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Evolutions of the momentum density, deformation tensor and the nonlocal term of the Camassa–Holm equation

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ABSTRACT

Methods originally developed to study the finite time blow-up problem of the regular solutions of the three dimensional incompressible Euler equations are used to investigate the regular solutions of the Camassa–Holm equation. We obtain results on the relative behaviors of the momentum density, the deformation tensor and the nonlocal term along the trajectories. In terms of these behaviors, we get new types of asymptotic properties of global solutions, blow-up criterion and blow-up time estimate for local solutions. More precisely, certain ratios of the quantities are shown to be vaguely monotonic along the trajectories of global solutions. Finite time blow-up of the accumulated momentum density is necessary and sufficient for the finite time blow-up of the solution. An upper estimate of the blow-up time and a blow-up criterion are given in terms of the initial short time trajectorial behaviors of the deformation tensor and the nonlocal term.

1. Introduction

We consider the initial-value problem of the Camassa–Holm equation

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x P = 0, \quad t > 0,\ x \in \mathbb{R}, \\ P(t,x) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}} e^{-|x-y|}\left(u^2 + \dfrac{1}{2}u_x^2\right)(t,y)dy, \end{cases} \tag{1.1}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R} \tag{1.2}$$

where $u = u(t,x)$ is a scalar. The symbols P, $P(u)$, $P(t,x)$ and $P(u(t,x))$ will be used interchangeably, as are P_{xx}, $\partial_x^2 P$ etc. The momentum density is defined by $m := u - u_{xx}$, or equivalently

$$u(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|} m(t,y)dy. \tag{1.3}$$

The momentum density form of (1.1) is

$$\partial_t m + u\partial_x m = -2(\partial_x u)m, \tag{1.4}$$

as can be seen by applying $1-\partial_x^2$ to (1.1). In this article, we study regular solutions $u \in C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}))$ of (1.1)–(1.2) with $s > 5/2$, and even $s > 7/2$ in some theorems. For $s \geq 3$, u satisfies (1.1) almost everywhere if and only

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if the corresponding m satisfies (1.4) almost everywhere. For $s > 7/2$, the statement with 'almost everywhere' replaced by 'everywhere' holds.

The Camassa–Holm equation is a model for the unidirectional propagation of shallow water waves over a flat bottom. It was obtained by approximating the incompressible Euler equations under the special assumptions of the shallow water regime [1–3]. It was actually discovered much earlier [4] as an example of bi-Hamiltonian equation. It can model wave breaking phenomena and have peaked solitons called peakons [1]. The peakons capture a feature of water waves of great height, or precisely solutions of the largest amplitude to the free-boundary Euler equations [5–8]. Moreover, the shape of some peakons is stable under small perturbations, making these waves recognizable physically [9,10]. Numerical computations indicate that global solutions tend to a train of peakons moving at different speeds [11], but theoretically their asymptotic behavior is open. In this article, we use techniques originally developed for studying the incompressible Euler equations to investigate the behaviors of the momentum density, the deformation tensor (or velocity gradient) u_x and the nonlocal term P_x in local and global regular solutions of (1.1). This gives information on the asymptotic properties of global solutions, finite-time blow-up properties and the blow-up time of local solutions.

Not surprisingly, the Camassa–Holm equation is similar to the incompressible Euler equations. The momentum density, deformation tensor and the nonlocal term in the former are analogous to the vorticity, deformation tensor and the pressure term in the latter. These latter objects have been investigated partly for studying the open problem of the possibility of finite time blow-up of regular solutions of the three dimensional Euler equations, and finding conditions that can guarantee their global existence. Results in these directions include the Beale–Kato–Majda (BKM) criterion for finite time blow-up of regular solutions [12,13], sufficient conditions for global existence in terms of the direction of the vorticity [14,15], and more recently the works of Chae on a blow-up criterion in terms of these objects [16] and their dynamics with an eye on detecting possible absurdities arising from the assumption of global existence of regular solutions [17].

Compared to the 3D incompressible Euler equations, the well-posedness theory for the Camassa–Holm equation in regular function classes is better developed. For results on weak solutions, see Bressan and Constantin [18,19]. In sufficiently regular function classes, (1.1)–(1.2) is locally well-posed [20–22]. The same references contain sufficient conditions for global well-posedness, but in general that does not hold. In fact, a large class of regular initial data guarantee the finite time blow-up of regular solutions [1,20–24]. In the following theorem, we quote some of these results directly from the literature, though some of the conditions can be relaxed. For a Banach space X, for $k = 0, 1, \ldots,$ we say that v belongs to the set $C^k([0, \tilde{T}); X)$ if for all $T \in (0, \tilde{T})$, v is in the Banach space $C^k([0, T]; X)$.

