GROMOV-HAUSDORFF LIMITS OF KÄHLER MANIFOLDS WITH BISECTIONAL CURVATURE LOWER BOUND I

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ABSTRACT. Given a sequence of complete(compact or noncompact) Kähler manifolds M_i^n with bisectional curvature lower bound and noncollapsed volume, we prove that the pointed Gromov-Hausdorff limit is homeomorphic to a normal complex analytic space. The complex analytic structure is the natural "limit" of complex structure of M_i.

1. Introduction

In this paper, we consider the Gromov-Hausdorff limits of Kähler manifolds with bisectional curvature lower bound. The main interest is the degeneration of the complex structure. One motivation is from the uniformization conjecture of Yau which states that a complete noncompact Kähler manifold with positive bisectional curvature is biholomorphic to $\mathbb{C}^n$. Another motivation is from Alexandrov geometry or manifolds with sectional curvature lower bound, in particular, Perelman's stability theorem [30]. For Kähler manifolds with bounded Ricci curvature or Kähler-Einstein case, see the notable works [14][31].

Definition 1.1. [26] [32] *On a Kähler manifold M^n, we say the bisectional curvature is greater than or equal to K (simply denoted by $BK \geq K$), if*

$$\frac{R(X,\overline{X},Y,\overline{Y})}{||X||^2||Y||^2+|\langle X,\overline{Y}\rangle|^2} \geq K \tag{1.1}$$

for any two nonzero vectors $X, Y \in T^{1,0}M$.

Observe that the equality holds for complex space forms. Note that the bisectional curvature lower bound condition is weaker than the sectional curvature lower bound. It is stronger than the Ricci curvature lower bound. In fact, by taking the trace, we have $R_{i\bar{j}} \geq (n+1)Kg_{i\bar{j}}$.

Theorem 1.1. *Let (M_∞, p_∞) be the pointed Gromov-Hausdorff limit of a sequence of complete(compact or noncompact) Kähler manifolds (M_i^n, p_i) with $BK(M_i) \geq -1$ and $vol(B(p_i, 1)) \geq v > 0$. Then (M_∞, p_∞) is homeomorphic to a normal complex analytic space.*

Remark 1.1. *The complex analytic structure on M_∞ is induced from the limit of holomorphic functions on small balls of M_i. Note this is very similar to* [14]*, where holomorphic functions are replaced by holomorphic sections.*

Remark 1.2. *The conclusion of theorem 1.1 might be surprising at the first glance: the singularity of a normal complex analytic variety has real codimension at least* 4 *while the metric singularity might have codimension* 2*. To resolve this problem, we actually prove*

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that metric singularities with tangent cones splitting off $\mathbb{R}^{2n-2}$ are regular in the complex analytic sense. Compare with [14]*, where it was shown that complex analytic singularities are the same as metric singularities in the Kähler-Einstein case.*

It is a general fact that complex analytic spaces are locally contractible. See, for example, corollary 5.2 in [12]. Therefore, we conclude the following

Corollary 1.1. *The limit space M_∞ is locally contractible.*

Remark 1.3. *When the sectional curvature has a lower bound, the local contractibility of M_∞ was proved in [30][32].*

During the proof of theorem 1.1, we obtain a topological result for complete Kähler surfaces with positive bisectional curvature:

Corollary 1.2. *Let (M^2, p) be a complete noncompact Kähler surface with positive bisectional curvature and maximal volume growth. Then M is simply connected. Maximal volume growth means $vol(B(p, r)) \geq cr^4$ for some $c > 0$ and for all r.*

Remark 1.4. *This result is rather weak. However, according to the author's knowledge, it is new. Indeed, there are very few results on topology of complete noncompact Kähler manifolds with positive bisectional curvature, even with the assumption that the manifold has maximal volume growth. In a forthcoming paper* [23]*, we shall continue to study the uniformization conjecture by using the results here.*

Our strategy to theorem 1.1 is an extension of techniques in [22] to the negatively curved case. We need the Gromov-Hausdorff convergence theory by Cheeger-Colding [4][5], Cheeger-Colding-Tian [8]; adaptation of the heat flow theory by Ni-Tam [28] to negatively curved case (note that here we essentially require the bisectional curvature lower bound, due to a Bochner formula of the complex hessian); Hörmander's L^2-estimate [19][11]; three circle theorem for negatively curved case [21]. We also need to localize some argument in [14].

This paper is organized as follows. In section 2, we collect some preliminary results. Section 3 is an extension of Ni-Tam's maximum principle to the negatively curved case. The proof is similar to the nonnegatively curved case [28]. In section 4, we construct good holomorphic coordinates near special points of a Kähler manifold. Note this is crucial for that the complex analytic singularity has real codimension at least 4. Section 5 deals with the separation of points on the limit space. We construct holomorphic coordinates on M_∞ in section 6. The structure sheaf on M_∞ is introduced is section 7. Finally, we complete the proof of theorem 1.1 in section 8.

Here are some conventions in this paper. Let e_α be a local unitary frame of $T^{1,0}M$ and s be a smooth tensor on M. Define $\Delta s = s_{\alpha\overline{\alpha}} + s_{\overline{\alpha}\alpha}$. Note this is twice the Laplacian defined in [28]. Also define $|\nabla u|^2 = 2u_\alpha u_{\overline{\beta}} g^{\alpha\overline{\beta}}$. We will denote by $\Phi(u_1, ..., u_k|....)$ any nonnegative functions depending on $u_1, ..., u_k$ and some additional parameters such that when these parameters are fixed,

$$\lim_{u_k\to 0} \cdots \lim_{u_1\to 0} \Phi(u_1, ..., u_k|...) = 0.$$

Let $C(\cdot, \cdot, .., \cdot)$ and $c(\cdot, \cdot, .., \cdot)$ be large and small positive constants respectively, depending only on the parameters. The values might change from line to line.

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2. Preliminary results

First recall some convergence results for manifolds with Ricci curvature lower bound. Let (M_i^n, y_i, ρ_i) be a sequence of pointed complete Riemannian manifolds, where $y_i \in M_i^n$ and ρ_i is the metric on M_i^n. By Gromov's compactness theorem, if (M_i^n, y_i, ρ_i) have a uniform lower bound of the Ricci curvature, then a subsequence converges to some $(M_\infty, y_\infty, \rho_\infty)$ in the Gromov-Hausdorff topology. See [15] for the definition and basic properties of Gromov-Hausdorff convergence.

Definition 2.1. *Let $K_i \subset M_i^n \to K_\infty \subset M_\infty$ in the Gromov-Hausdorff topology. Assume $\{f_i\}_{i=1}^\infty$ are functions on M_i^n, f_∞ is a function on M_∞. Φ_i are ϵ_i-Gromov-Hausdorff approximations, $\lim\limits_{i\to\infty} \epsilon_i = 0$. If $f_i \circ \Phi_i$ converges to f_∞ uniformly, we say $f_i \to f_\infty$ uniformly over $K_i \to K_\infty$.*

In many applications, f_i are equicontinuous. The Arzela-Ascoli theorem applies to the case when the spaces are different. When $(M_i^n, y_i, \rho_i) \to (M_\infty, y_\infty, \rho_\infty)$ in the Gromov-Hausdorff topology, any bounded, equicontinuous sequence of functions f_i has a subsequence converging uniformly to some f_∞ on M_∞.

Let the complete pointed metric space (M_∞^n, y) be the Gromov-Hausdorff limit of a sequence of connected pointed Riemannian manifolds, $\{(M_i^n, p_i)\}$, with $Ric(M_i) \geq -(n-1)$ and $vol(B(p_i, 1)) \geq v > 0$. M_∞ is called a noncollapsed limit. A tangent cone at $y \in M_\infty^m$ is a complete pointed Gromov-Hausdorff limit $((M_\infty)_y, d_\infty, y_\infty)$ of $\{(M_\infty, r_i^{-1}d, y)\}$, where d, d_∞ are the metrics of $M_\infty, (M_\infty)_y$ respectively, $\{r_i\}$ is a positive sequence converging to 0. The following is theorem 5.2 in [5]:

Theorem 2.1. *Under the assumptions of the last paragraph, any tangent cone is a metric cone.*

Definition 2.2. *A point $y \in M_\infty$ is called k-weakly Euclidean, if some tangent cone splits off $\mathbb{R}^k$ isometrically. Let $\mathcal{WE}_k$ denote the k-weakly Euclidean points. We also call $\mathcal{WE}_n$ the set of regular points, denoted by $\mathcal{R}$. For any $\epsilon > 0$, let $\mathcal{R}_\epsilon$ be the set of points $y \in M_\infty$ such that there exists $\delta > 0$ with $d_{GH}(B(y, r), B_{\mathbb{R}^n}(0, r)) < \epsilon r$ for all $0 < r < \delta$. Let $\mathring{\mathcal{R}}_\epsilon$ be the interior of $\mathcal{R}_\epsilon$.*

In [5], the following theorem was proved:

Theorem 2.2. *The Hausdorff dimension of $M_\infty \backslash \mathcal{WE}_k$ is at most $k-1$.*

If in addition, M_i are all Kähler, then theorem 9.1 in [8] states

Theorem 2.3. $\mathcal{WE}_{2k-1} = \mathcal{WE}_{2k}$.

Hörmander's L^2 theory:

Theorem 2.4. *Let (X^n, ω) be a connected but not necessarily complete Kähler manifold with $Ric \geq -(n+1)\epsilon (\epsilon > 0)$. Assume X is Stein. Let φ be a C^∞ function on X with $\sqrt{-1}\partial\overline{\partial}\varphi \geq c\omega$ for some positive function $c > (n+1)\epsilon$ on X. Let g be a smooth $(0,1)$ form satisfying $\overline{\partial} g = 0$ and $\int_X \frac{|g|^2}{c-(n+1)\epsilon} e^{-\varphi}\omega^n < +\infty$, then there exists a smooth function f on X with $\overline{\partial} f = g$ and $\int_X |f|^2 e^{-\varphi}\omega^n \leq \int_X \frac{|g|^2}{c-(n+1)\epsilon} e^{-\varphi}\omega^n$.*

The proof can be found in [11], page 38-39. Also compare with lemma 4.4.1 in [19]. Note that the theorem also applies to singular metrics with positive curvature in the current sense.

Three circle theorem in [21]:

Theorem 2.5. *Let M be a complete noncompact Kähler manifold with holomorphic sectional curvature $H \geq -1$, $p \in M$. Let f be a holomorphic function on M. Let $M(r) = \sup_{B(p,r)} |f(x)|$. Then $\log M(r)$ is a convex function of $\log \frac{e^r-1}{e^r+1}$.*

3. A maximum principle for heat flow

In this section we extend Ni-Tam's maximum principle [28] to the negatively curved case. The proposition below is a modification of corollary 1.1 in [28].

Proposition 3.1. *Let (M^n, p) be a complete noncompact Kähler manifold with $BK \geq -1$. Let $r(x) = d(x, p)$. Let u be a nonnegative function on M satisfying $u(x) \leq \exp(a + br(x))$ for some constants $a, b > 0$. Let*

$$v(x, t) = \int_M H(x, y, t)u(y)dy. \tag{3.1}$$

H is the heat kernel on M. Then given any $1 > \delta > 0, T > 0$, there exist $C_1 > 0, C_2 > 0$ depending only on n, δ, a, b, T such that for any $x \in M$ with $r = r(x) > C_2$,

$$\frac{1}{2} \inf_{B(x,\delta r)} u \leq v(x, t) \leq C_1 + \sup_{B(x,\delta r)} u \tag{3.2}$$

for $0 \leq t \leq T$. The latter inequality holds for all r.

