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Filtrations and homological degrees of FI-modules ☆

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Let $\Bbbk$ be a commutative Noetherian ring. In this paper we consider $\sharp$-filtered modules of the category $\mathscr{FI}$ firstly introduced in [12]. We show that a finitely generated $\mathscr{FI}$-module V is $\sharp$-filtered if and only if its higher homologies all vanish, and if and only if a certain homology vanishes. Using this homological characterization, we characterize finitely generated $\mathscr{C}$-modules V whose projective dimension $\mathrm{pd}(V)$ is finite, and describe an upper bound for $\mathrm{pd}(V)$. Furthermore, we give a new proof for the fact that V induces a finite complex of $\sharp$-filtered modules, and use it as well as a result of Church and Ellenberg in [1] to obtain another upper bound for homological degrees of V.

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1. Introduction

1.1. Motivation

The category $\mathcal{FI}$, whose objects are finite sets and morphisms are injections between them, has played a central role in representation stability theory introduced by Church and Farb in [3]. It has many interesting properties, which were used to prove quite a few stability phenomena observed in [1–3,5,14]. Among these properties, the existence of a *shift functor* Σ is extremely useful. For instances, it was applied to show the locally Noetherian property of $\mathcal{FI}$ over any Noetherian ring by Church, Ellenberg, Farb, and Nagpal in [4], and the Koszulity of $\mathcal{FI}$ over a field of characteristic 0 by Gan and the first author in [7]. Recently, Nagpal proved that for an arbitrary finitely generated representation V of $\mathcal{FI}$, when N is large enough, $\Sigma_N V$ has a special filtration, where Σ_N is the N-th iteration of Σ; see [12, Theorem A]. Church and Ellenberg showed that $\mathcal{FI}$-modules have Castlenuovo–Mumford regularity (for a definition in commutative algebra, see [6]), and gave an upper bound for the regularity; see [1, Theorem A].

The main goal of this paper is to use the shift functor to investigate homological degrees and special filtrations of $\mathcal{FI}$-modules. Specifically, we want to:

(1) obtain a homological characterization of $\sharp$-*filtered modules* firstly introduced by Nagpal in [12]; and
(2) use these $\sharp$-filtered modules, which play almost the same role as projective modules for homological calculations, to obtain upper bounds for projective dimensions and homological degrees of finitely generated $\mathcal{FI}$-modules.

In contrast to the combinatorial approach described in [1], our methods to realize these objectives are mostly conceptual and homological. We do not rely on any specific combinatorial structure of the category $\mathcal{FI}$. The main technical tools we use are the shift functor Σ and its induced cokernel functor D introduced in [4, Subsection 2.3] and [1, Section 3]. Therefore, it is hopeful that our approach, with adaptable modifications, can be applied to other combinatorial categories recently appearing in representation stability theory [15].

1.2. Notation

Before describing the main results, we first introduce necessary notation. Throughout this paper we let $\mathcal{C}$ be a skeletal category of $\mathcal{FI}$, whose objects are $[n] = \{1, 2, \dots, n\}$ for $n \in \mathbb{Z}_+$, the set of nonnegative integers. By convention, $[0] = \emptyset$. By $\Bbbk$ we mean a commutative Noetherian ring with identity. Given a set S, $\underline{S}$ is the free $\Bbbk$-module spanned by elements in S. Let $\underline{\mathcal{C}}$ be the $\Bbbk$-linearization of $\mathcal{C}$, which can be regarded as both a $\Bbbk$-linear category and a $\Bbbk$-algebra without identity.

A *representation* of $\mathcal{C}$, or a $\mathcal{C}$-*module*, is a covariant functor V from $\mathcal{C}$ to $\Bbbk$-Mod, the category of left $\Bbbk$-modules. Equivalently, a $\mathcal{C}$-module is a $\underline{\mathcal{C}}$-module, which by definition is a $\Bbbk$-linear covariant functor from $\underline{\mathcal{C}}$ to $\Bbbk$-Mod. It is well known that $\mathcal{C}$-Mod is an abelian category. Moreover, it has enough projectives. In particular, for $i \in \mathbb{Z}_+$, the $\Bbbk$-linearization $\underline{\mathcal{C}}(i,-)$ of the representable functor $\mathcal{C}(i,-)$ is projective.

A representation V of $\mathcal{C}$ is said to be *finitely generated* if there exists a finite subset S of V such that any submodule containing S coincides with V; or equivalently, there exists a surjective homomorphism

$$\bigoplus_{i\in\mathbb{Z}_+} \underline{\mathcal{C}}(i,-)^{\oplus a_i} \to V$$

such that $\sum_{i\in\mathbb{Z}_+} a_i < \infty$. It is said to be *generated in degrees* $\leqslant N$ if in the above surjection one can let $a_i = 0$ for all $i > N$. Obviously, V is finitely generated if and only if it is generated in degrees $\leqslant N$ for a certain $N \in \mathbb{Z}_+$ and the values of V on objects $i \leqslant N$ are finitely generated $\Bbbk$-modules. Since $\mathcal{C}$ is *locally Noetherian* by the fundamental result in [4], the category $\mathcal{C}$-mod of finitely generated $\mathcal{C}$-modules is abelian. In this paper we only consider finitely generated $\mathcal{C}$-modules over commutative Noetherian rings.

Given a finitely generated $\mathcal{C}$-module V and an object $i \in \mathbb{Z}_+$, we denote its value on i by V_i. For every $n \in \mathbb{Z}_+$, one can define a *truncation functor* $\tau_n : \mathcal{C}\text{-mod} \to \mathcal{C}\text{-mod}$ as follows: For $V \in \mathcal{C}$-mod,

$$(\tau_n V)_i := \begin{cases} 0, & i < n \\ V_i, & i \geqslant n \end{cases}$$

A finitely generated $\mathcal{C}$-module V is called *torsion* if there exists some $N \in \mathbb{Z}_+$ such that $\tau_N V = 0$. In other words, $V_i = 0$ for $i \geqslant N$.

The category $\mathcal{C}$ has a *self-embedding functor* $\iota : \mathcal{C} \to \mathcal{C}$ which is faithful and sends an object $i \in \mathbb{Z}_+$ to $i+1$. For a morphism $\alpha \in \mathcal{C}(i,j)$ (which is an injection from $[i]$ to $[j]$), $\iota(\alpha)$ is an injection from $[i+1]$ to $[j+1]$ defined as follows:

$$(\iota(\alpha))(r) = \begin{cases} 1, & r = 1 \in [i+1]; \\ \alpha(r-1)+1, & 1 \neq r \in [i+1]. \end{cases} \tag{1.1}$$

The functor ι induces a pull-back $\iota^* : \mathcal{C}\text{-mod} \to \mathcal{C}\text{-mod}$. The *shift functor* Σ is defined to be $\iota^* \circ \tau_1$. For details, see [4,7,11].

By the directed structure, the $\Bbbk$-linear category $\underline{\mathcal{C}}$ has a two-sided ideal

$$J = \bigoplus_{0 \leqslant i < j} \underline{\mathcal{C}}(i,j).$$

Therefore,

$$\underline{\mathcal{C}}_0 = \bigoplus_{i \in \mathbb{Z}_+} \underline{\mathcal{C}}(i,i)$$

is a $\underline{\mathcal{C}}$-module via identifying it with $\underline{\mathcal{C}}/J$.

Given a finitely generated $\mathcal{C}$-module V, its *torsion degree* is defined to be

$$\mathrm{td}(V) = \sup\{i \in \mathbb{Z}_+ \mid \mathrm{Hom}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}}(i,i), V) \neq 0\}$$

or $-\infty$ if $\mathrm{td}(V) = \emptyset$. In the latter case we say that V is *torsionless*. Its 0-th *homology* is defined to be

$$H_0(V) = V/JV \cong \underline{\mathcal{C}}_0 \otimes_{\underline{\mathcal{C}}} V.$$

Since $\underline{\mathcal{C}}_0 \otimes_{\underline{\mathcal{C}}} -$ is right exact, we define the s-th *homology*

$$H_s(V) = \mathrm{Tor}_s^{\underline{\mathcal{C}}}(\underline{\mathcal{C}}_0, V)$$

for $s \geqslant 1$. Note that this is a $\underline{\mathcal{C}}$-module since $\underline{\mathcal{C}}_0$ is a $(\underline{\mathcal{C}}, \underline{\mathcal{C}})$-bimodule. Moreover, it is finitely generated and torsion. For $s \in \mathbb{Z}_+$, the s-th *homological degree* is set to be

$$\mathrm{hd}_s(V) = \mathrm{td}(H_s(V)).$$

Sometimes we call the 0-th homological degree *generating degree*, and denote it by $\mathrm{gd}(V)$.

Remark 1.1. The above definition of torsion degrees seems mysterious, so let us give an equivalent but more concrete definition. Let V be a finitely generated $\mathcal{C}$-module and $i \in \mathbb{Z}_+$ be an object. If there exists a nonzero $v \in V_i$ and $\alpha_0 \in \mathcal{C}(i, i+1)$ such that $\alpha_0 \cdot v = 0$, we claim that for all $\alpha \in \mathcal{C}(i, i+1)$, one has $\alpha \cdot v = 0$. Indeed, since the symmetric group $S_{i+1} = \mathcal{C}(i+1, i+1)$ acts transitively on $\mathcal{C}(i, i+1)$ from the left side, for an arbitrary $\alpha \in \mathcal{C}(i, i+1)$, we can find an element $g \in \mathcal{C}(i+1, i+1)$ (which is unique) such that $\alpha = g\alpha_0$. Therefore, $\alpha \cdot v = g\alpha_0 \cdot v = 0$.