Theorem 1.1 *([20,22,25]). Let $u_0 \in H^s(\mathbb{R})$ for some $s > 3/2$.*

(a) *There is a maximal time $T^* = T(u_0) \in (0, \infty]$ so that on $[0, T^*)$, (1.1)–(1.2) has a unique solution*

$$u = u(\cdot, u_0) \in C([0, T^*); H^s(\mathbb{R})) \cap C^1([0, T^*); H^{s-1}(\mathbb{R})).$$

(b) *If the solution blows up in finite time, i.e. $T^* < \infty$, then*

$$\limsup_{t \nearrow T^*} \|u(t, \cdot)\|_{H^s(\mathbb{R})} = \infty.$$

(c) *The solution blows up at $T^* < \infty$ if and only if*

$$\liminf_{t \nearrow T^*} \left\{ \inf_{y \in \mathbb{R}} u_x(t, y) \right\} = -\infty.$$

For $s \geq 3$, u blows up at $T^* < \infty$ implies that

$$\lim_{t \nearrow T^*} \left\{ \inf_{y \in \mathbb{R}} [u_x(t, y)](T^* - t) \right\} = -2. \tag{1.5}$$

(d) *(a sufficient condition for finite time blow-up) Suppose $s \geq 3$. If there is an $x_0 \in \mathbb{R}$ such that $m_0 := u_0 - u_{0,xx} \geq 0$ on $(-\infty, x_0]$, $m_0 \leq 0$ on $[x_0, \infty)$ and m_0 changes sign, then the solution blows up in finite time.*

(e) *(sufficient conditions for global existence) Suppose $s \geq 3$. If m_0 does not change sign, or if there exists an $x_0 \in \mathbb{R}$ such that $m_0 \leq 0$ on $(-\infty, x_0]$ and $m_0 \geq 0$ on $[x_0, \infty)$, then the solution exists globally.*

We now state theorems for the momentum density, the deformation tensor and the nonlocal term in the Camassa–Holm equation similar to those for the corresponding terms in the Euler equations investigated in [12,13,16,17]. For a regular solution u of (1.1)–(1.2), with $s > 3/2$ so that $u(t, \cdot)$ is Lipschitz in the second variable, the trajectory starting from $a \in \mathbb{R}$ is the solution $X(t, a)$ of the problem

$$\begin{cases} \dfrac{d}{dt} X(t, a) = u(t, X(t, a)), & t > 0, \\ X(0, a) = a. \end{cases} \tag{1.6}$$

The following two theorems describe the relative behaviors of m, u_x and P_x or terms derived from them. More precisely, the ratios $-u_x/|m|$ and $-P_{xx}/u_x^2$ are 'vaguely monotonic' along the trajectories. They correspond to [17, Theorems 1.1, 1.2].

Theorem 1.2. *Let $u_0 \in H^s(\mathbb{R})$ with $s > 7/2$ and u be the solution of (1.1)–(1.2) given in Theorem 1.1(a). If $m(t, X(t, a)) \neq 0$, define*

$$\Phi_1(t,a) := \frac{-u_x(t, X(t,a))}{|m(t, X(t,a))|}, \qquad \Sigma_1(t) := \{a \in \mathbb{R} | u_x(t, X(t,a)) < 0\}.$$

Suppose $a \in \Sigma_1(0)$ and $m_0(a) \neq 0$. Then one of the following holds.

1. *m cannot be extended indefinitely along $X(t, a)$, and hence the solution of (1.1)–(1.2) blows up in finite time.*
2. *One of the following holds.*
   (a) *There exists $\tilde{t} \in (0, \infty)$ such that $u_x(\tilde{t}, X(\tilde{t}, a)) = 0$.*
   (b) *There exists a sequence $\{t_j\}_{j=1}^{\infty}$ with $t_1 < t_2 < \cdots < t_j < t_{j+1} \to \infty$ as $j \to \infty$ such that for $j = 1, 2, \ldots$, $\Phi_1(0, a) > \Phi_1(t_1, a) > \cdots > \Phi_1(t_j, a) > \Phi_1(t_{j+1}, a) > 0$, and for $t \in [0, t_j]$, $\Phi_1(t, a) \geq \Phi_1(t_j, a) > 0$.*

Theorem 1.3. *Let $u_0 \in H^s(\mathbb{R})$ with $s > 5/2$ and u be the solution of (1.1)–(1.2) given in Theorem 1.1(a). If $u_x(t, X(t, a)) \neq 0$, define*

$$\Phi_2(t,a) := \frac{-P_{xx}(t, X(t,a))}{u_x^2(t, X(t,a))},$$
$$\Sigma_2^+(t) := \{a \in \mathbb{R} | u_x(t, X(t,a)) > 0, \Phi_2(t,a) > 1\},$$
$$\Sigma_2^-(t) := \{a \in \mathbb{R} | u_x(t, X(t,a)) < 0, \Phi_2(t,a) < 1\}.$$

For $a \in \Sigma_2^+(0) \cup \Sigma_2^-(0)$, one of the following holds.