Remark 3.1. *The theorem also holds for compact manifolds.*

Proof. Let $v = vol(B(p, 1))$. Recall the heat kernel estimate [27], there exists $C(n) > 0$ with

$$H(x, y, t) \leq C(n)\frac{1}{\sqrt{vol(B(x, \sqrt{t}))vol(B(y, \sqrt{t}))}} \exp(-\frac{d^2(x, y)}{8t} + C(n)t). \tag{3.3}$$

By volume comparison,

$$vol(B(x, \sqrt{t})) \geq \frac{1}{C(n)} \exp(-8nr(x))v \min(t^n, 1), \tag{3.4}$$

$$vol(B(y, \sqrt{t})) \geq \frac{1}{C(n)} \exp(-8n(r(x) + d(x, y))v \min(t^n, 1), \tag{3.5}$$

$$\begin{aligned} \int_{M\setminus B(x,\delta r(x))} H(x, y, t)dy &\leq \frac{C(n)}{v \min(1, t^n)} \int_{M\setminus B(x,\delta r(x))} \exp(8n(r(x) + d(x, y)) - \frac{d^2(x, y)}{8t} + C(n)t)dy \\ &\leq \frac{C(n, T)}{\min(1, t^n)} \exp(80nr(x)) \int_{\delta r(x)}^{\infty} \exp(16n\lambda - \frac{\lambda^2}{8t})d\lambda \\ &\leq \frac{1}{2} \end{aligned} \tag{3.6}$$

for $r(x) \geq C_2(n, T, \delta)$. As u is of exponential growth, by (3.3), we find that

$$\int_{M\setminus B(x,\delta r(x))} H(x, y, t)u(y)dy \leq C_1(n, T, \delta, a, b). \tag{3.7}$$

Now

$$
\begin{aligned}
v(x,t) &= \int_{B(x,\delta r(x))} H(x,y,t)u(y)dy + \int_{M\setminus B(x,\delta r(x))} H(x,y,t)u(y)dy \\
&\leq \sup_{B(x,\delta r)} u + C_1;
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
v(x,t) &= \int_{B(x,\delta r(x))} H(x,y,t)u(y)dy + \int_{M\setminus B(x,\delta r(x))} H(x,y,t)u(y)dy \\
&\geq (\inf_{B(x,\delta r)} u)\int_{B(x,\delta r)} H(x,y,t)dy \\
&\geq (1-\int_{M\setminus B(x,\delta r)} H(x,y,t)dy)\inf_{B(x,\delta r)} u \\
&\geq \frac{1}{2}\inf_{B(x,\delta r)} u.
\end{aligned} \tag{3.9}
$$

□

Theorem 3.1. *Let (M^n, p) be a complete Kähler manifold. Let $r(x) = d(x,p)$. Assume the bisectional curvature is bounded from below by $-\epsilon$ for some $1 > \epsilon > 0$. Let u be a smooth function on M with compact support. Let*

$$
v(x,t) = \int_M H(x,y,t)u(y)dy. \tag{3.10}
$$

Here $H(x,y,t)$ is the heat kernel of M. Let $\eta(x,t)_{\alpha\overline{\beta}} = v_{\alpha\overline{\beta}}$ and $\lambda(x)$ be the minimal eigenvalue for $\eta(x,0) - \epsilon|\nabla u(x)|^2 g_{\alpha\overline{\beta}}$. Let

$$
\lambda(x,t) = \exp(8n\epsilon t)\int_M H(x,y,t)\lambda(y)dy. \tag{3.11}
$$

Then $\eta(x,t) - \lambda(x,t)g_{\alpha\overline{\beta}} - \epsilon|\nabla v(x,t)|^2 g_{\alpha\overline{\beta}} + Ktg_{\alpha\overline{\beta}}$ is a nonnegative $(1,1)$ tensor for $t \in [0,T]$, provided the following conditions are satisfied:

$$
8n\epsilon T < \frac{1}{2}; \tag{3.12}
$$

$$
\frac{1}{2}K > 8n\epsilon^2 \exp(8n\epsilon T)\sup|\nabla u(x)|^2 + 8n^2\epsilon. \tag{3.13}
$$

Remark 3.2. *We shall prove the theorem for the case when M is noncompact. The proof for the compact case is even simpler.*

Proof. During the proof, $C, C_i (i \geq 1)$ will be large positive constants. The dependence will be clear from the context. Following [28], we establish some bounds for v and its derivatives.

Lemma 3.1.

$$
(\frac{\partial}{\partial t} - \Delta)\eta_{\gamma\overline{\delta}} = 2R_{\beta\overline{\alpha}\gamma\overline{\delta}}\eta_{\alpha\overline{\beta}} - (R_{\gamma\overline{p}}\eta_{p\overline{\delta}} + R_{p\overline{\delta}}\eta_{\gamma\overline{p}}). \tag{3.14}
$$

For any $a > 0$,

$$
\liminf_{r\to\infty} \int_0^T \int_{B(p,r)} |\nabla v(x,t)|^2 \exp(-ar^2(x))dxdt < \infty, \tag{3.15}
$$

$$
\liminf_{r\to\infty} \int_0^T \int_{B(p,r)} ||\eta||^2(x,t) \exp(-ar^2(x))dxdt < \infty. \tag{3.16}
$$

Proof. (3.14) follows from direct computation. As u has compact support, $|u| \leq C$. Then by the definition of v, $|v(x,t)| \leq C$ for all $x \in M, t \geq 0$. Note

$$(\Delta - \frac{\partial}{\partial t})v^2 = 2|\nabla v|^2. \tag{3.17}$$

We multiply (3.17) by the standard cutoff function φ^2 supported in $B(p, 2r)$ with $\varphi = 1$ in $B(p,r)$ and $|\nabla\varphi| \leq \frac{5}{r}$. By integration by parts and volume comparison, we find

$$\int_0^T \int_{B(p,r)} |\nabla v|^2 \leq C_1(r^{-2}\int_0^{2T}\int_{B(p,2r)} v^2 + \int_{B(p,2r)} u^2) \leq C_2(T+1)e^{50n\epsilon r} \tag{3.18}$$

for $r \geq 1$. Then (3.15) follows. For the last equation, we have

$$(\Delta - \frac{\partial}{\partial t})|\nabla v|^2 = 4(|v_{ij}|^2 + |v_{i\bar{j}}|^2 + R_{i\bar{j}}v_j v_{\bar{i}}) \geq 2|\nabla^2 v|^2 - 8n\epsilon|\nabla v|^2. \tag{3.19}$$

By integration by parts as before,

$$\int_0^T \int_{B(p,r)} |\nabla^2 v|^2 \leq C_3((r^{-2} + 8n\epsilon)\int_0^{2T}\int_{B(p,2r)} |\nabla v|^2 + \int_{B(p,2r)} |\nabla u|^2) \leq C_4(T+1)e^{100n\epsilon r} \tag{3.20}$$

for $r \geq 1$. Then (3.16) follows. □

Note (3.19) implies that

$$(\Delta - \frac{\partial}{\partial t})(e^{-8n\epsilon t}|\nabla v(x,t)|^2) \geq 2e^{-8n\epsilon t}|\nabla^2 v|^2. \tag{3.21}$$

Combining this with

$$|\nabla|\nabla v||^2 \leq |\nabla^2 v|^2, \tag{3.22}$$

we find

$$(\Delta - \frac{\partial}{\partial t})(e^{-4n\epsilon t}|\nabla v(x,t)|) \geq 0. \tag{3.23}$$

By the maximum principle in [24] or theorem 1.2 in [29], (3.15) and (3.23),

$$e^{-8n\epsilon t}|\nabla v(x,t)|^2 \leq \max |\nabla u|^2. \tag{3.24}$$

At a point $x \in M$, we can diagonalize η so that $\eta_{\alpha\bar{\beta}} = \lambda_\alpha \delta_{\alpha\beta}$. By direct calculations on page 477 of [28],

$$\begin{aligned}(\Delta - \frac{\partial}{\partial t})||\eta||^2 &= 2|v_{\alpha\bar{\beta}s}|^2 + 2|v_{\alpha\bar{\beta}\bar{s}}|^2 + 4R_{\alpha\bar{p}}v_{p\bar{\delta}}v_{\delta\alpha} - 4R_{\alpha\bar{\beta}q\bar{p}}v_{p\bar{q}}v_{\beta\bar{\alpha}} \\ &= 2|v_{\alpha\bar{\beta}s}|^2 + 2|v_{\alpha\bar{\beta}\bar{s}}|^2 + 2R_{\alpha\bar{\alpha}\beta\bar{\beta}}(\lambda_\alpha - \lambda_\beta)^2 \\ &\geq 2|v_{\alpha\bar{\beta}s}|^2 + 2|v_{\alpha\bar{\beta}\bar{s}}|^2 - 100\epsilon||\eta||^2\end{aligned} \tag{3.25}$$

This implies that

$$(\Delta - \frac{\partial}{\partial t})(e^{-100\epsilon t}||\eta||^2) \geq (2|v_{\alpha\bar{\beta}s}|^2 + 2|v_{\alpha\bar{\beta}\bar{s}}|^2)e^{-100\epsilon t}. \tag{3.26}$$

A direct calculation shows

$$|\nabla||\eta|||^2 \leq |v_{\alpha\bar{\beta}s}|^2 + |v_{\alpha\bar{\beta}\bar{s}}|^2. \tag{3.27}$$

Then

$$(\Delta - \frac{\partial}{\partial t})(e^{-50\epsilon t}||\eta||) \geq 0. \tag{3.28}$$

By (3.16), we proved the following lemma:

Lemma 3.2. $\|\eta(y,t)\| \leq e^{50\epsilon t} \max_{x\in M} \|\eta(x,0)\|$.

Let $\phi(x) = \exp(r(x))$. Define

$$\phi(x,t) = e^{8n\epsilon t}\int_M H(x,y,t)\phi(y)dy. \tag{3.29}$$

Then

$$(\frac{\partial}{\partial t} - \Delta)\phi = 8n\epsilon\phi \tag{3.30}$$

and

$$\phi(x,t) \geq ce^{c_1 r} \tag{3.31}$$

for $0 \leq t \leq T$, by proposition 3.1. Here c, c_1 are positive constants. Given any $\tau > 0$, consider

$$(\tilde{\eta})_{\alpha\overline{\beta}} = \eta(x,t) + (-\lambda(x,t) - \epsilon|\nabla v(x,t)|^2 + Kt + \tau\phi(x,t))g_{\alpha\overline{\beta}}. \tag{3.32}$$

At $t = 0$, $\tilde{\eta} > 0$. Also, for $0 \leq t \leq T$, if R is sufficiently large, by (3.24), lemma 3.2 and (3.31), we have $\tilde{\eta} > 0$ on $\partial B(p,R)$. Suppose at some $t_0 \in [0,T]$, $\tilde{\eta}(x_0,t_0) < 0$ for $x_0 \in \overline{B(p,R)}$. Then there exists $0 \leq t_1 < T$ with $\tilde{\eta}(x,t) \geq 0$ for $x \in B(p,R)$ and $0 \leq t \leq t_1$. Moreover, the minimum eigenvalue of $\tilde{\eta}(x_1,t_1)$ is zero for some $x_1 \in B(p,R)$(note x_1 cannot be on the boundary). Now we apply the maximal principle. Let us assume

$$\tilde{\eta}(x_1,t_1)_{\gamma\overline{\gamma}} = 0 \tag{3.33}$$

for $\gamma \in T^{1,0}_{x_1}M, |\gamma| = 1$. We may diagonalize $\tilde{\eta}$ at (x_1,t_1) and assume γ is one of the basis of the holomorphic tangent space. Then at (x_1,t_1),

$$(\frac{\partial}{\partial t} - \Delta)\tilde{\eta}_{\gamma\overline{\gamma}} \leq 0. \tag{3.34}$$

On the other hand, by (3.14),

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)\eta_{\gamma\overline{\gamma}} &= 2\sum_\alpha R_{\gamma\overline{\gamma}\alpha\overline{\alpha}}\eta_{\alpha\overline{\alpha}} - 2\sum_\alpha R_{\gamma\overline{\gamma}\alpha\overline{\alpha}}\eta_{\gamma\overline{\gamma}} \\ &= 2\sum_\alpha R_{\gamma\overline{\gamma}\alpha\overline{\alpha}}(\tilde{\eta}_{\alpha\overline{\alpha}} - \tilde{\eta}_{\gamma\overline{\gamma}}) \\ &\geq -2\epsilon\sum_\alpha \tilde{\eta}_{\alpha\overline{\alpha}} \\ &\geq -8n\epsilon(\|\eta\| - \lambda + Kt + \tau\phi).\end{aligned} \tag{3.35}$$

Note by (3.11), (3.19) and (3.30),

$$\begin{aligned}&(\frac{\partial}{\partial t} - \Delta)(-\lambda(x,t) - \epsilon|\nabla v(x,t)|^2 + Kt + \tau\phi(x,t))g_{\gamma\overline{\gamma}} \\ &\geq -8n\epsilon\lambda + 8n\epsilon\tau\phi + \epsilon(2\|\eta\|^2 - 8n\epsilon|\nabla v|^2) + K.\end{aligned} \tag{3.36}$$

Hence at (x_1,t_1),

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)\tilde{\eta}_{\gamma\overline{\gamma}} \geq{}& -8n\epsilon(\|\eta\| - \lambda + Kt + \tau\phi) + \\ & -8n\epsilon\lambda + 8n\epsilon\tau\phi + \epsilon(2\|\eta\|^2 - 8n\epsilon|\nabla v|^2) + K \\ \geq{}& 2\epsilon(\|\eta\| - 2n)^2 - 8n^2\epsilon - 8n\epsilon^2|\nabla v|^2 + (1 - 8n\epsilon t)K \\ >{}& 0,\end{aligned} \tag{3.37}$$

according to (3.24), (3.12) and (3.13). This contradicts (3.34). The theorem follows if we first let $R \to \infty$, then $\tau \to 0$.

□

Corollary 3.1. *Under the assumption of theorem 3.1,* $\eta(x,t)_{\alpha\overline{\beta}} \geq (\lambda(x,t) - Kt)g_{\alpha\overline{\beta}}$.

4. Construct good holomorphic coordinates on manifolds

In this section, we construct good holomorphic coordinates around certain points on a manifold. This will be crucial for that the complex analytic singularity has codimension at least 4.

