BOUNDS ON HOMOLOGICAL INVARIANTS OF VI-MODULES

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Abstract. We give bounds for various homological invariants (including Castelnuovo-Mumford regularity, degrees of local cohomology, and injective dimension) of finitely generated VI-modules in the non-describing characteristic case. It turns out that the formulas of these bounds for VI-modules are the same as the formulas of corresponding bounds for FI-modules.

1. Introduction

Let $\mathbf{k}$ be a commutative Noetherian ring, and let F be a finite field whose order q is invertible in $\mathbf{k}$ (called the non-describing characteristic case). The category VI has the finite dimensional F-vector spaces as its objects and the injective F-linear maps as its morphisms. By definition, a VI-module is a covariant functor from VI to the category of $\mathbf{k}$-modules.

The purpose of this paper is to prove bounds for various homological invariants of finitely generated VI-modules; these homological invariants have been shown to be finite in a recent paper [12] of Nagpal. Surprisingly, though the combinatorial structure and representation theory of VI seem to be more complicated than that of the category FI, whose objects are finite sets and morphisms are injections, homological invariants of VI-modules and FI-modules are bounded by the same formulas. That is, many results on FI-modules proved in [1, 2, 3, 8] can be extended to VI-modules. In particular, we obtain upper bounds of Castelnuovo-Mumford regularity, degrees of local cohomology, and injective dimension of finitely generated VI-modules M, and these upper bounds are in terms of the first two homological degrees of M, which measure the degrees of the generators and relations of M.

The key ingredients in our arguments are the shift theorem of Nagpal [12, Theorem 4.34(a)] and (a modification of) the approach for FI-modules described in [8, 10] (in particular, we avoid an argument in the proof of [8, Theorem 2.4] which used a result [10, Corollary 2.12] that may not hold for VI-modules). For FI-modules, other approaches have been found by various authors [1, 2, 3], but at this moment it is not obvious to us if their arguments can also be adapted to VI-modules.

We collect the main results of this paper in the following theorem. For definitions and notations, see the next section.

Theorem 1.1. *Let $\mathbf{k}$ be a commutative Noetherian ring such that q is invertible in $\mathbf{k}$. Let M be a finitely generated* VI*-module over $\mathbf{k}$. Let $t_i(M)$ be the degree of the i-th* VI*-homology $\mathrm{H}_i^{\mathrm{VI}}(M)$ of M. Let $h_i(M)$ be the degree of the i-th local cohomology $\mathrm{R}^i\,\Gamma(M)$ of M. Then one has:*

1. $h_0(M) \leqslant t_0(M) + t_1(M) - 1$ *and* $h_i(M) \leqslant 2t_0(M) - 2i$ *for* $i \geqslant 1$.
2. *The Castelnuovo-Mumford regularity of M satisfies:*
$$\mathrm{reg}(M) \leqslant \max\{h_0(M), 2t_0(M) - 1\};$$
$$\mathrm{reg}(M) \leqslant t_0(M) + t_1(M) - 1.$$
3. $\Sigma^X M$ *is semi-induced if* $\dim_F(X) \geqslant t_0(M) + t_1(M)$.

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(4) If $\mathbf{k}$ *is a field of characteristic 0, then the injective dimension of* M *satisfies:*

$$\operatorname{inj}\dim(M) \leqslant \max\{t_0(M), t_0(M)+t_1(M)-1\}.$$

From this theorem one can deduce more bounds, all of which are in terms of $t_0(M)$ and $t_1(M)$, for a finitely generated VI-module M; for example:

- if $\mathbf{k}$ is a field, then $\dim_{\mathbf{k}}(M(F^n))$ is a polynomial in q^n for $n \geqslant t_0(M)+t_1(M)$;
- if $\mathbf{k}$ is an algebraically closed field of characteristic zero, then the sequence $\{M(F^n)\}$ of $\mathbf{k}[\mathbf{GL}_n(F)]$-modules is representation stable in the sense of [6, Definition 1.5] starting at $n = \max\{t_0(M)+t_1(M), 2t_0(M)\}$.

This paper is, admittedly, not self-contained since our main goal is to explain how the shift theorem in [12] and the approach in [8, 10] can be used to get explicit bounds for homological invariants of finitely generated VI-modules; we avoid repeating definitions and preliminary results in those papers unless it is necessary. However, we do try to make it readable for anyone who is somewhat familiar with any of the existing papers on homology of FI-modules or VI-modules. It is recommended to read [12] before turning to the present paper.

The paper is organized as follows. In Section 2, we introduce the notations and prove some preliminary results which are used later in the paper. Formulas for upper bounds of Castelnuovo-Mumford regularity and injective dimension are proved in Section 3. In the last section, we recall the construction of a finite complex given by the shift theorem in [12]; it was shown in [12] that this complex computes local cohomology. Using results from the previous sections, we obtain upper bounds of degrees of local cohomology and certain other invariants.

