

FINITENESS OF REGULARITY OF FI_d -MODULES

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ABSTRACT. We prove by an inductive argument that any finitely generated FI_d -module over a commutative Noetherian ring has finite (Castelnuovo-Mumford) regularity. Our inductive argument is applicable also to the categories OI_d , FI^m , and OI^m .

1. INTRODUCTION

Let d be an integer ≥ 1 . The main result of this paper is:

Theorem 1.1. *Any finitely generated FI_d -module over a commutative Noetherian ring has finite regularity.*

When $d = 1$ (that is, for FI -modules) this was first proved by Church and Ellenberg [1, Theorem A]. When $d > 1$ and the ring is a field of characteristic zero, this was proved recently by Sam and Snowden in [9, Corollary 1.11] and [10, Theorem 1.1]. Let us mention that the results of Church and Ellenberg, and of Sam and Snowden, are stronger than what Theorem 1.1 says in their respective cases; we refer the reader to [1] and [10] for their precise statements.

Our strategy for proving Theorem 1.1 is by induction on generating degree. The key tools are the shift functor Σ and the derivative functor D . For FI -modules, a simplified version of this inductive argument is in our previous paper [5]. However, the argument in [5] does not generalize in a straightforward way to FI_d -modules where $d > 1$. The new ideas in our present paper which allow us to do this are the introduction of a collection of functors D_S which interpolates between Σ and D , and the use of a tree as a bookkeeping device to carry out the induction. A nontrivial ingredient in our proof is the theorem of Sam and Snowden [8, Corollary 7.1.5] that any finitely generated FI_d -module over a commutative Noetherian ring is Noetherian.

Besides finiteness of regularity, our inductive argument can potentially be used to derive other properties of finitely generated FI_d -modules over a commutative Noetherian ring. Moreover, it can be applied to a few other categories (for example, OI_d , FI^m , and OI^m); in particular, the analogue of Theorem 1.1 holds for these categories.

This paper is organized as follows. In Section 2, we recall the definition of the notion of FI_d -modules. We also recall the definitions of regularity and generating degree. In Section 3, we recall the definitions of the shift functor Σ and the derivative functor D . We also introduce the functors D_S and prove their basic properties. In Section 4, we construct a tree which plays a crucial role in our inductive argument. The proof of Theorem 1.1 is given in Section 5. In Section 6, we formulate the conditions under which our inductive argument can be applied.

2. NOTATIONS AND RECOLLECTIONS

We denote by \mathcal{C} the category FI_d introduced by Sam and Snowden in [8]. Let us recall its definition. For any integer $n \geq 0$, let $[n] = \{1, \dots, n\}$ whereby $[0] = \emptyset$. By a d -coloring on a set Z , we mean a map from Z to $[d]$. The category \mathcal{C} has objects the finite sets. The morphisms of \mathcal{C} from a finite X to a finite set Y are the pairs (f, δ) where $f : X \rightarrow Y$ is any injection and δ is any d -coloring on $Y \setminus f(X)$. The composition of a morphism $(f, \delta) : X \rightarrow Y$ with a morphism $(g, \varepsilon) : Y \rightarrow Z$ is defined to be the morphism $(h, \zeta) : X \rightarrow Z$ where $h = g \circ f$ and

$$\zeta(z) = \begin{cases} \delta(y) & \text{if } z = g(y), \\ \varepsilon(z) & \text{else.} \end{cases}$$

For any finite set X , let $|X|$ denotes the number of elements of X .

Let k be a commutative ring. A \mathcal{C} -module (over k) is a covariant functor from \mathcal{C} to the category of k -modules. Denote by $\mathcal{C}\text{-Mod}$ the category of \mathcal{C} -modules.

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Let V be any \mathcal{C} -module. For any object X of \mathcal{C} , we write V_X for $V(X)$. We say that V is *finitely generated* if there exists objects X_1, \dots, X_n of \mathcal{C} and a finite subset E of $V_{X_1} \sqcup \dots \sqcup V_{X_n}$ such that the only \mathcal{C} -submodule of V containing E is V itself.

