Counting the number of permutations with longest increasing subsequence of a certain length

Zhiming Feng

Tsinghua High School

Abstract

This essay mainly involves a method to count the number of permutations of size n with length of the longest increasing subsequence equal to 2. The question is raised as the research project of Algebraic Combinatorics in 2016 Tsinghua Math Camp. This essay will show the relationship between a permutation's longest increasing subsequence and its corresponding Robinson-Schensted map. Moreover, the essay will develop a general formula to count the number with the help of the hook length formula.

The key point of this essay is the Robinson-Schensted map. This marvelous map builds a bijection between pairs of standard Young tableaux with n boxes and S_n (the permutation group of n), which change a problem of permutation into a problem of standard Young tableau.

Different from linear algebra and calculus, combinatory doesn't involve complex computing procedure and complicated partial differential equations. In this essay you will not see gigantic matrixes and notations like $\Gamma^k{}_{ij}$. It is all pure thoughts and ideas that push me forward and enable me to continuously transform the original question into its equivalent but friendlier form for me to tackle with.

Key words: Robinson-Schensted map; longest increasing subsequence

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1 The Problem

Here is the problem that is raised as the research project of Algebraic Combinatorics in 2016 Tsinghua Math Camp: Count the number of permutations of size n with length of the longest increasing subsequence equal to 2.

This essay is going to show a new combinatorical method to this widely solved problem.

2 The hook length formula

The hook length formula is a formula to count the number of Young tableau. Let us recall the definition of Young tableau and hook length.

Definition 2.1

A Young diagram is a finite collection of boxes arranged in left-justified rows, with the row length in non-increasing order. A Young tableau is obtained by filling in the boxes of the Young diagram with number 1~n, following the rule that the numbers are left-to-right and up-to-down increasing.

Definition 2.2

Set the coordinate of the box at the i^{th} row and the j^{th} column in a Young diagram as (i,j). The hook length of this box is the number of boxes in the same Young diagram of which their coordinate (a,b) satisfy: $a=i$, $b\geq j$ or $b=j, a\geq i$.

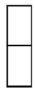
Let us recall the hook length formula.

<u>Theorem 2.1</u>

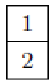
Set γ as a Young diagram, then the number of Young tableau in the shape of γ is $\frac{n!}{\prod_{s\in\gamma} h(s)}$

n is the number of boxes that γ contains, s is the box of γ, h(s) is the hook length of s.

When dealing with the problem, we only need to prove a special case of Theorem 2.1 which is partitions of length 2.

<u>Proof</u> Using induction. When $n=2$, there is only one two-row Young diagram 

According to Theorem 2.1, the number of Young tableau in the shape of this diagram should be

$2!/2*1=1$. And there is exactly one Young tableau of this shape which is 

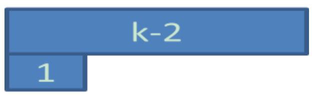
As a result, Theorem 2.1 holds true for partitions of length 2 when $n=2$.

Assume Theorem 2.1 holds true when $n=k-1$, consider the case of $n=k$. Three subcases are supposed to discuss.

Subcase 1 There is only one box in the second row of Young diagram

Because k is the biggest among 1~k, the possible places for k to appear in the corresponding Young tableau of this subcase are either the last box of the first row or the only box in the second row. Erase k, two kinds of Young tableau with (k-1) boxes will be got.

One is 

The other is 

Here (k-1), (k-2) and 1 refer to the number of boxes in that row. Same goes for the following graph.

Meanwhile, these two kinds of Young tableaux with (k-1) boxes can be changed into the corresponding Young tableau of this subcase by adding the box filled with k at the tail of the first row. Thus, a bijection has been built up between the corresponding Young tableau of this subcase and two kinds of Young tableaux with (k-1) boxes.

Therefore, there would be an equation like:

Attention should be paid that 

cannot be calculated by the assumption of induction. However it is obvious to tell that the result is 1 according to Definition 2.1.

According to the assumption of induction, the left hand side

$=1+\frac{(k-1)!(k-2)}{(k-1)!}=1+k-2=k-1=\frac{k!(k-1)}{k!}$

As a result, Theorem 2.1 holds true for this subcase.

