Let $M \in M_n(\mathbb{Z})$ be an integer matrix with characteristic polynomial $f(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ and $v = (x_1, \ldots, x_n)^t \in \mathbb{Z}^n \setminus \{0\}$. If the set of vectors $\{v, Mv, \ldots, M^{n-1}v\}$ is linearly independent, then there exists an integer matrix B, such that*

$$\widetilde{M} := \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} = B^{-1}MB \tag{3.3}$$

and $B^{-1}v = (0, \ldots, 0, 1)^t$.

Proof. Let $B = \big[M^{n-1}v, M^{n-2}v, \ldots, Mv, v\big]$, then B is invertible and $B(0, \ldots, 0, 1)^t = v$. Hence $B^{-1}v = (0, \ldots, 0, 1)^t$. To prove (3.3), we only need to show that $B\widetilde{M} = MB = \big[M^n v, M^{n-1}v, \ldots, M^2 v, Mv\big]$. Indeed,

$$\begin{aligned} B\widetilde{M} &= \big[M^{n-1}v, M^{n-2}v, \ldots, Mv, v\big] \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= \left[\sum_{i=1}^{n} -a_i M^{n-i} v, M^{n-1}v, \ldots, M^2 v, Mv\right] \\ &= \left[(-\sum_{i=1}^{n} a_i M^{n-i})v, M^{n-1}v, \ldots, M^2 v, Mv\right]. \end{aligned} \tag{3.4}$$

Since $f(x)$ is the characteristic polynomial of M, it follows that $f(M) = M^n + a_1 M^{n-1} + \cdots + a_{n-1}M + a_n I = 0$ and $M^n = -\sum_{i=1}^{n} a_i M^{n-i}$. By (3.4), we have $B\widetilde{M} = \big[M^n v, M^{n-1}v, \ldots, M^2 v, Mv\big] = MB$. $\square$

Theorem 3.2. *Let* $\mathcal{D} = \{0, 1, \ldots, q-1\}v$ *with* $q \geq 2, v \in \mathbb{Z}^n \setminus \{0\}$, *and let* $M \in M_n(\mathbb{Z})$ *be an expanding matrix. If* $\{v, Mv, \ldots, M^{n-1}v\}$ *is linearly independent, then*

(i) $\mu_{M,\mathcal{D}}$ *has infinitely many orthogonal exponentials if* $\gcd(q, \det(M)) > 1$*;*

(ii) $\mu_{M,\mathcal{D}}$ *is a spectral measure if* $q | \det(M)$*.*

Proof. Suppose $f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ is the characteristic polynomial of M. Let $\widetilde{M}$ be as in (3.3) and $\widetilde{v} = (0, \ldots, 0, 1)^t$. By Lemmas 2.3 and 3.1, it suffices to prove the theorem for the measure $\mu_{\widetilde{M},\widetilde{\mathcal{D}}}$ which is generated by $\widetilde{M}$ and $\widetilde{\mathcal{D}} = \{0, 1, \ldots, q-1\}\widetilde{v}$.

(i) For $\xi = (\xi_1, \ldots, \xi_n)^t \in \mathbb{R}^n$, note that

$$m_{\widetilde{\mathcal{D}}}(\xi) = \frac{1}{q} \sum_{\widetilde{d} \in \widetilde{\mathcal{D}}} e^{2\pi i \langle \widetilde{d}, \xi \rangle} = \frac{1}{q}(1 + e^{2\pi i \xi_n} + e^{2\pi i 2\xi_n} + \cdots + e^{2\pi i (q-1)\xi_n}).$$

Then

$$
\begin{aligned}
Z_{\widetilde{\mathcal{D}}}^n &:= \{\xi \in [0,1)^n : m_{\widetilde{\mathcal{D}}}(\xi) = 0\} \\
&= \{(\xi_1, \ldots, \xi_{n-1}, j/q)^t : \xi_1, \ldots, \xi_{n-1} \in [0,1), j = 1, \ldots, q-1\}.
\end{aligned}
$$

Suppose $\gcd(q, \det(M)) = s > 1$, let $\alpha := (0, \ldots, 0, 1/s)^t \in \mathbb{R}^n$, then $\alpha \in Z_{\widetilde{\mathcal{D}}}^n$ and

$$
\widetilde{M}^*\alpha = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \frac{-a_n}{s} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.
$$

As $a_n = (-1)^n \det(M) = (-1)^n \det(\widetilde{M})$, then $\frac{-a_n}{s} \in \mathbb{Z}$ and $\widetilde{M}^*\alpha \in \mathbb{Z}^n$. Lemma 2.2 implies that $\mu_{\widetilde{M},\widetilde{\mathcal{D}}}$ has infinitely many orthogonal exponentials.

