Coherent state superpositions, entanglement and gauge/gravity correspondence

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Abstract

We focus on two types of coherent states, the coherent states of multi graviton states and the coherent states of giant graviton states, in the context of gauge/gravity correspondence. We conveniently use a phase shift operator and its actions on the superpositions of these coherent states. We find N-state Schrodinger cat states which approach the one-row Young tableau states, with fidelity between them asymptotically reaches 1 at large N. The quantum Fisher information of these states is proportional to the variance of the excitation energy of the underlying states, and characterizes the localizability of the states in the angular direction in the phase space. We analyze the correlation and entanglement between gravitational degrees of freedom using different regions of the phase space plane in bubbling AdS. The correlation between two entangled rings in the phase space plane is related to the area of the annulus between the two rings. We also analyze two types of noisy coherent states, which can be viewed as interpolated states that interpolate between a pure coherent state in the noiseless limit and a maximally mixed state in the large noise limit.

1 Introduction

The gauge/gravity correspondence [1, 2, 3] is a nontrivial correspondence between a quantum system without gravity on the boundary and a quantum theory with gravity in the bulk. It provides a method for working on quantum gravity by quantum field theory on the boundary of the spacetime. The correspondence allows us to perform calculations related to superstring theory and quantum gravity from the quantum field theory side. On the other hand, the superstring theory provides the UV completion of the supergravity, and is hence a UV-complete quantum gravity. The correspondence also reveals the nature of the emergent spacetime [4, 5, 6, 7]. The bulk emerges dynamically from the quantum mechanical description that lives in fewer dimensions on the boundary.

The boundary system is described by a quantum field theory or quantum mechanics with a well-defined global time near the boundary. Hence, the boundary theory has a well-defined Hilbert space. Moreover, there is a correspondence between observables of the bulk spacetime and observables of the boundary. Progress on the detailed dictionary between the quantum field theory side and the gravity side, has been made tremendously. For states that are in the same Hilbert space, we can perform superpositions of these states. We can also perform other quantum operations [8] such as unitary transformations on these states.

In the gauge/gravity correspondence, the quantum field theory side of the duality is an example of a quantum system with many degrees of freedom. In such a many-body quantum system, quantum correlations and quantum entanglement are generic features [9]. Quantum correlations provide important resources for information processing, and are important in quantum information theory. There are states that are of interest both in the gravity side and in quantum information theory, such as coherent states and their superpositions and entanglement. Moreover, there are also Young tableau states, which are entangled states.

In the context of this correspondence, there are backreacted geometries that correspond to highly excited states in the quantum field theory side, such as the bubbling geometries [10, 11, 12]. The states in the Hilbert space of the quantum field theory are explicitly mapped to the gravity side. Analysis in the field theory side shows that these different configurations live in the same Hilbert space. Since they live in the same Hilbert space, one can perform quantum operations allowed by quantum mechanics, for example one can superpose states and compute transition probabilities between different states, e.g. [13, 14, 15, 16].

We focus on states which have interesting gravitational properties. One interesting type of states are coherent states [14]. Gravity dual of coherent states has been analyzed. It has also been shown that superposition of topologically trivial states can form topologically nontrivial states in the context of quantum gravity theory [14]. In this paper, we focus on two highly interesting classes of coherent states. One is the coherent state of multi traces, and another is the coherent state of multi columns of Young tableaux.

Another interesting type of states are Young tableau states. They are also entangled states in the tensor product of the multi-trace Hilbert spaces [14, 16]. These states contain nontrivial entanglement stored between different multi-trace Hilbert spaces.

The phase space also plays an useful role in identifying the states. The map between the states in the gravity side and in the field theory side can also be done by using Wigner function representation of the states in the phase space, e.g. [17, 18]. We work on correlations and entanglement between different regions of the phase space plane [19, 16]. These setups provide a laboratory for studying quantum gravitational questions.

The organization of this paper is as follows. In Section 2, we overview a class of single mode and multi mode coherent states, which are central in the concepts and for the preparation of later discussions. We define a generalized phase shift operator and parity operator, acting on the superpositions of coherent states. In Section 3, we find Schrodinger cat states, and in particular N-state Schrodinger cats which have a limit approaching Young tableau states. In Section 4, we analyze noisy coherent states by adding random traces on top of pure coherent states. And then in Section 5, we analyze a different class of coherent states, which are coherent states of multi columns of Young tableaux. In Section 6, we make use of quantum detection theory to distinguish microstates in the systems discussed in this paper. In Section 7, we analyze correlations and entanglement between different regions of the phase space, and point out that they can be understood in terms of connectivity between different regions of spacetime. Finally, we discuss our results and draw some conclusions in Section 8.

2 Class of pure coherent states

2.1 Class of single mode and multi mode pure coherent states

In this subsection, we discuss a class of single mode and multi mode coherent states. This class was constructed in [14], and analyzed in further details in [16]. Many later sections are closely related to the concept and method in this subsection, hence we overview them in details for the preparation of later discussions. There is also a different class of coherent states as coherent states of multi columns of Young tableaux in Sec. 5.

Consider the Hilbert space factorizes as $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots = \otimes_k \mathcal{H}_k$. Here $\mathcal{H}_k$ is the Hilbert space for mode $k \in \mathbb{Z}_{>0}$. The creation and annihilation operators for mode k are $a_k^\dagger$ and a_k. Their commutation relations are

$$[a_k,\ a_{k'}^\dagger] = k\delta_{kk'}, \tag{2.1}$$

with the appropriate normalization convention. The state in mode k, with occupation number l, is

$$t_k^l = (a_k^\dagger)^l|0\rangle_k, \tag{2.2}$$

where $|0\rangle_k$ is the vacuum of $\mathcal{H}_k$ and $t_k^0 = |0\rangle_k$. The state $\frac{1}{\sqrt{l!k^l}}(a_k^\dagger)^l|0\rangle_k$ has unit norm.

This construction works generally for systems having a similar Hilbert space, and in the context of half BPS sector of the gauge theory in which gauge invariant observables can be constructed from a complex matrix Y, the t_k corresponds to $\mathrm{Tr}(Y^k)$. In that case,

$$t_k = a_k^\dagger|0\rangle_k = \frac{1}{\sqrt{N^k}}\mathrm{Tr}(Y^k), \tag{2.3}$$

and $t_k^l = (a_k^\dagger)^l|0\rangle_k = (\frac{1}{\sqrt{N^k}}\mathrm{Tr}(Y^k))^l$. The t_k^l and t_k^0 are the analogs of the photon states $|l\rangle$ and $|0\rangle$ in quantum optics and quantum information theory. In the context of gauge theory, the prefactors involving N can be calculated by gauge theory computations, e.g. [20, 21, 22, 14]. In this paper, we work in the large N limit, because in this limit, the multi traces provide good orthogonality property. In the context of gauge/string duality, t_k is also a closed string state.

We consider coherent states generalized from the coherent states of photons [23, 24, 25]. A general *multi-mode* coherent state can be written as, in the language of the creation operator $a_k^\dagger$,

$$|Coh(\{\Lambda_k\})\rangle = \prod_k \exp(\Lambda_k \frac{a_k^\dagger}{k})|0\rangle_k \tag{2.4}$$

$$= \prod_k (\sum_{l_k=0}^{\infty} \frac{1}{l_k!}(\Lambda_k \frac{a_k^\dagger}{k})^{l_k})|0\rangle_k, \tag{2.5}$$

where $\Lambda_k \in \mathbb{C}$. In the language of t_k,

$$|Coh(\{\Lambda_k\})\rangle = \prod_k \exp(\Lambda_k \frac{t_k}{k}) \tag{2.6}$$

$$= \prod_k (\sum_{l_k=0}^{\infty} \frac{1}{l_k!}(\Lambda_k \frac{t_k}{k})^{l_k}). \tag{2.7}$$

The $\{\Lambda_k\}_{k=1,\ldots,\infty}$ in $Coh(\{\Lambda_k\})$ is a family of complex parameters for the modes $k \in \mathbb{Z}_{>0}$. These coherent states are at the level of multi-mode coherent states in the Hilbert space $\otimes_k \mathcal{H}_k$.

These are *pure* coherent states. They are the eigenstates of the annihilation operator a_k with eigenvalue Λ_k,

$$a_k|Coh(\{\Lambda_k\})\rangle = \Lambda_k|Coh(\{\Lambda_k\})\rangle. \tag{2.8}$$

We have our convention in the expression of the coherent states (2.4).

The single-mode coherent state is a special case. We denote $|Coh(\{\Lambda_k\})\rangle$ for the multi-mode one, and $|coh(\Lambda_k)\rangle_k = \exp(\Lambda_k \frac{a_k^\dagger}{k})|0\rangle_k$ for the single-mode one in mode k.

For a single mode k, the pure coherent state, with unit norm, is

$$\frac{1}{\sqrt{\mathcal{N}(\Lambda_k)}}|coh(\Lambda_k)\rangle_k = e^{-\frac{|\Lambda_k|^2}{2k}} \exp(\Lambda_k \frac{a_k^\dagger}{k})|0\rangle_k. \tag{2.9}$$

The subscript k denotes that the state is in mode-k subspace $\mathcal{H}_k$. The $\mathcal{N}(\Lambda_k)$ is a normalization factor $\langle 0|_k \exp(\bar{\Lambda}_k \frac{a_k}{k}) \exp(\Lambda_k \frac{a_k^\dagger}{k})|0\rangle_k = e^{\frac{|\Lambda_k|^2}{k}}$. The density matrix of a single-mode pure coherent state is

$$\rho_{coh(\Lambda_k)} = \frac{1}{\mathcal{N}(\Lambda_k)} \exp(\Lambda_k \frac{a_k^\dagger}{k})|0\rangle_k \langle 0|_k \exp(\bar{\Lambda}_k \frac{a_k}{k}) \tag{2.10}$$

and tr $\rho_{coh(\Lambda_k)} = 1$.

The normalization factor of the multi-mode coherent state is as follows. The normalization factor for $|Coh(\{\Lambda_k\})\rangle$ is

$$\mathcal{N}(\{\Lambda_k\}) = \langle Coh(\{\Lambda_k\})|Coh(\{\Lambda_k\})\rangle = \exp(\sum_{k=1}^{\infty} \frac{|\Lambda_k|^2}{k}). \tag{2.11}$$

A *special* multi-mode coherent state is when $\Lambda_k := (\Lambda)^k$, in which $\Lambda \in \mathbb{C}$. The amplitude in each mode is correlated, since $\Lambda_k = \Lambda^k$. This multi-mode state is

$$|Coh(\Lambda)\rangle = |Coh(\{\Lambda^k\})\rangle = \prod_{k=1}^{\infty} \exp(\Lambda^k \frac{a_k^\dagger}{k})|0\rangle_k = \prod_{k=1}^{\infty} |coh(\Lambda^k)\rangle. \tag{2.12}$$

We denote $\rho(\Lambda) = \rho(\{\Lambda^k\})$ and the normalization factor $\mathcal{N}(\Lambda) = \mathcal{N}(\{\Lambda^k\})$. The normalization for $|Coh(\Lambda)\rangle$ is

$$\mathcal{N}(\Lambda) = \langle Coh(\Lambda)|Coh(\Lambda)\rangle = \exp(\sum_{k=1}^{\infty} \frac{|\Lambda|^{2k}}{k}) = \frac{1}{(1-|\Lambda|^2)}. \tag{2.13}$$

The expansion we use is $-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$. The density matrix for the special multi-mode coherent state is

$$\rho(\Lambda) = \frac{1}{\mathcal{N}(\Lambda)} \prod_{k=1}^{\infty} \exp(\Lambda^k \frac{a_k^\dagger}{k})|0\rangle_k \langle 0|_k \exp(\bar{\Lambda}^k \frac{a_k}{k}). \tag{2.14}$$

The product is tensor product and tr $\rho(\Lambda) = 1$.

The inner product of two multi-mode coherent states is

$$\begin{aligned} \langle Coh(\Lambda_{(0)})|Coh(\Lambda_{(1)})\rangle &= \prod_{k=1}^{\infty} \langle 0|_k \exp(\bar{\Lambda}_{(0)}^k \frac{a_k}{k}) \exp(\Lambda_{(1)}^k \frac{a_k^\dagger}{k})|0\rangle_k \\ &= \frac{1}{(1-\bar{\Lambda}_{(0)}\Lambda_{(1)})}. \end{aligned} \tag{2.15}$$

Hence the inner product of two unit-norm multi-mode coherent states is

$$\begin{aligned}&\frac{1}{\sqrt{\mathcal{N}(\Lambda_{(0)})\mathcal{N}(\Lambda_{(1)})}}\langle Coh(\Lambda_{(0)})|Coh(\Lambda_{(1)})\rangle\\=&\frac{\left(1-|\Lambda_{(0)}|^2\right)^{1/2}\left(1-|\Lambda_{(1)}|^2\right)^{1/2}}{(1-\bar{\Lambda}_{(0)}\Lambda_{(1)})}.\end{aligned} \tag{2.16}$$

2.2 Phase shift operators

In this subsection, we consider unitary operations or unitary transformations on the states considered throughout this paper, in a general setting. We first discuss the unitary operations on the generalized coherent states. These unitary operations act similarly on other classes of states that we shall discuss in Sec. 3 and Sec. 5.

We first define a generalized phase shift operator. Consider the ordinary phase shift operator $\exp(i\theta_k\hat{N}_k)=\exp(i\frac{\theta_k}{k}a_k^\dagger a_k)$, which acts on the single-mode coherent state as

$$\exp(i\frac{\theta_k}{k}a_k^\dagger a_k)|coh(\Lambda_k)\rangle=|coh(\Lambda_k e^{i\theta_k})\rangle. \tag{2.17}$$

Note that

$$\exp(i\theta_k\hat{N}_k)|t_k^l\rangle=\exp(il\theta_k)|t_k^l\rangle=\exp(ikl\theta)|t_k^l\rangle. \tag{2.18}$$

Consider

$$\theta_k=k\theta. \tag{2.19}$$

In this case, for the multi-mode coherent state $|Coh(\Lambda)\rangle$, $\Lambda_k e^{i\theta_k}=\Lambda^k e^{ik\theta}$. The phase shift operator on the multi-mode coherent state is then

$$\exp(\sum_k i\frac{\theta_k}{k}a_k^\dagger a_k)|Coh(\Lambda)\rangle=|Coh(\Lambda e^{i\theta})\rangle. \tag{2.20}$$

Hence we define

$$\exp(i\theta\hat{E}):=\exp(\sum_k i\theta a_k^\dagger a_k) \tag{2.21}$$

to be the phase shift operator on $|Coh(\Lambda)\rangle$, where $\hat{E}=\sum_k a_k^\dagger a_k$ is the excitation energy operator. The θ can be viewed as a variable conjugate to the generator $\hat{E}$. The action is

$$\exp(i\theta\hat{E})|Coh(\Lambda)\rangle=|Coh(\Lambda e^{i\theta})\rangle \tag{2.22}$$

for any $|Coh(\Lambda)\rangle$. For any superposition of coherent states, the action is

$$\exp(i\theta\hat{E})(c_1|Coh(\Lambda_1)\rangle+c_2|Coh(\Lambda_2)\rangle)=c_1|Coh(\Lambda_1 e^{i\theta})\rangle+c_2|Coh(\Lambda_2 e^{i\theta})\rangle. \tag{2.23}$$

An interesting superposition is

$$|Coh(\Lambda)\rangle + e^{i\varphi}|Coh(\Lambda e^{i\chi})\rangle = (1 + e^{i\varphi I}e^{i\chi\hat{E}})|Coh(\Lambda)\rangle. \tag{2.24}$$

The phase shift operator can be used to define a generating function for a *general* state $|\Psi\rangle$,

$$\langle \exp(i\theta\hat{E})\rangle_{|\Psi\rangle} := \frac{\langle\Psi|\exp(i\theta\hat{E})|\Psi\rangle}{\|\Psi\|^2}. \tag{2.25}$$

We define the expectation value of the excitation energy

$$E_{|\Psi\rangle} := \langle\hat{E}\rangle_{|\Psi\rangle} \tag{2.26}$$

above the ground state, and the zero point energy is subtracted out in the definition of $\hat{E}$. In other words, for ground state, E is the energy subtracting the zero point energy. For the special case $|Coh(\Lambda)\rangle$, its generating function is

$$\langle \exp(i\theta\hat{E})\rangle_{Coh(\Lambda)} = \frac{(1-|\Lambda|^2)}{(1-|\Lambda|^2e^{i\theta})}. \tag{2.27}$$

By taking derivatives, we have $\langle\hat{E}\rangle_{Coh(\Lambda)} = \frac{|\Lambda|^2}{1-|\Lambda|^2}$, which is the same as the direct summation in Sec. 4.1 as in (4.16).