This observation tells us that α_0 (and hence all $\alpha \in \mathcal{C}(i, i+1)$) sends the $\mathcal{C}(i,i)$-module $\underline{\mathcal{C}}(i,i) \cdot v$ to 0. Indeed, for every $g \in \mathcal{C}(i,i)$, since $\alpha_0 g \in \mathcal{C}(i, i+1)$, one has $(\alpha_0 g) \cdot v = 0$ by the argument in the previous paragraph. But the $\mathcal{C}(i,i)$-module $\underline{\mathcal{C}}(i,i) \cdot v$ can be regarded as a $\mathcal{C}$-module in a natural way, so $\mathrm{Hom}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}}(i,i), V) \neq 0$. Conversely, if $\mathrm{Hom}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}}(i,i), V) \neq 0$, one can easily find a nonzero element $v \in V_i$ such that $\alpha \cdot v = 0$ for $\alpha \in \mathcal{C}(i, i+1)$.

The above observations immediately imply

$$\mathrm{td}(V) = \sup\{i \in \mathbb{Z}_+ \mid \exists\, 0 \neq v \in V_i \text{ and } \alpha \in \mathcal{C}(i, i+1) \text{ such that } \alpha \cdot v = 0\}.$$

Remark 1.2. Homologies of $\mathcal{FI}$-modules were defined to be homologies of a special complex in [4, Section 2.4]. Gan and the first author proved in [9] that homologies of this special complex coincide with ones defined in the above way. Since Tor is a classical homological construction, in this paper we take the above definition.

Since each $\Bbbk S_i = \underline{\mathcal{C}}(i,i)$ is a subalgebra of $\underline{\mathcal{C}}$ for $i \in \mathbb{Z}_+$, given a $\Bbbk S_i$-module T, it induces a $\underline{\mathcal{C}}$-module $\underline{\mathcal{C}} \otimes_{\Bbbk S_i} T$. We call these modules *basic $\sharp$-filtered modules.* A finitely generated $\mathcal{C}$-module V is called *$\sharp$-filtered* by Nagpal if it has a filtration

$$0 = V^{-1} \subseteq V^0 \subseteq \ldots \subseteq V^n = V$$

such that V^{i+1}/V^i is isomorphic to a basic $\sharp$-filtered module for $-1 \leqslant i \leqslant n-1$; see [12, Definition 1.10]. The reader will see that $\sharp$-filtered modules have similar homological behaviors as projective modules.

1.3. Main results

Now we are ready to state main results of this paper. The first result characterizes $\sharp$-filtered modules by homological degrees.

Theorem 1.3 *(Homological characterizations of $\sharp$-filtered modules). Let $\Bbbk$ be a commutative Noetherian ring and let V be a finitely generated $\mathcal{C}$-module. Then the following statements are equivalent:*

(1) *V is $\sharp$-filtered;*
(2) $\mathrm{hd}_s(V) = -\infty$ *for all* $s \geqslant 1$*;*
(3) $\mathrm{hd}_1(V) = -\infty$*;*
(4) $\mathrm{hd}_s(V) = -\infty$ *for some* $s \geqslant 1$*.*

Remark 1.4. This theorem was also independently proved almost at the same time by Ramos in [13, Theorem B] via a different approach.

Using these homological characterizations, one can deduce an upper bound for projective dimensions of finitely generated $\mathcal{C}$-modules whose projective dimension is finite.

Theorem 1.5 *(Upper bounds of projective dimensions). Let $\Bbbk$ be a commutative Noetherian ring whose finitistic dimension* $\mathrm{findim}\,\Bbbk$ *is finite,*[1] *and let V be a finitely generated $\mathcal{C}$-module with* $\mathrm{gd}(V) = n$*. Then the projective dimension* $\mathrm{pd}(V)$ *is finite if and only if*

[1] By definition, the finitistic dimension is the supremum of projective dimensions of finitely generated $\Bbbk$-modules whose projective dimension is finite. The famous finitistic dimension conjecture asserts that if $\Bbbk$ is a finite dimensional algebra, then $\mathrm{findim}\,\Bbbk < \infty$. However, the finitistic dimension of an arbitrary commutative Noetherian ring might be infinity.

for $0 \leqslant i \leqslant n$, *one has*

$$V^i/V^{i-1} \cong \underline{\mathcal{C}} \otimes_{\Bbbk S_i} (V^i/V^{i-1})_i$$

and

$$\mathrm{pd}_{\Bbbk S_i}((V^i/V^{i-1})_i) < \infty,$$

where V^i *is the submodule of* V *generated by* $\bigoplus_{j \leqslant i} V_j$. *Moreover, in that case*

$$\mathrm{pd}(V) = \max\{\mathrm{pd}_{\Bbbk S_i}((V^i/V^{i-1})_i)\}_{i=0}^n = \max\{\mathrm{pd}_{\Bbbk}((V^i/V^{i-1})_i)\}_{i=0}^n \leqslant \mathrm{findim}\,\Bbbk.$$

Remark 1.6. This theorem asserts that finitely generated $\mathcal{C}$-modules which are not $\sharp$-filtered have infinite projective dimension. Moreover, if the finitistic dimension of $\Bbbk$ is 0, or in particular the global dimension $\mathrm{gldim}\,\Bbbk$ is 0, then the projective dimension of a $\mathcal{C}$-module is either 0 or infinity. This special result has been pointed out in [8, Corollary 1.6] for fields of characteristic 0.

Another important application of Theorem 1.3 is to prove the fact that every finitely generated $\mathcal{C}$-module can be approximated by a finite complex of $\sharp$-filtered modules, which was firstly proved by Nagpal in [12, Theorem A]. We give a new proof based on the conclusion of Theorem 1.3 as well as the shift functor.

Theorem 1.7 *($\sharp$-Filtered complexes). Let* $\Bbbk$ *be a commutative Noetherian ring and let* V *be a finitely generated* $\mathcal{C}$*-module. Then there exists a complex*

$$F^\bullet: \quad 0 \to V \to F^0 \to F^1 \to \ldots \to F^n \to 0$$

satisfying the following conditions:

(1) each F^i *is a $\sharp$-filtered module with* $\mathrm{gd}(F^i) \leqslant \mathrm{gd}(V) - i$;
(2) $n \leqslant \mathrm{gd}(V)$;
(3) the homology in each degree of the complex is a torsion module, including the homology at V.

In particular, $\Sigma_d V$ *is a $\sharp$-filtered module for* $d \gg 0$.

Remark 1.8. This complex of $\sharp$-filtered modules generalizes the finite injective resolution described in [8], where it was used by Gan and the first author to give a homological proof for the uniform representation stability phenomenon observed and proved in [2,3]. For an arbitrary field, it also implies the polynomial growth of finitely generated $\mathcal{C}$-modules; see [4, Theorem B] and Remark 3.14.

Homological characterizations of ♯-filtered modules also play a very important role in estimating homological degrees of finitely generated $\mathcal{C}$-modules V. Relying on an existing upper bound described in [1, Theorem A], we obtain another upper bound for homological degrees of $\mathcal{C}$-modules, removing the unnecessary assumption that $\Bbbk$ is a field of characteristic 0 in [11, Theorem 1.17].

Theorem 1.9 *(Castlenuovo–Mumford regularity). Let $\Bbbk$ be a commutative Noetherian ring and let V be a finitely generated $\mathcal{C}$-module. Then for $s \geqslant 1$,*

$$\mathrm{hd}_s(V) \leqslant \max\{2\,\mathrm{gd}(V) - 1,\, \mathrm{td}(V)\} + s. \tag{1.2}$$

Remark 1.10. Theorem A in [1] asserts that

$$\mathrm{hd}_s(V) \leqslant \mathrm{hd}_0(V) + \mathrm{hd}_1(V) + s - 1$$

for $s \geqslant 1$. The conclusion of the above theorem refines this bound for torsionless modules since in that case $\mathrm{td}(V) = -\infty$ and Corollary 3.4 tells us that by a certain reduction one can always assume that $\mathrm{gd}(V) < \mathrm{hd}_1(V)$. Furthermore, it is more practical since it is easier to find $\mathrm{td}(V)$ than $\mathrm{hd}_1(V)$. The reader may refer to [11, Example 5.20].

Remark 1.11. Given a finitely generated $\mathcal{C}$-module V, there exists a short exact sequence

$$0 \to V_T \to V \to V_F \to 0$$

such that V_T is torsion and V_F is torsionless. Note that $\mathrm{gd}(V_F) \leqslant \mathrm{gd}(V)$ and $\mathrm{td}(V_T) = \mathrm{td}(V)$. Using the long exact sequence induced by this short exact sequence, one intuitively sees that the torsion part V_T contributes to the term $\mathrm{td}(V)$ in inequality (1.2), and V_F contributes to the term $2\,\mathrm{gd}(V) - 1$ in inequality (1.2). Indeed, if V is a torsionless $\mathcal{C}$-module, then we have $\mathrm{hd}_s(V) \leqslant 2\,\mathrm{gd}(V) + s - 1$ for $s \geqslant 1$. On the other hand, if V is a torsion module, then $\mathrm{hd}_s(V) \leqslant \mathrm{td}(V) + s$ for $s \geqslant 0$.

1.4. Organization

The paper is organized as follows. In Section 2 we introduce some elementary but important properties of the shift functor Σ and its induced cokernel functor D. In particular, if V is a torsionless $\mathcal{C}$-module, then an *adaptable projective resolution* gives rise to an adaptable projective resolution of DV; see Definition 2.8 and Proposition 2.11. This observation provides us a useful technique to estimate homological degrees of V. Those ♯-filtered modules are studied in details in Section 3. We characterize ♯-filtered modules by homological degrees, and use it to prove that every finitely generated $\mathcal{C}$-module becomes ♯-filtered after applying the shift functor enough times. In the last section we prove all main results mentioned before.

2. Preliminary results

Throughout this section let $\Bbbk$ be a commutative Noetherian ring, and let $\mathcal{C}$ be the skeletal subcategory of $\mathcal{FI}$ with objects $[n]$, $n \in \mathbb{Z}_+$.