For any object Y of \mathcal{C} , let $M(Y)$ be the \mathcal{C} -module defined by

$$M(Y) = k\mathrm{Hom}_{\mathcal{C}}(Y, -),$$

that is, $M(Y)$ is the composition of the functor $\mathrm{Hom}_{\mathcal{C}}(Y, -)$ followed by the free k -module functor. Note that $M(Y)$ is a projective \mathcal{C} -module, and every projective \mathcal{C} -module is a direct summand of some $\bigoplus_{i \in I} M(Y_i)$.

For any \mathcal{C} -module V , let $(JV)_X$ be the k -submodule of V_X spanned by the images of the maps $\alpha_* : V_Y \rightarrow V_X$ for all morphisms $\alpha : Y \rightarrow X$ of \mathcal{C} with $|Y| < |X|$; this defines a \mathcal{C} -submodule JV of V . Define a functor $H_0 : \mathcal{C}\text{-Mod} \rightarrow \mathcal{C}\text{-Mod}$ by $H_0(V) = V/JV$. The functor H_0 is right exact. For any integer $a \geq 1$, let H_a be the a -th left derived functor of H_0 .

For any integer $a \geq 0$, we define the a -th homological degree $\mathrm{hd}_a(V)$ of V by

$$\mathrm{hd}_a(V) = \begin{cases} \sup \{|X| \mid (H_a(V))_X \neq 0\} & \text{if } H_a(V) \neq 0, \\ -1 & \text{else.} \end{cases}$$

We define the *regularity* $\mathrm{reg}(V)$ of V by

$$\mathrm{reg}(V) = \sup \{\mathrm{hd}_a(V) - a \mid a \geq 0\}.$$

We define the *generating degree* $\mathrm{gd}(V)$ of V by

$$\mathrm{gd}(V) = \mathrm{hd}_0(V).$$

For example, one has $\mathrm{gd}(M(Y)) = |Y|$. It is easy to see that for any \mathcal{C} -module V , there exists a surjective homomorphism $P \rightarrow V$ where P is of the form $\bigoplus_{i \in I} M(Y_i)$ and $\mathrm{gd}(P) = \mathrm{gd}(V)$.

We should mention that our conventions for homological degrees, generating degrees, and regularity differ from some papers; see [5, Remark 1.9].

3. SHIFT FUNCTORS

Following along the same lines as [1] and [2] for FI-modules, we first define the shift functor for FI_d -modules and study its basic properties.

Fix a one-element set $\{\star\}$ and define a functor $\iota : \mathcal{C} \rightarrow \mathcal{C}$ by $X \mapsto X \sqcup \{\star\}$. If $(f, \delta) : X \rightarrow Y$ is a morphism of \mathcal{C} , then $\iota(f, \delta) : X \sqcup \{\star\} \rightarrow Y \sqcup \{\star\}$ is the morphism $(f \sqcup \mathrm{id}_{\{\star\}}, \delta)$.

Definition 3.1. The *shift functor* $\Sigma : \mathcal{C}\text{-Mod} \rightarrow \mathcal{C}\text{-Mod}$ is defined by

$$\Sigma V = V \circ \iota \quad \text{for every } V \in \mathcal{C}\text{-Mod},$$

that is,

$$(\Sigma V)_X = V_{X \sqcup \{\star\}} \quad \text{for every finite set } X.$$

For any finite set X and $\ell \in [d]$, we denote by

$$\mathrm{incl}_X : X \hookrightarrow X \sqcup \{\star\} \quad \text{and} \quad \mathrm{col}_\ell : \{\star\} \rightarrow [d]$$

the inclusion map and the d -coloring $\mathrm{col}_\ell(\star) = \ell$, respectively. The pair $(\mathrm{incl}_X, \mathrm{col}_\ell)$ is a morphism of \mathcal{C} from X to $\iota(X)$.

For each $\ell \in [d]$, there is a natural morphism of functors $\eta_\ell : \mathrm{id}_{\mathcal{C}} \rightarrow \iota$ whose component at each object X of \mathcal{C} is the morphism

$$(\mathrm{incl}_X, \mathrm{col}_\ell) : X \longrightarrow \iota(X).$$

For any \mathcal{C} -module V , the horizontal composition of morphisms of functors $\mathrm{id}_V \circ \eta_\ell$ gives a natural \mathcal{C} -module homomorphism

$$\theta_\ell : V \longrightarrow \Sigma V.$$

Lemma 3.2. *For any object Y of \mathcal{C} , there is an isomorphism*

$$\left(\bigoplus_{\ell \in [d]} M(Y) \right) \oplus \left(\bigoplus_{y \in Y} M(Y \setminus \{y\}) \right) \longrightarrow \Sigma M(Y)$$

whose restriction to the direct summand $M(Y)$ indexed by $\ell \in [d]$ is $\theta_\ell : M(Y) \rightarrow \Sigma M(Y)$.