(ii) If $q | \det(M)$, let $u = (\frac{-a_n}{q}, 0, \ldots, 0)^t$, $\widetilde{S} = \{0, 1, \ldots, q-1\}u$, then $|\widetilde{D}| = |\widetilde{S}| = q$ and $\widetilde{S} \subset \mathbb{Z}^n$. Moreover,

$$
\widetilde{M}^{-1}\widetilde{v} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{-1}{a_n} \\ 1 & 0 & \cdots & 0 & \frac{-a_1}{a_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{-a_{n-2}}{a_n} \\ 0 & 0 & \cdots & 1 & \frac{-a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{a_n} \\ \frac{-a_1}{a_n} \\ \vdots \\ \frac{-a_{n-2}}{a_n} \\ \frac{-a_{n-1}}{a_n} \end{bmatrix}.
$$

It yields that

$$
H = \frac{1}{\sqrt{q}} \left[e^{2\pi i \langle \widetilde{M}^{-1}\widetilde{d}, \widetilde{s} \rangle} \right]_{\widetilde{d} \in \widetilde{\mathcal{D}}, \widetilde{s} \in \widetilde{S}} = \frac{1}{\sqrt{q}} \left[e^{2\pi i \frac{k\ell}{q}} \right]_{k,\ell \in \{0,1,\ldots,q-1\}}
$$

is unitary. Hence $(\widetilde{M}, \widetilde{\mathcal{D}}, \widetilde{S})$ is a Hadamard triple, and $\mu_{\widetilde{M},\widetilde{\mathcal{D}}}$ is a spectral measure by Lemma 2.1. □

In particular, if the characteristic polynomial of M is of the simple form: $f(x) = x^n + c$, we obtain a sufficient and necessary condition for $\mu_{M,\mathcal{D}}$ to have infinitely many orthogonal exponentials or to be a spectral measure. The following lemma was due to Dai, He and Lau [5].

Lemma 3.3. *Let $\mu = \mu_1 * \mu_2$ be the convolution of two probability measures μ_i, $i = 1, 2$, and they are not Dirac measures. Suppose that $E(\Lambda)$ is an orthogonal set of μ_1 with $0 \in \Lambda$, then $E(\Lambda)$ is also an orthogonal set of μ, but it cannot be a spectrum of μ.*

Theorem 3.4. *Under the same assumption of Theorem 3.2 and the matrix M with characteristic polynomial $f(x) = x^n + c$, then*

(i) $\mu_{M,\mathcal{D}}$ has infinitely many orthogonal exponentials if and only if $\gcd(q, \det(M)) > 1$;
(ii) $\mu_{M,\mathcal{D}}$ is a spectral measure if and only if $q | \det(M)$.

Proof. Likewise Theorem 3.2, we consider the conjugate situation where the matrix

$$\widetilde{M} := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c & 0 & 0 & \cdots & 0 \end{bmatrix} = B^{-1}MB$$

and digit set $\widetilde{\mathcal{D}} = \{0, 1, \dots, q-1\}\widetilde{v}$ where $\widetilde{v} = (0, \dots, 0, 1)^t$ as in (3.3). Then the sufficiencies of (i) and (ii) come from Theorem 3.2, we now prove the necessities.