2.1. Functor Σ

The shift functor $\Sigma : \mathcal{C}\text{-mod} \to \mathcal{C}\text{-mod}$ has been defined in the previous section. We list certain properties.

Proposition 2.1. *Let V be a $\mathcal{C}$-module. Then one has:*

(1) $\Sigma(\underline{\mathcal{C}}(i,-)) \cong \underline{\mathcal{C}}(i,-) \oplus \underline{\mathcal{C}}(i-1,-)^{\oplus i}$.
(2) If $\mathrm{gd}(V) \leqslant n$, *then* $\mathrm{gd}(\Sigma V) \leqslant n$; *conversely, if* $\mathrm{gd}(\Sigma V) \leqslant n$, *then* $\mathrm{gd}(V) \leqslant n+1$.
(3) The $\mathcal{C}$-module V is finitely generated if and only if so is ΣV.
(4) If V is torsionless, so is ΣV.

Proof. Statement (1) is well known, and it immediately implies the first half of (2) since Σ is an exact functor. For the second half of (2), one may refer to the proof of [11, Lemma 3.4]. Statement (3) is immediately implied by (2) as a $\mathcal{C}$-module is finitely generated if and only if its generating degree is finite and its value on each object is a finitely generated $\Bbbk$-module.

To prove (4), one observes that V is torsionless if and only if the following conditions hold: for $i \in \mathbb{Z}_+$, $0 \neq v \in V_i$, and $\alpha \in \mathcal{C}(i, i+1)$, one always has $\alpha \cdot v \neq 0$. If ΣV is not torsionless, we can find a nonzero element $v \in (\Sigma V)_i$ and $\alpha \in \mathcal{C}(i, i+1)$ for a certain $i \in \mathbb{Z}_+$ such that $\alpha \cdot v = 0$. But $(\Sigma V)_i = V_{i+1}$, and $\iota(\mathcal{C}(i, i+1)) \subseteq \mathcal{C}(i+1, i+2)$ where ι is the self-embedding functor inducing Σ. Therefore, by regarding v as an element in V_{i+1} one has $\iota(\alpha) \cdot v = 0$. Consequently, V is not torsionless either. The conclusion follows from this contradiction. □

2.2. Homological degrees under shift

The following lemma is a direct application of [11, Proposition 4.5].

Lemma 2.2. *Let V be a finitely generated $\mathcal{C}$-module. Then for $s \geqslant 0$,*

$$\mathrm{hd}_s(V) \leqslant \max\{\mathrm{hd}_0(V) + 1, \ldots, \mathrm{hd}_{s-1}(V) + 1, \mathrm{hd}_s(\Sigma V) + 1\}.$$

If V is a torsion module, then $\Sigma_d V = 0$ for a large enough d. Thus the above lemma can be used to estimate homological degrees of torsion modules.

Proposition 2.3. *[11, Theorem 1.5] If V is a finitely generated torsion $\mathcal{C}$-module, then for $s \in \mathbb{Z}_+$, one has*

$$\mathrm{hd}_s(V) \leqslant \mathrm{td}(V) + s.$$

2.3. Functor D

The functor D was introduced in [1,2]. Here we briefly mention its definition. Since the family of inclusions

$$\{\pi_i : [i] \to [i+1],\, r \mapsto r+1 \mid i \geqslant 0\} \tag{2.1}$$

gives a natural transformation π from the identity functor $\mathrm{Id}_{\mathcal{C}}$ to the self-embedding functor ι, we obtain a natural transformation π^* from the identity functor on $\mathcal{C}$-mod to Σ, which induces a natural map $\pi^*_V : V \to \Sigma V$ for each $\mathcal{C}$-module V. The functor D is defined to be the cokernel of this map. Clearly, D is a right exact functor. That is, it preserves surjection.

The following properties of D play a key role in our approach.

Proposition 2.4. *Let D be the functor defined as above.*

(1) The functor D preserves projective $\mathcal{C}$-modules. Moreover, $D\underline{\mathcal{C}}(i,-) \cong \underline{\mathcal{C}}(i-1,-)^{\oplus i}$.
(2) A $\mathcal{C}$-module V is torsionless if and only if there is a short exact sequence

$$0 \to V \to \Sigma V \to DV \to 0.$$

(3) Let V be a finitely generated $\mathcal{C}$-module. Then:

$$\mathrm{gd}(DV) = \begin{cases} -\infty & \text{if } \mathrm{gd}(V) = 0 \text{ or } -\infty \\ \mathrm{gd}(V) - 1 & \text{if } \mathrm{gd}(V) \geqslant 1. \end{cases}$$

Proof. Statements (1) and (2) have been established in [1, Lemma 3.6], so we only give a proof of (3). Take a surjection $P \to V \to 0$ such that P is a projective $\underline{\mathcal{C}}$-module and $\mathrm{gd}(P) = \mathrm{gd}(V) = n$. Since D is a right exact functor, we get a surjection $DP \to DV \to 0$. When $n = 0$ or $-\infty$, we know that $DP = 0$, and hence $DV = 0$, so $\mathrm{gd}(DV) = -\infty$. If $n > 0$, then DP is a projective module with $\mathrm{gd}(DP) = n - 1$ by (1). Consequently, $\mathrm{gd}(DV) \leqslant n - 1$. We finish the proof by showing that $\mathrm{gd}(DV) \geqslant n - 1$ as well.

Let V' be the submodule of V generated by $\bigoplus_{i \leqslant n-1} V_i$. This is a proper submodule of V since $\mathrm{gd}(V) = n$. Let $V'' = V/V'$, which is not zero. Moreover, V'' is a $\mathcal{C}$-module generated in degree n. Applying the right exact functor D to $V \to V'' \to 0$ one gets a surjection $DV \to DV'' \to 0$. But one easily sees that $(DV'')_i = 0$ for $i < n - 1$ and $(DV'')_{n-1} \neq 0$. Consequently, $\mathrm{gd}(DV'') \geqslant n - 1$. This forces $\mathrm{gd}(DV) \geqslant n - 1$. □

Remark 2.5. If V is not torsionless, we have an exact sequence

$$0 \to V_T \to V \to V_F \to 0$$

such that $V_T \neq 0$ is torsion and V_F is torsionless. It induces a short exact sequence

$$0 \to \Sigma V_T \to \Sigma V \to \Sigma V_F \to 0.$$

Using snake lemma, one obtains exact sequences

$$0 \to K \to V \to \Sigma V \to DV \to 0$$

and

$$0 \to K \to V_T \to \Sigma V_T \to DV_T \to 0.$$

In particular, K is a torsion module.

An immediate consequence is:

Corollary 2.6. *A short exact sequence $0 \to W \to M \to V \to 0$ of finitely generated torsionless $\mathcal{C}$-modules gives rise to the following commutative diagram such that all rows and columns are short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & \Sigma W & \longrightarrow & DW & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \Sigma M & \longrightarrow & DM & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V & \longrightarrow & \Sigma V & \longrightarrow & DV & \longrightarrow & 0. \end{array}$$

Proof. Since W and M are torsionless. By [1, Lemma 3.6], $W \to \Sigma W$ and $M \to \Sigma M$ are injective, and one gets a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & \Sigma W & \longrightarrow & DW & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & M & \longrightarrow & \Sigma M & \longrightarrow & DM & \longrightarrow & 0, \end{array}$$

which by the snake lemma induces the following exact sequence

$$0 \to \ker \alpha \to V \to \Sigma V \to DV \to 0.$$

But since V is torsionless, the map $V \to \Sigma V$ is injective. Therefore, $\ker \alpha = 0$. The conclusion follows. □

The next lemma asserts that the functor D “almost” commutes with Σ.

Lemma 2.7. *Let V be a finitely generated $\mathcal{C}$-module. Then $\Sigma DV \cong D\Sigma V$.*

Proof. It is sufficient to construct a natural isomorphism between ΣD and $D\Sigma$. This has been done by Ramos in [13, Lemma 3.5]. Note that in the setting of that paper, the self-embedding functor is defined in a way different from ours; see [13, Definition 2.20]. Therefore, we have to slightly modify the proof in [13, Lemma 3.5]. In our setting,

$$(\Sigma DV)_n = V_{n+2}/\pi_{n+1}(V_{n+1});$$
$$(D\Sigma V)_n = V_{n+2}/(\iota(\pi_n))(V_{n+1})$$

where ι is the self-embedding functor and π_n is defined in (2.1). By (1.1) and (2.1), we have

$$\pi_{n+1} : [n+1] \to [n+2], \quad i \mapsto i+1;$$
$$\iota(\pi_n) : [n+1] \to [n+2], \quad i \mapsto \begin{cases} 1, & i = 1; \\ i+1, & 2 \leqslant i \leqslant n+1. \end{cases}$$

Now the reader can see that $(D\Sigma V)_n$ and $(\Sigma DV)_n$ are isomorphic under the action of an bijection $\alpha_{n+2} : [n+2] \to [n+2]$ which permutes 1 and 2 and fixes all other elements. Moreover, the family of such bijections $\{\alpha_n \mid n \geqslant 0\}$ gives a natural isomorphism between ΣD and $D\Sigma$. □

2.4. Adaptable projective resolutions

A standard way to compute homologies and hence homological degrees is to use a suitable projective resolution.

Definition 2.8. Let V be a finitely generated $\underline{\mathcal{C}}$-module. A projective resolution

$$\ldots \to P^s \to P^{s-1} \to \ldots \to P^0 \to V \to 0$$

of V is said to be *adaptable* if for every $s \geqslant -1$, $\mathrm{gd}(P^{s+1}) = \mathrm{gd}(Z^s)$, where Z^s is the s-th cycle and by convention $Z^{-1} = V$.