Let $0 < \gamma \leq 2\pi$. Let $(X, (0, o)) = (\mathbb{C}^{n-1}, 0) \times (Z, o)$ where (Z, o) is a complex one dimensional cone with cone angle α satisfying $2\pi \geq \alpha \geq \gamma$. The metric on (Z, o) is given by the standard metric $dr^2 + r^2 d\theta^2 (0 \leq \theta < \alpha)$. On X, there is a global holomorphic chart $(z_1, ..., z_{n-1}, z_n)$: $z_1, ..., z_{n-1}$ are standard coordinates on $(\mathbb{C}^{n-1}, 0)$, $z_n(r, \theta) = r^{\frac{2\pi}{\alpha}} e^{\frac{2\pi\theta\sqrt{-1}}{\alpha}}$. It is clear that the coordinate functions are Lipschitz on each compact set of X. Let $K_r \subset \mathbb{C}^n$ be the image of $(z_1, ..., z_{n-1}, z_n)$ on $B_X((0, o), r)$. Then

$$K_r = \{(z_1, ..., z_n) \in \mathbb{C}^n | |z_1|^2 + \cdots + |z_{n-1}|^2 + |z_n|^{\frac{\alpha}{\pi}} \leq r^2\} \tag{4.1}$$

Below is the main result in this section:

Proposition 4.1. *Let $a > 0$. There exist $\tilde{\epsilon} = \tilde{\epsilon}(n, \gamma) > 0, \delta = \delta(n) > 0$ so that the following hold. Assume (M^n, x) is a complete Kähler manifold with $BK \geq -\frac{\epsilon^3}{a^2}$ for some $0 < \epsilon < \tilde{\epsilon}$ and $d_{GH}(B(x, \frac{a}{\epsilon}), B_X((0, o), \frac{a}{\epsilon})) < \epsilon a$, then there exists a holomorphic chart $(w_1, ..., w_n)$ containing $B(x, \delta a)$ so that*

- *$w_s(x) = 0 (1 \leq s \leq n)$.*
- *Up to an isometry of $(X, (0, o))$, on $B(x, \delta a)$, we have: for $1 \leq i \leq n-1$, w_i is $a\Phi(\epsilon|n, \gamma)$ close to z_i under the Gromov-Hausdorff approximation; w_n is $a^{\frac{2\pi}{\alpha}}\Phi(\epsilon|n, \gamma)$ close to z_n. In particular, on $B(x, \delta a)$, $|w_i| \leq C(n, \gamma)a (1 \leq i \leq n-1)$ and $|w_n| \leq C(n, \gamma)a^{\frac{2\pi}{\alpha}}$.*
- *The image of $(w_1, ..., w_n)$ contains the domain $K_{(\delta - \Phi(\epsilon|n,\gamma))a}$.*

Proof. It is clear that the proposition is independent of a. We may assume that a is sufficiently large, to be determined. Let $a = 100R$. Let $r(y)$ be the distance from y to x. We shall assume $\tilde{\epsilon}$ is sufficiently small. The value will be fixed later. We first construct the weight function for Hörmander's L^2 estimate. The argument follows from a slight modification of [22]. The completeness, we include most of the details. Set

$$A = B(x, 5R) \backslash B(x, \frac{1}{5R}). \tag{4.2}$$

By the volume convergence theorem [9] or theorem 5.9 in [5], A satisfies the almost maximal volume condition (see (4.8) or (4.10) in [4]). By Cheeger-Colding theory [4]((4.43) and (4.82)), there exists a smooth function ρ on M so that

$$\int_A |\nabla\rho - \nabla\frac{1}{2}r^2|^2 + |\nabla^2\rho - g|^2 < \Phi(\epsilon|R, n, \gamma); \tag{4.3}$$

$$|\rho - \frac{r^2}{2}| < \Phi(\epsilon|R, n, \gamma) \tag{4.4}$$

on A. Let $F(r)$ be the Green function on $2n$ dimensional real space form with $Ric = -(n+1)\frac{\epsilon^3}{a^2}$. Then $F'(r) < 0$. As $\epsilon \to 0$, up to a factor,

$$F \to r^{2-2n}, n > 1; F \to \log r, n = 1. \tag{4.5}$$

According to (4.20)-(4.23) in [4],

$$\rho = \frac{1}{2}(F^{-1}\mathcal{G})^2; \Delta\mathcal{G}(y) = 0, y \in B(x, 10R)\backslash B(x, \frac{1}{10R}); \tag{4.6}$$

$$\mathcal{G} = F(r) \tag{4.7}$$

on $\partial(B(x, 10R)\backslash B(x, \frac{1}{10R}))$. Now

$$|\nabla\rho(y)| = |F^{-1}\mathcal{G}||(F^{-1})'(\mathcal{G})||\nabla\mathcal{G}(y)|. \tag{4.8}$$

By (4.4)-(4.7) and Cheng-Yau's gradient estimate [10],

$$|\nabla\rho(y)| \leq C(n)r(y) \tag{4.9}$$

for $y \in A$ and sufficiently small ϵ depending only on n, R, γ. Now consider a smooth function $\overline{\varphi}$: $\mathbb{R}^+ \to \mathbb{R}^+$ given by $\overline{\varphi}(t) = t$ for $t \geq 2$; $\overline{\varphi}(t) = 0$ for $0 \leq t \leq 1$; $|\overline{\varphi}|, |\overline{\varphi}'|, |\overline{\varphi}''| \leq C(n)$. Let

$$u(y) = \frac{1}{R^2}\overline{\varphi}(R^2\rho(y)). \tag{4.10}$$

We set $u(y) = 0$ for $y \in B(x, \frac{1}{5R})$. Then u is smooth on $B(x, 4R)$.

Claim 4.1. $\int_{B(x,4R)} |\nabla u - \nabla\frac{1}{2}r^2|^2 + |\nabla^2 u - g|^2 < \Phi(\epsilon|R, n, \gamma); |u - \frac{r^2}{2}| < \Phi(\epsilon|R, n, \gamma)$ *and* $|\nabla u| \leq C(n)r$ *on* $B(x, 4R)$.

Proof. We have

$$\nabla u(y) = \overline{\varphi}'(R^2\rho(y))\nabla\rho(y); \tag{4.11}$$

$$\nabla^2 u(y) = R^2\overline{\varphi}''(R^2\rho(y))\nabla\rho \otimes \nabla\rho + \overline{\varphi}'(R^2\rho(y))\nabla^2\rho. \tag{4.12}$$

The proof follows from a routine calculation, by (4.3), (4.4) and (4.9). □

Now consider a smooth function φ: $\mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(t) = t$ for $0 \leq t \leq 1$; $\varphi(t) = 0$ for $t \geq 2$; $|\varphi|, |\varphi'|, |\varphi''| \leq C(n)$. Let $H(z, y, t)$ be the heat kernel on M and set

$$h(y) = 5R^2\varphi(\frac{u(y)}{5R^2}), h_t(z) = \int_M H(z, y, t)h(y)dy. \tag{4.13}$$

Claim 4.2. *Assume ϵ is sufficiently small, depending only on R, γ, n. Then* $(h_1)_{\alpha\overline{\beta}}(z) \geq c(n, \gamma)g_{\alpha\overline{\beta}} > 0$ *on* $B(x, \frac{R}{10})$.

Proof. Let $\lambda(y)$ be the lowest eigenvalue of $h_{\alpha\overline{\beta}} - \frac{\epsilon^3}{a^2}|\nabla h|^2 g_{\alpha\overline{\beta}}$. Let

$$\lambda(z, t) = e^{8n\frac{\epsilon^3}{a^2}} \int_M H(z, y, t)\lambda(y)dy. \tag{4.14}$$

By corollary 3.1,

$$(h_1)_{\alpha\overline{\beta}}(z) \geq (\lambda(z, 1) - K)g_{\alpha\overline{\beta}}, \tag{4.15}$$

provided the following inequalities are satisfied:

$$8n\frac{\epsilon^3}{a^2} < \frac{1}{2}, \tag{4.16}$$

$$\frac{1}{2}K > 8n(\frac{\epsilon^3}{a^2})^2 \exp(8n\frac{\epsilon^3}{a^2}) \sup |\nabla h|^2 + 8n^2\frac{\epsilon^3}{a^2}. \tag{4.17}$$

From (4.13) and claim 4.1, it is clear that $|\nabla h| \leq C(n)R$ on M. If ϵ is very small, we can make K small and (4.16), (4.17) hold. To prove claim 4.2, it suffices to prove that

$\lambda(z,1) \geq c(n,\gamma) > 0$ for $z \in B(x, \frac{R}{10})$. The proof is almost the same as in claim 1 in [22]. We skip the details here. □

Claim 4.3. *There exist $\epsilon_0 = \epsilon_0(n) > 0$, $R \geq C_0(n) > 100$, $\epsilon = \epsilon(n, R, \gamma)$ sufficiently small so that*

$$\min_{y \in \partial B(x, \frac{R}{20})} h_1(y) > 4 \sup_{y \in B(x, \epsilon_0 R)} h_1(y). \tag{4.18}$$

Also $0 \leq h_1(y) \leq C(n,\gamma)R^2$ on $B(x,R)$.

Proof. According to (4.4), this is a consequence of proposition 3.1. □

Now we freeze the value $R = C_0(n)$ in claim 4.3. Then ϵ depends only on n and γ. We might make ϵ even smaller later. Let Ω be the connected component of $\{y \in B(x, \frac{R}{20}) | h_1(y) < 2 \sup\limits_{y \in B(x, \epsilon_0 R)} h_1(y)\}$ containing $B(x, \epsilon_0 R)$. Then Ω is relatively compact in $B(x, \frac{R}{20})$ and Ω is a Stein manifold by claim 4.2.

Lemma 4.1. *There exist complex harmonic functions $w_i'(1 \leq i \leq n)$ on $B(x, 2R)$ so that the following hold.*

- *Up to an isometry of $(X, (0, o))$, on $B(x, 2R)$, we have for $1 \leq i \leq n$, w_i' is $\Phi(\epsilon|n,\gamma)$ close to z_i under the Gromov-Hausdorff approximation.*
- $\int_{B(x,R)} |\overline{\partial} w_i'|^2 \leq \Phi(\epsilon|n,\gamma)$ *for $1 \leq i \leq n$.*

Proof. First we construct w_i' for $1 \leq i \leq n-1$. The construction is similar to proposition 1 in [22]. For completeness, we include the details. According to Cheeger-Colding theory [4](also equation (1.23) in [6]), there exist real harmonic functions $b_1, ..., b_{2n-2}$ on $B(x, 4R)$ so that

$$⨍_{B(x,2R)} \sum_s |\nabla(\nabla b_s)|^2 + \sum_{s,l} |\langle \nabla b_s, \nabla b_l \rangle - \delta_{sl}|^2 \leq \Phi(\epsilon|n,\gamma) \tag{4.19}$$

and

$$b_s(x) = 0 (1 \leq s \leq 2n-2); |\nabla b_s| \leq C(n) \tag{4.20}$$

on $B(x, 2R)$. Moreover, the map $F(y) = (b_1(y), ..., b_{2n-2}(y))$ is a $\Phi(\epsilon|n,\gamma)$ approximation to the Euclidean factor of X. According to the argument above lemma 9.14 in [8](see also (20) in [20]), after a suitable orthogonal transformation, we may assume

$$⨍_{B(x,2R)} |J\nabla b_{2j-1} - \nabla b_{2j}|^2 \leq \Phi(\epsilon|n,\gamma) \tag{4.21}$$

for $1 \leq j \leq n-1$. Set $w_j' = b_{2j-1} + \sqrt{-1} b_{2j}$. Then

$$⨍_{B(x,2R)} |\overline{\partial} w_j'|^2 \leq \Phi(\epsilon|n,\gamma). \tag{4.22}$$

By composing with an isometry of $(X, (0, o))$, we may assume w_j' is close to z_j.

Now we construct the function w_n'. It is clear that z_n is Lipschitz on $\partial B((0, o), 2R)$. We can transplant it to $\partial B(x, 2R)$ as a Lipschitz function h_n'. Basically we first transplant the values to a δ-net, then extend to a Lipschitz function by Macshane lemma (see, for example, (8.2) in [2]). One can also directly apply lemma 10.7 in [2]. We may assume h_n' is very close to $z_n|_{\partial B((0,o),2R)}$. Following Ding [13], we solve the Dirichlet problem $\Delta w_n' = 0$ with boundary data $w_n' = h_n'$. By using the same arguments as in theorem 2.1 of [13](replace b_i

in (2.3) of [13] by the Green function on the space form with $Ric = -(n+1)\frac{\epsilon^3}{a^2}$), we find that w_n' is close to z_n up to $\Phi(\epsilon|n,\gamma)$ error.

Next we prove that w_n' is almost holomorphic or anti-holomorphic on $B(x,R)$. More precisely, we prove

$$\int_{B(x,R)} |Dw_n'|^2 \le \Phi(\epsilon|n,\gamma) \tag{4.23}$$

where $D = \partial$ or $\overline{\partial}$. We always assume ϵ is as small as we want. Let $S = \{y \in X | z_n(y) = 0\}$. That is, S is the set of singular points of X. Fix small $\epsilon' > 0$. Given any point $y' \in B(x, \frac{3}{2}R)\backslash B(S,\epsilon' R)(B(S,\epsilon' R)$ is just the distance neighborhood of S), we can find $y \in B_X((0,o),\frac{3}{2}R)$ with y close to y' up to distance $\Phi(\epsilon|n,\gamma)$. Since X is flat outside S, there exist $\delta'' = \delta''(n,\epsilon',\gamma)$ and a holomorphic chart $(a_1,...,a_n)$ in $B(y,2\delta'' R)$ with $a_i = z_i$ for $1 \le i \le n-1$ and the metric is given by $\omega = \frac{\sqrt{-1}}{2}\sum_{i=1}^{n} da_i \wedge \overline{da_i}$. This means that each a_i is a parallel coordinate function. Furthermore, we can require that a_n is a function depending only on z_n. Thus we can regard z_n as a function of a_n.