1. *The solution of (1.1)–(1.2) blows-up in finite time.*
2. *One of the following holds.*
   (a) *There exists $\tilde{t} \in (0, \infty)$ such that $u_x(\tilde{t}, X(\tilde{t}, a)) = 0$.*
   (b) *Either there exists $T_1 \in (0, \infty)$ such that $\Phi(T_1, a) = 1$, or there exists a sequence $\{t_j\}_{j=1}^{\infty}$ with $t_1 < t_2 < \cdots < t_j < t_{j+1} \to \infty$ as $j \to \infty$ such that one of the following holds:*
      i. *If $a \in \Sigma_2^+(0)$, for $j = 1, 2, \ldots$, we have $\Phi_2(0, a) > \Phi_2(t_1, a) > \cdots > \Phi_2(t_j, a) > \Phi_2(t_{j+1}, a) > 1$, and $\Phi_2(t, a) \geq \Phi_2(t_j, a) > 1$.*
      ii. *If $a \in \Sigma_2^-(0)$, for $j = 1, 2, \ldots$, we have $\Phi_2(0, a) < \Phi_2(t_1, a) < \cdots < \Phi_2(t_j, a) < \Phi_2(t_{j+1}, a) < 1$, and $\Phi_2(t, a) \leq \Phi_2(t_j, a) < 1$.*

In Theorem 1.2, we require $s > 7/2$ as we need m_x to make sense pointwise and (1.4) to hold everywhere (see (2.2)). In Theorem 1.3, $s > 5/2$ is enough as we only need u_{xx} and P_{xx} to be meaningful pointwise. From the proofs of the theorems, even for solutions blowing up in finite time, the vaguely monotonic behavior of Φ_1 and Φ_2 indicated in scenario 2(b) still holds until the blow-up time if scenario 2(a) does not hold.

The following theorem corresponds to the BKM blow-up criterion for the Euler equations [12]. We record it as it highlights the correspondence between the momentum density here and the vorticity in the Euler equations. Moreover, it will be used in Theorem 1.5.

Theorem 1.4. *Let $s \geq 3$ and $u_0 \in H^s(\mathbb{R})$. Suppose that for any $T \in (0, T^*)$, $u \in C([0, T], H^s(\mathbb{R})) \cap C^1([0, T], H^{s-1}(\mathbb{R}))$ is the unique solution of (1.1)–(1.2). Then T^* is the maximal time of existence of u if and only if $\int_0^{T^*} \|m(t, \cdot)\|_{L^\infty(\mathbb{R})} dt = \infty$.*

Anther equivalent condition is $\int_0^{T^*} \|u_x(\tau, \cdot)\|_{L^\infty(\mathbb{R})} d\tau = \infty$. It corresponds to the result in [13] for the Euler equation and will be proved after the proof of the theorem.

The next theorem gives an upper estimate of the blow-up time and a blow-up criterion in terms of the short time behavior of u_x and P_{xx} along the trajectories. It corresponds to [16, Theorem 2.1].

Theorem 1.5. *Let $u_0 \in H^s(\mathbb{R})$ with $s > 7/2$ and u be the solution of (1.1)–(1.2) given in Theorem 1.1(a). Let*

$$S = \{a \in \mathbb{R} | m_0(a) \neq 0, u_0'(a) < 0, u_0'(a)^2 - P_{xx}(0, a) < 0\}. \tag{1.7}$$

