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An asymptotic property of the Camassa–Holm equation

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Asymptotic densities are used to study an asymptotic property of the dispersionless Camassa–Holm equation. An asymptotic density of a global solution is a weak limit of its scaled momentum density along a sequence of time increasing to infinity. For a global solution with non-negative compactly supported initial momentum density, we show that if the asymptotic density is unique, then it is a positive combination of Dirac measures supported in a bounded interval in the non-negative axis with zero as the only possible accumulation point. In other words, if the scaled momentum density does not oscillate as time goes to infinity, the solution behaves as a combination of peakons moving to the right at different speeds. In contrast to many investigations on the topic, our approach is not spectral theoretic and hence is independent of the structure of the isospectral problem associated with the equation.

1. Introduction

We consider the Cauchy problem of the dispersionless Camassa–Holm equation

$$\begin{cases} u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & x \in \mathbb{R}, t > 0, \\ u(0, x) = u_0(x) & x \in \mathbb{R} \end{cases} \tag{1.1}$$

where $u = u(t, x)$ is a scalar. Let $h = u - u_{xx}$ be the momentum density. The same problem in terms of h is

$$\begin{cases} h_t + uh_x + 2u_x h = 0, & x \in \mathbb{R}, t > 0, \\ h(0, x) = h_0(x) & x \in \mathbb{R} \end{cases} \tag{1.2}$$

with $h_0 = u_0 - u_{0,xx}$. For $s \geq 3$, $u \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R}))$ satisfies (1.1) if and only if the corresponding h satisfies (1.2). The Camassa–Holm equation is a model for the unidirectional propagation of water waves in irrotational flow over a flat bed in the shallow water moderate amplitude regime [1–3]. It can also model the propagation of axially symmetric waves in a hyperelastic rod [4]. The local well-posedness for strong solutions are given in [5–7], together with conditions for their global existence and finite time blow-up. For the well-posedness theory for weak solutions, see Bressan and Constantin [8,9]. The equation has received much attention since Camassa and Holm [1] showed that it can model wave breaking phenomena and have peaked solitary wave solutions called peakons, weak solutions of the form $ce^{-|x-ct|}$ with $c > 0$ ('antipeakon' if $c < 0$). They are smooth except at the crest where they are continuous but with different one-sided tangents, similar to the traveling waves of the greatest height which are solutions to the equations for water waves [10–13]. There are multipeakon/antipeakon solutions of the form $\sum_{i=1}^{N} m_i(t)e^{|x-q_i(t)|}$ [1,14], where the $m_i(t)$'s and $q_i(t)$'s

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satisfy a Hamiltonian system. Without causing confusion, we will use the same phrase 'multipeakon/antipeakon' whatever the signs of m_i's are. The shapes of peakons and multipeakon/antipeakons are stable under small perturbations, making them recognizable physically [15–18]. The Camassa–Holm equation is an integrable infinite dimensional Hamiltonian system. It is the compatibility condition for an isospectral problem and a linear evolution equation for the corresponding eigenfunctions, and can be investigated through direct and inverse spectral theory [1,14,19–21].

Numerical computations (see for instance [14,22,23]) indicate that some global solutions evolve into a train of peakons moving at different speeds. Whether this is true in general has been an open problem (see for instance [24]) until the recent work of Eckhardt and Teschl [21]. Previously, Beals, Sattinger and Szmigielski [25] settled the problem for multipeakon/antipeakon solutions by determining the limits of $m_i(t)$'s and $\dot{q}_i(t)$'s (see the expression for such solutions in the last paragraph) and deduced that in large time they tend to superpositions of noninteracting peakons and antipeakons. The problem was also settled for a simplified flow by Loubet [26] and a class of low regularity solutions by Li [27]. The works of El Dika and Molinet on the stability of multipeakon/antipeakons [16,17] can be interpreted as results on this problem. They show that solutions with initial values close to a superposition of peakon and antipeakon profiles will stay close to the superposition of the profiles each of which translated with a (varying) velocity close to its signed height [16, Lemma 4.1] [17, Theorem 2.1]. Recently Eckhardt and Teschl [21] study the direct and inverse spectral theory for the isospectral problem associated with the equation, only assuming the weight to be a finite signed Borel measure. Using the known time evolution of the spectral quantities, they obtain the asymptotic behavior of the solutions and conclude that the peakon train phenomenon appears in general.

In this article, a method involving asymptotic densities used in [28–30] is employed to give another deduction of the phenomenon. In contrast to many of the studies mentioned above, our approach is not spectral theoretic and hence is independent of the structure of the isospectral problem associated with the equation, or even the existence of such a problem. This may be of value in investigating analogous problems for some other equations. To explain our result, we first fix our notations and define asymptotic densities. Let $[a, b] \subset \mathbb{R}$ be a finite interval. Let $\mathcal{M}[a, b]$ be the space of regular Borel measures on $[a, b]$. We will identify such measures with those on $\mathbb{R}$ supported in $[a, b]$. Let $C_c(\mathbb{R})$ be the space of continuous functions with compact support on $\mathbb{R}$. Obviously, the restrictions of $C_c(\mathbb{R})$ functions to a finite interval $[a, b]$ gives $C[a, b]$. For a Banach space X, $k = 0, 1, \dots, T^* \in (0, \infty]$, we say that v belongs to the set $C^k([0, T^*); X)$ if for all $T < T^*$, v is in the Banach space $C^k([0, T]; X)$.

Definition 1.1. Let $s \geq 3$ and $u \in C([0, \infty); H^s(\mathbb{R}))$ be a global solution of (1.1).

(a) For $t > 0$, $\tilde{h}(t, y) := th(t, ty)$ is called the scaled momentum density of u.