We define the variance of the excitation energy $\hat{E}$ on a general state, to be $(\Delta E)^2$, by

$$(\Delta E)^2_{|\Psi\rangle} := \langle\hat{E}^2\rangle_{|\Psi\rangle} - \langle\hat{E}\rangle^2_{|\Psi\rangle}. \tag{2.28}$$

For a general state $|\Psi\rangle$, $E_{|\Psi\rangle} = \langle\hat{E}\rangle_{|\Psi\rangle}$ and $(\Delta E)^2_{|\Psi\rangle}$ are two important physical quantities charactering that state. These two quantities can be calculated by the generating function $\langle\exp(i\theta\hat{E})\rangle_{|\Psi\rangle}$. Note that θ is also the phase variable or the angular variable of the phase space from our perspectives.

Using Heisenberg uncertainty relation, θ is the conjugate variable for the generator $\hat{E}$ in $\exp(i\theta\hat{E})$, and hence we are subject to a Heisenberg uncertainty relation between ΔE and $\Delta\theta$, i.e. $\Delta E\cdot\Delta\theta \gtrsim \frac{1}{2}$.

The $(\Delta E)^2$ is related to the quantum Fisher information (QFI) of the state. The quantum Fisher information $4(\Delta E)^2$ characterizes how well the state can be localized in the θ direction, or how well the state can be resolved and distinguished in the θ direction. The concept of quantum Fisher information is useful for quantum precision measurement [26, 27, 28]. In quantum information theory, the quantum Fisher information of the state $|\Psi\rangle$ can be written as

$$\mathrm{QFI}(|\Psi\rangle) = 4(\Delta E)^2_{|\Psi\rangle}. \tag{2.29}$$

And $\Delta\theta_{|\Psi\rangle} \gtrsim \frac{1}{2\Delta E_{|\Psi\rangle}} = \frac{1}{\sqrt{\mathrm{QFI}(|\Psi\rangle)}}$. Small $\Delta\theta_{|\Psi\rangle}$ means that the state is localized on the θ direction, such as $|Coh(\Lambda)\rangle$, and big $\Delta\theta_{|\Psi\rangle}$ means that the state is delocalized on the θ direction, such as $|t_k^l\rangle$. States with symmetry on θ direction has big $\Delta\theta_{|\Psi\rangle}$. A state with a greater $4(\Delta E)^2$ has a greater localizabiliy and resolution in the θ direction. Hence, $4(\Delta E)^2_{|\Psi\rangle}$ measures the quantum Fisher information of the state and the localizability of the state in the angular direction in phase space. We shall calculate $(\Delta E)^2_{Cat_\pm(\Lambda)}$, $(\Delta E)^2{}_{\Phi_{N,n}(\Lambda)}$, $(\Delta E)^2_{cat_\pm(\Lambda_k)}$ of the states in Sec. 3 and Sec. 5, also because this quantity shows the energy fluctuation of the underlying superposition state. We can, in particular, explore states with large quantum Fisher information.

2.3 Parity operators

We now define a parity operator associated to this phase shift operator in Sec. 2.2. The parity operator is a special case of the phase shift operator when the phase rotation is π. The parity operator is thus

$$\hat{P} = \exp(\sum_k i\pi a_k^\dagger a_k) = \exp(i\pi\hat{E}), \tag{2.30}$$

and

$$\hat{P}|Coh(\Lambda)\rangle = |Coh(-\Lambda)\rangle, \tag{2.31}$$

for any coherent state and their superpositions thereof. We also have $\hat{P}^2 = I$ and the following relations

$$\hat{P} = \prod_{k=1}^{\infty}(\hat{P}_k)^k, \qquad \hat{P}_k = \exp(i\frac{\pi}{k}a_k^\dagger a_k), \tag{2.32}$$

$$\hat{P}_k|coh(\Lambda_k)\rangle = |coh(-\Lambda_k)\rangle, \qquad (\hat{P}_k)^2 = I. \tag{2.33}$$

For a general state $|\Psi\rangle$,

$$\langle\hat{P}\rangle_{|\Psi\rangle} := \frac{\langle\Psi|\hat{P}|\Psi\rangle}{\|\Psi\|^2}. \tag{2.34}$$

For the coherent state,

$$\langle\hat{P}\rangle_{Coh(\Lambda)} = \frac{1-|\Lambda|^2}{1+|\Lambda|^2}. \tag{2.35}$$

Note $0 \le \langle\hat{P}\rangle_{Coh(\Lambda)} \le 1$ which is an interesting property, and

$$(\Delta P)^2_{Coh(\Lambda)} = \frac{4|\Lambda|^2}{(1+|\Lambda|^2)^2}. \tag{2.36}$$

For a general state $|\Psi\rangle$, because $\hat{P}$ is a Hermitian operator, it is easy to infer that

$$-1 \le \langle\hat{P}\rangle_{|\Psi\rangle} \le 1. \tag{2.37}$$

2.4 Multi-parameter multi-mode coherent states and the merging of two bumps

The multi-parameter coherent states can be defined as

$$|\Psi\rangle = |Coh(x_1, x_2, \ldots, x_m)\rangle = B_+(x_1, x_2, ..., x_m)|0\rangle := \prod_{i=1}^{m} B_{+,x_i}|0\rangle, \tag{2.38}$$

where $B_{+,x_i} = \exp(\sum_k {x_i}^k \frac{a_k^\dagger}{k})$, and $|Coh(x_1, x_2, \ldots, x_m)\rangle$ corresponds to m bumps [14, 16] near the edge of the black disk. The shape of the bumps is calculated by the expectation value of the chiral field $\langle\hat{\phi}(\theta)\rangle_{|\Psi\rangle}$, where $\hat{\phi}(\theta) = \sum_{k>0}(a_k \exp(-ik\theta) + a_k^\dagger \exp(ik\theta))$ [14]. Each bump is localized in the angular direction $\arg(x_i)$.

Consider that the two bumps with an angular distance seperation 2θ, merge into a single bump. Consider states $|\Psi_1\rangle = |Coh(\Lambda_1, \Lambda_2)\rangle$ and $|\Psi_2\rangle = |Coh(\Lambda_3)\rangle$, where $\Lambda_1 = \Lambda$, $\Lambda_2 = \Lambda e^{2i\theta}$, $\Lambda_3 = \frac{2|\Lambda|}{1+|\Lambda|^2}\Lambda e^{i\theta}$. So that the total excitation energy of the two states is the same, i.e. $E = \langle\hat{E}\rangle_{|\Psi_1\rangle} = \langle\hat{E}\rangle_{|\Psi_2\rangle}$, where $\hat{E} = \sum_k k\hat{N}_k$. We assume that the angle θ is very small for simplicity of the calculation. In the context of gauge/gravity duality, these two states belong to the same Hilbert space and have the same excitation energy, measured at the asymptotic infinity of the spacetime. The transition amplitude calculated by the field theory side between these two states is

$$A = \frac{\langle\Psi_1|\Psi_2\rangle}{|\Psi_1||\Psi_2|} \simeq \frac{1}{E^{\frac{1}{2}}} \frac{16(1 + E^2 \sin^2\theta)^{\frac{1}{2}}}{9(1 + \frac{16}{9}E^2 \sin^2\frac{\theta}{2})}, \tag{2.39}$$

for large $E \gg 1$. This expression has taken into account small θ and small E^{-1} limits. This is the transition from a pair of bumps to a single bump that the two merged into. The above amplitude is a real number, because we have made the final state to be in the middle of the angular separation, for simplicity of the calculation.

By the map of gauge/gravity duality, this is a transition amplitude between two gravitational states. Transition amplitudes between backreacted geometries were also calculated in [14, 13, 15, 16]. Similar computations of tunneling probabilities between different fuzzball geometries were also computed in the context of quantum gravity in [29].

3 Superposition states and cat states

3.1 Superposition states and Schrodinger cat states

These coherent states in Sec. 2.1 are in the same Hilbert space, hence we can perform superpositions of these states. We can also perform various other quantum operations

[8] on these states. Consider the Schrodinger cat states

$$
\begin{aligned}
|Cat_{\pm}(\Lambda)\rangle &= \frac{1}{\sqrt{N_{\pm}}}(|Coh(\Lambda)\rangle \pm |Coh(-\Lambda)\rangle) &(3.1)\\
&= \frac{1}{\sqrt{N_{\pm}}}(1 \pm \hat{P})|Coh(\Lambda)\rangle, &(3.2)
\end{aligned}
$$

where $|Cat_{\pm}(\Lambda)\rangle$ has unit norm, and

$$N_{+} = 4(1-|\Lambda|^4)^{-1}\ ,\ N_{-} = 4|\Lambda|^2(1-|\Lambda|^4)^{-1}. \tag{3.3}$$

The $|Cat_{\pm}(\Lambda)\rangle$ are orthogonal to each other, i.e. $\langle Cat_{-}(\Lambda)|Cat_{+}(\Lambda)\rangle = 0$. We have that $\langle \hat{P}\rangle_{|Cat_{\pm}(\Lambda)\rangle} = \pm 1$. They form orthogonal basis of a qubit, and these states can be used to construct qubits. In the context of gauge/gravity duality, on the gravity side, the cat states mean that there is a half percent change that there is a bump at one end of a black disk, and another half percent chance that there is a bump at the opposite end of the black disk.

The Schrodinger cat states are the superposition states of macroscopic quantum states, which was proposed in thought experiment by Schrodinger [30]. These cat states are similar to the superposition of photonic coherent states, in e.g. [31, 32]. Such Schrodinger cat states have been prepared in photonic experiments, see [31, 32] and references therein.

The generating function of the cat states is

$$\langle \exp(i\theta\hat{E})\rangle_{Cat_{\pm}(\Lambda)} = \frac{1}{N_{\pm}}\left(\frac{2}{1-e^{i\theta}|\Lambda|^2} \pm \frac{2}{1+e^{i\theta}|\Lambda|^2}\right). \tag{3.4}$$

The excitation energy is given by

$$
\begin{aligned}
\langle \hat{E}\rangle_{Cat_{+}(\Lambda)} &= \frac{2|\Lambda|^4}{(1-|\Lambda|^2)(1+|\Lambda|^2)}, &(3.5)\\
\langle \hat{E}\rangle_{Cat_{-}(\Lambda)} &= \frac{1+|\Lambda|^4}{(1-|\Lambda|^2)(1+|\Lambda|^2)}, &(3.6)
\end{aligned}
$$

and

$$(\Delta E)^2_{Cat_{\pm}(\Lambda)} = \frac{4|\Lambda|^4}{(1-|\Lambda|^2)^2(1+|\Lambda|^2)^2}. \tag{3.7}$$

Now we consider entanglement between *light* states and *heavy* states. In the context of gauge/gravity duality, the heavy states can be viewed as backgrounds and the light states can be viewed as probes. Consider the EPR state [33]

$$\Psi = \frac{1}{\sqrt{2}}(|\phi_{+}\rangle|Cat_{+}(\Lambda)\rangle + |\phi_{-}\rangle|Cat_{-}(\Lambda)\rangle), \tag{3.8}$$

where $|\phi_+\rangle, |\phi_-\rangle$ have unit norm and are orthogonal to each other, i.e. $\langle\phi_-|\phi_+\rangle = 0$. Here, $|\phi_+\rangle, |\phi_-\rangle$ are light states and $|Cat_+(\Lambda)\rangle$, $|Cat_-(\Lambda)\rangle$ are heavy states.

For example, we can make graviton number states, involving only one mode,

$$|\phi_+\rangle = \frac{1}{\sqrt{k^{l_1} l_1!}} |t_k^{l_1}\rangle, \quad |\phi_-\rangle = \frac{1}{\sqrt{k^{l_2} l_2!}} |t_k^{l_2}\rangle, \tag{3.9}$$

with $l_1 \neq l_2$.

In the above case, each term, such as $\frac{1}{\sqrt{k^{l_i} l_i!}} |t_k^{l_i}\rangle |Coh(\Lambda)\rangle$, can be viewed as an excitation of a light state, e.g. $|t_k^{l_i}\rangle$, on the background of a heavy state, e.g. $|Coh(\Lambda)\rangle$. In this term, the background is a bump at the edge of the disk, and the light state is l_i Kaluza-Klein (KK) gravitons each with momentum k. This state is similar to the micro-macro entangled states for photonic coherent states as in e.g. [31].

Another example, involving two modes, is

$$|\phi_+\rangle = \frac{1}{\sqrt{k^{l_1} l_1!}} |t_{k_1}^{l_1}\rangle |t_{k_2}^{0}\rangle, \quad |\phi_-\rangle = \frac{1}{\sqrt{k^{l_2} l_2!}} |t_{k_1}^{0}\rangle |t_{k_2}^{l_2}\rangle, \tag{3.10}$$

with $k_1 \neq k_2$, $l_1 \neq 0$, $l_2 \neq 0$. A special case, involving two modes, is

$$|\phi_+\rangle = \frac{1}{\sqrt{k_1}} |t_{k_1}^{1}\rangle |t_{k_2}^{0}\rangle, \quad |\phi_-\rangle = \frac{1}{\sqrt{k_2}} |t_{k_1}^{0}\rangle |t_{k_2}^{1}\rangle, \tag{3.11}$$

with $k_1 \neq k_2$.

3.2 N-state Schrodinger cats

Now we consider the superposition of N distinct macroscopic quantum states, generalizing the superposition of only two states in the preceding section 3.1. Define the N-state Schrodinger cats

$$\Phi_{N,n}(\Lambda) = \frac{1}{\sqrt{\mathcal{N}_{N,n}}} \sum_{m=0}^{N-1} |Coh(\Lambda e^{i\frac{2\pi}{N}m})\rangle e^{-i\frac{2\pi}{N}mn}, \tag{3.12}$$

for $n = 0, ..., N-1$. In this superposition, there are N multi-mode coherent states. We make $\|\Phi_{N,n}(\Lambda)\| = 1$ and require $0 < |\Lambda| < 1$. The normalization factor is

$$\begin{aligned} \mathcal{N}_{N,n} &= \sum_{0\leq m,m'\leq N-1} (1 - |\Lambda|^2 e^{i\frac{2\pi}{N}(m-m')})^{-1} e^{-i\frac{2\pi}{N}(m-m')n} \\ &= \sum_{k=0}^{\infty} (|\Lambda|^{2k} \sum_{0\leq m,m'\leq N-1} e^{i\frac{2\pi}{N}(k-n)(m-m')}) = \frac{N^2 |\Lambda|^{2n}}{1-|\Lambda|^{2N}}, \end{aligned} \tag{3.13}$$

where in the last line we used that

$$\sum_{m,m'=0}^{N-1} e^{i\frac{2\pi}{N}(m-m')(k-n)} = \begin{cases} N^2 & k-n \equiv 0 \bmod N \\ 0 & \text{otherwise} \end{cases} . \tag{3.14}$$

There is a discrete symmetry of this state, which is the symmetry of the finite group Z_N. The state $|\Phi_{N,n}(\Lambda)\rangle$ is the n-th irreducible representation of the finite group Z_N. The action of Z_N is generated by $\tau : \theta \to \theta + \frac{2\pi}{N}$.

