ON CENTRAL STABILITY

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Abstract. The notion of central stability was first formulated for sequences of representations of the symmetric groups by Putman. A categorical reformulation was subsequently given by Church, Ellenberg, Farb, and Nagpal using the notion of FI-modules, where FI is the category of finite sets and injective maps. We extend the notion of central stability from FI to a wide class of categories, and prove that a module is presented in finite degrees if and only if it is centrally stable. We also introduce the notion of d-step central stability, and prove that if the ideal of relations of a category is generated in degrees at most d, then every module presented in finite degrees is d-step centrally stable.

1. Introduction

Let R be a commutative ring. For any small category $\mathcal{C}$, a $\mathcal{C}$-*module* over R is, by definition, a (covariant) functor from $\mathcal{C}$ to the category of R-modules. A *morphism* of $\mathcal{C}$-modules over R is, by definition, a natural transformation of functors. The category of $\mathcal{C}$-modules over R is an abelian category. One can define, in a natural way, the notion of a *finitely generated* $\mathcal{C}$-module over R.

Let $\mathcal{FI}$ be the category of finite sets and injective maps, and denote by $\mathbb{Z}_+$ the set of non-negative integers. For any $\mathcal{FI}$-module V over R and $n \in \mathbb{Z}_+$, we write V_n for $V([n])$, where $[n] := \{1, \ldots, n\}$. Since the automorphism group of $[n]$ in the category $\mathcal{FI}$ is the symmetric group S_n, an $\mathcal{FI}$-module V over R gives rise to a sequence $\{V_n\}$ where V_n is a representation of S_n. It was shown by Church, Ellenberg, Farb, and Nagpal (see [3] and [4]) that many interesting sequences of representations of symmetric groups arise in this way from finitely generated $\mathcal{FI}$-modules. A principle result they proved is that any finitely generated $\mathcal{FI}$-module over a commutative noetherian ring is noetherian. They deduced as a consequence that if V is a finitely generated $\mathcal{FI}$-module over a commutative noetherian ring, then the sequence $\{V_n\}$ admits an inductive description, in the sense that for all sufficiently large integer N, one has

$$(1.1) \qquad V_n \cong \operatorname*{colim}_{\substack{S \subset [n] \\ |S| \leqslant N}} V(S) \quad \text{for each } n \in \mathbb{Z}_+,$$

where the colimit is taken over the poset of all subsets S of $[n]$ such that $|S| \leqslant N$.

To provide the context for our present article, let us briefly describe the way in which (1.1) is proved in [4]. Suppose that V is a finitely generated $\mathcal{FI}$-module over a noetherian ring. For any finite set T, there is a canonical homomorphism of R-modules

$$(1.2) \qquad \operatorname*{colim}_{\substack{S \subset T \\ |S| \leqslant N}} V(S) \longrightarrow V(T).$$

Since V is finitely generated, there exists an integer N' such that V is generated by $\sqcup_{i \leqslant N'} V_i$. It is easy to see that the homomorphism (1.2) is surjective if $N \geqslant N'$. To prove the injectivity of (1.2) when N is sufficiently large, Church, Ellenberg, Farb, and Nagpal constructed a (Koszul) complex $\widetilde{S}_{-*} V$ of $\mathcal{FI}$-modules which has the property that:

$$(H_1(\widetilde{S}_{-*} V))(T) = \operatorname{Ker} \operatorname*{colim}_{\substack{S \subset T \\ |T|-2 \leqslant |S| \leqslant |T|-1}} V(S) \to V(T) .$$

They verified that one has:

$$(1.3) \qquad \operatorname*{colim}_{\substack{S \subset T \\ |T|-2 \leqslant |S| \leqslant |T|-1}} V(S) = \operatorname*{colim}_{\substack{S \subset T \\ |S| < |T|}} V(S).$$

From the noetherian property of V, they proved that there exists an integer N'' such that $(H_1(\check{\mathbb{S}}_{-*}V))(T) = 0$ if $|T| \geqslant N''$. As a consequence,

$$(1.4) \qquad\qquad \operatorname*{colim}_{\substack{S \subset T \\ |S| < |T|}} V(S) = V(T) \quad \text{if } |T| > \max\{N'; N''\};$$

Set $N = \max\{N'; N''\}$. The homomorphism in (1.2) is an isomorphism if $|T| \leqslant N$, for T is terminal in the poset $\{S \mid S \subset T\}$. It now follows by an induction on $|T|$ that (1.2) is an isomorphism too if $|T| > N$, for:

$$\operatorname*{colim}_{\substack{S \subset T \\ |S| \leqslant N}} V(S) = \operatorname*{colim}_{\substack{U \subset T \\ |U| < |T|}} \operatorname*{colim}_{\substack{S \subset U \\ |S| \leqslant N}} V(S) = \operatorname*{colim}_{\substack{U \subset T \\ |U| < |T|}} V(U) = V(T);$$

where the first isomorphism is routine, the second isomorphism is by the induction hypothesis, and the third isomorphism is by (1.4).

It was subsequently proved by Church and Ellenberg that for any $\mathcal{FI}$-module V over an arbitrary commutative ring, if V is presented in finite degrees (in a suitable sense), then there are integers N' and N'' such that V is generated by $\sqcup_{i \leqslant N'} V_i$, and $(H_1(\check{\mathbb{S}}_{-*}V))(T) = 0$ if $|T| \geqslant N''$. Consequently, using the same arguments as above, they showed that the isomorphism (1.1) holds for all N sufficiently large. This extends the result of their joint work with Farb and Nagpal to $\mathcal{FI}$-modules which are not necessarily noetherian (since every finitely generated $\mathcal{FI}$-module over a commutative noetherian ring is presented in finite degrees).

The left-hand side of (1.3) is isomorphic to the central stabilization construction of Putman [11, §1]. In their joint work, Putman and Sam generalized the isomorphism in (1.1) to modules over complemented categories with a generator of which $\mathcal{FI}$ is an example. A complemented category with a generator X is the data of a symmetric monoidal category and an object X satisfying a list of axioms. Their proof is similar to the one for $\mathcal{FI}$ described above.

A goal of our present paper is to give a very simple and transparent proof of a generalization of the above results. In particular, our proof does not require consideration of the complex $\check{\mathbb{S}}_{-*}V$ or similar complexes. Moreover, the setting for our generalization is much simpler than the one of complemented categories with a generator used by Putman and Sam [12], and include many more examples; our generalization is, in fact, motivated by the fact that several combinatorial categories studied by Sam and Snowden in [14] do not fall within the framework of [12]. Another goal of our paper is to explain the role of the quadratic property of $\mathcal{FI}$ in the isomorphism (1.3).

