BGP-REFLECTION FUNCTORS AND LUSZTIG'S SYMMETRIES OF MODIFIED QUANTIZED ENVELOPING ALGEBRAS

JIE XIAO AND MINGHUI ZHAO

Abstract. Let $\mathbf{U}$ be the quantized enveloping algebra and $\dot{\mathbf{U}}$ its modified form. Lusztig gives some symmetries on $\mathbf{U}$ and $\dot{\mathbf{U}}$. Since the realization of $\mathbf{U}$ by the reduced Drinfeld double of the Ringel-Hall algebra, one can apply the BGP-reflection functors to the double Ringel-Hall algebra to obtain Lusztig's symmetries on $\mathbf{U}$ and their important properties, for instance, the braid relations. In this paper, we define a modified form $\dot{\mathcal{H}}$ of the Ringel-Hall algebra and realize the Lusztig's symmetries on $\dot{\mathbf{U}}$ by applying the BGP-reflection functors to $\dot{\mathcal{H}}$.

1. Introduction

Let $\mathbf{U}$ be the quantized enveloping algebra associated to a symmetrizable generalized Cartan matrix. Lusztig introduces some symmetries T_i acting on an integrable $\mathbf{U}$-module and then on the quantized enveloping algebra $\mathbf{U}$ ([1][2][3]). Let $\dot{\mathbf{U}}$ be the modified quantized enveloping algebra obtained from $\mathbf{U}$ by modifying the Cartan part $\mathbf{U}^0$ to $\oplus_{\lambda\in P}\mathbb{Q}(v)\mathbf{1}_\lambda$. This algebra has same representations with $\mathbf{U}$. Lusztig also introduces some symmetries T_i acting on the modified quantized enveloping algebra $\dot{\mathbf{U}}$ ([3]).

Let $\mathcal{H}_q^*(\Lambda)$ be the Ringel-Hall algebra associated to a finite dimensional hereditary algebra Λ. Then the composition subalgebra $\mathcal{C}_q^*(\Lambda)$ realizes the positive part $\mathbf{U}^+$ of the quantized enveloping algebra by the Ringel-Green Theorem ([4][5]). One can extend the Ringel-Green theorem to the Drinfeld double version and realize the whole $\mathbf{U}$ by the reduced Drinfeld double of the composition algebra ([6]). These work give a connection between the representation theory of finite dimensional hereditary algebras and quantized enveloping algebras.

Via the Ringel-Hall algebra approach, one can apply the BGP-reflection functors to the quantum enveloping algebras $\mathbf{U}^+$ and $\mathbf{U}$ to obtain Lusztig's symmetries and their properties in a conceptual way ([7][8]). This method gives a precise construction of Lusztig's symmetries not only in the quantum enveloping algebras, also for the whole Drinfeld doubles of Ringel-Hall algebras ([9][10]).

In this paper, we define a modified form $\dot{\mathcal{H}}_q^*(\Lambda)$ of the Ringel-Hall algebra $\mathcal{H}_q^*(\Lambda)$. We apply the BGP-reflection functors to obtain Lusztig's symmetries on $\dot{\mathcal{H}}_q^*(\Lambda)$. Viewing the modified quantized enveloping algebra $\dot{\mathbf{U}}$ as a subalgebra of $\dot{\mathcal{H}}_q^*(\Lambda)$, we get a precise construction of Lusztig's symmetries on $\dot{\mathbf{U}}$. From this construction,

Date: February 19, 2017.
2000 *Mathematics Subject Classification.* 16G20, 17B37.
Key words and phrases. BGP-reflection functors, Lusztig's symmetries, Ringel-Hall algebras.
This work was supported by NSF of China (Grant No. 11131001).

we can obtain important properties of Lusztig's symmetries, for instance, the braid relations.

In Section 2, we first give the basic notation of quantized enveloping algebras and modified quantized enveloping algebras; then we recall the definition of Lusztig's symmetries on $\mathbf{U}$ and $\dot{\mathbf{U}}$. In Section 3, we recall the definition of the Ringel-Hall algebra $\mathcal{H}_q^*(\Lambda)$ and define a modified form $\dot{\mathcal{H}}_q^*(\Lambda)$ of it. In Section 4, we recall the BGP-reflection functors and define the corresponding maps from $\dot{\mathcal{H}}_q^*(\Lambda)$ to $\dot{\mathcal{H}}_q^*(\sigma_i\Lambda)$ induced by them. We prove in Section 6 that these maps induce algebra isomorphisms from $\dot{\mathbf{U}}$ to itself, which coincide to the Lusztig's symmetries on $\dot{\mathbf{U}}$ and satisfy the braid relations. In Section 5, we define Lusztig's symmetries on $\dot{\mathcal{H}}_q^*(\Lambda)$ and find the precise relation between these symmetries and the maps induced by the BGP-reflection functors.

2. Quantized enveloping algebras and their modified forms

2.1. **Quantized enveloping algebras.** Denote by $\mathbb{Q}$ the field of rational numbers and $\mathbb{Z}$ the ring of integers. Let I be a finite index set with $|I| = n$ and $A = (a_{ij})_{i,j\in I}$ be a generalized Cartan matrix. Denote by $r(A)$ the rank of A. Let $P^\vee$ be a free abelian group of rank $2n-r(A)$ with a $\mathbb{Z}$-basis $\{h_i|i\in I\}\cup\{d_s|s=1,\dots,n-r(A)\}$ and $\mathfrak{h} = \mathbb{Q}\otimes_{\mathbb{Z}} P^\vee$ be the $\mathbb{Q}$-linear space spanned by $P^\vee$. We call $P^\vee$ the dual weight lattice and $\mathfrak{h}$ the Cartan subalgebra. We also define the weight lattice to be $P = \{\lambda\in\mathfrak{h}^*|\lambda(P^\vee)\subset\mathbb{Z}\}$.

Set $\Pi^\vee = \{h_i|i\in I\}$ and choose a linearly independent subset $\Pi = \{\alpha_i|i\in I\}\subset\mathfrak{h}^*$ satisfying $\alpha_j(h_i) = a_{ij}$ and $\alpha_j(d_s) = 0$ or 1 for $i,j\in I$, $s = 1,\dots,n-\mathrm{rank}A$. The elements of Π are called simple roots, and the elements of $\Pi^\vee$ are called simple coroots. The quintuple $(A,\Pi,\Pi^\vee,P,P^\vee)$ is called a Cartan datum associated with the generalized Cartan matrix A. Let W be the Weyl group generated by simple reflections s_i for all $i\in I$. There exists a bilinear form $(-,-)$ on $\mathfrak{h}^*$ ([11]).

We recall the definition of the quantized enveloping algebras. Assume that $A = (a_{ij})_{i,j\in I}$ is a symmetrizable generalized Cartan matrix and $D = \mathrm{diag}(\varepsilon_i|i\in I)$ is its symmetrizing matrix.

Fix an indeterminate v. For $n\in\mathbb{Z}$, we set

$$[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}},$$

and $[0]_v! = 1$, $[n]_v! = [n]_v[n-1]_v\cdots[1]_v$ for $n\in\mathbb{Z}_{>0}$. For nonnegative integers $m\geq n\geq 0$, the analogues of binomial coefficients are given by

$$\begin{bmatrix} m \\ n \end{bmatrix}_v = \frac{[m]_v!}{[n]_v![m-n]_v!}.$$

Then $[n]_v$ and $\begin{bmatrix} m \\ n \end{bmatrix}_v$ are elements of the field $\mathbb{Q}(v)$.

The quantized enveloping algebra $\mathbf{U}$ associated with a Cartan datum $(A,\Pi,\Pi^\vee,P,P^\vee)$ is an associative algebra over $\mathbb{Q}(v)$ with $\mathbf{1}$ generated by the elements E_i, $F_i (i\in I)$ and $K_\mu (\mu\in P^\vee)$ subject to the following relations:

(1) $$K_0 = \mathbf{1}, K_\mu K_{\mu'} = K_{\mu+\mu'} \text{ for all } \mu,\mu'\in P^\vee;$$

(2) $$K_\mu E_i K_{-\mu} = v^{\alpha_i(\mu)}E_i \text{ for all } i\in I,\ \mu\in P^\vee;$$

$$K_\mu F_i K_{-\mu} = v^{-\alpha_i(\mu)} E_i \text{ for all } i \in I,\ \mu \in P^\vee; \tag{3}$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{\tilde{K}_i - \tilde{K}_{-i}}{v_i - v_i^{-1}} \text{ for all } i, j \in I; \tag{4}$$

for $i \neq j$, setting $b = 1 - a_{ij}$,

$$\sum_{k=0}^{b} (-1)^k E_i^{(k)} E_j E_i^{(b-k)} = 0; \tag{5}$$

for $i \neq j$, setting $b = 1 - a_{ij}$,

$$\sum_{k=0}^{b} (-1)^k F_i^{(k)} F_j F_i^{(b-k)} = 0. \tag{6}$$

Here, $\tilde{K}_\nu = \Pi_{i \in I} K_{\varepsilon_i \nu_i h_i}$ for $\nu = \sum_{i \in I} \nu_i h_i$, $v_i = v^{\varepsilon_i}$ and $E_i^{(n)} = E_i^n/[n]_{v_i}!$, $F_i^{(n)} = F_i^n/[n]_{v_i}!$.

Let $\mathbf{U}^+$ (resp. $\mathbf{U}^-$) be the subalgebra of $\mathbf{U}$ generated by the elements E_i (resp. F_i) for $i \in I$, and let $\mathbf{U}^0$ be the subalgebra of $\mathbf{U}$ generated by K_μ for $\mu \in P^\vee$. We know that the quantized enveloping algebra has the triangular decomposition

$$\mathbf{U} \cong \mathbf{U}^- \otimes \mathbf{U}^0 \otimes \mathbf{U}^+.$$

Let $\mathbf{f}$ be the associative algebra defined by Lusztig in [3], which is generated by $\theta_i (i \in I)$ subject to the following relations

$$\sum_{k=0}^{b} (-1)^k \theta_i^{(k)} \theta_j \theta_i^{(b-k)} = 0,$$

where $i \neq j$, $b = 1 - a_{ij}$ and $\theta_i^{(n)} = \theta_i^n/[n]_{v_i}!$. There exist well-defined $\mathbb{Q}(v)$-algebra monomorphisms $\mathbf{f} \to \mathbf{U}(x \mapsto x^+)$ and $\mathbf{f} \to \mathbf{U}(x \mapsto x^-)$ with image $\mathbf{U}^+$ and $\mathbf{U}^-$ respectively satisfying $E_i = \theta_i^+$ and $F_i = \theta_i^-$.

2.2. **Modified quantized enveloping algebras.** Let us recall the definition of the modified form $\dot{\mathbf{U}}$ of $\mathbf{U}$ in [3].

If $\lambda', \lambda'' \in P$, we set

$${}_{\lambda'}\mathbf{U}_{\lambda''} = \mathbf{U} \Big/ \left(\sum_{\mu \in P^\vee} (K_\mu - v^{\lambda'(\mu)}) \mathbf{U} + \sum_{\mu \in P^\vee} \mathbf{U}(K_\mu - v^{\lambda''(\mu)}) \right).$$

Let $\pi_{\lambda',\lambda''} : \mathbf{U} \to_{\lambda'} \mathbf{U}_{\lambda''}$ be the canonical projection and

$$\dot{\mathbf{U}} = \bigoplus_{\lambda', \lambda'' \in P} {}_{\lambda'}\mathbf{U}_{\lambda''}.$$

Consider the weight space decomposition $\mathbf{U} = \oplus_\beta \mathbf{U}(\beta)$, where β runs through $\mathbb{Z}I$ and $\mathbf{U}(\beta) = \{x \in \mathbf{U} | K_\mu x K_\mu^{-1} = v^{\beta(\mu)} x \text{ for all } \mu \in P^\vee\}$. The image of summands $\mathbf{U}(\beta)$ under $\pi_{\lambda',\lambda''}$ form the weight space decomposition ${}_{\lambda'}\mathbf{U}_{\lambda''} = \oplus_\beta {}_{\lambda'}\mathbf{U}_{\lambda''}(\beta)$. Note that ${}_{\lambda'}\mathbf{U}_{\lambda''}(\beta) = 0$ unless $\lambda' - \lambda'' = \beta$.

There is a natural associative $\mathbb{Q}(v)$-algebra structure on $\dot{\mathbf{U}}$ inherited from that of $\mathbf{U}$. It is defined as follows: for any $\lambda_1', \lambda_1'', \lambda_2', \lambda_2'' \in P$, $\beta_1, \beta_2 \in \mathbb{Z}I$ such that $\lambda_1' - \lambda_1'' = \beta_1, \lambda_2' - \lambda_2'' = \beta_2$ and any $x \in \mathbf{U}(\beta_1), y \in \mathbf{U}(\beta_2)$,

$$\pi_{\lambda_1',\lambda_1''}(x)\pi_{\lambda_2',\lambda_2''}(y) = \begin{cases} \pi_{\lambda_1',\lambda_2''}(xy) & \text{if } \lambda_1'' = \lambda_2' \\ 0 & \text{otherwise} \end{cases} .$$

Let $\mathbf{1}_\lambda = \pi_{\lambda,\lambda}(\mathbf{1})$, where $\mathbf{1}$ is the unit element of $\mathbf{U}$. Then they satisfy $\mathbf{1}_\lambda \mathbf{1}_{\lambda'} = \delta_{\lambda,\lambda'}\mathbf{1}_\lambda$. In general, there is no unit element in the algebra $\dot{\mathbf{U}}$. However the family $(\mathbf{1}_\lambda)_{\lambda\in P}$ can be regarded locally as the unit element in $\dot{\mathbf{U}}$.

Note that ${}_{\lambda'}\mathbf{U}_{\lambda''} = \mathbf{1}_{\lambda'}\dot{\mathbf{U}}\mathbf{1}_{\lambda''}$. We define $\dot{\mathbf{U}}\mathbf{1}_\lambda = \oplus_{\lambda'\in P}\mathbf{1}_{\lambda'}\dot{\mathbf{U}}\mathbf{1}_\lambda$. Then $\dot{\mathbf{U}} = \oplus_{\lambda\in P}\dot{\mathbf{U}}\mathbf{1}_\lambda$.

2.3. **Lusztig's symmetries on $\dot{\mathbf{U}}$.** In [3], Lusztig introduces some symmetries on $\mathbf{U}$, which is now called Lusztig's symmetries.

Fix $i \in I$. Define $T_i : \mathbf{U} \to \mathbf{U}$ on the generators as follows:

$$\begin{aligned} &T_i(E_i) = -F_i\tilde{K}_i, T_i(F_i) = -\tilde{K}_{-i}E_i;\\ &T_i(E_j) = \sum_{r+s=-\alpha_j(h_i)} (-1)^r v_i^{-r} E_i^{(s)} E_j E_i^{(r)} \text{for } j \neq i;\\ &T_i(F_j) = \sum_{r+s=-\alpha_j(h_i)} (-1)^r v_i^{r} F_i^{(r)} F_j F_i^{(s)} \text{for } j \neq i;\\ &T_i(K_\mu) = K_{\mu-\alpha_i(\mu)h_i}. \end{aligned}$$

Lusztig also introduces symmetries $T_i : \dot{\mathbf{U}} \to \dot{\mathbf{U}}$ induced by the symmetries on $\mathbf{U}$. We write the following formulas:

$$\begin{aligned} &T_i(E_i\mathbf{1}_\lambda) = -v_i^{-\lambda(h_i)}F_i\mathbf{1}_{s_i\lambda};\\ &T_i(F_i\mathbf{1}_\lambda) = -v_i^{-(2-\lambda(h_i))}E_i\mathbf{1}_{s_i\lambda};\\ &T_i(E_j\mathbf{1}_\lambda) = \sum_{r+s=-\alpha_j(h_i)} (-1)^r v_i^{-r} E_i^{(s)} E_j E_i^{(r)}\mathbf{1}_{s_i\lambda} \text{ for } j \neq i;\\ &T_i(F_j\mathbf{1}_\lambda) = \sum_{r+s=-\alpha_j(h_i)} (-1)^r v_i^{r} F_i^{(r)} F_j F_i^{(s)}\mathbf{1}_{s_i\lambda} \text{ for } j \neq i. \end{aligned}$$

3. Ringel-Hall algebras and their modified form

3.1. **Ringel-Hall algebras.** In this subsection, we recall the definition of Ringel-Hall algebras, following the notations in [12], [8] and [10].