The action of the phase shift operator is:

$$\exp(i\theta\hat{E})\ \Phi_{N,n}(\Lambda) = \Phi_{N,n}(\Lambda e^{i\theta}). \tag{3.15}$$

The Z_N action $\tau : \theta \to \theta + \frac{2\pi}{N}$ is:

$$\tau\ \Phi_{N,n}(\Lambda) = \Phi_{N,n}(\Lambda e^{i\frac{2\pi}{N}}). \tag{3.16}$$

Hence

$$\tau = e^{i\frac{2\pi}{N}\hat{E}}. \tag{3.17}$$

The action of the parity operator is:

$$\hat{P}\ \Phi_{N,n}(\Lambda) = (-1)^n \Phi_{N,n}(\Lambda) = \Phi_{N,n}(-\Lambda). \tag{3.18}$$

Hence for even N, we have $\tau^{N/2}\ \Phi_{N,n}(\Lambda) = \hat{P}\ \Phi_{N,n}(\Lambda)$, and

$$\tau^{N/2} = \hat{P}. \tag{3.19}$$

There is a periodicity in n, for $n' = n \bmod N$, they are the equivalent state. Hence $n = 0, ..., N-1$. Note that this N does not denote the rank of the gauge group.

This state includes the states in preceding sections 2.2 and 3.1 as the special cases for small N. For $N = 1$,

$$\Phi_{1,n}(\Lambda) = \frac{1}{\sqrt{\mathcal{N}_{1,n}}}|Coh(\Lambda)\rangle, \tag{3.20}$$

and $n = 0$. For $N = 2$,

$$\Phi_{2,n}(\Lambda) = \frac{1}{\sqrt{\mathcal{N}_{2,n}}}(|Coh(\Lambda)\rangle + (-1)^n|Coh(-\Lambda)\rangle). \tag{3.21}$$

and $n = 0, 1$. So $\Phi_{2,n=0} = |Cat_+(\Lambda)\rangle$, $\Phi_{2,n=1} = |Cat_-(\Lambda)\rangle$.

For $\Phi_{N,n}(\Lambda)$ at large N, we can make a large N limit. We have that $\theta_m = \frac{2\pi}{N}m$, and $\lim_{N\to\infty} \frac{1}{N}\sum_{m=0}^{N-1} = \frac{1}{2\pi}\int_0^{2\pi} d\theta$. Hence

$$\lim_{N\to\infty} \Phi_{N,n}(\Lambda) = \frac{1}{\sqrt{\mathcal{N}_n}}\frac{N}{2\pi}\int_0^{2\pi} d\theta e^{-in\theta}|Coh(\Lambda e^{i\theta})\rangle. \tag{3.22}$$

Here $\mathcal{N}_n = \lim_{N\to\infty} \mathcal{N}_{N,n} = N^2|\Lambda|^{2n}$.

For general N, n,

$$\langle \exp(i\theta\hat{E})\rangle_{\Phi_{N,n}(\Lambda)} = \frac{e^{in\theta}(1-|\Lambda|^{2N})}{1-|\Lambda|^{2N}e^{iN\theta}}, \tag{3.23}$$

$$\langle \hat{E}\rangle_{\Phi_{N,n}(\Lambda)} = \frac{n+(N-n)|\Lambda|^{2N}}{1-|\Lambda|^{2N}}, \tag{3.24}$$

$$(\Delta E)^2{}_{\Phi_{N,n}(\Lambda)} = \frac{N^2|\Lambda|^{2N}}{(1-|\Lambda|^{2N})^2}. \tag{3.25}$$

We denote the Young tableau state with a single row of length n as $|\Delta_n\rangle$ [14, 16]. These states can be written as Schur polynomial operators [11]. We have the following proposition:

Proposition 3.1. *The generalized Schrodinger cat state $|\Phi_{N,n}(\Lambda)\rangle$ approaches the Young tableau state $|\Delta_n\rangle$ with fidelity approaching* 1*, in the large N limit.*

Proof. In the large N limit, $\theta_m = \frac{2\pi}{N}m$, $\lim_{N\to\infty}\frac{1}{N}\sum_{m=0}^{N-1} = \frac{1}{2\pi}\int_0^{2\pi} d\theta$. Hence

$$\lim_{N\to\infty} \Phi_{N,n}(\Lambda) = \frac{1}{\sqrt{\mathcal{N}_n}}\frac{N}{2\pi}\int_0^{2\pi} d\theta e^{-in\theta}|Coh(\Lambda e^{i\theta})\rangle, \tag{3.26}$$

where $\mathcal{N}_n = \lim_{N\to\infty}\mathcal{N}_{N,n} = N^2|\Lambda|^{2n}$. We have $\|\Phi_{N,n}(\Lambda)\| = 1$. The one-row Young tableau state is [16]

$$|\Delta_n\rangle = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\mathrm{d}w}{w} w^{-n}|Coh(w)\rangle, \tag{3.27}$$

where $\mathcal{C}$ can be any path that encloses 0. The $|\Delta_n\rangle$ has unit norm. It has property

$$|Coh(w)\rangle = \sum_n w^n|\Delta_n\rangle, \quad \langle\Delta_n|Coh(w)\rangle = w^n. \tag{3.28}$$

We define $w = \Lambda e^{i\theta} = |\Lambda|e^{i\theta+i\theta_0}$. Hence

$$\lim_{N\to\infty} \Phi_{N,n}(\Lambda) = \frac{1}{\sqrt{\mathcal{N}_n}} N|\Lambda|^n e^{in\theta_0}|\Delta_n\rangle. \tag{3.29}$$

Alternatively, we compute the inner product between $|\Phi_{N,n}(\Lambda)\rangle$ and $|\Delta_n\rangle$, and then take the large N limit. By (3.28),

$$\langle\Delta_n|\Phi_{N,n}(\Lambda)\rangle = \frac{1}{\sqrt{\mathcal{N}_{N,n}}}\sum_{m=0}^{N-1}\Lambda^n = \frac{N\Lambda^n}{\sqrt{\mathcal{N}_{N,n}}}. \tag{3.30}$$

The fidelity F between $|\Phi_{N,n}(\Lambda)\rangle$ and $|\Delta_n\rangle$ is

$$F^{1/2} = |\langle\Delta_n|\Phi_{N,n}(\Lambda)\rangle| = \frac{N|\Lambda|^n}{\sqrt{\mathcal{N}_{N,n}}}. \tag{3.31}$$

Hence, we find that $\lim_{N\to\infty}\Phi_{N,n}(\Lambda)=e^{in\theta_0}|\Delta_n\rangle$, $\lim_{N\to\infty}\langle\Delta_n|\Phi_{N,n}(\Lambda)\rangle=e^{in\theta_0}$ and the fidelity F approaches 1. Hence, at large N, the N-state schrodinger cat state approaches the one-row Young tableau state, with fidelity between the two states asymptotically reaches 1. □

The fact that the N-state cat has limit to the Young tableau state, is of significance. The superposition of the angularly localized states produces an angularly de-localized state. From the behavior of Eq. (3.25) with respect to N, we see that the state $\Phi_{N,n}(\Lambda)$ with a bigger N is more de-localized than the state with a smaller N. In the large N, it approaches a state very different from each typical state in the superposition. This is another very interesting example of a superposition pure state. Other aspects of the superposition states were discussed in [14, 16].

4 Noisy coherent states

4.1 A class of single mode and multi mode noisy coherent states

Now we consider noisy coherent states with our generalization. These are generalizations of the original construction of the noisy coherent states of photons [34]. The noisy coherent states were originally constructed directly using the P-representation of the density matrix operator [34]. In quantum information theory, the noisy coherent states can be generated experimently from pure coherent states through the action of Gaussian additive noise. We use the creation and annihilation operators in our case to derive the states to illustrate their relation to the specific class of coherent states using multi traces in Sec. 2.1.

Now we derive the mixed density matrix by adding random multi trace states. Consider $|coh(\Lambda_k)\rangle_k$ which is a pure coherent state for mode k. We define $a_k=d_k+\Lambda_k$, $a_k^\dagger=d_k^\dagger+\bar{\Lambda}_k$, and $\frac{1}{k}[d_k,\ d_k^\dagger]=1$. Hence,

$$\begin{aligned} d_k|coh(\Lambda_k)\rangle_k &= 0, & (4.1)\\ d_k|coh(\Gamma_k)\rangle_k &= (\Gamma_k-\Lambda_k)|coh(\Gamma_k)\rangle_k, & (4.2)\end{aligned}$$

and $\frac{1}{\mathcal{N}(\Lambda_k)}\langle coh(\Gamma_k)|_k(d_k^\dagger)^l(d_k)^l|coh(\Gamma_k)\rangle_k=|\Gamma_k-\Lambda_k|^{2l}$. We define

$$|t_k^l(\Lambda_k)\rangle=\frac{1}{\sqrt{\mathcal{N}(\Lambda_k)}}(d_k^\dagger)^l|coh(\Lambda_k)\rangle_k. \tag{4.3}$$

The state $|t_k^l(\Lambda_k)\rangle$ can be viewed as the product of the multi trace with coherent state. The conjugate is $\langle\bar{t}_k^l(\bar{\Lambda}_k)|=\frac{1}{\sqrt{\mathcal{N}(\Lambda_k)}}\langle coh(\Lambda_k)|_k(d_k)^l$. Its special cases are $|t_k^l(0)\rangle=|t_k^l\rangle$

and $|t_k^0(\Lambda_k)\rangle = \frac{1}{\sqrt{\mathcal{N}(\Lambda_k)}}|coh(\Lambda_k)\rangle_k$. The $\hat{n}_k = \frac{1}{k}d_k^\dagger d_k$ is the particle number operator for the $d_k^\dagger$ excitations, which can be considered as random thermal particles, and $\hat{n}_k|t_k^l(\Lambda_k)\rangle = l|t_k^l(\Lambda_k)\rangle$. Hence $\langle\hat{n}_k\rangle$ is the quantity that measures the average thermal particle number.

Consider a mixed state density matrix,

$$\rho(\Lambda_k)_{\rm mix} = \sum_{l=0}^{\infty} \frac{\langle\hat{n}_k\rangle^l}{(1+\langle\hat{n}_k\rangle)^{l+1}} \frac{|t_k^l(\Lambda_k)\rangle}{\sqrt{k^l l!}} \frac{\langle\bar{t}_k^l(\bar{\Lambda}_k)|}{\sqrt{k^l l!}} \tag{4.4}$$

$$= \frac{1}{\langle\hat{n}_k\rangle}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\frac{(d_k)^l|t_k^0(\Lambda_k)\rangle\langle\bar{t}_k^0(\bar{\Lambda}_k)|(d_k^\dagger)^l}{l!k^l\langle\hat{n}_k\rangle^l} \tag{4.5}$$

$$= \int d^2\Gamma_k\,\frac{1}{\pi k\langle\hat{n}_k\rangle}\exp(-\frac{|\Gamma_k-\Lambda_k|^2}{k\langle\hat{n}_k\rangle})|t_k^0(\Gamma_k)\rangle\langle\bar{t}_k^0(\Gamma_k)|. \tag{4.6}$$

The distribution $p_k(l) = \frac{\langle\hat{n}_k\rangle^l}{(1+\langle\hat{n}_k\rangle)^{l+1}}$ in (4.4) is a Bose-Einstein distribution of random multi traces on top of a coherent state. The random traces are the noise. Hence,

$$\rho(\Lambda_k)_{\rm mix} = \int d^2\Gamma_k\frac{1}{\mathcal{N}(\Gamma_k)}\,\frac{1}{\pi\mu_k}\exp(-|\Gamma_k-\Lambda_k|^2/\mu_k)|coh(\Gamma_k)\rangle\langle coh(\bar{\Gamma}_k)|, \tag{4.7}$$

where

$$\mu_k = k\langle\hat{n}_k\rangle. \tag{4.8}$$

Denote $p(\Gamma_k,\Lambda_k) = \frac{1}{\pi\mu_k}\exp(-|\Gamma_k-\Lambda_k|^2/\mu_k)$, and we have $\int p(\Gamma_k,\Lambda_k)d^2\Gamma_k = 1$. The density operator $\rho(\Lambda_k)_{\rm mix}$ is a superposition of the density operators with a distribution function $p(\Gamma_k,\Lambda_k)$ in the coherent state basis. This is a superposition mixed state, instead of a superposition pure state.

The $\hat{N}_k = \frac{1}{k}a_k^\dagger a_k$ is the total particle number operator in mode k. The mean particle number in the pure state in mode k is

$$\langle\hat{N}_k\rangle_{coh(\Gamma_k)} = \frac{1}{\mathcal{N}(\Gamma_k)}\langle 0|_k\exp(\bar{\Gamma}_k\frac{a_k}{k})\left(\frac{1}{k}a_k^\dagger a_k\right)\exp(\Gamma_k\frac{a_k^\dagger}{k})|0\rangle_k = \frac{|\Gamma_k|^2}{k}. \tag{4.9}$$

In the mixed state,

$$\langle\hat{N}_k\rangle_{\rho(\Lambda_k)_{\rm mix}} = {\rm tr}\;(\rho(\Lambda_k)_{\rm mix}\frac{1}{k}a_k^\dagger a_k) = \int d^2\Gamma_k\frac{1}{\pi\mu_k}\exp(-|\Gamma_k-\Lambda_k|^2/\mu_k)\frac{|\Gamma_k|^2}{k}$$

$$= \frac{|\Lambda_k|^2}{k}+\frac{\mu_k}{k} = \langle\hat{N}_k\rangle_{coh(\Lambda_k)}+\langle\hat{n}_k\rangle. \tag{4.10}$$

The $\hat{E}_k = a_k^\dagger a_k$ is the excitation energy operator in the mode k . The excitation energy in mode k is E_k. The total excitation energy operator is $\hat{E} = \sum_k\hat{E}_k = \sum_k a_k^\dagger a_k$.

The generating function of the mixed state is $\langle \exp(i\theta\hat{E})\rangle_{\rho_{\text{mix}}} = \text{tr}(e^{i\theta\hat{E}}\rho_{\text{mix}})$. We have in mode k,

$$\langle \hat{E}_k\rangle = k\langle \hat{N}_k\rangle = |\Lambda_k|^2 + \mu_k. \tag{4.11}$$

The fidelity between the pure coherent state $\rho_{\text{pure}} = \rho_{_{coh(\Lambda_k)}}$ and noisy coherent state $\rho_{\text{mix}} = \rho(\Lambda_k)_{\text{mix}}$ is

$$\text{tr}(\rho_{\text{mix}}\rho_{\text{pure}}) = \frac{1}{\mathcal{N}(\Lambda_k)}\langle coh(\Lambda_k)|\rho(\Lambda_k)_{\text{mix}}|coh(\Lambda_k)\rangle = \frac{1}{1+\frac{\mu_k}{k}} = \frac{1}{1+\langle \hat{n}_k\rangle}. \tag{4.12}$$

Consider in each mode k, the signal is centered at Λ_k. We define a special case of multi mode noisy coherent states, by centering the signal at Λ_k for each mode k. We can introduce noise in each mode k, in the multi-mode case. This is to introduce distribution in each mode, resulting in a joint distribution function $P(\{\Gamma_k\})$ in the joint parameter space,

$$P(\{\Gamma_k\}) = \prod_{k=1}^{\infty}\frac{1}{\pi\mu_k}\exp(\sum_{k=1}^{\infty} -|\Gamma_k - \Lambda_k|^2/\mu_k). \tag{4.13}$$

The general multi mode noisy coherent states can be written as

$$\rho(\{\Lambda_k\})_{\text{mix}} = \prod_{k=1}^{\infty}\int d^2\Gamma_k\,\frac{1}{\mathcal{N}(\Gamma_k)}\,p(\Gamma_k,\Lambda_k)\exp(\Gamma_k\frac{a_k^\dagger}{k})|0\rangle_k\langle 0|_k\exp(\bar{\Gamma}_k\frac{a_k}{k}). \tag{4.14}$$

The fidelity between the multi-mode pure coherent state and noisy multi-mode coherent state is

$$\langle Coh(\Lambda)|\rho(\{\Lambda_k\})_{\text{mix}}|Coh(\Lambda)\rangle = \prod_{k=1}^{\infty}\frac{1}{1+\frac{\mu_k}{k}} = \prod_{k=1}^{\infty}\frac{1}{1+\langle\hat{n}_k\rangle}. \tag{4.15}$$

If we consider $\frac{|\Lambda_k|^2}{k} = \frac{|\Lambda|^{2k}}{k}$, the total exitation energy $E = \langle\hat{E}\rangle$ of the noisy multi-mode coherent state is

$$\langle\hat{E}\rangle = \sum_k k\langle\hat{N}\rangle_{\text{mix},k} = \sum_k(|\Lambda|^{2k}+\mu_k) = \frac{|\Lambda|^2}{1-|\Lambda|^2} + \sum_k\mu_k. \tag{4.16}$$

The total excitation energy, in the case $\mu_k = 0$, recovers exactly the calculation for the pure coherent state case $\langle\hat{E}\rangle_{Coh(\Lambda)} = \frac{|\Lambda|^2}{1-|\Lambda|^2}$ in Sec 2.2.