Outline of the paper. This paper is organized as follows.

In Section 2, we define our generalization of the notion of central stability and introduce the notion of d-step central stability. We show that in the special case of complemented categories with a generator studied by Putman and Sam, our notion of central stability is indeed equivalent to their notion of central stability. We also show that for the category $\mathcal{FI}$, it is equivalent to the inductive description (1.1).

In Section 3, we give a reminder of a key lemma from Morita theory. We then prove our first main result, that a module is presented in finite degrees if and only if it is centrally stable. We deduce that if every finitely generated module is neotherian, then every finitely generated module is centrally stable. We prove that the converse of the preceding statement holds under a local finiteness assumption on the category when the base ring is a commutative noetherian ring. To illustrate the wide applicability of our results, we recall examples of combinatorial categories introduced by Sam and Snowden in [14] which fall within our framework but are not complemented categories with a generator.

In Section 4, we prove our second main result, that if the ideal of relations of a category is generated in degrees at most d, then every module presented in finite degrees is d-step centrally stable; for example, if the category is quadratic, then every module presented in finite degrees is 2-step centrally stable.

To apply our second main result, one needs to have a practical way to check that the ideal of relations of a category is generated in degrees at most d. In Section 5, we give sufficiency conditions which allow one to do this.

2. Generalities

2.1. **Notations and definitions.** We prefer to formulate our main results in the more familiar language of modules over algebras. Throughout this paper, we denote by R a commutative ring and $\mathbb{Z}_+$ the set of non-negative integers.

Let $\mathcal{A}$ be an R-linear category, i.e. a category enriched over the category of R-modules. We assume that $\mathrm{Ob}(\mathcal{A}) = \mathbb{Z}_+$ and $\mathrm{Hom}_{\mathcal{A}}(m, n) = 0$ if $m > n$. We set

(2.1) $$A := \bigoplus_{m,n \in \mathrm{Ob}(\mathcal{A})} \mathrm{Hom}_{\mathcal{A}}(m, n).$$

There is a natural structure of a (non-unital) R-algebra on A. (In the terminology of [1], A is a $\mathbb{Z}$-*algebra*.) We call A the *category algebra* of $\mathcal{A}$. For each $n \in \mathrm{Ob}(\mathcal{A})$, we denote by e_n the identity endomorphism of n. For any $m, n \in \mathrm{Ob}(\mathcal{A})$ with $m \leqslant n$, we set

$$e_{m,n} := e_m + e_{m+1} + \cdots + e_n \in A.$$

One has: $Ae_{m,n} = Ae_m \oplus Ae_{m+1} \oplus \cdots \oplus Ae_n$.

A *graded A-module* is an A-module V such that $V = \bigsqcup_{n \in \mathrm{Ob}(\mathcal{A})} e_n V$. Observe that if V is a graded A-module, then any A-submodule of V is also a graded A-module.

A graded A-module V is *finitely generated* if it is finitely generated as an A-module. Equivalently, V is finitely generated if for some $N \in \mathrm{Ob}(\mathcal{A})$, there is an exact sequence $\bigsqcup_{i \in I} Ae_{0,N} \to V \to 0$ where the indexing set I is finite. A graded A-module V is *noetherian* if every A-submodule of V is finitely generated.

A graded A-module V is *finitely presented* if for some $N \in \mathrm{Ob}(\mathcal{A})$, there is an exact sequence of the form

$$\bigsqcup_{j \in J} Ae_{0,N} \longrightarrow \bigsqcup_{i \in I} Ae_{0,N} \longrightarrow V \longrightarrow 0$$

where both the indexing sets I and J are finite.

A graded A-module V is *presented in finite degrees* if for some $N \in \mathrm{Ob}(\mathcal{A})$, there is an exact sequence

(2.2) $$\bigsqcup_{j \in J} Ae_{0,N} \longrightarrow \bigsqcup_{i \in I} Ae_{0,N} \longrightarrow V \longrightarrow 0$$

where the indexing sets I and J may be finite or infinite. The smallest N for which such an exact sequence exists is called the *presentation degree* of V, and we denote it by $\mathrm{prd}(V)$.

Remark 2.1. It is easy to see that if V is a graded A-module presented in finite degrees, then for every $N \geqslant \mathrm{prd}(V)$, there exists an exact sequence of the form (2.2), where I and J may depend on N.

The main definitions of this paper are as follows.

Definition 2.2. A graded A-module V is called *centrally stable* if for all N sufficiently large, one has

(2.3) $$Ae \otimes_{eAe} eV \cong V \quad \text{where } e = e_{0,N}.$$

Definition 2.3. Let d be an integer $\geqslant 1$. A graded A-module V is called *d-step centrally stable* if for all N sufficiently large, one has

$$Ae \otimes_{eAe} eV \cong \bigsqcup_{n \geqslant N-(d-1)} e_n V \quad \text{where } e = e_{N-(d-1),N}.$$

2.2. Inductive descriptions.

We now explain the relation of our definition of central stability given above to the notion of central stability given by Putman and Sam in [12] for complemented categories with a generator, and the relation to the inductive description (1.1) in the special case of $\mathcal{FI}$-modules as given by Church, Farb, Ellenberg, and Nagpal [2, 4].

Let $\mathcal{C}$ be a small category such that $\mathrm{Ob}(\mathcal{C}) = \mathbb{Z}_+$, and $\mathrm{Hom}_{\mathcal{C}}(m, n) = \emptyset$ if $m > n$. We denote by $\mathcal{A}_{\mathcal{C}}$ the R-linear category with $\mathrm{Ob}(\mathcal{A}_{\mathcal{C}}) = \mathbb{Z}_+$ and $\mathrm{Hom}_{\mathcal{A}_{\mathcal{C}}}(m, n)$ the free R-module with basis $\mathrm{Hom}_{\mathcal{C}}(m, n)$ for each $m, n \in \mathbb{Z}_+$. Let $A_{\mathcal{C}}$ be the category algebra of $\mathcal{A}_{\mathcal{C}}$; see (2.1).

Recall that a $\mathcal{C}$-module over R is, by definition, a (covariant) functor from $\mathcal{C}$ to the category of R-modules. For any $\mathcal{C}$-module V over R and $n \in \mathbb{Z}_+$, we write V_n for $V(n)$. If V is a $\mathcal{C}$-module over R, then $\bigsqcup_{n \in \mathrm{Ob}(\mathcal{C})} V_n$ is a graded $A_{\mathcal{C}}$-module. This defines an equivalence from the category of $\mathcal{C}$-modules over R to the category of graded $A_{\mathcal{C}}$-modules. We say that a $\mathcal{C}$-module V over R has a certain property (such as centrally stable) if the graded $A_{\mathcal{C}}$-module $\bigsqcup_{n \in \mathrm{Ob}(\mathcal{C})} V_n$ has the property.

