Brocard Configuration in $q-$period Loop Broken Line

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Abstract: In this paper we study the generalization of the Brocard configurations. The definition of q-period broken line is given and then the new necessary and sufficient condition about the existence of two conjugate Brocard points of the $q-$period loop broken line is investigated. Based on the past research, a geometric method for the construction of all possible loop broken lines with conjugate Brocard points is developed.

Keywords: Brocard configurations, Loop broken line, Inversion

1. Introduction

1.1 Background

Brocard point is feature point in a triangle which is first observed by *Crelle* in 1816[1],Their definition is as follows:

In any $\Delta A_1A_2A_3$ there is exactly one point T_1 such that

$$\angle T_1A_1A_2 = \angle T_1A_2A_3 = \angle T_1A_3A_1 = \omega_1,$$

and exactly one point T_2 such that

$$\angle T_2A_2A_1 = \angle T_2A_3A_2 = \angle T_2A_1A_3 = \omega_2.$$

Then T_1 and T_2 are defined as the *first* and *second Brocard point* of $\Delta A_1A_2A_3$, respectively. They are collectively called *conjugate Brocard point.*

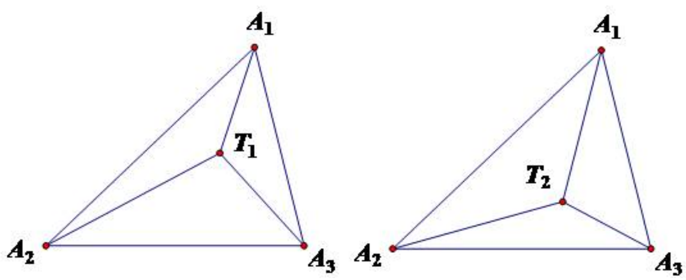

Fig.1.

After being investigated for more than a century, many theorems of Brocard configurations have been found. One issue with such past research has been limited on triangles and quadrilaterals,with general polygons or broken lines being largely ignored.

To facilitate understanding, the definitions of a loop broken line[4] and its Brocard points are given. *Unless stated otherwise, any $'n'$ appearing in this article is not less than 5, with the subscripts in modulus n.*

Definition 1.1 If any three of n distinct points $A_1, A_2, \cdots, A_n$ on a plane are non-collinear, draw the segments $A_1A_2, A_2A_3, \cdots, A_nA_1$; the figure is called a *closed broken line*, denoted by $A_1A_2\cdots A_n$.

Definition 1.2 A closed broken line is defined as a *loop broken line* if any of its angles are less than π, and there exists a fixed point M; when a point P moves forward along the broken line, M is always on the left (or right) side of the forward direction. For a loop broken line, this kind of point M is called a same-side point of the loop broken line.

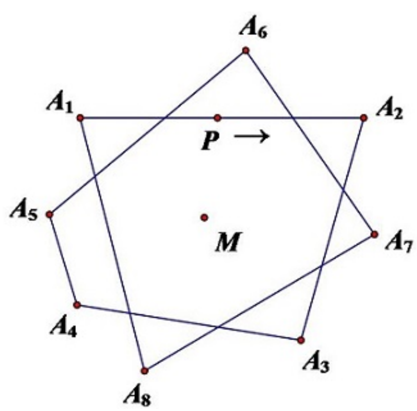

Fig.2.

Definition 1.3 For a loop broken line $A_1A_2\cdots A_n$, one of its same-side points, denoted T_1, is the *first Brocard point* if $\angle T_1A_iA_{i+1} = \omega_1$ is valid for all $1 \le i \le n$; ω_1 is known as the *first Brocrd angle.*

Similarly, one of its same-side point, denoted T_2, is the *second Brocard point* if $\angle T_2A_iA_{i-1} = \omega_2$ is valid for all $1 \le i \le n$. ω_2 is known as the *second Brocrd angle.*

If $A_1A_2\cdots A_n$ has both the first and the second Brocard points T_1 and T_2, then they are collectively called the conjugate Brocard points of $A_1A_2\cdots A_n$, ω_1 and ω_2 are collectively called the *conjugate Brocard angles.*

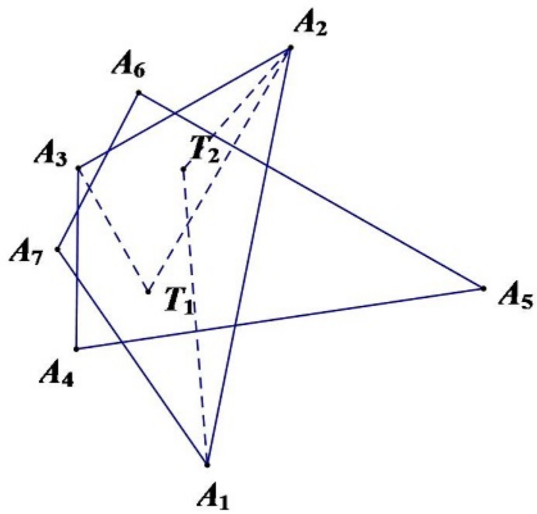

Fig.3.