In the limit $\mu_k \to 0$, $\Lambda_k = \Lambda^k$, since the distribution function becomes delta function,

$$\begin{aligned}\rho(\{\Lambda^k\})_{\text{mix}}|_{\mu_k\to 0} &= \prod_{k=1}^{\infty}\frac{1}{\mathcal{N}(\Lambda^k)}\exp(\Lambda^k\frac{a_k^\dagger}{k})|0\rangle_k\langle 0|_k\exp(\bar{\Lambda}^k\frac{a_k}{k}) \\ &= \frac{1}{\mathcal{N}(\Lambda)}|Coh(\Lambda)\rangle\langle Coh(\Lambda)|,\end{aligned} \tag{4.17}$$

which recovers the multi mode pure coherent state as in (2.14).

We can turn on noise on one mode or on a few modes. This is related to the partial thermalization in these modes on the subspace $\mathcal{H}_k$ of the Hilbert space. In this partial thermalization, a subset of the degree of freedom in the quantum system reaches a maximal entropy constrained by a given amount of energy of the excitation; the gravity dual of such partial thermalization is analyzed in [35]. Such maximal entropy state is related to black holes [36] or small black holes, in the deconfined phase. In the confined phase, they are gas of thermal radiations. The distinguishing between pure microstates and thermal states have been analyzed in [37, 38, 39, 17, 40] by methods of gauge/gravity duality and quantum gravitational theory.

Here we used coherent state basis to analyse the superposition of density matrices, which yields a mixed state that is nearby to the pure coherent state, by adding noise to the pure coherent state. There is also another basis analyzing superposition of nearby pure states [14, 41, 42], which is related to code subspaces [41].

Now we analyze the single mode noisy coherent states from more perspectives and later in relation to gauge/gravity duality. We could first make an expansion. In each mode,

$$\rho(\Lambda_k)_{\text{mix}} = \sum_{i,j=0}^{\infty} \left(\int d^2\Gamma_k \frac{p(\Gamma_k, \Lambda_k)}{\mathcal{N}(|\Gamma_k|)} (\Gamma_k)^i (\bar{\Gamma}_k)^j \right) \frac{1}{i! j! k^{i+j}} (a_k^\dagger)^i |0\rangle_k \langle 0|_k (a_k)^j. \tag{4.18}$$

We denote $c_{i,j}$ the density matrix elements in this basis $\frac{1}{\sqrt{k^i i!}}(a_k^\dagger)^i|0\rangle_k$. The von Neumann entropy associated to the density matrix is

$$s(\rho(\Lambda_k)_{\text{mix}}) = -\text{Tr}_{\mathcal{H}_k} \, \rho(\Lambda_k)_{\text{mix}} \log \rho(\Lambda_k)_{\text{mix}}. \tag{4.19}$$

4.2 Relation to mixed thermal states

We can consider small $|\Lambda_k|/\sqrt{\mu_k}$ regime. The density matrix is approximately diagonal in this regime, with diagonal matrix elements

$$c_{i,i} = \int d^2\Gamma_k \frac{e^{-|\Gamma_k - \Lambda_k|^2/\mu_k}}{\pi \mu_k \mathcal{N}(|\Gamma_k|)} |\Gamma_k|^{2i}. \tag{4.20}$$

We can explicitly integrate this in the above approximation, which gives $c_{i,i} \propto e^{-\beta k i}$, where we make $\ln \beta = -\ln \mu_k + \text{const}$. Since $\sum_i c_{i,i} = 1$, we have approximately

$$c_{i,i} \simeq (1 - e^{-\beta k}) e^{-\beta k i}. \tag{4.21}$$

These could be expanded in the trace basis $|t_k^i\rangle = (a_k^\dagger)^i|0\rangle_k$, $\langle \bar{t}_k^j| = \langle 0|_k (a_k)^j$. Hence We can denote $|t_k^i\rangle = \frac{1}{(\sqrt{N^k})^i}(\text{Tr}(Y^k))^i|0\rangle$, $\langle \bar{t}_k^i| = \frac{1}{(\sqrt{N^k})^i}\langle 0|(\text{Tr}(\bar{Y}^k))^i$ in the context of gauge theory. And hence in this approximation, we have

$$\rho \simeq (1-e^{-\beta k})\sum_{i=0}^{\infty}\frac{e^{-\beta ki}}{k^i i!(\sqrt{N^k})^{2i}}(\mathrm{Tr}(Y^k))^i|0\rangle\langle 0|(\mathrm{Tr}(\bar{Y}^k))^i. \tag{4.22}$$

All trace numbers and $U(1)$ charges, which are also the R-charges in the context of gauge/gravity duality, are allowed, hence it is in a grand canonical ensemble. It is reminiscent to the ensemble of microstates of one-charge black hole in AdS. In the confined phase, this can also be viewed as a thermal gas of multi gravitons. While, in the deconfined phase, they would be related to black holes [36, 43, 44] or small black holes.

5 Coherent states of Young tableaux

In this general section 5, we consider another class of coherent states, different from the class in Sec 2. The class in Sec 2 could be viewed as coherent states of multi traces. This class is for coherent states of multi columns of Young tableaux (YT).

5.1 Coherent states of single-row Young tableaux

We denote Δ_n as a Young tableau with a single row of length n. Consider the creation and annihilation operators $A^\dagger, A$ on the single-row Young tableau states $|\Delta_n\rangle$

$$A^\dagger|\Delta_n\rangle=\sqrt{n+1}|\Delta_{n+1}\rangle, \qquad A|\Delta_n\rangle=\sqrt{n}|\Delta_{n-1}\rangle, \tag{5.1}$$

with $[A,A^\dagger]=1$ and $A|\Delta_0\rangle=0$. The first equation in (5.1) can also be written as

$$A^\dagger|\Delta_{n-1}\rangle=\sqrt{n}|\Delta_n\rangle. \tag{5.2}$$

Hence $|\Delta_n\rangle=\frac{(A^\dagger)^n}{\sqrt{n!}}|\Delta_0\rangle$ and $\|\Delta_n\|=1$. The action of $A^\dagger$ is adding one column on the Young tableau. The action of A is removing one column from the Young tableau. This is in the large N limit.

We can construct the coherent states of Young tableaux,

$$|\Lambda\rangle=e^{-\frac{|\Lambda|^2}{2}}e^{\Lambda A^\dagger}|\Delta_0\rangle=e^{-\frac{|\Lambda|^2}{2}}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\Lambda^n|\Delta_n\rangle. \tag{5.3}$$

We have $A|\Lambda\rangle=\Lambda|\Lambda\rangle$ and $\||\Lambda\rangle\|=1$. Here the range of Λ is $0<|\Lambda|<\infty$. Note that this state (5.3) is different from $|Coh(\Lambda)\rangle$ because of the coefficients in the expansion. The $|\Lambda\rangle$ and the YT state $|\Delta_n\rangle$ are entangled states in the tensor product of the multi-trace Hilbert spaces $\otimes_k\mathcal{H}_k$ in Sec. 2.

The inner product is

$$\langle \Lambda | \Lambda' \rangle = e^{-\frac{1}{2}|\Lambda|^2 - \frac{1}{2}|\Lambda'|^2 + \bar{\Lambda}\Lambda'}, \qquad |\langle \Lambda | \Lambda' \rangle| = e^{-\frac{1}{2}|\Lambda - \Lambda'|^2}. \tag{5.4}$$

The number operator is $\hat{N} = A^\dagger A$. The excitation energy operator is $\hat{E} = A^\dagger A$, with the zero point ground state energy subtracted out in the definition. One box of Young tableau has one unit of energy. The phase shift operator is $\exp(i\theta \hat{N}) = \exp(i\theta A^\dagger A) = \exp(i\theta \hat{E})$, and the action is $\exp(i\theta \hat{E})|\Lambda\rangle = |\Lambda e^{i\theta}\rangle$. The parity operator is $\hat{P} = \exp(i\pi \hat{E})$.

The generating function is

$$\langle \exp(i\theta \hat{E}) \rangle_{|\Lambda\rangle} = e^{|\Lambda|^2 (e^{i\theta} - 1)}. \tag{5.5}$$

Hence $\langle \hat{E} \rangle_{|\Lambda\rangle} = |\Lambda|^2$ and $(\Delta E)^2_{|\Lambda\rangle} = |\Lambda|^2$.

5.2 Coherent states of k-row Young tableaux

We can define coherent states of k-row Young tableaux. We denote $\Delta_{n,k}$ as a Young tableau with n columns each with a column-length k. In the context of gauge theory, this state can be written as a Schur polynomial operator labelled by a Young tableau $\Delta_{n,k}$ with n columns and k rows [11].

The $\Delta_{n,k}$ is a Young tableau with n columns each with a column-length k. Consider the creation and annihilation operators $A_k^\dagger, A_k$ on the multi-row Young tableau states $|\Delta_{n,k}\rangle$,

$$A_k^\dagger |\Delta_{n,k}\rangle = \sqrt{k(n+1)} |\Delta_{n+1,k}\rangle, \qquad A_k |\Delta_{n,k}\rangle = \sqrt{kn} |\Delta_{n-1,k}\rangle, \tag{5.6}$$

with $\frac{1}{k}[A_k, A_k^\dagger] = 1$ and $A_k |\Delta_{0,k}\rangle = 0$. The first equation in (5.6) can also be written as

$$A_k^\dagger |\Delta_{n-1,k}\rangle = \sqrt{kn} |\Delta_{n,k}\rangle. \tag{5.7}$$

In other words, $\frac{1}{\sqrt{k}} A_k^\dagger$ and $\frac{1}{\sqrt{k}} A_k$ play the role of ordinary creation and annihilation operators, with the $\frac{1}{\sqrt{k}}$ factor due to our particular convention of the definition. This is in the large N limit. Hence $|\Delta_{n,k}\rangle = \frac{(A_k^\dagger)^n}{\sqrt{k^n n!}} |\Delta_{0,k}\rangle$ and $\|\Delta_{n,k}\| = 1$. The action of $A_k^\dagger$ is adding one column of length-k on the Young tableau. The action of A_k is removing one column of length-k from the Young tableau. The YT state $|\Delta_{n,k}\rangle$ is an entangled state in the tensor product of the multi-trace Hilbert spaces $\otimes_k \mathcal{H}_k$ in Sec. 2. These states contain nontrivial entanglement stored between different multi-trace Hilbert spaces $\mathcal{H}_k$. See detailed analysis of this statement in [16]. The entanglement entropy of these states, entangled in $\otimes_k \mathcal{H}_k$, were computed in [16, 14].

The action of the excitation energy operator $\hat{E} = \sum_k \hat{E}_k$ and the phase shift operator $\exp(i\theta \hat{E})$ on the YT states are as follows: $\hat{E}|\Delta_{n,k}\rangle = kn |\Delta_{n,k}\rangle$, $\exp(i\theta \hat{E})|\Delta_{n,k}\rangle = e^{ikn\theta} |\Delta_{n,k}\rangle$, and $\langle \exp(i\theta \hat{E}) \rangle_{|\Delta_{n,k}\rangle} = e^{ikn\theta}$. We also have $\hat{P}_k |\Delta_{n,k}\rangle = (-1)^n |\Delta_{n,k}\rangle$ and

$\exp(i\pi\hat{E})|\Delta_{n,k}\rangle = (-1)^{kn}|\Delta_{n,k}\rangle$. The phase rotation of the matrix field induces a phase rotation for each box of the Young tableau.

This class of coherent states are also described in [45, 46], and our case is a special case of them. We can construct the coherent states of multi-row (k-row) Young tableaux,

$$|\Lambda_k\rangle_k = e^{-\frac{|\Lambda_k|^2}{2k}} e^{\frac{1}{k}\Lambda_k A_k^\dagger}|\Delta_{0,k}\rangle = e^{-\frac{|\Lambda_k|^2}{2k}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{k^n n!}}(\Lambda_k)^n|\Delta_{n,k}\rangle. \tag{5.8}$$

The subscript k means 'mode' k. We have $A_k|\Lambda_k\rangle_k = \Lambda_k|\Lambda_k\rangle_k$ and $\||\Lambda_k\rangle_k\| = 1$. Here the range of Λ_k is $0 < |\Lambda_k| < \infty$. The amplitude of the coherent states, in the conventional notation of quantum optics, is $\frac{\Lambda_k}{\sqrt{k}}$. We sometimes also denote $\frac{\Lambda_k}{\sqrt{k}} = z_k$. In the expansion of this coherent state, the $|\Delta_{n,k}\rangle$ and $|\Delta_{0,k}\rangle$ are the analogs of the photon states $|n\rangle$ and $|0\rangle$ in quantum optics and quantum information theory.

We can refer to the giant gravitons wrapping internal sphere directions, as sphere giant gravitons, and those wrapping AdS directions, as dual giant gravitons. Each column is a sphere giant graviton with momentum k. This state (5.8) is a coherent state of length-k columns, and in other words a coherent state of momentum-k sphere giant gravitons, with an average sphere giant graviton number $\langle\hat{E}\rangle/k$. The variance of the sphere giant graviton number is $(\Delta E)^2/k^2$. It has a dual interpretation that it is also a state of k dual giant gravitons with a fluctuating size and an average size $\langle\hat{E}\rangle/k$. The variance of the size of the dual giant gravitons is $(\Delta E)^2/k^2 = \mathrm{QFI}/(4k^2)$, which is related to the quantum Fisher information.

There is a relation between YT states and fermions. The i-th fermion excitation-energy is $E_i = c_i + i$, where i is the i-th row and c_i is the row-length of the i-th row. Here, the average row-length is $\langle\hat{E}\rangle/k$. So the average fermion excitation-energy of the above coherent state is $\langle\hat{E}\rangle/k + (k-1)/2$. The state $|\Lambda_k\rangle_k$ can also be viewed as a k-fermion droplet or blob centered around $\Lambda_k/\sqrt{k}$, with k fermions in that droplet. This state has very good localizability in phase space. The special case of single-row, in the preceding section, corresponds to $k = 1, \Lambda = \Lambda_1$.