Subcase 2 The length of the two rows of the Young diagram is the same.

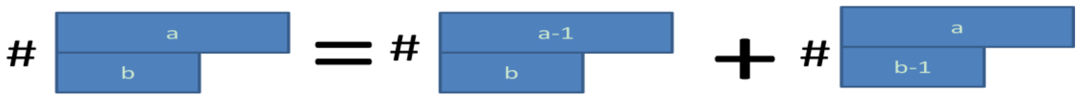
This subcase will only appear when k is even. In this subcase there is only one place for k to appear in the corresponding Young tableau which is the last box of the second row. Thus, the equation appears to be in this way:

$LHS=\frac{2(k-1)!}{(k/2+1)!(k/2-1)!}$ then multiply $k/2$ on both sides of the fraction, $\frac{k!}{(k/2+1)!(k/2)!}$ will be got.

As a result, Theorem 2.1 holds true for this subcase.

Subcase 3 Other cases

For other cases, the equation appears to be in this way where $a+b=k$:



$$LHS = \frac{(k-1)!(a-b)}{a!b!} + \frac{(k-1)!(a-b+2)}{(a+1)!(b-1)!}$$

$$= \frac{(k-1)!(a^2-ab+a-b)}{(a+1)!b!} + \frac{(k-1)!(ab-b^2+2b)}{(a+1)!b!}$$

$$= \frac{(k-1)!(a+b)(a-b+1)}{(a+1)!b!} = \frac{k!(a-b+1)}{(a+1)!b!}$$

As a result, Theorem 2.1 holds true for this subcase

According to above, Theorem 2.1 holds true for all three subcases when n=k. By induction, we could tell that Theorem 2.1 holds true for partitions of length 2.

3 The Robinson-Schensted map

Let us recall the Robinson-Schensted map.

Theorem 3.1

For every permutation $\sigma \in S_n$, there will be a corresponding Robinson-Schensted map of which:

(1) This map has two rows

(2) The first row consists of (n+1) Young tableaux p0~pn. Tableau p0=$\varnothing$ and the way to generate tableau p(k) is to insert σ(k) into tableau p(k-1), if σ(k) is larger than every number in the first row of tableau p(k-1) then it is supposed to be inserted at the last column of the first row. Otherwise pick out the smallest number which is larger than σ(k), σ(k) replaces its place and that number will go into the second row to apply the same insertion rule until a number is placed at the last row of a column.

(3) The second row consists of (n+1) Young tableaux q0~qn. Tableau q0=$\varnothing$ and the way to generate tableau q (k) is to add k to tableau q (k-1) at the bonus place of tableau p (k) compare with tableau p (k-1).

Proof

In [1], Schensted had offered an outline of the proof of Theorem 3.1. Following is a more detailed discussion.

The thing need to be proved is that after operating the insertion rule on tableau p (k-1) and tableau q (k-1), they are still Young tableaux, for every $k \in \{1, 2\ldots n\}$

First prove that tableau p (k-1) is still a Young tableau after an operation. The proof will be divided into two parts. Part one is to prove that the shape still satisfies the feature of a Young diagram. Consider the possible effects that an operation will bring to the length of the rows in tableau p (k-1). If the inserted number is larger than any other number in that row, then the length of that row will increase by one otherwise the length will remain the same. Therefore, the only illegal case we need to prove doesn't exist is that before the operation the length of the two rows are the same while after the operation the length of upper row doesn't change but the length of the below row increase by one. Set the largest number of the upper row before the operation is x, the largest number of the below row before the operation is y, the number that inserts to the below row is z. According to the insertion rule, z must come from the upper row and z is larger than any other numbers in the below row. Therefore, the relationship between x, y and z goes like this: $y< z\leq x$. However, according to Definition 2.1, there should be $y>x$, which is a paradox.