For any $M, N \in \mathrm{Ob}(\mathcal{C})$ with $M \leqslant N$, we write $\mathcal{C}_{M,N}$ for the full subcategory of $\mathcal{C}$ on the set of objects $\{M, M+1, \ldots, N\}$. Then the category of $\mathcal{C}_{M,N}$-modules over R is equivalent to the category of $eA_{\mathcal{C}}e$-modules

where $e = e_{M,N}$. Let $\iota_{M,N} : \mathcal{C}_{M,N} \to \mathcal{C}$ be the inclusion functor. We define the *restriction* functor $\mathrm{Res}_{M,N}$ along $\iota_{M,N}$ by

$$\mathrm{Res}_{M,N} : (\text{category of } \mathcal{C}\text{-modules over } R) \longrightarrow (\text{category of } \mathcal{C}_{M,N}\text{-modules over } R);$$
$$V \longmapsto V \circ \iota_{M,N}.$$

The *left Kan extension functor* $\mathrm{Lan}_{M,N}$ along $\iota_{M,N}$ is a left adjoint functor to $\mathrm{Res}_{M,N}$; for every $\mathcal{C}_{M,N}$-module V over R and $n \in \mathrm{Ob}(\mathcal{C})$, one has:

$$(2.4) \qquad (\mathrm{Lan}_{M,N} V)_n = \operatorname*{colim}_{\substack{\alpha : s \to n \\ M \leqslant s \leqslant N}} V_s;$$

where $V_s = V(s)$ and the colimit is taken over the (comma) category whose objects are the morphisms $\alpha : s \to n$ in $\mathcal{C}$ such that $M \leqslant s \leqslant N$; see [7, Theorem 2.3.3] or [10, Section X.3, (10)].

The following proposition gives a reformulation for the notions of central stability and d-step central stability for $\mathcal{C}$-modules over R.

Proposition 2.4. *Let V be a $\mathcal{C}$-module over R. For every $M, N \in \mathrm{Ob}(\mathcal{C})$ with $M \leqslant N$, one has:*

$$(2.5) \qquad \bigoplus_{n \in \mathrm{Ob}(\mathcal{C})} (\mathrm{Lan}_{M,N}(\mathrm{Res}_{M,N} V))_n \cong Ae \otimes_{eAe} e \bigoplus_{n \in \mathrm{Ob}(\mathcal{C})} V_n \qquad \text{where } A = A_{\mathcal{C}} \text{ and } e = e_{M,N}.$$

In particular, V is centrally stable if and only if for all sufficiently large N, one has

$$V_n \cong \operatorname*{colim}_{\substack{\alpha : s \to n \\ s \leqslant N}} V_s \qquad \text{for each } n \in \mathbb{Z}_+.$$

Moreover, V is d-step centrally stable if and only if for all sufficiently large N, one has

$$V_n \cong \operatorname*{colim}_{\substack{\alpha : s \to n \\ N-(d-1) \leqslant s \leqslant N}} V_s \qquad \text{for each } n \geqslant N - (d-1).$$

Proof. Let $e = e_{M,N}$. For any $\mathcal{C}$-module V over R, one has

$$\left(\mathrm{Res}_{M,N} V\right)_{\bigoplus_{M \leqslant n \leqslant N} n} = e \bigoplus_{n \in \mathrm{Ob}(\mathcal{C})} V_n.$$

Since the functor:

$$(\text{category of } eAe\text{-modules}) \longrightarrow (\text{category of graded } A\text{-modules});$$
$$W \longmapsto Ae \otimes_{eAe} W;$$

is left adjoint to the functor:

$$(\text{category of graded } A\text{-modules}) \longrightarrow (\text{category of } eAe\text{-modules});$$
$$V \longmapsto eV;$$

it follows that for any $\mathcal{C}_{M,N}$-module W over R, one has

$$\left(\mathrm{Lan}_{M,N} W\right)_{\bigoplus_{M \leqslant n \leqslant N} n} \cong Ae \otimes_{eAe} \bigoplus_{M \leqslant n \leqslant N} W_n.$$

We have proven the isomorphism (2.5). The remaining statements now follow from the formula (2.4). $\qquad\square$

A complemented category with a generator, in the sense of Putman and Sam [12], is the data of a symmetric monoidal category $\mathtt{A}$ and an object X of $\mathtt{A}$ satisfying a list of axioms. We do not recall those axioms here since we will not need them; however, a consequence of the axioms is that the full subcategory $\mathcal{C}$ of $\mathtt{A}$ on the set of objects X^n for $n \in \mathbb{Z}_+$ is a skeleton of $\mathtt{A}$, and $\mathrm{Hom}_{\mathcal{C}}(X^m, X^n) = \emptyset$ if $m > n$. We may identify the set of objects of $\mathcal{C}$ with $\mathbb{Z}_+$ in the obvious way. Following Putman and Sam [12, Theorem E], an $\mathtt{A}$-module V over R is *centrally stable* if for all sufficiently large N, the functor V is the left Kan extension of the restriction of V to the full subcategory of $\mathtt{A}$ spanned by the objects isomorphic to X^n for some $n \leqslant N$.

Corollary 2.5. *Let $\mathtt{A}$ be a complemented category with a generator X. Suppose that $\mathcal{C}$ is the skeleton of $\mathtt{A}$ spanned by the objects X^n for all $n \in \mathbb{Z}_+$. Let V be an $\mathtt{A}$-module over R, and regard V also as a $\mathcal{C}$-module by restriction along the inclusion functor $\mathcal{C} \to \mathtt{A}$. Then V is centrally stable in the sense of Putman and Sam [12, Theorem E] if and only if V is centrally stable as a $\mathcal{C}$-module over R.*

Proof. This is immediate from (2.5). $\qquad\square$

The full subcategory of $\mathcal{FI}$ spanned by the objects $[n]$ for all $n \in \mathbb{Z}_+$ is a skeleton of $\mathcal{FI}$. In the following corollary, we identify the set of objects of this skeleton with $\mathbb{Z}_+$ in the obvious way.