(b) Suppose supp $\tilde{h}(t, \cdot) \subset [a, b]$ for all $t \geq 1$. $\mu \in \mathcal{M}[a, b]$ is called an *asymptotic density* associated with the initial momentum density h_0 if there is a sequence $t_k \to \infty$ as $k \to \infty$ for which

$$\tilde{h}(t_k, \cdot) \rightharpoonup \mu \quad \text{as } t_k \to \infty,$$

where the convergence is the weak-∗ convergence in $\mathcal{M}[a, b]$, i.e. for all $\psi \in C[a, b]$, $\int_a^b \tilde{h}(t, y)\psi(y)dy \to \langle \mu, \psi \rangle$.

Notice that asymptotic densities associated with h_0 may not be unique.

Suppose that $u_0 \in H^s(\mathbb{R})$ with $s \geq 3$ and $h_0 \geq 0$ is compactly supported. Let u be the global strong solution to (1.1) (see Theorem 2.1). From Lemma 2.4 the scaled momentum density $\tilde{h}(t, \cdot)$ must be supported in some finite interval $[a, b]$ for t sufficiently large. From Lemma 2.3, $\|\tilde{h}(t, \cdot)\|_{L^1(\mathbb{R})}$ is constant in time. Hence the existence of an asymptotic density associated with h_0 is guaranteed. The following is the main result of this article.

Theorem 1.1. *Let $s \geq 3$ and $u \in C([0, \infty); H^s(\mathbb{R})) \cap C^1([0, \infty); H^{s-1}(\mathbb{R}))$ be a global solution to (1.1). Let $h = u - u_{xx}$ be the momentum density. Suppose $h_0(\cdot) = h(0, \cdot) \geq 0$ has compact support. For $t > 0$, let $\tilde{h}(t, y) = th(t, ty)$. Suppose that there is a unique asymptotic density μ associated with h_0, that is, $\tilde{h}(t, \cdot) \rightharpoonup \mu$ as $t \to \infty$. Then there exist finitely or countably infinitely many $m_i, \alpha_i \in [0, \infty)$ such that*

$$\mu = \sum_i m_i \delta_{\alpha_i}, \tag{1.3}$$

where δ_{α_i} is the delta function supported at α_i, and

(a) $\alpha_i \neq \alpha_j$ if $i \neq j$, and in case there are infinitely many i's, $\alpha_i \to 0$ as $i \to \infty$;

(b) $m_i > 0$ and $\sum_i m_i = \|h_0\|_{L^1(\mathbb{R})}$;

(c) for any i, $\alpha_i \in [0, M]$, where $M = \|u\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})}$;

(d) for all i, $\alpha_i \leq (3/4)m_i$.

Notice that $\mu = \sum_i m_i \delta_{\alpha_i}$ is the asymptotic density of a positive combination of non-interacting peakons $v(t, x) = \sum_i (m_i/2)e^{-|x-\alpha_i t|}$, which is not necessarily a solution though we still use the phrase 'asymptotic density' for convenience. Indeed it is straightforward to check that in the distribution sense, $g(t, x) := (v - v_{xx})(t, x) = \sum_i m_i \delta_{\alpha_i t}(x)$. Hence

$\tilde{g}(t,x) = tg(t,tx) = \sum_i tm_i\delta_{\alpha_i t}(tx) = \sum_i m_i\delta_{\alpha_i}(x)$, with the last equality following from (in suggestive integral notations)

$$\int_{\mathbb{R}} tm_i\delta_{\alpha_i t}(ty)\phi(y)dy = \int_{\mathbb{R}} m_i\delta_{\alpha_i t}(z)\phi\left(\frac{z}{t}\right)dz = m_i\phi(\alpha_i) = \langle m_i\delta_{\alpha_i}, \phi\rangle$$

for all $\phi \in C_c(\mathbb{R})$. As it is independent of t, it is also the asymptotic density of v. Hence Theorem 1.1 says that if $\tilde{h}(t,\cdot)$ does not oscillate as $t \to \infty$, then a global strong solution satisfying the hypothesis behaves as a positive combination of peakons moving to the right at different speeds in large time. We have not proved that a larger m_i corresponds to a larger α_i, meaning that a higher peakon moves faster.

This approach of asymptotic density was introduced by Chen and Frid [28] to study the asymptotic behavior of the entropy solutions of conservation laws. It has been used to discuss the asymptotic behavior of the vorticity of the two dimensional incompressible Euler equation by Iftimie, Lopes and Nussenzveig [29,30]. See also [31] for an exposition. The incompressible Euler equations and the Camassa–Holm equation are similar, with the vorticity for the former corresponding to the momentum density for the latter (see e.g. [32,33]). There are also differences. The two dimensional Euler flow preserves volume and the vorticity is transported along the particle trajectories, but the same do not hold for the Camassa–Holm flow and its momentum density. Nonetheless, we show that the approach in [28,31,29,30] can still be used to study the asymptotic properties of the momentum density and the Camassa–Holm flow.

We mention here some other works on asymptotic properties. Xin and Zhang [34,35] proved that under certain assumptions, a $C([0,\infty)\times\mathbb{R}) \cap L^\infty([0,\infty), H^1(\mathbb{R}))$ weak solution tends to 0 pointwise as $t \to \infty$. An asymptotic result is implicit in [36, Theorem 1.4], which implies that for a smooth solution with fast decaying initial value, $\lim_{x\to\pm\infty} e^{\pm x}u(t,x) = c_\pm(t)$ with $c_+(t) > 0$ increasing and $c_-(t) < 0$ decreasing. Another kind of asymptotic behavior is derived in [32], where certain ratios involving the momentum density, the deformation tensor and the nonlocal term are shown to be vaguely monotonic along the trajectories.

We give some preliminaries in Section 2 and prove Theorem 1.1 in Section 3.