As we mentioned before, if ϵ is sufficiently small, $B(y,2\delta'' R)$ is close to $B(y',2\delta'' R)$ as we want. According to Cheeger-Colding theory [4], we can find $0 < \delta' = \delta'(n,\gamma,\epsilon',\delta'') << \delta''$ and complex harmonic functions $(a_1'',...,a_n'')$ on $B(y',2\delta' R)$ with a_i'' close to a_i up to error $\epsilon'\delta' R$. Furthermore,

$$⨍_{B(y',\delta' R)} \sum_{1\le i,j\le n} (|\langle da_i'', \overline{da_j''}\rangle - 2\delta_{ij}|^2 + |\langle da_i'', da_j''\rangle|^2) < \epsilon'; |da_i''| \le C(n). \tag{4.24}$$

By assume δ' be sufficiently small, we may assume

$$|z_n(t) - \frac{\partial z_n}{\partial a_n}(y)a_n(t) - (z_n(y) - \frac{\partial z_n}{\partial a_n}(y)a_n(y))| < \epsilon'\delta' R \tag{4.25}$$

for any $t \in B(y,2\delta' R)$. This merely says z_n is almost linear in terms of a_n on $B(y,2\delta' R)$. For notational convenience, we set $\lambda_1(y) = z_n(y) - \frac{\partial z_n}{\partial a_n}(y)a_n(y)$. Recall the definition of z_n in the second paragraph of this section. Since z_n depends only on a_n and the metric $\omega = \frac{\sqrt{-1}}{2}\sum_{i=1}^{n} da_i \wedge \overline{da_i}$ on $B(y,2\delta'' R)$,

$$|dz_n(y)| = |\frac{\partial z_n}{\partial a_n}(y)da_n| = \sqrt{2}\frac{2\pi}{\alpha} r(y)^{\frac{2\pi}{\alpha}-1} \le C(n,\gamma). \tag{4.26}$$

We have used that R depends only on n,γ.

Let $a_j'(1 \le j \le n-1)$ be the restriction of w_j' on $B(y',2\delta' R)$. Let $a_n' = a_n''$. By the sentence below (4.22), we may assume a_j' is close to $z_j (1 \le j \le n-1)$ up to error $2\epsilon'\delta' R$. Since $a_j = z_j$ for $1 \le j \le n-1$ on $B(y,\delta'' R)$, by the sentence above (4.24), we find that on $B(y,\delta' R)$,

$$|a_j' - a_j''| \le 10\epsilon'\delta' R \tag{4.27}$$

As a_j' and a_j'' are harmonic, gradient estimate says on $B(y',\delta' R)$,

$$|da_j' - da_j''| \le C(n,\gamma)\epsilon'. \tag{4.28}$$

Claim 4.4.
(4.29)

$$⨍_{B(y',\delta' R)} \sum_{1\le i,j\le n} |\langle da_i', \overline{da_j'}\rangle - 2\delta_{ij}|^2 < C(n,\gamma)\epsilon'; ⨍_{B(y',\delta' R)} |\overline{\partial} a_j'|^2 < C(n,\gamma)\epsilon', 1 \le j \le n-1;$$

$$\fint_{B(y',\delta' R)} \sum_{1\leq i,j\leq n} |\langle da'_i, da'_j\rangle|^2 < C(n,\gamma)\epsilon'; \fint_{B(y',\delta' R)} |Da'_n|^2 < C(n,\gamma)\epsilon' \tag{4.30}$$

for $D = \partial$ or $\overline{\partial}$. D can only be one of them, if ϵ' is sufficiently small.

Proof. (4.29) and the first inequality in (4.30) follow from (4.22), (4.28), (4.24). For the last inequality, one can apply the same argument as in (4.21). If (4.30) holds for $D = \partial$ and $D = \overline{\partial}$, $\fint_{B(y',\delta' R)} |da'_n|^2 \leq C(n,\gamma)\epsilon'$. This contradicts (4.29). □

Recall the function $\lambda_1(y)$ defined below (4.25). Set

$$\tilde{z}_n(s) = \frac{\partial z_n}{\partial a_n}(y)a'_n(s) + \lambda_1(y) \tag{4.31}$$

for $s \in B(y', \frac{3}{2}\delta' R)$. Then $\tilde{z_n}$ is harmonic. By (4.26) and (4.30),

$$\fint_{B(y',\delta' R)} |D\tilde{z}_n|^2 < C(n,\gamma)\epsilon'. \tag{4.32}$$

Claim 4.5. $|d\tilde{z}_n - dw'_n| < C(n,\gamma)\epsilon'$ *on* $B(y', \delta' R)$. *Thus*

$$\fint_{B(y',\delta' R)} |d\tilde{z}_n - dw'_n|^2 < C(n,\gamma)\epsilon'. \tag{4.33}$$

Proof. As a_n is close to a'_n up to error $C(n)\epsilon'\delta' R$, by (4.25) and (4.31), z_n is close to $\tilde{z}_n$ up to error $C(n,\gamma)\epsilon'\delta' R$. By the paragraph above (4.23), z_n is also close to w'_n up to error $\Phi(\epsilon|n,\gamma)$. We can make this as small as we want. Thus we may assume $|\tilde{z}_n - w'_n| \leq C(n,\gamma)\epsilon'\delta' R$ on $B(y', \frac{3}{2}\delta' R)$. Cheng-Yau's gradient estimate implies the desired claim. □

By claim 4.5 and (4.32), we find

$$\fint_{B(y',\delta' R)} |Dw'_n|^2 < C(n,\gamma)\epsilon'. \tag{4.34}$$

Let $S' \in B(x, \frac{3}{2}R)$ be the preimage of S under the Gromov-Haudorff approximation. This is rough, but enough for purpose. If $\epsilon' << 1$, the type of D does not change when y' moves in $B(x, \frac{3}{2}R)\backslash B(S', 2\epsilon' R)$. We can consider covering of $\overline{B(x, \frac{5}{4}R)\backslash B(S', 2\epsilon' R)}$ by balls $B(y', \delta' R)$ so that each point belongs only to at most $C(n,\gamma)$ balls. This implies that

$$\fint_{B(x,R)\backslash B(S',2\epsilon' R)} |Dw'_n|^2 < C(n,\gamma)\epsilon'. \tag{4.35}$$

Gradient estimate says $|dw'_n| \leq C(n,\gamma)$ on $B(x,R)$. The volume convergence theorem [9] says

$$Vol(B(S', 2\epsilon' R) \cap B(x,R)) \leq Vol(B(S, 3\epsilon' R) \cap B_X((0,o),R)) + \Phi(\epsilon|n,\gamma). \tag{4.36}$$

Therefore, we have

$$\int_{B(S',2\epsilon' R)\cap B(x,R)} |dw'_n|^2 \leq C(n,\gamma)\epsilon'^2 \tag{4.37}$$

(4.35) and (4.37) imply

$$\int_{B(x,R)} |Dw'_n|^2 \leq C(n,\gamma)\epsilon'. \tag{4.38}$$

Given any $\epsilon' > 0$, we can find small $\epsilon > 0$ so that the inequalities above all hold. By taking the conjugate of w'_n if necessary, we conclude the proof of lemma 4.1.

□

Now we are ready to solve the $\overline{\partial}$-problem $\overline{\partial} f_i = \overline{\partial} w_i'$. By claim 4.2, claim 4.3, theorem 2.4, lemma 4.1 and the definition of Ω below claim 4.3,

$$\int_\Omega |f_i|^2 e^{-h_1} \leq \frac{1}{c(n,\gamma)} \int_\Omega |\overline{\partial} w_i'|^2 e^{-h_1} < \Phi(\epsilon|\gamma, n). \tag{4.39}$$

As w_i' is harmonic, f_i is harmonic. Therefore, mean value theorem [25] and gradient estimate imply that

$$|f_i|, |\nabla f_i| \leq \Phi(\epsilon|\gamma, n) \tag{4.40}$$

on $B(x, \frac{1}{2}\epsilon_0 R)$. Set $w_i = w_i' - f_i$. We can do a perturbation so that $w_i(x) = 0$. Next we prove that $(w_1, ..., w_n)$ is a holomorphic chart on $B(x, \frac{\epsilon_0}{4} R)$.

Claim 4.6. $\int_{B(x, \frac{\epsilon_0 R}{2})} \sum_{1\leq i\leq n-1; 1\leq j\leq n} |\langle dw_i(y'), \overline{dw_j(y')}\rangle - 2\delta_{ij}|^2 + ||dw_n(y')| - \sqrt{2}\frac{2\pi}{\alpha} r(y')^{\frac{2\pi}{\alpha}-1}|dy' < \Phi(\epsilon|n, \gamma)$.

Proof. By the definition of z_n right above proposition 4.1 and (4.26), $|\frac{\partial z_n}{\partial a_n}(y)| = \frac{2\pi}{\alpha} r(y)^{\frac{2\pi}{\alpha}-1}$. The proof follows from (4.19), (4.29), (4.31), (4.40) and claim 4.5. □

Recall K_r is defined in (4.1). By claim 4.6 and that w_i are holomorphic, we have

Claim 4.7. $|\frac{1}{(2\sqrt{-1})^n} \int_{B(x, \frac{\epsilon_0 R}{2})} dw_1 \wedge \overline{dw_1} \wedge \cdots \wedge dw_n \wedge \overline{dw_n} - vol(K_{\frac{\epsilon_0 R}{2}})| \leq \Phi(\epsilon|n, \gamma)$.

Set $w = (w_1, ..., w_n)$. By lemma 4.1, $w^{-1}(K_{\frac{\epsilon_0 R}{2} - \Phi(\epsilon|n,\gamma)})$ is relatively compact in $B(x, \frac{\epsilon_0 R}{2})$. Take the connected component K' of $w^{-1}(K_{\frac{\epsilon_0 R}{2} - \Phi(\epsilon|n,\gamma)})$ containing $B(x, \frac{\epsilon_0 R}{4})$. Then $w : K' \to K_{\frac{\epsilon_0 R}{2} - \Phi(\epsilon|n,\gamma)}$ is proper. Claim 4.7 implies that if ϵ is sufficiently small, the degree of w is 1. Thus w is generically one to one on K'. In particular, it is surjective. By the first conclusion of lemma 4.1, $w(B(x, \frac{1}{4}\epsilon_0 R))$ contains $K_{\frac{1}{4}\epsilon_0 R - \Phi(\epsilon|n,\gamma)}$. Observe w is a finite map, as the preimage of a point is a subvariety which is compact in the Stein manifold Ω. According to proposition 14.7 on page 87 of [17], w is an isomorphism on $B(x, \frac{1}{4}\epsilon_0 R)$. Now we can find the values of $\tilde{\epsilon}$ and δ required in proposition 4.1. This concludes the proof. □

Corollary 4.1. *Let (Y^n, x) be a complete Kähler manifold with bisectional curvature bounded from below by -1 and $vol(B(x, 1)) \geq v > 0$. Then there exist $0 < \epsilon' << 1$, $\delta_5, \delta_6 < 1$ depending only on n, v so that the following hold. If $d_{GH}(B(x, r), B_W(o, r)) < \epsilon' r$ for some metric cone (W, o) and $0 < r < \epsilon'$, then there exists a smooth function u on $B(x, 2\delta_5 r)$ with*

$$0 \leq u \leq C(n, v)\delta_5^2 r^2; u_{\alpha\overline{\beta}} \geq c(n, v) g_{\alpha\overline{\beta}} > 0; \tag{4.41}$$

$$\min_{y\in\partial B(x,\delta_5 r)} u(y) > 4 \sup_{y\in B(x,\delta_6 r)} u(y). \tag{4.42}$$

Proof. The proof is just a rescaled version of some arguments above. Let $0 < \delta_4 << 1$ depend only on n, v, to be determined. Set $(Y', x', g') = (Y, x, \frac{g}{\delta_4^2 r^2})$. Then $BK(Y') \geq -r^2\delta_4^2 \geq -\epsilon'^2$, $vol(B(x', \frac{1}{\delta_4})) \geq \frac{c(n)v}{\delta_4^{2n}}$ and

$$d_{GH}(B(x', \frac{1}{\delta_4}), B_W(o, \frac{1}{\delta_4})) < \frac{\epsilon'}{\delta_4}.$$

Observe Cheeger-Colding estimates (4.3) and (4.4) hold for the annulus $B(x', \frac{1}{5\delta_4}) \backslash B(x', \frac{\delta_4}{100})$, if ϵ' is sufficiently small depending on n, v, δ_4. By the same argument from (4.3) to claim 4.3, we find a function h_1 on $B(x', \frac{1}{100\delta_4})$, $1 >> \epsilon_0, \delta_4 > 0$ depending only on n, v satisfying

$$(h_1)_{\alpha\overline{\beta}} \geq c(n, v) g_{\alpha\overline{\beta}} > 0,$$

$$\min_{y\in\partial B(x',\frac{1}{2000\delta_4})} h_1(y) > 4 \sup_{y\in B(x',\frac{\epsilon_0}{\delta_4})} h_1(y); 0 \leq h_1(y) \leq C(n,v)\frac{1}{\delta_4^2}.$$

Now we freeze the value of $\delta_4 = \delta_4(n,v)$ and $\epsilon' = \epsilon'(n,v,\delta_4) = \epsilon'(n,v)$. We can think h_1 is defined on $B(x,\frac{r}{100})$. Take $u = \delta_4^2 r^2 h_1$, $\delta_5 = \frac{1}{2000}$, $\delta_6 = \epsilon_0$. This concludes the proof. □