Corollary 2.6. *Suppose that $\mathcal{C}$ is the skeleton of $\mathcal{FI}$ spanned by the objects $[n]$ for all $n \in \mathbb{Z}_+$. Let V be an $\mathcal{FI}$-module over R, and regard V also as a $\mathcal{C}$-module by restriction along the inclusion functor $\mathcal{C} \to \mathcal{FI}$. Then V is centrally stable as a $\mathcal{C}$-module over R if and only if for all sufficiently large N, the isomorphism (1.1) holds.*

Proof. By Proposition 2.4, it suffices to show that

$$\operatorname*{colim}_{\substack{\alpha:s\to n \\ s\leqslant N}} V_s \cong \operatorname*{colim}_{\substack{S\subset[n] \\ |S|\leqslant N}} V(S).$$

But this is immediate from the observation that the natural functor:

$$\{ \ | \ \ \in \operatorname{Hom}_{\mathcal{FI}}([s],[n]) \text{ and } s \leqslant N\} \longrightarrow \{S \mid S \subset [n] \text{ and } |S| \leqslant N\};$$
$$\longmapsto \operatorname{Im}(\ );$$

is final; see [10, Theorem IX.3.1]. (Some authors refer to final functors as cofinal functors. See, for example, [7, Definition 2.5.1].) $\qquad\square$

Let us also mention that for a principal ideal domain R, Dwyer defined the notion of a *central coefficient system* in [6] to mean a sequence $_n$ of $\operatorname{GL}_n(R)$-modules and maps $F_n : {}_n \to {}_{n+1}$ such that F_n is a $\operatorname{GL}_n(R)$-map (when $_{n+1}$ is considered as a $\operatorname{GL}_n(R)$-module by restriction) and the image of $F_{n+1}F_n$ is invariant under the action of the permutation matrix $s_{n+2} \in \operatorname{GL}_{n+2}(R)$ which interchanges the last two standard basis vectors of R^{n+2}. Suppose that $\mathcal{C}$ is the skeleton of $\mathcal{FI}$ spanned by the objects $[n]$ for all $n \in \mathbb{Z}_+$. A central coefficient system defines a $\mathcal{C}$-module V over R with $V_n = {}_n$ and such that the standard inclusion $[n] \rightarrowtail [n+1]$ induces the map F_n. If V is finitely generated as a $\mathcal{C}$-module over R, then by [4, Theorem C] it is a centrally stable $\mathcal{C}$-module over R.

3. Central stability

3.1. **Key lemma.** The following lemma is a standard result in Morita theory; see, for instance, [5, Theorem 6.4.1]. We recall its proof here since it plays a key role in the proof of our main results.

Lemma 3.1. *Let A be any (possibly non-unital) ring, and e an idempotent element of A. If V is an A-module such that, for some indexing sets I and J, there is an exact sequence*

(3.1)
$$\bigvee_{j\in J} Ae \longrightarrow \bigvee_{i\in I} Ae \longrightarrow V \longrightarrow 0;$$

then $Ae \otimes_{eAe} eV \cong V$.

Proof. Applying the right-exact functor $Ae \otimes_{eAe} e(-)$ to (3.1), we obtain the first row in the following commuting diagram:

(3.2)
$$\bigsqcup_{j\in J} Ae \otimes_{eAe} eAe \longrightarrow \bigsqcup_{i\in I} Ae \otimes_{eAe} eAe \longrightarrow Ae \otimes_{eAe} eV \longrightarrow 0$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$\bigsqcup_{j\in J} Ae \longrightarrow \bigsqcup_{i\in I} Ae \longrightarrow V \longrightarrow 0$$

Both the rows in (3.2) are exact. Since the two leftmost vertical maps are isomorphisms, the rightmost vertical map is also an isomorphism. $\qquad\square$

3.2. **Central stability and presentation in finite degrees.** Let $\mathcal{A}$ be an R-linear category such that $\operatorname{Ob}(\mathcal{A}) = \mathbb{Z}_+$ and $\operatorname{Hom}_{\mathcal{A}}(m,n) = 0$ if $m > n$. Denote by A the category algebra of $\mathcal{A}$; see (2.1).

Theorem 3.2. *Let V be a graded A-module. Then V is presented in finite degrees if and only if it is centrally stable. Moreover, if V is presented in finite degrees, then the isomorphism (2.3) holds if and only if $N \geqslant \operatorname{prd}(V)$.*

Proof. Suppose that V is presented in finite degrees, and $N \geqslant \operatorname{prd}(V)$. Then there is an exact sequence (2.2) with $e = e_{0,N}$; see Remark 2.1. Hence, by Lemma 3.1, one has $Ae \otimes_{eAe} eV \cong V$.

Conversely, suppose that N is an integer such that $Ae \otimes_{eAe} eV \cong V$, where $e = e_{0,N}$. There is an exact sequence $\bigsqcup_{j \in J} eAe \to \bigsqcup_{i \in I} eAe \to eV \to 0$ for some indexing sets I and J. Applying the functor $Ae \otimes_{eAe} (-)$, we obtain an exact sequence of the form (2.2) with $e = e_{0,N}$. Therefore, V is presented in finite degrees and $N \geqslant \operatorname{prd}(V)$. $\qquad\square$

Corollary 3.3. *If every finitely generated graded A-module is noetherian, then every finitely generated graded A-module is centrally stable.*

Proof. If V is a finitely generated graded A-module, then V is finitely presented, hence centrally stable by Theorem 3.2. $\qquad\square$

The above proofs are very simple in comparison to the proofs for the special cases given in [2, Theorem B], [4, Theorem C], and [12, Theorem E]. Let us emphasize, however, that the crux is to recognize that the notion of central stability in [2, 4, 12] can be reformulated in Morita theory; *this was not a priori obvious.*

3.3. **Central stability and noetherian property.** We prove a partial converse to Corollary 3.3.

Let $\mathcal{A}$ be an R-linear category such that $\operatorname{Ob}(\mathcal{A}) = \mathbb{Z}_+$ and $\operatorname{Hom}_{\mathcal{A}}(m, n) = 0$ if $m > n$. Denote by A the category algebra of $\mathcal{A}$; see (2.1). We say that A is *locally finite* if $\operatorname{Hom}_{\mathcal{A}}(m, n)$ is a finitely generated R-module for all $m, n \in \mathbb{Z}_+$.

Corollary 3.4. *Suppose that R is a commutative noetherian ring and A is locally finite. If every finitely generated graded A-module is centrally stable, then every finitely generated graded A-module is noetherian.*

Proof. Let V be a finitely generated graded A-module and U a (graded) A-submodule of V. We want to prove that U is finitely generated as a graded A-module. Since V/U is a finitely generated graded A-module, it is centrally stable, hence presented in finite degrees by Theorem 3.2. Thus, for some $N \in \operatorname{Ob}(\mathcal{A})$, there is a commuting diagram

$$
\begin{array}{ccccccccc}
\bigsqcup_{j \in J} Ae_{0,N} & \longrightarrow & \bigsqcup_{i \in I} Ae_{0,N} & \longrightarrow & V/U & \longrightarrow & 0 \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \| & & \\
0 \longrightarrow U & \longrightarrow & V & \longrightarrow & V/U & \longrightarrow & 0.
\end{array}
$$

It suffices to prove that both $\operatorname{Im}(g)$ and $\operatorname{Coker}(g)$ are finitely generated graded A-modules.