Define $\hat{x}_k = \frac{1}{\sqrt{2k}}(A_k + A_k^\dagger)$, $\hat{p}_k = \frac{1}{\sqrt{2k}i}(A_k - A_k^\dagger)$, then $\langle\hat{x}_k\rangle = \frac{1}{\sqrt{k}}\mathrm{Re}\ \Lambda_k$, $\langle\hat{p}_k\rangle = \frac{1}{\sqrt{k}}\mathrm{Im}\ \Lambda_k$. Hence the state $|\Lambda_k\rangle_k$ is a blob or a droplet centered around $(\frac{1}{\sqrt{k}}\mathrm{Re}\ \Lambda_k, \frac{1}{\sqrt{k}}\mathrm{Im}\ \Lambda_k)$ in the phase space, and subject to the Heisenberg uncertainty principle.

The inner product is

$$\langle\Lambda_k|\Lambda_k'\rangle_k = e^{-\frac{1}{2k}|\Lambda_k|^2 - \frac{1}{2k}|\Lambda_k'|^2 + \frac{1}{k}\bar{\Lambda}_k\Lambda_k'}, \qquad |\langle\Lambda_k|\Lambda_k'\rangle| = e^{-\frac{1}{2k}|\Lambda_k - \Lambda_k'|^2}. \tag{5.9}$$

The number operator is $\hat{N}_k = \frac{1}{k}A_k^\dagger A_k$. The excitation energy operator is $\hat{E}_k = A_k^\dagger A_k$, with the zero point ground state energy subtracted out in the definition. One box of Young tableau has one unit of energy. We have that $\theta_k = k\theta$. The phase shift operator is $\exp(i\theta_k\hat{N}_k) = \exp(i\frac{\theta_k}{k}\hat{E}_k)$. The action is

$$\exp(i\frac{\theta_k}{k}\hat{E}_k)|\Lambda_k\rangle_k = |\Lambda_k e^{i\theta_k}\rangle_k. \tag{5.10}$$

The parity operator for the k-row state is $\hat{P}_k = \exp(i\frac{\pi}{k}\hat{E}_k)$, and the action is $\hat{P}_k|\Lambda_k\rangle_k = |-\Lambda_k\rangle_k$.

The generating function is

$$\langle \exp(i\frac{\theta_k}{k}\hat{E}_k)\rangle_{|\Lambda_k\rangle_k} = \langle \exp(i\theta\hat{E})\rangle_{|\Lambda_k\rangle_k} = e^{\frac{|\Lambda_k|^2}{k}(e^{i\theta_k}-1)}. \tag{5.11}$$

Hence $\langle \hat{E}\rangle_{|\Lambda_k\rangle_k} = |\Lambda_k|^2$, $\langle \hat{N}_k\rangle_{|\Lambda_k\rangle_k} = \frac{|\Lambda_k|^2}{k} = |z_k|^2$, and $(\Delta E)^2_{|\Lambda_k\rangle_k} = k|\Lambda_k|^2$, $(\Delta N)^2_{|\Lambda_k\rangle_k} = \frac{|\Lambda_k|^2}{k} = \langle \hat{N}_k\rangle_{|\Lambda_k\rangle_k}$.

In the context of gauge/string duality, $|\Delta_{n,k}\rangle$, or $|\Delta_n\rangle = |\Delta_{n,1}\rangle$ as a special case, is also a D-brane state, and more specifically, a bound state of k dual giant gravitons with size n. In the geometric dual, the state $|\Delta_{n,k}\rangle$ is also a black ring with area k located around the radial location n in the phase space plane. These interpretations are used in Sec. 7.

This class of coherent states is similar to the first type of coherent states discussed in 2.1. The two different types differ as the difference between the multi trace case and the multi column case. Because the Young tableau has expansion in terms of products of multi-traces, this coherent state $|\Lambda_k\rangle_k$ for fixed k is at the level of 'multi-mode' in the multi-trace basis, like the multi-mode $|Coh(\Lambda)\rangle$ in Sec. 2.

5.3 Multi mode coherent states of Young tableaux

Now we denote $\mathcal{H}_k$ as the Hilbert space of all column length k tableaux, an analog of 'mode' k, and $k \in \mathbb{Z}_{>0}$. The $|\Delta_{0,k}\rangle = |0\rangle$ is the vacuum state. Consider $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots = \otimes_k \mathcal{H}_k$. The 'multi-mode' coherent states of YT is

$$\begin{aligned}\Psi(\{\Lambda_k\}) &= \prod_k |\Lambda_k\rangle_k \\ &= \exp(-\sum_k \frac{|\Lambda_k|^2}{2k})\exp(\sum_k \frac{1}{k}\Lambda_k A_k^\dagger)|0\rangle. \end{aligned} \tag{5.12}$$

These are coherent states involving arbitrary-shape Young tableaux, with allowed shapes [47, 48, 49]. The k denotes and labels the column length. If we include all possible k and all possible shapes, each individual configuration is very reminiscent of a microstate configuration of a fuzzball [39].

For multi-mode, we define phase shift operator as $\exp(i\sum_k \frac{\theta_k}{k}\hat{E}_k)$. We consider $\frac{\theta_k}{k} = \theta$. Hence

$$\exp(i\sum_k \frac{\theta_k}{k}\hat{E}_k) = \exp(i\theta\hat{E}), \tag{5.13}$$

where $\hat{E}_k = A_k^\dagger A_k$ and $\hat{E} = \sum_k \hat{E}_k$. The action of the phase shift operator is

$$\exp(i\theta\hat{E})|\Lambda_k\rangle_k = |\Lambda_k e^{ik\theta}\rangle_k. \tag{5.14}$$

The symplectic form in the configuration space of the droplets can be written [50, 45] as

$$\Omega = \sum_k \frac{1}{k} \delta\Lambda_k \wedge \delta\bar{\Lambda}_k, \tag{5.15}$$

where k indexes column lengthes. The $\frac{1}{k}$ factor is due to our convention, since $\frac{1}{\sqrt{k}}\Lambda_k$ is the amplitude or the complex parameter of the coherent states in the phase space. This captures the deformation modes of the droplet geometries, and hence plays important role in the deformation space of microstate geometries, and also has implications for the stretched horizon of the superstar geometry [45].

For example, the 'two-mode' coherent states of YT is

$$\Psi = e^{-\frac{|\Lambda_{k_1}|^2}{2k_1}} e^{\frac{1}{k_1}\Lambda_{k_1} A^\dagger_{k_1}} e^{-\frac{|\Lambda_{k_2}|^2}{2k_2}} e^{\frac{1}{k_2}\Lambda_{k_2} A^\dagger_{k_2}} |0\rangle. \tag{5.16}$$

Here we use the column number to classify the 'modes'. We can alternatively use the corners of YT to classify the 'modes', e.g. [41, 42]. The former is more convenient for creating D-brane state, while the later is more convenient for creating closed string state. See also other closely related bases describing different degrees of freedom using Young tableaux [51, 52, 53, 45, 54].

5.4 Cat states

Similar to Sec. 3 where we consider the Schrodinger cat states for the first type of coherent states, in this section we analyze the cat states for the second type of coherent states. For a general k, we have the cat states

$$|cat_\pm(\Lambda_k)\rangle_k = \frac{1}{\sqrt{N_{k,\pm}}}(|\Lambda_k\rangle_k \pm |-\Lambda_k\rangle_k) \tag{5.17}$$

$$= \frac{1}{\sqrt{N_{k,\pm}}}(1 \pm \hat{P}_k)|\Lambda_k\rangle_k, \tag{5.18}$$

where $|cat_\pm(\Lambda_k)\rangle_k$ has unit norm and here $N_{k,\pm} = 2(1 \pm e^{-\frac{2}{k}|\Lambda_k|^2})$. This is a superposition of two blobs. They are centered around $(\frac{1}{\sqrt{k}}\text{Re } \Lambda_k, \frac{1}{\sqrt{k}}\text{Im } \Lambda_k)$ and $(-\frac{1}{\sqrt{k}}\text{Re } \Lambda_k, -\frac{1}{\sqrt{k}}\text{Im } \Lambda_k)$ respectively in the phase space plane. This phase space plane is the droplet plane in the gravity side. The larger the $\frac{1}{k}|\Lambda_k|^2$, the more far apart they are. We have $\langle \hat{P}_k \rangle_{cat_\pm(\Lambda_k)} = \pm 1$, $(\Delta P_k)^2_{cat_\pm(\Lambda_k)} = 0$. These states can be used to construct qubits.

The generating function is

$$\langle \exp(i\frac{\theta_k}{k}\hat{E}_k)\rangle_{|cat_\pm(\Lambda_k)\rangle_k} = \frac{e^{-\frac{1}{k}|\Lambda_k|^2}}{(1 \pm e^{-\frac{2}{k}|\Lambda_k|^2})}[\exp(\frac{1}{k}|\Lambda_k|^2 e^{i\theta_k}) \pm \exp(-\frac{1}{k}|\Lambda_k|^2 e^{i\theta_k})]. \tag{5.19}$$

We have that:

$$
\begin{aligned}
\langle \hat{E} \rangle_{cat_+(\Lambda_k)} &= |\Lambda_k|^2 \tanh(\frac{|\Lambda_k|^2}{k}), \quad \langle \hat{E} \rangle_{cat_-(\Lambda_k)} = |\Lambda_k|^2 \tanh(\frac{|\Lambda_k|^2}{k})^{-1}, \quad (5.20)\\
(\Delta E)^2_{cat_+(\Lambda_k)} &= k|\Lambda_k|^2 \tanh(\frac{|\Lambda_k|^2}{k}) + |\Lambda_k|^4 \cosh(\frac{|\Lambda_k|^2}{k})^{-2}, \quad (5.21)\\
(\Delta E)^2_{cat_-(\Lambda_k)} &= k|\Lambda_k|^2 \tanh(\frac{|\Lambda_k|^2}{k})^{-1} - |\Lambda_k|^4 \sinh(\frac{|\Lambda_k|^2}{k})^{-2}. \quad (5.22)
\end{aligned}
$$

For big $|\Lambda_k| \gg 1$, we have that $\langle \hat{E} \rangle_{cat_\pm(\Lambda_k)} \simeq |\Lambda_k|^2 \pm 2|\Lambda_k|^2 e^{-\frac{2|\Lambda_k|^2}{k}}$, and $(\Delta E)^2_{cat_\pm(\Lambda_k)} \simeq k|\Lambda_k|^2 \pm 4|\Lambda_k|^4 e^{-\frac{2|\Lambda_k|^2}{k}}$. For big $|\Lambda_k| \gg 1$, $\frac{\Delta E}{\langle \hat{E} \rangle}|_{cat_\pm(\Lambda_k)} \simeq \frac{\sqrt{k}}{|\Lambda_k|}$. Hence, for relatively big coherent state amplitude, the cat_+, cat_- are more close to the coherent state.

Moreover, it has been known that the expression of the cat state can be approximated by squeezed state:

$$
\begin{aligned}
|cat_+(\Lambda_k)\rangle_k &\sim S(\xi)|\Delta_{0,k}\rangle, \quad (5.23)\\
|cat_-(\Lambda_k)\rangle_k &\sim S(\xi)|\Delta_{1,k}\rangle, \quad (5.24)
\end{aligned}
$$

where $S(\xi) = e^{-\frac{\xi}{k}(A_k^\dagger A_k^\dagger - A_k A_k)}$, with high fidelity [32, 31], and we have here generalized them to new states defined using Young tableaux, from those happen in quantum information theory [32, 31]. This approximation is practically useful, since certain calculation can be simplified with it.

5.5 Noisy coherent states of multi-row Young tableaux

The preceding sec. 4.1 is on a class of noisy coherent states of multi traces. In this section, we discuss a different class, which is a class of noisy coherent states of Young tableaux. We define $A_k = d_k + \Lambda_k$, $A_k^\dagger = d_k^\dagger + \bar{\Lambda}_k$, and $\frac{1}{k}[d_k,\ d_k^\dagger] = 1$. Hence,

$$
\begin{aligned}
d_k|\Lambda_k\rangle_k &= 0, \quad (5.25)\\
d_k|\Gamma_k\rangle_k &= (\Gamma_k - \Lambda_k)|\Gamma_k\rangle. \quad (5.26)
\end{aligned}
$$

We have $\langle \Gamma_k|(d_k^\dagger)^l (d_k)^l|\Gamma_k\rangle_k = |\Gamma_k - \Lambda_k|^{2l}$. The thermal particle number operator is $\hat{n}_k = \frac{1}{k} d_k^\dagger d_k$.

Consider a mixed state density matrix,

$$
\begin{aligned}
\rho(\Lambda_k)_{\text{mix}} &= \sum_{l=0}^{\infty} \frac{\langle \hat{n}_k \rangle^l}{(1+\langle \hat{n}_k \rangle)^{l+1}} (d_k^\dagger)^l |\Lambda_k\rangle_k \langle \Lambda_k|_k (d_k)^l \quad (5.27)\\
&= \int d^2\Gamma_k \frac{1}{\pi k \langle \hat{n}_k \rangle} \exp(-\frac{|\Gamma_k - \Lambda_k|^2}{k\langle \hat{n}_k \rangle}) |\Gamma_k\rangle_k \langle \Gamma_k|_k. \quad (5.28)
\end{aligned}
$$

The distribution $p(l) = \frac{\langle \hat{n}_k \rangle^l}{(1+\langle \hat{n}_k \rangle)^{l+1}}$ in (5.27) is Bose-Einstein distribution of random columns on top of a coherent state. The random columns are the noise.

Hence, the density matrix of noisy coherent states of Young tableaux is

$$\begin{aligned} \rho_{\text{mix}}(\Lambda_k) &= \int d^2\Gamma_k \; p(\Gamma_k, \Lambda_k) \frac{1}{\mathcal{N}(\Gamma_k)} \exp(\frac{1}{k}\Gamma_k A_k^\dagger) |\Delta_{0,k}\rangle \langle \Delta_{0,k}| \exp(\frac{1}{k}\bar{\Gamma}_k A_k), \\ p(\Gamma_k, \Lambda_k) &= \frac{1}{\pi \mu_k} \exp(-|\Gamma_k - \Lambda_k|^2/\mu_k). \end{aligned} \tag{5.29}$$

The fidelity between the pure coherent state $\rho_{\text{pure}} = \rho_{|\Lambda_k\rangle_k}$ and noisy coherent state $\rho_{\text{mix}} = \rho(\Lambda_k)_{\text{mix}}$ is

$$\text{tr}(\rho_{\text{mix}}\rho_{\text{pure}}) = \langle \Lambda_k | \rho(\Lambda_k)_{\text{mix}} | \Lambda_k \rangle_k = \frac{1}{1 + \frac{\mu_k}{k}} = \frac{1}{1 + \langle \hat{n} \rangle_k}. \tag{5.30}$$

The total particle number operator in mode k is $\hat{N}_k = \frac{1}{k} A_k^\dagger A_k$. In the mixed state,

$$\begin{aligned} \langle \hat{N}_k \rangle_{\text{mix}} &= \text{tr}\ (\rho_{\text{mix}}(\Lambda_k) \frac{1}{k} A_k^\dagger A_k) = \int d^2\Gamma_k \frac{1}{\pi \mu_k} \exp(-|\Gamma_k - \Lambda_k|^2/\mu_k)\ |\Gamma_k|^2 \\ &= \frac{1}{k}|\Lambda_k|^2 + \frac{1}{k}\mu_k = \langle \hat{N}_k \rangle_{\text{pure}} + \langle \hat{n}_k \rangle. \end{aligned} \tag{5.31}$$

The generating function for the mixed state is $\langle \exp(i\frac{\theta_k}{k}\hat{E}_k) \rangle_{\rho_{\text{mix}}} = \text{tr}(e^{i\frac{\theta_k}{k}\hat{E}_k}\rho_{\text{mix}})$. The excitation energy is

$$\langle \hat{E}_k \rangle = k\langle \hat{N}_k \rangle = |\Lambda_k|^2 + k\langle \hat{n}_k \rangle = |\Lambda_k|^2 + \mu_k. \tag{5.32}$$

$$(\Delta E_k)^2 = k\langle \hat{E}_k \rangle + 2|\Lambda_k|^2 \mu_k + \mu_k^2. \tag{5.33}$$

Of course, the $k = 1$ case, is a special case in the preceding section 5.1.