2. Generalities

For convenience and to avoid confusion due to differences in notations, we adopt the notations of [12], but our convention for the degree of the zero VI-module is $-\infty$. In particular, for any VI-module M and $X \in \mathrm{Ob}(\mathrm{VI})$, we denote:

M_n : the $\mathbf{k}[\mathbf{GL}_n(F)]$-module $M(F^n)$;
$M_{\prec d}$: the smallest VI-submodule of M containing M_n for all $n < d$;
$\Gamma(M)$: the maximal torsion submodule of M;
$\mathrm{R}^i\,\Gamma$: the i-th right derived functor of Γ;
$h_i(M)$: the degree of $\mathrm{R}^i\,\Gamma(M)$;
$\mathrm{H}_0^{\mathrm{VI}}$: the VI-homology functor defined by $\mathrm{H}_0^{\mathrm{VI}}(M)(X) = (M/M_{\prec d})(X)$ where $d = \dim_F(X)$;
$\mathrm{H}_i^{\mathrm{VI}}$: the i-th left derived functor of $\mathrm{H}_0^{\mathrm{VI}}$;
$t_i(M)$: the degree of $\mathrm{H}_i^{\mathrm{VI}}(M)$;
Σ^X : the functor defined by $(\Sigma^X M)(Z) = M(X+Z)$ for every $Z \in \mathrm{Ob}(\mathrm{VI})$;
Δ^X : the cokernel of the natural transformation $\mathrm{id} \to \Sigma^X$;
$\mathbf{U}_X$: the VI-group defined in [12, §4.2];
$\bar{\Sigma}^X$: the functor defined by $\bar{\Sigma}^X M = \left(\Sigma^X M\right)_{\mathbf{U}_X}$;
$\bar{\Delta}^X$: the functor defined by $\bar{\Delta}^X M = \left(\Delta^X M\right)_{\mathbf{U}_X}$;
κ^X : the kernel of the natural transformation $\mathrm{id} \to \bar{\Sigma}^X$;
$\mathcal{I}(V)$: the VI-module induced from a VB-module V,

where VB is the category whose objects are the finite dimensional F-vector spaces and whose morphisms are the bijective F-linear maps.

Let us mention that $t_i(M)$ here are called *homological degrees* and denoted by $\mathrm{hd}_i(M)$ in [8, 10]; in particular, $t_0(M)$ coincides with the *generating degree* $\mathrm{gd}(M) = \mathrm{hd}_0(M)$ of M. The degree $h_0(M)$ of $\Gamma(M)$ here is called *torsion degree* $\mathrm{td}(M)$ of M in [8, 10].

We remind the reader that finitely generated VI-modules are Noetherian ([13, Theorem A] and [14, Corollary 8.3.3]).

First, we show how to make a simple but useful reduction: for every finitely generated VI-module M, one can find a submodule M' such that:

- $t_0(M') < t_1(M')$,
- M/M' is *semi-induced* (see [12, §3.1]).

The positive VI-homology and all local cohomology of any semi-induced VI-module vanish ([12, Theorem 1.3 and Proposition 3.1]). Consequently, for many of our purposes, we can replace M by M' and hence assume that $t_0(M) < t_1(M)$. For FI-modules, a similar reduction was used in [8, 10].

Lemma 2.1. *Let M be a* VI*-module generated in degree d. If $t_1(M) \leqslant d$, then M is induced from d.*

Proof. Let $P = \mathcal{I}(M_d)$ and let N be the kernel of the natural surjection $P \to M$. The short exact sequence $0 \to N \to P \to M \to 0$ gives a long exact sequence

$$\cdots \to \mathrm{H}_1^{\mathrm{VI}}(M) \to \mathrm{H}_0^{\mathrm{VI}}(N) \to \mathrm{H}_0^{\mathrm{VI}}(P) \to \mathrm{H}_0^{\mathrm{VI}}(M) \to 0.$$

Since the map $\mathrm{H}_0^{\mathrm{VI}}(P) \to \mathrm{H}_0^{\mathrm{VI}}(M)$ is an isomorphism, it follows that the map $\mathrm{H}_1^{\mathrm{VI}}(M) \to \mathrm{H}_0^{\mathrm{VI}}(N)$ is surjective. Hence $t_0(N) \leqslant t_1(M) \leqslant d$. But $N_n = 0$ for every $n \leqslant d$. Therefore $N = 0$. □

Lemma 2.2. *Let M be a* VI*-module presented in finite degrees. If $d \geqslant t_1(M)$, then $Q := M_{\preceq d}/M_{\prec d}$ is induced from d, and one has canonical isomorphims:*

$$\Gamma(M_{\prec d}) = \Gamma(M),$$
$$\mathrm{H}_i^{\mathrm{VI}}(M_{\prec d}) = \mathrm{H}_i^{\mathrm{VI}}(M) \quad \text{for every } i \geqslant 1.$$

In particular, if $t_0(M) \geqslant t_1(M)$ and $d = t_1(M)$, then $t_0(M_{\prec d}) < t_1(M_{\prec d})$ and $M/M_{\prec d}$ is semi-induced.