By the Snake Lemma, there is an isomorphism $\operatorname{Coker}(g) \cong \operatorname{Coker}(f)$, so $\operatorname{Coker}(g)$ is a finitely generated graded A-module.

Since A is locally finite and V is finitely generated as a graded A-module, it follows that $e_{0,N} V$ is a finitely generated R-module, hence $e_{0,N} V$ is a noetherian R-module. But $\operatorname{Im}(g)$ is generated as a graded A-module by $e_{0,N} \operatorname{Im}(g)$, and $e_{0,N} \operatorname{Im}(g)$ is an R-submodule of the noetherian R-module $e_{0,N} V$, hence $\operatorname{Im}(g)$ is finitely generated as a graded A-module. $\qquad\square$

3.4. **Examples.** Let us mention some examples of combinatorial categories introduced by Sam and Snowden [14] which fall within our framework. The central stability of modules presented in finite degrees for these categories follow from Theorem 3.2 (and were not previously known).

In the following examples, the set of objects of the category $\mathcal{C}$ is $\mathbb{Z}_+$. For any $n \in \mathbb{Z}_+$, we set $[n] := \{1, \ldots, n\}$.

(i) $\mathcal{C} = \mathcal{FI}_a$ where a is an integer $\geqslant 1$. For any $m, n \in \mathbb{Z}_+$, a morphism $m \to n$ in $\mathcal{FI}_a$ is a pair (f, c) where $f : [m] \to [n]$ is an injective map and $c : [n] \setminus \operatorname{Im}(f) \to [a]$ is any map. The composition of $(f_1, c_1) : m \to n$ and $(f_2, c_2) : n \to \ell$ is defined to be $(f_2 \circ f_1, c) : m \to \ell$ where $c(i) = c_1(j)$ if $i = f_2(j)$ for some $j \in [n] \setminus \operatorname{Im}(f_1)$, and $c(i) = c_2(i)$ if $i \in [\ell] \setminus \operatorname{Im}(f_2)$.

(ii) $\mathcal{C} = \mathcal{OI}_a$ where a is an integer $\geqslant 1$. For any $m, n \in \mathbb{Z}_+$, a morphism $m \to n$ in $\mathcal{OI}_a$ is a pair (f, c) where $f : [m] \to [n]$ is a strictly increasing map and $c : [n] \setminus \operatorname{Im}(f) \to [a]$ is any map. The composition of morphisms is defined in the same way as above for $\mathcal{FI}_a$.

(iii) $\mathcal{C} = \mathcal{FS}^{\mathrm{op}}$, the opposite of the category $\mathcal{FS}$. For any $m, n \in \mathbb{Z}_+$, a morphism $n \to m$ in $\mathcal{FS}$ is a surjective map $[n] \to [m]$. The composition of morphisms in $\mathcal{FS}$ is defined to be the composition of maps.

It was proved by Sam and Snowden [14] that for the categories $\mathcal{C}$ in (i)-(iii) above, every finitely generated graded $\mathcal{C}$-module over a commutative noetherian ring is noetherian (and hence they are centrally stable by Corollary 3.3).

4. d-step central stability

4.1. Ideal of relations. Suppose V is a finitely generated $\mathcal{FI}$-module over a commutative noetherian ring. The isomorphism (1.1) says that, provided N is sufficiently large, V can be recovered from its restriction to the full subcategory of $\mathcal{FI}$ on the set of objects $\{S \in \mathrm{Ob}(\mathcal{FI}) \mid |S| \leqslant N\}$; using the isomorphism (1.3), we see that in fact we can recover $V(S)$ for $|S| > N$ using only the restriction of V to the full subcategory of $\mathcal{FI}$ on the set of objects $\{S \in \mathrm{Ob}(\mathcal{FI}) \mid N - 1 \leqslant |S| \leqslant N\}$. The purpose of the present section and the next is to show that this is a consequence of the quadratic property of $\mathcal{FI}$. More generally, we shall prove that if the ideal of relations of a category algebra A is generated in degrees $\leqslant d$ (in the sense of Definition 4.1 below), then any graded A-module which is presented in finite degrees is d-step centrally stable.

Let $\mathcal{A}$ be an R-linear category such that $\mathrm{Ob}(\mathcal{A}) = \mathbb{Z}_+$ and $\mathrm{Hom}_{\mathcal{A}}(m; n) = 0$ if $m > n$. Denote by A the category algebra of $\mathcal{A}$; see (2.1). For any $m \in \mathbb{Z}_+$, let

$$\mathfrak{A}(m; m) := \mathrm{End}_{\mathcal{A}}(m);$$

for any $n \geqslant m + 1$, let

$$\mathfrak{A}(m; n) := \mathrm{Hom}_{\mathcal{A}}(n-1; n) \otimes_{\mathrm{End}_{\mathcal{A}}(n-1)} \cdots \otimes_{\mathrm{End}_{\mathcal{A}}(m+1)} \mathrm{Hom}_{\mathcal{A}}(m; m+1).$$

We define a R-linear category $\widetilde{\mathfrak{A}}$ with $\mathrm{Ob}(\widetilde{\mathfrak{A}}) = \mathbb{Z}_+$ by $\mathrm{Hom}_{\widetilde{\mathcal{A}}}(m; n) = \mathfrak{A}(m; n)$ if $m \leqslant n$, and $\mathrm{Hom}_{\widetilde{\mathcal{A}}}(m; n) = 0$ if $m > n$. There is a natural R-linear functor from $\widetilde{\mathfrak{A}}$ to $\mathcal{A}$ which is the identity map on the set of objects. For any $n \geqslant m$, let $\mathfrak{P}(m; n)$ be the kernel of the map $\mathfrak{A}(m; n) \to \mathrm{Hom}_{\mathcal{A}}(m; n)$. One has $\mathfrak{P}(m; n) = 0$ whenever n is m or $m + 1$.