The average *size* of the dual giant gravitons is $\langle \hat{E} \rangle / k = \frac{1}{k}|\Lambda_k|^2 + \frac{1}{k}\mu_k$. The variance of the size of the dual giant gravitons is $(\Delta E)^2/k^2 = \langle \hat{E} \rangle / k + \frac{2}{k^2}|\Lambda_k|^2\mu_k + \frac{1}{k^2}\mu_k^2$. Note that the shift $|\Lambda_k|^2 \to |\Lambda_k|^2 + \mu_k$ in the size of the dual giant graviton, from a pure state to a mixed state, is very reminiscent to the shift from a black disk to a gray disk corresponding to the superstar geometry which may be viewed as a mixed state of dual giant gravitons [37, 17]. There is a giant/dual-giant duality between sphere giants and dual giants, and hence the $\langle \hat{E} \rangle / k$ is also the *number* of the sphere giant gravitons. The notation $\langle \hat{N} \rangle = \langle \hat{E} \rangle / k$ sphere giants here is the same as the notation N_c giants in [17, 19].

In the small $|\Lambda_k|/\sqrt{\mu_k}$ regime, we have a limit to mixed thermal state which is approximately diagonal. Similar to Sec. 4.2, we have then the resulting density matrix

$$\rho \simeq (1 - e^{-\beta k}) \sum_{n=0}^{\infty} e^{-\beta k n} |\Delta_{n,k}\rangle \langle \Delta_{n,k}|. \tag{5.34}$$

5.6 Relation to mixed thermal state and its purification

The above mixed density matrix (5.34) have a purification in doubled Hilbert space. Consider a *pure* state $|g\rangle$ in $\mathcal{H}_k^L \otimes \mathcal{H}_k^R$,

$$
\begin{aligned}
|g\rangle &= \frac{1}{\sqrt{\mathcal{N}}} \sum_{n=0}^{\infty} e^{-\frac{\beta}{2}kn} |\Delta_{n,k}\rangle_L |\Delta_{n,k}\rangle_R \\
&= \frac{1}{\sqrt{\mathcal{N}}} \sum_{n=0}^{\infty} \frac{e^{-\frac{\beta}{2}kn}}{k^n n!} (A_{k;L}^\dagger)^n (A_{k;R}^\dagger)^n |0\rangle_{k;L} |0\rangle_{k;R},
\end{aligned}
\tag{5.35}
$$

where $\mathcal{N} = (1 - e^{-\beta k})^{-1}$. Here $|\Delta_{n,k}\rangle_L \in \mathcal{H}_k^L \subset \mathcal{H}^L$ and $|\Delta_{n,k}\rangle_R \in \mathcal{H}_k^R \subset \mathcal{H}^R$.

This state can be derived from a squeezed state,

$$
|g\rangle = \frac{1}{\sqrt{\mathcal{N}}} e^{-\frac{\xi}{k}(A_{k;L}^\dagger A_{k;R}^\dagger - A_{k;L} A_{k;R})} |0\rangle_L |0\rangle_R, \qquad \tanh \xi = -e^{-\frac{\beta}{2}k}. \tag{5.36}
$$

Here $S(\xi) = e^{-\frac{\xi}{k}(A_{k;L}^\dagger A_{k;R}^\dagger - A_{k;L} A_{k;R})}$ is the squeeze operator. The $\frac{1}{k}$ factor is due to our convention of the definition. The normalization factor is $\mathcal{N} = (1 - \tanh^2 \xi)^{-1} = (1 - e^{-\beta k})^{-1}$. Here $1/T = \beta = \frac{2}{k} \ln |\tanh \xi|^{-1} \geq 0$. In relating (5.36) and (5.35), we use a formula

$$
\begin{aligned}
& e^{-\frac{\xi}{k}(A_{k;L}^\dagger A_{k;R}^\dagger - A_{k;L} A_{k;R})} |0\rangle_L |0\rangle_R \\
= \; & e^{-\frac{1}{k}\tanh \xi \; A_{k;L}^\dagger A_{k;R}^\dagger} (\cosh \xi)^{-\frac{1}{k} A_{k;L}^\dagger A_{k;L} - \frac{1}{k} A_{k;R}^\dagger A_{k;R} - 1} e^{\frac{1}{k}\tanh \xi \; A_{k;L} A_{k;R}} |0\rangle_L |0\rangle_R.
\end{aligned}
\tag{5.37}
$$

We can view $e^{iH_{\text{in}}} = e^{-\frac{\xi}{k}(A_{k;L}^\dagger A_{k;R}^\dagger - A_{k;L} A_{k;R})}$, where the interacting Hamiltonian coupling the two copies of the Hilbert space is $H_{\text{int}} = i\frac{\xi}{k} A_{k;L}^\dagger A_{k;R}^\dagger - i\frac{\xi}{k} A_{k;L} A_{k;R}$.

Its density matrix on $\mathcal{H}_k^L \otimes \mathcal{H}_k^R$ is $|g\rangle\langle g|$. The partial trace gives a mixed density matrix in $\mathcal{H}_k^L$,

$$
\begin{aligned}
\rho_L = \rho_{\text{mix}} &= \text{tr}_{\mathcal{H}_k^R} |g\rangle\langle g| \\
&= (1 - e^{-\beta k}) \sum_{n=0}^{\infty} \frac{e^{-\beta k n}}{k^n n!} (A_k^\dagger)^n |0\rangle_k \langle 0|_k (A_k)^n \\
&= (1 - e^{-\beta k}) \sum_{n=0}^{\infty} e^{-\beta k n} |\Delta_{n,k}\rangle\langle\Delta_{n,k}|,
\end{aligned}
\tag{5.38}
$$

which is the same as (5.34), hence $|g\rangle$ is a purification of (5.34). This also describes thermal sphere giant gravitons with size k. This can be viewed as a reduced density matrix in the subspace of the Hilbert space, labeled by the 'mode' k. This can also be viewed as being from the noisy coherent state in the large noise limit. In the gravity dual, in the confined phase, they can be viewed as a gas of thermal giant gravitons [55, 17].

6 Quantum state discrimination and quantum detection

Imagining a measurement experiment, that Alice send signals of two quantum states ρ_0, ρ_1, which are linearly independent, but may or may not be orthogonal. She sends them with a priori probabilities η_0, η_1, respectively. Bob choose projectors Π_0, Π_1 to measure these two states respectively. For each measurement, Bob has to decide which state he has measured. There is a probability that his decision is not correct. Bob wants to maximize the correct decision probability P_c and minimize the error probability $P_e = 1 - P_c$.

We consider quantum state discrimination [56, 57, 58, 59] of the states in our system. For example, we can use Helstrom's quantum detection and estimation theory [58], see [59] for a review and references therein. We first consider the binary system of two pure coherent states $\rho(\Lambda_{(0)})$ and $\rho(\Lambda_{(1)})$ of the first type in Sec. 2.1, and then the binary system of $|\Lambda_{(0)}\rangle_k$ and $|\Lambda_{(1)}\rangle_k$ of the second type in Sec 5.2.

The correct decision probability P_c is [58, 59]

$$\begin{aligned} P_c &= \eta_0 \mathrm{Tr}[\Pi_0 \rho_0] + \eta_1 \mathrm{Tr}[\Pi_1 \rho_1] \\ &= \eta_0 + \mathrm{Tr}[\Pi_1(\eta_1 \rho_1 - \eta_0 \rho_0)] =: \eta_0 + \mathrm{Tr}[\Pi_1 D], \end{aligned} \tag{6.1}$$

where Π_0, Π_1 are projectors and $\eta_0 + \eta_1 = 1$, $\Pi_0 + \Pi_1 = I$. The error probability is complementary to the correct decision probability and is hence

$$P_e = \eta_1 \mathrm{Tr}[\Pi_0 \rho_1] + \eta_0 \mathrm{Tr}[\Pi_1 \rho_0] = 1 - P_c. \tag{6.2}$$

We maximize P_c,

$$\max\ P_c = \eta_0 + \max_{\{\Pi_0,\ \Pi_1\}} \mathrm{Tr}[\Pi_1 D]. \tag{6.3}$$

The D has eigen-decomposition

$$D = \eta_1 \rho_1 - \eta_0 \rho_0 = \sum_{\lambda_m > 0} \lambda_m |\lambda_m\rangle\langle\lambda_m| + \sum_{\lambda_m < 0} \lambda_m |\lambda_m\rangle\langle\lambda_m|. \tag{6.4}$$

To maximize P_c, we have $\Pi_1 = \sum_{\lambda_m > 0} |\lambda_m\rangle\langle\lambda_m|$ and $\mathrm{Tr}[\Pi_1 D] = \sum_{\lambda_m > 0} \lambda_m$, and hence

$$\max\ P_c = \eta_0 + \sum_{\lambda_m > 0} \lambda_m. \tag{6.5}$$

In the case there are multiple degenerate positive eigenvalues, all these degenerate positive eigenvalues should be added.

Here, $\rho_0 = \rho(\Lambda_{(0)})$ and $\rho_1 = \rho(\Lambda_{(1)})$ are two pure multi-mode coherent states, as (2.14). The inner product of these two pure coherent states is

$$\begin{aligned} & \frac{1}{\sqrt{\mathcal{N}(\Lambda_{(0)})\mathcal{N}(\Lambda_{(1)})}}\langle Coh(\Lambda_{(0)})|Coh(\Lambda_{(1)})\rangle \\ & = \frac{\left(1-|\Lambda_{(0)}|^2\right)^{1/2}\left(1-|\Lambda_{(1)}|^2\right)^{1/2}}{(1-\bar{\Lambda}_{(0)}\Lambda_{(1)})}. \end{aligned} \tag{6.6}$$

The eigenvalues of $D = \eta_1\rho(\Lambda_{(1)}) - \eta_0\rho(\Lambda_{(0)})$ are

$$\lambda_\pm = \frac{1}{2}(\eta_1 - \eta_0 \pm \sqrt{1 - 4\eta_0\eta_1\mathcal{N}_{(0)}^{-1}\mathcal{N}_{(1)}^{-1}|\langle Coh(\Lambda_{(0)})|Coh(\Lambda_{(1)})\rangle|^2}). \tag{6.7}$$

Hence,

$$\begin{aligned} \max\ P_c &= \eta_0 + \lambda_+ \\ &= \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\eta_0\eta_1\frac{\left(1-|\Lambda_{(0)}|^2\right)\left(1-|\Lambda_{(1)}|^2\right)}{|1-\bar{\Lambda}_{(0)}\Lambda_{(1)}|^2}}. \end{aligned} \tag{6.8}$$

This is also Helstrom's bound.

In the special case $\Lambda_{(1)} = -\Lambda_{(0)} \neq 0,$ if $\eta_0 = \eta_1 = \frac{1}{2}$,

$$\max\ P_c = \frac{1}{2} + \frac{|\Lambda_{(1)}|}{1+|\Lambda_{(1)}|^2}, \qquad \min\ P_e = \frac{1}{2} - \frac{|\Lambda_{(1)}|}{1+|\Lambda_{(1)}|^2}. \tag{6.9}$$

In the special case $\Lambda_{(0)} = 0, \Lambda_{(1)} \neq 0$, and hence

$$\max\ P_c = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\eta_0\eta_1\left(1-|\Lambda_{(1)}|^2\right)} \tag{6.10}$$

$$= \frac{1}{2} + \frac{1}{2}|\Lambda_{(1)}| \ \leq 1, \quad \text{if } \eta_0 = \eta_1 = \frac{1}{2}, \tag{6.11}$$

and in this case

$$\min\ P_e = \frac{1}{2} - \frac{1}{2}|\Lambda_{(1)}|. \tag{6.12}$$

In both above cases, the quantum error probability P_e can be decreased to zero, when $\Lambda_{(1)} = 0$.

Similarly we can use two blobs $|\Lambda_{(0)}\rangle_k, |\Lambda_{(1)}\rangle_k$ centered at two different locations in the phase space, and we have

$$\begin{aligned} \max\ P_c &= \eta_0 + \lambda_+ \\ &= \frac{1}{2}(1 + \sqrt{1 - 4\eta_0\eta_1|\langle\Lambda_{(0)}|\Lambda_{(1)}\rangle_k|^2}) \\ &= \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\eta_0\eta_1 e^{-\frac{1}{k}|\Lambda_{(0)}-\Lambda_{(1)}|^2}}. \end{aligned} \tag{6.13}$$

If the two are far apart, the maximal probability can be very close to 1.

This method can be useful for discriminating the complicated microstates in the quantum systems discussed in this paper, and also those complicated microstates of fuzzballs. The distinguishing between different microstates of fuzzballs is highly important for understanding the information of the microstates and the total entropy. The ideas in quantum measurement theory would be useful for this purpose. Hence we consider quantum detection in this section. The quantum detection theory is particularly needed and powerful when the states to distinguish in the measurement are non-orthogonal.

7 Entanglement and correlation in Hilbert spaces

7.1 Entanglement and correlation in phase space in single Hilbert space

Quantum states with entanglement and correlation in different regions of phase space are very common, see e.g. the general and formal discussions [60]. Now we want to make correlation and entanglement between different regions in the phase space. We could have multi droplets and multi rings in the phase space plane. In this section, we first consider two entangled rings in single copy of Hilbert space and their correlation.

We consider the entangled state of rings in single copy of Hilbert space

$$|\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|\Delta_{n_1,k_1}\rangle|\Delta_{n_2,k_2}\rangle \pm |\Delta_{n_2,k_1}\rangle|\Delta_{n_1,k_2}\rangle). \tag{7.1}$$

This state is in $\mathcal{H}_{k_1} \otimes \mathcal{H}_{k_2}$. The first term means that there is k_1 dual giants at location n_1 and k_2 dual giants at location n_2. The second term means that there is k_1 dual giants at location n_2 and k_2 dual giants at location n_1. From the quantum measurement point of view, this is an entangled state between a first set of k_1 dual giants and a second set of k_2 dual giants. The Young tableau states describe both the giant graviton states and the multi black ring states which are geometric spacetime backgrounds. The multi black ring geometries are in terms of multi black rings in the phase space plane. We make an approximation that the radii of the two black rings are much larger than their widths. The areas of the black rings are k_i. The locations of the black rings are centered at $|z_i|^2 = n_i + r_0^2$. For two nearby rings labeled by i and i', we have $|z_i|^2 - |z_{i'}|^2 = n_i - n_{i'}$. So this state is also an entangled state of two black rings centered at $|z_1|$ and $|z_2|$ respectively in the phase space plane.

We consider the Hilbert space of the two rings to be $\mathcal{H}_A \otimes \mathcal{H}_B = \mathcal{H}_S$, where $\mathcal{H}_A =$

$\mathcal{H}_{k_1}, \mathcal{H}_B = \mathcal{H}_{k_2}$. Consider the density matrix of this entangled state ρ_S in $\mathcal{H}_A \otimes \mathcal{H}_B$,

$$\rho_S = |\Psi_\pm\rangle\langle\Psi_\pm|. \tag{7.2}$$

$$\rho_A = \mathrm{tr}_{\mathcal{H}_B}\ \rho_S = \frac{1}{2}|\Delta_{n_1,k_1}\rangle\langle\Delta_{n_1,k_1}| + \frac{1}{2}|\Delta_{n_2,k_1}\rangle\langle\Delta_{n_2,k_1}|. \tag{7.3}$$

$$\rho_B = \mathrm{tr}_{\mathcal{H}_A}\ \rho_S = \frac{1}{2}|\Delta_{n_1,k_2}\rangle\langle\Delta_{n_1,k_2}| + \frac{1}{2}|\Delta_{n_2,k_2}\rangle\langle\Delta_{n_2,k_2}|. \tag{7.4}$$

The correlation between A and B is

$$\begin{aligned} C_{\rho_S}(\mathcal{O}_A, \mathcal{O}_B) : &= \mathrm{tr}\ (\rho_S \mathcal{O}_A \mathcal{O}_B) - \mathrm{tr}\ (\rho_A \mathcal{O}_A)\mathrm{tr}\ (\rho_B \mathcal{O}_B) \\ &= \mathrm{tr}\ ((\rho_S - \rho_A \otimes \rho_B)\mathcal{O}_A \mathcal{O}_B), \end{aligned} \tag{7.5}$$

where $\mathcal{O}_A$ acts on $\mathcal{H}_A$, $\mathcal{O}_B$ acts on $\mathcal{H}_B$, and $\mathcal{O}_A\mathcal{O}_B$ acts on $\mathcal{H}_A \otimes \mathcal{H}_B$.