Corollary 4.2. *Let (M^n, x) be a complete Kähler manifold with $BK \geq -1$. Let $(X,(0,o)) = (\mathbb{C}^{n-1},0)\times(Z,o)$, where (Z,o) is a real two dimensional cone with cone angle α. Assume $vol(B(x,1)) \geq v > 0$. Then there exist $0 < \tilde{\epsilon}', \delta_0 << 1$ depending only on n, v so that the following hold. If $0 < r < \tilde{\epsilon}'$ and $d_{GH}(B(x,r), B_X((0,o),r)) < \tilde{\epsilon}'^2 r$, then there exists a holomorphic chart $(w_1,...,w_n)$ on $B(x,\delta_0 r)$ such that*

- *$w_i(x) = 0 (1 \leq i \leq n)$.*
- *On $B(x,\delta_0 r)$, $|w_i| \leq C(n,v)r$.*
- *$(w_1,...,w_n)(B(x,\frac{1}{3}\delta_0 r)) \subset K_{\frac{5}{12}\delta_0 r}$. $(w_1,...,w_n)(B(x,\frac{4}{5}\delta_0 r)) \subset K_{\frac{8}{9}\delta_0 r}$. Recall that K_r is defined in (4.1).*
- *$(w_1,...,w_n)(B(x,\delta_0 r))$ contains the domain $K_{\frac{9}{10}\delta_0 r}$. $(w_1,...,w_n)(B(x,\frac{2}{3}\delta_0 r))$ contains the domain $K_{\frac{1}{2}\delta_0 r}$.*

Proof. Corollary 4.2 is a rescaled version of proposition 4.1. First, note that if $\tilde{\epsilon}'$ is sufficiently small, then $\alpha > c(n,v) > 0$. Set $(M',x',g') = (M,x,\frac{g}{\tilde{\epsilon}'^2 r^2})$. Then $BK(M') \geq -\tilde{\epsilon}'^2 r^2 \geq -\tilde{\epsilon}'^4$,

$$d_{GH}(B(x',\frac{1}{\tilde{\epsilon}'}), B_X((0,o),\frac{1}{\tilde{\epsilon}'})) < \tilde{\epsilon}'.$$

According to proposition 4.1, if $\tilde{\epsilon}' = \tilde{\epsilon}'(n,v)$ is sufficiently small, then the following hold. There exist $\delta = \delta(n,v) > 0$ and a holomorphic chart $(\tilde{w}_1,...,\tilde{w}_n)$ on $B(x',\delta)$ with

- $\tilde{w}_i(x') = 0$ for $1 \leq i \leq n$.
- On $B(x',\delta)$, $|\tilde{w}_i| \leq C(n,v)$.
- $(\tilde{w}_1,...,\tilde{w}_n)(B(x',\frac{1}{3}\delta)) \subset K_{\frac{5}{12}\delta}$. $(\tilde{w}_1,...,\tilde{w}_n)(B(x',\frac{4}{5}\delta)) \subset K_{\frac{8}{9}\delta}$.
- $(\tilde{w}_1,...,\tilde{w}_n)(B(x',\delta))$ contains the domain $K_{\frac{9}{10}\delta}$. $(\tilde{w}_1,...,\tilde{w}_n)(B(x',\frac{2}{3}\delta))$ contains the domain $K_{\frac{1}{2}\delta}$.

We may think $\tilde{w}_i$ are functions on $B(x,\delta\tilde{\epsilon}' r)$. Set $w_j = \tilde{\epsilon}' r \tilde{w}_j$ for $1 \leq j \leq n-1$, $w_n = (\tilde{\epsilon}' r)^{\frac{2\pi}{\alpha}}\tilde{w}_n$, $\delta_0 = \delta\tilde{\epsilon}'$. The proof is complete. □

Corollary 4.3. *Under the assumptions of proposition 4.1, there exist $\rho = \rho(n,\gamma) > 0$ and an open set Ω_x with $B(x,\delta a) \supset \Omega_x \supset B(x,\rho a)$, such that Ω_x biholomorphic to a ball in the Euclidean space. In particular, Ω_x is contractible.*

Now we prove corollary 1.2:

Proof. Assume M is not simply connected. Let γ be a smooth closed curve on M which represents a nonzero element in $\pi_1(M)$. By the second variation of arc length, one finds that γ cannot minimize the length in its free homotopy class. Thus we can take a sequence of smooth closed curves $\gamma_i \to \infty$ on M with $[\gamma_i] = [\gamma] \in \pi_1(M)$ and the length $|\gamma_i| \leq |\gamma|$. Let $q_i \in \gamma_i$. Let $r_i = d(p,q_i) \to \infty$.

Consider the blow down sequence $(M_i,p_i) = (M,p,\frac{g}{(r_i)^2})$. By passing to a subsequence, we may assume $(M_i,p_i) \to (X,p_\infty)$ in the pointed Gromov-Hausdorff sense. We may think that the $q_i \in \gamma_i \subset (M_i,p_i)$ and $q_i \to q_\infty \in (X,p_\infty)$. Then $d(p_\infty,q_\infty) = 1$ on X. By Cheeger-Colding [5], (X,p_∞) is a metric cone. Thus the tangent cone at q_∞ splits off a line. This means that given any $\epsilon > 0$, there exists $\delta > 0$ with $B(q_\infty,\delta)$ $\epsilon\delta$-Gromov-Hausdorff close to a ball in $(\mathbb{C},0)\times(Z,o)$ centered at $(0,o)$. Here (Z,o) is a complex one dimensional cone with cone angle $\alpha > 0$. Then for i sufficiently large, $B(q_i, \delta)$ is also $\epsilon\delta$-Gromov-Hausdorff close to a ball in $(\mathbb{C}, 0)\times(Z, o)$ centered at $(0, o)$. We may assume ϵ is so small that corollary 4.3 can be applied. Then there exists a contractible neighborhood Ω_i of q_i which contains a fixed size metric ball centered at q_i. As the length of γ_i is converging to zero in (M_i, p_i), eventually $\gamma_i \subset \Omega_i$. Hence γ_i is contractible. Contradiction! □

5. Separation of points

Proposition 5.1. *Let $v, R > 0$. There exists $\epsilon'_1 = \epsilon'_1(n, v) > 0$ so that the following hold. Let (Y'^n, q') be a complete Kähler manifold with bisectional curvature lower bound $-\frac{(\epsilon'_1)^3}{R^2}$. Assume $vol(B(q', \frac{R}{\epsilon'_1})) \geq \frac{vR^{2n}}{(\epsilon'_1)^{2n}}$. Assume also*

$$d_{GH}(B(q', \frac{1}{\epsilon'_1}R), B_W(o, \frac{1}{\epsilon'_1}R)) \leq \epsilon'_1 R \tag{5.1}$$

for some metric cone (W, o) centered at o. Then there exist $N' = N'(v, n) \in \mathbb{N}, 1 > \delta'_1 > 5\delta'_2 > c(v, n) > 0$ and holomorphic functions $g'^1, ..., g'^{N'}$ on $B(q', \delta'_1 R)$ with $g'^j(q') = 0$ and

$$\min_{x\in\partial B(q', \frac{1}{3}\delta'_1 R)} \sum_{j=1}^{N'} |g'^j(x)|^2 > 2 \sup_{x\in B(q', \delta'_2 R)} \sum_{j=1}^{N'} |g'^j(x)|^2. \tag{5.2}$$

Furthermore, for all j,

$$\frac{\sup_{x\in B(q', \frac{1}{2}\delta'_1 R)} |g'^j(x)|^2}{\sup_{x\in B(q', \frac{1}{3}\delta'_1 R)} |g'^j(x)|^2} \leq C(n, v). \tag{5.3}$$

Proof. The proof is a minor modification of proposition 3 in [22]. The key is an induction argument which involves the stratification of singular sets. Note we need to apply the three circle theorem 2.5. □

The next corollary is a rescaled version of proposition 5.1.

Corollary 5.1. *Let (Y^n, q) be a complete Kähler manifold with $BK \geq -1$ and $vol(B(q, 1)) > v > 0$. Then there exist $\epsilon_1 = \epsilon_1(n, v) > 0$ so that the following hold. Assume*

$$d_{GH}(B(q, r), B(o, r)) < \epsilon_1^2 r$$

for some metric cone (W, o) and $0 < r < \epsilon_1$. Then there exist $N = N(v, n) \in \mathbb{N}, 1 > \delta_1 > 5\delta_2 > c(v, n) > 0$ and holomorphic functions $g^1, ..., g^N$ on $B(q, \delta_1 r)$ with $g^j(q) = 0$ and

$$\min_{x\in\partial B(q, \frac{1}{3}\delta_1 r)} \sum_{j=1}^{N} |g^j(x)|^2 > 2 \sup_{x\in B(q, \delta_2 r)} \sum_{j=1}^{N} |g^j(x)|^2.$$

Furthermore, for all j,

$$\frac{\sup_{x\in B(q, \frac{1}{2}\delta_1 r)} |g^j(x)|^2}{\sup_{x\in B(q, \frac{1}{3}\delta_1 r)} |g^j(x)|^2} \leq C(n, v).$$

The proof is similar to corollary 4.2. It suffices to scale the metric by $\frac{1}{\epsilon_1^2 r^2}$. We omit the details. Now we come to the separation of points. The following proposition uses the same notations as in theorem 1.1.

Proposition 5.2. *Let $x \in M_\infty$, $r(x) = d(x, p_\infty)$. There exist $\epsilon_2 > 0, \delta_3 > 0, 1 > \gamma_1 > 0$ depending only on $n, r(x), v$ so that the following hold. Consider a sequence $x_i \to x$, $x_i \in M_i$. Let (X, o) be a metric cone centered at o. If $0 < R < \delta_3$ and $d_{GH}(B(x, R), B_X(o, R)) < \epsilon_2 R$, then for sufficiently large i and any two points $y_i^1 \neq y_i^2 \in B(x_i, \gamma_1 R)$ with $d(y_i^1, y_i^2) > d > 0$, there exists a holomorphic function f_i on $B(x_i, 2\gamma_1 R)$ with $f_i(y_i^1) = 0, f_i(y_i^2) = 1$ and $|f_i| \leq C(n, v, r(x), d, R)$.*

Remark 5.1. *The point is that all constants are independent of i. Thus, limit functions separate near points on M_∞.*

Proof. The volume comparison theorem says $vol(B(x_i, 1)) \geq c(n, v, r(x)) > 0$. By corollary 4.1, we can find small positive constants $\gamma_0, \gamma_1, \epsilon_2, \delta_3$ depending only on $n, v, r(x)$ and a function h_i with

$$C(n, v, r(x))\gamma_0^2 R^2 \geq h_i \geq 0 \tag{5.4}$$

on $B(x_i, \gamma_0 R)$. Moreover,

$$(h_i)_{\alpha\overline{\beta}} \geq c(n, v, r(x))g_{\alpha\overline{\beta}} > 0, \tag{5.5}$$

$$\min_{y\in\partial B(x_i, \frac{\gamma_0 R}{2})} h_i(y) > 4 \sup_{y\in B(x_i, 3\gamma_1 R)} h_i(y). \tag{5.6}$$

Let Ω_i be the connected component of $\{z \in B(x_i, \frac{\gamma_0 R}{2}) | h_i(z) < 2 \sup_{y\in B(x_i, 3\gamma_1 R)} h_i(y)\}$ containing $B(x_i, 3\gamma_1 R)$. Then Ω_i is relatively compact in $B(x_i, \frac{1}{2}\gamma_0 R)$. Ω_i is a Stein manifold.

Let $\epsilon'' > 0$ be a small constant depending only on $n, v, r(x)$. For any point y in $B(x_i, R)$, there exists $\frac{d}{10} > r_y > 0$ with

$$d_{GH}(B(y, r_y), B_{X_y}(o_y, r_y)) < \epsilon'' r_y. \tag{5.7}$$

Here (X_y, o_y) is a metric cone. We may assume ϵ'' and r_y are so small that corollary 5.1 can be applied. Now we freeze the value of ϵ''. By Gromov compactness theorem, we may also assume

$$\frac{d}{10} > r_y > c(n, v, d, r(x)) > 0. \tag{5.8}$$

Thus for $j = 1, 2$, there exist $N = N(v, n, r(x)) \in \mathbb{N}, 1 > \delta_1 > 5\delta_2 > c(v, n, r(x)) > 0$ and holomorphic functions $g_{ij}^1, ..., g_{ij}^N$ on $B(y_i^j, \delta_1 r_{y_i^j})$ with $g_{ij}^s(y_i^j) = 0$ and

$$\min_{z\in\partial B(y_i^j, \frac{1}{3}\delta_1 r_{y_i^j})} \sum_{s=1}^N |g_{ij}^s(z)|^2 > 2 \sup_{z\in B(y_i^j, \delta_2 r_{y_i^j})} \sum_{s=1}^N |g_{ij}^s(z)|^2. \tag{5.9}$$

Furthermore, for all s,

$$\frac{\sup_{z\in B(y_i^j, \frac{1}{2}\delta_1 r_{y_i^j})} |g_{ij}^s(z)|^2}{\sup_{z\in B(y_i^j, \frac{1}{3}\delta_1 r_{y_i^j})} |g_{ij}^s(z)|^2} \leq C(n, v, r(x)). \tag{5.10}$$

By normalization, we can also assume

$$\max_s \sup_{z\in B(y_i^j, \delta_2 r_{y_i^j})} |g_{ij}^s(z)| = 2. \tag{5.11}$$

Note by three circle theorem 2.5 and (5.10),

$$\max_{z\in B(y_i^j,\frac{1}{2}\delta_1 r_{y_i^j})}|g_{ij}^s(z)|\leq C(n,v,r(x)). \tag{5.12}$$

Set

$$F_i^j=\sum_{s=1}^N|g_{ij}^s|^2. \tag{5.13}$$

Let λ be a standard cut-off function: $\mathbb{R}^+\to\mathbb{R}^+$ given by $\lambda(t)=1$ for $0\leq t\leq 1$; $\lambda(t)=0$ for $t\geq 2$; $|\lambda'|,|\lambda''|\leq C(n)$. Consider

$$v_i^j(z)=4n\log F_i^j(z)\lambda(F_i^j(z)) \tag{5.14}$$

on $B(y_i^j,\frac{1}{3}\delta_1 r_{y_i^j})$. By (5.9) and (5.11), v_i^j is compactly supported on $B(y_i^j,\frac{1}{3}\delta_1 r_{y_i^j})$. We extend it to zero outside.