Part two is to prove that the number still satisfies the rule of Young tableau after operation. First check the row. If an inserted number is the larger than any other number in a row, then it will be placed at the last column which still makes this row left-to-right increasing. Otherwise there will be a replacement. Set the inserted number as a, the number that is replaced as b. Assume in tableau p (k-1), b is between $\{c\}$ and $\{d\}$ (notation "{ }" means it could be nonexistent). Since b is the smallest number that is larger than a, $\{c\}$ is smaller than b thus smaller than a. After the operation, a is between $\{c\}$ and $\{d\}$ and their relationship goes like $\{c\} <a<b< \{d\}$, which is still left-to-right increasing. Then check the column. First prove that every number in tableau p (k-1) cannot jump to its right column after the operation. This obviously goes true for those who don't move. For those who move, assume they can jump to their right columns , set one of them as e. Suppose in tableau p (k-1), e stays at the i^{th} row the j^{th} column then after the operation it will be at the ***(i+1)***th row the k^{th} column where $k>j$. According to the insertion rule, ***e*** is larger than the number at the ***(i+1)***th row the j^{th} column. However, according to Definition 2.1, the number at the ***(i+1)***th row the j^{th} column should be larger than the number at i^{th} row the j^{th} column before the operation, which is ***e*** itself. Therefore it is a paradox. So ***k*** should be no larger than ***j***. Consider the number right under the ***e*** after the operation. It is not smaller than then the number that has been replaced by ***e*** therefore is larger than ***e***. Consider the number right above e after the operation. If $k<j$, ***e*** is larger than the number at the i^{th} row the k^{th} column before the operation thus is larger than the number at the i^{th} row the k^{th} column after the operation. If $j=k$, there is only one possibility that the number at the i^{th} row the j^{th} column after the operation replaces ***e***. Then according to the insertion rule, it is clear to tell that ***e*** is larger then it. As a result, after the operation the number is still up-to-down increasing.

After checking the shape, the row and the column, it can be concluded that tableau p (k-1) is still a Young tableau after the operation.

A corollary has to be introduced before proving tableau q (k-1) is still a Young tableau after the operation.

<u>**Corollary 3.1.1**</u> **In the Robinson-Schensted map, the shape of tableau p (k) and the shape of tableau q (k) are the same, for every** $k \in \{0, 1 \ldots n\}$.

<u>**Proof**</u>

Because tableau p0=tableau q0=∅ and the bonus box tableau q(k) has compare with tableau q(k-1) is the same as the bonus box tableau p(k) has compare with tableau p(k-1). Therefore the shape of tableau p (k) is the same as tableau q (k) for every $k \in \{0, 1 \ldots n\}$.

Since we have proved that tableau p(k-1) is still a Young tableau after the operation, according to Corollary 3.1.1，the shape of tableau q(k-1) still satisfies the feature of a Young diagram after the operation. Meanwhile, according to the insertion rule, the inserted number for tableau q(k-1) is ***k***, which is larger than every number in tableau q(k-1), and the inserted place must be the last column of a row therefore the row and the column are both checked. As a result, tableau q (k-1) is still a Young tableau after the operation.

<u>Corollary 3.1.2</u> **The Robinson-Schensted map is invertible. Different pair of tableau p (n) and tableau q (n) will be correspondent with different permutation $\sigma \in S_n$ and each pair is correspondent with only one permutation.**