In 1998, Professor Xiong Zengrun extended the Brocard points to the loop broken lines[3]. He put forward a necessary and sufficient condition for a loop broken line to have a first Brocard point:

$$\frac{a_i}{a_{i+1}\sin A_{i+1}} + \cot A_{i+2} = \cot\omega_1$$

is valid for all $1 \le i \le n$, herein $a_i = A_iA_{i+1}$.

It is well known that any general triangle has conjugate Brocard points. *But a loop broken line with first Brocard point may not have conjugate ones* (see the figures in chapter 4). We investigated the existence conditions of conjugate Brocard points, something that the past research is yet to consider.

Obviously we cannot construct the second Brocard points by simply using reflections. However, by simulating the symmetrical conclusion of Xiong's, we can get a necessary and sufficient condition for a loop broken line to have conjugate Brocard points:

$$\begin{cases} \dfrac{a_i}{a_{i+1}sinA_{i+1}} + \cot A_{i+2} = \cot\omega_1 \\ \dfrac{a_{i+1}}{a_i sinA_{i+1}} + \cot A_i = \cot\omega_2 \end{cases}$$

is valid for all $1 \le i \le n$.

Further progression from these formulas remains impractical. Additionally, it is difficult to generalize concise results.

Therefore we added a necessary and sufficient condition based on the geometric method of *inversion*: we will prove the existence of the conjugate Brocard point iff in the harmonic $n-$point range we defined (see Chapter 2). Furthermore, we gave *a geometric method of constructing all the loop broken line which has the conjugate Brocard point.* Then we investigated the figure of several loop broken lines sharing common conjugate Brocard points, which are called $q-$period loop broken line.

Besides, we found that the conjugate Brocard points and their associative feature points in loop broken lines have similar geometric structures as those in triangles, most of them can be generalized. Due to limited space, these results will be examined in detail at a later date.

1.2 Main results

Firstly, we list the main results in the paper below. The definitions of the notations of Theorem 1 and 2 can be found in Chapter 2, while those of Theorem 3 can be found in Chapter 4.

Theorem 1 *If conjugate Brocard points T_1 and T_2 exist in a loop broken line $B_1B_2\cdots B_n$, then its conjugate Brocard angles are equal. (i.e.$\omega_1 = \omega_2$).*
Theorem 2 *A loop broken line $B_1B_2\cdots B_n$ has conjugate Brocard points iff there exist an interger p where $1 \le p \le n-1$ and $gcd(n,p) = 1$, and $B_1B_2\cdots B_n$ can be noted as $B[n,p]$.*
Theorem 3 *For an integer p where $1 \le p \le n-1$ and $gcd(n,p) = 1$, if each period of a $B(n,p,q)$ has conjugate Brocard points and can be denoted as a $B_i[n,p]$, then $B(n,p,q)$ has conjugate Brocard points iff the conjugate Brocard points and circumcircle of each period coincide.*

2. Definitions

In this chapter several definitions are given in order to simplify the proof.

Definition 2.1 If any three of n distinct points $A_1, A_2, \cdots, A_n$ on a plane are non-collinear, they are defined as an *n-point set*, denoted by $A(n)$ where A can be replaced by any letters corresponding to the points.For example,for $X_1, X_2, \cdots, X_n$,we use $X(n)$.
Definition 2.2 If n distinct points $A_1, A_2, \cdots, A_n$ lie on a circle in an arbitrary order, they are defined as an *inscribed n-point set*, denoted by $A\overline{\{n\}}$. An $A\overline{\{n\}}$ is especially defined as a *inscribed n-point range* if the subscripts of the points increase (or decrease) in counterclockwise order, denoted by $A\{n\}$.

Unless stated otherwise, we use the subscripts of $A\{n\}$ increase in counterclockwise order for simplicity.
Definition 2.3 For an $A\{n\}$, if

$$\frac{A_iA_{i+3} \cdot A_{i+1}A_{i+2}}{A_iA_{i+1} \cdot A_{i+2}A_{i+3}} = f(n)$$

is valid for any $1 \le i \le n$, herein

$$f(n) = |\frac{\sin\frac{3\pi}{n}}{\sin\frac{\pi}{n}}| = 2\cos\frac{2\pi}{n} + 1$$ [1]

then this $A\{n\}$ is called a *harmonic n-point range*, denoted by $A[n]$.
Definition 2.4 If an $A\{n\}$ coincides with the vertices of a regular n-gon, then the $A\{n\}$ is defined as a *regular n-point range*, denoted by $A\langle n\rangle$.
Definition 2.5 For an $A(n)$,integer p such that $gcd(n,p) = 1$ where $1 \le p \le n-1$ and $k = 1, 2, \cdots, n$,mark the vertices of $A(n)$ by denoting B_k to the point A_{kp}. It is clear that every point of $A(n)$ is marked exactly once by the residue theory in number theory.

[1] Actually we get the value of $f(n)$ from a regular n-point range. I.e. $A\langle n\rangle$ is a special type of $A[n]$.