Proof. This follows easily from the following observation: for each morphism

$$(f, \delta) : Y \rightarrow X \sqcup \{\star\},$$

either $\star \notin f(Y)$ in which case $f(Y) \subset X$ and there is a unique $\ell \in [d]$ such that $\delta(\star) = \ell$, or there is a unique $y \in Y$ such that $f(y) = \star$ in which case $f(Y \setminus \{y\}) \subset X$. \square

For any \mathcal{C} -module V and any set S , we write

$$V^S = \bigoplus_{\ell \in S} V.$$

If $S \subset [d]$, let

$$\Theta_S : V^S \rightarrow \Sigma V$$

be the natural homomorphism whose restriction to the direct summand V indexed by $\ell \in S$ is $\theta_\ell : V \rightarrow \Sigma V$. Define the functor

$$D_S : \mathcal{C}\text{-Mod} \rightarrow \mathcal{C}\text{-Mod}$$

to be the cokernel of Θ_S , that is, $D_S V$ is the cokernel of $\Theta_S : V^S \rightarrow \Sigma V$.

If V is a \mathcal{C} -module and $S \subset T \subset [d]$, then $V^S \subset V^T$ and there is a natural surjective homomorphism $D_S V \rightarrow D_T V$. One has

$$\Sigma V = D_S V \quad \text{where } S = \emptyset.$$

Define the *derivative functor* $D : \mathcal{C}\text{-Mod} \rightarrow \mathcal{C}\text{-Mod}$ by

$$DV = D_S V \quad \text{where } S = [d].$$

The derivative functor for FI-modules was first introduced by Church and Ellenberg in [1].

Lemma 3.3. *Let $S \subset [d]$.*

(a) *If P is a projective \mathcal{C} -module, then $D_S P$ is a projective \mathcal{C} -module.*

(b) *If V is a \mathcal{C} -submodule of a projective \mathcal{C} -module P , then $\Theta_S : V^S \rightarrow \Sigma V$ is injective.*

Proof. Any projective \mathcal{C} -module is a direct summand of a \mathcal{C} -module of the form $\bigoplus_{i \in I} M(Y_i)$, so it suffices to prove the assertions when P is $M(Y)$, in which case the assertions are immediate from Lemma 3.2. \square

Lemma 3.4. *Let $S \subset [d]$. For any \mathcal{C} -module V , one has $\text{gd}(D_S V) \leq \text{gd}(V)$. If $V \neq 0$, then $\text{gd}(DV) \leq \text{gd}(V) - 1$.*

Proof. The case $V = 0$ is trivial, so suppose that $V \neq 0$. Let $P \rightarrow V$ be a surjective homomorphism where P is of the form $\bigoplus_{i \in I} M(Y_i)$ and $\text{gd}(P) = \text{gd}(V)$. Since D_S is right exact, we have a surjective homomorphism $D_S P \rightarrow D_S V$, hence $\text{gd}(D_S V) \leq \text{gd}(D_S P)$. By Lemma 3.2, we have $\text{gd}(D_S P) \leq \text{gd}(P)$ and $\text{gd}(DP) \leq \text{gd}(P) - 1$. Hence,

$$\text{gd}(D_S V) \leq \text{gd}(D_S P) \leq \text{gd}(P) = \text{gd}(V)$$

and

$$\text{gd}(DV) \leq \text{gd}(DP) \leq \text{gd}(P) - 1 = \text{gd}(V) - 1.$$

\square

The proof of the following lemma is a slight modification of the proof of [6, Proposition 2.4].

Lemma 3.5. *Let $S \subset [d]$. For any \mathcal{C} -module V , one has $\text{gd}(D_S V) \geq \text{gd}(V) - 1$. In particular, if $\text{gd}(V) = \infty$, then $\text{gd}(D_S V) = \infty$.*

Proof. There is a natural surjective homomorphism $D_S V \rightarrow DV$, so $\text{gd}(D_S V) \geq \text{gd}(DV)$. It suffices to prove that $\text{gd}(DV) \geq \text{gd}(V) - 1$.