<u>Proof</u>

First consider the possible place for n to appear in tableau q (n). Claim that it must be at the last column of a row otherwise it cannot be a Young tableau after erasing n. Denote X_n to be the number that inserts into tableau p (n-1) which means $\sigma(n)=X_n$. If in tableau q (n), n is at the first row the last column then in tableau p (n), X_n is also at the first row the last column. If in tableau q (n), n is at the i^{th} row the last column where $i\neq 1$. Then it is clear to tell that several replacements happened when X_n inserting to tableau p (n-1). Denote the number at the i^{th} row the last column in tableau p(n) to be Y_i, then in tableau p(n-1) Y_i must be at the $(i-1)^{th}$ row and the number that replace Y_i is the largest number that is smaller than Y_i in the $(i-1)^{th}$ row of tableau p(n). Denote it to be $Y_{(i-1)}$. Using the same method, which is pick out the largest number that is smaller than $Y_{(i-1)}$ in the $(i-2)^{th}$ row of tableau p(n), can find out $Y_{(i-2)}$. Repeat the same process until finding out Y_1 in the first row of tableau p (n). It is obvious to tell that Y_1 is X_n. Erase n in tableau q (n) will obtain tableau q (n-1). If n is at the first row the last column in tableau q (n), then erasing the box at the first row the last column in tableau p(n) will obtain tableau p(n-1). If n is at the i^{th} row the last column in tableau q (n) where $i\neq 1$. Then the operation goes like replace $Y_{(i-1)}$ with Y_i , replace $Y_{(i-2)}$ with $Y_{(i-1)}$ etc. Until Y_1 is replaced by Y_2 then erase the box where Y_i used to stay. In this way, tableau p(n-1) will be obtained. Repeat this process, we could draw out the whole map and get the permutation $X_1X_2...X_{(n-1)}X_n$. Since $Y_1, Y_2..., Y_i$ are something of the largest, they are unique. Therefore the X_n being generated by them is also unique. As a result, the permutation which the pair of tableau p(n) and tableau q(n) is correspondent with is unique.

Following is going to prove that different pairs of tableau p(n) and tableau q(n) is correspondent with different permutations. Theorem 3.1 has actually built up a function that reflect tableau p(i) to tableau p (i+1) (as well as for tableau q(i) to tableau q(i+1)) for every $i \in\{0,1,...(n-1)\}$. Denote this as g. While the operation mentioned in the above paragraph is a function that reflect tableau p(i+1) to tableau p(i) (as well as for tableau q(i+1) to tableau q(i)) for every $i \in \{0,1,\ldots(n-1)\}$. Denote this as h. The relationship between g and h goes like $g \bullet h$*(tableau p(i+1))=tableau p(i+1)*; $h \bullet g$*(tableau pi)=tableau pi*. Therefore, h equals to g^{-1}. As a result, $h^{(n)}$ equals the inverse of $g^{(n)}$. Theorem 3.1 tells that $g^{(n)}$ is injective, thus $h^{(n)}$ is injective which means different pairs of tableau p(n) and tableau q(n) is correspondent with different permutations.

With Theorem 3.1 and Corollary 3.1.2, the essence of the Robinson-Schensted map could be discovered.

<u>Corollary 3.1.3</u> The Robinson-Schensted map is a bijection between a permutation $\sigma \in S_n$ and a pair of Young tableaux with n boxes.

4 The Robinson-Schensted map and the longest increasing subsequnce

<u>Definition 4.1</u>

If $\sigma \in S_n$, define the length of the first row of its Robinson-Schensted map as the length of the first row of tableau p (n).

<u>Theorem 4.1</u>

For every permutation, the length of its longest increasing subsequence is the length of the first row of its corresponding Robinson-Schensted map.

<u>Proof</u>

In [2], Simon Rubinstein-Salzedo has proved Theorem 4.1 using the basic subsequence. Recall that for a positive integer j, the j^{th} basic subsequence is a chronology of the numbers that, at some point in the Robinson-Schensted map, occupy the j^{th} column in the first row. Note that each basic subsequence is decreasing, it can be concluded that the length of the longest increasing subsequence of a permutation is no larger than the number of the basic subsequence, which is the length of the first row of the permutation's corresponding Robinson-Schensted map.

Suppose there are altogether ***r*** basic subsequences. The way to find an increasing subsequence whose length is ***r*** is: First randomly pick out a number in the r^{th} basic subsequence, call it X_r. Then pick out the number which is at the left of X_r when X_r first appear in the tableau, call it $X_{(r-1)}$. Because X_r first appear at the first row the r^{th} column, $X_{(r-1)}$ must be at the first row the $(r-1)^{th}$ column which means $X_{(r-1)}$ belongs to the $(r-1)^{th}$ basic subsequence. Meanwhile, according to the insertion rule, $X_{(r-1)} < X_r$. Repeat this process until we pick out ***r*** numbers from ***r*** basic subsequences respectively, their relationship goes like: $X_1 > X_2 > \ldots > X_{(r-1)} > X_r$. As a result, $X_1X_2 \ldots\ldots X_{(r-1)}X_r$ is a increasing subsequence of length ***r*** which makes it the longest increasing subsequence.