Let B_k be the point A_{kp} for each $k = 1, 2, \cdots, n$. It is clear that every point of $A(n)$ is redenoted exactly once. Hereinafter when B_k appears it always refer to this way of redenoting $A(n)$.

If $B_1 B_2 \cdots B_n$ is a loop broken line, then it is called the *p-th broken line* of $A(n)$, denoted by $B(n, p)$. There is no confusion because we do not consider different values of p at the same time.

We can replace $A(n)$ in the definition by $A\overline{\{n\}}$, $A\{n\}$, $A[n]$ and $A\langle n\rangle$ to define *pseudo inscribed p-th broken line, inscribed p-th broken line, harmonic p-th broken line* and *regular p-th broken line*, respectively, also denoted similarly by $B\overline{\{n, p\}}$, $B\{n, p\}$, $B[n, p]$ and $B\langle n, p\rangle$.[2]

3. Auxiliary Theorems

Through our study we found some special properties of the point range from one loop broken line with conjugate Brocard points. We will give several auxiliary theorems and corollaries which are useful in later proofs.

3.1 Preliminaries

Definition 3.1 Given a circle $\odot O$ whose radius r not zero; if X and X' lie on a line through O, and $\overrightarrow{OX} \cdot \overrightarrow{OX'} = r^2$, then the transformation of determining either when the other is given is called *inversion*[1], which is denoted by $I(O, r^2)$. $\odot O$ is called *base circle*, r is called *inversion radius*, O is called *inversion center*. Hereinafter an inversion is usually denoted by $I(O)$ for short.

Definition 3.2 Let α and β be set of points, then α and β are said to be *mutually inverse* with regard to an inversion $I(O)$, if $I(O)$ transform α into β. We denote by $J(\odot O, X)$ the inverse of X with regard to circle $\odot O$.

Hereinafter, $I(O) : X \to X'$ implies that X and X' are mutually inverse with regard to $I(O)$, $I(O) : \odot C \to \odot C'$ implies that $\odot C$ and $\odot C'$ are mutually inverse with regard to $I(O)$.

Definition 3.3 Let A, B be two fixed points. A point C moves such that AC and BC has a constant ratio t, then the locus of C is a circle whose center is collinear with A, B. The locus is called an *Apollonius circle*[1] whose *base points* are A, B. We take an arbitrary point C_0 on the locus to denote it by $\tau(A, B, C_0)$.

3.2 Theorems of point range

In this section a theorem depicting the harmonic n-point range is given. We discovered and proved the theorem along with the corollaries mainly by inversion.

Theorem 3.4 The sufficient and necessary condition for an $A\{n\}$ to be an $A[n]$ is it can be transformed into an $A'\langle n\rangle$ by an inversion $I(S) : A_i \to A'_i$where $i = 1, 2, \cdots, n$,similarly hereinafter.

[2]From the definiton it can be seen that $B(n, p)$ and $B(n, n-p)$ are the same broken line with different orientation. Therefore a $B(n, p)$ with conjugate Brocard points is equivalent to both $B(n, p)$ and $B(n, n-p)$ with first Brocard point.

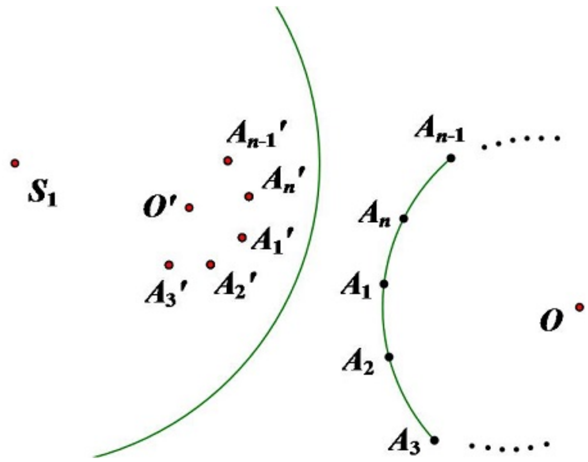

Fig.4.

Proof. We need two lemmas.

Lemma 3.5 Let A, B, C be three points on the plane. Construct $I(S, r^2) : X \to X'$(where $X = A, B, C, \cdots$). Then $A'B' = C'B'$ if and only if S is on $\tau(A, C, B)$.

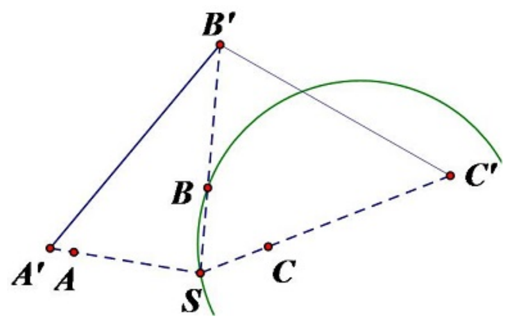

Fig.5.