2. Preliminaries

We recall some facts and derive some properties of the scaled momentum density. We first quote a theorem on the existence of global strong solutions [5–7] for the Camassa–Holm equation.

Theorem 2.1. (a) *([6, p. 304], [7, Theorem 3.1]) Let $u_0 \in H^s(\mathbb{R})$, $s \geq 3$. There exists a maximal time $T^* = T^*(u_0)$ so that for all $T < T^*$, (1.1) admits a unique solution $u = u(\cdot, u_0) \in C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}))$.*

(b) *([5, Theorem 4.1], [7, Theorem 4.1]) Suppose in addition that $h_0 = u_0 - u_0''$ does not change sign on $\mathbb{R}$. Then $T^* = \infty$, i.e. the solution in* (a) *exists globally in time.*

Lemma 2.2. *Let $s \geq 3$, $T > 0$ and $u \in C([0,T]; H^s(\mathbb{R}))$ be a solution to (1.1).*

(a) *([5, Theorem 3.1]) Then*

$$\begin{cases} \dfrac{d\eta(t,x)}{dt} = u(t,\eta(t,x)), & t \in (0,T), x \in \mathbb{R}, \\ \eta(0,x) = x, & x \in \mathbb{R} \end{cases} \tag{2.1}$$

has a unique solution $\eta \in C^1([0,T)\times\mathbb{R},\mathbb{R})$. For any $t \in [0,T)$, $\eta(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is an increasing diffeomorphism.

(b) *If supp $h_0 \subset [a,b]$, then for all $t \in (0,T)$, supp $h(t,\cdot) \subset [\eta(t,a), \eta(t,b)]$.*

(c) *If $h_0 \geq 0$, then for $t \in [0,T)$, $h(t,\cdot) \geq 0$.*

Proof. (b) and (c) follows from $h_0(x) = h(t,\eta(t,x))\eta_x^2(t,x)$ for $(t,x) \in [0,T)\times\mathbb{R}$ [5, Lemma 3.2]. □

Lemma 2.3. *Let $s \geq 3$ and $u \in C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R}))$ a solution to (1.1). Suppose that $h_0 \in L^1(\mathbb{R})$ and $h_0 \geq 0$. Then for $t \geq 0$,*

$$\|\tilde{h}(t,\cdot)\|_{L^1(\mathbb{R})} = \|h(t,\cdot)\|_{L^1(\mathbb{R})} = \|h_0(\cdot)\|_{L^1(\mathbb{R})}. \tag{2.2}$$

Proof. From $h_0 \geq 0$ and Lemma 2.2(c), $h(t,\cdot) \geq 0$ for $t > 0$. The first equality of (2.2) results from a change of variable. For the second, use (1.2) and integrate by parts to get

$$\frac{d}{dt}\int_{\mathbb{R}} h(t,x)dx = \int_{\mathbb{R}}(-uh_x - 2u_xh) = \int_{\mathbb{R}} -u_xhdx = 0. \quad \square \tag{2.3}$$

Lemma 2.4. *Let $s \geq 3$, $u_0 \in H^s(\mathbb{R})$ with supp h_0 compact and $h_0 \geq 0$. Let $u \in C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R}))$ be the global solution of (1.1) guaranteed by Theorem 2.1. Then there is an interval $[a,b]$ with $-\infty < a \leq 0 \leq M := \|u\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})} \leq b < \infty$, such that supp $\tilde{h}(t,\cdot) \subset [a,b]$ for $t \geq 1$.*

Proof. That $u_0 \in H^s(\mathbb{R})$, $s \geq 3$ implies that $h_0 \in H^{s-2}(\mathbb{R}) \subset L^\infty(\mathbb{R})$. The compactness of supp h_0 further implies that $h_0 \in L^1(\mathbb{R})$. From the hypothesis and Lemma 2.2(c), $h(t,\cdot) \geq 0$ for all $t \geq 0$. From the definition of h,

$$u(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}h(t,y)dy. \tag{2.4}$$

Lemma 2.3 and (2.4) imply that for all $(t,x) \in [0,\infty) \times \mathbb{R}$,

$$0 \leq u(t,x) \leq \frac{1}{2}\|h(t,\cdot)\|_{L^1(\mathbb{R})} = \frac{1}{2}\|h_0\|_{L^1(\mathbb{R})} < \infty. \tag{2.5}$$

Hence $M := \|u\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}$ is well defined. Let $\mathrm{supp}(h_0) \subset [c,d]$. By Lemma 2.2, supp $h(t,\cdot) \subset [c, d+Mt]$, or

$$\mathrm{supp}\, \tilde{h}(t,\cdot) \subset \left[\frac{c}{t}, \frac{d}{t} + M\right]. \tag{2.6}$$

Therefore there are $a, b \in \mathbb{R}$, with $-\infty < a \leq 0 \leq M \leq b < \infty$ such that supp $\tilde{h}(t,\cdot) \subset [a,b]$ for $t \geq 1$. □

Lemma 2.5 (*See e.g. [37], [31, Lemma 9.3]*)**.** *Let $\mu \in \mathcal{M}[a,b]$ be a nonnegative measure. Then there exist a nonnegative continuous measure $\nu \in \mathcal{M}[a,b]$, countably (including finitely) many real numbers $m_i > 0$ and $\alpha_i \in [a,b]$ such that*

$$\mu = \nu + \sum_i m_i \delta_{\alpha_i}. \tag{2.7}$$

In particular, given $\epsilon > 0$, there is a $\delta > 0$ such that for an interval $I \subset [a,b]$ with $|I| < \delta$, $\nu(I) < \epsilon$.