Definition 4.2

If $\sigma \in S_n$, define the length of the first column of its Robinson-Schensted map is the length of the first column of tableau p (n).

Similarly, we have:

Theorem 4.2

For every permutation, the length of its longest decreasing subsequence is the length of the first column of its corresponding Robinson-Schensted map.

Theorem 4.1 enables the initial problem to be transformed into: Count the number of permutation $\sigma \in S_n$ which the length of the first row of the corresponding Robinson-Schented map is 2.

5 Problem Transformations

The following theorems can further transform the problem.

Theorem 5.1

For permutation $\sigma \in S_n$, the number of their Robinson-Schensted maps whose length of the first row is k is the square of the number of the Young tableau which is made up by n boxes and

the length of the first row is k.

Proof

Construct a set A= ***{Young tableau made up by n boxes | the length of the first row of the Young tableau is k}***. According to Definition 4.1, tableau p (n) should belong to A while tableau q (n) should belong to A as well. Therefore, there are altogether $|A|^2$ pairs of tableau p (n) and tableau q(n). According to Corollary 3.1.2, every one of this $|A|^2$ pairs is correspondent with a different and unique permutation which means a different and unique Robinson-Schensted map. Therefore, the number of the Robinson-Schensted map is $|A|^2$.

Corollary 5.1.1

For permutation $\sigma \in S_n$, the number of their Robinson-Schensted maps whose length of the first column is k is the square of the number of the Young tableau which is made up by n boxes and the length of the first column is k.

With the help of Theorem 5.1, the problem could be further transformed into: Count the number of the Young tableau which is made up by n boxes and the length of the first row is 2, and then take square of the result.

The following theorems are helpful to continue transforming the problem.

Theorem 5.2

For permutation $\sigma \in S_n$, the number of the permutation whose length of the longest increasing subsequence is k equals to the number of the permutation whose length of the longest decreasing subsequence is k.

Proof

Assume the length of the longest increasing subsequence of $\sigma(1)\sigma(2)\ldots\sigma(n)$ is k. Consider permutation $\sigma(n)\ldots\sigma(2)\sigma(1)$, it still belongs to S_n while it has the feature that the length of the longest decreasing subsequence is k. Assume the length of the longest decreasing subsequence of $\sigma'(1)\sigma'(2)\ldots\sigma'(n)$ is k. Consider permutation $\sigma'(n)\ldots\sigma'(2)\sigma'(1)$, it still belongs to S_n while it has the feature that the length of the longest increasing subsequence is k. Thus a bijection has been built up between permutations whose length of the longest increasing subsequence is k and permutations whose length of the longest decreasing subsequence is k. Therefore, their numbers are the same.

Theorem 5.3

For Young tableau, the number of the tableau whose length of the first row is k equals to the number of the tableau whose length of the first column is k.

Proof

Denote the number of the tableau whose length of the first row is k as X while the number of the tableau whose length of the first column is k as Y. According to Theorem 5.1, the number of the Robinson-Schensted map whose length of the first row is k is X^2, meanwhile, according to Corollary 5.1.1, the number of the Robinson-Schensted map whose length of the first column is k is Y^2. With the help of Theorem 4.1, Theorem 4.2 and Theorem 5.2, the relationship between X and Y should be $X^2=Y^2$, which is $X=Y$.

With the help of Theorem 5.3, the problem could be finally transformed into: Count the number of the Young tableau which is made up by n boxes and the length of the first column is 2(partitions of length 2), and then take square of the result.

6 Analysis to the problem transformations

The purpose of this transformation is that there are two disadvantages when dealing with the problem "Count the number of the Young tableau which is made up by n boxes and the length of the first row is 2, and then take square of the result."

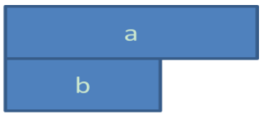
1. It is more difficult to prove that Theorem 2.1 holds true for Young diagram which the length of the first row is 2 compare with proving the "partition of length 2" case.
2. It is difficult to use Theorem 2.1 to count the number of the Young tableau which is made up by n boxes and the length of the first row is 2. Because the only information those diagrams contain is the length of their first row is 2. The situations of their rest rows are uncertain which means the product of the hook length is uncertain. Therefore it is impossible to

develop a general formula of the number of the Young tableau in that shape.