Proof. If $d > t_0(M)$, then $M_{\prec d} = M$ and we are done. We prove the lemma by downwards induction on d. The short exact sequence $0 \to M_{\prec d} \to M_{\preceq d} \to Q \to 0$ gives a long exact sequence

$$\cdots \to \mathrm{H}_1^{\mathrm{VI}}(M_{\preceq d}) \to \mathrm{H}_1^{\mathrm{VI}}(Q) \to \mathrm{H}_0^{\mathrm{VI}}(M_{\prec d}) \to \cdots .$$

By induction hypothesis, $t_1(M_{\preceq d}) = t_1(M)$, so $t_1(Q) \leqslant d$. By Lemma 2.1, we deduce that Q is induced from d. It now follows easily from [12, Propositions 3.4 and 4.21] and induction hypothesis that:

$$\Gamma(M_{\prec d}) = \Gamma(M_{\preceq d}) = \Gamma(M),$$
$$\mathrm{H}_i^{\mathrm{VI}}(M_{\prec d}) = \mathrm{H}_i^{\mathrm{VI}}(M_{\preceq d}) = \mathrm{H}_i^{\mathrm{VI}}(M) \quad \text{for every } i \geqslant 1.$$

□

Remark 2.3. Let M be a VI-module presented in finite degrees. By [12, Proposition 3.8], there is a short exact sequence $0 \to N \to P \to M \to 0$ such that P is an induced VI-module with $t_0(P) = t_0(M)$. By [12, Proposition 3.4], we obtain the exact sequence:

$$0 \to \mathrm{H}_1^{\mathrm{VI}}(M) \to \mathrm{H}_0^{\mathrm{VI}}(N) \to \mathrm{H}_0^{\mathrm{VI}}(P) \to \mathrm{H}_0^{\mathrm{VI}}(M) \to 0.$$

So,

$$t_1(M) \leqslant t_0(N) \leqslant \max\{t_0(P), t_1(M)\} = \max\{t_0(M), t_1(M)\}.$$

In particular, if $t_0(M) \leqslant t_1(M)$, then $t_0(N) = t_1(M)$.

Given a finitely generated VI-module M and $X \in \mathrm{Ob}(\mathrm{VI})$, we have an exact sequence:

$$0 \to \kappa^X M \to M \to \bar{\Sigma}^X M \to \bar{\Delta}^X M \to 0.$$

The analogue of this sequence has played a central role in the representation theory of FI, and as we will see, it also plays a crucial role for us to obtain bounds of homological invariant of VI-modules. The next lemmas give us some preliminary results related to this sequence.

Lemma 2.4. *Let $X \in \mathrm{Ob}(\mathrm{VI})$, and let M be a* VI*-module presented in finite degrees. If $\dim_F(X) > 0$ and $t_0(M) \leqslant t_1(M)$, then $t_1(\bar{\Delta}^X M) \leqslant t_1(M) - 1$.*

Proof. By Remark 2.3, there is a short exact sequence $0 \to N \to P \to M \to 0$ such that P is an induced VI-module with $t_0(P) = t_0(M)$, and $t_0(N) = t_1(M)$. This gives an exact sequence

$$\bar{\Delta}^X N \to \bar{\Delta}^X P \to \bar{\Delta}^X M \to 0.$$

By [12, Corollary 4.19], one has $t_0(\bar{\Delta}^X N) \leqslant t_0(N) - 1$. Let K be the kernel of the map $\bar{\Delta}^X P \to \bar{\Delta}^X M$. Then

$$t_0(K) \leqslant t_0(\bar{\Delta}^X N) \leqslant t_0(N) - 1 = t_1(M) - 1.$$

The short exact sequence $0 \to K \to \bar{\Delta}^X P \to \bar{\Delta}^X M \to 0$ gives a long exact sequence

$$\cdots \to \mathrm{H}_1^{\mathrm{VI}}(\bar{\Delta}^X P) \to \mathrm{H}_1^{\mathrm{VI}}(\bar{\Delta}^X M) \to \mathrm{H}_0^{\mathrm{VI}}(K) \to \cdots.$$

By [12, Proposition 4.12], we know that $\bar{\Delta}^X P$ is induced, so by [12, Proposition 3.4], we have $\mathrm{H}_1^{\mathrm{VI}}(\bar{\Delta}^X P) = 0$. Therefore,

$$t_1(\bar{\Delta}^X M) \leqslant t_0(K) \leqslant t_1(M) - 1.$$

□

By definition, $\kappa^X M$ is the kernel of the natural map $M \to \bar{\Sigma}^X M$. The next lemma asserts that $\kappa^X M$ coincides with the kernel of another natural map $M \to \Sigma^X M$.