Lemma 5.1.

$$\sqrt{-1}\partial\overline{\partial}v_i^j(z)\geq -C(n,v,R,r(x),d)\omega_i \tag{5.15}$$

where ω_i is the Kähler metric on M_i. Moreover, $e^{-v_i^j}$ is not locally integrable at y_i^j.

Proof. The proof is similar to lemma 1 in [22]. We skip it here. Note (5.8) is crucial. □

Therefore, there exists $\xi=\xi(n,v,R,d,r(x))>0$ with

$$\sqrt{-1}\partial\overline{\partial}\psi_i\geq 5(n+1)\omega_i \tag{5.16}$$

on Ω_i, where $\psi_i=\xi h_i+v_i^1+v_i^2$. Then $\psi_i\leq C(n,v,r(x),d,R)$.

Now consider a function $\mu_i(z)=1$ for $z\in B(y_i^1,\frac{d}{4})$; μ_i has compact support in $B(y_i^1,\frac{d}{2})$; $|\nabla\mu_i|\leq C(n,d)$. By theorem 2.4, we can solve the equation $\overline{\partial}w_i=\overline{\partial}\mu_i$ in Ω_i (defined below (5.6)) with

$$\int_{\Omega_i}|w_i|^2e^{-\psi_i}\leq\int_{\Omega_i}|\overline{\partial}\mu_i|^2e^{-\psi_i}\leq C(n,v,R,r(x),d). \tag{5.17}$$

Set $f_i=\mu_i-w_i$ on $B(x,3\gamma_1 R)$. By lemma 5.1, $w_i(y_i^1)=w_i(y_i^2)=0$. As $\mu_i(y_i^1)=1,\mu_i(y_i^2)=0$, $f_i(y_i^1)=1$, $f_i(y_i^2)=0$. Note $\int_{\Omega_i}|f_i|^2\leq 2\int_{\Omega_i}(|\mu_i|^2+|w_i|^2)\leq C(n,v,R,d,r(x))$. By mean value inequality, $|f_i|\leq C(n,v,R,d,r(x))$ on $B(x_i,2\gamma_1 R)$. □

6. Construction of local coordinates on the limit space

Recall $\mathcal{WE}_{2n-2}=\{x\in M_\infty|\text{there exists a tangent cone splitting off }\mathbb{R}^{2n-2}\}$. For $x\in\mathcal{WE}_{2n-2}$, let C_x be a tangent cone at x which splits off $\mathbb{R}^{2n-2}$. Then

$$C_x(0,o)=(\mathbb{R}^{2n-2},0)\times(Z_x,o) \tag{6.1}$$

where Z_x is a real two dimensional cone with cone angle α satisfying $2\pi\geq\alpha\geq\gamma$. Here $\gamma=\gamma(r(x),v,n)>0$, $r(x)=d(x,p_\infty)$. For sufficiently large i, we can find $\epsilon_2>0, 1>>r'_x>0, x_i\in M_i, x_i\to x$ and

$$d_{GH}(B(x_i,r'_x),B_{C_x}((0,o),r'_x))<\epsilon_2 r'_x \tag{6.2}$$

so that the conditions of proposition 5.2 are satisfied. Let $\gamma_1=\gamma_1(n,v,r(x))>0$ be the constant in proposition 5.2. It is straightforward to see that

$$d_{GH}(B(x_i,\gamma_1 r'_x),B_{C_x}((0,o),\gamma_1 r'_x))<10\epsilon_2 r'_x \tag{6.3}$$

By shrinking the values of r'_x and ϵ_2 if necessary, we may assume that corollary 4.2 can be applied to $B(x_i, \gamma_1 r'_x)$. Then there exists a holomorphic chart $(w^i_{x1}, ..., w^i_{xn})$ on $B(x_i, \delta_0\gamma_1 r'_x)$ for $\delta_0 = \delta_0(n, v, r(x)) << 1$. Also

$$|w^i_{xs}| \leq C(n, v, r(x))r'_x. \tag{6.4}$$

Gradient estimate says

$$|dw^i_{xs}| \leq C(n, v, r(x)) \tag{6.5}$$

on $B(x_i, \frac{5}{6}\delta_0\gamma_1 r'_x)$. Now Arzela-Ascoli theorem implies that a subsequence of w^i_{xs} converges uniformly to w^∞_{xs} on $B(x, \frac{5\delta_0\gamma_1 r'_x}{6})$.

Lemma 6.1. *$(w^\infty_{x1}, ..., w^\infty_{xn})$ is injective on $B(x, \frac{4}{5}\delta_0\gamma_1 r'_x)$.*

Proof. Assume $q_1 \neq q_2 \in B(x, \frac{4}{5}\delta_0\gamma_1 r'_x)$ and $w^\infty_{xs}(q_1) = w^\infty_{xs}(q_2)$ for $1 \leq s \leq n$. Let $d = d(q_1, q_2) > 0$. Consider sequences $M_i \ni q^i_1 \to q_1, M_i \ni q^i_2 \to q_2$. We may assume $d(q^i_1, q^i_2) > \frac{d}{2} > 0$. According to proposition 5.2, we find f_i holomorphic on $B(x_i, 2\gamma_1 r'_x)$ with

$$f_i(q^i_1) = 0; f_i(q^i_2) = 1; |f_i| \leq C(n, r'_x, v, r(x), d). \tag{6.6}$$

As w^i_{xs} is a holomorphic chart on $B(x_i, \delta_0\gamma_1 r'_x)$, we may write $f_i(z) = g_i(w^i_{x1}(z), ..., w^i_{xn}(z))$ on $B(x_i, \delta_0\gamma_1 r'_x)$.

By corollary 4.2, the image of $(w^i_{x1}, ..., w^i_{xn})$ contains $K_{\frac{9}{10}\delta_0\gamma_1 r'_x}$ in $\mathbb{C}^n$. Then g_i is well defined on $K_{\frac{9}{10}\gamma_1\delta_0 r'_x}$. From the standard Cauchy integral estimate, we have

Claim 6.1. *$|\frac{\partial g_i}{\partial w^i_{xs}}| \leq C(n, v, r'_x, r(x), d)$ on $K_{\frac{8\delta_0\gamma_1 r'_x}{9}}$. In particular, g_i has a convergent subsequence. Also note by corollary 4.2, $(w^i_{x1}, ..., w^i_{xn})(B(x_i, \frac{4}{5}\delta_0\gamma_1 r'_x)) \subset K_{\frac{8}{9}\gamma_1\delta_0 r'_x}$.*

On the one hand, f_i has a convergent subsequence, say $f_i \to f_\infty$ uniformly on $B(x, \frac{3\gamma_1 r'_x}{2})$. Therefore, $f_\infty(q_1) = 0, f_\infty(q_2) = 1$. On the other hand, by claim 6.1 and that w^i_{xs} are convergent, after taking further subsequence, $f_i = g_i(w^i_{x1}, ..., w^i_{xn})$ converges uniformly to $f_\infty = g_\infty(w^\infty_{x1}, ..., w^\infty_{xn})$ on $B(x, \frac{4}{5}\delta_0\gamma_1 r'_x)$. Then

$$f_\infty(q_1) = g_\infty(w^\infty_{x1}(q_1), ..., w^\infty_{xn}(q_1)) = g_\infty(w^\infty_{x1}(q_2), ..., w^\infty_{xn}(q_2)) = f_\infty(q_2).$$

This is a contradiction. □

Let $\Omega_\infty = (w^\infty_{x1}, ..., w^\infty_{xn})^{-1}(K_{\frac{1}{2}\delta_0\gamma_1 r'_x})$.

Claim 6.2. *$(w^\infty_{x1}, ..., w^\infty_{xn})$ is a homeomorphism from Ω_∞ to $K_{\frac{1}{2}\delta_0\gamma_1 r_x}$.*

Proof. Ω_∞ is open in M_∞, as $(w^\infty_{x1}, ..., w^\infty_{xn})$ is continuous. According to corollary 4.2, $(w^i_{x1}, ..., w^i_{xn})^{-1}(K_{\frac{1}{2}\delta_0\gamma_1 r'_x}) \subset B(x_i, \frac{2}{3}\delta_0\gamma_1 r'_x)$. Then $\Omega_\infty \subset B(x, \frac{3}{4}\delta_0\gamma_1 r'_x)$. Lemma 6.1 implies that $(w^\infty_{x1}, ..., w^\infty_{xn})$ is injective on Ω_∞. It suffices to prove the surjectivity. For any $y \in K_{\frac{1}{2}\delta_0\gamma_1 r_x}$, let $z_i = (w^i_{x1}, ..., w^i_{xn})^{-1}(y) \in B(x_i, \frac{2}{3}\delta_0\gamma_1 r'_x)$. We may assume a subsequence of z_i converges to $z \in B(x, \frac{3}{4}\gamma_1\delta_0 r'_x)$. Then $y = (w^\infty_{x1}(z), ..., w^\infty_{xn}(z))$. This concludes the proof. □

Corollary 4.2 says $(w^i_{x1}, ..., w^i_{xn})(B(x_i, \frac{1}{3}\gamma_1\delta_0 r'_x)) \subset K_{\frac{5\delta_0\gamma_1 r'_x}{12}}$. Therefore

$$B(x, \frac{1}{3}\gamma_1\delta_0 r'_x) \subset \Omega_\infty.$$

We conclude that $(w^\infty_{x1}, ..., w^\infty_{xn})$ is a coordinate system on $B(x, \frac{1}{3}\gamma_1\delta_0 r'_x)$. Let

$$\tilde{\gamma}_1 = \gamma_1\delta_0. \tag{6.7}$$

Note $\tilde{\gamma}_1$ depends only on $n, v, r(x)$. Set

$$G = \bigcup_{x\in\mathcal{WE}_{2n-2}} B(x, \frac{1}{5}\tilde{\gamma}_1 r'_x). \tag{6.8}$$

Then G is open. The complement has codimension at least 4 by theorem 2.2 and theorem 2.3. Take a locally finite covering of G, say

$$G = \bigcup_{j\in\mathbb{N}} B(x^j, \frac{\tilde{\gamma}_1 r'_{x^j}}{5}). \tag{6.9}$$

By taking a subsequence, we may assume that $w^i_{x^j_i s}$ converge to $w^\infty_{x^j s}$ for $j \in \mathbb{N}$. Here $M_i \ni x^j_i \to x^j$.

Claim 6.3. $(w^\infty_{x^j 1}, ..., w^\infty_{x^j n})$ *form a holomorphic atlas on G.*

Proof. It suffices to prove the transition functions are holomorphic. One can just look at the transition functions on M_i for charts given by $w^i_{x^j_i s}$. By Cauchy estimates as in claim 6.1, one proves that the transition functions are holomorphic with uniform bound. Thus their limits are still holomorphic. □

From claim 6.3, G has a holomorphic structure. Let $x \in M_\infty$, $M_i \ni x_i$ and $x_i \to x$. Let $r_x, \epsilon_2 > 0$ satisfy $d_{GH}(B(x_i, r_x), B_X(o, r_x)) < \epsilon_2 r_x$ for some metric cone (X, o). We assume proposition 5.2 is satisfied. We have the following proposition.

Proposition 6.1. *For any* $y \in B(x, \frac{1}{2}\gamma_1 r_x) \cap G$*, there exist n sequences of holomorphic functions* $\lambda^i_j (1 \leq j \leq n)$ *on* $B(x_i, \gamma_1 r_x)$ *so that* $\lambda^i_j \to \lambda^\infty_j$ *uniformly on* $B(x, \frac{1}{2}\gamma_1 r_x)$ *and* $(\lambda^\infty_1, ..., \lambda^\infty_n)$ *forms a holomorphic coordinate around y.*

Proof. By the definition of G, (6.4), (6.5) and lemma 6.1, we can find a sequence $y_i \in M_i$ with $y_i \to y$ so that the following hold.

- there exist holomorphic charts $(w^i_1, ..., w^i_n)$ on $B(y_i, 5\delta)$ for some $\delta > 0$;
- $w^i_j \to w^\infty_j$ uniformly on $B(y, 4\delta)$;
- $(w^\infty_1, .., w^\infty_n)$ is a holomorphic chart on $B(y, 4\delta) \subset G$;
- $|w^i_j|$ is uniformly bounded on $B(y_i, 5\delta)$ for all i. Say $|w^i_j| \leq C$;
- $B(y_i, 10\delta) \subset B(x_i, \frac{1}{2}\gamma_1 r_x)$, $B(y, 10\delta) \subset B(x, \frac{1}{2}\gamma_1 r_x)$.
- $w^i_j(y_i) = 0$ for all i and j.
- For sufficiently large i, $(w^i_1, ..., w^i_n)(B(y_i, \delta)) \supset B_{\mathbb{C}^n}(0, \delta')$ for some $\delta' > 0$.
- $(w^\infty_1, ..., w^\infty_n)(B(y, \delta)) \supset B_{\mathbb{C}^n}(0, \delta')$.
- There exists $\delta'' > 0$ with $(w^i_1, ..., w^i_n)(B(y, \delta'')) \subset B_{\mathbb{C}^n}(0, \frac{\delta'}{2})$.