Suppose that there is an $\epsilon \in (0, 1]$ and an $a \in \mathbb{R}$ such that

$$\sup_{0 \leq t \leq 1/(-2\epsilon u_0'(a))} \sqrt{2(u_x^2 - P_{xx})_+(t, X(t,a))} \leq -2(1-\epsilon) u_0'(a), \tag{1.8}$$

then the momentum density of u cannot be extended past $t_*(a) = 1/(-2\epsilon u_0'(a))$ along the trajectory $X(t, a)$. Moreover, there exists $T^* \leq t_*(a)$ such that

$$\limsup_{t \to T^*} \|u(t, \cdot)\|_{H^s(\mathbb{R})} = \infty, \tag{1.9}$$

and

$$\int_0^{T^*} \|m(t,\cdot)\|_{L^\infty(\mathbb{R})} dt = \infty. \tag{1.10}$$

Furthermore

$$T^* \leq -\frac{1}{2 \inf\limits_{a\in S} \epsilon u_0'(a)}. \tag{1.11}$$

From Theorem 1.5, we can also obtain information on a global solution u. As (1.8) fails for every $\epsilon \in (0, 1]$, choosing $\epsilon = 1$ shows that u_x^2 speeds past P_{xx} along $(t, X(t, a))$ in the time period $(0, -1/[2u_0'(a)])$.

Though the theorems here are similar to those for the Euler equations, there are two factors favorable for our investigations. First, as the Camassa–Holm equation is of spatial dimension one and the three quantities we study are scalars, the results are probably more transparent. Second, the better developed well-posedness theory for the Camassa–Holm equation in regular function classes provides a better foundation for studying the regular solutions. The theorems here describe actual behaviors of the solutions, in addition to being blow-up criteria and dichotomies. For example, combining conditions for finite time blow-up like Theorems 1.1(d) and 1.4, we know a class of solutions of the Camassa–Holm equation with accumulated momentum density blowing up in finite time. On the other hand, there is no known example of a local regular solution to the 3D incompressible Euler equation with similar behavior for the accumulated vorticity. Similarly, for the solutions with initial data satisfying the hypothesis of Theorem 1.1(e), we know that the dynamics described in scenario 2 of Theorems 1.2 and 1.3 hold.

In Section 2, we prove Theorems 1.2 and 1.3. Theorems 1.4 and 1.5 will be proved in Section 3.

2. Behavior of global regular solutions

Let u be the solution of (1.1)–(1.2) given by Theorem 1.1(a). We prove Theorems 1.2 and 1.3 in this section. For a smooth $f(t, x) : (0, T) \times \mathbb{R} \to \mathbb{R}$, we write $f'(t, X(t, a)) = \frac{Df}{Dt}(t, X(t, a)) = \frac{d}{dt} f(t, X(t, a)) = (f_t + uf_x)(t, X(t, a))$. We will clarify if there is any possible confusion. We prove a lemma before proving Theorem 1.2.

Lemma 2.1. *Let $s > 7/2$. Let $u_0'(a) < 0$ and $\epsilon > 0$ be such that*

$$u_0'(a)|m_0(a)| \leq -\frac{1}{2}\epsilon |m_0(a)|^2.$$

Let $T_* = 1/(\epsilon|m_0(a)|)$. Then either $m(t, X(t, a))$ blows up in $(0, T_*]$, or there is a $t \in (0, T_*)$ such that

$$u_x(t, X(t, a))|m(t, X(t, a))| > -\frac{1}{2}\epsilon |m(t, X(t, a))|^2.$$

Proof. Suppose that $m(t, X(t, a))$ does not blow-up in $(0, T_*]$ and that for all $t \in (0, T_*)$,

$$u_x(t, X(t, a))|m(t, X(t, a))| \leq -\frac{1}{2}\epsilon |m(t, X(t, a))|^2. \tag{2.1}$$

Multiply (1.4) at $(t, X(t, a))$ by $2m(t, X(t, a))$ to get

$$\frac{D|m|^2}{Dt} = -4u_x|m|^2 \quad \text{and hence } \frac{D|m|}{Dt} = -2u_x|m| \tag{2.2}$$

along $(t, X(t, a))$. Eqs. (2.1) and (2.2) gives $|m(t, X(t, a))|' \geq \epsilon |m(t, X(t, a))|^2$. Hence $|m(t, X(t, a))| \geq |m_0(a)|/(1 - \epsilon|m_0(a)|t) \to \infty$ within $(0, T_*]$, contradictory to our assumption. □

Proof of Theorem 1.2. Eq. (2.2) implies that

$$|m(t, X(t, a))| = |m_0(a)| \exp\left(-\int_0^t 2u_x(\tau, X(\tau, a))d\tau\right).$$

Hence $m(t, X(t, a)) \neq 0$ if and only if $m_0(a) \neq 0$ (or see [20]). This property was exploited to investigate the propagation speed of a localized disturbance in [26,27]. Choosing $\epsilon = -2u_0'(a)/|m_0(a)|$ in Lemma 2.1, we conclude that either $m(t, X(t, a))$ and hence $\|m(t,\cdot)\|_{L^\infty(\mathbb{R})}$ and $\|u(t,\cdot)\|_{H^s(\mathbb{R})}$ blows-up in $(0, T_* = -1/2u_0'(a)]$, or there is a $t_1 \in (0, T_*)$ such that

$$\Phi_1(t_1, a) = \frac{-u_x(t_1, X(t_1, a))}{|m(t_1, X(t_1, a))|} < \frac{\epsilon}{2} = \frac{-u_0'(a)}{|m_0(a)|} = \Phi_1(0, a).$$