3. Proof of the main theorem

We prove Theorem 1.1 in this section. We will obtain information on the asymptotic density μ by testing it with $\psi \in C_c(\mathbb{R})$. To do that, we first derive an equation satisfied by $\tilde{h}$. Integrating the equation against a primitive of ψ and passing to the limit, one of the resulting terms is $\langle \mu, y\psi(y)\rangle$. We obtain the desired information of this term by estimating the other terms. The proof will be given in several lemmas. The assumption of the uniqueness of asymptotic density will not be imposed until Lemma 3.4. In the following, let $s \geq 3$ and $u \in C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R}))$ be a solution of (1.1), with $h_0 \geq 0$ compactly supported.

Lemma 3.1. *Suppose that h satisfies (1.2). Then $\tilde{h}$ satisfies*

$$\frac{\partial}{\partial t}\tilde{h}(t,y) - \frac{\partial}{\partial y}\left[\frac{y\tilde{h}(t,y)}{t}\right] + \frac{u(t,ty)}{t}\frac{\partial}{\partial y}\tilde{h}(t,y) + 2u_x(t,ty)\tilde{h}(t,y) = 0, \tag{3.1}$$

where u_x is the partial derivative of u with respect to the spatial variable.

Proof. (3.1) follows from (1.2) and a straightforward calculation:

$$\frac{\partial}{\partial t}\tilde{h}(t,y) = h(t,ty) + th_t(t,ty) + tyh_x(t,ty),$$
$$-\frac{\partial}{\partial y}\left[\frac{y\tilde{h}(t,y)}{t}\right] = -h(t,ty) - tyh_x(t,ty),$$
$$\frac{u(t,ty)}{t}\frac{\partial}{\partial y}\tilde{h}(t,y) = tu(t,ty)h_x(t,ty),$$
$$2u_x(t,ty)\tilde{h}(t,y) = 2tu_x(t,ty)h(t,ty). \quad \square$$

We record here two formulas frequently used later. From (2.4), we get

$$u(t,tx) = \frac{1}{2}\int_{\mathbb{R}} e^{-|tx-y|}h(t,y)dy = \frac{1}{2}\int_{\mathbb{R}} e^{-t|x-z|}\tilde{h}(t,z)dz. \tag{3.2}$$

Differentiate (2.4) with respect to the spatial variable to get

$$u_x(t,\xi) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{sgn}(y-\xi)e^{-|\xi-y|}h(t,y)dy = \frac{1}{2}\int_{\mathbb{R}} \mathrm{sgn}(tz-\xi)e^{-|\xi-tz|}\tilde{h}(t,z)dz.$$

Hence

$$(u_x)(t,tx) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{sgn}\,(z-x)e^{-t|x-z|}\tilde{h}(t,z)dz. \tag{3.3}$$

Proposition 3.2. *Suppose $h_0 \geq 0$ and has compact support. Let $\psi \in C_c(\mathbb{R})$, and $\phi \in C(\mathbb{R})$ be given by $\phi(x) = \int_{-\infty}^{x} \psi(y)dy$. Let $t_k \to \infty$ be a sequence of time such that $\tilde{h}(t_k, \cdot) \rightharpoonup \mu$, an asymptotic density with $\mu = \nu + \sum_i m_i \delta_{\alpha_i}$, ν continuous and nonnegative, $m_i > 0$, $\alpha_i \in [a, b]$ as given by Lemma 2.5. Then*

$$\limsup_{k\to\infty} \left| \int_{\mathbb{R}} \phi(y)u(t_k, t_k y)\frac{\partial}{\partial y}[\tilde{h}(t_k, y)]dy + 2\int_{\mathbb{R}} t_k\phi(y)u_x(t_k, t_k y)\tilde{h}(t_k, y)dy \right| \leq \frac{3}{4}\sum_i |\psi(\alpha_i)|m_i^2. \tag{3.4}$$

Notice that the terms on the left hand side of (3.4) are the third and fourth terms in (3.1) multiplied by $t\phi(y)$. ϕ will give a test function ψ after integration by parts.

Proof. Fix $\psi \in C_c(\mathbb{R})$. We divide the proof into the following steps.
Step 1. From the properties of μ and ν, one can easily obtain the following (compare [31, pp. 160–161]). Given $\epsilon > 0$, there is an $N = N(\epsilon)$ such that

$$\sum_{i>N} m_i < \frac{\epsilon}{4}. \tag{3.5}$$

There is a $\delta = \delta(\epsilon) > 0$ such that if I is an interval with $|I| < \delta$,

$$\nu(I) < \epsilon/4, \tag{3.6}$$

$$\mu([\alpha_i - 2\delta, \alpha_i + 2\delta]) < m_i(1+\epsilon), \quad i = 1, \ldots, N, \tag{3.7}$$

$$[\alpha_i - \delta, \alpha_i + \delta] \cap [\alpha_j - \delta, \alpha_j + \delta] = \emptyset \quad \text{for } i \neq j,\ 1 \leq i, j \leq N. \tag{3.8}$$

$$|\psi(y) - \psi(\alpha_i)| < \epsilon \quad \text{for } y \in [\alpha_i - 2\delta, \alpha_i + 2\delta],\ 1 \leq i \leq N. \tag{3.9}$$