Lemma 2.5. *For any $X \in \mathrm{Ob}(\mathrm{VI})$, the kernel of the natural transformation* $\mathrm{id} \to \Sigma^X$ *is κ^X.*

Proof. Let M be any VI-module. The $\mathbf{U}_X$-action on M is trivial and the natural map $M \to \Sigma^X M$ is $\mathbf{U}_X$-equivariant. Since the order of the group $\mathbf{U}_X(Z)$ is invertible in $\mathbf{k}$ for every $Z \in \mathrm{Ob}(\mathrm{VI})$, the lemma follows. □

As in [12], we drop the superscript X in Σ^X, Δ^X, $\bar{\Sigma}^X$, $\bar{\Delta}^X$ and κ^X when $\dim_F(X) = 1$.

We now consider κM. By the previous lemma, κM is also the kernel of the natural map $M \to \Sigma M$. Explicitly, for any object Z of VI, the value $(\kappa M)(Z)$ of κM on Z consists of those elements $v \in M(Z)$ such that $\iota_*(v) = 0$ for the standard inclusion $\iota : Z \to X + Z$ (where $X + Z$ is the direct sum of Z with the one-dimensional F-vector space X used to define the shift functor Σ). Since the group $\mathbf{GL}(X + Z)$ acts transitively from the left on the set of morphisms from Z to $X + Z$, one knows that:

$$(\kappa M)(Z) = \{v \in M(Z) \mid f_*(v) = 0 \text{ for every morphism } f : Z \to X + Z\}.$$

Therefore, $(\kappa M)(Z)$ is a VI-submodule of M, and one has the following decomposition:

$$\kappa M = \bigoplus_{Z \in \mathrm{Ob}(\mathrm{VI})} (\kappa M)(Z)$$

as VI-modules. From this observation, one deduces the following simple but key fact. (A similar fact for FI-modules was shown in [8, Lemma 2.1].)

Lemma 2.6. *For any* VI*-module M, one has $h_0(M) = t_0(\kappa M)$.*

Proof. By Lemma 2.5 and the above observation, one has $h_0(M) = \deg(\kappa M) = t_0(\kappa M)$. □

We also need the following technical lemma to establish our main results.

Lemma 2.7. *Let M be a* VI*-module and $X \in \mathrm{Ob}(\mathrm{VI})$. If $\bar{\Sigma}^X M$ is semi-induced, then:*

$$\kappa^X M = \Gamma(M),$$
$$t_1(\bar{\Delta}^X M) \leqslant t_0(M),$$
$$t_{i+1}(\bar{\Delta}^X M) = t_i(M/\Gamma(M)) \quad \textit{for each } i \geqslant 1.$$

Proof. By [12, Proposition 4.27 and Corollary 4.22], we have $\bar{\Sigma}^X \Gamma(M) = \Gamma(\bar{\Sigma}^X M) = 0$. Hence, $\kappa^X \Gamma(M) = \Gamma(M)$. Since $M/\Gamma(M)$ is torsion-free, the map $M/\Gamma(M) \to \Sigma^X(M/\Gamma(M))$ is injective, so by [12, Lemma 4.7], the map $M/\Gamma(M) \to \bar{\Sigma}^X(M/\Gamma(M))$ is injective. Hence, $\kappa^X(M/\Gamma(M)) = 0$. Since κ^X is a left exact functor, it follows that $\kappa^X M = \Gamma(M)$.

By [12, Proposition 3.10], we know that $\bar{\Sigma}^X M$ is homology acyclic. The short exact sequence

$$0 \to M/\Gamma(M) \to \bar{\Sigma}^X M \to \bar{\Delta}^X M \to 0$$

gives a long exact sequence:

$$\cdots \to 0 \to \mathrm{H}_2^{\mathrm{VI}}(\bar{\Delta}^X M) \to \mathrm{H}_1^{\mathrm{VI}}(M/\Gamma(M)) \to 0 \to \mathrm{H}_1^{\mathrm{VI}}(\bar{\Delta}^X M) \to \mathrm{H}_0^{\mathrm{VI}}(M/\Gamma(M)) \to \cdots.$$

Hence, $t_1(\bar{\Delta}^X M) \leqslant t_0(M/\Gamma(M)) \leqslant t_0(M)$, and $t_{i+1}(\bar{\Delta}^X M) = t_i(M/\Gamma(M))$ for each $i \geqslant 1$. □

3. Bounds on regularity and injective dimensions

The main task of this section is to prove formulas for upper bounds of Castelnuovo-Mumford regularity and, when $\mathbf{k}$ is a field of characteristic zero, a formula for the injective dimension. First, let us make the following definition:

Definition 3.1. For any VI-module M, the *Castelnuovo-Mumford regularity* of M, denoted by $\mathrm{reg}(M)$, is the supremum of $\{t_i(M) - i : i \geqslant 1\}$.