(i) Let $\mathcal{Z}_0 := \{\xi = (\xi_1, \dots, \xi_n)^t \in \mathbb{R}^n : m_{\widetilde{\mathcal{D}}}(\xi) = 0\}$ be the zero set of the mask function $m_{\widetilde{\mathcal{D}}}$. Then

$$\mathcal{Z}_0 = \left\{ \left(\xi_1, \dots, \xi_{n-1}, k + \frac{i}{q}\right)^t \in \mathbb{R}^n : \xi_1, \dots, \xi_{n-1} \in \mathbb{R}, k \in \mathbb{Z}, 1 \le i \le q-1 \right\}.$$

For $1 \le j \le n$, by letting $\xi_n = k + \frac{i}{q}$ as above, we observe that

$$\mathcal{Z}_j := \widetilde{M}^{*j}(\mathcal{Z}_0) = \left\{ \begin{pmatrix} -c\xi_{n-j+1} \\ \vdots \\ -c\xi_{n-1} \\ -c\xi_n \\ \xi_1 \\ \vdots \\ \xi_{n-j} \end{pmatrix} : \xi_1, \dots, \xi_{n-1} \in \mathbb{R}, \xi_n = k + \frac{i}{q}, k \in \mathbb{Z}, 1 \le i \le q-1 \right\}$$

$$\subset \left\{ \begin{pmatrix} \xi'_1 \\ \vdots \\ \xi'_{j-1} \\ k' - \frac{ci}{q} \\ \xi'_{j+1} \\ \vdots \\ \xi'_n \end{pmatrix} : \xi'_1, \dots, \xi'_{j-1}, \xi'_{j+1}, \dots, \xi'_n \in \mathbb{R}, k' \in \mathbb{Z}, 1 \le i \le q-1 \right\}.$$

By (2.2), the zero set $\mathcal{Z}(\widehat{\mu}_{\widetilde{M},\widetilde{\mathcal{D}}})$ of the Fourier transform $\widehat{\mu}_{\widetilde{M},\widetilde{\mathcal{D}}}$ can be written as

$$\mathcal{Z}(\widehat{\mu}_{\widetilde{M},\widetilde{\mathcal{D}}}) = \bigcup_{j=1}^{\infty} \mathcal{Z}_j.$$

If $\gcd(q, \det(M)) = 1$, it follows from $c = (-1)^n \det(M)$ that

$$\{k' - \frac{ci}{q} : k' \in \mathbb{Z}, 1 \le i \le q-1\} \subset \{k + \frac{i}{q} : k \in \mathbb{Z}, 1 \le i \le q-1\}.$$

Hence $\mathcal{Z}_n \subset \mathcal{Z}_0$, which shows that

$$\mathcal{Z}(\widehat{\mu}_{\widetilde{M},\widetilde{\mathcal{D}}}) = \bigcup_{j=1}^{n} \mathcal{Z}_j. \tag{3.5}$$

We now prove that there are only finite mutually orthogonal exponentials in $L^2(\mu_{\widetilde{M},\widetilde{\mathcal{D}}})$. If otherwise, there exists a mutually orthogonal set $E(\Lambda)$ with $\Lambda = \{\tau_\ell\}_{\ell=1}^{\infty}$ for $\mu_{\widetilde{M},\widetilde{\mathcal{D}}}$, we may assume $\tau_1 = 0$ so that $\Lambda \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu}_{\widetilde{M},\widetilde{\mathcal{D}}})$. By (3.5), there exists $1 \le j_0 \le n$ such that $\mathcal{Z}_{j_0}$ contains infinitely many elements of Λ. Without loss of generality, we let $\mathcal{Z}_1$ contain infinite elements of Λ, say $\{\tau_{\ell_i}\}_{i=1}^{\infty}$. According to the form of $\mathcal{Z}_1$ above, there exist an $i_0 \in \{1, \dots, q-1\}$ and a subsequence $\{\tau_{\ell_{i_j}}\}_{j=1}^{\infty}$ satisfying

$$\left\{\tau_{\ell_{i_j}}\right\}_{j=1}^{\infty} \subset \left\{ \left(k - \frac{ci_0}{q}, \xi_2, \dots, \xi_n\right)^t : \xi_2, \dots, \xi_n \in \mathbb{R}, k \in \mathbb{Z} \right\}.$$

Then for $j \ge 2$, the difference

$$\tau_{\ell_{i_j}} - \tau_{\ell_{i_1}} \in \left\{ (k_1, \xi_2', \dots, \xi_n')^t : \xi_2', \dots, \xi_n' \in \mathbb{R}, k_1 \in \mathbb{Z} \right\}.$$