Lemma 2.9. *Let $0 \to W \to P \to V \to 0$ be a short exact sequence of finitely generated $\mathcal{C}$-modules such that P is projective and $\mathrm{gd}(V) = \mathrm{gd}(P)$. Then*

$$\mathrm{gd}(W) \leqslant \max\{\mathrm{hd}_1(V), \mathrm{gd}(V)\} = \max\{\mathrm{gd}(V), \mathrm{gd}(W)\}.$$

Proof. The conclusions hold for $V = 0$ by convention, so we assume that V is nonzero. The given short exact sequence gives rise to

$$0 \to H_1(V) \to H_0(W) \to H_0(P) \to H_0(V) \to 0.$$

Clearly,

$$\mathrm{gd}(W) = \mathrm{td}(H_0(W)) \leqslant \max\{\mathrm{td}(H_1(V)), \mathrm{td}(H_0(P))\} = \max\{\mathrm{hd}_1(V), \mathrm{gd}(V)\}.$$

Moreover, if $\mathrm{hd}_1(V) \leqslant \mathrm{gd}(V)$, then $\mathrm{gd}(W) \leqslant \mathrm{gd}(V)$ as well. If $\mathrm{hd}_1(V) > \mathrm{gd}(V)$, then $\mathrm{hd}_1(V) = \mathrm{gd}(W)$. The equality follows from this observation. □

Given a finitely generated $\mathcal{C}$-module V, the following corollary relates generating degrees of components in an adaptable projective resolution of V to homological degrees of V. That is:

Corollary 2.10. *Let V be a finitely generated $\underline{\mathcal{C}}$-module, and let*

$$\ldots \to P^s \to P^{s-1} \to \ldots \to P^0 \to V \to 0$$

be an adaptable projective resolution of V. Let $d_s = \mathrm{gd}(P^s)$. Then

$$d_s \leqslant \max\{\mathrm{hd}_0(V), \ldots, \mathrm{hd}_{s-1}(V), \mathrm{hd}_s(V)\}$$

and

$$\max\{d_0, \ldots, d_{s-1}, d_s\} = \max\{\mathrm{hd}_0(V), \ldots, \mathrm{hd}_{s-1}(V), \mathrm{hd}_s(V)\}.$$

Proof. We use induction on s. If $s = 0$, nothing needs to show. Suppose that the conclusion holds for $s = n \geqslant 0$, and consider $s = n + 1$.

Consider the short exact sequence

$$0 \to Z^n \to P^n \to Z^{n-1} \to 0.$$

By Lemma 2.9 and the definition of adaptable projective resolutions,

$$d_{n+1} = \mathrm{gd}(Z^n) \leqslant \max\{\mathrm{gd}(Z^{n-1}), \mathrm{hd}_1(Z^{n-1})\} = \max\{d_n, \mathrm{hd}_{n+1}(V)\}$$

since $V = Z^{-1}$ by convention. However, by induction,

$$d_n \leqslant \max\{\mathrm{hd}_0(V), \ldots, \mathrm{hd}_n(V)\}.$$

The last two inequalities imply the conclusion for $n + 1$. This establishes the inequality.

To show the equality, one observes that the inequality we just proved implies that

$$\max\{d_0, \ldots, d_{s-1}, d_s\} \leqslant \max\{\mathrm{hd}_0(V), \ldots, \mathrm{hd}_{s-1}(V), \mathrm{hd}_s(V)\}.$$

However, one observes from the definition of homologies that $d_i \geqslant \mathrm{hd}_i(V)$ for $i \in \mathbb{Z}_+$. Thus we also have

$$\max\{d_0, \ldots, d_{s-1}, d_s\} \geqslant \max\{\mathrm{hd}_0(V), \ldots, \mathrm{hd}_{s-1}(V), \mathrm{hd}_s(V)\}. \quad \square$$

Using the functor D, one may relate the homological degrees of a finitely generated $\mathcal{C}$-module V to those of DV.

Proposition 2.11. *Let V be a finitely generated torsionless $\underline{\mathcal{C}}$-module, and let*

$$\ldots \to P^s \to P^{s-1} \to \ldots \to P^0 \to V \to 0$$

be an adaptable projective resolution of V. Then it induces an adaptable projective resolution

$$\ldots \to DP^s \to DP^{s-1} \to \ldots \to DP^0 \to DV \to 0$$

such that for $s \in \mathbb{Z}_+$

$$\mathrm{gd}(DP^s) = \begin{cases} -\infty & \text{if } \mathrm{gd}(P^s) = 0 \text{ or } -\infty \\ \mathrm{gd}(P^s) - 1 & \text{if } \mathrm{gd}(P^s) \geqslant 1. \end{cases}$$

Proof. Let $P^\bullet \to V \to 0$ be the resolution. Since V and all cycles are torsionless, by Corollary 2.6 one gets a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^\bullet & \longrightarrow & \Sigma P^\bullet & \longrightarrow & DP^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V & \longrightarrow & \Sigma V & \longrightarrow & DV & \longrightarrow & 0. \end{array}$$

The conclusion then follows from Proposition 2.4. $\square$

As an immediate consequence of the above result, we have:

Corollary 2.12. *Let V be a finitely generated torsionless $\mathcal{C}$-module. Then for $s \in \mathbb{Z}_+$*

$$\max\{\mathrm{hd}_0(V), \ldots, \mathrm{hd}_s(V)\} \geqslant \max\{\mathrm{hd}_0(DV), \ldots, \mathrm{hd}_s(DV)\} + 1.$$

Moreover, the equality holds if $\mathrm{gd}(V) \geqslant 1$.

Proof. The conclusion holds trivially if $\mathrm{gd}(V) = 0$ or $-\infty$ since in that case $DV = 0$. So we assume $\mathrm{gd}(V) \geqslant 1$, and $DV \neq 0$. Let $P^\bullet \to V \to 0$ be an adaptable projective resolution. By Corollary 2.10 and Proposition 2.11, one has

$$\max\{\mathrm{gd}(V), \ldots, \mathrm{hd}_s(V)\} = \max\{\mathrm{gd}(P^0), \ldots, \mathrm{gd}(P^s)\}$$

and

$$\max\{\mathrm{gd}(DV), \ldots, \mathrm{hd}_s(DV)\} = \max\{\mathrm{gd}(DP^0), \ldots, \mathrm{gd}(DP^s)\}.$$

Moreover, $\mathrm{gd}(P^i) \geqslant \mathrm{gd}(DP^i) + 1$ for $i \in \mathbb{Z}_+$, and the equality holds if $\mathrm{gd}(P^i) \geqslant 1$. The desired inequality and equality follow from these observations. □

3. Filtrations of $\mathcal{FI}$-modules

In the previous section we use adaptable projective resolutions to estimate homological degrees of finitely generated $\mathcal{C}$-modules. However, since finitely generated $\mathcal{C}$-modules usually have infinite projective dimension, the resolutions are of infinite length. For the purpose of estimating homological degrees, $\sharp$-filtered modules play a more subtle role since we will show that every finitely generated $\mathcal{C}$-module V gives rise to a complex of $\sharp$-filtered modules which is of finite length. Moreover, we will see that these special modules, coinciding with projective modules when $\Bbbk$ is a field of characteristic 0, have similar homological properties as projective modules.

3.1. A homological characterization of $\sharp$-filtered modules

Recall that a finitely generated $\mathcal{C}$-module is $\sharp$-*filtered* if there exists a chain

$$0 = V^{-1} \subseteq V^0 \subseteq \ldots \subseteq V^n = V$$

such that V^i/V^{i-1} is isomorphic to $\underline{\mathcal{C}} \otimes_{\Bbbk S_i} T_i$ for $0 \leqslant i \leqslant n$, where S_i is the symmetric group on i letters, and T_i is a finitely generated $\Bbbk S_i$-module.

An important fact of $\mathcal{FI}$, which can be easily observed, is:

Lemma 3.1. *For $n \in \mathbb{Z}_+$, the $\mathcal{C}$-module $\underline{\mathcal{C}} 1_n$ is a right free $\Bbbk S_n$-module.*

Proof. Note that for $n \in \mathbb{Z}_+$ and $m \geqslant n$, the group $S_n = \mathcal{C}(n, n)$ acts freely on $\mathcal{C}(n, m)$ from the right side. The conclusion follows. □

This elementary observation implies that higher homologies of $\sharp$-filtered modules vanish.

Lemma 3.2. *If V is a $\sharp$-filtered module, then $H_s(V) = 0$ for all $s \geqslant 1$.*

Proof. Firstly we consider a special case: V is basic. That is, $V = \underline{\mathcal{C}} \otimes_{\Bbbk S_i} V_i$ for some $i \in \mathbb{Z}_+$. Let

$$0 \to W_i \to P_i \to V_i \to 0$$

be a short exact sequence of $\Bbbk S_i$-modules such that P_i is projective. Since $\underline{\mathcal{C}}$ is a right projective $\Bbbk S_i$-module, we get an exact sequence

$$0 \to W = \underline{\mathcal{C}} \otimes_{\Bbbk S_i} W_i \to P = \underline{\mathcal{C}} \otimes_{\Bbbk S_i} P_i \to V = \underline{\mathcal{C}} \otimes_{\Bbbk S_i} V_i \to 0.$$

Note that the middle term is a projective $\underline{\mathcal{C}}$-module. By applying $\underline{\mathcal{C}}_0 \otimes_{\underline{\mathcal{C}}} -$ one recovers the first exact sequence, so $H_1(V) = 0$. Replacing V by W, one deduces that $H_2(V) = H_1(W) = 0$. The conclusion follows by recursion.

For the general case, one may take a filtration for V, each component of which is a basic $\sharp$-filtered module. The conclusion follows from a standard homological method: short exact sequences induce long exact sequences on homologies. $\square$

The following lemma was proved in [12, Lemma 2.2]. Here we give two proofs from the homological viewpoint.