Furthermore, as $\tilde{h}(t_k, \cdot) \rightharpoonup \mu$, there is a $K_0 > 0$ such that for any integer $k > K_0$,

$$\int_{[\alpha_i - 2\delta, \alpha_i + 2\delta]} \tilde{h}(t_k, y)dy < m_i(1+\epsilon) \tag{3.10}$$

and for any interval $I \subset \mathbb{R} - \bigcup_{i=1}^{\infty}[\alpha_i - \delta/2, \alpha_i + \delta/2]$ with $|I| < \delta$,

$$\int_I \tilde{h}(t_k, y)dy < \epsilon. \tag{3.11}$$

For $i = 1, \ldots, N$, let $E_i = [\alpha_i - \delta, \alpha_i + \delta]$, $F_i = [\alpha_i - 2\delta, \alpha_i + 2\delta]$, $E = \cup_{i=1}^N E_i$.
Step 2. Define

$$\begin{aligned} B_k &:= \int_{\mathbb{R}} \phi(y)u(t_k, t_k y)\frac{\partial}{\partial y}[\tilde{h}(t_k, y)]dy + 2\int_{\mathbb{R}} t_k\phi(y)u_x(t_k, t_k y)\tilde{h}(t_k, y)dy \\ &= -\int_{\mathbb{R}} \psi(y)u(t_k, t_k y)\tilde{h}(t_k, y)dy + \int_{\mathbb{R}} t_k\phi(y)u_x(t_k, t_k y)\tilde{h}(t_k, y)dy \\ &:= C_k + D_k. \end{aligned} \tag{3.12}$$

By (3.2),

$$\begin{aligned} C_k &= -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \psi(y)e^{-t_k|y-z|}\tilde{h}(t_k, y)\tilde{h}(t_k, z)dydz \\ &= -\frac{1}{2}\left(\sum_{i=1}^N \iint_{\substack{|z-y|<\delta/2 \\ y\in E_i}}\right) - \frac{1}{2}\iint_{\substack{|z-y|<\delta/2 \\ y\in E^C}} - \frac{1}{2}\iint_{|z-y|>\delta/2} \psi(y)e^{-t_k|y-z|}\tilde{h}(t_k, y)\tilde{h}(t_k, z)dydz \\ &= C_{k1} + C_{k2} + C_{k3}. \end{aligned} \tag{3.13}$$

Using (2.2), we have

$$|C_{k3}| \leq \frac{1}{2}\iint_{|z-y|>\delta/2} |\psi(y)|e^{-t_k\delta/2}\tilde{h}(t_k, y)\tilde{h}(t_k, z)dydz \leq \frac{1}{2}e^{-t_k\delta/2}\|\psi\|_{L^\infty(\mathbb{R})}\|h_0\|^2_{L^1(\mathbb{R})}. \tag{3.14}$$

From (3.11) and (2.2),

$$\begin{aligned} |C_{k2}| &\leq \frac{1}{2}\iint_{\substack{|z-y|<\delta/2 \\ y\in E^C}} |\psi(y)|\tilde{h}(t_k, y)\tilde{h}(t_k, z)dydz \\ &\leq \frac{\epsilon}{2}\int_{y\in E^c} |\psi(y)|\tilde{h}(t_k, y)dy \leq \frac{\epsilon}{2}\|\psi\|_{L^\infty(\mathbb{R})}\|h_0\|_{L^1(\mathbb{R})}. \end{aligned} \tag{3.15}$$

From (3.9) and (3.10),

$$\begin{aligned} |C_{k1}| &\le \frac{1}{2}\sum_{i=1}^{N}\iint_{\substack{|z-y|<\delta/2\\ y\in E_i}} |\psi(y)|\tilde{h}(t_k,y)\tilde{h}(t_k,z)dydz \\ &\le \frac{1}{2}\sum_{i=1}^{N}\iint_{\substack{y\in E_i\\ z\in F_i}} |\psi(y)|\tilde{h}(t_k,y)\tilde{h}(t_k,z)dydz \\ &\le \frac{1}{2}\sum_{i=1}^{N}(|\psi(\alpha_i)|+\epsilon)m_i^2(1+\epsilon)^2. \end{aligned} \tag{3.16}$$

Step 3. We estimate D_k in (3.12) as we have done for C_k. Using (3.3),

$$D_k = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \phi(y)t_k \mathrm{sgn}(z-y)e^{-t_k|z-y|}\tilde{h}(t_k,y)\tilde{h}(t_k,z)dydz. \tag{3.17}$$

As it is symmetric in y and z, there is $\xi=\xi(y,z)$ between y and z such that

$$\begin{aligned} D_k &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \phi(z)t_k \mathrm{sgn}(y-z)e^{-t_k|z-y|}\tilde{h}(t_k,y)\tilde{h}(t_k,z)dydz \\ &= \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} (\phi(y)-\phi(z))t_k \mathrm{sgn}(z-y)e^{-t_k|z-y|}\tilde{h}(t_k,y)\tilde{h}(t_k,z)dydz \\ &= \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} \psi(\xi)t_k(y-z)\mathrm{sgn}(z-y)e^{-t_k|z-y|}\tilde{h}(t_k,y)\tilde{h}(t_k,z)dydz \\ &= \frac{1}{4}\left(\sum_{i=1}^{N}\iint_{\substack{|z-y|<\delta/2\\ y\in E_i}}\right)+\frac{1}{4}\iint_{\substack{|z-y|<\delta/2\\ y\in E^c}}+\frac{1}{4}\iint_{|z-y|>\delta/2}\psi(\xi)t_k(y-z)\mathrm{sgn}(z-y)e^{-t_k|z-y|}\tilde{h}(t_k,y)\tilde{h}(t_k,z)dydz \\ &:= D_{k1}+D_{k2}+D_{k3}. \end{aligned} \tag{3.18}$$

We estimate the D_{ki}'s below. Notice that for $\eta>0$, $\eta e^{-\eta}\le \eta/(\eta^2/2)=2/\eta$. Hence for $|y-z|>\delta/2$,

$$t_k|y-z|e^{-t_k|y-z|} \le \frac{2}{t_k|y-z|} < \frac{4}{t_k\delta}.$$

Together with (2.2),

$$|D_{k3}| \le \frac{1}{4}\|\psi\|_{L^\infty}\iint_{|y-z|>\delta/2} t_k|y-z|e^{-t_k|z-y|}\tilde{h}(t_k,y)\tilde{h}(t_k,z)dydz \le \frac{\|\psi\|_{L^\infty(\mathbb{R})}}{t_k\delta}\|h_0\|^2_{L^1(\mathbb{R})}. \tag{3.19}$$