The case $\text{gd}(V) \leq 0$ is trivial, so suppose that $\text{gd}(V) > 0$. It suffices to prove that for any non-negative integer $n < \text{gd}(V)$, one has $\text{gd}(DV) \geq n$. To this end, let V' be the quotient of V by its \mathcal{C} -submodule generated by all the elements of V_X for every object X of \mathcal{C} with $|X| \leq n$. Since D is right exact, we have a surjective homomorphism $DV \rightarrow DV'$, so $\text{gd}(DV) \geq \text{gd}(DV')$. We shall show that $\text{gd}(DV') \geq n$.

If X is any object of \mathcal{C} such that $|X| < n$, then $(\Sigma V')_X = 0$ and so $(DV')_X = 0$. Therefore, we only need to see that $DV' \neq 0$. Since $n < \text{gd}(V)$, we know that $V' \neq 0$. Let m be the minimal integer such that $V'_{[m]} \neq 0$. Then $m > n \geq 0$. We have $V'_{[m-1]} = 0$ and $(\Sigma V')_{[m-1]} \neq 0$, so $(DV')_{[m-1]} \neq 0$. \square

Corollary 3.6. *For any nonzero \mathcal{C} -module V , one has $\text{gd}(DV) = \text{gd}(V) - 1$.*

Proof. Immediate from Lemma 3.4 and Lemma 3.5. \square

Lemma 3.7. *Let $S \subset [d]$, and let n be an integer ≥ 0 . For any \mathcal{C} -module V such that $\Theta_S : V^S \rightarrow \Sigma V$ is injective, one has*

$$\mathrm{hd}_n(D_S V) \leq \max\{\mathrm{hd}_n(V), \mathrm{hd}_{n-1}(V), \dots, \mathrm{hd}_0(V)\},$$

and

$$\mathrm{hd}_n(V) \leq \max\{\mathrm{hd}_n(D_S V) + 1, \mathrm{hd}_{n-1}(D_S V) + 2, \dots, \mathrm{hd}_0(D_S V) + n + 1\}.$$

Proof. We use induction on n . The base case $n = 0$ follows from Lemma 3.4 and Lemma 3.5.

Suppose $n > 0$. Let $0 \rightarrow W \rightarrow P \rightarrow V \rightarrow 0$ be a short exact sequence where P is a projective \mathcal{C} -module with $\mathrm{gd}(P) = \mathrm{gd}(V)$. We have a commuting diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W^S & \longrightarrow & P^S & \longrightarrow & V^S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Sigma W & \longrightarrow & \Sigma P & \longrightarrow & \Sigma V & \longrightarrow & 0. \end{array}$$

Since $\Theta_S : V^S \rightarrow \Sigma V$ is injective, it follows from the Snake Lemma that $0 \rightarrow D_S W \rightarrow D_S P \rightarrow D_S V \rightarrow 0$ is a short exact sequence. By Lemma 3.3, we know that $D_S P$ is a projective \mathcal{C} -module, and $\Theta_S : W^S \rightarrow \Sigma W$ is injective. We have:

$$\begin{aligned} \mathrm{hd}_n(D_S V) &\leq \mathrm{hd}_{n-1}(D_S W) \\ &\leq \max\{\mathrm{hd}_{n-1}(W), \mathrm{hd}_{n-2}(W), \dots, \mathrm{hd}_0(W)\} \quad (\text{by induction hypothesis}) \\ &\leq \max\{\mathrm{hd}_n(V), \mathrm{hd}_{n-1}(V), \dots, \mathrm{hd}_1(V), \mathrm{hd}_0(P)\} \\ &= \max\{\mathrm{hd}_n(V), \mathrm{hd}_{n-1}(V), \dots, \mathrm{hd}_1(V), \mathrm{hd}_0(V)\}. \end{aligned}$$