Let k be a finite field and Λ be a finite dimensional hereditary k-algebra. According to [12], we can identity Λ with the tensor algebra of a k-species. A valued graph $(\Gamma, \mathbf{d})$ is a finite set Γ together with nonnegative integers d_{ij} for all $i, j \in \Gamma$ such that $d_{ii} = 0$ and there exist positive integers $\{\varepsilon_i\}_{i\in\Gamma}$ satisfying

$$d_{ij}\varepsilon_j = d_{ji}\varepsilon_i \text{ for } i, j \in \Gamma.$$

Given a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$, there is a valued graph $(\Gamma, \mathbf{d})$ corresponding to it.

An orientation Ω of a valued graph $(\Gamma, \mathbf{d})$ is given by an order on each edge $\{i, j\}$, which is indicated by an arrow $i \to j$. We call $Q = (\Gamma, \mathbf{d}, \Omega)$ a valued quiver.

We assume that $Q = (\Gamma, \mathbf{d}, \Omega)$ is connected and contains no cycles. Let $\mathcal{S} = (F_i, {}_iM_j)_{i,j\in\Gamma}$ be a reduced k-species of type Q, that is, for all $i, j \in \Gamma$, ${}_iM_j$ is an

F_i-F_j-bimodule, where F_i and F_j are finite extensions of k in an algebraic closure and $\dim({}_iM_j)_{F_j} = d_{ij}$ and $\dim_k(F_i) = \varepsilon_i$. A k-representation $(V_i, {}_j\varphi_i)$ of $\mathcal{S}$ is given by vector spaces $(V_i)_{F_i}$ for any $i \in \Gamma$ and F_j-linear mapping ${}_j\varphi_i : V_i \otimes_i M_j \to V_j$ for any $i \to j$. Such a representation is called finite dimensional if $\sum_{i\in\Gamma} \dim_k V_i < \infty$. We denote by rep-$\mathcal{S}$ the category of finite dimensional representations of $\mathcal{S}$ over k. Let Λ be the tensor algebra of $\mathcal{S}$. Then the category rep-$\mathcal{S}$ is equivalent to the module category mod-Λ of finite dimensional modules over Λ.

Given three modules L, M and N in mod-Λ, denote by g^L_{MN} the number of Λ-submodules W of L such that $W \simeq N$ and $L/W \simeq N$ in mod-Λ. Let $v = \sqrt{|k|} \in \mathbb{C}$, $\mathcal{P}$ be the set of isomorphism classes of finite dimensional (nilpotent) Λ-modules and ind$(\mathcal{P})$ be the set of isomorphism classes of indecomposable finite dimensional (nilpotent) Λ-modules. The Ringel-Hall algebra $\mathcal{H}_q(\Lambda)$ of Λ is by definition the $\mathbb{Q}(v)$-space with basis $\{u_{[M]} | [M] \in \mathcal{P}\}$ whose multiplication is given by

$$u_{[M]}u_{[N]} = \sum_{[L]\in\mathcal{P}} g^L_{MN} u_{[L]}.$$

It is easily seen that $\mathcal{H}_q(\Lambda)$ is associative $\mathbb{Q}(v)$-algebra with unit $u_{[0]}$, where 0 denotes the zero module.

For each representation $V = (V_i, {}_j\varphi_i)$ in rep-$\mathcal{S}$, the dimension vector of V is defined to be $\underline{\dim}V = (\dim_{F_i} V_i)_{i\in\Gamma} \in \mathbb{N}^\Gamma$. For $V, W \in$ rep-$\mathcal{S}$, The Euler form is defined by

$$\langle \underline{\dim}V, \underline{\dim}W\rangle = \sum_{i\in\Gamma} \varepsilon_i a_i b_i - \sum_{i\to j} d_{ij}\varepsilon_j a_i b_j,$$

where $\underline{\dim}V = (a_1, \ldots, a_n)$ and $\underline{\dim}W = (b_1, \ldots, b_n)$. It is well known that

$$\langle \underline{\dim}V, \underline{\dim}W\rangle = \dim_k \mathrm{Hom}_\Lambda(V, W) - \dim_k \mathrm{Ext}_\Lambda(V, W).$$

Further, the symmetric Euler form is defined as

$$(\underline{\dim}V, \underline{\dim}W) = \langle \underline{\dim}V, \underline{\dim}W\rangle + \langle \underline{\dim}W, \underline{\dim}V\rangle.$$

Both $\langle -, -\rangle$ and $(-, -)$ are well defined on the Grothendieck group $G(\Lambda)$ of mod-Λ. In fact, the Grothendieck group $G(\Lambda)$ with the symmetric Euler form is a Cartan datum.

Let $I \subset \mathcal{P}$ be the set of isomorphism classes of (nilpotent) simple Λ-modules, which can be identified with Γ. Then the Euler form and the symmetric Euler form are defined on $\mathbb{Z}I$. We also identify $\mathbb{N}^\Gamma$ with $\mathbb{N}I$ and regard $\underline{\dim}V$ as an element in $\mathbb{N}I$ for each representation $V = (V_i, {}_j\varphi_i)$ in rep-$\mathcal{S}$. For each $\alpha \in \mathcal{P}$, we fix a representation V_α in the isomorphism class α and let $M(\alpha)$ be the corresponding Λ-module. For $\alpha, \beta \in \mathcal{P}$, we set

$$\langle \alpha, \beta\rangle = \langle \underline{\dim}V_\alpha, \underline{\dim}V_\beta\rangle$$

and

$$(\alpha, \beta) = (\underline{\dim}V_\alpha, \underline{\dim}V_\beta).$$

Note that for $\alpha, \beta \in \mathcal{P}$, $(\alpha, \beta) = (\sum_{i\in I} a_i\alpha_i, \sum_{i\in I} b_i\alpha_i)$, where $\underline{\dim}V_\alpha = \sum a_i i$ and $\underline{\dim}V_\beta = \sum b_i i$. Hence, we also use α to express the element $\sum_{i\in I} a_i\alpha_i$ in P and the element $\sum_{i\in I} a_i h_i$ in $P^\vee$.

The twisted Ringel-Hall algebra $\mathcal{H}_q^*(\Lambda)$ is defined as follows. Set $\mathcal{H}_q^*(\Lambda) = \mathcal{H}_q(\Lambda)$ as $\mathbb{Q}(v)$-vector space and define the multiplication by

$$u_{[M]} * u_{[N]} = v^{\langle \underline{\dim} M, \underline{\dim} N\rangle} \sum_{[L]\in\mathcal{P}} g_{MN}^L u_{[L]}.$$

The composition algebra $\mathcal{C}_q^*(\Lambda)$ is a subalgebra of $\mathcal{H}_q^*(\Lambda)$ generated by $u_i = u_{[S_i]}$, $i \in I$, where S_i is the (nilpotent) simple module corresponding to $i \in I$. For any Λ-module M, we denote

$$\langle M\rangle = v^{-\dim M + \dim \mathrm{End}_\Lambda(M)} u_{[M]}.$$

Note that $\{\langle M\rangle | M \in \mathcal{P}\}$ is a $\mathbb{Q}(v)$-basis of $\mathcal{H}_q^*(\Lambda)$.

Then we consider the generic form of Ringel-Hall algebras. Let Q be a valued quiver and Λ_k the corresponding finite dimensional hereditary algebra of a k-species which is of type Q. Denote by $\mathcal{H}_q^*(\Lambda_k)$ the twisted Ringel-Hall algebra of Λ_k. Let $\mathcal{K}$ be a set of finite fields k such that the set $\{q_k = |k| | k \in \mathcal{K}\}$ is infinite and R be an integral domain containing $\mathbb{Q}$ and an element v_{q_k} such that $v_{q_k}^2 = q_k$ for each $k \in \mathcal{K}$. For each $k \in \mathcal{K}$, we consider the composition algebra $\mathcal{C}_q^*(\Lambda_k)$ which is the R-subalgebra of $\mathcal{H}_q^*(\Lambda_k)$ generated by the elements $u_i(k)$. Consider the direct product

$$\mathcal{H}^*(Q) = \prod_{k\in\mathcal{K}} \mathcal{H}_q^*(\Lambda_k)$$

and the elements $v = (v_{q_k})_{k\in\mathcal{K}}$, $v^{-1} = (v_{q_k}^{-1})_{k\in\mathcal{K}}$ and $u_i = (u_i(k))_{k\in\mathcal{K}}$. By $\mathcal{C}^*(Q)_{\mathcal{A}}$ we denote the subalgebra of $\mathcal{H}^*(Q)$ generated by v, v^{-1} and u_i over $\mathbb{Q}$, where $\mathcal{A} = \mathbb{Q}[v, v^{-1}]$. We may regard it as the $\mathcal{A}$-algebra generated by u_i where v is considered as an indeterminate. Finally, denote by $\mathcal{C}^*(Q) = \mathbb{Q}(v) \otimes \mathcal{C}^*(Q)_{\mathcal{A}}$ the generic twisted composition algebra of type Q.

Remark 3.1. *If Q is a Dynkin quiver, then the generic composition algebra of Q can be defined directly using Hall polynomials.*

Then we have the following well-known result of Green and Ringel ([4][5]).

Theorem 3.2. *Let Q be a valued quiver, A be the associated generalized Cartan matrix, and $\mathbf{f}$ be the Lusztig's algebra of type A. Then the correspondence $u_i \mapsto \theta_i$, $i \in I$ induces an algebra isomorphism from $\mathcal{C}^*(Q)$ to $\mathbf{f}$.*

3.2. **Double Ringel-Hall algebras.** Let Λ be a finite dimensional hereditary algebra. In [6], the reduced Drinfeld double $\mathcal{D}(\Lambda)$ of Λ is defined. As an associative algebra, $\mathcal{D}(\Lambda)$ is generated by $\langle u_\alpha(+)\rangle$, $\langle u_\alpha(-)\rangle(\alpha \in \mathcal{P})$ and $K_\mu(\mu \in P^\vee)$ subject to the following relations ([8]):

$$K_0 = \langle u_0(+)\rangle = \langle u_0(-)\rangle = \mathbf{1},\ K_\mu K_{\mu'} = K_{\mu+\mu'}; \tag{7}$$

$$\langle u_\alpha(+)\rangle\langle u_\beta(+)\rangle = v^{-\langle\beta,\alpha\rangle}\sum_{\lambda\in\mathcal{P}} g^\lambda_{\alpha\beta}\langle u_\lambda(+)\rangle; \tag{8}$$

$$\langle u_\alpha(-)\rangle\langle u_\beta(-)\rangle = v^{-\langle\beta,\alpha\rangle}\sum_{\lambda\in\mathcal{P}} g^\lambda_{\alpha\beta}\langle u_\lambda(-)\rangle; \tag{9}$$

$$K_\mu\langle u_\beta(+)\rangle = v^{\beta(\mu)}\langle u_\beta(+)\rangle K_\mu; \tag{10}$$

$$K_\mu\langle u_\beta(-)\rangle = v^{-\beta(\mu)}\langle u_\beta(-)\rangle K_\mu; \tag{11}$$

$$\begin{aligned}&\sum_{\alpha,\alpha'\in\mathcal{P}} v^{\langle\alpha',\alpha\rangle+(\alpha,\alpha)}\frac{a_{\alpha'}}{a_{\lambda'}}g^{\lambda'}_{\alpha'\alpha}\tilde{K}_{-\alpha}\langle u_{\alpha'}(-)\rangle r'_\alpha(\langle u_\lambda(+)\rangle)\\ = &\sum_{\alpha,\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle+(\beta,\beta)}\frac{a_\alpha}{a_\lambda}g^\lambda_{\alpha\beta}\tilde{K}_\beta\langle u_\alpha(+)\rangle r_\beta(\langle u_{\lambda'}(-)\rangle),\end{aligned} \tag{12}$$

where $\alpha,\beta,\lambda,\lambda'\in\mathcal{P}$, $\mu,\mu'\in P^\vee$ and

$$r'_\alpha(\langle u_\lambda(+)\rangle) = \sum_{\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle+(\alpha,\beta)}g^\lambda_{\alpha\beta}\frac{a_\alpha a_\beta}{a_\lambda}\langle u_\beta(+)\rangle;$$

$$r_\alpha(\langle u_\lambda(-)\rangle) = \sum_{\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle+(\alpha,\beta)}g^\lambda_{\alpha\beta}\frac{a_\alpha a_\beta}{a_\lambda}\langle u_\beta(-)\rangle.$$

From the definition of $\mathcal{D}(\Lambda)$, we have two algebra monomorphisms $(+):\mathcal{H}^*_q(\Lambda)\to\mathcal{D}(\Lambda)$ mapping $\langle M(\lambda)\rangle$ to $u_\lambda(+)$ and $(-):\mathcal{H}^*_q(\Lambda)\to\mathcal{D}(\Lambda)$ mapping $\langle M(\lambda)\rangle$ to $u_\lambda(-)$ for all $\lambda\in\mathcal{P}$.

Consider the weight space decomposition $\mathcal{D}(\Lambda)=\oplus_\beta\mathcal{D}(\Lambda)(\beta)$, where β runs through $\mathbb{Z}I$ and $\mathcal{D}(\Lambda)(\beta)=\{x\in\mathcal{D}(\Lambda)|K_\mu xK_\mu^{-1}=v^{\beta(\mu)}x \text{ for all } \mu\in P^\vee\}$.

Let $\mathcal{D}_c(\Lambda)$ be the subalgebra of $\mathcal{D}(\Lambda)$ generated by $\langle u_i(\pm)\rangle(i\in I)$ and $K_\mu(\mu\in P^\vee)$. In [6], the Green-Ringel Theorem 3.2 is extended to the Drinfeld double version and $\mathcal{D}_c(\Lambda)$ realizes the corresponding quantum enveloping algebra $\mathbf{U}$.