An upper bound (in terms of $t_0(M)$ and $t_1(M)$) of Castelnuovo-Mumford regularity of FI-modules presented in finite degrees was first obtained by Church and Ellenberg in [2]. In [8], the second author provided a new proof as well as another upper bound in terms of $t_0(M)$ and $h_0(M)$. Subsequently, an alternative proof was given by the first author in [3]. Motivated by the work of [8] and [10], in a short note [1], Church simplified the original proof in [2]. The following theorem asserts that all these upper bounds for FI-modules also hold for finitely generated VI-modules. Its proof uses the shift theorem of Nagpal [12] for finitely generated VI-modules; we do not know if the shift theorem and our results below can be extended to VI-modules presented in finite degrees.

Theorem 3.2. *Let M be a finitely generated* VI*-module. Then:*

$$h_0(M) \leqslant t_0(M) + t_1(M) - 1,$$
$$\mathrm{reg}(M) \leqslant \max\{h_0(M),\, 2t_0(M) - 1\},$$
$$\mathrm{reg}(M) \leqslant t_0(M) + t_1(M) - 1.$$

Proof. If $M = 0$, then $h_0(M)$ and $\mathrm{reg}(M)$ are both $-\infty$, so there is nothing to prove. Suppose $M \neq 0$. We use induction on $t_0(M)$.

We may assume that $t_0(M) < t_1(M)$ for by Lemma 2.2, if $t_0(M) \geqslant t_1(M)$, we can replace M by $M_{\prec d}$ where $d = t_1(M)$. Hence, by Remark 2.3, there is a short exact sequence $0 \to N \to P \to M \to 0$ such that P is an induced VI-module with $t_0(P) = t_0(M)$, and $t_0(N) = t_1(M)$. By [12, Propositions 4.8 and 4.21], we have $\kappa P = 0$. Therefore, by [12, Proposition 4.17], we have an exact sequence:

$$0 \to \kappa M \to \bar{\Delta} N \to \bar{\Delta} P \to \bar{\Delta} M \to 0.$$

We break this into two short exact sequences:

$$0 \to \kappa M \to \bar{\Delta} N \to \bar{\Delta} N/\kappa M \to 0,$$
$$0 \to \bar{\Delta} N/\kappa M \to \bar{\Delta} P \to \bar{\Delta} M \to 0.$$

They give two long exact sequences:

$$\cdots \to \mathrm{H}_1^{\mathrm{VI}}(\bar{\Delta} N/\kappa M) \to \mathrm{H}_0^{\mathrm{VI}}(\kappa M) \to \mathrm{H}_0^{\mathrm{VI}}(\bar{\Delta} N) \to \cdots, \tag{1}$$
$$\cdots \to \mathrm{H}_2^{\mathrm{VI}}(\bar{\Delta} M) \to \mathrm{H}_1^{\mathrm{VI}}(\bar{\Delta} N/\kappa M) \to 0 \to \cdots, \tag{2}$$

where we used [12, Propositions 3.4 and 4.5] to see that $\mathrm{H}_1^{\mathrm{VI}}(\bar{\Delta} P) = 0$. By [12, Corollary 4.19], we have:

$$t_0(\bar{\Delta} N) \leqslant t_0(N) - 1 = t_1(M) - 1, \tag{3}$$
$$t_0(\bar{\Delta} M) \leqslant t_0(M) - 1. \tag{4}$$

By Lemma 2.4, we have:

$$t_1(\bar{\Delta} M) \leqslant t_1(M) - 1. \tag{5}$$

Hence,

$$\begin{aligned}
h_0(M) &= t_0(\kappa M) && \text{by Lemma 2.6}\\
&\leqslant \max\{t_0(\bar{\Delta} N), t_1(\bar{\Delta} N/\kappa M)\} && \text{by (1)}\\
&\leqslant \max\{t_1(M) - 1, t_2(\bar{\Delta} M)\} && \text{by (2) and (3)}\\
&\leqslant \max\{t_1(M) - 1, \mathrm{reg}(\bar{\Delta} M) + 2\}\\
&\leqslant \max\{t_1(M) - 1, t_0(\bar{\Delta} M) + t_1(\bar{\Delta} M) + 1\} && \text{by induction hypothesis}\\
&\leqslant \max\{t_1(M) - 1, (t_0(M) - 1) + (t_1(M) - 1) + 1\} && \text{by (4) and (5)}\\
&\leqslant t_0(M) + t_1(M) - 1.
\end{aligned}$$