Proof. By the definition of inversion, $\triangle SA'B' \backsim \triangle SBA$, hence $\dfrac{A'B'}{AB} = \dfrac{SA'}{SB} = \dfrac{r^2}{SA \cdot SB}$, similarly $\dfrac{C'B'}{CB} = \dfrac{r^2}{SC \cdot SB}$, thus $A'B' = C'B' \Leftrightarrow \dfrac{SA}{AB} = \dfrac{SC}{CB} \Leftrightarrow S$ is on $\tau(A, C, B)$.

Lemma 3.6 Let A_1, A_2, A_3, A_4 lies on a circle $\odot O$. Denote $t = \dfrac{A_1A_4 \cdot A_2A_3}{A_1A_2 \cdot A_3A_4}$, Γ_1 the circle $\tau(A_1, A_3, A_2)$, and Γ_2 the circle $\tau(A_2, A_4, A_3)$. Then

$$\Gamma_1, \Gamma_2 \quad are \begin{cases} \text{intersecting,if} \quad 0 < t < 3 \\ \text{tangent,if} \quad t = 3 \\ \text{seperated,if} \quad t > 3 \end{cases}$$

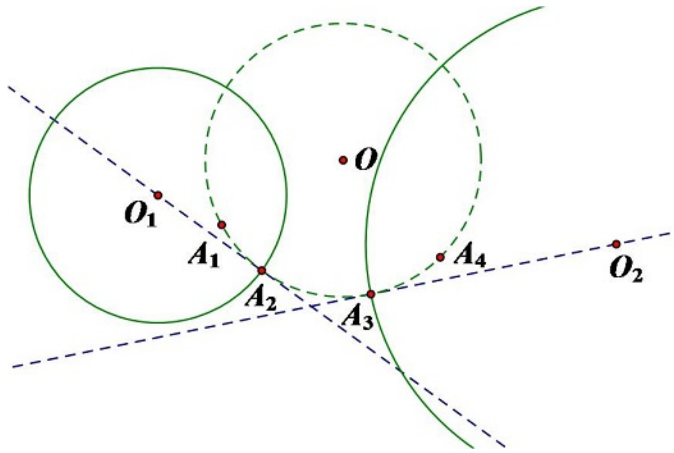

Fig.6.

Proof. We transform the proposition into the following figure.

Given a $\triangle ABC$, let P, Q lies on the sides AB, AC respectively such that $AP = AQ$. Let $\odot O$ passes through P, Q; also both on a tangent to AB, AC. BQ meet $\odot O$ again at R, CP meet $\odot O$ again at S. Denote $u = BP + CQ - BC$ and $v = \dfrac{PQ \cdot RS}{PR \cdot QS}$; then the relations between u, v are shown below:

$$u \begin{cases} < 0 \Leftrightarrow v > 3 \\ = 0 \Leftrightarrow v = 3 \\ > 0 \Leftrightarrow v < 3 \end{cases}$$

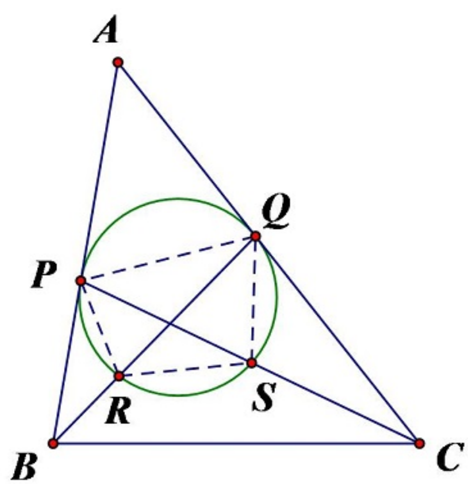

Fig.7.

Let the tangent line of $\odot O$ which passes through A_2, A_3 be l_2, l_3 respectively. It is known that the centers of Γ_1, Γ_2, denoted by O_1, O_2, lies on l_2, l_3 respectively. Then the positional relationship between Γ_1, Γ_2 is determined by $O_1A_2 + O_2A_3 - O_1O_2$. Thats why we make the transformation.

Denote $BC = a$, $CA = b$, $AB = c$, $s = \dfrac{a+b+c}{2}$, then $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$. By the law of sines,

$$\begin{aligned} v &= \frac{\sin \angle BPS}{\sin \angle BQP} \cdot \frac{\sin \angle CQR}{\sin \angle CPQ} = \frac{\sin \angle APC}{\sin \angle QPC} \cdot \frac{\sin \angle AQB}{\sin \angle PQB} \\ &= \frac{AC}{CQ} \cdot \frac{PQ}{AQ} \cdot \frac{AB}{BP} \cdot \frac{QP}{AP} = \frac{4bc \sin^2 \frac{A}{2}}{CQ \cdot BP} \end{aligned}$$

If $u=0$, it is obvious that $\odot O$ is the inscribed circle of $\triangle ABC$. Thus $BP=s-b$, $CQ=s-c$, we get that

$$\frac{PS\cdot QR}{PR\cdot QS}=\frac{4bc\sin^2\frac{A}{2}}{(s-b)(s-c)}=4$$

by the Ptolemy′s theorem we get $v=3$.