3.3. **Another definition of $\dot{\mathbf{U}}$ and a similar form of $\mathcal{H}^*(\Lambda)$.** In [3], Lusztig gives another definition of $\dot{\mathbf{U}}$ as follows. $\dot{\mathbf{U}}$ can be viewed as the algebra generated by the symbols $x^+\mathbf{1}_\zeta x'^-$ and $x^-\mathbf{1}_\zeta x'^+$ with $x\in\mathbf{f}_\nu, x'\in\mathbf{f}_{\nu'}$ for various $\nu,\nu'\in\mathbb{N}I$ and $\zeta\in P$; these symbols are subject to the following relations (13) to (19):

$$(\theta_i^{(a)})^+\mathbf{1}_\zeta(\theta_j^{(b)})^- = (\theta_j^{(b)})^-\mathbf{1}_{\zeta+a\alpha_i+b\alpha_j}(\theta_i^{(a)})^+ \text{if } i\neq j; \tag{13}$$

(14)
$$(\theta_i^{(a)})^+\mathbf{1}_{-\zeta}(\theta_i^{(b)})^- = \sum_{t\geq 0}\begin{bmatrix} a+b-\zeta(h_i)\\ t\end{bmatrix}_{v_i}(\theta_i^{(b-t)})^-\mathbf{1}_{-\zeta+(a+b-t)\alpha_i}(\theta_i^{(a-t)})^+;$$

$$(\theta_i^{(b)})^-\mathbf{1}_\zeta(\theta_i^{(a)})^+ = \sum_{t\geq 0}\begin{bmatrix} a+b-\zeta(h_i)\\ t\end{bmatrix}_{v_i}(\theta_i^{(a-t)})^+\mathbf{1}_{\zeta-(a+b-t)\alpha_i}(\theta_i^{(b-t)})^+; \tag{15}$$

$$x^+\mathbf{1}_\zeta = \mathbf{1}_{\zeta+\nu}x^+, x^-\mathbf{1}_\zeta = \mathbf{1}_{\zeta-\nu}x^- \text{for } x\in\mathbf{f}_\nu; \tag{16}$$

$$(x^+\mathbf{1}_\zeta)(\mathbf{1}_{\zeta'}x'^-) = \delta_{\zeta,\zeta'}x^+\mathbf{1}_\zeta x'^-, (x^-\mathbf{1}_\zeta)(\mathbf{1}_{\zeta'}x'^+) = \delta_{\zeta,\zeta'}x^-\mathbf{1}_\zeta x'^+; \tag{17}$$

$$(x^+\mathbf{1}_\zeta)(\mathbf{1}_{\zeta'}x'^+) = \delta_{\zeta,\zeta'}\mathbf{1}_{\zeta+\nu}(xx')^+,$$
$$(x^-\mathbf{1}_\zeta)(\mathbf{1}_{\zeta'}x'^-) = \delta_{\zeta,\zeta'}\mathbf{1}_{\zeta-\nu}(xx')^-\text{for } x\in\mathbf{f}_\nu; \tag{18}$$

$$(rx+r'x')^+\mathbf{1}_\zeta = rx^+\mathbf{1}_\zeta + r'x'^+\mathbf{1}_\zeta, (rx+r'x')^-\mathbf{1}_\zeta = rx^-\mathbf{1}_\zeta + r'x'^-\mathbf{1}_\zeta$$
$$\text{for } x,x'\in\mathbf{f}_\nu \text{ and } r,r'\in\mathbb{Q}(v). \tag{19}$$

Let k be a finite field and Λ a finite dimensional hereditary k-algebra. For each $\nu\in\mathbb{N}I$, set

$$\mathcal{H}_q^*(\Lambda)_\nu = \text{span}\{u_{[M]}|\underline{\dim}M=\nu\}.$$

Similarly, we can define $\dot{\mathcal{H}}_q^*(\Lambda)$ as follows. $\dot{\mathcal{H}}_q^*(\Lambda)$ is the algebra generated by the symbols $x^+\mathbf{1}_\zeta x'^-$ and $x^-\mathbf{1}_\zeta x'^+$ with $x\in\mathcal{H}_q^*(\Lambda)_\nu, x'\in\mathcal{H}_q^*(\Lambda)_{\nu'}$ for various $\nu,\nu'\in\mathbb{N}I$ and $\zeta\in P$; these symbols are subject to the following relations (20) to (24):

$$\sum_{\alpha,\alpha'\in\mathcal{P}} v^{\langle\alpha',\alpha\rangle+(\alpha,\alpha)+(\zeta,-\alpha)}\frac{a_{\alpha'}}{a_{\lambda'}}g^{\lambda'}_{\alpha'\alpha}(-1)^{tr\alpha'}v^{m(\alpha')}\langle M(\alpha')\rangle^-\mathbf{1}_{\zeta+\alpha'}(r'_\alpha(\langle M(\lambda)\rangle))^+ =$$
$$\sum_{\alpha,\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle+(\beta,\beta)+(\zeta,\beta)}\frac{a_\alpha}{a_\lambda}g^{\lambda}_{\alpha\beta}(-1)^{tr(\lambda'-\beta)}v^{m(\lambda'-\beta)}\langle M(\alpha)\rangle^+\mathbf{1}_{\zeta-\alpha}(r_\beta(\langle M(\lambda')\rangle))^-$$
$$\text{for all } \lambda,\lambda'\in\mathcal{P}; \tag{20}$$

$$x^+\mathbf{1}_\zeta = \mathbf{1}_{\zeta+\nu}x^+, x^-\mathbf{1}_\zeta = \mathbf{1}_{\zeta-\nu}x^-\text{for } x\in\mathcal{H}_q^*(\Lambda)_\nu; \tag{21}$$

$$(x^+\mathbf{1}_\zeta)(\mathbf{1}_{\zeta'}x'^-) = \delta_{\zeta,\zeta'}x^+\mathbf{1}_\zeta x'^-, (x^-\mathbf{1}_\zeta)(\mathbf{1}_{\zeta'}x'^+) = \delta_{\zeta,\zeta'}x^-\mathbf{1}_\zeta x'^+; \tag{22}$$

$$(x^+\mathbf{1}_\zeta)(\mathbf{1}_{\zeta'}x'^+) = \delta_{\zeta,\zeta'}\mathbf{1}_{\zeta+\nu}(xx')^+,$$
$$(x^-\mathbf{1}_\zeta)(\mathbf{1}_{\zeta'}x'^-) = \delta_{\zeta,\zeta'}\mathbf{1}_{\zeta-\nu}(xx')^-\text{for } x\in\mathcal{H}_q^*(\Lambda)_\nu; \tag{23}$$

$$(rx+r'x')^+\mathbf{1}_\zeta = rx^+\mathbf{1}_\zeta + r'x'^+\mathbf{1}_\zeta, (rx+r'x')^-\mathbf{1}_\zeta = rx^-\mathbf{1}_\zeta + r'x'^-\mathbf{1}_\zeta$$
$$\text{for } x,x'\in\mathcal{H}_q^*(Q)_\nu \text{ and } r,r'\in\mathbb{Q}(v). \tag{24}$$

Here a_λ is the order of the automorphism group of V_λ for $\lambda\in\mathcal{P}$, $tr\alpha=\sum_{i\in I}a_i$, $m(\alpha)=\sum_{i\in I}a_i\varepsilon_i$ if $\alpha=\sum_{i\in I}a_i\alpha_i$, and

$$r_\alpha(\langle M(\lambda)\rangle) = \sum_{\beta\in\mathcal{P}} v^{\langle\beta,\alpha\rangle+(\beta,\alpha)}g^\lambda_{\beta\alpha}\frac{a_\beta a_\alpha}{a_\lambda}\langle M(\beta)\rangle;$$

$$r'_\alpha(\langle M(\lambda)\rangle) = \sum_{\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle+(\beta,\alpha)}g^\lambda_{\alpha\beta}\frac{a_\alpha a_\beta}{a_\lambda}\langle M(\beta)\rangle.$$

Similarly to the case of modified form of quantum group, we have the following direct sums decompositions

$$\dot{\mathcal{H}}_q^*(\Lambda) = \bigoplus_{\zeta\in P}\{x^+1_\zeta x'^-|x,x'\in\mathcal{H}_q^*(\Lambda)\}$$

and

$$\dot{\mathcal{H}}_q^*(\Lambda) = \bigoplus_{\zeta\in P}\{x^-1_\zeta x'^+|x,x'\in\mathcal{H}_q^*(\Lambda)\}.$$

Let $\dot{\mathcal{C}}_q^*(\Lambda)$ be the composition algebra, which is a subalgebra of $\dot{\mathcal{H}}_q^*(\Lambda)$ generated by $u_i^+1_\zeta u_j^-$ and $u_i^-1_\zeta u_j^+$ for all $i,j\in I$ and $\zeta\in P$.

Similarly to the Ringel-Hall algebra case we can consider the generic form

$$\dot{\mathcal{H}}^*(Q) = \prod_{k\in\mathcal{K}} \dot{\mathcal{H}}^*(\Lambda_k)$$

and its generic composition subalgebra $\dot{\mathcal{C}}^*(Q)$ generated by $u_i^+ 1_\zeta u_j^-$ and $u_i^- 1_\zeta u_j^+$ for all $i, j \in I$ and $\zeta \in P$, which is isomorphic to the corresponding modified quantum enveloping algebra $\dot{\mathbf{U}}$. If a formula in $\dot{\mathcal{C}}_q^*(\Lambda)$ is independent of the choice of the field, it can be viewed as a formula in $\dot{\mathcal{C}}^*(Q) \simeq \dot{\mathbf{U}}$.

4. BGP-reflection functors and Lusztig's symmetries

In this section we apply the BGP-reflection functors to the Ringel-Hall algebras and obtain an alternative construction of Lusztig's symmetries on modified quantum enveloping algebras.

4.1. **BGP-reflection functors.** Let $Q = (\Gamma, \mathbf{d}, \Omega)$ be a valued quiver, $\mathcal{S} = (F_i, {}_iM_j)_{i,j\in\Gamma}$ be a k-species of type Q and p be a sink or source of $(\Gamma, \mathbf{d}, \Omega)$. We define a new orientation $\sigma_p\Omega$ of $(\Gamma, \mathbf{d})$ by reversing the direction of arrows along all edges containing p and $\sigma_p Q = (\Gamma, \mathbf{d}, \sigma_p\Omega)$. Let $\sigma_p\mathcal{S}$ be the k-species obtained from $\mathcal{S}$ by replacing ${}_rM_s$ by its k-dual for $r = p$ or $s = p$. Then $\sigma_p\mathcal{S}$ is a reduced k-species of type $\sigma_p Q$. Assume Λ is the corresponding finite dimensional hereditary algebra to $\mathcal{S}$. We denote by $\sigma_i\Lambda$ the corresponding finite dimensional hereditary algebra to $\sigma_i\mathcal{S}$.

Now, we recall the definition of the Bernstein-Gelfand-Ponomarev (BGP) reflection functors $\sigma_p^\pm$: rep-$\mathcal{S}$ →rep-$\sigma_p\mathcal{S}$ ([13] [12] [8]).

Let p be a sink of Ω. For any $V = (V_i, {}_j\varphi_i) \in$ rep-$\mathcal{S}$, define $\sigma_p^+ V = W = (W_i, {}_j\psi_i)$ as follows. Let

$$W_i = V_i \quad \text{for } i \neq p,$$

and W_p be the kernel of

$$\bigoplus\nolimits_{j\to p} V_j \otimes_j M_p \xrightarrow{({}_p\varphi_j)_j} V_p\ ,$$

that is, we have the following exact sequence of vector spaces

$$0 \longrightarrow W_p \xrightarrow{({}_j\kappa_p)_j} \bigoplus\nolimits_{j\to p} V_j \otimes_j M_p \xrightarrow{({}_p\varphi_j)_j} V_p\ .$$

Let

$${}_j\psi_i = {}_j\varphi_i \quad \text{for } i \neq p,$$

and

$${}_j\psi_p = {}_j\bar{\kappa}_p : W_p \otimes_p M_j \to W_j,$$

where ${}_j\bar{\kappa}_p$ corresponds to ${}_j\kappa_p$ under the natural isomorphism

$$\mathrm{Hom}_{F_j}(W_p \otimes_p M_j, W_j) \simeq \mathrm{Hom}_{F_p}(W_p, W_j \otimes_j M_p).$$

For any morphism $f = (f_i) : V \to V'$ in rep-$\mathcal{S}$, define $\sigma_p^+ f = g = (g_i)$ as follows. Let

$$g_i = h_i \quad \text{for } i \neq p$$

and $g_p : W_p \to W'_p$ be the restriction of $\oplus_{j\to p}(f_j \otimes 1)$, that is, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & W_p \arrow[r, "({}_j\kappa_p)_j"] \arrow[d, "g_p"] & \bigoplus_{j\to p} V_j \otimes_j M_p \arrow[r, "({}_p\varphi_j)_j"] \arrow[d, "\bigoplus_{j\to p}(f_j\otimes 1)"] & V_p \arrow[d, "f_p"] \\
0 \arrow[r] & W'_p \arrow[r, "({}_j\kappa'_p)_j"] & \bigoplus_{j\to p} V'_j \otimes_j M_p \arrow[r, "({}_p\varphi'_j)_j"] & V'_p
\end{tikzcd}$$

Similarly, if p is a source of Ω, we can define σ_p^- from rep-$\mathcal{S}$ to rep-$\sigma_p\mathcal{S}$.

For $i \in \Gamma$, let rep-$\mathcal{S}\langle i\rangle$ be the full subcategory of rep-$\mathcal{S}$ containing all representations which do not have V_i as a direct summand, where V_i is the simple representation with $\underline{\dim} V_i = i$. If i is a sink or source, then rep-$\mathcal{S}\langle i\rangle$ is closed under direct summands and extensions. If i is a sink (resp. source), then σ_i^+ : rep-$\mathcal{S}\langle i\rangle \simeq$ rep-$\sigma_i\mathcal{S}\langle i\rangle$ (resp. σ_i^+ : rep-$\mathcal{S}\langle i\rangle \simeq$ rep-$\sigma_i\mathcal{S}\langle i\rangle$) is an equivalence.

4.2. **Construction of Lusztig's symmetries.** Assume i is a sink of Q. We first define a map $\mathcal{T}_i$ from $\dot{\mathcal{H}}_q^*(\Lambda)$ to $\dot{\mathcal{H}}_q^*(\sigma_i\Lambda)$.

For $\lambda \in \mathcal{P}$, assume that $V_\lambda = V_{\lambda_0} \oplus tV_i$ and V_{λ_0} contains no direct summand isomorphic to V_i. Then $\mathrm{Hom}(V_{\lambda_0}, V_i) = 0$ and $\mathrm{Ext}(V_i, V_{\lambda_0}) = 0$. In this case

$$\langle M(\lambda)\rangle = v^{\langle\lambda_0, ti\rangle} u_i^{(t)} \langle M(\lambda_0)\rangle$$

in $\mathcal{H}_q^*(\Lambda)$. We define a map $\mathcal{T}_i : \dot{\mathcal{H}}_q^*(\Lambda) \to \dot{\mathcal{H}}_q^*(\sigma_i\Lambda)$ given by

(25)
$$\mathcal{T}_i(\langle M(\lambda)\rangle^+ \mathbf{1}_\zeta \langle M(\lambda')\rangle^-) = (-1)^{p_1} v^{q_1} u_i^{-(t)} \langle M(\sigma_i^+\lambda_0)\rangle^+ \mathbf{1}_{s_i\zeta} u_i^{+(t')} \langle M(\sigma_i^+\lambda'_0)\rangle^-$$

where $p_1 = t + t' - \lambda'_0(h_i)$ and $q_1 = -\langle ti, \lambda_0\rangle - t^2\varepsilon_i + t\varepsilon_i - (\zeta, t\alpha_i) + \langle\lambda'_0, t'i\rangle - (\lambda'_0, i) + t'^2\varepsilon_i - t'\varepsilon_i + (\zeta, t'\alpha_i)$;

(26)
$$\mathcal{T}_i(\langle M(\lambda')\rangle^- \mathbf{1}_\zeta \langle M(\lambda)\rangle^+) = (-1)^{p_2} v^{q_2} u_i^{+(t')} \langle M(\sigma_i^+\lambda'_0)\rangle^- \mathbf{1}_{s_i\zeta} u_i^{-(t)} \langle M(\sigma_i^+\lambda_0)\rangle^+$$

where $p_2 = t + t' - \lambda'_0(h_i)$ and $q_2 = t^2\varepsilon_i + t\varepsilon_i + \langle\lambda_0, ti\rangle - (\zeta, t\alpha_i) - \langle t'i, \lambda'_0\rangle - (\lambda'_0, i) - t'^2\varepsilon_i - t'\varepsilon_i + (\zeta, t'\alpha_i)$.

In fact, the definition of $\mathcal{T}_i$ is induced by the following formulas:

$$\begin{aligned}
\mathcal{T}_i(\langle M(\lambda)\rangle^+ \mathbf{1}_\zeta) &= \langle M(\sigma_i^+\lambda)\rangle^+ \mathbf{1}_{s_i\zeta} \\
\mathcal{T}_i(\langle M(\lambda)\rangle^- \mathbf{1}_\zeta) &= (-1)^{\lambda(h_i)} v^{-(\lambda, i)} \langle M(\sigma_i^+\lambda)\rangle^- \mathbf{1}_{s_i\zeta}
\end{aligned}$$

if V_λ contains no direct summand isomorphic to V_i and

$$\begin{aligned}
\mathcal{T}_i(u_i^+ \mathbf{1}_\zeta) &= -v^{-(\zeta,\alpha_i)} u_i^- \mathbf{1}_{s_i\zeta} \\
\mathcal{T}_i(u_i^- \mathbf{1}_\zeta) &= -v^{(\zeta,\alpha_i)-2\varepsilon_i} u_i^+ \mathbf{1}_{s_i\zeta}.
\end{aligned}$$

Note that, by the relation (24) in the definition of $\dot{\mathcal{H}}^*(\Lambda)$, we can define $\mathcal{T}_i$ on all the generators of $\dot{\mathcal{H}}^*(\Lambda)$. If we can prove that $\mathcal{T}_i$ keeps the relations (20) to (23), then $\mathcal{T}_i$ induces a map from $\dot{\mathcal{H}}^*(\Lambda)$ to $\dot{\mathcal{H}}^*(\sigma_i\Lambda)$. This is the first main result of this section.

Theorem 4.1. *Let i be a sink. The formula (25) and (26) induces a $\mathbb{Q}(v)$-algebra isomorphism $\mathcal{T}_i : \dot{\mathcal{H}}^*(\Lambda) \simeq \dot{\mathcal{H}}^*(\sigma_i\Lambda)$*

The proof of Theorem 4.1 will be given in the last section.