Next, by [12, Theorem 4.34(a)], if X is a finite dimensional F-vector space whose dimension is sufficiently large, then $\bar{\Sigma}^X M$ is semi-induced; we choose and fix such a nonzero X. From the long exact sequence associated to the short exact sequence $0 \to \Gamma(M) \to M \to M/\Gamma(M) \to 0$, we see that:

$$\mathrm{reg}(M) \leqslant \max\{\mathrm{reg}(\Gamma(M)), \mathrm{reg}(M/\Gamma(M))\}.$$

By [7, Corollary 5.1], we have:

$$\mathrm{reg}(\Gamma(M)) \leqslant h_0(M).$$

By [12, Corollary 4.19], we have:

$$t_0(\bar{\Delta}^X M) \leqslant t_0(M) - 1. \tag{6}$$

Hence,

$$\begin{aligned}
\mathrm{reg}(M/\Gamma(M)) &\leqslant \mathrm{reg}(\bar{\Delta}^X M) + 1 && \text{by Lemma 2.7}\\
&\leqslant t_0(\bar{\Delta}^X M) + t_1(\bar{\Delta}^X M) && \text{by induction hypothesis}\\
&\leqslant 2t_0(M) - 1 && \text{by Lemma 2.7 and (6).}
\end{aligned}$$

It follows from the above that:

$$\mathrm{reg}(M) \leqslant \max\{h_0(M),\ 2t_0(M) - 1\}. \tag{7}$$

Finally, since we have:

$$h_0(M) \leqslant t_0(M) + t_1(M) - 1,$$
$$2t_0(M) - 1 < t_0(M) + t_1(M) - 1,$$

it follows from (7) that $\mathrm{reg}(M) \leqslant t_0(M) + t_1(M) - 1$. □

Remark 3.3. The finiteness of Castelnuovo-Mumford regularity of finitely generated VI-modules in non-describing characteristic case was proved by Nagpal in [12, Theorem 5.13]. Furthermore, he gave an upper bound of this invariant in terms of the degrees of the local cohomology of M which in general are hard to obtain.

Remark 3.4. Let us also mention that for a closely related category VIC, Miller and Wilson [11, Theorem 2.26] proved an upper bound for the degrees of syzygies of VIC-modules over a field of characteristic zero; their bound grow exponentially.

In the rest of this section, we study the injective dimension of finitely generated VI-modules when $\mathbf{k}$ is a field of characteristic zero. For any VI-module M, denote by $\operatorname{inj}\dim(M)$ the injective dimension of M. We adopt the convention that the zero module has injective dimension $-\infty$.

First, we consider the special case of finitely generated torsion VI-modules M; in this special case, there is a non-negative integer N such that $M(Z)=0$ for all objects Z with $\dim_F(Z)>N$.

Lemma 3.5. *Assume that* $\mathbf{k}$ *is a field of characteristic zero. Let* M *be a finitely generated torsion* VI-*module. Then* $\operatorname{inj}\dim(M)\leqslant \deg(M)$.

Proof. As shown in the proof of [5, Lemma 3.2], there exists an exact sequence

$$0\to M\to E^0\to E^1\to\cdots\to E^n\to 0$$

where each E^i is a finitely generated torsion injective VI-module such that $\deg(E^i)\leqslant\deg(M)-i$. Now the conclusion follows by an easy induction. □

Now we are ready to extend the result of [8, Corollary 3.7] for FI-modules to VI-modules.

Corollary 3.6. *Assume that* $\mathbf{k}$ *is a field of characteristic zero. Let* M *be a finitely generated* VI-*module. Then:*

$$\operatorname{inj}\dim(M)\leqslant \max\{h_0(M),\, t_0(M),\, 2t_0(M)-1\}, \tag{8}$$

$$\operatorname{inj}\dim(M)\leqslant \max\{t_0(M),\, t_0(M)+t_1(M)-1\}. \tag{9}$$

Proof. We use induction on $t_0(M)$. The base case of the zero module is trivial. Suppose M is nonzero. We have:

$$\operatorname{inj}\dim(M)\leqslant \max\{\operatorname{inj}\dim(\Gamma(M)),\, \operatorname{inj}\dim(M/\Gamma(M))\}.$$

By Lemma 3.5,

$$\operatorname{inj}\dim(\Gamma(M))\leqslant h_0(M).$$

By [12, Theorem 4.34(a)], we can choose a nonzero $X\in \operatorname{Ob}(\mathrm{VI})$ such that $\bar{\Sigma}^X M$ is semi-induced. Since $\mathbf{k}$ is a field characteristic zero, $\bar{\Sigma}^X M$ is injective (by [4, Theorem 1.5]). By Lemma 2.7, we have a short exact sequence $0\to M/\Gamma(M)\to\bar{\Sigma}^X M\to\bar{\Delta}^X M\to 0$. Hence,

$$\begin{aligned}\operatorname{inj}\dim(M/\Gamma(M)) &\leqslant \max\{0,\operatorname{inj}\dim(\bar{\Delta}^X M)+1\}\\ &\leqslant \max\{t_0(M),\operatorname{inj}\dim(\bar{\Delta}^X M)+1\}.\end{aligned}$$