We have

$$\begin{aligned} & \mathrm{tr}\ ((\rho_S - \rho_A \otimes \rho_B)\mathcal{O}_A \mathcal{O}_B) \\ = & \frac{1}{4}(\langle\mathcal{O}_A\rangle_{|\Delta_{n_1,k_1}\rangle} - \langle\mathcal{O}_A\rangle_{|\Delta_{n_2,k_1}\rangle})(\langle\mathcal{O}_B\rangle_{|\Delta_{n_1,k_2}\rangle} - \langle\mathcal{O}_B\rangle_{|\Delta_{n_2,k_2}\rangle}) \\ & \pm\frac{1}{2}[\langle\Delta_{n_1,k_1}|\mathcal{O}_A|\Delta_{n_2,k_1}\rangle\langle\Delta_{n_2,k_2}|\mathcal{O}_B|\Delta_{n_1,k_2}\rangle + \langle\Delta_{n_2,k_1}|\mathcal{O}_A|\Delta_{n_1,k_1}\rangle\langle\Delta_{n_1,k_2}|\mathcal{O}_B|\Delta_{n_2,k_2}\rangle]. \end{aligned} \tag{7.6}$$

In the last line, the sign is $\pm$ for $|\Psi_\pm\rangle$ respectively.

With $\mathcal{O}_A = \hat{E} = \sum_k A_k^\dagger A_k, \mathcal{O}_B = \hat{E} = \sum_k A_k^\dagger A_k$, we have

$$\begin{aligned} C_{\rho_S}(\hat{E}_A, \hat{E}_B) &= \mathrm{tr}\ ((\rho_S - \rho_A \otimes \rho_B)\hat{E}_A \hat{E}_B) \\ &= \frac{1}{4}(\langle\hat{E}_A\rangle_{|\Delta_{n_1,k_1}\rangle} - \langle\hat{E}_A\rangle_{|\Delta_{n_2,k_1}\rangle})(\langle\hat{E}_A\rangle_{|\Delta_{n_1,k_2}\rangle} - \langle\hat{E}_A\rangle_{|\Delta_{n_2,k_2}\rangle}) \\ &= \frac{1}{4}k_1k_2(n_1 - n_2)^2 \\ &= \frac{1}{4}k_1k_2(|z_1|^2 - |z_2|^2)^2. \end{aligned} \tag{7.7}$$

In the above, $\langle\hat{E}\rangle_{|\Delta_{n_i,k_i}\rangle} = k_i n_i$.

The area, i.e. area(annulus), between the two black rings is proportional to $|z_1|^2 - |z_2|^2 = n_1 - n_2 = \frac{1}{2\pi\hbar}$ area(annulus) [10]. Hence

$$\begin{aligned} C_{\rho_S}(\hat{E}_A, \hat{E}_B) &= \mathrm{tr}\ ((\rho_S - \rho_A \otimes \rho_B)\hat{E}_A \hat{E}_B) \\ &= \frac{1}{16\pi^2\hbar^2}k_1k_2\ \mathrm{area(annulus)}^2. \end{aligned} \tag{7.8}$$

The amount of correlation is proportional to the area squared of the annulus between the two rings. This is a geometric interpretation of the correlation and entanglement between the two rings. This annulus configuration can be viewed as a 'bridge' between the two rings, in which the two rings are entangled. Similar scenario has also been pointed out in [19, 16].

7.2 States in doubled Hilbert space and purification

Now we want to make correlation and entanglement between two copies of the phase space, and further relate this to the connectivity between the two copies of the spacetime [19]. This is different from the preceding Sec 7.1 for single copy. The doubled Hilbert space has been considered in [19]. We are interested in gray rings and doubled Hilbert space, since the gray rings carry entropy and one can perform a purification of the gray ring states to a pure state in doubled Hilbert space.

Here $|\Psi_1\rangle, |\Psi_2\rangle$ are pure states in single copy of Hilbert space:

$$|\Psi_1\rangle = \sqrt{\frac{k_1}{N}}|\Delta_{n_1,k_1}\rangle + \sqrt{\frac{k_2}{N}}|\Delta_{n_2,k_2}\rangle, \tag{7.9}$$

$$|\Psi_2\rangle = \sqrt{\frac{k_1}{N}}|\Delta_{n_1,k_1}\rangle + \sqrt{\frac{k_2}{N}}|\Delta_{n_3,k_2}\rangle, \tag{7.10}$$

where $k_1 + k_2 = N$, and $|\Psi_1| = |\Psi_2| = 1$. We make an approximation that $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are approximately orthogonal, and this means that $k_1 \ll N$. The YT has better orthogonality than the coherent state, and as a result the separations between n_1, n_2, n_3 needn't be large. This gives the pure state in doubled system

$$|\Psi\rangle = \alpha|\Psi_1\rangle|\Psi_1\rangle + \sqrt{1-\alpha^2}|\Psi_2\rangle|\Psi_2\rangle \tag{7.11}$$

which is in $\mathcal{H}^L \otimes \mathcal{H}^R$. We have $|\Psi| = 1$. The density matrix ρ of the doubled system is

$$\rho = |\Psi\rangle\langle\Psi|. \tag{7.12}$$

The density matrix after partial tracing from the above density matrix (7.12) is

$$\rho_L = \mathrm{tr}_{\mathcal{H}^R}\ |\Psi\rangle\langle\Psi| \simeq \alpha^2|\Psi_1\rangle\langle\Psi_1| + (1-\alpha^2)|\Psi_2\rangle\langle\Psi_2|, \tag{7.13}$$

which is derived by assuming the orthogonality between $|\Psi_1\rangle$ and $|\Psi_2\rangle$. And similarly we have $\rho_R = \mathrm{tr}_{\mathcal{H}^L}\ |\Psi\rangle\langle\Psi|$. The ρ can be viewed as a purification of ρ_L. Hence,

$$\begin{aligned}\rho_L &= \mathrm{tr}_{\mathcal{H}^R}\ |\Psi\rangle\langle\Psi| \\ &\simeq \frac{k_1}{N}|\Delta_{n_1,k_1}\rangle\langle\Delta_{n_1,k_1}| + \frac{k_2}{N}\alpha^2|\Delta_{n_2,k_2}\rangle\langle\Delta_{n_2,k_2}| + \frac{k_2}{N}(1-\alpha^2)|\Delta_{n_3,k_2}\rangle\langle\Delta_{n_3,k_2}| \\ &\quad + \sqrt{\frac{k_1 k_2}{N^2}}(\alpha^2|\Delta_{n_2,k_2}\rangle + (1-\alpha^2)|\Delta_{n_3,k_2}\rangle)\langle\Delta_{n_1,k_1}|\ + \mathrm{h.c.}\end{aligned} \tag{7.14}$$

Here $k_1 + k_2 = N$ and $\mathrm{tr}\rho_L = 1$. This state corresponds to a black droplet with filling fraction 1 and two gray rings located around $|z_2|$ and $|z_3|$ respectively, with filling fraction which is not 1, but α^2 and $(1-\alpha^2)$ respectively. And we have $|z_2|^2 = n_2 + r_0^2$ and $|z_3|^2 = n_3 + r_0^2$. This is a superposed mixed state. We make an approximation that the radii of the two rings are much larger than their widths. Note that the locations of the rings are centered at $|z_i|^2 = n_i + r_0^2$, and $|z_i|^2 - |z_{i'}|^2 = n_i - n_{i'}$ for two nearby rings.

The correlation between left and right systems is:

$$\begin{aligned} C_\rho(\mathcal{O}_L, \mathcal{O}_R) : &= \text{tr } (\rho \mathcal{O}_L \mathcal{O}_R) - \text{tr } (\rho_L \mathcal{O}_L) \text{tr } (\rho_R \mathcal{O}_R) \\ &= \text{tr } ((\rho - \rho_L \otimes \rho_R) \mathcal{O}_L \mathcal{O}_R), \end{aligned} \tag{7.15}$$

where $\mathcal{O}_L$ acts on $\mathcal{H}^L$, $\mathcal{O}_R$ acts on $\mathcal{H}^R$, and $\mathcal{O}_L \mathcal{O}_R$ acts on $\mathcal{H}^L \otimes \mathcal{H}^R = \mathcal{H}^S$. We denote the total system as $S = L \cup R$, and $\rho = \rho_S$.

We can have $\mathcal{O}_L := \mathcal{O} = \exp(i\theta \hat{E}) = \exp(i\theta \sum_k \hat{E}_k)$, where $\hat{E}_k = A_k^\dagger A_k$. And we have $\hat{E}_L = \sum_k \hat{E}_{k;L} = \sum_k A_{k;L}^\dagger A_{k;L}$, $\hat{E}_R = \sum_k \hat{E}_{k;R} = \sum_k A_{k;R}^\dagger A_{k;R}$. We define

$$F_{\theta_L, \theta_R} = \text{tr } ((\rho - \rho_L \otimes \rho_R) \exp(i\theta_L \hat{E}_L) \exp(i\theta_R \hat{E}_R)). \tag{7.16}$$

We can calculate $\mathcal{O}_L := \mathcal{O}$, where $\mathcal{O} = \hat{E} = \sum_k \hat{E}_k = \sum_k A_k^\dagger A_k$, and

$$\text{tr } ((\rho - \rho_L \otimes \rho_R) \hat{E}_L \hat{E}_R) = -\partial_{\theta_L} \partial_{\theta_R} F_{\theta_L, \theta_R}|_{\theta_L = \theta_R = 0}. \tag{7.17}$$

With the density matrix above, for a general $\mathcal{O}$,

$$\begin{aligned} & \text{tr } ((\rho - \rho_L \otimes \rho_R) \mathcal{O}_L \mathcal{O}_R) \\ \simeq\ & \alpha^2 (1 - \alpha^2) [(\langle \mathcal{O}_L \rangle_{|\Psi_1\rangle} - \langle \mathcal{O}_L \rangle_{|\Psi_2\rangle})(\langle \mathcal{O}_R \rangle_{|\Psi_1\rangle} - \langle \mathcal{O}_R \rangle_{|\Psi_2\rangle})] \\ & + \alpha \sqrt{1 - \alpha^2} [\langle \Psi_1 | \mathcal{O}_L | \Psi_2 \rangle \langle \Psi_1 | \mathcal{O}_R | \Psi_2 \rangle + \langle \Psi_2 | \mathcal{O}_L | \Psi_1 \rangle \langle \Psi_2 | \mathcal{O}_R | \Psi_1 \rangle]. \end{aligned} \tag{7.18}$$

This is a special case of the more general formalism in [19].

For $\mathcal{O} = \exp(i\theta \hat{E})$, we have that

$$\begin{aligned} & \text{tr } ((\rho - \rho_L \otimes \rho_R) \exp(i\theta_L \hat{E}_L) \exp(i\theta_R \hat{E}_R)) \\ \simeq\ & \alpha^2 (1 - \alpha^2) \frac{{k_2}^2}{N^2} [(e^{ik_2 n_2 \theta_L} - e^{ik_2 n_3 \theta_L})(e^{ik_2 n_2 \theta_R} - e^{ik_2 n_3 \theta_R})]. \end{aligned} \tag{7.19}$$

We denote $S = L \cup R$ and

$$\begin{aligned} \langle \hat{E}_S \rangle_\rho &= \langle \hat{E}_L + \hat{E}_R \rangle_\rho = \langle \hat{E}_L \rangle_{\rho_L} + \langle \hat{E}_R \rangle_{\rho_R}, \\ \langle \hat{E}_L \rangle_\rho &= \langle \hat{E}_R \rangle_\rho \simeq \frac{{k_1}^2}{N} n_1 + \frac{{k_2}^2}{N} (\alpha^2 n_2 + (1 - \alpha^2) n_3). \end{aligned} \tag{7.20}$$

On the YT states, we have $\langle \hat{E}_L \rangle_{|\Psi_i\rangle} = \frac{{k_1}^2}{N} n_1 + \frac{{k_2}^2}{N} n_{i+1}$, $\langle \hat{E}_L \rangle_{|\Psi_i\rangle} - \langle \hat{E}_L \rangle_{|\Psi_{i'}\rangle} = \frac{{k_2}^2}{N} (n_{i+1} - n_{i'+1})$.

We hence have

$$\begin{aligned} \text{tr } ((\rho - \rho_L \otimes \rho_R) \hat{E}_L \hat{E}_R) &= -\partial_{\theta_L} \partial_{\theta_R} F_{\theta_L, \theta_R}|_{\theta_L = \theta_R = 0} \\ &\simeq \alpha^2 (1 - \alpha^2) \frac{{k_2}^2}{N^2} {k_2}^2 (n_2 - n_3)^2. \end{aligned} \tag{7.21}$$

This is always non-zero as long as $n_2 \neq n_3$. The locations of the rings are centered at $|z_i|^2$, and $|z_i|^2 - |z_{i'}|^2 = n_i - n_{i'}$.

Since $n_2 - n_3 = |z_2|^2 - |z_3|^2 = \frac{1}{2\pi\hbar}$ area(annulus) [10], the correlation between the left and right is

$$\begin{aligned} C_\rho(\hat{E}_L, \hat{E}_R) &= \text{tr } ((\rho - \rho_L \otimes \rho_R)\hat{E}_L\hat{E}_R) \\ &\simeq \frac{1}{4\pi^2\hbar^2}\alpha^2(1-\alpha^2)\frac{{k_2}^4}{N^2} \text{ area(annulus)}^2. \end{aligned} \tag{7.22}$$

It is related to the area of annulus between two gray rings. There is a square because we have the product of two operators.

This is related to the superstar spacetime [61, 17, 62]. The microstates include gray rings whose filling fraction is smaller than 1. The gray rings are the analog of black hole horizon, in the sense they encode information of the microstates and carry entropy.

These density matrices are similar to that of the superstar geometry. We have to sum over the gray rings and hence in the geometric dual the gray rings are glued so that they disappear after the gluing. After the gluing, since there is no gray droplets left, it becomes a pure state in the doubled system. This supports the proposal that the doubled system is glued along the gray droplets [19]. This indicates that the above doubled system is glued along the gray rings. The 'bridge' or 'tunnel' configuration may be viewed as the annulus between the gray rings [19].

8 Discussion

In this paper, we focused on two types of coherent states. One type [14] is the coherent states of multi traces, and the other type is the coherent states of multi columns. The first type can be viewed as the coherent states of multi gravitons. In the second type, each column can be viewed as a giant graviton or a D-brane. Hence, the second type can be viewed as the coherent states of giant gravitons.

Among other things, we also analyzed interesting states in the phase space, the bumps on the edge of droplet, blobs and rings. The bumps are coherent states of multi-traces. While the rings are Young tableau states. Adding the bumps does not change the topology of the droplet configuration, while adding blobs or rings changes the topology.