Let i be a sink. For $j \in I$, if $i = j$, we have $\mathcal{T}_i(u_i^+\mathbf{1}_\zeta) \in \dot{\mathcal{C}}_q^*(\sigma_i\Lambda)$ and $\mathcal{T}_i(u_i^-\mathbf{1}_\zeta) \in \dot{\mathcal{C}}_q^*(\sigma_i\Lambda)$ since $u_i^+\mathbf{1}_\zeta$ and $u_i^-\mathbf{1}_\zeta$ are contained in $\dot{\mathcal{C}}_q^*(\sigma_i\Lambda)$. If $i \neq j$, we have $\mathcal{T}_i(u_j^+\mathbf{1}_\zeta) = \langle M(\sigma_i^+(j))\rangle^+\mathbf{1}_{s_i\zeta}$. Note that $V_{\sigma_i^+(j)}$ is an exceptional object in rep-$\sigma_i\mathcal{S}$. Hence $\langle M(\sigma_i^+(j))\rangle \in \dot{\mathcal{C}}_q^*(\sigma_i\Lambda)$. Hence $\mathcal{T}_i(u_j^+\mathbf{1}_\zeta) \in \dot{\mathcal{C}}_q^*(\sigma_i\Lambda)$. Similarly we have $\mathcal{T}_i(u_j^-\mathbf{1}_\zeta) \in \dot{\mathcal{C}}_q^*(\sigma_i\Lambda)$. Hence $\mathcal{T}_i$ induces an $\mathbb{Q}(v)$-algebra homomorphism from $\dot{\mathcal{C}}_q^*(\Lambda)$ to $\dot{\mathcal{C}}_q^*(\sigma_i\Lambda)$. Note the formula (25) and (26) are independent of the choice of the field. We can consider them as formulas in $\dot{\mathcal{C}}^*(Q)$ and $\dot{\mathcal{C}}^*(\sigma_iQ)$. Since both $\dot{\mathcal{C}}^*(Q)$ and $\dot{\mathcal{C}}^*(\sigma_iQ)$ are isomorphic to $\dot{U}$, $\mathcal{T}_i$ induces a endomorphism on $\dot{U}$, if we identify $\dot{\mathcal{C}}^*(Q)$ and $\dot{\mathcal{C}}^*(\sigma_iQ)$ with $\dot{U}$.

Assume i is a source. For $\lambda \in \mathcal{P}$, assume that $V_\lambda = V_{\lambda_0} \oplus tV_i$ and V_{λ_0} contains no direct summand isomorphic to V_i. Then $\mathrm{Hom}(V_i, V_{\lambda_0}) = 0$ and $\mathrm{Ext}(V_{\lambda_0}, V_i) = 0$. In this case

$$\langle M(\lambda)\rangle = v^{\langle ti,\lambda_0\rangle}\langle M(\lambda_0)\rangle u_i^{(t)}$$

in $\mathcal{H}_q^*(\Lambda)$. We define a map $\mathcal{T}_i' : \dot{\mathcal{H}}_q^*(\Lambda) \to \dot{\mathcal{H}}_q^*(\sigma_i\Lambda)$ given by

$$\mathcal{T}_i'(\langle M(\lambda)\rangle^+\mathbf{1}_\zeta\langle M(\lambda')\rangle^-) = (-1)^{p_1}v^{q_1}\langle M(\sigma_i^+\lambda_0)\rangle^+u_i^{-(t)}\mathbf{1}_{s_i\zeta}\langle M(\sigma_i^+\lambda_0')\rangle^-u_i^{+(t')}$$

where $p_1 = t - t' - \lambda_0'(h_i)$ and $q_1 = \langle ti,\lambda\rangle + t\varepsilon_i + (\zeta, t\alpha_i) - (\lambda_0', i) - t'\varepsilon_i - t'^2\varepsilon_i - (\zeta, t'\alpha_i) - \langle\lambda_0', t'i\rangle$;

$$\mathcal{T}_i'(\langle M(\lambda')\rangle^-\mathbf{1}_\zeta\langle M(\lambda)\rangle^+) = (-1)^{p_2}v^{q_2}\langle M(\sigma_i^+\lambda_0')\rangle^-u_i^{+(t')}\mathbf{1}_{s_i\zeta}\langle M(\sigma_i^+\lambda_0)\rangle^+u_i^{-(t)}$$

where $p_2 = t - t' - \lambda_0'(h_i)$ and $q_2 = -t^2\varepsilon_i + t\varepsilon_i + (\zeta, t\alpha_i) - \langle\lambda_0, ti\rangle - (\lambda_0', i) - t'\varepsilon_i - (\zeta, t'\alpha_i) + \langle t'i, \lambda'\rangle$.

By a similar way, we can prove that $\mathcal{T}_i'$ induces a $\mathbb{Q}(v)$-algebra homomorphism from $\dot{\mathbf{U}}$ to $\dot{\mathbf{U}}$.

Now assume i is a sink of Q. Then i is a source of σ_iQ. We can easily check that $\mathcal{T}_i\mathcal{T}_i' = 1$ and $\mathcal{T}_i'\mathcal{T}_i = 1$. Hence $\mathcal{T}_i$ is a $\mathbb{Q}(v)$-algebra isomorphism with $\mathcal{T}_i'$ as its inverse.

Hence, we have the following theorem.

Theorem 4.2. *Let i be a sink. The formula (25) and (26) induces a $\mathbb{Q}(v)$-algebra isomorphism $\mathcal{T}_i : \dot{\mathbf{U}} \simeq \dot{\mathbf{U}}$.*

Then we will prove that $\mathcal{T}_i$ coincides with T_i.

Proposition 4.3 ([8])**.** *Let $i \neq j \in I$ and $n = a_{ij}$.*

(1) If i is a sink, then in $\mathcal{H}_q^*(\Lambda)$ we have

$$\langle M(\lambda)\rangle = \sum_{t=0}^{n}(-1)^t v_i^{-t}u_i^{(t)}u_ju_i^{(n-t)}$$

where $\lambda \in \mathcal{P}$ is the unique isomorphism class of indecomposable representation with the dimension vector $j + ni$.

(2) If i is a source, then in $\mathcal{H}_q^*(\Lambda)$ we have

$$\langle M(\lambda)\rangle = \sum_{t=0}^{n}(-1)^t v_i^{-t}u_i^{(n-t)}u_ju_i^{(t)}$$

where $\lambda \in \mathcal{P}$ is the unique isomorphism class of indecomposable representation with the dimension vector $j + ni$.

Since i is a sink in Q, i is a source in $\sigma_i Q$, and $V_{\sigma_i^+(j)}$ is a unique indecomposable module in rep-$\sigma_i\mathcal{S}$ with dimension vector $j+ni$ where $n=a_{ij}$. Thus by the Proposition 4.3,

$$\langle M(\sigma_i^+(j))\rangle^+\mathbf{1}_{s_i\zeta}=\sum_{t=0}^{n}(-1)^t v_i^{-t}u_i^{+(n-t)}u_j^+u_i^{+(t)}\mathbf{1}_{s_i\zeta}.$$

Hence

$$\mathcal{T}_i(u_j^+\mathbf{1}_\zeta)=\sum_{t=0}^{n}(-1)^t v_i^{-t}u_i^{+(n-t)}u_j^+u_i^{+(t)}\mathbf{1}_{s_i\zeta}=T_i(u_j^+\mathbf{1}_\zeta).$$

Similarly we can check $\mathcal{T}_i=T_i$ on other generators.

Hence, we have the following theorem.

Theorem 4.4. *If i is a sink, then the isomorphism $\mathcal{T}_i:\dot{\mathbf{U}}\to\dot{\mathbf{U}}$ coincides with T_i.*

4.3. **Braid group relations.** Let $A=(a_{ij})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix. If $d(i,j)=a_{ij}a_{ji}\leq 3$, then the order $m(i,j)$ of s_is_j is finite ([11]). In fact, we have

$$m(i,j)=\begin{cases}2 & \text{if } d(i,j)=0;\\ 3 & \text{if } d(i,j)=1;\\ 4 & \text{if } d(i,j)=2;\\ 6 & \text{if } d(i,j)=3;\\ \infty & \text{if } d(i,j)\geq 4.\end{cases}$$

The braid group of type A is defined by the generators $\{\kappa_i\}_{i\in I}$ and relations

$$\kappa_i\kappa_j\cdots=\kappa_j\kappa_i\ldots$$

for $i\neq j$ with $m(i,j)\leq+\infty$ factors on both sides, where $m(i,j)$ is the order of s_is_j in W, that is,

$$\kappa_i\kappa_j=\kappa_j\kappa_i \quad \text{if } m(i,j)=2;$$
$$\kappa_i\kappa_j\kappa_i=\kappa_j\kappa_i\kappa_j \quad \text{if } m(i,j)=3;$$
$$\kappa_i\kappa_j\kappa_i\kappa_j=\kappa_j\kappa_i\kappa_j\kappa_i \quad \text{if } m(i,j)=4;$$
$$\kappa_i\kappa_j\kappa_i\kappa_j\kappa_i\kappa_j\kappa_i=\kappa_j\kappa_i\kappa_j\kappa_i\kappa_j\kappa_i\kappa_j \quad \text{if } m(i,j)=6. \tag{27}$$

Let Λ be a finite dimensional hereditary algebra, and A be the corresponding generalized Cartan matrix. In [8], the Lusztig's symmetries on $\mathcal{D}_c(\Lambda)$ are constructed as follows.

Theorem 4.5. *Let i be a sink. For all $\lambda\in\mathcal{P}$ and $\mu\in P^\vee$, we write $V_\lambda\simeq V_{\lambda_0}\oplus tV_i$ where V_{λ_0} contain no direct summand isomorphic to V_i. Then the map $\tilde{\mathcal{T}}_i$ is defined as follows:*

$$\tilde{\mathcal{T}}_i(\langle u_\lambda(+)\rangle)=v^{\langle\lambda,ti\rangle}\tilde{K}_{ti}\langle u_i(-)\rangle^{(t)}\langle u_{\sigma_i^+\lambda_0}(+)\rangle; \tag{28}$$

$$\tilde{\mathcal{T}}_i(\langle u_\lambda(-)\rangle)=v^{\langle\lambda,ti\rangle}\tilde{K}_{-ti}\langle u_i(+)\rangle^{(t)}\langle u_{\sigma_i^+\lambda_0}(-)\rangle; \tag{29}$$

$$\tilde{\mathcal{T}}_i(K_\mu)=K_{s_i(\mu)}, \tag{30}$$

induces a $\mathbb{Q}(v)$-algebra isomorphism: $\mathcal{D}_c(\Lambda)\simeq\mathcal{D}_c(\sigma_i\Lambda)$.

In [8], the following theorem is proved.

Theorem 4.6. *For any $i\neq j\in I$ such that $m=m(i,j)\leq+\infty$, $\tilde{\mathcal{T}}_i$ and $\tilde{\mathcal{T}}_j$ satisfy braid group relations (27) of type A as maps on $\mathcal{D}_c(\Lambda)$.*

Let Λ be a finite dimensional hereditary algebra. Similarly to the the relation between $\dot{\mathbf{U}}$ and $\mathbf{U}$, We consider the relation between $\dot{\mathcal{H}}_q^*(\Lambda)$ and $\mathcal{D}(\Lambda)$. For any $\zeta \in P$, we have a surjective linear mapping

$$\begin{array}{rcl} \pi_\zeta : \mathcal{D}(\Lambda) & \to & \dot{\mathcal{H}}_q^*(\Lambda)\mathbf{1}_\zeta \\ \langle u_\alpha(+)\rangle\langle u_\beta(-)\rangle K_\mu & \mapsto & (-1)^{tr(\beta)}v^{m(\beta)+\zeta(\mu)}\langle M(\alpha)\rangle^+\langle M(\beta)^-\rangle\mathbf{1}_\zeta \end{array}$$

where $\beta = \sum_{i\in I} b_i\alpha_i$, $tr(\beta) = \sum_{i\in I} b_i$ and $m(\beta) = \sum_{i\in I} b_i\varepsilon_i$. The kernel of π_ζ is

$$\sum_{\mu\in P^\vee} \mathcal{D}(\Lambda)(K_\mu - v^{\zeta(\mu)}).$$

For any $\zeta, \zeta' \in P$, $\beta \in \mathbb{Z}I$ and any $x \in \mathcal{D}(\Lambda), y \in \mathcal{D}(\Lambda)(\beta)$,

$$\pi_\zeta(x)\pi_{\zeta'}(y) = \begin{cases} \pi_{\zeta'}(xy) & \text{if } \zeta = \zeta' + \beta \\ 0 & \text{otherwise} \end{cases}.$$

Our main result in this subsection is the following.

Theorem 4.7. *Let Λ be a finite dimensional hereditary algebra, and A be the corresponding generalized Cartan matrix. For any $i \neq j \in I$ such that $m = m(i,j) \leq +\infty$, $\mathcal{T}_i$ and $\mathcal{T}_j$ satisfy braid group relations (27) of type A as maps on $\dot{\mathcal{C}}_q^*(\Lambda)$.*

Proof. For all $\lambda \in \mathcal{P}$ and $\mu \in P^\vee$, we write $V_\lambda \simeq V_{\lambda_0} \oplus tV_i$ where V_{λ_0} contain no direct summand isomorphic to V_i. We need to check that for any $\zeta \in P$

$$\pi_{s_i\zeta}(\tilde{\mathcal{T}}_i(\langle u_\lambda(+)\rangle)) = \mathcal{T}_i(\pi_\zeta(\langle u_\lambda(+)\rangle)); \tag{31}$$

$$\pi_{s_i\zeta}(\tilde{\mathcal{T}}_i(\langle u_\lambda(-)\rangle)) = \mathcal{T}_i(\pi_\zeta(\langle u_\lambda(-)\rangle)); \tag{32}$$

$$\pi_{s_i\zeta}(\tilde{\mathcal{T}}_i(K_\mu)) = \mathcal{T}_i(\pi_\zeta(K_\mu)). \tag{33}$$

First

$$\begin{aligned} \pi_{s_i\zeta}(\tilde{\mathcal{T}}_i(\langle u_\lambda(+)\rangle)) &= \pi_{s_i\zeta}(v^{\langle\lambda,ti\rangle}\tilde{K}_{ti}\langle u_i(-)\rangle^{(t)}\langle u_{\sigma_i^+\lambda_0}(+)\rangle) \\ &= v^{\langle\lambda,ti\rangle+(\sigma_i^+\lambda_0-ti,ti)}\pi_{s_i\zeta}(\langle u_i(-)\rangle^{(t)}\langle u_{\sigma_i^+\lambda_0}(+)\rangle\tilde{K}_{ti}) \\ &= v^{\langle\lambda,ti\rangle+(\sigma_i^+\lambda_0-ti,ti)+(s_i\zeta,t\alpha_i)}(-1)^t v^{m(ti)}u_i^{-(t)}\langle M(\sigma_i^+\lambda_0)\rangle^+\mathbf{1}_{s_i\zeta} \\ &= (-1)^t v^{-\langle ti,\lambda_0\rangle-t^2\varepsilon_i+t\varepsilon_i-(\zeta,t\alpha_i)}u_i^{-(t)}\langle M(\sigma_i^+\lambda_0)\rangle^+\mathbf{1}_{s_i\zeta} \\ &= \mathcal{T}_i(\langle M(\sigma_i^+\lambda_0)\rangle^+\mathbf{1}_\zeta) \\ &= \mathcal{T}_i(\pi_\zeta(\langle u_\lambda(+)\rangle)). \end{aligned}$$

Hence we have formula (31). Similarly, we can get formula (32) and (33). Then Theorem 4.6 implies this theorem. $\square$

5. Lusztig's symmetries on the modified form of Ringel-Hall algebras

5.1. **The structure of Ringel-Hall algebras.** First we recall the structure of the Ringel-Hall algebra considered in [14] and [9].

We consider a bilinear form $\psi : \mathcal{H}_q^*(\Lambda) \times \mathcal{H}_q^*(\Lambda)$ as

$$\psi(\langle M(\beta)\rangle, \langle M(\beta')\rangle) = \frac{|V_\beta|}{a_\beta}\delta_{\beta\beta'}$$

for $\beta, \beta' \in \mathcal{P}$.