Notice that when $\eta>0$, $\eta e^{-\eta}\le 1$. Hence from (3.11) and (2.2),

$$\begin{aligned} |D_{k2}| &\le \frac{1}{4}\|\psi\|_{L^\infty}\iint_{\substack{|y-z|<\delta/2\\ y\in E^c}} t_k|y-z|e^{-t_k|z-y|}\tilde{h}(t_k,y)\tilde{h}(t_k,z)dydz \\ &\le \frac{1}{4}\|\psi\|_{L^\infty}\int_{y\in E^c}\tilde{h}(t_k,y)\left(\int_{[y-\delta/2,y+\delta/2]}\tilde{h}(t_k,z)dz\right)dy \\ &\le \frac{\epsilon}{4}\|\psi\|_{L^\infty}\int_{\mathbb{R}}\tilde{h}(t_k,y)dy = \frac{\epsilon}{4}\|\psi\|_{L^\infty(\mathbb{R})}\|h_0\|_{L^1(\mathbb{R})}. \end{aligned} \tag{3.20}$$

From (3.9) and (3.10),

$$|D_{k1}| \le \frac{1}{4}\sum_{i=1}^{N}(|\psi(\alpha_i)|+\epsilon)\int_{y\in E_i}\tilde{h}(t_k,y)dy\int_{z\in F_i}\tilde{h}(t_k,z)dz \le \frac{1}{4}\sum_{i=1}^{N}(|\psi(\alpha_i)|+\epsilon)m_i^2(1+\epsilon)^2. \tag{3.21}$$

Step 4. From (3.12)–(3.21),

$$|B_k| \le \|\psi\|_{L^\infty}\|h_0\|_{L^1}\left(\frac{3\epsilon}{4}+\frac{e^{-t_k\delta/2}\|h_0\|_{L^1}}{2}+\frac{\|h_0\|_{L^1}}{t_k\delta}\right)+\frac{3}{4}\sum_{i=1}^{N}(|\psi(\alpha_i)|+\epsilon)m_i^2(1+\epsilon)^2. \tag{3.22}$$

Let $k \to \infty$ to get

$$\limsup_{k\to\infty} |B_k| \le \frac{3\epsilon}{4} \|\psi\|_{L^\infty} \|h_0\|_{L^1} + \frac{3}{4} \sum_{i=1}^{N} (|\psi(\alpha_i)| + \epsilon) m_i^2 (1+\epsilon)^2.$$

As $\epsilon > 0$ is arbitrary, the proposition is proved. □

Lemma 3.3. *Let $\psi \in C_c(\mathbb{R})$, $\phi \in C^1(\mathbb{R})$ be given by $\phi(x) = \int_{-\infty}^{x} \psi(y)dy$. Let*

$$A[t;\psi] := -\int_{\mathbb{R}} \phi(y) \frac{\partial}{\partial y}[y\tilde{h}(t,y)]dy,$$
$$B[t;\psi] := -\int_{\mathbb{R}} \phi(y) u(t,ty) \frac{\partial}{\partial y}[\tilde{h}(t,y)]dy - 2\int_{\mathbb{R}} t\phi(y) u_x(t,ty)\tilde{h}(t,y)dy.$$

Then

$$\limsup_{t\to\infty} (B[t;\psi] - A[t;\psi]) \ge 0. \tag{3.23}$$

Proof. *Step* 1. Let $f(t) := \int_{\mathbb{R}} \phi(y)\tilde{h}(t,y)dy$. Then from (3.1),

$$\begin{aligned} f'(t) &= \int_{\mathbb{R}} \phi(y) \frac{\partial}{\partial t}[\tilde{h}(t,y)]dy \\ &= \frac{1}{t}\int_{\mathbb{R}} \phi(y)\frac{\partial}{\partial y}[y\tilde{h}(t,y)]dy - \frac{1}{t}\int_{\mathbb{R}} \phi(y)u(t,ty)\frac{\partial}{\partial y}[\tilde{h}(t,y)]dy - \frac{1}{t}\int_{\mathbb{R}} 2t\phi(y)u_x(t,ty)[\tilde{h}(t,y)]dy \\ &= -\frac{1}{t}A[t;\psi] + \frac{1}{t}B[t;\psi]. \end{aligned}$$

Integrate from t to t^2, we get

$$f(t^2) - f(t) = \int_t^{t^2} \frac{B[s;\psi] - A[s;\psi]}{s} ds.$$

Step 2. We claim that $L = \limsup_{s\to\infty} (B[s;\psi] - A[s;\psi]) < \infty$. To see this, recall from Lemma 2.4 that for all $t \in [1,\infty)$, supp $\tilde{h}(t,\cdot) \subset [a,b]$, with $-\infty < a \le 0 \le \|u\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})} \le b < \infty$. For such t's, from (2.2),

$$\begin{aligned} |A[t;\psi]| &= \left| -\int_{\mathbb{R}} \phi(y)\frac{\partial}{\partial y}[y\tilde{h}(t,y)]dy \right| = \left| \int_{\mathbb{R}} \psi(y) y\tilde{h}(t,y)dy \right| \\ &\le \|\psi\|_{L^\infty} \left| \int_{[a,b]} y\tilde{h}(t,y)dy \right| \\ &\le \|\psi\|_{L^\infty(\mathbb{R})} \max\{|a|,|b|\} \|h_0\|_{L^1(\mathbb{R})}. \end{aligned} \tag{3.24}$$