Hence $\tau_{\ell_{i_j}} - \tau_{\ell_{i_1}} \notin \mathcal{Z}_1$ for all $j \ge 2$. Since $(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu}_{\widetilde{M},\widetilde{\mathcal{D}}})$, it follows that

$$\left\{\tau_{\ell_{i_j}} - \tau_{\ell_{i_1}}\right\}_{j=2}^{\infty} \subset \bigcup_{j=2}^{n} \mathcal{Z}_j.$$

By continuing the above process n times, finally we obtain an infinite sequence of differences $\{\tau_{\ell_m} - \tau_{\ell^*}\}_{m=1}^{\infty}$ such that

$$\tau_{\ell_m} - \tau_{\ell^*} \in \left\{ (k_1, k_2, \dots, k_n)^t : k_i \in \mathbb{Z}, i = 1, \dots, n \right\}.$$

That means $\tau_{\ell_m} - \tau_{\ell^*} \notin \mathcal{Z}(\widehat{\mu}_{\widetilde{M},\widetilde{\mathcal{D}}})$, which is impossible. Therefore, there are only finite mutually orthogonal exponentials, and $\mu_{\widetilde{M},\widetilde{\mathcal{D}}}$ is not a spectral measure.

(ii) As $c = (-1)^n \det(M)$, we only need to prove the $\mu_{\widetilde{M},\widetilde{\mathcal{D}}}$ is a non-spectral measure for $\gcd(q, c) = d$ where $1 < d < q$. Let $q = q'd$, $c = c'd$, then $\gcd(c', q') = 1$. Denote by

$$\widetilde{\mathcal{D}} = \{0, 1, \dots, d-1\}\widetilde{v} \oplus d\{0, 1, \dots, q'-1\}\widetilde{v} := \widetilde{\mathcal{D}}_1 \oplus \widetilde{\mathcal{D}}_2.$$

Let

$$\mu_1 = \delta_{\widetilde{M}^{-1}\widetilde{\mathcal{D}}} * \cdots * \delta_{\widetilde{M}^{-n}\widetilde{\mathcal{D}}} * \delta_{\widetilde{M}^{-(n+1)}\widetilde{\mathcal{D}}_1} * \delta_{\widetilde{M}^{-(n+2)}\widetilde{\mathcal{D}}} * \cdots \quad \text{and} \quad \mu_2 = \delta_{\widetilde{M}^{-(n+1)}\widetilde{\mathcal{D}}_2}$$

where $\delta_E = \frac{1}{\#E}\Sigma_{e\in E}\delta_e$ for a finite set E and Dirac mass measure δ_e at the point e, and the infinite convolutions converge in the weak sense. Then we have $\mu_{\widetilde{M},\widetilde{\mathcal{D}}} = \mu_1 * \mu_2$.

It can be seen that

$$\mathcal{Z}(\widehat{\delta}_{\widetilde{M}^{-1}\widetilde{\mathcal{D}}}) = \{(-c\frac{m}{q}, \eta_1, \ldots, \eta_{n-1})^t : m \in \mathbb{Z}, q \nmid m, \eta_1, \ldots, \eta_{n-1} \in \mathbb{R}\}$$

and

$$\mathcal{Z}(\widehat{\mu}_2) = \{(c^2\frac{m'}{dq'}, \eta'_1, \ldots, \eta'_{n-1})^t : m' \in \mathbb{Z}, q' \nmid m', \eta'_1, \ldots, \eta'_{n-1} \in \mathbb{R}\}.$$

Write $c^2\frac{m'}{dq'} = -c\frac{-cm'}{q}$. Since $\frac{-cm'}{q} = \frac{-cm'}{dq'} = \frac{-c'm'}{q'}$, $\gcd(c', q') = 1$ and $q' \nmid m'$, it follows that $q \nmid -cm'$, and $\mathcal{Z}(\widehat{\mu}_2) \subset \mathcal{Z}(\widehat{\delta}_{\widetilde{M}^{-1}\widetilde{\mathcal{D}}})$. Hence $\mathcal{Z}(\widehat{\mu}_{\widetilde{M},\widetilde{\mathcal{D}}}) = \mathcal{Z}(\widehat{\mu}_1)$. This shows that any orthogonal set $E(\Lambda)$ of $\mu_{\widetilde{M},\widetilde{\mathcal{D}}}$ is an orthogonal set of μ_1, proving that $\mu_{\widetilde{M},\widetilde{\mathcal{D}}}$ is a non-spectral measure by Lemma 3.3. □