We could use more matrices [53, 45] to construct the coherent states of these two types for the multi graviton states and giant graviton states. They have a rich entanglement structure. With more matrices, we can construct more unitary operations generalizing the phase shift operation, since different types of matrices can have independent phase rotations.

The gauge/gravity duality enables us to analyze the aspects of superposition and entanglement for the quantum gravity side. The gravitational aspects of the gravitational superposition states have also been discussed in [14, 63, 64, 65]. The cat states in Newtonian gravity and the associated fluctuations of gravitational potential and the effect on the probes in the spacetime background have been discussed in [65]. These probes have time-dependent dynamics. We can also study various probes on the backgrounds similar to the ones related to this paper, such as multi gravitons, closed strings, giant magnons and probe branes.

We found N-state Schrodinger cat states which approach the one-row Young tableau states with fidelity between them asymptotically reaches 1 at large N. This is of significance that angularly localized states superpose to form angularly delocalized state in quantum gravity. These superposition cat states have high variance of excitation energy, when the underlying coherent state amplitude is big. The N-state Schrodinger cat states also correspond to the irreducible representations of the cyclic group.

We conveniently used a phase shift operator, which also provides a generating function for the excitation energy operator. The phase shift operator also gives an associated parity operator. The quantum Fisher information of these states is proportional to the variance of the excitation energy of the underlying states. Some states like the first type of coherent states have high quantum Fisher information, and as a result have good localizability in the angular direction in the phase space. Some other states have relatively low quantum Fisher information and as a result are delocalized in the angular direction in the phase space.

We also analyzed correlation and entanglement between gravitational degrees of freedom using the phase space. In the context of single sided Hilbert space, the correlation between two entangled ring shaped droplets in phase space is related to the area of the annulus between the two rings. In the context of the doubled Hilbert space, the correlation between two sets of rings entangled in the doubled Hilbert space is related to the area of the annulus between the two rings, where the annulus plays the role of the 'bridge'.

The approach of correlation and entanglement in phase space [60] is convenient for working on questions with gravitational degrees of freedom using different regions of the phase space. The setup in this context provides a laboratory for studying these quantum gravitational questions.

In the context of gauge/gravity duality, it has been observed that the entanglement between different parts of the bulk spacetime or universe is related to the connectivity between the different parts [66, 67, 19, 16]. Our findings are in agreement with this scenario.

We considered two types of noisy coherent states and their derivations from adding noise. These are generalizations of the noisy coherent states of photons. The noisy coherent state can be viewed as an interpolated state. It interpolates between a pure coherent state in the noiseless limit, and a maximally mixed state in the large noise limit. The later has well-known gravity dual interpretation as thermal gas or black holes. For the noisy coherent state of the second type, the random columns are similar to a gas of thermal giant gravitons. The thermal distributions play the role of the noise. Moreover, we have a purification of the limit of the noisy coherent states. The noisy coherent states contain both quantum information and noise, and are widely used in quantum information theory.

The multitude of these different microstates in the quantum systems discussed in this paper are similar to the microstates of fuzzballs [39, 68, 69, 70], which have provided important insights into the information paradox. The quantum detection and quantum measurement theory are useful for distinguishing different quantum states of the quantum system.

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References

[1] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2** (1998), 231 [hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428** (1998), 105 [arXiv:hep-th/9802109 [hep-th]].

[3] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998), 253 [arXiv:hep-th/9802150 [hep-th]].

[4] G. T. Horowitz and J. Polchinski, "Gauge/gravity duality," In: D. Oriti, Approaches to quantum gravity, Cambridge University Press [gr-qc/0602037].

[5] M. Rangamani and T. Takayanagi, "Holographic Entanglement Entropy," Lect. Notes Phys. **931** (2017), pp.1-246 [arXiv:1609.01287 [hep-th]].

[6] D. Berenstein, "Large N BPS states and emergent quantum gravity," JHEP **01** (2006), 125 [arXiv:hep-th/0507203 [hep-th]].

[7] R. de Mello Koch and J. Murugan, "Emergent Spacetime," In: G. Ellis et. al., Foundations of Space and Time: Reflections on Quantum Gravity, Cambridge University Press [arXiv:0911.4817 [hep-th]].

[8] M. A. Nielsen, I. L. Chuang, "Quantum Computation and Quantum Information," Cambridge University Press, Cambridge, U.K., 2000.

[9] R. Horodecki, P. Horodecki, M. Horodecki, et al. "Quantum entanglement," Rev. Mod. Phys. **81** (2009) 865 [quant-ph/0702225].

[10] H. Lin, O. Lunin and J. M. Maldacena, "Bubbling AdS space and 1/2 BPS geometries," JHEP **10** (2004), 025 [arXiv:hep-th/0409174 [hep-th]].

[11] S. Corley, A. Jevicki and S. Ramgoolam, "Exact correlators of giant gravitons from dual N=4 SYM theory," Adv. Theor. Math. Phys. **5** (2002), 809-839 [arXiv:hep-th/0111222 [hep-th]].

[12] D. Berenstein, "A Toy model for the AdS / CFT correspondence," JHEP **07** (2004), 018 [arXiv:hep-th/0403110 [hep-th]].

[13] T. W. Brown, R. de Mello Koch, S. Ramgoolam and N. Toumbas, "Correlators, Probabilities and Topologies in N=4 SYM," JHEP **03** (2007), 072 [arXiv:hep-th/0611290 [hep-th]].

[14] D. Berenstein and A. Miller, "Superposition induced topology changes in quantum gravity," JHEP **11** (2017), 121 [arXiv:1702.03011 [hep-th]].

[15] P. Diaz, H. Lin and A. Veliz-Osorio, "Graph duality as an instrument of Gauge-String correspondence," J. Math. Phys. **57** (2016) no.5, 052302 [arXiv:1505.04837 [hep-th]].

[16] H. Lin and K. Zeng, "Detecting topology change via correlations and entanglement from gauge/gravity correspondence," J. Math. Phys. **59** (2018) no.3, 032301 [arXiv:1705.10776 [hep-th]].

[17] V. Balasubramanian, J. de Boer, V. Jejjala and J. Simon, "The Library of Babel: On the origin of gravitational thermodynamics," JHEP **12** (2005), 006 [arXiv:hep-th/0508023 [hep-th]].

[18] G. Mandal, "Fermions from half-BPS supergravity," JHEP **08** (2005), 052 [arXiv:hep-th/0502104 [hep-th]].

[19] J. Simon, "Correlations vs connectivity in R-charge," JHEP **10** (2018), 048 [arXiv:1805.11279 [hep-th]].

[20] S. Corley and S. Ramgoolam, "Finite factorization equations and sum rules for BPS correlators in N=4 SYM theory," Nucl. Phys. B **641** (2002), 131-187 [arXiv:hep-th/0205221 [hep-th]].

[21] C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, "A New double scaling limit of N=4 superYang-Mills theory and PP wave strings," Nucl. Phys. B **643** (2002), 3-30 [arXiv:hep-th/0205033 [hep-th]].

[22] R. de Mello Koch, N. Ives and M. Stephanou, "Correlators in Nontrivial Backgrounds," Phys. Rev. D **79** (2009), 026004 [arXiv:0810.4041 [hep-th]].

[23] R. J. Glauber, "The Quantum theory of optical coherence," Phys. Rev. **130** (1963), 2529-2539.

[24] W. M. Zhang, D. H. Feng and R. Gilmore, "Coherent States: Theory and Some Applications," Rev. Mod. Phys. **62** (1990) 867.

[25] J. Perina, Z. Hradil, B. Jurco, "Quantum Optics and Fundamentals of Physics," Kluwer Academic Publ., 1994.

[26] S. L. Braunstein and C. M. Caves, "Statistical distance and the geometry of quantum states," Phys. Rev. Lett. **72** (1994), 3439.

[27] P. Hyllus, W. Laskowski, R. Krischek, C. Schwemmer, W. Wieczorek, H. Weinfurter, L. Pezze, A. Smerzi, "Fisher information and multiparticle entanglement," Phys. Rev. A **85** (2012), 022321.

[28] G. Toth, "Multipartite entanglement and high-precision metrology," Phys. Rev. A **85** (2012), 022322.

[29] S. D. Mathur, "Tunneling into fuzzball states," Gen. Rel. Grav. **42** (2010), 113-118 [arXiv:0805.3716 [hep-th]].

[30] E. Schrodinger, "Die gegenwartige Situation in der Quantenmechanik.", Naturwiss. **23** (1935), 807-812.

[31] U. L. Andersen, J. S. Neergaard-Nielsen, "Heralded generation of a micro-macro entangled state," Phys. Rev. A **88** (2013) 022337.

[32] A. E. B. Nielsen, K. Molmer, "Transforming squeezed light into a large-amplitude coherent-state superposition," Phys. Rev. A **76** (2007) 043840.

[33] A. Einstein, B. Podolsky and N. Rosen, "Can quantum mechanical description of physical reality be considered complete?," Phys. Rev. **47** (1935), 777-780.

[34] C. W. Helstrom, "Quantum Detection and Estimation Theory," Mathematics in Science and Engineering **123**, Academic Press, New York, 1976.

[35] D. Berenstein, "Submatrix deconfinement and small black holes in AdS," JHEP **09** (2018), 054 [arXiv:1806.05729 [hep-th]].

[36] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," Adv. Theor. Math. Phys. **2** (1998), 505 [arXiv:hep-th/9803131 [hep-th]].

[37] V. Balasubramanian, B. Czech, V. E. Hubeny, K. Larjo, M. Rangamani and J. Simon, "Typicality versus thermality: An Analytic distinction," Gen. Rel. Grav. **40** (2008), 1863-1890 [arXiv:hep-th/0701122 [hep-th]].

[38] K. Skenderis and M. Taylor, "Anatomy of bubbling solutions," JHEP **09** (2007), 019 [arXiv:0706.0216 [hep-th]].

[39] S. D. Mathur, "The Quantum structure of black holes," Class. Quant. Grav. **23** (2006), R115 [arXiv:hep-th/0510180 [hep-th]].

[40] V. Balasubramanian, B. Czech, K. Larjo, D. Marolf and J. Simon, "Quantum geometry and gravitational entropy," JHEP **12** (2007), 067 [arXiv:0705.4431 [hep-th]].

[41] D. Berenstein and A. Miller, "Code subspaces for LLM geometries," Class. Quant. Grav. **35** (2018) no.6, 065003 [arXiv:1708.00035 [hep-th]].

[42] R. de Mello Koch, "Geometries from Young Diagrams," JHEP **11** (2008), 061 [arXiv:0806.0685 [hep-th]].

[43] V. Balasubramanian, J. de Boer, V. Jejjala and J. Simon, "Entropy of near-extremal black holes in AdS(5)," JHEP **05** (2008), 067 [arXiv:0707.3601 [hep-th]].

[44] R. Fareghbal, C. N. Gowdigere, A. E. Mosaffa and M. M. Sheikh-Jabbari, "Nearing Extremal Intersecting Giants and New Decoupled Sectors in N = 4 SYM," JHEP **08** (2008), 070 [arXiv:0801.4457 [hep-th]].

[45] V. Balasubramanian, D. Berenstein, A. Lewkowycz, A. Miller, O. Parrikar and C. Rabideau, "Emergent classical spacetime from microstates of an incipient black hole," JHEP **01** (2019), 197 [arXiv:1810.13440 [hep-th]].

[46] A. Dhar, G. Mandal and N. V. Suryanarayana, "Exact operator bosonization of finite number of fermions in one space dimension," JHEP **01** (2006), 118 [arXiv:hep-th/0509164 [hep-th]].

[47] W. Fulton, "Young tableaux," London Mathematical Society Student Texts **35**, Cambridge University Press, Cambridge, 1997.

[48] B. E. Sagan, "The symmetric group. Representations, combinatorial algorithms, and symmetric functions," Graduate Texts in Mathematics **203**, Springer, 2001.

[49] W. Fulton, J. Harris, "Representation theory," Graduate Texts in Mathematics, **129**, Springer-Verlag, New York, 1991.

[50] L. Grant, L. Maoz, J. Marsano, K. Papadodimas and V. S. Rychkov, "Minisuperspace quantization of Bubbling AdS and free fermion droplets," JHEP **08** (2005), 025 [arXiv:hep-th/0505079 [hep-th]].

[51] R. de Mello Koch, E. Gandote and A. L. Mahu, "Scrambling in Yang-Mills," [arXiv:2008.12409 [hep-th]].

[52] S. Ramgoolam and M. Sedlak, "Quantum Information Processing and Composite Quantum Fields," JHEP **01** (2019), 170 [arXiv:1809.05156 [hep-th]].

[53] H. Lin and K. Zeng, "A construction of quarter BPS coherent states and Brauer algebras," Adv. Theor. Math. Phys. **24** (2020) 5, 1111 [arXiv:1709.10093 [hep-th]].

[54] A. Dhar, G. Mandal and M. Smedback, "From gravitons to giants," JHEP **03** (2006), 031 [arXiv:hep-th/0512312 [hep-th]].

[55] J. Armas, T. Harmark, N. A. Obers, M. Orselli and A. V. Pedersen, "Thermal Giant Gravitons," JHEP **11** (2012), 123 [arXiv:1207.2789 [hep-th]].

[56] Y. C. Eldar, A. Megretski, G. C. Verghese, "Optimal detection of symmetric mixed quantum states," IEEE Trans. Inf. Theory **50** (2004) 1198-1207.

[57] A. Assalini, G. Cariolaro, G. Pierobon, "Efficient optimal minimum error discrimination of symmetric quantum states," Phys. Rev. A **81** (2010) 012315.

[58] C. W. Helstrom, "Quantum detection and estimation theory," J. Statist. Phys. **1** (1969), 231-252.

[59] J. A. Bergou, "Discrimination of Quantum States," J. Mod. Opt. **57** (2010) 160-180.

[60] A. M. Ozorio de Almeida, "Entanglement in Phase Space," Lect. Notes Phys. **768** (2009), 157.

[61] R. C. Myers and O. Tafjord, "Superstars and giant gravitons," JHEP **11** (2001), 009 [arXiv:hep-th/0109127 [hep-th]].

[62] L. D'Errico, W. Mueck and R. Pettorino, "Stretched horizon and entropy of superstars," JHEP **05** (2007), 063 [arXiv:hep-th/0703223 [hep-th]].

[63] D. Berenstein and A. Miller, "Can Topology and Geometry be Measured by an Operator Measurement in Quantum Gravity?," Phys. Rev. Lett. **118** (2017) no.26, 261601 [arXiv:1605.06166 [hep-th]].

[64] D. Berenstein and A. Miller, "Reconstructing spacetime from the hologram, even in the classical limit, requires physics beyond the Planck scale," Int. J. Mod. Phys. D **25** (2016) no.12, 1644012 [arXiv:1605.05288 [hep-th]].

[65] C. Anastopoulos and B. L. Hu, "Probing a Gravitational Cat State," Class. Quant. Grav. **32** (2015) no.16, 165022.

[66] J. Maldacena and L. Susskind, "Cool horizons for entangled black holes," Fortsch. Phys. **61** (2013), 781-811 [arXiv:1306.0533 [hep-th]].

[67] M. Van Raamsdonk, "Building up spacetime with quantum entanglement," Gen. Rel. Grav. **42** (2010), 2323-2329 [arXiv:1005.3035 [hep-th]].

[68] I. Bena and N. P. Warner, "Black holes, black rings and their microstates," Lect. Notes Phys. **755** (2008), 1-92 [arXiv:hep-th/0701216 [hep-th]].

[69] K. Skenderis and M. Taylor, "The fuzzball proposal for black holes," Phys. Rept. **467** (2008), 117-171 [arXiv:0804.0552 [hep-th]].

[70] V. Balasubramanian, J. de Boer, S. El-Showk and I. Messamah, "Black Holes as Effective Geometries," Class. Quant. Grav. **25** (2008), 214004 [arXiv:0811.0263 [hep-th]].