Lemma 3.3. *Let V be a finitely generated $\mathcal{C}$-module generated in one degree. If $\mathrm{hd}_1(V) \leqslant \mathrm{gd}(V)$, then V is a $\sharp$-filtered module.*

Proof. The conclusion holds trivially for $V = 0$, so we assume $\mathrm{gd}(V) = n \geqslant 0$. Consider the short exact sequence

$$0 \to W \to P \to V \to 0$$

where P is a projective $\mathcal{C}$-module with $\mathrm{gd}(P) = n$. Since $\mathrm{hd}_1(V) \leqslant n$, one knows that $\mathrm{gd}(W) \leqslant n$ by Lemma 2.9. If $\mathrm{gd}(W) < n$, then $W = 0$ since $W_i = 0$ for all $i < n$. Thus $V \cong P$ is clearly a $\sharp$-filtered module. Now we consider the case that $\mathrm{gd}(W) = n$.

Since $0 \to W_n \to P_n \to V_n \to 0$ is a short exact sequence of $\Bbbk S_n$-modules and $\underline{\mathcal{C}}$ is a right projective $\Bbbk S_n$-module, we obtain a short exact sequence

$$0 \to \underline{\mathcal{C}} \otimes_{\Bbbk S_n} W_n \to \underline{\mathcal{C}} \otimes_{\Bbbk S_n} P_n \to \underline{\mathcal{C}} \otimes_{\Bbbk S_n} V_n \to 0.$$

Note that W, P, and V are all generated in degree n. Via the multiplication map we get a commutative diagram such that all vertical maps are surjective:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{\mathcal{C}} \otimes_{\Bbbk S_n} W_n & \longrightarrow & \underline{\mathcal{C}} \otimes_{\Bbbk S_n} P_n & \longrightarrow & \underline{\mathcal{C}} \otimes_{\Bbbk S_n} V_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W & \longrightarrow & P & \longrightarrow & V & \longrightarrow & 0.
\end{array}$$

But the middle vertical map is actually an isomorphism. This forces the other two vertical maps to be isomorphisms by snake lemma, and the conclusion follows. □

Proof. If $V = 0$, nothing needs to show. Otherwise, let $\mathrm{gd}(V) = n \geqslant 0$. Since V is generated in degree n, there is a short exact sequence

$$0 \to K \to \tilde{V} = \underline{\mathcal{C}} \otimes_{\Bbbk S_n} V_n \to V \to 0$$

which induces an exact sequence

$$0 \to H_1(V) \to H_0(K) \to H_0(\tilde{V}) \to H_0(V) \to 0$$

by the previous lemma. Note that the map $H_0(\widetilde{V}) \to H_0(V)$ is an isomorphism. Consequently, $H_0(K) \cong H_1(V)$. In particular, $\mathrm{gd}(K) = \mathrm{hd}_1(V) \leqslant \mathrm{gd}(V) = n$. But it is clear that $K_i = 0$ for all $i \leqslant n$. Therefore, the only possibility is that $K = 0$. □

A useful result is:

Corollary 3.4. *Let V be a finitely generated $\mathcal{C}$-module. Then there exists a short exact sequence*

$$0 \to U \to V \to W \to 0$$

such that $H_s(U) = H_s(V)$ for $s \geqslant 1$ and W is $\sharp$-filtered. Moreover, if V is not $\sharp$-filtered, one always has $\mathrm{hd}_1(U) > \mathrm{gd}(U)$.

Proof. Suppose that V is nonzero. If $\mathrm{hd}_1(V) > \mathrm{gd}(V)$, then we can let $U = V$ and $W = 0$, so the conclusion holds trivially. Otherwise, one has a short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

where V' is the submodule of V generated by $\bigoplus_{i \leqslant \mathrm{gd}(V)-1} V_i$, which might be 0. Then $V'' \neq 0$. The long exact sequence

$$\ldots \to H_1(V) \to H_1(V'') \to H_0(V') \to H_0(V) \to H_0(V'') \to 0$$

tells us that

$$\mathrm{hd}_1(V'') \leqslant \max\{\mathrm{gd}(V'), \mathrm{hd}_1(V)\} \leqslant \mathrm{gd}(V) = \mathrm{gd}(V'').$$

But the previous lemma asserts that V'' is $\sharp$-filtered.

If $\mathrm{gd}(V') < \mathrm{hd}_1(V')$, then the above short exact sequence is what we want. Otherwise, we can continue this process for V'. Since $\mathrm{gd}(V') < \mathrm{gd}(V)$, it must stop after finitely many steps. Since V is not $\sharp$-filtered, in the last step we must get a submodule U of V with

$\mathrm{hd}_1(U) > \mathrm{gd}(U)$. Moreover, since W is $\sharp$-filtered, one easily deduces that $H_s(U) \cong H_s(V)$ for $s \geqslant 1$. $\square$

A finitely generated $\mathcal{C}$-module is $\sharp$-filtered if and only if its higher homologies vanish, and if and only if its first homology vanishes.

Theorem 3.5. *Let $\Bbbk$ be a commutative Noetherian ring and let V be a finitely generated $\mathcal{C}$-module. Then the following statements are equivalent:*

(1) V is a $\sharp$-filtered module;
(2) $H_i(V) = 0$ for all $i \geqslant 1$;
(3) $H_1(V) = 0$.

Proof. (1) ⇒ (2): This is Lemma 3.2.

(2) ⇒ (3): Trivial.

(3) ⇒ (1): Suppose that $H_1(V) = 0$. That is, $\mathrm{hd}_1(V) = -\infty$. If V is not $\sharp$-filtered, then by Corollary 3.4, there exists a short exact sequence

$$0 \to U \to V \to W \to 0$$

such that $H_1(U) \cong H_1(V) = 0$ and $\mathrm{hd}_1(U) > \mathrm{gd}(U)$. This is absurd. $\square$

An extra bonus of this characterization is that: the category of finitely generated $\sharp$-filtered modules is closed under taking kernels and extensions.

Corollary 3.6. *Let $0 \to U \to V \to W \to 0$ be a short exact sequence.*

(1) If both V and W are $\sharp$-filtered, so is U.
(2) If both U and W are $\sharp$-filtered, so is V.

Proof. Use the long exact sequence induced by the given short exact sequence and the above theorem. $\square$

3.2. Properties of $\sharp$-filtered modules

In this subsection we explore certain important properties of $\sharp$-filtered modules.

For a $\sharp$-filtered module V, one knows that it has a filtration by basic $\sharp$-filtered modules from the definition. The following result tells us an explicit construction of such a filtration.

Proposition 3.7. *Let V be a $\sharp$-filtered module with $\mathrm{gd}(V) = n$. Then there exists a chain of $\mathcal{C}$-modules*

$$0 = V^{-1} \subseteq V^0 \subseteq V^1 \subseteq \ldots \subseteq V^n = V$$

such that for $-1 \leqslant s \leqslant n-1$, V^s *is the submodule of* V *generated by* $\bigoplus_{i \leqslant s} V_i$ *and* V^{s+1}/V^s *is 0 or a basic* $\sharp$*-filtered module.*

Proof. We prove the statement by induction on $\mathrm{gd}(V)$. The conclusion holds obviously for $n = 0$ or $n = -\infty$. Suppose that $n \geqslant 1$. We have a short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

such that V' is the submodule generated by $\bigoplus_{i \leqslant n-1} V_i$. Then V'' is generated in degree n. The conclusion follows immediately after we show that V'' is a $\sharp$-filtered module. However, we can apply the same argument as in the proof of Corollary 3.4. □

Remark 3.8. So far we do not use any special property of the category $\mathcal{FI}$ to prove the above results in this section. Therefore, all these results hold for $\Bbbk$-linear categories $\underline{\mathcal{D}}$ satisfying the following conditions:

(1) objects of $\underline{\mathcal{D}}$ are parameterized by nonnegative integers;
(2) there is no nonzero morphisms from bigger objects to smaller objects;
(3) $\underline{\mathcal{D}}$ is *locally finite*; that is, $\underline{\mathcal{D}}(x, y)$ are finitely generated $\Bbbk$-modules for $x, y \in \mathbb{Z}_+$;
(4) $\underline{\mathcal{D}}(x, y)$ is a right projective $\underline{\mathcal{D}}(x, x)$-module for $x, y \in \mathbb{Z}_+$.

Now we begin to use some special properties of $\mathcal{FI}$ to deduce more results. The following proposition tells us that the property of being $\sharp$-filtered is preserved by functors Σ and D.

Proposition 3.9. *Let* V *be a finitely generated* $\mathcal{C}$*-module. If* V *is* $\sharp$*-filtered, then it is torsionless. Moreover,* ΣV *and* DV *are also* $\sharp$*-filtered.*

Proof. Clearly, we can suppose that V is nonzero. Since V is a $\sharp$-filtered module, it has a filtration such that each component of which has the form $\underline{\mathcal{C}} \otimes_{\Bbbk S_i} T_i$, where each T_i is a finitely generated $\Bbbk S_i$-module. Since V is a torsionless module if and only if $\mathrm{Hom}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}}_0, V)$ is 0, to show that V is torsionless, it suffices to verify $\mathrm{Hom}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}}_0, \underline{\mathcal{C}} \otimes_{\Bbbk S_i} T_i) = 0$; that is, each $\underline{\mathcal{C}} \otimes_{\Bbbk S_i} T_i$ is torsionless. By [12, Lemma 2.2], there is a natural embedding

$$0 \to \underline{\mathcal{C}} \otimes_{\Bbbk S_i} M_i \to \Sigma(\underline{\mathcal{C}} \otimes_{\Bbbk S_i} M_i).$$

By Proposition 2.4, this $\sharp$-filtered module must be torsionless.