Example 3.1. Let

$$M = \begin{bmatrix} 2 & 6 & 4 \\ -1 & 2 & 2 \\ -1 & -1 & -4 \end{bmatrix}$$

and $\mathcal{D} = \{0, 1, \ldots, q-1\}v$ where $q \geq 2, v = (0, 0, 1)^t$. Then M is an expanding matrix and $\{v, Mv, M^2v\}$ is linearly independent. Moreover, the characteristic polynomial of M is $f(x) = x^3 + 36$. Hence by Theorem 3.4, we have

(i) $\mu_{M,\mathcal{D}}$ has infinitely many orthogonal exponentials if and only if $\gcd(36, q) > 1$;

(ii) $\mu_{M,\mathcal{D}}$ is a spectral measure if and only if $q|36$.

On the other hand, if the set $\{v, Mv, \ldots, M^{n-1}v\}$ is linearly dependent, by using some techniques of linear algebra, we can reduce the dimension and get the similar results as Theorems 3.2 and 3.4.

Lemma 3.5. *Let $v \in \mathbb{Z}^n \setminus \{0\}$ and $M \in M_n(\mathbb{Z})$. If $\{v, Mv, \ldots, M^{n-1}v\}$ is linearly dependent with rank $r < n$, then there exists a unimodular matrix $B \in M_n(\mathbb{Z})$ such that $Bv = (x_1, \ldots, x_r, 0, \ldots, 0)^t$ and*

$$\widetilde{M} = BMB^{-1} = \begin{bmatrix} M_1 & C \\ 0 & M_2 \end{bmatrix} \tag{3.6}$$

where $M_1 \in M_r(\mathbb{Z})$, $M_2 \in M_{n-r}(\mathbb{Z})$ and $C \in M_{r,n-r}(\mathbb{Z})$.

Proof. Let $A = [M^{r-1}v, \ldots, Mv, v] = [a_{ij}]_{1\leq i\leq n, 1\leq j\leq r}$. It is well known that by a series of elementary (unimodular) row operations on A which is equivalent to multiple a unimodular B from the left side of A, we can get $BA = \begin{bmatrix} A'_{r,r} \\ 0_{n-r,r} \end{bmatrix} \in M_{n,r}(\mathbb{Z})$, i.e.,

$$BA = B\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nr} \end{bmatrix} = \begin{bmatrix} a'_{11} & a'_{12} & \cdots & a'_{1r} \\ 0 & a'_{22} & \cdots & a'_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a'_{rr} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{3.7}$$

Now we show that the matrix B is the desired one. As $v = (a_{1r}, a_{2r}, \ldots, a_{nr})^t$, it follows from (3.7) that $Bv = (a'_{1r}, \ldots a'_{rr}, 0, \ldots, 0)^t := (x_1, \ldots, x_r, 0, \ldots, 0)^t$.

Suppose

$$BMB^{-1} = \begin{bmatrix} M_{r,r} & C_{r,n-r} \\ D_{n-r,r} & N_{n-r,n-r} \end{bmatrix}.$$

Then

$$BM = \begin{bmatrix} M_{r,r} & C_{r,n-r} \\ D_{n-r,r} & N_{n-r,n-r} \end{bmatrix} B.$$

Multiplying $A = [M^{r-1}v, \ldots, Mv, v]$ from both right sides of the above identity, we have

$$\begin{aligned} BMA &= \begin{bmatrix} M_{r,r} & C_{r,n-r} \\ D_{n-r,r} & N_{n-r,n-r} \end{bmatrix} BA \\ &= \begin{bmatrix} M_{r,r} & C_{r,n-r} \\ D_{n-r,r} & N_{n-r,n-r} \end{bmatrix} \begin{bmatrix} A'_{r,r} \\ 0_{n-r,r} \end{bmatrix} = \begin{bmatrix} M_{r,r}A'_{r,r} \\ D_{n-r,r}A'_{r,r} \end{bmatrix}. \end{aligned} \tag{3.8}$$