If $P^{'}$,$Q^{'}$ satisfy $u^{'}=BP^{'}+CQ^{'}-BC<0$, it is obvious that $BP^{'}<BP$, $CQ^{'}<CQ$, therefore

$$\frac{P^{'}S^{'}\cdot Q^{'}R^{'}}{P^{'}R^{'}\cdot Q^{'}S^{'}}=\frac{4bc\sin^2\frac{A}{2}}{CQ^{'}\cdot BP^{'}}>\frac{4bc\sin^2\frac{A}{2}}{CQ\cdot BP}=4$$

Thus $v^{'}=v=\frac{P^{'}Q^{'}\cdot R^{'}S^{'}}{P^{'}R^{'}\cdot Q^{'}S^{'}}>3$. By similar argument we can prove the case of $u^{'}>0$.

Then we return to the original problem.

Sufficiency:

For an $A^{'}\langle n\rangle$, let S be an arbitrary point not on its circumcircle. Construct $I(S):A_i\to A_i^{'}$, by the definition of inversion, it is easy to obtain that the ratio of the product of opposite sides in a quadrangle in inversion is a constant (then we simplify it as *inversion invariance*). From the conclusion we get

$$\frac{A_iA_{i+3}\cdot A_{i+1}A_{i+2}}{A_iA_{i+1}\cdot A_{i+2}A_{i+3}}=\frac{A_i{'}A_{i+3}{'}\cdot A_{i+1}{'}A_{i+2}{'}}{A_i{'}A_{i+1}{'}\cdot A_{i+2}{'}A_{i+3}{'}}=f(n)$$

Therefore the inverse of $A^{'}\langle n\rangle$ must be an $A[n]$.

Necessity:

Denote by Γ_1 the circle $\tau(A_1,A_3,A_2)$; and Γ_2 the circle $\tau(A_2,A_4,A_3)$.

Since $f(n)=|\frac{\sin\frac{3\pi}{n}}{\sin\frac{\pi}{n}}|=|3-4\cos^2\frac{\pi}{n}|<3$, by Lemma 3.6 we have Γ_1,Γ_2 intersecting at two points S_1,S_2; take either of them to be the inversion center, we may assume it as S_1.

Construct $I(S_1):A_i\to A_i^{'}$. Note that S_1 lies on Γ_1 and Γ_2, by Lemma 3.5 we get $A_1^{'}A_2^{'}=A_2^{'}A_3^{'}$ and $A_2^{'}A_3^{'}=A_3^{'}A_4^{'}$. From the inversion invariance follows $\frac{A_1^{'}A_4^{'}}{A_1^{'}A_2^{'}}=f(n)$, thus $A_1^{'},A_2^{'},A_3^{'},A_4^{'}$ are 4 adjacent vertices of a regular n-point range.

We claim that $I(S_1)$ invert $A[n]$ into $A^{'}\langle n\rangle$. Suppose that $A_1^{'},A_2^{'},\cdots,A_l^{'}$ are l adjacent vertices of a regular n-point range. The proposition already holds for $l=4$. Assume the proposition already holds for $i=4,5,\cdots,l$. Then we follow on with the case of $l+1$. From inversion invariance follows $\frac{A_{l-2}^{'}A_{l+1}^{'}\cdot A_{l-1}^{'}A_l^{'}}{A_{l-2}{'}A_{l-1}^{'}\cdot A_l^{'}A_{l+1}^{'}}=f(n)$, hence $\frac{A_{l-2}^{'}A_{l+1}^{'}}{A_l{'}A_{l+1}^{'}}=f(n)$, thus $A_{l+1}^{'}$ is the intersection of $\odot O$ and an Apollonius circle whose base points are $A_{l-2}^{'},A_l^{'}$.

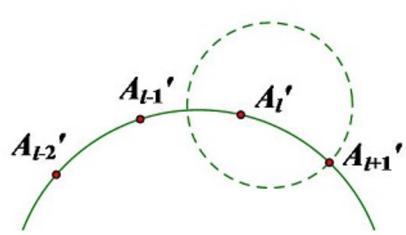

Fig.8.

It is clear that the next adjacent vertice of the regular n-point range is one of the intersections. Note that an inversion does not change the arrangement of $A_1^{'}, A_2^{'}, \cdots, A_l^{'}, A_{l+1}^{'}$, therefore $A_{l+1}^{'}$ must be the next adjacent point of the regular n-point range. The proposition holds for $l+1$; by induction we have completed the proof. Thus an $A[n]$ can be inverted into an $A^{'}\langle n\rangle$.

Corollary 3.7 There exist exactly two points S_1, S_2 as the center of inversion, which transform an $A[n]$ into an $A^{'}\langle n\rangle$. We refer to them as the *first* and the *second isodynamic point*. It is can be clearly observed that S_1, S_2 are inverse with regard to $\odot O$. Without loss of generality we let S_1 is always outside $\odot O$.