To prove the second statement, we use induction on the generating degree $n = \mathrm{gd}(V)$. For $n = 0$, the conclusion is implied by [12, Lemma 2.2]. For $n \geqslant 1$, one considers the short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

where V' is the submodule of V generated by $\bigoplus_{i \leqslant n-1} V_i$. By Proposition 3.7, all terms in this short exact sequence are $\sharp$-filtered. Moreover, V'' is a basic $\sharp$-filtered module. Since we just proved that they are all torsionless, by Corollary 2.6, we have a commutative diagram each row or column of which is an short exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & \Sigma V' & \longrightarrow & DV' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V & \longrightarrow & \Sigma V & \longrightarrow & DV & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V'' & \longrightarrow & \Sigma V'' & \longrightarrow & DV'' & \longrightarrow & 0. \end{array}$$

By induction hypothesis, $\Sigma V'$ is $\sharp$-filtered. Moreover, $\Sigma V''$ is $\sharp$-filtered by [12, Lemma 2.2], so ΣV is $\sharp$-filtered as well. The same reasoning tells us that DV are also $\sharp$-filtered. □

3.3. Finitely generated $\mathcal{C}$-modules become $\sharp$-filtered under enough shifts

The main task of this subsection is to use functor D as well as the homological characterization of $\sharp$-filtered modules to show that for every finitely generated $\mathcal{C}$-module V, $\Sigma_N V$ is $\sharp$-filtered for $N \gg 0$.

As the starting point, we consider $\mathcal{C}$-modules generated in degree 0.

Lemma 3.10. *Every torsionless $\mathcal{C}$-module generated in degree 0 is $\sharp$-filtered.*

Proof. Since V is torsionless, one gets a short exact sequence

$$0 \to V \to \Sigma V \to DV \to 0.$$

By Proposition 2.4, $DV = 0$, so $V \cong \Sigma V$. In particular, one has

$$V_0 \cong V_1 \cong V_2 \cong \dots.$$

Thus $V \cong \underline{\mathcal{C}} \otimes_{\underline{\mathcal{C}}(0,0)} V_0$. □

Lemma 3.11. *Let V be a finitely generated torsionless $\mathcal{C}$-module. If DV is $\sharp$-filtered, then* $\mathrm{gd}(V) \geqslant \mathrm{hd}_1(V)$.

Proof. One may assume that V is nonzero. If $\mathrm{gd}(V) = 0$, then V is $\sharp$-filtered by Lemma 3.10 and the conclusion holds. For $\mathrm{gd}(V) \geqslant 1$, since $\mathrm{hd}_1(DV) = -\infty$, by Corollary 2.12,

$$\max\{\mathrm{gd}(V), \mathrm{hd}_1(V)\} = \max\{\mathrm{gd}(DV) + 1, \mathrm{hd}_1(DV) + 1\} = \mathrm{gd}(DV) + 1 = \mathrm{gd}(V),$$

where the last equality follows from Proposition 2.4. This implies the desired result. □

The following lemma, similar to Proposition 2.11, shows the connections between a finitely generated $\mathcal{C}$-module V and DV.

Lemma 3.12. *Let V be a finitely generated torsionless $\mathcal{C}$-module. If DV is $\sharp$-filtered, so is V.*

Proof. We use induction on $\mathrm{gd}(V)$. The conclusion for $\mathrm{gd}(V) = 0$ has been established in Lemma 3.10. Suppose that it holds for all finitely generated $\mathcal{C}$-modules with generating degree at most n, and let V be a finitely generated $\mathcal{C}$-module with $\mathrm{gd}(V) = n + 1$. As before, consider the exact sequence $0 \to V' \to V \to V'' \to 0$ where V' is the submodule generated by $\bigoplus_{i \leqslant n} V_i$.

By considering the long exact sequence

$$\ldots \to H_1(V) \to H_1(V'') \to H_0(V') \to H_0(V) \to H_0(V'') \to 0$$

and using the fact $\mathrm{gd}(V') \leqslant n$ and $n + 1 = \mathrm{gd}(V) \geqslant \mathrm{hd}_1(V)$ which is proved in Lemma 3.11, one deduces that $\mathrm{hd}_1(V'') \leqslant n + 1 = \mathrm{gd}(V'')$. Consequently, V'' is a $\sharp$-filtered module by Lemma 3.3, and hence is torsionless by Proposition 3.9. Moreover, DV'' is $\sharp$-filtered as well by Proposition 3.9.

Since V'' is torsionless, the above exact sequence gives rise to a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V' & \longrightarrow & \Sigma V' & \longrightarrow & DV' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V & \longrightarrow & \Sigma V & \longrightarrow & DV & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V'' & \longrightarrow & \Sigma V'' & \longrightarrow & DV'' & \longrightarrow & 0.
\end{array}$$

Note that both DV'' and DV are $\sharp$-filtered. By Corollary 3.6, DV' is $\sharp$-filtered as well. But clearly V' is torsionless since it is a submodule of the torsionless module V. By induction hypothesis, V' is $\sharp$-filtered, so is V. The conclusion follows from induction. □

Now we are ready to show the following result.

Theorem 3.13. *Let $\Bbbk$ be a commutative Noetherian ring and let V be a finitely generated $\mathcal{C}$-module. If $d \gg 0$, then $\Sigma_d V$ is $\sharp$-filtered.*

Proof. Again, assume that V is nonzero. Since $d \gg 0$, one may suppose that $\mathrm{td}(V) < d$. Therefore, $\Sigma_d V$ is torsionless. We use induction on $\mathrm{gd}(V)$. If $\mathrm{gd}(V) = 0$, then $\Sigma_d V$ is generated in degree 0, so the conclusion holds by Lemma 3.10. Now suppose that the conclusion holds for all modules with generating degrees at most n. We then deal with $\mathrm{gd}(V) = n + 1$.

Consider the exact sequence

$$0 \to \Sigma_d V \to \Sigma_{d+1} V \to \Sigma_d DV \cong D\Sigma_d V \to 0.$$

If $\mathrm{gd}(\Sigma_d V) = 0$, nothing needs to show. So we suppose that $\mathrm{gd}(\Sigma_d V) \geqslant 1$. Note that $\mathrm{gd}(DV) = n$. Therefore, by induction hypothesis, $\Sigma_d DV$ is $\sharp$-filtered. That is, $D\Sigma_d V$ is $\sharp$-filtered. By Lemma 3.12, $\Sigma_d V$ is $\sharp$-filtered as well. □

Remark 3.14. This theorem gives another proof for the polynomial growth phenomenon observed in [4]. Since for a sufficiently large $N \in \mathbb{Z}_+$, $\Sigma_N V$ has a filtration each component of which is exactly of the form $M(W)$ as in [2, Definition 2.2.2], and each $M(W)$ satisfies the polynomial growth property, so is $\Sigma_N V$. Consequently, $\tau_N V$ satisfies the polynomial growth property.

Because $\sharp$-filtered modules coincide with projective modules when $\Bbbk$ is a field of characteristic 0, one has:

Corollary 3.15. *Let $\Bbbk$ be a field of characteristic 0 and let V be a finitely generated $\mathcal{C}$-module. Then for $d \gg 0$, $\Sigma_d V$ is projective.*

4. Proofs of main results

In this section we prove several main results mentioned in Section 1.

4.1. Proof of Theorem 1.3

Theorem 3.5 has established the equivalence of the first three statements in Theorem 1.3. In this subsection we show the equivalence between the first statement and the last one.

Lemma 4.1. *Let V be a finitely generated $\mathcal{C}$-module. If $H_2(V) = 0$, then V is torsionless.*

Proof. Consider a short exact sequence

$$0 \to W \to P \to V \to 0.$$

Since $H_1(W) = H_2(V) = 0$, W is $\sharp$-filtered. Applying Σ to it one gets

$$0 \to \Sigma W \to \Sigma P \to \Sigma V \to 0.$$

They induce the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & \Sigma W & \longrightarrow & DW & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & P & \longrightarrow & \Sigma P & \longrightarrow & DP & \longrightarrow & 0, \end{array}$$

and hence an exact sequence

$$0 \to \ker \alpha \to V \to \Sigma V \to DV \to 0.$$

Since W is $\sharp$-filtered, DW is $\sharp$-filtered as well, and is torsionless by Proposition 3.9. Therefore, $\ker \alpha$ as a submodule of DW is torsionless as well. However, as explained in Remark 2.5, the kernel of $V \to \Sigma V$ is torsion since it is a submodule of V_T, the torsion part of V. This happens if and only if the kernel is 0. That is, V is torsionless. □

The conclusion of this lemma can be strengthened.

Lemma 4.2. *Let V be a finitely generated $\mathcal{C}$-module. If $H_2(V) = 0$, then V is $\sharp$-filtered.*

Proof. We already know that V is torsionless from the previous lemma. Now we use induction on $\mathrm{gd}(V)$. If $\mathrm{gd}(V) = 0$, then the conclusion follows from Lemma 3.10. Otherwise, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & \Sigma W & \longrightarrow & DW & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P & \longrightarrow & \Sigma P & \longrightarrow & DP & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V & \longrightarrow & \Sigma V & \longrightarrow & DV & \longrightarrow & 0, \end{array}$$

where P is projective and W is $\sharp$-filtered. By Proposition 3.9, DW is $\sharp$-filtered as well. Moreover, $\mathrm{gd}(DV) < \mathrm{gd}(V)$. Therefore, by induction hypothesis, DV is $\sharp$-filtered. But by Lemma 3.12, V must be $\sharp$-filtered as well. □

The conclusion of Corollary 3.6 can be strengthened as follows:

Corollary 4.3. *Let $0 \to U \to V \to W \to 0$ be a short exact sequence of finitely generated $\mathcal{C}$-modules. If two terms are $\sharp$-filtered, so is the third one.*

Proof. It suffices to show that if U and V are $\sharp$-filtered, so is W. The long exact sequence

$$\ldots \to H_2(U) \to H_2(V) \to H_2(W) \to H_1(U) \to \ldots$$

tells us $H_2(W) = 0$ since $H_1(U) = H_2(V) = 0$. The conclusion follows from Lemma 4.2. □

Now we are ready to prove the first main theorem mentioned in Section 1.