Suppose scenarios 1 and 2(a) do not hold. The latter implies that $\Phi_1(t_1, a) > 0$, or equivalently $u_x(t_1, X(t_1, a)) < 0$ and hence $a \in \Sigma_1(t_1)$. Now repeat the above reasoning with initial time t_1. In Lemma 2.1, take

$$\epsilon = \frac{-2u_x(t_1, X(t_1, a))}{|m(t_1, X(t_1, a))|}.$$

Then either $m(t, X(t, a))$ blows up in $(t_1, t_1 - 1/\{2u_x(t_1, X(t_1, a))\}]$, or there exists a $t_2 \in (t_1, t_1 - 1/\{2u_x(t_1, X(t_1, a))\})$ such that $\Phi_1(t_2, a) < \Phi_1(t_1, a)$. Hence as long as 1 and 2(a) do not hold, we can find an increasing sequence $\{t_j\}_{j=1}^{\infty}$ such that $\Phi_1(t_j, a) > \Phi_1(t_{j+1}, a) > 0$. Obviously, we can choose t_j such that for all $t \in (t_{j-1}, t_j]$ (and hence in $[0, t_j]$), $\Phi_1(t, a) \geq \Phi_1(t_j, a)$. Suppose $t_j \to t_\infty < \infty$. If $u_x(t_\infty, X(t_\infty, a)) = 0$, 2(a) holds. If $u_x(t_\infty, X(t_\infty, a)) < 0$, then by the above reasoning with t_∞ as the initial time, if 1 and 2(a) do not hold, there is a $\bar{t} > t_\infty$ such that $\Phi_1(t_\infty, a) > \Phi_1(\bar{t}, a) > 0$. Hence the original sequence $\{t_j\}$ can be modified to be continued past t_∞. Therefore if scenarios 1 and 2(a) do not hold, $\{t_j\} \to \infty$ and 2(b) holds. □

We prove several lemmas before proving Theorem 1.3. Assume $s > 5/2$ in the rest of this section.

Lemma 2.2. *Let $u_0'(a) > 0$, and $\epsilon > 0$ be such that*

$$(1+\epsilon)u_0'(a)^2 \leq -P_{xx}(0, a).$$

Let $T_* = 1/[\epsilon u_0'(a)]$. Then either the solution u of (1.1)–(1.2) blows-up in $(0, T_*]$, or there exists a $t \in (0, T_*)$ such that

$$(1+\epsilon)u_x^2(t, X(t, a)) > -P_{xx}(t, X(t, a)).$$

Proof. Suppose the solution u does not blow-up in $(0, T_*]$ and that on $(0, T_*)$,

$$(1+\epsilon)u_x^2(t, X(t, a)) \leq -P_{xx}(t, X(t, a)). \tag{2.3}$$

Eqs. (1.1), (1.6) and (2.3) implies that for $t \in (0, T_*)$,

$$\begin{aligned} \frac{d}{dt}u_x(t, X(t, a)) &= (u_{xt} + uu_{xx})(t, X(t, a)) = (-P_{xx} - u_x^2)(t, X(t, a)) \\ &\geq \epsilon u_x^2(t, X(t, a)). \end{aligned}$$

It follows that $u_x(t, X(t, a)) \geq u_0'(a)/[1 - \epsilon u_0'(a)t] \to \infty$ no later than $T_* = 1/(\epsilon u_0'(a))$ (or u may have blown-up even earlier before $u_x(t, X(t, a))$ has the chance to). The Sobolev inequality gives

$$|u_x(t, X(t, a))| \leq \|u_x(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C\|u(t, \cdot)\|_{H^s(\mathbb{R})}.$$

Hence $\|u(t, \cdot)\|_{H^s(\mathbb{R})} \to \infty$ no later than T_*, contradictory to the non-blow-up of u on $(0, T_*]$. The lemma is proved. □