After transforming into the "partitions of length 2" case, those two disadvantages automatically vanish because it is easy to prove that Theorem 2.1 holds true and a general formula of the number of the Young tableau in this shape could be developed.

7 Calculations

Firstly, according to Theorem2.1

\# 

$= \frac{n!(a-b+1)}{(a+1)!b!}$

Where ***a*** refers to the number of boxes in the first row, ***b*** refers to the number of boxes in the second row. Hence, ***a*** and ***b*** has a relationship: ***n=a+b***.

If ***n*** is even, followings are the ways to divide ***n*** into the sum of two positive integers: *n=(n-1)+1=(n-2)+2=…=(n/2)+(n/2)*

Because ***a*** is larger than ***b***, so the possible value of ***a*** could be every integer from *(n/2)* to *(n-1)*.

If ***n*** is odd, followings are the ways to divide ***n*** into the sum of two positive integers: *n=(n-1)+1=(n-2)+2=……(n+1/2)+(n-1/2)*

Because ***a*** is larger than ***b***, so the possible value of ***a*** could be every integer from *(n+1/2)* to *(n-1)*.

Therefore, the formula appears to be:

$$\sum_{a\geq\frac{n}{2}}^{n-1}\left(\binom{n}{a}\times\frac{2a+1-n}{a+1}\right)^2$$

8 Verifications

When $n=3$, apply the formula, the result is 4 which means there are four permutations belong to S_3 whose length of the longest increasing subsequence is 2.

It can be discovered that amid all six permutations belong to S_3, only (2 3), (1 2), (1 2 3) and (1 3 2) have a longest increasing subsequence of length 2.

When $n=4$, apply the formula, the result is 13 which means there are thirteen permutations belong to S_4 whose length of the longest increasing subsequence is 2.

It can be discovered that amid all twenty four permutations belong to S_4, only

(2 4), (1 4), (1 2 4 3), (1 2 4), (1 3 4 2), (1 3 4), (1 3), (1 3 2 4), (1 4 2), (1 4 3), (1 4 2 3), (1 2)(3 4) and (1 3)(2 4) have a longest increasing subsequence of length 2.

9 Extensions

There could be the next problem: Count the number of permutations of size n with length of the longest increasing subsequence equal to 3.

The main idea of dealing with this problem is the same with dealing with the initial problem. However, there will be more obstacles when length 2 is changed into length 3.

The first obstacle appears at the time when using induction to prove Theorem 2.1 holds true for partitions of length 3. There are more subcases to discuss:

Subcase 1 There is only one box in the third row of the Young diagram.

Subcase 2 The length of the second row and the third row of the Young diagram are the same.

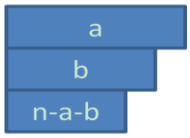
Subcase 3 The length of all three rows are the same.

Subcase 4 Other cases

The reason for distinguishing subcase 2, subcase 3 and subcase 4 is because their bijection between the (k-1) case and the k case are different. The reason for picking subcase 1 out is because if the only one box in the third row is erased, a case of partitions of length 2 will appear, which is something cannot be computed by the assumption. However, since Theorem 2.1 has been proved to be true for partitions of length 2 case, the hook length formula could be used directly to compute.

The second obstacle appears when trying to divide n into the sum of three positive integers. In this case there are two free variables but those two variables are not completely free for they have to share a certain relationship which is the length of second row is no larger than the length of the first row. Therefore, there will be two variables and two "Σ" in the final formula.

The ***a*** and ***b*** mentioned shown in the below graph are respectively the length of the first row and the length of the second row.