Let $\mathfrak{d}_0(\Lambda)=\mathcal{C}_q^*(\Lambda)$. We can define $\mathfrak{d}_m(\Lambda)$ and $L_{\pi_m}(\Lambda)$ inductively. For $m\geq 1$, assume $\mathfrak{d}_{m-1}(\Lambda)$ has been constructed. Let $\pi_m\in\mathbb{Z}I$ have smallest trace such that $\mathfrak{d}_{m-1}(\Lambda)_{\pi_m}\neq\mathcal{H}_q^*(\Lambda)_{\pi_m}$. Then $L_{\pi_m}(\Lambda)$ is defined as follow:

$$L_{\pi_m}(\Lambda):=\{x\in\mathcal{H}_q^*(\Lambda)_{\pi_m}|\psi(x,\mathfrak{d}_{m-1}(\Lambda)_{\pi_m})=0\}.$$

We define $\mathfrak{d}_m(\Lambda)$ as the subalgebra of $\mathcal{H}_q^*(\Lambda)$ generated by $\mathfrak{d}_{m-1}(\Lambda)$ and $L_{\pi_m}(\Lambda)$. Hence there is a chain of subalgebras of $\mathcal{H}_q^*(\Lambda)$

$$\mathfrak{d}_0(\Lambda)\subset\mathfrak{d}_1(\Lambda)\subset\ldots\mathfrak{d}_m(\Lambda)\subset\ldots\subset\mathcal{H}_q^*(\Lambda).$$

For $m\geq 1$, let $\eta_m=\dim L_{\pi_m}$. There exists a bases $\{x_{(m,p)}|1\leq p\leq\eta_m\}$ of L_{π_m} and nonzero numbers $\chi_{(m,p)}\in\mathbb{Q}(v), 1\leq p\leq\eta_m$ such that

$$\psi(x_{(m,p)},\chi_{(m,p)}x_{(m,q)})=\frac{-1}{v-v^{-1}}\delta_{pq}.$$

Set $x_i=u_i$ and $J=\{(m,p)|m\geq 1, 1\leq p\leq\eta_m\}$. The elements in the set $\{x_j|j\in I\cup J\}$ generate the Ringel-Hall algebra $\mathcal{H}_q^*(\Lambda)$.

Let $y_i=-v_i^{-1}u_i$ for all $i\in I$ and $y_j=\chi_j x_j$ for all $j\in J$. By [14] and [9], the double Ringel-Hall algebra $\mathcal{D}(\Lambda)$ is generated by the elements $x_i(+),y_i(-),i\in I\cup J$ and $K_\mu,\mu\in P^\vee$ subject to the following relations:

$$K_0=\mathbf{1}, K_\mu K_{\mu'}=K_{\mu+\mu'}\ \text{ for all } \mu,\mu'\in P^\vee; \tag{34}$$

$$K_\mu x_i(+)K_{-\mu}=v^{\delta_i(\mu)}x_i(+)\ \text{ for all } i\in I\cup J,\ \mu\in P^\vee; \tag{35}$$

$$K_\mu y_i(-)K_{-\mu}=v^{-\delta_i(\mu)}y_i(-)\ \text{ for all } i\in I\cup J,\ \mu\in P^\vee; \tag{36}$$

$$x_i(+)y_j(-)-y_j(-)x_i(+)=\delta_{ij}\frac{\tilde{K}_{\delta_i}-\tilde{K}_{-\delta_i}}{v_i-v_i^{-1}}\ \text{ for all } i,j\in I\cup J; \tag{37}$$

for $i\in I$, $j\in I\cup J$ and $i\neq j$, setting $b=1-a_{ij}$,

$$\sum_{k=0}^{b}(-1)^k x_i(+)^{(k)}x_j(+)x_i(+)^{(b-k)}=0, \tag{38}$$

and

$$\sum_{k=0}^{b}(-1)^k y_i(-)^{(k)}y_j(-)y_i(-)^{(b-k)}=0; \tag{39}$$

for any $i,j\in I\cup J$ with $(\delta_i,\delta_j)=0$,

$$x_i(+)x_j(+)=x_j(+)x_i(+),\qquad y_i(-)y_j(-)=y_j(-)y_i(-). \tag{40}$$

Here, $\delta_i=\alpha_i$ for $i\in I$, $\delta_j=\pi_m$ for $j=(m,p)\in J$ and $a_{ij}=2\frac{(\delta_i,\delta_j)}{(\delta_i,\delta_i)}$.

Note that $\tilde{A}=(a_{ij})_{ij\in I\cup J}$ is a Borcherds-Cartan matrix. We can define a modified quantized enveloping algebra $\dot{\mathbf{U}}(\tilde{A})$ of the generalized Kac-Moody algebra associated to $\tilde{A}$. $\dot{\mathbf{U}}(\tilde{A})$ is generated by the elements $E_i\mathbf{1}_\zeta, F_i\mathbf{1}_\zeta$ for all $i\in I\cup J$ and $\zeta\in P$ subject to the following relations:

$$\mathbf{1}_\zeta\mathbf{1}_{\zeta'}=\delta_{\zeta\zeta'}\mathbf{1}_\zeta\ \text{ for all } \zeta,\zeta'\in P; \tag{41}$$

$$E_i\mathbf{1}_\zeta=\mathbf{1}_{\zeta+\delta_i}E_i, F_i\mathbf{1}_\zeta=\mathbf{1}_{\zeta-\delta_i}F_i \text{ for all } i\in I\cup J,\ \zeta\in P; \tag{42}$$

(43)

$$(E_i\mathbf{1}_{\zeta-\delta_j})(F_j\mathbf{1}_\zeta)-(F_j\mathbf{1}_{\zeta+\delta_i})(E_i\mathbf{1}_\zeta)=\delta_{ij}(-1)^{tr\delta_j}v^{-m(\delta_j)}\frac{v^{(\zeta,\delta_i)}-v^{-(\zeta,\delta_i)}}{v_i-v_i^{-1}} \text{ for all } i,j\in I\cup J;$$

for $i\in I$, $j\in I\cup J$ and $i\neq j$, setting $b=1-a_{ij}$,

$$\sum_{k=0}^{b}(-1)^k(E_i^{(k)}\mathbf{1}_{\zeta+(b-k)\delta_i+\delta_j})(E_j\mathbf{1}_{\zeta+(b-k)\delta_i})(E_i^{(b-k)}\mathbf{1}_\zeta)=0, \tag{44}$$

and

$$\sum_{k=0}^{b}(-1)^k(F_i^{(k)}\mathbf{1}_{\zeta-(b-k)\delta_i-\delta_j})(F_j\mathbf{1}_{\zeta-(b-k)\delta_i})(F_i^{(b-k)}\mathbf{1}_\zeta)=0; \tag{45}$$

for any $i,j\in I\cup J$ with $(\delta_i,\delta_j)=0$,

$$(E_i\mathbf{1}_{\zeta+\delta_j})(E_j\mathbf{1}_\zeta)=(E_j\mathbf{1}_{\zeta+\delta_i})(E_i\mathbf{1}_\zeta),\quad (F_i\mathbf{1}_{\zeta-\delta_j})(F_j\mathbf{1}_\zeta)=(F_j\mathbf{1}_{\zeta-\delta_i})(F_i\mathbf{1}_\zeta), \tag{46}$$

where

$$E_i^{(k)}\mathbf{1}_\zeta=\frac{1}{[k]_{v_i}!}\prod_{s=1}^{k}E_i\mathbf{1}_{\zeta+(k-s)\delta_i},$$

$$F_i^{(k)}\mathbf{1}_\zeta=\frac{1}{[k]_{v_i}!}\prod_{s=1}^{k}F_i\mathbf{1}_{\zeta-(k-s)\delta_i}.$$

Since there exists a map $\pi_\zeta:\mathcal{D}(\Lambda)\to\dot{\mathcal{H}}_q^*(\Lambda)\mathbf{1}_\zeta$ for any $\zeta\in P$, the algebra $\dot{\mathcal{H}}_q^*(\Lambda)$ is generated by the elements $x_i^+\mathbf{1}_\zeta, y_i^-\mathbf{1}_\zeta$ for all $i\in I\cup J$ and $\zeta\in P$ subject to the relations (41) to (46). Hence, we have an isomorphism $\iota:\dot{\mathcal{H}}_q^*(\Lambda)\simeq\dot{\mathbf{U}}(\tilde{A})$ mapping $x_i^+\mathbf{1}_\zeta$ (resp. $y_i^-\mathbf{1}_\zeta$) to $E_i\mathbf{1}_\zeta$ (resp. $F_i\mathbf{1}_\zeta$).

There is an operator τ on $\mathcal{H}_q^*(\Lambda)$ defined as follows:

$$\begin{aligned}\tau\langle M(\lambda)\rangle &= (-1)^{tr\alpha}v^{-\tau(\alpha)}\\ &\times\Bigg(\delta_{\lambda 0}+\sum_{m\geq 1}(-1)^m\sum_{\pi\in\mathcal{P},\lambda_1,\dots,\lambda_m\in\mathcal{P}\setminus\{0\}}v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}\times\\ &\quad\frac{a_{\lambda_1}\dots a_{\lambda_m}}{a_\lambda}g^{\lambda}_{\lambda_1,\dots,\lambda_m}g_\pi^{\lambda_1,\dots,\lambda_m}\langle M(\pi)\rangle\Bigg)\end{aligned}$$

where $\lambda\in\mathcal{P}$, $u_\lambda\in\mathcal{H}_q^*(\Lambda)_\alpha,\alpha=\sum_i k_i\alpha_i\in\mathbb{N}[I]$, $tr\alpha=\sum_i k_i$ and $\tau(\alpha)=((\alpha,\alpha)-\sum_i k_i(i,i))/2$.

5.2. **Lusztig's symmetries on the modified form of the Ringel-Hall algebras.** We first recall the definition of Lusztig's symmetries of $\mathcal{D}(\Lambda)$ defined in [9].

For all $i \in I$, define $\tilde{T}_i : \mathcal{D}(\Lambda) \to \mathcal{D}(\Lambda)$ on generators as follows

$$\begin{aligned}
&\tilde{T}_i(x_i(+)) = -y_i(-)\tilde{K}_i, \tilde{T}_i(y_i(-)) = -\tilde{K}_{-i}x_i(+);\\
&\tilde{T}_i(x_j(+)) = \sum_{r+s=-a_{ij}} (-1)^r v_i^{-r} x_i(+)^{(s)} x_j(+) x_i(+)^{(r)} \text{for } i \neq j \in I \cup J;\\
&\tilde{T}_i(y_j(-)) = \sum_{r+s=-a_{ij}} (-1)^r v_i^{r} y_i(-)^{(r)} y_j(-) y_i(-)^{(s)} \text{for } i \neq j \in I \cup J;\\
&\tilde{T}_i(K_\mu) = K_{\mu-\alpha_i(\mu)h_i} \text{for } \mu \in P^\vee.
\end{aligned}$$

Under the maps

$$\pi_\zeta : \mathcal{D}(\Lambda) \to \dot{\mathcal{H}}_q^*(\Lambda)\mathbf{1}_\zeta,$$

Lusztig's symmetries $\tilde{T}_i$ of $\mathcal{D}(\Lambda)$ induce Lusztig's symmetries $T_i : \dot{\mathcal{H}}_q^*(\Lambda) \to \dot{\mathcal{H}}_q^*(\Lambda)$. From the formulas above, we get

$$\begin{aligned}
&T_i(x_i^+\mathbf{1}_\zeta) = -v_i^{-\zeta(h_i)}\tilde{y}_i^-\mathbf{1}_{s_i\zeta} \ \text{ for } \zeta \in P;\\
&T_i(\tilde{y}_i^-\mathbf{1}_\zeta) = -v_i^{-(2-\zeta(h_i))}x_i^+\mathbf{1}_{s_i\zeta} \ \text{ for } \zeta \in P;\\
&T_i(x_j^+\mathbf{1}_\zeta) = \sum_{r+s=-a_{ij}} (-1)^r v_i^{-r} x_i^{+(s)} x_j^+ x_i^{+(r)} \mathbf{1}_{s_i\zeta} \ \text{ for } i \neq j \in I \cup J;\\
&T_i(\tilde{y}_j^-\mathbf{1}_\zeta) = \sum_{r+s=-a_{ij}} (-1)^r v_i^{r} \tilde{y}_i^{-(r)} \tilde{y}_j^- \tilde{y}_i^{-(s)} \mathbf{1}_{s_i\zeta} \ \text{ for } i \neq j \in I \cup J
\end{aligned}$$

where $\tilde{y}_i = (-1)^{tr\delta_i} v^{m(\delta_i)} y_i$ for all $i \in I \cup J$. Note that $\pi_\zeta(y_i(-)) = \tilde{y}_i^-\mathbf{1}_\zeta$.

We define

$$\psi_\zeta^\pm(x^\pm\mathbf{1}_\zeta, x'^\pm\mathbf{1}_\zeta) = \psi(x, x')$$

for every $\zeta \in P$. Let $\mathcal{H}_q^*(\Lambda)\langle i\rangle$ be the subspace of $\mathcal{H}_q^*(\Lambda)$ spanned by the elements in the set

$$\{\langle M(\alpha)\rangle | \alpha \in \mathcal{P}, V_\alpha \in \text{rep-}\mathcal{S}\langle i\rangle\}$$

and $\mathfrak{d}_m(\Lambda)\langle i\rangle = \mathfrak{d}_m(\Lambda) \cap \mathcal{H}_q^*(\Lambda)\langle i\rangle$.

Proposition 5.1. *Let $i \in I$ be a sink. For all $\mu \in P$ and all $x, x' \in \mathcal{H}_q^*(\Lambda)\langle i\rangle_\mu$, we have*

$$\psi_\zeta^\pm(x^\pm\mathbf{1}_\zeta, x'^\pm\mathbf{1}_\zeta) = \psi_{s_i\zeta}^\pm(T_i(x^\pm\mathbf{1}_\zeta), T_i(x'^\pm\mathbf{1}_\zeta))$$

Proof. In [10], it is proved that

$$\psi(x, x') = \psi(\tilde{T}_i(x), \tilde{T}_i(x')).$$

From the definition of $\psi_\zeta^\pm(-,-)$,

$$\begin{aligned}
&\psi_{s_i\zeta}^\pm(T_i(x^\pm\mathbf{1}_\zeta), T_i(x'^\pm\mathbf{1}_\zeta))\\
= \ &\psi_{s_i\zeta}^\pm(\tilde{T}_i(x)^\pm\mathbf{1}_{s_i\zeta}, \tilde{T}_i(x')^\pm\mathbf{1}_{s_i\zeta})\\
= \ &\psi(\tilde{T}_i(x), \tilde{T}_i(x'))\\
= \ &\psi(x, x')\\
= \ &\psi_\zeta^\pm(x^\pm\mathbf{1}_\zeta, x'^\pm\mathbf{1}_\zeta).
\end{aligned}$$

□

5.3. **Relation between the Lusztig's symmetries and the BGP-reflection functors.** In this subsection, we consider the relation between the Lusztig's symmetries and the BGP-reflection functors. The method is similar to these in [10].