It is known that $\{M^{r-1}v, M^{r-2}v, \ldots, Mv, v\}$ is a maximal linear independent group as the rank of $\{M^{n-1}v, M^{n-2}v, \ldots, Mv, v\}$ is $r < n$. Then there exist $b_0, b_1, \ldots, b_{r-1} \in \mathbb{R}$ such that

$$M^r v = b_0 v + b_1 Mv + \cdots + b_{r-1} M^{r-1} v$$

and by (3.7),

$$BM^r v = (c_1, \ldots, c_r, 0, \ldots, 0)^t$$

where $c_r = b_0 a'_{rr} \neq 0$. Moreover,

$$BMA = BM[M^{r-1}v, \ldots, Mv, v] = [BM^r v, \ldots, BM^2 v, BMv]$$

$$= \begin{bmatrix} c_1 & a'_{11} & \cdots & a'_{1r-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r-1} & 0 & \cdots & a'_{r-1r-1} \\ c_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} := \begin{bmatrix} Q_{r,r} \\ 0_{n-r,r} \end{bmatrix}. \tag{3.9}$$

From (3.8) and (3.9), it follows that $D_{n-r,r}A'_{r,r} = 0$, and $D_{n-r,r} = 0$ as $A'_{r,r}$ is invertible. Thus we have

$$BMB^{-1} = \begin{bmatrix} M_{r,r} & C_{r,n-r} \\ 0 & N_{n-r,n-r} \end{bmatrix}.$$

Since B is a unimodular matrix, B^{-1} and BMB^{-1} are also integer matrices. Therefore $M_{r,r}$, $N_{n-r,n-r}$ and $C_{r,n-r}$ are integer matrices as desired. □

Example 3.2. Let

$$M = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

and $v = (1, 1, 2)^t$, then $Mv = 4v$, $Bv = (1, 0, 0)^t$ and

$$BMB^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Theorem 3.6. *Let $M \in M_n(\mathbb{Z})$ be an expanding matrix and $\mathcal{D} = \{0, 1, \ldots, q-1\}v$ where $q \geq 2$ and $v \in \mathbb{Z}^n \setminus \{0\}$. If $\{v, Mv, \ldots, M^{n-1}v\}$ is linearly dependent with rank $r < n$, and M, M_1 are expressed as in (3.6), then*

(i) $\mu_{M,\mathcal{D}}$ has infinitely many orthogonal exponentials if $\gcd(q, \det(M_1)) > 1$;

(ii) $\mu_{M,\mathcal{D}}$ is a spectral measure if $q | \det(M_1)$, where M_1 is as in Lemma 3.5;

In particular, if the characteristic polynomial of M_1 is $f(x) = x^r + c$, then

(iii) $\mu_{M,\mathcal{D}}$ has infinitely many orthogonal exponentials if and only if $\gcd(q, \det(M_1)) > 1$;

(iv) $\mu_{M,\mathcal{D}}$ is a spectral measure if and only if $q | \det(M_1)$.

Proof. By Lemma 3.5, there exists a unimodular matrix $B \in M_n(\mathbb{Z})$ such that $Bv = (x_1, \ldots, x_r, 0, \ldots, 0)^t := \widetilde{v}$ and

$$\widetilde{M} = BMB^{-1} = \begin{bmatrix} M_1 & C \\ 0 & M_2 \end{bmatrix}.$$

Let $\widetilde{\mathcal{D}} = B\mathcal{D} = \{0, 1, \ldots, q-1\}\widetilde{v}$. Hence it suffices to consider the conjugate case for $\widetilde{M}$ and $\widetilde{\mathcal{D}}$. For $j = 1, 2, \ldots,$ we have

$$\widetilde{M}^{*-j} = \begin{bmatrix} M_1^{*-j} & 0 \\ \times & M_2^{*-j} \end{bmatrix}.$$