Definition 4.1. Let d be an integer $\geqslant 1$. We say that *the ideal of relations of A is generated in degrees $\leqslant d$* if, whenever $n \geqslant m + 2$, the map $\mathfrak{A}(m; n) \to \mathrm{Hom}_{\mathcal{A}}(m; n)$ is surjective, and whenever $n \geqslant m + d$, one has:

(4.1) $$\mathfrak{P}(m; n) = \sum_{r=m}^{n-d} \mathfrak{A}(r+d; n) \otimes_{\mathrm{End}_{\mathcal{A}}(r+d)} \mathfrak{P}(r; r+d) \otimes_{\mathrm{End}_{\mathcal{A}}(r)} \mathfrak{A}(m; r).$$

Remark 4.2. If $d = 1$, so that the ideal of relations of A is generated in degrees $\leqslant 1$, then $\mathfrak{P}(m; n) = 0$ for every m and n, and so $\widetilde{\mathfrak{A}}$ and $\mathcal{A}$ are the same. In most interesting examples, one has $d \geqslant 2$.

Remark 4.3. If $d = 2$, so that the ideal of relation of A is generated in degrees $\leqslant 2$, then A is a quadratic algebra whose degree k component is $\bigoplus_{m \in \mathbb{Z}_+} \mathrm{Hom}_{\mathcal{A}}(m; m+k)$ for each $k \in \mathbb{Z}_+$. Many combinatorial categories such as $\mathcal{FI}_a$, $\mathcal{OI}_a$ and $\mathcal{FS}^{\mathrm{op}}$ are quadratic; see Section 5 below.

Example 4.4. Fix a finite totally ordered set Ω. Recall (see [8]) that the plactic monoid M on Ω is the monoid generated by Ω with defining relations

$$xzy = zxy \quad \text{if } x \leqslant y < z;$$
$$yxz = yzx \quad \text{if } x < y \leqslant z.$$

An element $w \in M$ is said to be of length $\grave{}(w) = n$ if w is a product of n elements of Ω; it is clear that $\grave{}(w)$ is well-defined. Now define $\mathcal{C}$ to be the category with $\mathrm{Ob}(\mathcal{C}) = \mathbb{Z}_+$ and

$$\mathrm{Hom}_{\mathcal{C}}(m; n) = \{w \in M \mid \grave{}(w) = n - m\}.$$

The composition of morphisms in $\mathcal{C}$ is given by the product in M. Then the category algebra $A_{\mathcal{C}}$ is not quadratic but has ideal of relations generated in degrees $\leqslant 3$.

It is plain that the preceding example can be generalized to any monoid with a presentation whose defining relations do not change the length of words.

4.2. d-step central stability. The following theorem is the second main result of this paper.

Theorem 4.5. *Let d be an integer $\geqslant 1$. Suppose the ideal of relations of A is generated in degrees $\leqslant d$. If a graded A-module V is presented in finite degrees, then V is d-step centrally stable.*

To prove Theorem 4.5, we need the following lemma.

Lemma 4.6. *Suppose the ideal of relations of A is generated in degrees $\leqslant d$. If V is a graded A-module, and $n > N \geqslant m + d$, then*

$$e_n A f \otimes_{fAf} fV \cong e_n A e \otimes_{eAe} eV \quad \text{where } e = e_{m,N}; \quad f = e_{m+1,N}.$$

Proof. Observe that

$$e_n A e = e_n A e_m \oplus e_n A f; \qquad eV = e_m V \oplus fV:$$

We have a natural map

$$\Phi : e_n A f \otimes_{fAf} fV \to e_n A e \otimes_{eAe} eV:$$

We shall construct a map Ψ inverse to Φ. First, define the maps

$$\mathbb{F}(m;n) \otimes_R e_m V$$
$$\downarrow \mu$$
$$\mathrm{Hom}_{\mathcal{A}}(m+1;n) \otimes_{\mathrm{End}_{\mathcal{A}}(m+1)} \mathrm{Hom}_{\mathcal{A}}(m;m+1) \otimes_R e_m V \xrightarrow{\;\widetilde{\theta}\;} e_n A f \otimes_{fAf} fV$$
$$\downarrow \nu$$
$$\mathrm{Hom}_{\mathcal{A}}(m;n) \otimes_R e_m V$$

where   and   are the obvious maps defined using composition of morphisms, and θ is defined by $\theta(\ \otimes\ \otimes x) =\ \otimes\ x$. The map   is surjective and its kernel is the image of   . Using (4.1) and $\mathrm{Hom}_{\mathcal{A}}(m+1;m+d) \subset fAf$, we see that $\theta\ = 0$. Therefore, θ factors uniquely through   to give a map

$$: \mathrm{Hom}_{\mathcal{A}}(m;n) \otimes_R e_m V \to e_n A f \otimes_{fAf} fV:$$

Now define

$$\widetilde{\Psi} : e_n A e \otimes_R eV \to Af \otimes_{fAf} fV$$

by

$$\widetilde{\Psi}(\ \otimes x) = \begin{cases} (\ \otimes x) & \text{if } \ \in e_n A e_m \text{ and } x \in e_m V; \\ \ \otimes x & \text{if } \ \in e_n A f \text{ and } x \in fV; \\ 0 & \text{otherwise}: \end{cases}$$

It is plain that $\widetilde{\Psi}$ descends to a map

$$\Psi : e_n A e \otimes_{eAe} eV \to Af \otimes_{fAf} fV;$$

and Ψ is an inverse to Φ. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We can now prove Theorem 4.5.

Proof of Theorem 4.5. By Theorem 3.2, for all N sufficiently large, one has

$$Ae_{0,N} \otimes_{e_{0,N} A e_{0,N}} e_{0,N} V \cong V;$$

in particular, for each $n \in \mathbb{Z}_+$, one has $e_n A e_{0,N} \otimes_{e_{0,N} A e_{0,N}} e_{0,N} V \cong e_n V$.

If $n > N \geqslant d-1$, then by Lemma 4.6,

$$e_n A e_{N-(d-1),N} \otimes_{e_{N-(d-1),N} A e_{N-(d-1),N}} e_{N-(d-1),N} V \cong e_n A e_{N-d,N} \otimes_{e_{N-d,N} A e_{N-d,N}} e_{N-d,N} V$$
$$\vdots$$
$$\cong e_n A e_{0,N} \otimes_{e_{0,N} A e_{0,N}} e_{0,N} V$$

If $N - (d-1) \leqslant n \leqslant N$, the map

$$e_n A e_{N-(d-1),N} \otimes_{e_{N-(d-1),N} A e_{N-(d-1),N}} e_{N-(d-1),N} V \to e_n V; \qquad \otimes x \mapsto\ x$$

has an inverse defined by $x \mapsto e_n \otimes x$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. Sufficiency conditions

5.1. **Sufficiency conditions.** To apply Theorem 4.5, we need to be able to check if the ideal of relations of A is generated in degrees $\leqslant d$. In this section, we provide sufficiency conditions which allow one to check this.