We also have:

$$\begin{aligned} \mathrm{hd}_n(V) &\leq \mathrm{hd}_{n-1}(W) \\ &\leq \max\{\mathrm{hd}_{n-1}(D_S W) + 1, \dots, \mathrm{hd}_0(D_S W) + n\} \quad (\text{by induction hypothesis}) \\ &\leq \max\{\mathrm{hd}_n(D_S V) + 1, \dots, \mathrm{hd}_1(D_S V) + n, \mathrm{hd}_0(D_S P) + n\} \\ &\leq \max\{\mathrm{hd}_n(D_S V) + 1, \dots, \mathrm{hd}_1(D_S V) + n, \mathrm{hd}_0(P) + n\} \quad (\text{by Lemma 3.4}) \\ &= \max\{\mathrm{hd}_n(D_S V) + 1, \dots, \mathrm{hd}_1(D_S V) + n, \mathrm{hd}_0(V) + n\} \\ &\leq \max\{\mathrm{hd}_n(D_S V) + 1, \dots, \mathrm{hd}_1(D_S V) + n, \mathrm{hd}_0(D_S V) + n + 1\} \quad (\text{by Lemma 3.5}). \end{aligned}$$

\square

Proposition 3.8. *Let V be a \mathcal{C} -module, and let $S \subset [d]$. Suppose that $\Theta_S : V^S \rightarrow \Sigma V$ is injective. Then*

$$\mathrm{reg}(D_S V) \leq \mathrm{reg}(V) \leq \mathrm{reg}(D_S V) + 1.$$

In particular, one has

$$\mathrm{reg}(\Sigma V) \leq \mathrm{reg}(V) \leq \mathrm{reg}(\Sigma V) + 1.$$

Proof. Immediate from Lemma 3.7. \square

4. A TREE OF QUOTIENT MODULES

Let V be a \mathcal{C} -module. For any $S \subset [d]$, let $K_S V$ be the kernel of $\Theta_S : V^S \rightarrow \Sigma V$. Let KV be $K_S V$ where $S = [d]$.

If $S \subset T \subset [d]$, then $V^S \subset V^T$ and $K_S V = K_T V \cap V^S$; in particular $K_S V \subset K_T V$. We have:

$$K_S V \subset KV \quad \text{and} \quad K_S V \cap K_{S'} V = K_{S \cap S'} V$$

for any $S, S' \subset [d]$.

Definition 4.1. Let V be a \mathcal{C} -module, and let $S \subset [d]$. We say that S is a *nil maximal subset* for V if $K_S V = 0$ and $K_T V \neq 0$ for every $T \subset [d]$ which properly contains S .

For every \mathcal{C} -module V , one has $K_S V = 0$ when $S = \emptyset$, so a nil maximal subset for V exists.

Lemma 4.2. *Let V be a \mathcal{C} -module, and let S be a nil maximal subset for V . Let T and T' be subsets of $[d]$ such that $S \subset T \subset T'$. If $K_T V = K_{T'} V$, then $T = T'$.*

Proof. Suppose, on the contrary, that $T \neq T'$. Then there exists an element $t' \in T' \setminus T$. Let $S' = S \cup \{t'\}$. Since S is a nil maximal subset for V , one has $K_{S'}V \neq 0$. But

$$K_{S'}V = K_{S'}V \cap K_{T'}V = K_{S'}V \cap K_TV = K_{S' \cap T}V = K_SV = 0,$$

a contradiction. \square

Lemma 4.3. *Let V be a \mathcal{C} -module, and let $S \subset T \subset [d]$. Then the kernel of the natural surjective homomorphism $D_SV \rightarrow D_TV$ is isomorphic to $V^T/(K_TV + V^S)$.*

Proof. Immediate from applying the Snake Lemma to the following commuting diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V^S/K_SV & \longrightarrow & \Sigma V & \longrightarrow & D_SV & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V^T/K_TV & \longrightarrow & \Sigma V & \longrightarrow & D_TV & \longrightarrow & 0. \end{array}$$

\square

Let V be a \mathcal{C} -module, and let S be a nil maximal subset for V . By adding the elements of $[d] \setminus S$ into S one by one in any order, we obtain a strictly increasing chain of sets:

$$S = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_{d-|S|} = [d],$$

where $|S_i| = |S| + i$ for each i . By Lemma 4.2, we have a corresponding strictly increasing chain of \mathcal{C} -modules:

$$0 = K_{S_0}V \subset K_{S_1}V \subset K_{S_2}V \subset \cdots \subset K_{S_{d-|S|}}V = KV.$$

For each $i \in [d - |S|]$, define

$$V^{(i)} = \frac{V^{S_i}}{K_{S_i}V + V^{S_{i-1}}}.$$

Then $V^{(i)}$ is isomorphic to the kernel of the natural homomorphism $D_{S_{i-1}}V \rightarrow D_{S_i}V$ by Lemma 4.3.