Lemma 2.3. *Let $u_0'(a) < 0$, and $\epsilon > 0$ be such that*

$$(1-\epsilon)u_0'(a)^2 \geq -P_{xx}(0, a).$$

Let $T_* = -\frac{1}{\epsilon u_0'(a)}$. Then either the solution u of (1.1)–(1.2) blows-up in $(0, T_*]$, or there exists a $t \in (0, T_*)$ such that

$$(1-\epsilon)u_x^2(t, X(t, a)) < -P_{xx}(t, X(t, a)).$$

Proof. The proof is similar to that of Lemma 2.2. Suppose the solution u does not blow-up in $(0, T^*]$ and that on $(0, T_*)$,

$$(1-\epsilon)u_x^2(t, X(t, a)) \geq -P_{xx}(t, X(t, a)). \tag{2.4}$$

Then (1.1), (1.6) and (2.4) gives $(d/dt)u_x(t, X(t, a)) \leq -\epsilon u_x^2(t, X(t, a))$. Hence $u_x(t, X(t, a)) \leq u_0'(a)/[1+u_0'(a)\epsilon t]$, implying that $u_x(t, X(t, a)) \to -\infty$ no later than T_*. The Sobolev inequality implies that $\|u(t, \cdot)\|_{H^s(\mathbb{R})} \to \infty$ on $(0, T_*]$, contradictory to our assumption. □

Lemma 2.4. *Suppose $u_0'(a) > 0$ and*

$$u_0'(a)^2 < -P_{xx}(0, a). \tag{2.5}$$

Let

$$T_* = -\frac{u_0'(a)}{P_{xx}(0, a) + u_0'(a)^2}. \tag{2.6}$$

Then either the solution of (1.1)–(1.2) blows-up in $(0, T_*]$, or there exists a $t \in (0, T_*)$ such that $\Phi_2(t, a) < \Phi_2(0, a)$.

Proof. Let $\epsilon = -[P_{xx}(0,a)/u_0'(a)^2] - 1$ in Lemma 2.2. If the solution of (1.1) and (1.2) does not blow-up in $(0, T_*]$, the lemma gives a $t \in (0, T_*)$ with

$$\Phi_2(0,a) = \frac{-P_{xx}(0,a)}{u_0'(a)^2} > \frac{-P_{xx}(t, X(t,a))}{u_x(t, X(t,a))^2} = \Phi_2(t,a). \quad \square$$

Similarly choosing $\epsilon = (P_{xx}(0,a)/u_0'(a)^2) + 1$ in Lemma 2.3 gives

Lemma 2.5. *Suppose $u_0'(a) < 0$ and $u_0'(a)^2 > -P_{xx}(0,a)$. Let*

$$T_* = -\frac{u_0'(a)}{P_{xx}(0,a) + u_0'(a)^2}.$$

Then either the solution of (1.1)–(1.2) blows-up in $(0, T_*]$, or there exists a $t \in (0, T_*)$ such that $\Phi_2(t,a) > \Phi_2(0,a)$.

Proof of Theorem 1.3. Suppose scenarios 1 and 2(a) do not hold. Suppose that $a \in \Sigma_2^+(0)$. We will show that 2(b) holds. The argument is similar for $a \in \Sigma_2^-(0)$. Now $a \in \Sigma_2^+(0)$ means that $u_0'(a) > 0$ and $\Phi_2(0,a) > 1$, the latter being equivalent to (2.5). Hence Lemma 2.4 shows that there exists $t_1 \in (0, T_*)$, T_* given by (2.6), such that

$$\Phi_2(t_1, a) < \Phi_2(0, a).$$

If $\Phi_2(s,a) \le 1$ for some $s \in (0, t_1]$, then 2(b) holds. Suppose $\Phi_2(t,a) > 1$ for all $t \in (0, t_1]$. Then from (1.1), for $t \in (0, t_1)$,

$$\frac{d}{dt} u_x(t, X(t,a)) = (u_{xt} + u u_{xx})(t, X(t,a)) = (-u_x^2 - P_{xx})(t, X(t,a)) > 0,$$

which together with $u_0'(a) > 0$ implies that $u_x(t_1, X(t_1, a)) > 0$. Hence $a \in \Sigma_2^+(t_1)$. Apply Lemma 2.4 with initial time t_1 to see that if the solution does not blow-up in

$$\left(t_1, t_1 - \frac{u_x(t_1, X(t_1,a))}{P_{xx}(t_1, X(t_1,a)) + u_x(t_1, X(t_1,a))^2}\right],$$

then there exists

$$t_2 \in \left(t_1, t_1 - \frac{u_x(t_1, X(t_1,a))}{P_{xx}(t_1, X(t_1,a)) + u_x(t_1, X(t_1,a))^2}\right)$$

such that $\Phi_2(t_2, a) < \Phi_2(t_1, a)$. If $\Phi_2(t_2, a) \le 1$, then there exists $T_2 \in (t_1, t_2]$ such that $\Phi_2(T_2, a) = 1$ and 2(b) holds. Otherwise $a \in \Sigma_2^+(t_2)$, and we can continue the process using Lemma 2.4.