By [12, Corollary 4.19], $t_0(\bar{\Delta}^X M)\leqslant t_0(M)-1$. By Lemma 2.7, $t_1(\bar{\Delta}^X M)\leqslant t_0(M)$. By Theorem 3.2,

$$h_0(\bar{\Delta}^X M)\leqslant t_0(\bar{\Delta}^X M)+t_1(\bar{\Delta}^X M)-1\leqslant 2t_0(M)-2.$$

Hence, by induction hypothesis, we have:

$$\begin{aligned}\operatorname{inj}\dim(\bar{\Delta}^X M)+1 &\leqslant \max\{h_0(\bar{\Delta}^X M),t_0(\bar{\Delta}^X M),2t_0(\bar{\Delta}^X M)-1\}+1\\ &\leqslant \max\{t_0(M),2t_0(M)-1\}.\end{aligned}$$

Putting the above inequalities together, we deduce (8).

We now prove (9). We may suppose M is nonzero. By Theorem 3.2, we have $h_0(M) \leqslant t_0(M) + t_1(M) - 1$. We consider two cases.

Case 1: $t_0(M) < t_1(M)$. Then $2t_0(M) - 1 < t_0(M) + t_1(M) - 1$, and we are done by (8).

Case 2: $t_0(M) \geqslant t_1(M)$. By Lemma 2.2, we reduce to the previous case by replacing M by $M_{\prec d}$ where $d = t_1(M)$, so we are also done. Indeed, from Lemma 2.2, we know that $M/M_{\prec d}$ is semi-induced, hence injective (by [4, Theorem 1.5]), and hence:

$$\operatorname{inj\,dim}(M) \leqslant \max\{0, \operatorname{inj\,dim}(M_{\prec d})\} \leqslant \max\{t_0(M), \operatorname{inj\,dim}(M_{\prec d})\}.$$

□

Remark 3.7. The finiteness of injective dimension of finitely generated VI-modules over fields of characteristic zero was proved in [12, Theorem 5.25] but no explicit upper bound was given there.

4. Bounds on local cohomology

In this section, we give upper bounds for degrees of the local cohomology of finitely generated VI-modules. We first recall, for any finitely generated VI-module M, the construction of a complex $I^\bullet$ in the proof of [12, Theorem 4.34(b)].

Let $I^0 = M$ and let $C^0 = M$. For each $i \geqslant 1$:

- choose a nonzero $X_i \in \mathrm{Ob}(\mathrm{VI})$ such that $\bar{\Sigma}^{X_i} C^{i-1}$ is semi-induced;
- let $I^i = \bar{\Sigma}^{X_i} C^{i-1}$ and let $C^i = \bar{\Delta}^{X_i} C^{i-1}$;
- let $d^{i-1} : I^{i-1} \to I^i$ be the composition of the natural maps: $I^{i-1} \to C^{i-1} \to I^i$.

The existence of $X_1, X_2, \ldots$ is guaranteed by [12, Theorem 4.34(a)]. By construction, I^i is semi-induced for every $i \geqslant 1$. By Lemma 2.7, we have an exact sequence

$$0 \to \Gamma(C^{i-1}) \to C^{i-1} \to I^i \to C^i \to 0.$$

Hence, $d^i \circ d^{i-1} = 0$ and $\mathrm{H}^i(I^\bullet) = \Gamma(C^i)$.

By [12, Corollary 4.19], we have

$$t_0(C^i) \leqslant t_0(M) - i \quad \text{for each } i \geqslant 0. \tag{10}$$

Hence, $C^i = 0$ when $i > t_0(M)$, and $I^i = 0$ when $i > t_0(M) + 1$. In particular, $I^\bullet$ is a finite complex.

Each cohomology $\mathrm{H}^i(I^\bullet)$ is a VI-module. The following theorem establishes an upper bound for the degree of the VI-module $\mathrm{H}^i(I^\bullet)$. The FI counterpart of this result was proved in [8, Proposition 3.1] (with a slight difference because a different indexing was used there).

Theorem 4.1. *Let M be a finitely generated* VI*-module, and let $I^\bullet$ be the complex constructed above. Then:*

$$\deg \mathrm{H}^0(I^\bullet) = h_0(M),$$
$$\deg \mathrm{H}^i(I^\bullet) \leqslant 2t_0(M) - 2i \quad \textit{for each } i \geqslant 1.$$

Proof. We have $\deg \mathrm{H}^0(I^\bullet) = \deg \Gamma(C^0) = h_0(M)$. For each $i \geqslant 1$, we have:

$$\begin{aligned}
\deg \mathrm{H}^i(I^\bullet) &= \deg \Gamma(C^i) \\
&= h_0(C^i) \\
&\leqslant t_0(C^i) + t_1(C^i) - 1 && \text{by Theorem 3.2} \\
&\leqslant t_0(C^i) + t_0(C^{i-1}) - 1 && \text{by Lemma 2.7} \\
&\leqslant (t_0(M) - i) + (t_0(M) - i + 1) - 1 && \text{by (10)} \\
&= 2t_0(M) - 2i.
\end{aligned}$$