Lemma 4.4. *Suppose $\text{reg}(DV) < \infty$. If $\text{reg}(V^{(i)}) < \infty$ for every $i \in [d - |S|]$, then $\text{reg}(V) < \infty$.*

Proof. We have the sequence of natural surjective homomorphisms

$$D_SV = D_{S_0}V \rightarrow D_{S_1}V \rightarrow \cdots \rightarrow D_{S_{d-|S|}}V = DV$$

and every kernel of a homomorphism in this sequence has finite regularity. Since DV has finite regularity, it follows successively that $D_{S_{d-|S|-1}}V, \dots, D_{S_1}V$ and D_SV have finite regularity. But $K_SV = 0$, so we can apply Proposition 3.8 to deduce that V has finite regularity. \square

We have a natural isomorphism

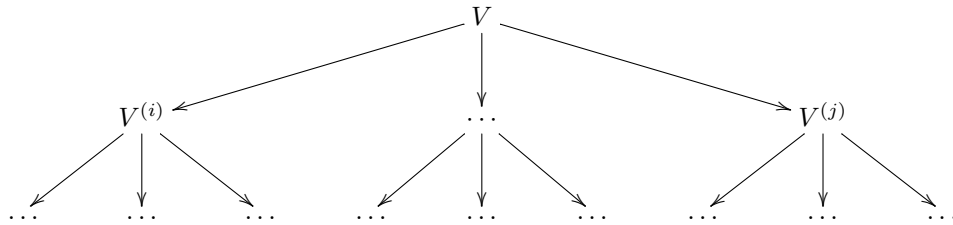
$$V^{(i)} \cong \frac{V^{S_i}/V^{S_{i-1}}}{(K_{S_i}V + V^{S_{i-1}})/V^{S_{i-1}}}.$$

Since $V^{S_i}/V^{S_{i-1}} \cong V$, the \mathcal{C} -module $V^{(i)}$ is a quotient module of V . Since

$$(K_{S_i}V + V^{S_{i-1}})/V^{S_{i-1}} \cong K_{S_i}V/(K_{S_i}V \cap V^{S_{i-1}}) \cong K_{S_i}V/K_{S_{i-1}}V \neq 0,$$

the kernel of the quotient map $V \rightarrow V^{(i)}$ is nonzero.

Let us call $V^{(i)}$ a *child* of V if $V^{(i)} \neq 0$. We can repeat the above procedure to construct the children of each child of V . Recursively, we get a tree of quotient modules, depicted as below:



The arrows in this tree are the quotient maps. We denote this tree by $\mathcal{T}(V)$. It has a natural poset structure with V the minimal element. Each \mathcal{C} -module U in $\mathcal{T}(V)$ has at most d children, and the quotient map from U to any child of U has nonzero kernel.

Lemma 4.5. *Let U be a \mathcal{C} -module in $\mathcal{T}(V)$, and suppose that $\text{reg}(DU) < \infty$. If every child of U has finite regularity, then U has finite regularity.*

Proof. Immediate from Lemma 4.4. □

5. PROOF OF MAIN RESULT

Let V be a \mathcal{C} -module, and let U and U' be \mathcal{C} -modules in the tree $\mathcal{T}(V)$. We say that U' is a *descendent* of U if there is a path in $\mathcal{T}(V)$ from U to U' .

We shall use the following simple observation:

Lemma 5.1. *Let V be a \mathcal{C} -module. The following statements are equivalent:*

- (1) *The tree $\mathcal{T}(V)$ is finite.*
- (2) *Every chain in $\mathcal{T}(V)$ has finite length.*

Proof. It is clear that (1) \Rightarrow (2). We show that (2) \Rightarrow (1). Suppose that $\mathcal{T}(V)$ is an infinite tree. We construct an infinite chain in the following way. Let V be the first element in the chain. Since V has finitely many children and infinitely many descendants, there must be a child V' of V with infinitely many descendants. Let V'' be the second element in the chain, and repeat the procedure. Recursively, we get an infinite chain. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let k be a commutative Noetherian ring. We want to prove that every finitely generated \mathcal{C} -module V has finite regularity. We use induction on $\text{gd}(V)$. The base case $\text{gd}(V) = -1$ is trivial. Suppose now that $\text{gd}(V) \geq 0$.