In conclusion either there exists a t such that $\Phi_2(t, X(t,a)) = 1$ and 2(b) holds, or there is a strictly increasing sequence $\{t_j\}_{j=1}^{\infty}$ such that $\Phi_2(t_j, a) > \Phi_2(t_{j+1}, a)$. Obviously, the t_j's can be chosen such that for $t \in (t_{j-1}, t_j]$, $\Phi_2(t,a) \ge \Phi_2(t_j, a)$. Suppose $t_j \to t_\infty < \infty$ as $j \to \infty$. If $\Phi_2(t_\infty, a) = 1$, 2(b) holds. If $\Phi_2(t_\infty, a) > 1$, we can apply Lemma 2.4 to get a $\bar{t} > t_\infty$ such that $\Phi_2(t_\infty, a) > \Phi_2(\bar{t}, a)$, and the original sequence of t_j's can be modified to get past t_∞. In either case, 2(b) holds. $\square$

3. Finite-time blow-up of regular solutions

We prove Theorems 1.4 and 1.5 in this section.

Proof of Theorem 1.4. Suppose the solution u blows up at $T^* < \infty$. From (1.3), $u_x(t,x) = (1/2)\int_{\mathbb{R}} \operatorname{sgn}(y-x) e^{-|x-y|} m(t,y)dy$ and hence $\|u_x(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \|m(t,\cdot)\|_{L^\infty(\mathbb{R})}$. Together with (1.5), we have

$$\begin{aligned}\int_0^{T^*} \|m(t,\cdot)\|_{L^\infty(\mathbb{R})} dt &\ge \int_0^{T^*} \|u_x(t,\cdot)\|_{L^\infty(\mathbb{R})} dt \\ &\ge \int_0^{T^*} \left|\inf_{y\in\mathbb{R}} u_x(t,y)\right| dt = \infty. \end{aligned} \tag{3.1}$$

Conversely, $\int_0^{T^*} \|m(t)\|_{L^\infty} dt = \infty$ implies that $\limsup_{t\nearrow T^*} \|m(t,\cdot)\|_{L^\infty} = \infty$. As $\|m(t,\cdot)\|_{L^\infty} \le C\|u(t,\cdot)\|_{H^s}$, $\limsup_{t\nearrow T^*} \|u(t,\cdot)\|_{H^s} = \infty$, and the solution cannot be extended past T^*. $\square$

Remark. Another necessary and sufficient condition is $\int_0^{T^*} \|u_x(t)\|_{L^\infty} dt = \infty$. The necessity follows from (3.1). Conversely if the condition holds, the Sobolev inequality implies that $\limsup_{t\nearrow T^*} \|u(t)\|_{H^s} \ge C \limsup_{t\nearrow T^*} \|u_x(t)\|_{L^\infty} = \infty$. Hence u blows up at T^*.

Lemma 3.1. *If* $m(t,x)\neq 0$, *define* $\Psi(t,x)=1/|m(t,x)|$. *Then*

$$\left[\frac{D^2\Psi}{Dt^2}-2(u_x^2-P_{xx})\Psi\right](t,X(t,a))=0. \tag{3.2}$$

Proof. From (2.2), m nonzero at one point on a trajectory implies that it is nonzero everywhere along it. In the following calculations, the quantities are evaluated along $(t,X(t,a))$. Differentiate (1.1) to get $u_{tx}+uu_{xx}=-u_x^2-P_{xx}$. Together with (2.2), we have

$$\begin{aligned}\frac{1}{2}\frac{D^2|m|^2}{Dt^2} &= \frac{D}{Dt}\left(\frac{1}{2}\frac{D|m|^2}{Dt}\right)=\frac{D(-2u_x|m|^2)}{Dt}\\ &= -2\frac{Dm^2}{Dt}u_x-2m^2\frac{Du_x}{Dt}=8u_x^2m^2-2m^2(u_{xt}+uu_{xx})\\ &= (10u_x^2+2P_{xx})m^2.\end{aligned} \tag{3.3}$$

On the other hand, from (2.2),

$$\frac{D^2|m|^2}{Dt^2}=2\frac{D^2|m|}{Dt^2}|m|+2\left(\frac{D|m|}{Dt}\right)^2=2\frac{D^2|m|}{Dt^2}|m|+8u_x^2m^2.$$