Consider smooth cut-off functions τ_i with $\tau_i = 1$ in $B(y_i, 2\delta)$, τ_i have compact support in $B(y_i, 3\delta)$, $|\nabla\tau_i| \leq \frac{20}{\delta}$. Let

$$h^i_j = \tau_i w^i_j. \tag{6.10}$$

Note h^i_j is holomorphic on $B(y_i, 2\delta)$. Recall the function h_i in proposition 5.2(replace R by r_x) satisfies (5.4), (5.5) and (5.6). Also recall the Stein manifold Ω_i right below (5.6). Let

λ be a standard cut-off function: $\mathbb{R}^+ \to \mathbb{R}^+$ given by $\lambda(t) = 1$ for $0 \le t \le 1$; $\lambda(t) = 0$ for $t \ge 2$; $|\lambda'|, |\lambda''| \le C(n)$. Define

$$F_i = \sum_{j=1}^{n} |w_j^i|^2. \tag{6.11}$$

Given a constant $\xi > 0$, set

$$\Psi_i = \xi h_i + 8n \log(F_i)\lambda(\frac{4F_i}{\delta'^2}). \tag{6.12}$$

Extend $\log(F_i)\lambda(\frac{4F_i}{\delta'^2})$ to zero outside $B(y_i, \delta)$. Similar as in (5.16), we can find a large constant ξ independent of i with

$$(\Psi_i)_{\alpha\overline{\beta}} \ge 5(n+1)(g_i)_{\alpha\overline{\beta}} \tag{6.13}$$

on Ω_i. Here g_i is the Kähler metric on M_i. We solve the $\overline{\partial}$-problem on Ω_i

$$\overline{\partial} f_j^i = \overline{\partial} h_j^i \tag{6.14}$$

satisfying

$$\int_{\Omega_i} |f_j^i|^2 e^{-\Psi_i} \le \int_{\Omega_i} |\overline{\partial} h_j^i|^2 e^{-\Psi_i} \tag{6.15}$$

Below $C_1, C_2, ...$ will be large constants independent of i. It is straightforward to verify that

$$\int_{\Omega_i} |\overline{\partial} h_j^i|^2 e^{-\Psi_i} \le C_1. \tag{6.16}$$

On $B(y_i, 3\delta)$, we can write $f_j^i = f_j^i(w_1^i, ..., w_n^i)$. We have

$$\int_{\Omega_i} |f_j^i|^2 e^{-\Psi_i} \le C_1, \tag{6.17}$$

Note for each fixed i, the volume form $(\frac{1}{2\sqrt{-1}})^n dw_1^i \wedge \overline{dw_1^i} \wedge \cdots \wedge dw_n^i \wedge \overline{dw_n^i}$ is equivalent to the volume form of g_i on $B(y_i, 3\delta)$. Since $w_j^i(y_i) = 0$, the local integrability near y_i implies

$$\frac{\partial f_j^i}{\partial w_s^i}(0, ..., 0) = 0 \tag{6.18}$$

for $1 \le j, s \le n$. Here $(0, .., 0) = (w_1^i(y_i), ..., w_n^i(y_i))$. Set $\lambda_j^i = h_j^i - f_j^i$ on $\Omega_i \supset B(x, 2\gamma_1 r_x)$. Note h_j^i is uniformly bounded. (6.17) implies that

$$\int_{\Omega_i} |\lambda_j^i|^2 \le C_2. \tag{6.19}$$

Mean value inequality implies that $|\lambda_j^i| \le C_3$ on $B(x_i, \gamma_1 r_x)$. As λ_j^i is holomorphic, by taking subsequences, we may assume $\lambda_j^i \to \lambda_j^\infty$ uniformly on $B(x, \frac{1}{2}\gamma_1 r_x)$. Recall $h_j^i = w_j^i$ on $B(y_i, 2\delta)$. According to (6.18),

$$\frac{\partial \lambda_j^i}{\partial w_s^i}(0, ..., 0) = \delta_{js}. \tag{6.20}$$

Letting $i \to \infty$, we obtain that

$$\frac{\partial \lambda_j^\infty}{\partial w_s^\infty}(0, ..., 0) = \delta_{js}. \tag{6.21}$$

This proves that $(\lambda_1^\infty, ..., \lambda_n^\infty)$ forms a holomorphic coordinate around y. □

7. Holomorphic functions on limit space

Definition 7.1. *Let $\mathcal{F}$ be the sheaf on M_∞ so that for any open set U of M_∞, $\Gamma(U, \mathcal{F})$ consists of all holomorphic functions on $U \cap G$ which are locally bounded on U.*

Below we use the same notions as in proposition 5.2. We shall replace R by r_x. Then, for some metric cone (X, o),

$$d_{GH}(B(x_i, r_x), B_X(o, r_x)) < \epsilon_2 r_x. \tag{7.1}$$

Lemma 7.1. *Let $x \in M_\infty$. Consider $x_i \in M_i$ with $x_i \to x$. If $f \in \Gamma(B(x, r_x), \mathcal{F})$, then there exists f_i holomorphic and uniformly bounded on $B(x_i, \frac{\gamma_1}{2} r_x)$ so that $f_i \to f$ uniformly on $B(x, \frac{\gamma_1}{2} r_x) \cap G$. Conversely, if f_i is holomorphic on $B(x_i, r_x)$ and $f_i \to f$ uniformly, then $f_{B(x, \frac{\gamma_1}{2} r_x)} \in \Gamma(B(x, \frac{\gamma_1}{2} r_x), \mathcal{F})$.*

Proof. Let f_i be holomorphic on $B(x_i, \gamma_1 r_x)$ and $f_i \to f$ uniformly on $B(x, \gamma_1 r_x)$. One just need to prove f is holomorphic on $G \cap B(x, \frac{1}{2}\gamma_1 r_x)$. This follows from the same argument as in lemma 6.1.

Now assume $f \in \Gamma(B(x, r_x), \mathcal{F})$. We shall use some cut-off argument similar as in [14]. Let $\Sigma = M_\infty \backslash G$ and Σ_i be the preimage of Σ in M_i by the Gromov-Hausdorff approximation (here is Σ_i need not be precisely defined). We are going to transplant f to $B(x_i, \frac{3}{4} r_x) \backslash B(\Sigma_i, d_i))$ for some $d_i \to 0$. By modifying the locally finite covering $B_j = B(x^j, \frac{1}{5}\tilde{\gamma}_1 r'_{x^j})$ in (6.9), we can find a partition of unity of G, φ_j, subordinate to B_j, smooth with respect to holomorphic structure on G. On B_j, we may write

$$\varphi_j = \varphi_j(w^\infty_{x^j 1}, ..., w^\infty_{x^j n}, \overline{w^\infty_{x^j 1}}, ..., \overline{w^\infty_{x^j n}}). \tag{7.2}$$

Define

$$\varphi_{ij} = \varphi_j(w^i_{x_i^j 1}, ..., w^i_{x_i^j n}, \overline{w^i_{x_i^j 1}}, ..., \overline{w^i_{x_i^j n}}). \tag{7.3}$$

Here we use the notations right below (6.9). Then on any compact set K of G, $\varphi_{ij} \to \varphi_j$ uniformly. If we replace φ_{ij} by $\frac{\varphi_{ij}}{\sum\limits_s \varphi_{is}}$, then $\sum\limits_j \varphi_{ij} = 1$ on K_i for sufficiently large i. Here K_i is the preimage of K in M_i.

G is dense in $B'_j = B(x^j, \frac{1}{3}\tilde{\gamma}_1 r'_{x^j})$. Note by the sentence above (6.7), there is a holomorphic chart on B'_j. Then f extends to a holomorphic function on $B'_j \cap B(x, r_x)$. It is clear the extension glues on the intersections of B'_j.

On $B'_j \cap B(x, r_x)$, write $f = f^j(w^\infty_{x^j 1}, ..., w^\infty_{x^j n})$ where f^j is holomorphic. Define

$$f_{ij} = f^j(w^i_{x_i^j 1}, ..., w^i_{x_i^j n}) \tag{7.4}$$

on $B(x_i^j, \frac{1}{5}\tilde{\gamma}_1 r'_{x_i^j}) \cap B(x_i, \frac{3}{4} r_x)$. Note this is well defined for sufficiently large i. Also $f_{ij} \to f$ on $B_j \cap B(x, \frac{3}{4} r_x)$. Now define a function

$$u_i = \sum_j \varphi_{ij} f_{ij}. \tag{7.5}$$

It is clear $u_i \to f$ uniformly on each compact set of $G \cap B(x, \frac{3}{4} r_x)$.

Claim 7.1. $|\overline{\partial} u_i| \to 0$ *uniformly on each compact set K of $G \cap B(x, \frac{3}{4} r_x)$. $|du_i|$ is uniformly bounded on the preimage of K in M_i.*

Proof. By definition, f_{ij} are holomorphic. Let $z \in B_{j_0}$. Consider a sequence $z_i \in B(x_i^{j_0}, \frac{1}{5}\tilde{\gamma}_1 r'_{x_i^{j_0}})$ with $z_i \to z$. Thus

$$
\begin{aligned}
|\overline{\partial}(\sum_j \varphi_{ij}(z_i) f_{ij}(z_i))| &= |\sum_j f_{ij}(z_i)\overline{\partial}\varphi_{ij}(z_i)| \\
&\leq \sum_j |f_{ij}(z_i) - f(z)||\overline{\partial}\varphi_{ij}(z_i)| + |f(z)||\overline{\partial}(\sum_j \varphi_{ij})(z_i)| \\
&\to 0.
\end{aligned}
\tag{7.6}
$$

The second assertion follows similarly. □

By the same argument as in proposition 3.5 of [14], we can find a smooth cut off function β_i on $B(x_i, r_x)$, satisfying $1 - \beta_i$ has compact support in a $\Phi(\frac{1}{i})$-neighborhood of Σ_i; equals 1 in a small neighborhood of Σ_i; $0 \leq \beta_i \leq 1$; $\int_{B(x_i, r_x)} |\nabla \beta_i|^2 \to 0$. We may also assume that $\beta_i \to 1$ sufficiently slow outside Σ. Define the function $g_i = u_i \beta_i$. Then we can make that on $B(x_i, \frac{2}{3} r_x)$,

$$
|g_i| \leq 2 \sup_{B(x, \frac{3}{4} r_x) \cap G} |f| + 1
\tag{7.7}
$$

Routine calculation shows

Claim 7.2. $\int_{B(x_i, \frac{2}{3} r_x)} |\overline{\partial} g_i|^2 \to 0$.

Let the function h_i satisfy (5.4), (5.5) and (5.6) with R replaced by r_x. Let $C = C(n, v, r(x)) > 0$ satisfy

$$
(Ch_i)_{\alpha\overline{\beta}} \geq 4(n+1) g_{\alpha\overline{\beta}} > 0.
\tag{7.8}
$$

Let Ω_i be the connected component of $\{z \in B(x_i, \frac{\gamma_0 r_x}{2}) | h_i(z) < 2 \max_{y \in B(x_i, 3\gamma_1 r_x)} h_i(y)\}$ containing $B(x_i, 3\gamma_1 r_x)$. Then Ω_i is relatively compact in $B(x_i, \frac{1}{2}\gamma_0 r_x) \subset B(x_i, \frac{2}{3} r_x)$ and Ω_i is a Stein manifold. Now we solve the $\overline{\partial}$-problem

$$
\overline{\partial} g'_i = \overline{\partial} g_i
\tag{7.9}
$$

on $\Omega_i \supset B(x_i, 3\gamma_1 r_x)$ with

$$
\int_{\Omega_i} |g'_i|^2 e^{-Ch_i} \leq \int_{\Omega_i} |\overline{\partial} g_i|^2 e^{-Ch_i} \to 0.
\tag{7.10}
$$

Therefore

$$
\int_{\Omega_i} |g'_i|^2 \to 0.
\tag{7.11}
$$

Then by (7.7), the holomorphic function $f_i = g_i - g'_i$ satisfies

$$
\int_{B(x, 3\gamma_1 r_x)} |f_i|^2 \leq C(n, v, r(x)) \sup_{B(x, \frac{3}{4} r_x)} (1 + |f|^2).
\tag{7.12}
$$

Mean value inequality and the gradient estimate imply

$$
|df_i|, |f_i| \leq C(n, v, r(x), r_x) \sup_{B(x, \frac{3}{4} r_x)} (1 + |f|)
\tag{7.13}
$$

on $B(x_i, \gamma_1 r_x)$. For any sequence $M_i \ni z_i \to z \in G \cap B(x, \frac{3}{4}\gamma_1 r_x)$, $dg_i(z_i) = du_i(z_i)$ for all sufficiently large i. Then by claim 7.1, $|dg'_i(z_i)| \leq |du_i(z_i)| + |df_i(z_i)|$ which has an upper bound independent of i. By (7.11), we obtain that $|g'_i| \to 0$ on each compact set of $G \cap B(x, \gamma_1 r_x)$. That is, $f_i \to f$ uniformly on each compact set of $G \cap B(x, \gamma_1 r_x)$. The convergence must be uniform on $G \cap B(x, \frac{1}{2}\gamma_1 r_x)$, since f_i is bounded and equicontinuous. $\square$

Corollary 7.1. *Let U be an open set of M_∞ and $f \in \Gamma(U, \mathcal{F})$. Then f extends to a continuous function on U.*

Proof. The problem is local. For $x \in U$, we can find r_x satisfying the conditions of proposition 5.2(r_x replaces R) and $B(x, 2r_x) \subset U$. The corollary follows from the first statement of lemma 7.1. $\square$

8. Completion of the proof of theorem 1.1

In this section, we shall apply some localized argument in [14]. Given $x \in M_\infty$, consider a sequence $x_i \in M_i$ converging to x. We still follow the notations in proposition 5.2 with R replaced by r_x. Then, r_x satisfies (7.1). We may also assume $r_x \geq c(n, v, r(x)) > 0$ by Gromov compactness theorem.