Proposition 5.2. *Let $i \in I$ be a sink. For each $x, x' \in \mathcal{H}_q^*(\Lambda)\langle i\rangle$, we have*

$$\psi_\zeta^{\pm}(x^{\pm}\mathbf{1}_\zeta, x'^{\pm}\mathbf{1}_\zeta) = \psi_{s_i\zeta}^{\pm}(\mathcal{T}_i(x^{\pm}\mathbf{1}_{s_i\zeta}), \mathcal{T}_i(x'^{\pm}\mathbf{1}_{s_i\zeta})).$$

Proof. Let $V_\beta, V_{\beta'} \in$ rep-$Q\langle i\rangle$. Then

$$\begin{aligned}
& \psi_{s_i\zeta}^{+}(\mathcal{T}_i(\langle M(\beta)\rangle^{+}\mathbf{1}_\zeta), \mathcal{T}_i(\langle M(\beta')\rangle^{+}\mathbf{1}_\zeta)) \\
= \ & \psi_{s_i\zeta}^{+}(\langle M(\sigma_i^{+}\beta)\rangle^{+}\mathbf{1}_{s_i\zeta}, \langle M(\sigma_i^{+}\beta')\rangle^{+}\mathbf{1}_{s_i\zeta}) \\
= \ & \psi(\langle M(\sigma_i^{+}\beta)\rangle, \langle M(\sigma_i^{+}\beta'\rangle) \\
= \ & \frac{|V_{\sigma_i^{+}\beta}|}{a_{\sigma_i^{+}\beta}}\delta_{\sigma_i^{+}\beta\sigma_i^{+}\beta'} \\
= \ & \frac{|V_\beta|}{a_\beta}\delta_{\beta\beta'} \\
= \ & \psi(\langle M(\beta)\rangle, \langle M(\beta'\rangle) \\
= \ & \psi_\zeta^{+}(\langle M(\beta)\rangle^{+}\mathbf{1}_\zeta, \langle M(\beta')\rangle^{+}\mathbf{1}_\zeta).
\end{aligned}$$

Hence we have

$$\psi_\zeta^{+}(x^{+}\mathbf{1}_\zeta, x'^{+}\mathbf{1}_\zeta) = \psi_{s_i\zeta}^{+}(\mathcal{T}_i(x^{+}\mathbf{1}_{s_i\zeta}), \mathcal{T}_i(x'^{+}\mathbf{1}_{s_i\zeta})).$$

Similarly we can prove that

$$\psi_\zeta^{-}(x^{-}\mathbf{1}_\zeta, x'^{-}\mathbf{1}_\zeta) = \psi_{s_i\zeta}^{-}(\mathcal{T}_i(x^{-}\mathbf{1}_{s_i\zeta}), \mathcal{T}_i(x'^{-}\mathbf{1}_{s_i\zeta})).$$

□

Theorem 5.3. *Let $i \in I$ be a sink. Then for each $m \geq 1$, $\mathcal{T}_i T_i^{-1}$ induces bijective maps from $L_{\pi_m}(\Lambda)^{\pm}\mathbf{1}_\zeta$ to $L_{\pi_m}(\sigma_i\Lambda)^{\pm}\mathbf{1}_\zeta$.*

Proof. We first prove the theorem for $L_{\pi_m}^{+}(\Lambda)\mathbf{1}_\zeta$. By the definition we have

$$L_{\pi_m}(\Lambda) = \{x \in \mathcal{H}_q^*(\Lambda)_{\pi_m} | \psi(x, \mathfrak{d}_{m-1}(\Lambda)_{\pi_m}) = 0\}.$$

By [10], we have $L_{\pi_m}(\Lambda) \subset {}^{\tau}\mathcal{H}_q^*(\Lambda)\langle i\rangle_{\pi_m}$, $\mathfrak{d}_{m-1}(\Lambda)\langle i\rangle = \sum_{s\geq 1} {}^{\tau}\mathfrak{d}_{m-1}(\Lambda)\langle i\rangle x_i^s$ and $\psi(x, {}^{\tau}\mathfrak{d}_{m-1}(\Lambda)\langle i\rangle x_i^s) = 0$ for $x \in {}^{\tau}\mathcal{H}_q^*(\Lambda)\langle i\rangle$, where ${}^{\tau}\mathcal{H}_q^*(\Lambda)\langle i\rangle := \tau(\mathcal{H}_q^*(\Lambda)\langle i\rangle)$ and ${}^{\tau}\mathfrak{d}_m(\Lambda)\langle i\rangle := \tau(\mathfrak{d}_m(\Lambda)\langle i\rangle)$. Then we have

$$L_{\pi_m}(\Lambda) = \{x \in {}^{\tau}\mathcal{H}_q^*(\Lambda)\langle i\rangle_{\pi_m} | \psi(x, {}^{\tau}\mathfrak{d}_{m-1}(\Lambda)\langle i\rangle_{\pi_m}) = 0\}.$$

We have the following isomorphisms

$${}^{\tau}\mathfrak{d}_{m-1}(\Lambda)\langle i\rangle_{\pi_m}^{+}\mathbf{1}_\zeta \xrightarrow{T_i^{-1}} \mathfrak{d}_{m-1}(\Lambda)\langle i\rangle_{s_i\pi_m}^{+}\mathbf{1}_{s_i\zeta} \xrightarrow{\mathcal{T}_i} \mathfrak{d}_{m-1}(\sigma_i^{+}\Lambda)\langle i\rangle_{\pi_m}^{+}\mathbf{1}_\zeta.$$

The first isomorphism is showed in [9]. For the second one, we have proved that $\mathcal{T}_i$ is an isomorphism in Theorem 4.1. Hence we just need to show

$$\mathcal{T}_i(\mathfrak{d}_m(\Lambda)\langle i\rangle_{\pi_m}^{+}\mathbf{1}_\zeta) \subset \mathfrak{d}_m(\sigma_i^{+}\Lambda)\langle i\rangle_{s_i\pi_m}^{+}\mathbf{1}_{s_i\zeta}.$$

By [9], we know

$$\mathfrak{d}_m(\sigma_i^{+}\Lambda)\langle i\rangle_{s_i\pi_m} = \mathcal{H}^*(\sigma_i^{+}\Lambda)\langle i\rangle_{s_i\pi_m}.$$

Hence we have

$$\mathcal{T}_i(\mathfrak{d}_m(\Lambda)\langle i\rangle_{\pi_m}^{+}\mathbf{1}_\zeta) \subset \mathcal{H}^*(\sigma_i^{+}\Lambda)\langle i\rangle_{s_i\pi_m}^{+}\mathbf{1}_{s_i\zeta} = \mathfrak{d}_m(\sigma_i^{+}\Lambda)\langle i\rangle_{s_i\pi_m}^{+}\mathbf{1}_{s_i\zeta}.$$

Take any $x \in L_{\pi_m}(\Lambda)$. Then $\psi(x, {}^\tau\mathfrak{d}_{m-1}(\Lambda)\langle i\rangle_{\pi_m}) = 0$. By Proposition 5.1 and Proposition 5.2 we have

$$\begin{aligned} 0 &= \psi(x, {}^\tau\mathfrak{d}_{m-1}(\Lambda)\langle i\rangle_{\pi_m}) \\ &= \psi^+_\lambda(x^+\mathbf{1}_\zeta, {}^\tau\mathfrak{d}_{m-1}(\Lambda)\langle i\rangle^+_{\pi_m}\mathbf{1}_\zeta) \\ &= \psi^+_\lambda(\mathcal{T}_iT_i^{-1}(x^+\mathbf{1}_\zeta), \mathcal{T}_iT_i^{-1}({}^\tau\mathfrak{d}_{m-1}(\Lambda)\langle i\rangle^+_{\pi_m}\mathbf{1}_\zeta)) \\ &= \psi^+_\lambda(\mathcal{T}_iT_i^{-1}(x^+\mathbf{1}_\zeta), {}^\tau\mathfrak{d}_{m-1}(\sigma_i^+\Lambda)\langle i\rangle^+_{\pi_m}\mathbf{1}_\zeta). \end{aligned}$$

Hence $\mathcal{T}_iT_i^{-1}(x^+\mathbf{1}_\zeta) \in L_{\pi_m}(\Lambda)^+\mathbf{1}_\zeta$. Conversely, $\mathcal{T}_iT_i^{-1}(x^+\mathbf{1}_\zeta) \in L_{\pi_m}(\Lambda)^+\mathbf{1}_\zeta$ implies $x^+\mathbf{1}_\zeta \in L_{\pi_m}(\Lambda)^+\mathbf{1}_\zeta$. Hence $\mathcal{T}_iT_i^{-1}$ induces bijective maps from $L^+_{\pi_m}(\Lambda)\mathbf{1}_\zeta$ to $L^+_{\pi_m}(\sigma_i\Lambda)\mathbf{1}_\zeta$.

Similarly, we can prove that $\mathcal{T}_iT_i^{-1}$ induces bijective maps from $L^-_{\pi_m}(\Lambda)\mathbf{1}_\zeta$ to $L^-_{\pi_m}(\sigma_i\Lambda)\mathbf{1}_\zeta$. □

As in Section 5.1, by choosing the basis $\{x_{(m,p)}|1 \le p \le \eta_m\}$ of L_{π_m} for all m, we get a set of generators $G = \{x_i^+\mathbf{1}_\zeta, y_i^-\mathbf{1}_\zeta | i \in I \cup J, \zeta \in P^\vee\}$ of $\dot{\mathcal{H}}^*_q(\Lambda)$ and $\dot{\mathcal{H}}^*_q(\Lambda)$ is generated by these elements subject to the relations (41) to (46). If $i \in I$ is a sink, the theorem above implies that the image of G under $\mathcal{T}_iT_i^{-1}$ becomes a set of generators of $\dot{\mathcal{H}}^*_q(\sigma_i\Lambda)$ subject to the same relations. Hence, we also have an isomorphism $\iota' : \dot{\mathcal{H}}^*_q(\sigma_i\Lambda) \simeq \dot{\mathbf{U}}(\tilde{A})$ mapping $\mathcal{T}_iT_i^{-1}(x_i^+\mathbf{1}_\zeta)$ (resp. $\mathcal{T}_iT_i^{-1}(y_i^-\mathbf{1}_\zeta)$) to $E_i\mathbf{1}_\zeta$ (resp. $F_i\mathbf{1}_\zeta$). Under the isomorphisms ι and ι', the maps $\mathcal{T}_i$ and T_i induce maps on $\dot{\mathbf{U}}(\tilde{A})$, which are also denoted by $\mathcal{T}_i$ and T_i respectively. Then we have the following theorem.

Theorem 5.4. *Let $i \in I$ be a sink. Then the isomorphisms $\mathcal{T}_i$ and T_i coincide as maps from $\dot{\mathbf{U}}(\tilde{A})$ to $\dot{\mathbf{U}}(\tilde{A})$.*

Proof. Under the isomorphisms ι and ι', we get a map $\mathcal{T}_iT_i^{-1}$ from $\dot{\mathbf{U}}(\tilde{A})$ to $\dot{\mathbf{U}}(\tilde{A})$. Note that $\mathcal{T}_iT_i^{-1}$ sends the generators $E_i\mathbf{1}_\zeta$ and $F_i\mathbf{1}_\zeta$ to themselves. Hence $\mathcal{T}_iT_i^{-1}$ is the identical map on $\dot{\mathbf{U}}(\tilde{A})$. So $\mathcal{T}_i$ and T_i coincide. □

5.4. **Braid group relations.** In [10], the following theorem is proved.

Theorem 5.5. *For any $i \neq j \in I$ such that $m = m(i,j) \le +\infty$, $\tilde{\mathcal{T}}_i$ and $\tilde{\mathcal{T}}_j$ satisfy braid group relations (27) of type A as maps on $\mathcal{D}(\Lambda)$.*

Similarly to the case in Section 4.3, we have the following theorem.

Theorem 5.6. *Let Λ be a finite dimensional hereditary algebra, and A be the corresponding generalized Cartan matrix. For any $i \neq j \in I$ such that $m = m(i,j) \le +\infty$, $\mathcal{T}_i$ and $\mathcal{T}_j$ satisfy braid group relations (27) of type A as maps on $\dot{\mathbf{U}}(\tilde{A})$.*

6. The proof of Theorem 4.1

Let i be a sink and we follow the method used in [8].

In this section, for $\alpha, \beta, \gamma \in \mathcal{P}$, we use the notation $\gamma = \alpha \oplus \beta$ and $\gamma \neq \alpha \oplus \beta$ to express $V_\gamma \simeq V_\alpha \oplus V_\beta$ and $V_\gamma \not\simeq V_\alpha \oplus V_\beta$ respectively.

From the definition of $\mathcal{T}_i$, we have the following proposition.

Proposition 6.1. *For any $\lambda, \lambda' \in \mathcal{P}$, we have*

$$\mathcal{T}_i(\langle M(\lambda)\rangle^+\mathbf{1}_\zeta) = \mathcal{T}_i(\mathbf{1}_{\zeta+\lambda}\langle M(\lambda)\rangle^+), \mathcal{T}_i(\langle M(\lambda)\rangle^-\mathbf{1}_\zeta) = \mathcal{T}_i(\mathbf{1}_{\zeta-\lambda}\langle M(\lambda)\rangle^-); \tag{47}$$

$$\mathcal{T}_i(\langle M(\lambda)\rangle^+\mathbf{1}_\zeta)\mathcal{T}_i(\mathbf{1}_{\zeta'}\langle M(\lambda')\rangle^-)=\delta_{\zeta,\zeta'}\mathcal{T}_i(\langle M(\lambda)\rangle^+\mathbf{1}_\zeta\langle M(\lambda')\rangle^-)$$
$$\mathcal{T}_i(\langle M(\lambda)\rangle^-\mathbf{1}_\zeta)\mathcal{T}_i(\mathbf{1}_{\zeta'}\langle M(\lambda')\rangle^+)=\delta_{\zeta,\zeta'}\mathcal{T}_i(\langle M(\lambda)\rangle^-\mathbf{1}_\zeta\langle M(\lambda')\rangle^+). \tag{48}$$

For the proof of other relations, we first give some lemmas.

Lemma 6.2. *For any $\lambda\in\mathcal{P}$ and $m\in\mathbb{N}$, we have*

$$\mathcal{T}_i(u_i^{+(m)}\mathbf{1}_\zeta)\mathcal{T}_i(\mathbf{1}_{\zeta'}\langle M(\lambda)\rangle^+)=\mathcal{T}_i(\delta_{\zeta,\zeta'}\mathbf{1}_{\zeta+\alpha}(u_i^{(m)}\langle M(\lambda)\rangle)^+). \tag{49}$$

Proof. We write $V_\lambda=V_{\lambda_0}\oplus tV_i$ as above, then

$$\begin{aligned}
&\mathcal{T}_i(\mathbf{1}_{\zeta+m\alpha_i}(u_i^{(m)}\langle M(\lambda)\rangle)^+)\\
=\;& v^{\langle\lambda_0,ti\rangle}\mathcal{T}_i(\mathbf{1}_{\zeta+m\alpha_i}(u_i^{(m)}u_i^{(t)}\langle M(\lambda_0)\rangle)^+)\\
=\;& v^{\langle\lambda_0,ti\rangle}\begin{bmatrix} s+t\\ m\end{bmatrix}_{v_i}\mathcal{T}_i(\mathbf{1}_{\zeta+m\alpha_i}(u_i^{(m+t)}\langle M(\lambda_0)\rangle)^+)\\
=\;& v^{\langle\lambda_0,ti\rangle}\begin{bmatrix} s+t\\ m\end{bmatrix}_{v_i}v^{-\langle\lambda_0,(m+t)i\rangle}\mathcal{T}_i(\mathbf{1}_{\zeta+m\alpha_i}(v^{\langle\lambda_0,(m+t)i\rangle}u_i^{(m+t)}\langle M(\lambda_0)\rangle)^+)\\
=\;& (-1)^{m+t}\begin{bmatrix} s+t\\ m\end{bmatrix}_{v_i}v^r\mathbf{1}_{s_i\zeta-m\alpha_i}u_i^{-(m+t)}\langle M(\sigma_i^+\lambda_0)\rangle^+,
\end{aligned}$$

where

$$\begin{aligned}
r\;=\;&\langle\lambda_0,ti\rangle-\langle\lambda_0,(m+t)i\rangle+\langle\lambda_0,(m+t)i\rangle\\
&(t+m)^2\varepsilon_i+t\varepsilon_i+m\varepsilon_i-(\zeta+m\alpha_i,(t+m)\alpha_i)\\
=\;&\langle\lambda_0,ti\rangle+(t+m)^2\varepsilon_i+t\varepsilon_i+m\varepsilon_i-(\zeta,(t+m)\alpha_i)-2m(t+m)\varepsilon_i\\
=\;&\langle\lambda_0,ti\rangle-m^2\varepsilon_i+t^2\varepsilon_i+t\varepsilon_i+m\varepsilon_i-(\zeta,(t+m)\alpha_i).
\end{aligned}$$