We claim that $B[t;\psi]$ is also bounded in $t \in [1,\infty)$. To see this, write

$$B[t;\psi] = -\int_{\mathbb{R}} \phi(y)\frac{\partial}{\partial y}[u(t,ty)\tilde{h}(t,y)]dy - \int_{\mathbb{R}} \phi(y) t u_x(t,ty)\tilde{h}(t,y)dy = I + II. \tag{3.25}$$

From (2.2) and (2.5),

$$|I| = \left| \int_{\mathbb{R}} \psi(y)u(t,ty)\tilde{h}(t,y)dy \right| \le \frac{1}{2} \|\psi\|_{L^\infty(\mathbb{R})} \|h_0\|^2_{L^1(\mathbb{R})}. \tag{3.26}$$

Using (3.3),

$$\begin{aligned} II &= -\int_{\mathbb{R}} \phi(y) t u_x(t,ty)\tilde{h}(t,y)dy \\ &= -\frac{1}{2}\iint_{|y-z|\ge 1} -\frac{1}{2}\iint_{|y-z|<1} \phi(y) t[\operatorname{sgn}(z-y)]e^{-t|y-z|}\tilde{h}(t,z)\tilde{h}(t,y)dzdy \\ &= II_1 + II_2. \end{aligned} \tag{3.27}$$

Notice that when $|y-z| \ge 1$, $te^{-t|y-z|} \le 1$. Hence by (2.2),

$$|II_1| \le \frac{1}{2}\iint_{[a,b]^2} |\phi(y)| te^{-t|y-z|}\tilde{h}(t,z)\tilde{h}(t,y)dzdy \le \frac{1}{2}\|\phi\|_{C[a,b]}\|h_0\|^2_{L^1(\mathbb{R})} \le \frac{1}{2}\|\psi\|_{L^1(\mathbb{R})}\|h_0\|^2_{L^1(\mathbb{R})}. \tag{3.28}$$

Using the symmetry of y and z in II_2, there is a $\xi = \xi(y,z)$ between y and z such that

$$\begin{aligned} II_2 &= -\frac{1}{2}\iint_{|y-z|<1} \phi(z)t[\mathrm{sgn}(y-z)]e^{-t|y-z|}\tilde{h}(t,z)\tilde{h}(t,y)dzdy \\ &= -\frac{1}{4}\iint_{|y-z|<1} [\phi(y)-\phi(z)]t[\mathrm{sgn}(z-y)]e^{-t|y-z|}\tilde{h}(t,z)\tilde{h}(t,y)dzdy \\ &= -\frac{1}{4}\iint_{|y-z|<1} \psi(\xi)t(y-z)[\mathrm{sgn}(z-y)]e^{-t|y-z|}\tilde{h}(t,z)\tilde{h}(t,y)dzdy. \end{aligned}$$

As $\eta e^{-\eta} \le 1$ for $\eta > 0$, (2.2) implies that

$$|II_2| \le \frac{1}{4}\iint_{|y-z|<1} |\psi(\xi)|t|y-z|e^{-t|y-z|}\tilde{h}(t,z)\tilde{h}(t,y)dzdy \le \frac{1}{4}\|\psi\|_{L^\infty(\mathbb{R})}\|h_0\|^2_{L^1(\mathbb{R})}. \tag{3.29}$$

From (3.25) to (3.29), $B[t;\psi]$ is bounded in t. Together with (3.24), $L \in (-\infty,\infty)$.

Step 3. From the last step, for any $\epsilon > 0$, there is a $K > 0$ such that when $s > K$, $B[s;\psi] - A[s;\psi] < L + \epsilon$. Hence when $t^2 > t > K$,

$$f(t^2) - f(t) < \int_t^{t^2} \frac{L+\epsilon}{s}ds = (L+\epsilon)\log t.$$

Now for $r \in [1,\infty)$, (2.2) gives

$$f(r) = \int_a^b \phi(y)\tilde{h}(r,y)dy \le \|\phi\|_{C[a,b]}\|h_0\|_{L^1} \le \|\psi\|_{L^1(\mathbb{R})}\|h_0\|_{L^1(\mathbb{R})}.$$

Hence

$$0 = \lim_{t\to\infty} \frac{f(t^2)-f(t)}{\log t} \le L + \epsilon.$$

As ϵ is arbitrary, we get the desired result. □

Lemma 3.4. *Suppose that $\{\tilde{h}(t,\cdot)\}$ has a unique asymptotic density $\mu \in \mathcal{M}[a,b]$. That is, $\tilde{h}(t,\cdot) \rightharpoonup \mu$ as $t \to \infty$. Suppose $\mu = \nu + \sum_i m_i\delta_{\alpha_i}$ with $\nu \ge 0$ a continuous measure and $m_i > 0$ (Lemma 2.5). Then for all $\psi \in C_c(\mathbb{R})$, $A[t;\psi] \to \langle y\mu, \psi(y)\rangle$ as $t \to \infty$, and*

$$|\langle y\mu, \psi(y)\rangle| \le \frac{3}{4}\sum_i |\psi(\alpha_i)|m_i^2. \tag{3.30}$$

Proof. As before, let $\phi(x) = \int_{-\infty}^x \psi(y)dy$. As $\tilde{h}(t,\cdot) \rightharpoonup \mu$ when $t \to \infty$,

$$A[t;\psi] = \int_{\mathbb{R}} \psi(y)y\tilde{h}(t,y)dy \to \langle y\mu, \psi(y)\rangle.$$

On the other hand, Proposition 3.2 gives $\limsup_{t\to\infty} |B[t;\psi]| \le \frac{3}{4}\sum_i |\psi(\alpha_i)|m_i^2$. Together with Lemma 3.3, we have

$$\langle y\mu, \psi(y)\rangle = \lim_{t\to\infty} A[t;\psi] \le \limsup_{t\to\infty} B[t;\psi] \le \frac{3}{4}\sum_i |\psi(\alpha_i)|m_i^2.$$