Proposition 4.4. *Let V be a finitely generated $\mathcal{C}$-module. Then V is $\sharp$-filtered if and only if $H_s(V) = 0$ for some $s \geqslant 1$.*

Proof. One direction is trivial. For the other direction, we can assume that $s \geqslant 2$. Take an adaptable projective resolution $P^\bullet \to 0$ and let Z^i be the i-th cycle. Then we have

$$0 = H_s(V) = H_1(Z^{s-1}).$$

Consequently, Z^{s-1} is a $\sharp$-filtered module. Applying the previous corollary to the short exact sequence

$$0 \to Z^{s-1} \to P^{s-2} \to Z^{s-2} \to 0$$

one deduces that Z^{s-2} is $\sharp$-filtered as well. The conclusion follows from recursion. □

4.2. Upper bounds for projective dimensions

In this subsection we prove Theorem 1.5. We need a well known result on representation theory of finite groups.

Lemma 4.5. *Let G be a finite group and $\Bbbk$ be a commutative Noetherian ring. Let V be a finitely generated $\Bbbk G$-module. If $\mathrm{pd}_{\Bbbk G}(M)$ is finite, then $\mathrm{pd}_{\Bbbk G}(M) = \mathrm{pd}_{\Bbbk}(M)$.*

Proof. The proof uses Eckmann–Shapiro Lemma. For details, see [10, Theorem 4.3]. □

If V is a basic $\sharp$-filtered module, then its projective dimension coincides with that of the finitely generated group representation inducing V. That is,

Lemma 4.6. *Let T be a finitely generated $\Bbbk S_i$-module. Then*

$$\mathrm{pd}_{\Bbbk S_i}(T) = \mathrm{pd}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}} \otimes_{\Bbbk S_i} T).$$

Proof. We claim that

$$\mathrm{pd}_{\Bbbk S_i}(T) \geqslant \mathrm{pd}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}} \otimes_{\Bbbk S_i} T).$$

If $\mathrm{pd}_{\Bbbk S_i}(T) = \infty$, the inequality holds. Otherwise, there is a projective resolution $Q^\bullet \to T \to 0$ of $\Bbbk S_i$-modules such that Z^s is projective for $s \geqslant \mathrm{pd}_{\Bbbk S_i}(T)$, where Z^s is the s-th cycle. Applying the exact functor $\underline{\mathcal{C}} \otimes_{\Bbbk S_i} -$ one gets a projective resolution of $\mathcal{C}$-modules

$$\underline{\mathcal{C}} \otimes_{\Bbbk S_i} Q^\bullet \to \underline{\mathcal{C}} \otimes_{\Bbbk S_i} T \to 0,$$

whose s-th cycle $\underline{\mathcal{C}} \otimes_{\Bbbk S_i} Z^s$ is a projective $\mathcal{C}$-module for $s \geqslant \mathrm{pd}_{\Bbbk S_i}(T)$. The claim is proved.

Now we show that

$$\mathrm{pd}_{\Bbbk S_i}(T) \leqslant \mathrm{pd}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}} \otimes_{\Bbbk S_i} T).$$

If $\mathrm{pd}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}} \otimes_{\Bbbk S_i} T) = \infty$, the inequality holds. Otherwise, there is a finite projective resolution of $\mathcal{C}$-modules

$$P^\bullet \to \underline{\mathcal{C}} \otimes_{\Bbbk S_i} T \to 0$$

such that each term of which is generated in degree i. Restricting this resolution to the object i one gets a finite resolution of $\Bbbk S_i$-modules

$$1_i P^\bullet \to 1_i(\underline{\mathcal{C}} \otimes_{\Bbbk S_i} T) = T \to 0.$$

The above inequality follows from this observation. $\square$

The following lemma tells us that to consider the projective dimension of a $\sharp$-filtered module, it is enough to consider its basic filtration components.

Lemma 4.7. *Let V be a finitely generated $\sharp$-filtered $\mathcal{C}$-module and suppose that* $\mathrm{pd}(V) < \infty$. *Let*

$$0 = V^{-1} \subseteq V^0 \subseteq V^1 \subseteq \ldots \subseteq V^n = V$$

be the filtration given in Proposition 3.7. Then for $0 \leqslant i \leqslant n$, one has

$$\mathrm{pd}(V^i / V^{i-1}) \leqslant \mathrm{pd}(V).$$

Proof. One uses induction on the length n, which is precisely $\mathrm{gd}(V)$. If $n = 0$ or $-\infty$, nothing needs to show. For $n \geqslant 1$, one has a short exact sequence

$$0 \to U \to V \to W \to 0$$

where U is the submodule generated by $\bigoplus_{i \leqslant n-1} V_i$. It suffices to show that both $\mathrm{pd}(U) \leqslant \mathrm{pd}(V)$ and $\mathrm{pd}(W) \leqslant \mathrm{pd}(V)$ since in that case by induction hypothesis one knows that each basic filtration component of U has projective dimension at most $\mathrm{pd}(U) \leqslant \mathrm{pd}(V)$.

Take two surjections $P^0 \to U$ and $Q^0 \to W$ such that $\mathrm{gd}(P^0) = \mathrm{gd}(U) \leqslant n - 1$ and Q^0 is generated in degree n. We get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U^{(1)} & \longrightarrow & V^{(1)} & \longrightarrow & W^{(1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P^0 & \longrightarrow & P^0 \oplus Q^0 & \longrightarrow & Q^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0.
\end{array}$$

Clearly, $U^{(1)}$, $V^{(1)}$, and $W^{(1)}$ are all $\sharp$-filtered modules (might be 0). Moreover, one still has $\mathrm{gd}(U^{(1)}) \leqslant n-1$ and $\mathrm{gd}(W^{(1)}) = n$ if $W^{(1)}$ is not 0.

Continuing this process, one gets a projective resolution $P^\bullet \oplus Q^\bullet \to V \to 0$. Since $\mathrm{pd}(V) < \infty$, there exists some $i \in \mathbb{Z}_+$ such that $V^{(i)}$ is projective. But from the short exact sequence

$$0 \to U^{(i)} \to V^{(i)} \to W^{(i)} \to 0$$

one sees that both $U^{(i)}$ and $W^{(i)}$ must be projective since $\mathrm{gd}(U^{(i)}) \leqslant n-1$ and $W^{(i)}$ is 0 or generated in degree n. This finishes the proof. □

Definition 4.8. The finitistic dimension of $\Bbbk$, denoted by $\operatorname{findim} \Bbbk$, is defined to be

$$\sup\{\mathrm{pd}_{\Bbbk}(T) \mid T \text{ is a finitely generated } \Bbbk\text{-module and } \mathrm{pd}_{\Bbbk}(T) < \infty\}.$$

Now we restate and prove Theorem 1.5.

Theorem 4.9. *Let $\Bbbk$ be a commutative Noetherian ring whose finitistic dimension is finite, and let V be a finitely generated $\mathcal{C}$-module with $\mathrm{gd}(V) = n$. Then the projective dimension $\mathrm{pd}(V)$ is finite if and only if for $0 \leqslant i \leqslant n$, one has*

$$V^i/V^{i-1} \cong \underline{\mathcal{C}} \otimes_{\Bbbk S_i} (V^i/V^{i-1})_i$$

and

$$\mathrm{pd}_{\Bbbk S_i}((V^i/V^{i-1})_i) < \infty,$$

where V^i is the submodule of V generated by $\bigoplus_{j \leqslant i} V_j$. Moreover, in that case

$$\mathrm{pd}(V) = \max\{\mathrm{pd}_{\Bbbk S_i}((V^i/V^{i-1})_i)\}_{i=0}^n = \max\{\mathrm{pd}_{\Bbbk}((V^i/V^{i-1})_i)\}_{i=0}^n \leqslant \operatorname{findim} \Bbbk.$$

Proof. If $\mathrm{pd}(V) < \infty$, then $H_s(V) = 0$ for $s \gg 0$. By Theorem 3.5, V must be a $\sharp$-filtered module. By Proposition 3.7,

$$V^i/V^{i-1} \cong \underline{\mathcal{C}} \otimes_{\Bbbk S_i} (V^i/V^{i-1})_i.$$

By Lemmas 4.6 and 4.7,

$$\mathrm{pd}_{\Bbbk S_i}((V^i/V^{i-1})_i) = \mathrm{pd}_{\underline{\mathcal{C}}}(V^i/V^{i-1}) \leqslant \mathrm{pd}_{\underline{\mathcal{C}}}(V) < \infty.$$

Conversely, if the structure of V has the given description, then the filtration and Lemma 4.6 tell us that

$$\mathrm{pd}_{\underline{\mathcal{C}}}(V) \leqslant \max\{\mathrm{pd}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}} \otimes_{\Bbbk S_i} (V^i/V^{i-1})_i)\}_{i=0}^n = \max\{\mathrm{pd}_{\Bbbk S_i}((V^i/V^{i-1})_i)\}_{i=0}^n,$$

which is finite.