First, let us show that the tree $\mathcal{T}(V)$ is finite. By Lemma 5.1, we have to show that every chain in $\mathcal{T}(V)$ has finite length. To this end, suppose on the contrary that there is an infinite chain

$$U_0 \rightarrow U_1 \rightarrow U_2 \rightarrow \cdots$$

For each integer $n \geq 1$, let $Z_n \subset U_0$ be the kernel of the composition of the quotient maps $U_0 \rightarrow U_1 \rightarrow \cdots \rightarrow U_n$. Since each quotient map $U_{n-1} \rightarrow U_n$ has a nonzero kernel, we have a strictly increasing chain $Z_1 \subset Z_2 \subset \cdots$ in U_0 . But U_0 is a finitely generated \mathcal{C} -module since it is a quotient module of V . By [8, Corollary 7.1.5], we know that U_0 is Noetherian, so it cannot have an infinite strictly increasing chain of submodules.

We now show that V has finite regularity. Suppose U is any \mathcal{C} -module in $\mathcal{T}(V)$. Then U is a quotient module of V , so it is finitely generated and $\text{gd}(U) \leq \text{gd}(V)$. By Lemma 3.4, we have $\text{gd}(DU) < \text{gd}(V)$. Hence, by induction hypothesis, we have $\text{reg}(DU) < \infty$. Let N be the number of layers in $\mathcal{T}(V)$, where V is at the first layer, the children of V are at the second layer, and so on. Since the \mathcal{C} -modules at the N -th layer have no child, we deduce from Lemma 4.5 that they have finite regularity. If a \mathcal{C} -module is at the $(N-1)$ -th layer, then its children (if any) are at the N -th layer. Applying Lemma 4.5 again, we deduce that the \mathcal{C} -modules at the $(N-1)$ -th layer have finite regularity. Recursively, we deduce that every \mathcal{C} -module in $\mathcal{T}(V)$, including V , has finite regularity. □

6. CONCLUDING REMARKS

The inductive argument for proving Theorem 1.1 used the following properties:

- (1) There is a shift functor $\Sigma : \mathcal{C}\text{-Mod} \rightarrow \mathcal{C}\text{-Mod}$, and for every \mathcal{C} -module V , there is a natural homomorphism $\Theta_{[d]} : V^{\oplus d} \rightarrow \Sigma V$.
- (2) For every nonzero \mathcal{C} -module V , one has $\text{gd}(DV) = \text{gd}(V) - 1$.
- (3) If k is a commutative Noetherian ring, then every finitely generated \mathcal{C} -module is Noetherian.
- (4) For every short exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ of \mathcal{C} -modules, if $\text{reg}(U)$ and $\text{reg}(W)$ are finite, so is $\text{reg}(V)$.
- (5) For every \mathcal{C} -module V and a nil maximal subset $S \subset [d]$ for V , if $\text{reg}(D_S V)$ is finite, so is $\text{reg}(V)$.

Replacing the property of finite regularity by an arbitrary property (P), one can use the same inductive argument to prove:

Theorem 6.1. *Suppose that k is a commutative Noetherian ring. Let (P) be a property of certain finitely generated \mathcal{C} -modules such that the zero \mathcal{C} -module has (P). Then every finitely generated \mathcal{C} -module has (P) if and only if the following two conditions hold:*

- (4') For every short exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ of finitely generated \mathcal{C} -modules, if both U and W have (P), so does V .
- (5') For every finitely generated \mathcal{C} -module V and a nil maximal subset $S \subset [d]$ for V , if $D_S V$ has (P), so does V .

The inductive argument can also be applied to other categories which possess the properties (1), (2) and (3) above, for example, the categories OI_d , FI^m , and OI^m , which we briefly discuss below. For each of these examples, we have an analogue of Theorem 1.1 and an analogue of Theorem 6.1.

Example 6.2 (The category OI_d). The category OI_d introduced by Sam and Snowden [8] is defined as follows. The objects of OI_d are the totally ordered finite sets. The morphisms are the pairs (f, δ) where f is any order-preserving injection and δ is any d -coloring on the complement of the image of f . Sam and Snowden [8, Theorem 7.1.2] proved that any finitely generated OI_d -module over a commutative Noetherian ring is Noetherian. There is a shift functor Σ for OI_d -modules (see [4, Example 5.8]) and it is easy to see that:

$$\Sigma M(X) \cong M(X)^{\oplus d} \oplus M(X \setminus \{x_1\}) \quad \text{for every object } X = \{x_1 < x_2 < \cdots < x_n\}.$$

Using this shift functor, one can prove in the same way that any finitely generated OI_d -module over a commutative Noetherian ring has finite regularity.