By applying proposition 5.2 and Gromov compactness theorem repeatedly, we can find some $m = N_0(n, v, r(x))$, $M = M(n, v, r(x)) > 0$, holomorphic functions g_i^s on $B(x_i, \gamma_1 r_x)(1 \leq s \leq N_0)$ with

$$g_i^s(x_i) = 0; |g_i^s| \leq M(n, v, r(x)); \tag{8.1}$$

$$\min_{y \in \partial B(x_i, \frac{1}{3}\gamma_1 r_x)} (\sum_{s=1}^{m} |g_i^s(y)|^2)^{\frac{1}{2}} \geq 2. \tag{8.2}$$

This merely means that we separate $\partial B(x_i, \frac{1}{3}\gamma_1 r_x)$ from x_i. Define $F_i^m = (g_i^1, ..., g_i^m)$. Below we will add more functions. That is, we increase the value m. By passing to subsequences, we always assume that $g_i^s \to g^s, F_i^m \to F^m$ on $B(x, \frac{1}{2}\gamma_1 r_x)$. We also assume that (8.1) is true for all $m \geq N_0$.

Let $|\cdot|$ be the standard norm on $\mathbb{C}^m$. By gradient estimate, $|dg_i^s| \leq C(n, v, r(x))$ on $B(x_i, \frac{1}{2}\gamma_1 r_x)$. Then there exists $\gamma_2 = \gamma_2(n, v, r(x))$ so that

$$|F_i^{N_0}(y)| < \frac{1}{10} \tag{8.3}$$

for $y \in B(x_i, \gamma_2 r_x)$. By applying proposition 5.2 and Gromov compactness theorem again, we find $\tau = \tau(n, v, r(x)) > 0$ and $N_1 = N_1(n, v, r(x)) > N_0$ with

$$|F_i^{N_1}(y)| < \frac{1}{5}, y \in B(x_i, \gamma_2 r_x), \tag{8.4}$$

$$|F_i^{N_1}| \geq 2\tau \tag{8.5}$$

on $B(x_i, \gamma_1 r_x) \backslash B(x_i, \frac{1}{2}\gamma_2 r_x)$. This can be achieved by rescaling $g_i^s (s > N_0)$ by small factors. We summarize the constructions above. For $m = N_1$, conditions (a)-(c) are valid:

(a). $g_i^s(x_i) = 0$ and $|g_i^s| \leq M(n, v, r(x))$ on $B(x_i, \gamma_1 r_x)$ for $1 \leq s \leq m$;

(b). $\min_{y \in \partial B(x_i, \frac{1}{3}\gamma_1 r_x)} |F_i^m(y)| \geq 2$; $|F_i^m(y)| < \frac{1}{5}$ for $y \in B(x_i, \gamma_2 r_x)$;

(c). $|F_i^m(y)| \geq 2\tau$ for $y \in B(x_i, \frac{1}{3}\gamma_1 r_x) \backslash B(x_i, \gamma_2 r_x)$.

When the value of m increases, we always assume F_i^m converges to F^m on $B(x, \frac{1}{2}\gamma_1 r_x)$, after taking subsequences. Furthermore, conditions (a)-(c) still hold. We further require that

(d). $|g_i^s| \leq \frac{\tau}{10^s}$ on $B(x_i, \gamma_1 r_x)$ for $s > N_1$.

This can be achieved if we rescale functions g_i^s for $s > N_1$.

Let Ω'_{im} be the connected component of $(F_i^m)^{-1}(B_{\mathbb{C}^m}(0,1))$ containing $B(x_i,\gamma_2 r_x)$. According to (b), $\Omega'_{im} \subset\subset B(x_i,\frac{1}{3}\gamma_1 r_x)$. Then F_i^m is a proper holomorphic map from Ω'_{im} to $B_{\mathbb{C}^m}(0,1)$. By the proper mapping theorem, the image $W_i^m \ni 0$ is an irreducible analytic set in $B_{\mathbb{C}^m}(0,1)$. We claim that W_i^m has complex dimension n. Indeed, if this is not true, pick a generic point $z \in W_i^m$ with $|z| < \tau$. Then $(F_i^m)^{-1}(z)$ has dimension greater than 0. By (c), $(F_i^m)^{-1}(z)$ is a compact analytic set in $B(x_i,\gamma_2 r_x)$. Note $B(x_i,\gamma_2 r_x)$ is contained in the Stein manifold Ω_i defined right below (5.6). Thus, $(F_i^m)^{-1}(z)$ consists of finitely many points. Contradiction.

By (a), there is a uniform gradient bound of g_i^s on $B(x_i,\frac{1}{3}\gamma_1 r_x)$. Then the image of $F_i^m(B(x_i,\frac{1}{3}\gamma_1 r_x))$ has uniform volume upper bound. Since g_i^s is convergent for each s, W_i^m is convergent in the Hausdorff metric sense to some W^m in $B_{\mathbb{C}^m}(0,1)$. By a theorem of Bishop [1], W^m is an analytic set of dimension n. By (b), we find $F^m(B(x,\gamma_2 r_x)) \subset W^m$.

We claim that after adding finitely many functions, $(F^m)^{-1}(z)$ is unique for generic $z \in B_{\mathbb{C}^m}(0,\tau) \cap W^m$. Note $W^m \cap B_{\mathbb{C}^m}(0,\tau)$ has finitely many irreducible components, say $W^{m1},...,W^{mj}$. Let $\Sigma'_m = F^m(B(x,\gamma_2 r_x)\backslash G)$. Then Σ'_m has codimension at least 4 in W^m, as F^m is Lipschitz. Therefore the regular points of $W^{mh}\backslash\Sigma'_m (1 \le h \le j)$ are connected. According to (c), the preimage of any point in $(W^{mh}\backslash\Sigma'_m) \cap B_{\mathbb{C}^m}(0,\tau)$ is a compact analytic subvariety in $G \cap B(x,\gamma_2 r_x)$. Thus we can separate it by adding only finitely many functions. We do this for all $1 \le h \le j$. Then $(F^m)^{-1}(z)$ is unique for generic $z \in B_{\mathbb{C}^m}(0,\tau)\cap W^m$. Say now $m = N_2$.

Next we prove that for some larger m, a small neighborhood of x is homeomorphic to $W^m \cap B_{\mathbb{C}^m}(0,\frac{\tau}{3})$. For any $k > l \ge N_2$, there exists a natural projection P_{kl}: $W^k \to W^l$. We have $P_{kl} \circ F^k = F^l$ on $B(x,\gamma_2 r_x)$. Let $z \in W^l \cap B_{\mathbb{C}^l}(0,\frac{1}{3}\tau)$. Then by (d), $P_{kl}^{-1}(z)$ is a compact analytic subvariety in $W^k \cap B_{\mathbb{C}^k}(0,\frac{\tau}{2})$. Hence it contains only finitely many points. Similar as on page 90 of [14], the number of $P_{kl}^{-1}(z)$ is actually bounded by the number of locally irreducible component of z in W^l. As on page 90 of [14], we may write $W^m \cap B_{\mathbb{C}^m}(0,\frac{\tau}{3})$ as a finite union of sets Z_α which are given by analytic variety minus analytic subvariety so that $(F^m)^{-1}(Z_\alpha)$ is a disjoint union of n_α copies of Z_α. By induction argument as on page 90 of [14], after adding finitely many functions, we find the preimage of $W^m \cap B_{\mathbb{C}^m}(0,\frac{\tau}{3})$ is unique. This proves that a small neighborhood of x is homeomorphic to $W^m \cap B_{\mathbb{C}^m}(0,\frac{\tau}{3})$. Say now $m = N_3$.

Next we prove that $W^m \cap B_{\mathbb{C}^m}(0,\frac{\tau}{3})$ is locally irreducible for $m \ge N_3$. If this is not true, we can find $z \subset W^m \cap B_{\mathbb{C}^m}(0,\frac{\tau}{3})$ and $\lambda > 0$ with $B_{\mathbb{C}^m}(z,\lambda) \cap W^m \subset B_{\mathbb{C}^m}(0,\frac{\tau}{3})$ and $B_{\mathbb{C}^m}(z,\lambda) \cap W^m$ is connected. Moreover, there exist holomorphic functions u, v on $B_{\mathbb{C}^m}(z,\lambda) \cap W^m$ with $uv = 0$, but u, v are not identically zero. Now $(F^m)^{-1}(B_{\mathbb{C}^m}(z,\lambda) \cap W^m)$ is a connected open set in $B(x,\gamma_2 r_x)$. It is clear that u, v are holomorphic on $G' = G \cap (F^m)^{-1}(B_{\mathbb{C}^m}(z,\lambda) \cap W^m)$. Recall $\mathcal{R}_\epsilon$ in definition 2.2. According to corollary 4.2, if $\epsilon = \epsilon(n)$ is sufficiently small, $\mathcal{R}_\epsilon$ is regular in the holomorphic sense. That is, for any $y \in \mathcal{R}_\epsilon$, there exists a holomorphic chart around y. Note $\mathcal{R}$ is dense. Assume at $y \in \mathcal{R} \cap G'$, $u(y) \neq 0$. Then v vanishes in a small neighborhood of y. By applying theorem 3.9 in [6] and the unique continuation of holomorphic functions, we find $v \equiv 0$. Contradiction.

Let S_m be the singular set of $W^m \cap B_{\mathbb{C}^m}(0,\frac{\tau}{3})$. We claim that for some larger m, $(F^m)^{-1}(S_m) \subset B(x,\gamma_2 r_x)\backslash G$. This is equivalent to saying that F^m maps $G \cap (F^m)^{-1}(B_{\mathbb{C}^m}(0,\frac{\tau}{3}))$ to the regular part of W^m. Note this is also equivalent to that the holomorphic structure on $G \cap (F^m)^{-1}(B_{\mathbb{C}^m}(0,\frac{\tau}{3}))$ is the same as the one induced from $W^m \cap B_{\mathbb{C}^m}(0,\frac{\tau}{3})$. S_m is a finite union of irreducible analytic sets in $B_{\mathbb{C}^m}(0,\frac{\tau}{3})$. Let $S_m^t (1 \le t \le l)$ be irreducible components so that $(F^m)^{-1}(S_m^t)$ intersects G. Pick a point $y \in (F^m)^{-1}(S_m^t) \cap G$. According to proposition 6.1, we can find sequences of holomorphic functions λ_j^i on $B(x_i,\gamma_1 r_x)$. Also $\lambda_j^i \to \lambda_j^\infty$

and λ_j^∞ form a holomorphic coordinate near y. If we add these functions to g_i^s(with certain normalizations), the dimension of S_m^t decreases. Then the claim follows from a standard induction.

For $x \in M_\infty$ as above, we consider the analytic structure in a neighborhood induced by F^m. Let O be the structure sheaf and O_x be the stalk at x. Now we prove that after adding finitely many functions, O_x is normal. There exists an open set $(F^m)^{-1}(B_{\mathbb{C}^m}(0, \frac{\tau}{3})) \supset U \ni x$ and a normalization $\hat{U} \to U \ni x$ so that $O(\hat{U})$ is a finite module over $O(U)$. Note by (14.11) on page 89 of [17], the natural map $\hat{U} \to U$ is a homeomorphism, as U is locally irreducible. Let us assume $O(\hat{U})$ is generated by $u_1, ..., u_k \in \Gamma(U, \mathcal{F})$ over $O(U)$. Thus they extend to continuous functions on U. According to lemma 7.1, there exist $\delta > 0, \epsilon_0 > 0$ and holomorphic functions u_j^i on $B(x_i, 2\delta)$ with $u_j^i \to u_j (1 \leq j \leq k)$ uniformly on $B(x, \epsilon_0) \cap G$. By adding these functions to g_i^s and shrinking the neighborhood of x, we find that O_x is normal.

As normal points are open (theorem 14.4 on page 87 of [17]), we proved that for any point $x \in M_\infty$, there exists a neighborhood $U_x \ni x$ so that U_x is a normal analytic variety with structure sheaf $O(x)$. Let $z \in U_x \cap U_y$. To prove that M_∞ is a normal complex analytic variety, it suffices to prove that $(O(x))_z = (O(y))_z$(stalk) for $z \in V \subset U_x \cap U_y$. Let $f \in \Gamma(V, O(x))$. Then $f|_{V \cap G} \in \Gamma(G \cap V, O(y))$. As $V \backslash G$ has real codimension 4 and $O(y)$ is normal, $f \in \Gamma(V, O(y))$. This completes the proof of theorem 1.1.

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