□

Nagpal [12] proved that the complex $I^\bullet$ computes the local cohomology of M. (For FI-modules, this was proved by Ramos and the second author in [9].) Therefore, one has:

Corollary 4.2. *Let M be a finitely generated* VI*-module. Then:*

$$h_i(M) \leqslant 2t_0(M) - 2i \quad \text{for each } i \geqslant 1.$$

Proof. By [12, Corollary 5.10], we have $\mathrm{R}^i\,\Gamma(M) = \mathrm{H}^i(I^\bullet)$. The corollary is now immediate from Theorem 4.1. □

Remark 4.3. The finiteness of $h_i(M)$ was established in [12, Theorem 5.9]. In [12, Question 1.12], Nagpal asked whether there exists an upper bound of $h_i(M)$ in terms of $t_0(M)$ and $t_1(M)$. The above corollary satisfactorily answers this question for finitely generated modules.

By [12, Remark 5.5], the VI-module $\Sigma^X M$ is semi-induced if and only if $\dim_F(X) \geqslant \max\{h_i(M)+1 \mid i \geqslant 0\}$, which is a finite number since $h_i(M) = -\infty$ when $i \gg 0$. Since it is not easy to find $h_i(M)$, the following corollary is more useful in practice.

Corollary 4.4. *Let M be a finitely generated* VI*-module and $X \in \mathrm{Ob}(\mathrm{VI})$. Then $\Sigma^X M$ is semi-induced if*

$$\dim_F(X) \geqslant \max\{h_0(M)+1,\, 2t_0(M)-1\}, \tag{11}$$

or if

$$\dim_F(X) \geqslant t_0(M) + t_1(M). \tag{12}$$

Proof. First, suppose that (11) holds. By Theorem 4.1, the complex $\Sigma^X I^\bullet$ is exact. For each $i \geqslant 1$, the VI-module I^i is semi-induced, so by [12, Corollary 4.4], the VI-module $\Sigma^X I^i$ is semi-induced. By [12, Corollary 4.23], it follows that $\Sigma^X M = \Sigma^X I^0$ is semi-induced.

Next, suppose that (12) holds. By Theorem 3.2, we have $t_0(M) + t_1(M) \geqslant h_0(M) + 1$. We now consider two cases.

Case 1: $t_1(M) > t_0(M)$. Then $t_0(M) + t_1(M) > 2t_0(M) - 1$, and we are done by (11).

Case 2: $t_1(M) \leqslant t_0(M)$. By Lemma 2.2, we reduce to the previous case by replacing M by $M_{\prec d}$ where $d = t_1(M)$, so we are also done. Indeed, from Lemma 2.2, we know that $M/M_{\prec d}$ (and hence $\Sigma^X(M/M_{\prec d})$) is semi-induced; therefore, if $\Sigma^X M_{\prec d}$ is semi-induced, so is $\Sigma^X M$. □

Remark 4.5. Suppose that $\mathbf{k}$ is a field. Let M be a finitely generated VI-module. It was shown in the proof of [12, Theorem 5.4] that if $X \in \mathrm{Ob}(\mathrm{VI})$ and $\Sigma^X M$ is semi-induced, then there is a polynomial P of degree $t_0(\Sigma^X M)$ such that $\dim_{\mathbf{k}}(M_n) = P(q^n)$ if $n \geqslant \dim_F(X)$. By Corollary 4.4, we deduce that $\dim_{\mathbf{k}}(M_n) = P(q^n)$ if $n \geqslant t_0(M) + t_1(M)$.

Remark 4.6. Suppose that $\mathbf{k}$ is an algebraically closed field of characteristic zero. Let M be a finitely generated VI-module, and choose $X \in \mathrm{Ob}(\mathrm{VI})$ such that $\Sigma^X M$ is semi-induced. The proof of [12, Theorem 5.14] shows that M (or more precisely, the consistent sequence $M_0 \to M_1 \to M_2 \to \cdots$ determined by M and the standard inclusion maps $F^0 \hookrightarrow F^1 \hookrightarrow F^2 \hookrightarrow \cdots$) is representation stable in the sense of [6, Definition 1.5] starting at $\max\{\dim_F(X), 2t_0(M)\}$. By Corollary 4.4, we deduce that M is representation stable starting at $\max\{t_0(M)+t_1(M), 2t_0(M)\}$. We point out that the proof of [12, Theorem 5.14] uses Pieri's formula which shows that any induced module $\mathcal{I}(V)$ is representation stable starting at $2\deg(V)$.

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