Let $\mathcal{C}$ be a small category such that $\mathrm{Ob}(\mathcal{C}) = \mathbb{Z}_+$, and $\mathrm{Hom}_{\mathcal{C}}(m, n) = \emptyset$ if $m > n$. Recall that $\mathcal{A}_{\mathcal{C}}$ is the R-linear category with $\mathrm{Ob}(\mathcal{A}_{\mathcal{C}}) = \mathbb{Z}_+$ and $\mathrm{Hom}_{\mathcal{A}_{\mathcal{C}}}(m, n)$ the free R-module with basis $\mathrm{Hom}_{\mathcal{C}}(m, n)$ for each $m, n \in \mathbb{Z}_+$. Let $A_{\mathcal{C}}$ be the category algebra of $\mathcal{A}_{\mathcal{C}}$; see (2.1).

Definition 5.1. We say that the *ideal of relations of $\mathcal{C}$ is generated in degrees $\leqslant d$* if the ideal of relations of $A_{\mathcal{C}}$ is generated in degrees $\leqslant d$.

Proposition 5.2. *Suppose $d \geqslant 2$. The ideal of relations of $\mathcal{C}$ is generated in degrees $\leqslant d$ if the following two conditions are satisfied:*

(i) The composition map $\mathrm{Hom}_{\mathcal{C}}(l, n) \times \mathrm{Hom}_{\mathcal{C}}(m, l) \to \mathrm{Hom}_{\mathcal{C}}(m, n)$ is surjective whenever $m < l < n$.

(ii) For every $\alpha_1, \alpha_2 \in \mathrm{Hom}_{\mathcal{C}}(m+1, n)$ and $\beta_1, \beta_2 \in \mathrm{Hom}_{\mathcal{C}}(m, m+1)$ satisfying

$$\alpha_1 \beta_1 = \alpha_2 \beta_2 \quad and \quad n > m + d,$$

there exists $\gamma \in \mathrm{Hom}_{\mathcal{C}}(m+d, n)$ and $\delta_1, \delta_2 \in \mathrm{Hom}_{\mathcal{C}}(m+1, m+d)$ such that the following diagram commutes:

(5.1)

Proof. Condition (i) implies that $\mathcal{A}_{\mathcal{C}}(m, n) \to \mathrm{Hom}_{\mathcal{A}_{\mathcal{C}}}(m, n)$ is surjective if $n \geqslant m + 2$. We shall prove, by induction on $n - m$, that (4.1) holds. The case $n - m = d$ is trivial. Now suppose $n - m > d$. Let

$$\mathcal{A}_{\mathcal{C}}(m, n) \xrightarrow{\pi_1} \mathrm{Hom}_{\mathcal{A}_{\mathcal{C}}}(m+1, n) \otimes_{\mathrm{End}_{\mathcal{A}_{\mathcal{C}}}(m+1)} \mathrm{Hom}_{\mathcal{A}_{\mathcal{C}}}(m, m+1) \xrightarrow{\pi_2} \mathrm{Hom}_{\mathcal{A}_{\mathcal{C}}}(m, n)$$

be the obvious maps defined by composition of morphisms. It is easy to see (and can be proved in the same way as [9, Lemma 6.1]) that $\mathcal{P}(m, n)$ is spanned over R by elements of the form

(5.2) $$\alpha_{n-1} \otimes \cdots \otimes \alpha_m - \alpha'_{n-1} \otimes \cdots \otimes \alpha'_m$$

such that $\alpha_i, \alpha'_i \in \mathrm{Hom}_{\mathcal{C}}(i, i+1)$ for $i = m, \ldots, n-1$ and $\alpha_{n-1} \cdots \alpha_m = \alpha'_{n-1} \cdots \alpha'_m$. By condition (ii), there exists $\gamma \in \mathrm{Hom}_{\mathcal{C}}(m+d, n)$ and $\delta_1, \delta_2 \in \mathrm{Hom}_{\mathcal{C}}(m+1, m+d)$ such that

$$\alpha_{n-1} \cdots \alpha_{m+1} = \gamma \delta_1, \qquad \alpha'_{n-1} \cdots \alpha'_{m+1} = \gamma \delta_2, \qquad \delta_1 \alpha_m = \delta_2 \alpha'_m.$$

We can choose

$$e = e_{n-1} \otimes \cdots \otimes e_{m+d} \in \mathcal{A}_{\mathcal{C}}(m+d, n),$$

where $e_i \in \mathrm{Hom}_{\mathcal{C}}(i, i+1)$ for $i = m+d, \ldots, n-1$, such that $e_{n-1} \cdots e_{m+d} = \gamma$. We can also choose

$$\mathrm{e} = \mathrm{e}_{m+d-1} \otimes \cdots \otimes \mathrm{e}_{m+1} \in \mathcal{A}_{\mathcal{C}}(m+1, m+d),$$

where $\mathrm{e}_i \in \mathrm{Hom}_{\mathcal{C}}(i, i+1)$ for $i = m+1, \ldots, m+d-1$, such that $\mathrm{e}_{m+d-1} \cdots \mathrm{e}_{m+1} = \delta_1$. Similarly, choose

$$\mathrm{e}' = \mathrm{e}'_{m+d-1} \otimes \cdots \otimes \mathrm{e}'_{m+1} \in \mathcal{A}_{\mathcal{C}}(m+1, m+d),$$

where $\mathrm{e}'_i \in \mathrm{Hom}_{\mathcal{C}}(i, i+1)$ for $i = m+1, \ldots, m+d-1$, such that $\mathrm{e}'_{m+d-1} \cdots \mathrm{e}'_{m+1} = \delta_2$. The element in (5.2) can be written as:

$$(\alpha_{n-1} \otimes \cdots \otimes \alpha_{m+1} - e \otimes \mathrm{e}) \otimes \alpha_m - (\alpha'_{n-1} \otimes \cdots \otimes \alpha'_{m+1} - e \otimes \mathrm{e}') \otimes \alpha'_m + e \otimes (\mathrm{e} \otimes \alpha_m - \mathrm{e}' \otimes \alpha'_m).$$

Since

$$\alpha_{n-1} \otimes \cdots \otimes \alpha_{m+1} - e \otimes \mathrm{e} \in \mathcal{P}(m+1, n),$$

$$\alpha'_{n-1} \otimes \cdots \otimes \alpha'_{m+1} - e \otimes \mathrm{e}' \in \mathcal{P}(m+1, n),$$

$$\mathrm{e} \otimes \alpha_m - \mathrm{e}' \otimes \alpha'_m \in \mathcal{P}(m, m+d),$$

the result follows by induction. $\qquad\qquad\square$

Corollary 5.3. *Suppose that $\mathcal{C}$ satisfies the two conditions in Proposition 5.2 for some $d \geqslant 2$. If a $\mathcal{C}$-module V is presented in finite degrees, then V is d-step centrally stable.*