Corollary 3.8 For an $A[n]$ and its p-th broken line , we have $\dfrac{B_iB_{i+3}\cdot B_{i+1}B_{i+2}}{B_iB_{i+1}\cdot B_{i+2}B_{i+3}} = f(n,p)$, herein $f(n,p) = |\dfrac{\sin\frac{3p\pi}{n}}{\sin\frac{p\pi}{n}}|$

Corollary 3.9 For an $A[n]$ and any $1 \leq i \leq n$, construct the tangent line through A_{i+k} and A_{i-k} with regard to $\odot O$ respectively and let P_{ik} be the intersection of them (if $A_{i-k} \equiv A_{i+k}$ then $P_{ik} \equiv A_{i-k}$) , then $P_{i1}, P_{i2}, \cdots, P_{is}, A_i$ are collinear; herein $s = \lfloor\dfrac{n}{2}\rfloor$. This line is refer to as the *harmonic diagonal* through A_i, denoted by l_i.

Proof. By theorem 3.4, let $A^{'}\langle n\rangle$ and $A[n]$ be mutually inverse with regard to $I(S_1)$. Let $A_i^{'}C_i^{'}$ be the diameter of $A^{'}\langle n\rangle$s circumcircle , and let $I(S_1): C_i \rightarrow C_i^{'}$

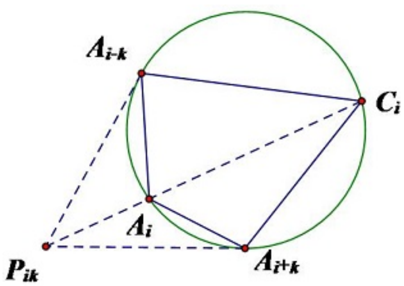

Fig.9.

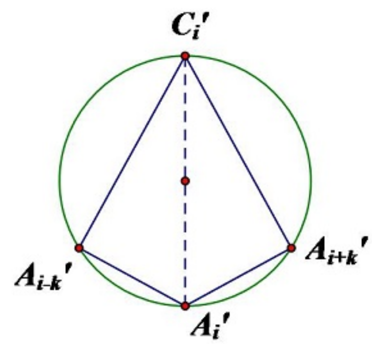

Fig.10.

Note that $A_i^{'}A_{i+k}^{'}C_i^{'}A_{i-k}^{'}$ is a harmonic quadrilateral[3]; by the inversion invariance we get that, $A_iA_{i+k}C_iA_{i-k}$ is also a harmonic quadrilateral. Therefore P_{ik}, A_i, C_i are collinear; $P_{i1}, P_{i2}, \cdots, P_{is}, A_i$ are also collinear.

Corollary 3.10 All of $A[n]$'s harmonic diagonals are concurrent at a point K, which is defined as the *conjugate barycenter* of $A[n]$.

[3]A cyclic quadrilateral that the products of opposite sides are equal.

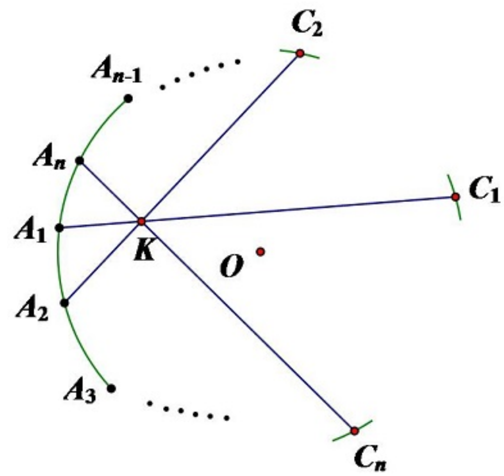

Fig.11.

3.3 Theorems of cyclic broken line

In this section we provide a cyclic broken line theorem; it will be particularly useful in proving the necessary condition in chapter 4.

Theorem 3.11 For any $B\{n,p\}$, there exist an integer $1 \le k \le n$ such that

$$\frac{B_kB_{k+3} \cdot B_{k+1}B_{k+2}}{B_kB_{k+1} \cdot B_{k+2}B_{k+3}} \le 3.$$

Proof. There are three mutually inequivalent orders of 4 adjacent points in a $B\{n,p\}$, which are shown in Fig.12. The directed arc,designated by the symbol $\frown$, $\widehat{XY}$ is that the arc of the circle on which the point X must be moved in counterclockwise direction in order coincide with Y.

However, the third case does not exist. According to the definition of $B\{n,p\}$,for any index i,the interior of $\widehat{B_iB_{i+1}}$(not including the end points of the arcs) should contain the same number of points in $A(n)$. Conversely, according to the figure, $\widehat{B_{k+2}B_{k+3}}$ contains less points in $A(n)$ than $\widehat{B_{k+1}B_{k+2}}$, leading to a contradiction. Therefore only the former two cases are considered.