Let $v' = (x_1, \dots, x_r)^t$ and $\mathcal{D}' = \{0, 1, \dots, q-1\}v'$. For $\xi = (\xi_1, \dots, \xi_n)^t \in \mathbb{R}^n$, we denote $\xi' = (\xi_1, \dots, \xi_r)^t$. It follows from (2.2) that

$$\begin{aligned} \hat{\mu}_{\widetilde{M},\widetilde{\mathcal{D}}}(\xi) &= \prod_{j=1}^{\infty} \frac{1}{|\widetilde{D}|} \sum_{d\in\widetilde{D}} e^{2\pi i\langle d, \widetilde{M^*}^{-j}\xi\rangle} = \prod_{j=1}^{\infty} \frac{1}{q} \sum_{k=0}^{q-1} e^{2\pi i\langle k\widetilde{v}, \widetilde{M^*}^{-j}\xi\rangle} \\ &= \prod_{j=1}^{\infty} \frac{1}{q} \sum_{k=0}^{q-1} e^{2\pi i\langle kv', M_1^{*-j}\xi'\rangle} = \hat{\mu}_{M_1,\mathcal{D}'}(\xi'). \end{aligned}$$

If $E(\Lambda)$ is an orthogonal system or an orthonormal basis for $L^2(\mu_{\widetilde{M},\widetilde{\mathcal{D}}})$, we let $\Lambda_r = \{\lambda' = (\lambda_1, \dots, \lambda_r)^t : \lambda = (\lambda_1, \dots, \lambda_n)^t \in \Lambda\}$, then $E(\Lambda_r)$ is an orthogonal system or an orthonormal basis for $L^2(\mu_{M_1,\mathcal{D}'})$. Conversely, if $E(\Lambda_r)$ is an orthogonal system or an orthonormal basis for $L^2(\mu_{M_1,\mathcal{D}'})$, we let

$$\Lambda := \left\{ \begin{pmatrix} \lambda' \\ \varphi(\lambda') \end{pmatrix} : \lambda' \in \Lambda_r \right\}$$

where $\varphi : \Lambda_r \to \mathbb{R}^{n-r}$ is an arbitrary single-valued function, then $E(\Lambda)$ is an orthogonal system or an orthonormal basis for $L^2(\mu_{\widetilde{M},\widetilde{\mathcal{D}}})$. Therefore, $\mu_{M_1,\mathcal{D}'}$ and $\mu_{\widetilde{M},\widetilde{\mathcal{D}}}$ have the same spectrality.

By the assumption, $\{v', M_1v', \dots, M_1^{r-1}v'\}$ is linearly independent. It follows from Theorems 3.2 and 3.4 that (i)–(iv) hold for $\mu_{M_1,\mathcal{D}'}$, hence hold for $\mu_{M,\mathcal{D}}$. □

If the rank of $\{v, Mv, \dots, M^{n-1}v\}$ is $r = 1$, then v becomes an eigenvector of M and $M_1 = \ell$ is the corresponding eigenvalue. Indeed, from Lemma 3.5, it can be seen that $Mv = B^{-1}BMB^{-1}Bv = B^{-1}\widetilde{M}Bv = B^{-1}(\ell x_1, 0, \cdots, 0)^t = \ell B^{-1}Bv = \ell v$. By Theorem 3.6, the following corollary is immediate.

Corollary 3.7. *Under the same assumption as Theorem 3.6 with the rank $r = 1$, in this case v is an eigenvector of M and $M_1 = \ell$ (as in (3.6)) is the corresponding eigenvalue, then*

(i) $\mu_{M,\mathcal{D}}$ has infinitely many orthogonal exponentials if and only if $\gcd(q, \ell) > 1$;
(ii) $\mu_{M,\mathcal{D}}$ is a spectral measure if and only if $q|\ell$.

Example 3.3. Let M, B and v be as in Example 3.2. Let $\mathcal{D}_1 = \{0, 1, 2, 3, 4, 5\}v$ and $\mathcal{D}_2 = \{0, 1\}v$, then $\mu_{M,\mathcal{D}_1}$ has infinitely many orthogonal exponentials but not a spectral measure and $\mu_{M,\mathcal{D}_2}$ is a spectral measure by Corollary 3.7.

Acknowledgment

Part of this work was done when the authors were visiting the Chinese University of Hong Kong, they would like to thank Professor Ka-Sing Lau for his support and valuable discussions.

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