Proof. By Proposition 5.2, the ideal of relations of $\mathcal{C}$ is generated in degrees $\leqslant d$. Hence, we may apply Theorem 4.5. □
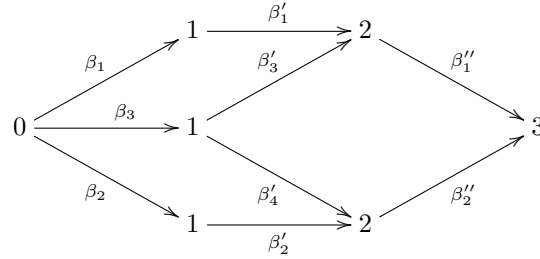
Remark 5.4. Suppose that $\mathcal{C}$ satisfies condition (i) in Proposition 5.2, and its ideal of relations is generated in degrees $\leqslant d$. In this case, condition (ii) might not hold, so it is not a necessary condition. For example, suppose that $d = 2$, and one has

$$\mathrm{Hom}_{\mathcal{C}}(0;1) = \{\beta_1;\ \beta_2;\ \beta_3\}; \quad \mathrm{Hom}_{\mathcal{C}}(1;2) = \{\beta'_1;\ \beta'_2;\ \beta'_3;\ \beta'_4\}; \quad \mathrm{Hom}_{\mathcal{C}}(2;3) = \{\beta''_1;\ \beta''_2\};$$

with the defining relations

$$\beta'_1\beta_1 = \beta'_3\beta_3; \quad \beta'_2\beta_2 = \beta'_4\beta_3; \quad \beta''_1\beta'_3 = \beta''_2\beta'_4;$$

as depicted in the following diagram:

Let $\alpha_1 = \beta''_1\beta'_1$ and $\alpha_2 = \beta''_2\beta'_2$. Then one has

$$\alpha_1\beta_1 = \beta''_1\beta'_1\beta_1 = \beta''_1\beta'_3\beta_3 = \beta''_2\beta'_4\beta_3 = \beta''_2\beta'_2\beta_2 = \alpha_2\beta_2;$$

On the other hand, it is easy to see that there do not exist γ, γ_1 and γ_2 such that the diagram in (5.1) (with $d = 2$, $m = 0$, $n = 3$) commutes.

5.2. **Examples.** Using Proposition 5.2, one can easily check, for example, that (a skeleton of) $\mathcal{FI}$ is quadratic. Let $\mathcal{C}$ be the full subcategory of $\mathcal{FI}$ on the set of objects $\{[n] \mid n \in \mathbb{Z}_+\}$. It is easy to see that $\mathcal{C}$ satisfies condition (i) in Proposition 5.2. We need to verify condition (ii). Thus, suppose that we have injective maps:

$$\alpha_1;\ \alpha_2 : [m+1] \longrightarrow [n] \quad \text{and} \quad \beta_1;\ \beta_2 : [m] \longrightarrow [m+1];$$

such that $\alpha_1 \circ \beta_1 = \alpha_2 \circ \beta_2$ and $n \geqslant m+2$. Since

$$|\mathrm{Im}(\alpha_1) \cup \mathrm{Im}(\alpha_2)| = |\mathrm{Im}(\alpha_1)| + |\mathrm{Im}(\alpha_2)| - |\mathrm{Im}(\alpha_1) \cap \mathrm{Im}(\alpha_2)| \leqslant (m+1) + (m+1) - m = m+2;$$

we can choose $S \subset [n]$ such that $\mathrm{Im}(\alpha_1) \cup \mathrm{Im}(\alpha_2) \subset S$ and $|S| = m+2$. Let $\gamma : [m+2] \to [n]$ be any injective map whose image is S. Then there exists a unique map γ_1 (respectively γ_2) such that $\gamma \circ \gamma_1 = \alpha_1$ (respectively $\gamma \circ \gamma_2 = \alpha_2$). Clearly, γ_1 and γ_2 are injective. We have:

$$\gamma \circ \gamma_1 \circ \beta_1 = \alpha_1 \circ \beta_1 = \alpha_2 \circ \beta_2 = \gamma \circ \gamma_2 \circ \beta_2;$$

Since γ is injective, it follows that $\gamma_1 \circ \beta_1 = \gamma_2 \circ \beta_2$. Therefore, it follows from Proposition 5.2 that $\mathcal{C}$ (or more precisely, the algebra $A_{\mathcal{C}}$) is quadratic.

Similarly, one can apply Proposition 5.2 to show that the categories $\mathcal{FI}_a$, $\mathcal{OI}_a$, $\mathcal{FS}^{\mathrm{op}}$, and $\mathcal{VI}(\mathbb{F})$ are quadratic, where for any field $\mathbb{F}$, the set of objects of $\mathcal{VI}(\mathbb{F})$ is $\mathbb{Z}_+$ and the morphisms $m \to n$ are the injective linear maps $\mathbb{F}^m \to \mathbb{F}^n$.

Remark 5.5. A twisted commutative algebra E is an associative unital graded algebra $E = \bigsqcup_{n \geqslant 0} E_n$ where each E_n is equipped with the structure of an S_n-module such that the multiplication map $E_n \otimes E_m \to E_{n+m}$ is $S_n \times S_m$-equivariant, and $yx = \sigma(xy)$ for every $x \in E_n$, $y \in E_m$ (where $\sigma \in S_{n+m}$ switches the first n and last m elements of $\{1; \ldots; n+m\}$); see [13, §8.1.2]. Any twisted commutative algebra E gives rise to an R-linear category $\mathcal{A}$ with $\mathrm{Ob}(\mathcal{A}) = \mathbb{Z}_+$ and

$$\mathrm{Hom}_{\mathcal{A}}(m;n) = RS_n \otimes_{RS_{n-m}} E_{n-m};$$

In particular, when E is the twisted commutative algebra with E_n the trivial S_n-module for every n (and with the obvious multiplication map), the R-linear category $\mathcal{A}$ we obtained is precisely the category $\mathcal{A}_{\mathcal{C}}$ associated to a skeleton $\mathcal{C}$ of $\mathcal{FI}$. One might ask if the category algebra of every category $\mathcal{A}$ obtained from

a twisted commutative algebra is quadratic. This is not true. For example, let d be any integer > 2, and let E be the twisted commutative algebra with E_n the trivial S_n-module if $n < d$, otherwise let $E_n = 0$; the multiplication map of E is defined in the obvious way. Then E gives rise to an R-linear category $\mathcal{A}$ with the property that the ideal of relations of its category algebra is generated in degrees $\leqslant d$, but the category algebra is not quadratic.

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