While

$$\begin{aligned}
&\mathcal{T}_i(u_i^{+(m)}\mathbf{1}_\zeta)\mathcal{T}_i(\mathbf{1}_\zeta\langle M(\lambda)\rangle^+)\\
=\;&(-1)^m v^{r_1}\mathbf{1}_{s_i\zeta-m\alpha_i}u_i^{-(m)}(-1)^t v^{r_2}u_i^{-(t)}\langle M(\sigma_i^+\lambda_0)\rangle^+\\
=\;&(-1)^{m+t}\begin{bmatrix} s+t\\ m\end{bmatrix}_{v_i}v^{r_1+r_2}\mathbf{1}_{s_i\zeta-m\alpha_i}u_i^{-(m+t)}\langle M(\sigma_i^+\lambda_0)\rangle^+,
\end{aligned}$$

where $r_1=-m^2\varepsilon_i+m\varepsilon_i-(\zeta,m\alpha_i)$ and $r_2=\langle\lambda_0,ti\rangle+t^2\varepsilon_i+t\varepsilon_i-(\zeta,t\alpha_i)$. Clearly, $r_1+r_2=r$. Hence we have formula (49) in Lemma 6.2. □

Lemma 6.3. *For any $\lambda\in\mathcal{P}$, we have*

$$\begin{aligned}
&-(u_i^-\langle M(\lambda)\rangle^+-\langle M(\lambda)\rangle^+u_i^-)\mathbf{1}_\zeta\\
=\;&\frac{v_i}{a_i}(v^{(\zeta,\alpha_i)}(r_i(\langle M(\lambda)\rangle))^+-v^{(\zeta+\lambda-\alpha_i,-\alpha_i)}(r'_i(\langle M(\lambda)\rangle))^+)\mathbf{1}_\zeta
\end{aligned}\tag{50}$$

and

$$\begin{aligned}
&-(\langle M(\lambda)\rangle^-u_i^+-u_i^+\langle M(\lambda)\rangle^-)\mathbf{1}_\zeta\\
=\;&\frac{v_i}{a_i}(v^{(\zeta-\lambda+\alpha_i,\alpha_i)}(r'_i(\langle M(\lambda)\rangle))^--v^{(\zeta,-\alpha_i)}(r_i(\langle M(\lambda)\rangle))^-)\mathbf{1}_\zeta.
\end{aligned}\tag{51}$$

Proof. Recall the relation (20)

$$\begin{aligned}
&\sum_{\alpha,\alpha'\in\mathcal{P}}v^{\langle\alpha',\alpha\rangle+(\alpha,\alpha)+(\zeta,-\alpha)}\frac{a_{\alpha'}}{a_{\lambda'}}g^{\lambda'}_{\alpha'\alpha}(-1)^{tr\alpha'}v^{m(\alpha')}\langle M(\alpha')\rangle^-\mathbf{1}_{\zeta+\alpha'}(r'_\alpha(\langle M(\lambda)\rangle))^+\\
=&\sum_{\alpha,\beta\in\mathcal{P}}v^{\langle\alpha,\beta\rangle+(\beta,\beta)+(\zeta,\beta)}\frac{a_\alpha}{a_\lambda}g^\lambda_{\alpha\beta}(-1)^{tr(\lambda'-\beta)}v^{m(\lambda'-\beta)}\langle M(\alpha)\rangle^+\mathbf{1}_{\zeta-\alpha}(r_\beta(\langle M(\lambda')\rangle))^-
\end{aligned}$$

in the definition of $\dot{\mathcal{H}}_q^*(\Lambda)$. Let $\lambda' = i$ in the above relation. We can get formula (50). Similarly, let $\lambda = i$ and $\lambda' = \lambda$. We get formula (51). $\square$

Lemma 6.4. *For any $\beta \in \mathcal{P}$ and $m \in \mathbb{N}$, we have*

$$\mathcal{T}_i(\langle M(\beta)\rangle^+ \mathbf{1}_\zeta)\mathcal{T}_i(\mathbf{1}_{\zeta'} u_i^{+(m)}) = \mathcal{T}_i(\delta_{\zeta,\zeta'}\mathbf{1}_{\zeta+\beta}(\langle M(\beta)\rangle u_i^{(m)})^+). \tag{52}$$

Proof. From the definition of $\mathcal{T}_i$, we only need to prove

$$\mathcal{T}_i(\langle M(\beta)\rangle^+ \mathbf{1}_\zeta)\mathcal{T}_i(\mathbf{1}_{\zeta} u_i^{+(m)}) = \mathcal{T}_i(\mathbf{1}_{\zeta+\beta}(\langle M(\beta)\rangle u_i^{(m)})^+).$$

By Lemma 6.2, it suffices to prove the lemma for the case V_β does not contain V_i as a direct summand. So we assume that V_i is not a direct summand of V_β.

First we have ([8])

$$\langle M(\beta)\rangle u_i = v^{(i,\beta)} u_i \langle M(\beta)\rangle + v^{-\langle i,\beta\rangle} \sum_{\alpha \neq \beta\oplus i} g_{\beta i}^{\alpha} \langle M(\alpha)\rangle.$$

Therefore

$$\begin{aligned}
& \mathcal{T}_i(\mathbf{1}_{\zeta+\beta}(\langle M(\beta)\rangle u_i)^+) \\
= \;& v^{(i,\beta)}\mathcal{T}_i(\mathbf{1}_{\zeta+\beta} u_i^+)\mathcal{T}_i(\langle M(\beta)\rangle^+ \mathbf{1}_{\zeta-\alpha_i}) \\
& + v^{-\langle i,\beta\rangle} \sum_{\alpha \neq \beta\oplus i} g_{\beta i}^{\alpha} \mathcal{T}_i(\mathbf{1}_{\zeta+\beta}\langle M(\alpha)\rangle^+) \\
= \;& -v^{(i,\beta)} v^{2\varepsilon_i} v^{-(\zeta+\beta,\alpha_i)} u_i^- \langle M(\sigma_i^+\beta)\rangle^+ \mathbf{1}_{s_i(\zeta-\alpha_i)} \\
& + v^{-\langle i,\beta\rangle} \sum_{\alpha \neq \beta\oplus i} g_{\beta i}^{\alpha} \langle M(\sigma_i^+\alpha)\rangle^+ \mathbf{1}_{s_i(\zeta-\alpha_i)}.
\end{aligned}$$

In the computation above, we use the fact that if $g_{\beta i}^{\alpha} \neq 0$ and $V_\alpha \neq V_\beta \oplus V_i$, then V_α contains no direct summand isomorphic to V_i. On the other hand,

$$\begin{aligned}
& \mathcal{T}_i(\langle M(\beta)\rangle^+ \mathbf{1}_\zeta)\mathcal{T}_i(\mathbf{1}_\zeta u_i^+) \\
= \;& -v^{2\varepsilon_i} v^{-(\zeta,\alpha_i)} \langle M(\sigma_i^+\beta)\rangle^+ \mathbf{1}_{s_i\zeta} u_i^-.
\end{aligned}$$

Thus, to prove

$$\mathcal{T}_i(\langle M(\beta)\rangle^+ \mathbf{1}_\zeta)\mathcal{T}_i(\mathbf{1}_\zeta u_i^+) = \mathcal{T}_i(\mathbf{1}_{\zeta+\beta}(\langle M(\beta)\rangle u_i)^+),$$

we only need to prove

$$\begin{aligned}
& -v^{2\varepsilon_i} v^{-(\zeta,\alpha_i)} \langle M(\sigma_i^+\beta)\rangle^+ \mathbf{1}_{s_i\zeta} u_i^- \\
= \;& -v^{(i,\beta)} v^{2\varepsilon_i} v^{-(\zeta+\beta,\alpha_i)} u_i^- \langle M(\sigma_i^+\beta)\rangle^+ \mathbf{1}_{s_i(\zeta-\alpha_i)} \\
& + v^{-\langle i,\beta\rangle} \sum_{\alpha \neq \beta\oplus i} g_{\beta i}^{\alpha} \langle M(\sigma_i^+\alpha)\rangle^+ \mathbf{1}_{s_i(\zeta-\alpha_i)}.
\end{aligned}$$

It is sufficient to prove that

$$\begin{aligned}
& \langle M(\sigma_i^+\beta)\rangle^+ u_i^- \mathbf{1}_{s_i(\zeta-\alpha_i)} - u_i^- \langle M(\sigma_i^+\beta)\rangle^+ \mathbf{1}_{s_i(\zeta-\alpha_i)} \\
= \;& -v^{-\langle i,\beta\rangle} v^{-2\varepsilon_i} v^{(\zeta,\alpha_i)} \sum_{\alpha \neq \beta\oplus i} g_{\beta i}^{\alpha} \langle M(\sigma_i^+\alpha)\rangle^+ \mathbf{1}_{s_i(\zeta-\alpha_i)}.
\end{aligned}$$

In rep-$\mathcal{S}$, V_i is a simple injective and $V_{\sigma_i^+\beta} \in$ rep-$\sigma_i\mathcal{S}$, so $g_{\gamma\sigma_i^+\alpha}^{\sigma_i^+\beta} = 0$ for all $V_\gamma \in$ rep-$\sigma_i\mathcal{S}$. By Lemma 6.3 we have

$$\begin{aligned}
& \langle M(\sigma_i^+\beta)\rangle^+ u_i^- \mathbf{1}_{s_i(\zeta-\alpha_i)} - u_i^- \langle M(\sigma_i^+\beta)\rangle^+ \mathbf{1}_{s_i(\zeta-\alpha_i)} \\
= & \frac{v_i}{a_i}(v^{(s_i(\zeta-\alpha_i),\alpha_i)}(r_i(\langle M(\sigma_i^+\beta)\rangle))^+ \\
& -v^{((s_i(\zeta-\alpha_i)+s_i\beta-\alpha_i,\alpha_i)}(r_i'(\langle M(\sigma_i^+\beta)\rangle))^+)\mathbf{1}_{s_i(\zeta-\alpha_i)} \\
= & \frac{v_i}{a_i}(v^{-(\zeta,\alpha_i)+2\varepsilon_i}(r_i(\langle M(\sigma_i^+\beta)\rangle))^+ \\
& -v^{(\zeta,\alpha_i)+(\beta,\alpha_i)}(r_i'(\langle M(\sigma_i^+\beta)\rangle))^+)\mathbf{1}_{s_i(\zeta-\alpha_i)} \\
= & -\frac{1}{a_i}v^{(\zeta,\alpha_i)+(\beta,\alpha_i)+\varepsilon_i}(r_i'(\langle M(\sigma_i^+\beta)\rangle))^+\mathbf{1}_{s_i(\zeta-\alpha_i)} \\
= & -\frac{1}{a_i}v^{(\zeta,\alpha_i)+(\beta,\alpha_i)+\varepsilon_i}\sum_\alpha \frac{a_{\sigma_i^+\alpha}a_i}{a_{\sigma_i^+\beta}}v^{\langle i,\sigma_i^+\alpha\rangle+(i,\sigma_i^+\alpha)}g_{i\sigma_i^+\alpha}^{\sigma_i^+\beta}\langle M(\sigma_i^+\alpha)\rangle^+\mathbf{1}_{s_i(\zeta-\alpha_i)} \\
= & -v^{(\zeta,\alpha_i)+(\beta,\alpha_i)+\varepsilon_i}\sum_\alpha v^{\langle i,\sigma_i^+\alpha\rangle+(i,\sigma_i^+\alpha)}g_{\beta i}^\alpha\langle M(\sigma_i^+\alpha)\rangle^+\mathbf{1}_{s_i(\zeta-\alpha_i)} \\
= & -v^{(\zeta,\alpha_i)-2\varepsilon_i-\langle i,\beta\rangle}\sum_\alpha g_{\beta i}^\alpha\langle M(\sigma_i^+\alpha)\rangle^+\mathbf{1}_{s_i(\zeta-\alpha_i)} \\
= & -v^{(\zeta,\alpha_i)-2\varepsilon_i-\langle i,\beta\rangle}\sum_\alpha g_{\beta i}^\alpha\langle M(\sigma_i^+\alpha)\rangle^+\mathbf{1}_{s_i(\zeta-\alpha_i)}.
\end{aligned}$$

In the computation, we use the following formula

$$g_{\beta i}^\alpha = \frac{a_\alpha}{a_\beta}g_{i\sigma_i^+\alpha}^{\sigma_i^+\beta}$$

for $i \in I$ be a sink and $V_\alpha, V_\beta \in$ rep-$\mathcal{S}\langle i\rangle$.

Then by induction, we get the formula (52). □

Proposition 6.5. *For $\alpha, \beta \in \mathcal{P}$, we have*

$$\mathcal{T}_i(\langle M(\alpha)\rangle^+\mathbf{1}_\zeta)\mathcal{T}_i(\mathbf{1}_{\zeta'}\langle M(\beta)\rangle^+) = \mathcal{T}_i(\delta_{\zeta,\zeta'}\mathbf{1}_{\zeta+\alpha}(\langle M(\alpha)\rangle\langle M(\beta)\rangle)^+). \tag{53}$$

Proof. By Lemma 6.2 and Lemma 6.4, we can assume that V_α and V_β do not contain V_i as a direct summand. In [15], Ringel points that σ_i^+ induces an $\mathbb{Q}(v)$-algebra isomorphism from $\mathcal{H}_q^*(\Lambda)\langle i\rangle$ to $\mathcal{H}_q^*(\sigma_i\Lambda)\langle i\rangle$ mapping $\langle M(\alpha)\rangle$ to $\langle M(\sigma_i^+\alpha)\rangle$, where $\mathcal{H}_q^*(\Lambda)\langle i\rangle$ is the subalgebra generated by $\langle M(\alpha)\rangle$ with $V_\alpha \in$ rep-$\mathcal{S}\langle i\rangle$. Hence we prove formula (53). □

Similarly, we have

Proposition 6.6. *For $\alpha, \beta \in \mathcal{P}$, we have*

$$\mathcal{T}_i(\langle M(\alpha)\rangle^-\mathbf{1}_\zeta)\mathcal{T}_i(\mathbf{1}_{\zeta'}\langle M(\beta)\rangle^-) = \mathcal{T}_i(\delta_{\zeta,\zeta'}\mathbf{1}_{\zeta+\alpha}(\langle M(\alpha)\rangle\langle M(\beta)\rangle)^-). \tag{54}$$

Then the most difficult defining relation (24) should be verified, that is, for an element $y \in \dot{\mathcal{H}}_q^*(\Lambda)$, which can be writen as

$$y = \sum_{x,x',\zeta} x^+\mathbf{1}_\zeta x'^-$$

and

$$y = \sum_{x,x',\zeta} x^-\mathbf{1}_\zeta x'^+,$$

we should verify that

$$\sum_{x,x',\zeta} T_i(x^+\mathbf{1}_\zeta x'^-) = \sum_{x,x',\zeta} T_i(x^-\mathbf{1}_\zeta x'^+).$$

Proposition 6.7. *For any $\lambda, \lambda' \in \mathcal{P}$, we have*

$$\sum_{\alpha,\alpha'\in\mathcal{P}} v^{\langle\alpha',\alpha\rangle+(\alpha,\alpha)+(\zeta,-\alpha)}\frac{a_{\alpha'}}{a_{\lambda'}}g^{\lambda'}_{\alpha'\alpha}(-1)^{tr\alpha'}v^{m(\alpha')}\mathcal{T}_i\left(\langle M(\alpha')\rangle^-\mathbf{1}_{\zeta+\alpha'}(r'_\alpha(\langle M(\lambda)\rangle))^+\right) =$$
$$\sum_{\alpha,\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle+(\beta,\beta)+(\zeta,\beta)}\frac{a_\alpha}{a_\lambda}g^{\lambda}_{\alpha\beta}(-1)^{tr(\lambda'-\beta)}v^{m(\lambda'-\beta)}\mathcal{T}_i\left(\langle M(\alpha)\rangle^+\mathbf{1}_{\zeta-\alpha}(r_\beta(\langle M(\lambda')\rangle))^-\right). \tag{55}$$

Proof. By Proposition 6.5 and Proposition 6.6, we may assume that V_λ and $V_{\lambda'}$ contain no direct summand isomorphic to V_i. Then V_α and $V_{\alpha'}$ also contain no direct summand isomorphic to V_i.