Now suppose that $\mathrm{pd}_{\underline{\mathcal{C}}}(V)$ is finite. Then Lemma 4.7 tells us

$$\mathrm{pd}_{\underline{\mathcal{C}}}(V) \geqslant \max\{\mathrm{pd}_{\underline{\mathcal{C}}}(V^i/V^{i-1})\}_{i=0}^n = \max\{\mathrm{pd}_{\Bbbk S_i}((V^i/V^{i-1})_i)\}_{i=0}^n.$$

Therefore,

$$\mathrm{pd}_{\underline{\mathcal{C}}}(V) = \max\{\mathrm{pd}_{\Bbbk S_i}((V^i/V^{i-1})_i)\}_{i=0}^n = \max\{\mathrm{pd}_{\Bbbk}((V^i/V^{i-1})_i)\}_{i=0}^n$$

by Lemma 4.5. Since all numbers in the last set must be finite, clearly $\mathrm{pd}_{\underline{\mathcal{C}}}(V) \leqslant \mathrm{findim}\,\Bbbk$ by the definition of finitistic dimensions. □

Remark 4.10. There does exist a finitely generated $\mathcal{C}$-module V whose projective dimension is exactly $\mathrm{findim}\,\Bbbk$. Indeed, let T be a $\Bbbk$-module with $\mathrm{pd}_{\Bbbk}(T) = \mathrm{findim}\,\Bbbk$, then

$$\mathrm{pd}_{\Bbbk S_i}(\Bbbk S_i \otimes_{\Bbbk} T) = \mathrm{pd}_{\Bbbk}(T) = \mathrm{findim}\,\Bbbk$$

and

$$\mathrm{pd}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}} \otimes_{\Bbbk S_i} (\Bbbk S_i \otimes_{\Bbbk} T)) = \mathrm{pd}_{\Bbbk S_i}(\Bbbk S_i \otimes_{\Bbbk} T) = \mathrm{findim}\,\Bbbk.$$

The following corollaries are immediate.

Corollary 4.11. *If* $\mathrm{gldim}\,\Bbbk < \infty$*, then the projective dimension of a finitely generated* $\mathcal{C}$*-module* V *is either* ∞ *or at most* $\mathrm{gldim}\,k$*.*

For instance, if $\Bbbk$ is $\mathbb{Z}$ or the polynomial ring of one variable over a field, then the projective dimension of a finitely generated $\mathcal{C}$-module V can only be 0, 1 or ∞.

Corollary 4.12. *If* $\mathrm{findim}\,\Bbbk = 0$*, then a finitely generated* $\mathcal{C}$*-module* V *has finite projective dimension if and only if* V *is projective.*

Remark 4.13. Actually, many important classes of rings have finitistic dimension 0. Examples includes semisimple rings, self-injective algebras, finite dimensional local algebras, etc.

4.3. Complexes of ♯-filtered modules

In this subsection we construct a finite complex of ♯-filtered modules for every finitely generated $\mathcal{C}$-module.

Theorem 4.14 *([12], Theorem A). Let $\Bbbk$ be a commutative Noetherian ring and let V be a finitely generated $\mathcal{C}$-module. Then there exists a complex*

$$F^\bullet: \quad 0 \to V \to F^0 \to F^1 \to \ldots \to F^n \to 0$$

satisfying the following conditions:

(1) each F^i is a ♯-filtered module with $\mathrm{gd}(F^i) \leqslant \mathrm{gd}(V) - i$;
(2) $n \leqslant \mathrm{gd}(V)$;
(3) the homology in each degree of the complex is a torsion module, including the homology at V.

Proof. One may assume that V is nonzero. Denote V by V^0 and let N_0 be a sufficiently large integer. There is a short exact sequence

$$0 \to V_T^0 \to V^0 \to V_F^0 \to 0$$

such that V_T^0 is torsion and V_F^0 is torsionless, which gives a short exact sequence

$$0 \to \Sigma_{N_0} V_T^0 \to \Sigma_{N_0} V^0 \to \Sigma_{N_0} V_F^0 \to 0.$$

Since N_0 is sufficiently large, we conclude that the first term in the above sequence is 0. Moreover, $\Sigma_{N_0} V_F^0 \cong \Sigma_{N_0} V$ is ♯-filtered by Theorem 3.13. Let $F^0 = \Sigma_{N_0} V_F^0$. The natural embeddings

$$V_F^0 \to \Sigma V_F^0 \to \Sigma_2 V_F^0 \to \ldots \to \Sigma_{N_0} V_F^0 = F^0$$

induces a map $\delta_{-1} : V^0 \to F^0$ which is the composite $V^0 \to V_F^0 \to F^0$. Let $V^1 = \operatorname{coker} \delta_{-1}$. Repeating the above construction we get a map $V^1 \to F^1$, which induces the map $\delta_0 : F^0 \to F^1$. In this way we construct a complex

$$F^\bullet: \quad V \to F^0 \to F^1 \to \ldots \to F^n \to \ldots$$

Note that for $i \geqslant 1$, we have

$$\mathrm{gd}(F^i) \leqslant \mathrm{gd}(V_F^i) \leqslant \mathrm{gd}(V^i) \leqslant \mathrm{gd}(F^{i-1}) - 1 \tag{4.1}$$

where the first inequality holds because $F^i = \Sigma_{N_i} V_F^i$, and the last inequality follows from the exact sequence

$$0 \to V_F^{i-1} \to F^{i-1} = \Sigma_{N_i} V_F^{i-1} \to V^i \to 0$$

and an argument similar to the proof of (3) in Proposition 2.4. Therefore, using induction one concludes that

$$\mathrm{gd}(F^i) \leqslant \mathrm{gd}(F^0) - i \leqslant \mathrm{gd}(V) - i$$

since $\mathrm{gd}(F^0) \leqslant \mathrm{gd}(V)$. This proves (1), which immediately implies (2). Moreover, one has

$$\mathrm{gd}(V^i) \leqslant \mathrm{gd}(V) - i \tag{4.2}$$

for $i \geqslant 1$.

Now we prove (3). Note that the image of the map $\delta_i : F^i \to F^{i+1}$ is precisely V_F^{i+1}. The commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & \to & \operatorname{im}\delta_{i-1} = V_F^i & \to & \ker\delta_i & \to & V_T^{i+1} \to 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F^i & = & F^i & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & V_T^{i+1} & \to & \operatorname{coker}\delta_{i-1} = V^{i+1} & \to & \operatorname{im}\delta_i = V_F^{i+1} & \to & 0
\end{array}$$

tells us that the homology at F^i is isomorphic to V_T^{i+1} for $i \geqslant 0$, a torsion module. A similar computation tells us that the homology at V is V_T. This finishes the proof. □

Remark 4.15. When $\Bbbk$ is a field of characteristic 0, $\sharp$-filtered modules coincide with projective modules, which have been shown to be injective as well. Therefore, the above construction generalizes the construction of finite injective resolutions described in [8].

Remark 4.16. We used the conclusion of Theorem 3.13 to prove the above theorem. But they are actually equivalent. Indeed, choosing a sufficiently large N and applying Σ_N to the complex $F^\bullet$, we get a complex $\Sigma_N F^\bullet$. Since all homologies in $F^\bullet$ are torsion modules, after applying Σ_N, these torsion modules all vanish. Consequently, $\Sigma_N F^\bullet$ is a right resolution of $\Sigma_N V$. By Lemma 3.9, each $\Sigma_N F^i$ is still $\sharp$-filtered. By Corollary 3.6, $\Sigma_N V$ is a $\sharp$-filtered module, as claimed by Theorem 3.13.

4.4. Another bound for homological degrees

In this subsection we use the complex of $\sharp$-filtered modules to obtain another bound for homological degrees. The proof of this result is almost the same as that of [11, Theorem 5.18] via replacing projective modules by $\sharp$-filtered modules. For the convenience of the reader, we still give enough details.

Lemma 4.17. *Let V be a finitely generated torsionless $\mathcal{C}$-module. Then for $s \geqslant 1$,*

$$\mathrm{hd}_s(V) \leqslant 2\,\mathrm{gd}(V) + s - 1.$$

Proof. By the proof of Theorem 4.14, there is a short exact sequence

$$0 \to V \to F \to W \to 0$$

where $F = \Sigma_N V$ is a $\sharp$-filtered module and $\mathrm{gd}(W) \leqslant \mathrm{gd}(V) - 1$. Using the long exact sequence

$$\ldots \to H_2(W) \to H_1(V) \to H_1(F) = 0 \to H_1(W) \to H_0(V) \to H_0(F) \to H_0(W) \to 0$$

one deduces that $\mathrm{hd}_s(V) = \mathrm{hd}_{s+1}(W)$ for $s \geqslant 1$ and $\mathrm{hd}_1(W) \leqslant \mathrm{gd}(V)$.

By [1, Theorem A], we have

$$\mathrm{hd}_s(V) = \mathrm{hd}_{s+1}(W) \leqslant \mathrm{gd}(W) + \mathrm{hd}_1(W) + s,$$

for $s \geqslant 1$. Consequently,

$$\mathrm{hd}_s(V) \leqslant \mathrm{gd}(V) - 1 + \mathrm{gd}(V) + s = 2\,\mathrm{gd}(V) + s - 1$$

as claimed. □

Now we can prove Theorem 1.9.

Theorem 4.18. *Let $\Bbbk$ be a commutative Noetherian ring and V be a finitely generated $\mathcal{C}$-module. Then for $s \geqslant 1$, we have*

$$\mathrm{hd}_s(V) \leqslant \max\{\mathrm{td}(V),\, 2\,\mathrm{gd}(V) - 1\} + s.$$

Proof. The short exact sequence

$$0 \to V_T \to V \to V_F \to 0$$

induces a long exact sequence

$$\ldots \to H_s(V_T) \to H_s(V) \to H_s(V_F) \to \ldots.$$

We deduce that

$$\mathrm{hd}_s(V) \leqslant \max\{\mathrm{hd}_s(V_T),\, \mathrm{hd}_s(V_F)\}.$$

Note that

$$\mathrm{hd}_s(V_T) \leqslant \mathrm{td}(V_T) + s = \mathrm{td}(V) + s,$$

by Corollary 2.3 and

$$\mathrm{hd}_s(V_F) \leqslant 2\,\mathrm{gd}(V_F) + s - 1 \leqslant 2\,\mathrm{gd}(V) + s - 1$$

by the previous lemma. The conclusion follows. □

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