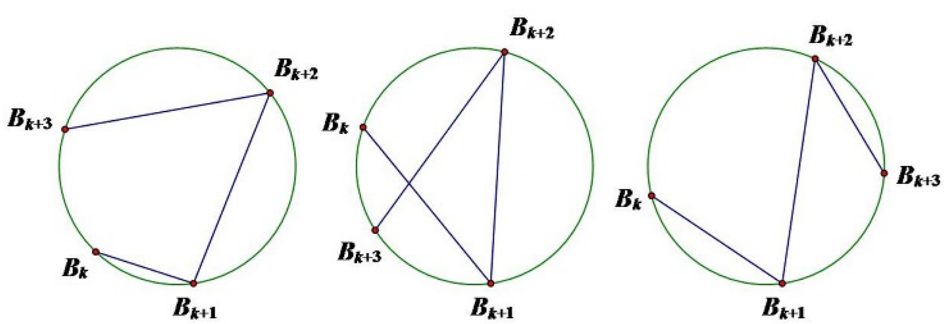

Fig.12.

If a k satisfies the second case in Fig.12, by Ptolemy′s theorem , $\frac{B_kB_{k+3} \cdot B_{k+1}B_{k+2}}{B_kB_{k+1} \cdot B_{k+2}B_{k+3}} < 1 \le 3$, which meets the condition.

If any of the integer k satisfies the first case in Fig.12, then contrarily we assume that

$$\frac{B_kB_{k+3} \cdot B_{k+1}B_{k+2}}{B_kB_{k+1} \cdot B_{k+2}B_{k+3}} > 3$$

holds for all $1 \leq k \leq n$.

Take a k arbitrarily. Denote the circle $\tau(B_k, B_{k+2}, B_{k+1})$ as Γ and let S be a point on Γ. Construct $I(S) : B_i \to B_i^{'}, \odot O \to \odot O^{'}$.

By lemma 3.5 we get that $B_k^{'}B_{k+1}^{'} = B_{k+1}^{'}B_{k+2}^{'}$. Note that when S tends to the intersection of $\odot O$ and Γ, the radius of $\odot O^{'}$ tends to infinity. Hence in that case the central angle of arc $\widehat{B_k B_{k+2}}$ tends to zero. Therefore we can take an S satisfying $\angle B_k^{'}O^{'}B_{k+1}^{'} < \dfrac{2p\pi}{n}$.

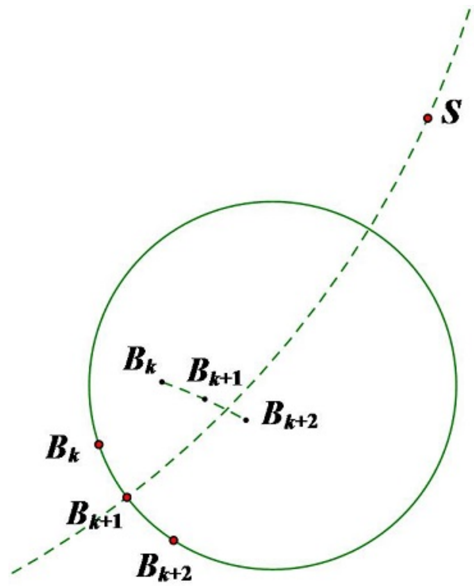

Fig.13.

From the inversion invariance we get that $\dfrac{B_k^{'}B_{k+3}^{'} \cdot B_{k+1}^{'}B_{k+2}^{'}}{B_k^{'}B_{k+1}^{'} \cdot B_{k+2}^{'}B_{k+3}^{'}} > 3$ is valid for all integer $1 \leq k \leq n$.

Notice a conclusion: Given A, B, C in an counterclockwise direction on $\odot O$. Suppose $B^{'}$ lies on $\odot O$ such that $\dfrac{AB^{'}}{BB^{'}} > \dfrac{AC}{BC}$, then $B^{'} \in \widehat{BC}$.

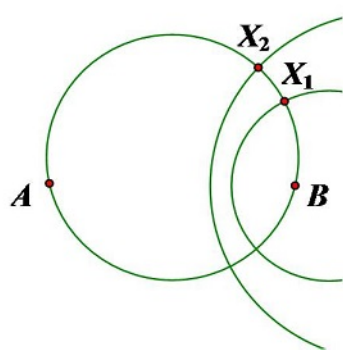

Fig.14.

We can see that when $t = \dfrac{\overline{AX}}{\overline{BX}}$ decreases, the intersection of $\odot O$ and $\tau(A, B, X)$ moves further from B in the direction of $\widehat{BA}$.

We claim that $B_{l+1}^{'}B_{l+2}^{'} < B_l^{'}B_{l+1}^{'}$ is true for all $l = k+1, k+2, \cdots, k+(n-2)$.

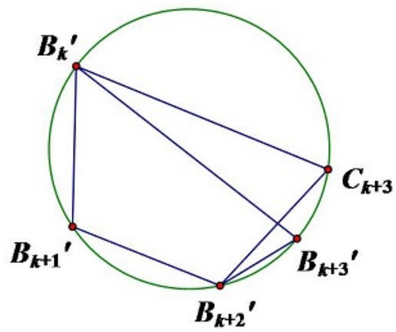

Fig.15.