Hence together with (3.3), we get

$$\frac{D^2|m|}{Dt^2}=\frac{1}{|m|}\left[\frac{1}{2}\frac{D^2|m|^2}{Dt^2}-4u_x^2|m|^2\right]=(6u_x^2+2P_{xx})|m|. \tag{3.4}$$

From the definition of Ψ, (2.2) and (3.4),

$$\begin{aligned}\frac{D^2\Psi}{Dt^2} &= -\frac{1}{|m|^2}\frac{D^2|m|}{Dt^2}+\frac{2}{|m|^3}\left(\frac{D|m|}{Dt}\right)^2=-\frac{1}{|m|}(6u_x^2+2P_{xx})+\frac{8u_x^2}{|m|}\\ &= (2u_x^2-2P_{xx})\frac{1}{|m|}=(2u_x^2-2P_{xx})\Psi.\end{aligned}$$

The lemma is proved. □

Proof of Theorem 1.5. Let u be the solution on the maximal interval of existence. Let $a\in S$. Then from (1.8) and (3.2),

$$\begin{aligned}\frac{d^2}{dt^2}\Psi(t,X(t,a)) &\le (2u_x^2-2P_{xx})_+(t,X(t,a))\Psi(t,X(t,a))\\ &\le (-2(1-\epsilon)u_0'(a))^2\Psi(t,X(t,a))\\ &= h^2\Psi(t,X(t,a)),\end{aligned} \tag{3.5}$$

where

$$h=h(a):=-2(1-\epsilon)u_0'(a)>0. \tag{3.6}$$

Multiply (3.5) by $\exp(ht)$ to get

$$\frac{d^2}{dt^2}[\Psi(t,X(t,a))\exp(ht)]-2h\frac{d}{dt}[\Psi(t,X(t,a))\exp(ht)]\le 0.$$

Multiply this by $\exp(-2ht)$ to get

$$\frac{d}{dt}\left\{\exp(-2ht)\frac{d}{dt}[\Psi(t,X(t,a))\exp(ht)]\right\}\le 0.$$

This can be integrated straightforwardly to get

$$\Psi(t,X(t,a))\le\Psi_0\exp(-ht)+(h\Psi_0+\Psi_0')\exp(-ht)\frac{\exp(2ht)-1}{2h}, \tag{3.7}$$

where $\Psi_0:=\Psi(0,a)$ and $\Psi_0'(a):=\frac{d}{dt}\big|_{t=0}\Psi(t,X(t,a))$. To see this, let

$$\phi(t):=\frac{d}{dt}\left\{\exp(-2ht)\frac{d}{dt}[\Psi(t,X(t,a))\exp(ht)]\right\}\le 0.$$

Integrate this on $[0, t]$ to get

$$\frac{d}{dt}(\Psi(t, X(t, a)) \exp(ht)) = \exp(2ht) \left(\int_0^t \phi(s)ds + \Psi_0' + h\Psi_0 \right).$$

Integrate this on $[0, t]$ and use $\int_0^t \phi(s)ds \leq 0$ to get (3.7).

From the definition of Ψ, (2.2), (3.6) and (3.7), denoting $\frac{d}{dt}\big|_{t=0} |m(t, X(t, a))|$ by $|m_0(a)|'$ (but $u_0'(a)$ still means the ordinary derivative of u_0 at a), we get

$$\begin{aligned} |m(t, X(t, a))| &\geq \left\{ \Psi_0 \exp(-ht) + (h\Psi_0 + \Psi_0') \exp(-ht) \frac{\exp(2ht) - 1}{2h} \right\}^{-1} \\ &= |m_0(a)| \exp(ht) \left\{ 1 - \left[\frac{|m_0(a)|'}{|m_0(a)|} - h(a) \right] \frac{\exp(2ht) - 1}{2h} \right\}^{-1} \\ &= |m_0(a)| \exp(ht) \left\{ 1 - [-2u_0'(a) - h(a)] \frac{\exp(2ht) - 1}{2h} \right\}^{-1} \\ &\geq \frac{|m_0(a)|}{1 + 2\epsilon u_0'(a)t}, \end{aligned} \tag{3.8}$$

where we have used $\frac{\exp(2ht)-1}{2h} \geq t$. Hence $|m(t, X(t, a))| \to \infty$ before t reaches $t_*(a) = 1/(-2\epsilon u_0'(a))$, or the solution has already blown up before it has the chance to. It follows that the maximal interval of existence of u is $[0, T^*)$ for some $T^* \leq t_*(a)$. Then (1.9) follows from Theorem 1.1(b). (1.10) follows from Theorem 1.4. That $T^* \leq t_*(a)$ implies the estimate (1.11) of T^*. The proof is completed. □

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