Example 6.3 (The category FI^m). The category FI^m is the product of m copies of the category FI ; it was introduced by Gadish in [3]. Its objects are m -tuples of finite sets (X_1, X_2, \dots, X_m) , and its morphisms are m -tuples of injections (f_1, f_2, \dots, f_m) . Using the rank function $(X_1, \dots, X_m) \mapsto |X_1| + \cdots + |X_m|$, one can define the homological degrees of FI^m -modules. In [7], Yu and the second author proved that any finitely generated FI^m -module over a commutative Noetherian ring is Noetherian. Moreover, the category of FI^m -modules has m distinct shift functors Σ_i , $i \in [m]$. Define the functor Σ from the category of FI^m -modules to itself by $\Sigma V = \Sigma_1 V \oplus \cdots \oplus \Sigma_m V$ for every FI^m -module V . One has ([7, Lemma 2.3]):

$$\Sigma M(X_1, \dots, X_m) \cong M(X_1, \dots, X_m)^{\oplus m} \oplus \left(\bigoplus_{i=1}^m \bigoplus_{x \in X_i} M(X_1, \dots, X_{i-1}, X_i \setminus \{x\}, X_{i+1}, \dots, X_m) \right).$$

Using the functor Σ , one can prove that any finitely generated FI^m -module over a commutative Noetherian ring has finite regularity.

Example 6.4 (The category OI^m). The category OI^m is the product of m copies of the category OI . In [7, Subsection 1.8], Yu and the second author explained that any finitely generated OI^m -module over a commutative Noetherian ring is Noetherian. Similarly to FI^m , the category of OI^m -modules has m distinct shift functors Σ_i , $i \in [m]$, and we define the functor Σ by $\Sigma V = \Sigma_1 V \oplus \cdots \oplus \Sigma_m V$ for every OI^m -module V . One can check that:

$$\Sigma M(X_1, \dots, X_m) \cong M(X_1, \dots, X_m)^{\oplus m} \oplus \left(\bigoplus_{i=1}^m M(X_1, \dots, X_{i-1}, X_i \setminus \{x_i\}, X_{i+1}, \dots, X_m) \right),$$

where x_i denotes the least element of X_i . Using the functor Σ , one can prove that any finitely generated OI^m -module over a commutative Noetherian ring has finite regularity.

REFERENCES

- [1] Church, Thomas; Ellenberg, Jordan S. Homology of FI -modules. To appear in *Geom. Topol.* arXiv:1506.01022.
- [2] Church, Thomas; Ellenberg, Jordan S.; Farb, Benson; Nagpal, Rohit. FI -modules over Noetherian rings. *Geom. Topol.* 18-5 (2014) 2951-2984. arXiv:1210.1854.
- [3] Gadish, Nir. Categories of FI type: a unified approach to generalizing representation stability and character polynomials. *J. Algebra* 480 (2017), 450-486. arXiv:1608.02664.
- [4] Gan, Wee Liang; Li, Liping. Koszulity of directed categories in representation stability theory. arXiv:1411.5308.
- [5] Gan, Wee Liang; Li, Liping. An inductive machinery for representations of categories with shift functors. arXiv:1610.09081.
- [6] Li, Liping; Yu, Nina. Filtrations and homological degrees of FI -modules. *J. Algebra* 472 (2017), 369-398. arXiv:1511.02977.
- [7] Li, Liping; Yu, Nina. FI^m -modules over Noetherian rings. Preprint.
- [8] Sam, Steven V.; Snowden, Andrew. Gröbner methods for representations of combinatorial categories. *J. Amer. Math. Soc.* 30 (2017), 159-203. arXiv:1409.1670.
- [9] Sam, Steven V.; Snowden, Andrew. GL -equivariant modules over polynomial rings in infinitely many variables. II. arXiv:1703.04516.
- [10] Sam, Steven V.; Snowden, Andrew. Regularity bounds for twisted commutative algebras. arXiv:1704.01630.

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