Let

$$L = \sum_{\alpha,\alpha'\in\mathcal{P}} v^{\langle\alpha',\alpha\rangle+(\alpha,\alpha)+(\zeta,-\alpha)}\frac{a_{\alpha'}}{a_{\lambda'}}g^{\lambda'}_{\alpha'\alpha}(-1)^{tr\alpha'}v^{m(\alpha')}\langle M(\alpha')\rangle^-\mathbf{1}_{\zeta+\alpha'}(r'_\alpha(\langle M(\lambda)\rangle))^+;$$

and

$$R = \sum_{\alpha,\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle+(\beta,\beta)+(\zeta,\beta)}\frac{a_\alpha}{a_\lambda}g^{\lambda}_{\alpha\beta}(-1)^{tr(\lambda'-\beta)}v^{m(\lambda'-\beta)}\langle M(\alpha)\rangle^+\mathbf{1}_{\zeta-\alpha}(r_\beta(\langle M(\lambda')\rangle))^-.$$

First consider L. We have

$$\begin{aligned} L &= \mathbf{1}_\zeta \sum_{\alpha,\alpha',\beta\in\mathcal{P}} v^{\langle\lambda',\alpha\rangle+(\zeta,-\alpha)+\langle\alpha,\lambda\rangle+(\alpha,\beta)}\frac{a_{\alpha'}a_\alpha a_\beta}{a_{\lambda'}a_\lambda}g^{\lambda'}_{\alpha'\alpha}g^{\lambda}_{\alpha\beta}(-1)^{tr\alpha'}v^{m(\alpha')}\langle M(\alpha')\rangle^-\langle M(\beta)\rangle)^+ \\ &= \mathbf{1}_\zeta \sum_{\alpha,\alpha',\beta\in\mathcal{P}} A_1B_1\langle M(\alpha')\rangle^-\langle M(\beta)\rangle)^+ \end{aligned}$$

where $A_1 = v^{\langle\lambda',\alpha\rangle+(\zeta,-\alpha)+\langle\alpha,\lambda\rangle+(\alpha,\beta)}(-1)^{tr\alpha'}v^{m(\alpha')}$ and $B_1 = \frac{a_{\alpha'}a_\alpha a_\beta}{a_{\lambda'}a_\lambda}g^{\lambda'}_{\alpha'\alpha}g^{\lambda}_{\alpha\beta}$.

Now assume $V_\beta = V_{\beta'}\oplus tV_i$, where $V_{\beta'}$ contains no direct summand isomorphic to V_i. Then we have $\langle M(\beta)\rangle = v^{\langle\beta',ti\rangle}u_i^{(t)}\langle M(\beta')\rangle$.

Then

$$\begin{aligned} &\mathcal{T}_i(L) \\ =\ &\mathbf{1}_{s_i\zeta}\sum_{\alpha,\alpha',\beta\in\mathcal{P}} A_1B_1\mathcal{T}_i(\langle M(\alpha')\rangle^-\mathbf{1}_{\zeta+\alpha'}\langle M(\beta)\rangle)^+) \\ =\ &\mathbf{1}_{s_i\zeta}\sum_{\alpha,\alpha',\beta'\in\mathcal{P},t} A_1B_1(-1)^{t-\alpha'(h_i)}v^{t^2\varepsilon_i+t\varepsilon_i+\langle\beta',ti\rangle-(\zeta+\alpha',t\alpha_i)-(\alpha',i)} \\ &\langle M(\sigma_i^+\alpha')\rangle^-u_i^{-(t)}\langle M(\sigma_i^+\beta')\rangle)^+ \\ =\ &\mathbf{1}_{s_i\zeta}\sum_{\alpha,\alpha',\beta'\in\mathcal{P},t} A_1B_1A_2\langle M(\sigma_i^+\alpha')\rangle^-u_i^{-(t)}\langle M(\sigma_i^+\beta')\rangle)^+ \end{aligned}$$

where $A_2 = (-1)^{t-\alpha'(h_i)}v^{t^2\varepsilon_i+t\varepsilon_i+\langle\beta',ti\rangle-(\zeta+\alpha',t\alpha_i)-(\alpha',i)}$.

Since i is a source of σ_iQ and $V_{\alpha'}$ contains no direct summand isomorphic to V_i, $\langle M(\sigma_i^+\alpha'\oplus ti)\rangle = v^{\langle ti,\alpha'\rangle}\langle M(\sigma_i^+\alpha')\rangle u_i^{(t)}$.

Hence we have

$$\mathcal{T}_i(L) = \mathbf{1}_{s_i\zeta}\sum_{\alpha,\alpha',\beta'\in\mathcal{P},t} A_1B_1A_2A_3\langle M(\sigma_i^+\alpha'\oplus ti)\rangle^-\langle M(\sigma_i^+\beta')\rangle)^+$$

where $A_3 = v^{-\langle ti,\alpha'\rangle}$.

Then we compute B_1.

If i is a sink and V_α, V_β contain no direct summand isomorphic to V_i, then $g^{\lambda}_{\alpha,\beta\oplus ti} = \sum_\gamma g^{\gamma}_{\alpha ti} g^{\lambda}_{\gamma\beta}$. If i is a source and V_α, V_β contain no direct summand isomorphic to V_i, then $g^{\lambda}_{\alpha\oplus ti,\beta} = \sum_\gamma g^{\gamma}_{ti\beta} g^{\lambda}_{\alpha\gamma}$.

Since V_α and $V_{\beta'}$ contain no direct summand isomorphic to V_i, we have

$$g^{\lambda}_{\alpha\beta} = \sum_\gamma g^{\gamma}_{\alpha ti} g^{\lambda}_{\gamma\beta'}.$$

Note that ([8])

$$a_\beta = v^{2\langle ti,\beta'\rangle} a_{\beta'} a_{ti}, a_{\sigma_i^+\alpha'\oplus ti} = v^{2\langle ti,\alpha'\rangle} a_{\alpha'} a_{ti}.$$

Then

$$\begin{aligned} B_1 &= \frac{a_{\alpha'} a_\alpha a_\beta}{a_{\lambda'} a_\lambda} g^{\lambda'}_{\alpha'\alpha} g^{\lambda}_{\alpha\beta} \\ &= \sum_\gamma v^{2\langle ti,\beta'\rangle} \frac{a_{\alpha'} a_\alpha a_{\beta'} a_{ti}}{a_{\lambda'} a_\lambda} g^{\lambda'}_{\alpha'\alpha} g^{\gamma}_{\alpha ti} g^{\lambda}_{\gamma\beta'}. \end{aligned}$$

We may assume V_γ contains no direct summand isomorphic to V_i. Hence we have

$$a_\alpha g^{\alpha ti}_{\gamma} = a_\gamma g^{\sigma_i^+\alpha}_{ti\sigma_i^+\gamma}.$$

Then

$$\begin{aligned} B_1 &= \sum_\gamma v^{2\langle ti,\beta'\rangle} \frac{a_{\alpha'} a_\gamma a_{\beta'} a_{ti}}{a_{\lambda'} a_\lambda} g^{\lambda'}_{\alpha'\alpha} g^{\sigma_i^+\alpha}_{ti\sigma_i^+\gamma} g^{\lambda}_{\gamma\beta'} \\ &= \sum_\gamma v^{2\langle ti,\beta'\rangle} \frac{a_{\sigma_i^+\alpha'} a_{\sigma_i^+\gamma} a_{\sigma_i^+\beta'} a_{ti}}{a_{\sigma_i^+\lambda'} a_{\sigma_i^+\lambda}} g^{\sigma_i^+\lambda'}_{\sigma_i^+\alpha'\sigma_i^+\alpha} g^{\sigma_i^+\alpha}_{ti\sigma_i^+\gamma} g^{\sigma_i^+\lambda}_{\sigma_i^+\gamma\sigma_i^+\beta'} \\ &= \sum_\gamma v^{2\langle ti,\beta'\rangle} v^{2\langle\alpha',ti\rangle} \frac{a_{\sigma_i^+\alpha'\oplus ti} a_{\sigma_i^+\gamma} a_{\sigma_i^+\beta'}}{a_{\sigma_i^+\lambda'} a_{\sigma_i^+\lambda}} g^{\sigma_i^+\lambda'}_{\sigma_i^+\alpha'\sigma_i^+\alpha} g^{\sigma_i^+\alpha}_{ti\sigma_i^+\gamma} g^{\sigma_i^+\lambda}_{\sigma_i^+\gamma\sigma_i^+\beta'} \\ &= \sum_\gamma A_4 \frac{a_{\sigma_i^+\alpha'\oplus ti} a_{\sigma_i^+\gamma} a_{\sigma_i^+\beta'}}{a_{\sigma_i^+\lambda'} a_{\sigma_i^+\lambda}} g^{\sigma_i^+\lambda'}_{\sigma_i^+\alpha'\sigma_i^+\alpha} g^{\sigma_i^+\alpha}_{ti\sigma_i^+\gamma} g^{\sigma_i^+\lambda}_{\sigma_i^+\gamma\sigma_i^+\beta'} \\ &= \sum_\gamma A_4 \frac{a_{\sigma_i^+\alpha'\oplus ti} a_{\sigma_i^+\gamma} a_{\sigma_i^+\beta'}}{a_{\sigma_i^+\lambda'} a_{\sigma_i^+\lambda}} g^{\sigma_i^+\lambda'}_{\sigma_i^+\alpha'\oplus ti,\sigma_i^+\gamma} g^{\sigma_i^+\lambda}_{\sigma_i^+\gamma\sigma_i^+\beta'} \end{aligned}$$

where $A_4 = v^{2\langle ti,\beta'\rangle} v^{2\langle\alpha',ti\rangle}$.

Then we compute $A = A_1A_2A_3A_4$.

$$\begin{aligned} A &= A_1A_2A_3A_4 \\ &= v^{\langle\lambda',\alpha\rangle+(\zeta,-\alpha)+\langle\alpha,\lambda\rangle+(\alpha,\beta)} (-1)^{tr\alpha'} v^{m(\alpha')} \\ &\quad (-1)^{t-\alpha'(h_i)} v^{t^2\varepsilon_i+t\varepsilon_i+\langle\beta',ti\rangle-(\zeta+\alpha',t\alpha_i)-(\alpha',i)} \\ &\quad v^{-\langle ti,\alpha'\rangle} \\ &\quad v^{2\langle ti,\beta'\rangle} v^{2\langle\alpha',ti\rangle} \\ &= (-1)^{tr(\sigma_i^+(\alpha'))+t} v^{(\zeta,-\alpha-t\alpha_i)+\langle\sigma_i^+(\lambda'),\sigma_i^+(\gamma)\rangle+\langle\sigma_i^+(\gamma),\sigma_i^+(\lambda)\rangle+(\sigma_i^+(\gamma),\sigma_i^+(\beta'))+m(\sigma_i^+(\alpha'))+t\varepsilon_i}. \end{aligned}$$

Let $\mu_1=\sigma_i^+\gamma$, $\mu_2=\sigma_i^+\beta'$ and $\mu_3=\sigma_i^+\alpha'\oplus ti$. Hence we have

$$\begin{aligned}\mathcal{T}_i(L) &= \mathbf{1}_{s_i\zeta}\sum_{\mu_1,\mu_2,\mu_3\in\mathcal{P}}(-1)^{tr\mu_3}v^{m(\mu_3)}v^{(s_i\zeta,-\mu_1)+\langle\sigma_i^+(\lambda'),\mu_1\rangle+\langle\mu_1,\sigma_i^+(\lambda)\rangle+(\mu_1,\mu_2)}\\ &\quad \frac{a_{\mu_3}a_{\mu_1}a_{\mu_2}}{a_{\sigma_i^+\lambda'}a_{\sigma_i^+\lambda}}g^{\sigma_i^+\lambda'}_{\mu_3,\mu_1}g^{\sigma_i^+\lambda}_{\mu_1\mu_2}\langle M(\mu_3)\rangle^-\langle M(\mu_2)\rangle\rangle^+\\ &= \mathbf{1}_{s_i\zeta}\sum_{\mu_1,\mu_2,\mu_3\in\mathcal{P}}(-1)^{tr\mu_3}v^{m(\mu_3)}v^{(s_i\zeta,-\mu_1)+\langle\mu_1+\mu_3,\mu_1\rangle+\langle\mu_1,\mu_1+\mu_2\rangle+(\mu_1,\mu_2)}\\ &\quad \frac{a_{\mu_3}a_{\mu_1}a_{\mu_2}}{a_{\sigma_i^+\lambda'}a_{\sigma_i^+\lambda}}g^{\sigma_i^+\lambda'}_{\mu_3,\mu_1}g^{\sigma_i^+\lambda}_{\mu_1\mu_2}\langle M(\mu_3)\rangle^-\langle M(\mu_2)\rangle\rangle^+\\ &= \mathbf{1}_{s_i\zeta}\sum_{\mu_1,\mu_3\in\mathcal{P}}(-1)^{tr\mu_3}v^{m(\mu_3)}v^{(s_i\zeta,-\mu_1)+\langle\mu_3,\mu_1\rangle+(\mu_1,\mu_1)}\\ &\quad \frac{a_{\mu_3}}{a_{\sigma_i^+\lambda'}}g^{\sigma_i^+\lambda'}_{\mu_3,\mu_1}\langle M(\mu_3)\rangle^-(r'_{\mu_1}(\langle M(\sigma_i^+\lambda)\rangle))^+.\end{aligned}$$

Similarly we have

$$\begin{aligned}\mathcal{T}_i(R) &= \mathbf{1}_{s_i\zeta}\sum_{\mu_4,\mu_5\in\mathcal{P}}(-1)^{tr(s_i\lambda'-\mu_5)}v^{m(s_i\lambda'-\mu_5)}v^{(s_i\zeta,\mu_5)+\langle\mu_4,\mu_5\rangle+(\mu_5,\mu_5)}\\ &\quad \frac{a_{\mu_4}}{a_{\sigma_i^+\lambda}}g^{\sigma_i^+\lambda}_{\mu_4\mu_5}\langle M(\mu_4)\rangle^+(r_{\mu_5}(\langle M(\sigma_i^+\lambda')\rangle))^-.\end{aligned}$$

By the first relation (20) in the definition of $\dot{\mathcal{H}}_q^*(\Lambda)$, we have $\mathcal{T}_i(L)=\mathcal{T}_i(R)$. □

Then Proposition 6.1, 6.5, 6.6 and 6.7 imply Theorem 4.1.

References

[1] Lusztig, G.: Quantum deformations of certain simple modules over enveloping algebras. *Adv. Math.*, **70**, 237–249 (1988)
[2] Lusztig, G.: Quantum groups at roots of 1. *Geom. Dedicata*, **35**, 89–114 (1990)
[3] Lusztig, G.: Introduction to quantum groups, Birkhauser, Boston, 1993
[4] Green, J. A.: Hall algebras, hereditary algebras and quantum groups. *Invent. Math.*, **120**, 361–377 (1995)
[5] Ringel, C. M.: Hall algebras and quantum groups. *Invent. Math.*, **101**, 583–592 (1990)
[6] Xiao, J.: Drinfeld double and Ringel-Green theory of Hall algebras. *J. Algebra*, **190**, 100–144 (1997)
[7] Sevenhant, B., Van den Bergh, M.: On the double of the Hall algebra of a quiver. *J. Algebra*, **221**, 135–160 (1999)
[8] Xiao, J., Yang, S.: BGP-Reflection Functors and Lusztig's Symmetries: A Ringel-Hall Algebra Approach to Quantum Groups. *J. Algebra*, **241**, 204–246 (2001)
[9] Deng, B., Xiao, J.: On double Ringel-Hall algebras. *J. Algebra*, **251**, 110–149 (2002)
[10] Deng, B., Xiao, J.: Ringel-Hall algebras and Lusztig's symmetries. *J. Algebra*, **255**, 357–372 (2002)
[11] Kac, V. G.: Infinite dimensional Lie algebras, Third edition, Cambridge Univ. Press, Cambridge, 1990
[12] Dlab, V., Ringel, C. M.: Indecomposable representation of graphs and algebras. *Mem. Amer. Math. Sco.*, **173**, (1976)
[13] Bernstein, I. N., Gelfand, I. M., Ponomarev, A.: Coxter functors and Gabriel's Theorem. *Lispehi Math. Nauk*, **28**, 19–33 (1972)
[14] Sevenhant, B., Van den Bergh, M.: A relation between a conjecture of Kac and the structure of the Hall algebra. *J. Pure Appl, Algebra*, **160**, 319–332 (2001)

[15] Ringel, C. M.: PBW-bases of quantum groups. *J. reine angew. Math.*, **470**, 51–88 (1996)

Department of Mathematics, Tsinghua University, Beijing 100084, P. R. China
E-mail address: jxiao@math.tsinghua.edu.cn

College of Science, Beijing Forestry University, Beijing 100083, P. R. China
E-mail address: zhaomh@bjfu.edu.cn