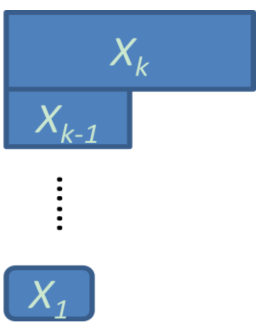
The result goes like：

$$\sum_{a\geq\frac{n}{3}}^{n-2}\ \sum_{b\geq\frac{n-a}{2}}^{min\{a,(n-a-1)\}}\left(\binom{n}{b}\times\binom{n-b}{a}\times\frac{(2a+b-n+2)(a-b+1)(2b+a-n+1)}{(a+1)(a+2)(b+1)}\right)^2$$

When n=3, apply the formula, the possible value for ***a*** is 1 while the possible value for ***b*** is also one. Thus the result appears to be 1. It means that there is only one permutation $\in S_3$ which the length of the longest increasing subsequence is 3. It is obvious to tell that there is only (1) $\in S_3$ who satisfies this feature.

It is spontaneous to make a step further to study the general case after finishing the above work. The general problem looks like: Count the number of permutations of size n with length of the longest increasing subsequence equal to k where k can be any positive integer. The process of solving this problem is going to be divided into two parts.

Part one is to count the number of Young tableaux which has a partition of length k. Denote $X_1+X_2+\ldots+X_k=n$, where $X_1, X_2\ldots X_k$ are all positive integers and they share the relationship: $X_1\leq X_2\leq\ldots\leq X_k$. In this case, the Young diagram appears to be:



The hook length product of the last row is $(X_1)!$

The hook length product of the row right above the last the row is $\frac{(X_2+1)!}{(X_2-X_1+1)}$

Generally, the hook length product of the row of which the length is X_i is $\frac{(X_{i+}(i-1))!}{\prod_{j=1}^{i-1}(X_i-X_j+i-j)}$

Therefore, according to Theorem 2.1, the number of Young tableaux in this shape is:

$$\frac{n!}{\prod_{i=1}^{k}(X_i+(i-1))!} \times \prod_{1\leq p<q\leq k}(X_p - X_q + p - q)$$

Part two is to decide the possible value of X_1, X_2... X_k. First stabilize X_1, which means the value of X_1 depends on the value of X_2... X_k. Consider the possible value of X_2. Because X_2 is larger than X_1, therefore the minimum value of X_2 is $\frac{n-(X_3+X_4+\cdots+X_k)}{2}$. There are cases when $\frac{n-(X_3+X_4+\cdots+X_k)}{2}$ is not an integer so the relationship goes like: $X_2 \geqslant \frac{n-(X_3+X_4+\cdots+X_k)}{2}$. It is known that $X_2 \leq X_3$, meanwhile X_i cannot be so large that it makes X_1 less than one. As a result, the maximum value of X_2 should be: $min\{X_3,(n-X_3-X_4-\cdots-X_k-1)\}$.

Generally, Because X_i is larger than $X_1, X_2, \ldots X_{i-1}$, the minimum value of X_i is: $\frac{n-(X_{i+1}+X_{i+2}+\cdots+X_k)}{i}$, It is known that $X_i \leq X_{i+1}$, meanwhile X_i cannot be so large that it makes some of $X_1 \ldots X_{i-1}$ less than one. Therefore, the maximum value of X_i is: $min\{X_{i+1},(n-i+1-X_{i+1}-X_{i+2}-\cdots-X_k)\}$.

As a result, the formula for the number of permutations of size n with length of the longest increasing subsequence equal to k appears to be:

$$\sum_{X_k\geq\frac{n}{k}}^{n-k+1} \cdots \sum_{X_i\geq\frac{n-(X_{i+1}+X_{i+2}+\cdots+X_k)}{i}}^{min\{X_{i+1},(n-i+1-X_3-X_4-\cdots-X_k)} \cdots \sum_{X_2\geq\frac{n-(X_3+X_4+\cdots+X_k)}{2}}^{min\{X_3,(n-X_3-X_4-\cdots-X_k-1)\}} \left(\frac{n!}{\prod_{i=1}^{k}(X_i+(i-1))!} \times \prod_{1\leq p<q\leq k}(X_p - X_q + p - q)\right)^2$$

Where X_1, X_2... X_k are all positive integers and such that:

$X_1+X_2+\ldots+X_k=n$, $X_1 \leq X_2 \leq \ldots \leq X_k$

References

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