Replacing ψ by $-\psi$, we get (3.30). □

Lemma 3.5. *Let μ, ν, α_i and m_i for $i = 1, 2, \ldots$ be as in Lemma 3.4. Then*

(a) *supp μ, supp $\nu \subset [0, M]$, $\alpha_i \in [0, M]$, recalling that $M := \|u\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}$.*

(b) *$\alpha_i \le (3/4)m_i$, $i = 1, 2, \ldots$.*

(c) *If there are infinitely many m_i, then $\alpha_i \to 0$ as $i \to \infty$.*

Proof. (a) Formula (2.6) and $\tilde{h}(t,\cdot) \rightharpoonup \mu$ implies that supp $\mu \in [0, M]$. Together with $m_i > 0$, $\nu \ge 0$ and (2.7), we get supp $\nu \subset [0, M]$ and $\alpha_i \in [0, M]$.

(b) Fix i. If $\alpha_i = 0$, the conclusion holds. Assume $\alpha_i > 0$. Then from (3.7), given $\epsilon > 0$, there is a $\delta > 0$ such that $[\alpha_i - \delta, \alpha_i + \delta] \subset (0,\infty)$ and $\mu[\alpha_i - \delta, \alpha_i + \delta] < m_i + \epsilon$. If $\alpha_j \in [\alpha_i - \delta, \alpha_i + \delta]$ and $j \ne i$, $m_i\delta_{\alpha_i} + m_j\delta_{\alpha_j} \le \mu$ implies that $m_j < \epsilon$.

Let $\psi \in C_c(\alpha_i - \delta, \alpha_i + \delta)$, $\psi \geq 0$ with $\max \psi = \psi(\alpha_i)$. Then by (3.30),

$$\begin{aligned} m_i\alpha_i\psi(\alpha_i) &= \langle m_i\delta_{\alpha_i}, y\psi(y)\rangle \leq \langle \mu, y\psi(y)\rangle \\ &\leq \frac{3}{4}\left(m_i^2\psi(\alpha_i) + \sum_{j\neq i}\psi(\alpha_j)m_j^2\right). \end{aligned} \tag{3.31}$$

If α_j is not in $[\alpha_i - \delta, \alpha_i + \delta]$, then $\psi(\alpha_j) = 0$. In any case,

$$\psi(\alpha_j)m_j^2 \leq \psi(\alpha_j)m_j\epsilon \leq \psi(\alpha_i)m_j\epsilon.$$

Together with (3.31), we get

$$m_i\alpha_i\psi(\alpha_i) \leq \frac{3}{4}\psi(\alpha_i)\left[m_i^2 + \epsilon\sum_{j\neq i} m_j\right].$$

Hence $m_i\alpha_i \leq (3/4)[m_i^2 + \epsilon \sum_{j\neq i} m_j]$. Let $\epsilon \to 0$ to get $\alpha_i \leq (3/4)m_i$.

(c) That $\mu = \nu + \sum_i m_i\delta_{\alpha_i}$ is a finite measure, $\nu \geq 0$ and $m_i > 0$ implies that $\sum_i m_i < \infty$. Hence $m_i \to 0$ as $i \to \infty$ and (b) implies that $\alpha_i \to 0$. □

Lemma 3.6. *Let μ, ν, α_i and m_i be as in Lemmas 3.4 and 3.5. Suppose μ does not have discrete parts over $(a', b') \subset \mathbb{R} - \{0\}$, that is, no α_i is in (a', b'). Then*

(a) $\mu|_{(a',b')} = 0$.

(b) $\nu = 0$, *or equivalently* $\mu = \sum_i m_i\delta_{\alpha_i}$.

Proof. The proof is given in [31, p. 167, Lemma 9.10, and the proof of Theorem 9.8]. We outline it here for completeness. For (a), let $\psi \in C_c(a', b')$. Then $\operatorname{supp} \psi \cap \{\alpha_1, \alpha_2, \ldots\} = \emptyset$. It follows from (3.30) that $\langle y\mu, \psi(y)\rangle = 0$. Hence $y\mu|_{(a',b')} = 0$. As $(a', b') \subset \mathbb{R} \setminus \{0\}$, $\mu|_{(a',b')} = 0$.

To prove (b), notice from Lemma 3.5(c) that any $\alpha > 0$ is not an accumulation point of $\{\alpha_i\}$. Hence there is a $\delta \in (0, \alpha)$ such that $\{\alpha_i\} \cap [(\alpha - \delta, \alpha + \delta) \setminus \{\alpha\}] = \emptyset$. By (a), $\nu|_{(\alpha-\delta,\alpha)\cup(\alpha,\alpha+\delta)} = 0$. As ν is a continuous measure, $\nu|_{(\alpha-\delta,\alpha+\delta)} = 0$. As $\alpha \in (0, \infty)$ is arbitrary and $\operatorname{supp} \nu \subset [0, M]$ by Lemma 3.5(a), we get $\nu|_{\mathbb{R}\setminus\{0\}} = 0$. Again ν continuous implies that $\nu = 0$. □

Proof of Theorem 1.1. That μ has the form (1.3) is proved in Lemma 3.6(b). Part (a) is proved in Lemma 3.5(c). To prove (b), notice from Lemma 3.6(b) and (2.2) that

$$\sum_i m_i = \sum_i m_i\delta_{\alpha_i}(\mathbb{R}) = \mu(\mathbb{R}) = \lim_{t\to\infty}\int_{\mathbb{R}} \tilde{h}(t, y)dy = \|h_0\|_{L^1(\mathbb{R})}.$$

(c) and (d) are Lemma 3.5(a)(b) respectively. □

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