In fact we only need to prove the case of $l = k+1$ and then complete the proof by induction. Take a point C_{k+3} on $\overset{\frown}{B'_{k+2}B'_{k+1}}$ such that $B'_{k+2}C_{k+3} = B'_{k+1}B'_{k+2}$. Since

$$\frac{B'_kB'_{k+3}}{B'_{k+2}B'_{k+3}} > 3 \cdot \frac{B'_kB'_{k+1}}{B'_{k+1}B'_{k+2}} = 3.$$

and $\dfrac{B'_kC_{k+3}}{B'_{k+2}C_{k+3}} < 3$, from the conclusion above we get that $B'_{k+2}B'_{k+3} < B'_{k+2}C_{k+3} = B'_{k+1}B'_{k+2}$.

Hence $\angle B'_{l+1}O'B'_{l+2} < \angle B'_lO'B'_{l+1} < \dfrac{2p\pi}{n}$ is valid for all $l = k+1, k+2, \cdots, k+(n-2)$. Thus

$$2p\pi = \sum_{i=0}^{n-1} \angle B_{k+i}{}'O'B_{k+i+1}{}' < n \cdot \angle B_k{}'O'B_{k+1}{}' < n \cdot \frac{2p\pi}{n}$$

which leads to a contradiction to the definition of $B\{n,p\}$.

Consequently the theorem is proved.

By combining Lemma 3.6 and Theorem 3.11 the following corollary is concluded.

Corollary 3.12 For any $B\{n,p\}$, there exists an integer $1 \le k \le n$ and a point S such that $B'_kB'_{k+1} = B'_{k+1}B'_{k+2} = B'_{k+2}B'_{k+3}$ in the construction of $I(S) : B_i \to B'_i$.

4. Main Results

In this chapter the proofs of three main theorems are provided.

4.1 Proof of Theorem 1

Theorem 1 *If conjugate Brocard points T_1 and T_2 exist in a loop broken line $B_1B_2\cdots B_n$, then its conjugate Brocard angles are equal. (i.e. $\omega_1 = \omega_2$).*

Proof. First we give the definition of directed area[4].

If T is a same-side point of a loop broken line $A_1A_2\cdots A_n$. In the rectangular coordinates system, suppose T is the original point and $A_i(x_i, y_i)$. Then the definition of the directed area of $\triangle TA_iA_{i+1}$ follows:

$$\overrightarrow{S_{\triangle TA_iA_{i+1}}} = \frac{1}{2}\begin{vmatrix} 0 & x_i & X_{i+1} \\ 0 & y_i & y_{i+1} \\ 1 & 1 & 1 \end{vmatrix}$$

By the definition of same-side point we get that for all the index i, $\overrightarrow{S_{\triangle TA_iA_{i+1}}}$ have positive (or negative) values. The the sum of all the directed area of $\triangle TA_iA_{i+1}$ is defined as the directed area of $A_1A_2\cdots A_n$,denoted as $\overrightarrow{S_{A_1A_2\cdots A_n}}$.

Let b_i be the length of B_iB_{i+1} By the law of sines in $\triangle T_1B_iB_{i+1}$, we get $\dfrac{T_1B_{i+1}}{\sin\omega_1} = \dfrac{b_i}{\sin\angle B_iT_1B_{i+1}} = \dfrac{b_i}{\sin B_{i+1}}$. Hence

$$|\overrightarrow{S_{B(n,p)}}| = \frac{1}{2}\sum_{i=1}^{n} T_1B_i\cdot b_i\cdot\sin\omega_1 = \frac{1}{2}\sum_{i=1}^{n}\frac{b_{i-1}b_i}{\sin B_i}\cdot\sin^2\omega_1,$$

we get that $\csc^2\omega_1 = \dfrac{\sum_{i=1}^{n}\frac{b_{i-1}b_i}{\sin B_i}}{2|\overrightarrow{S_{B(n,p)}}|}$. Similarly, we have $\csc^2\omega_2 = \dfrac{\sum_{i=1}^{n}\frac{b_{i-1}b_i}{\sin B_i}}{2|\overrightarrow{S_{B(n,p)}}|}$; only the proof for $\omega_1+\omega_2<\pi$ remains to be found.

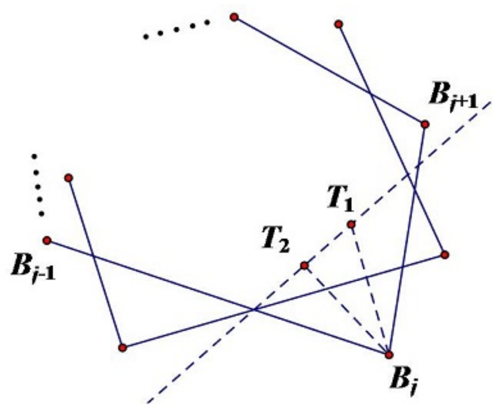

Fig.16.

Construct the line T_1T_2, $B_1B_2\cdots B_n$ is divided into two parts such that each part contains at least one vertice of $B_1B_2\cdots B_n$. Since T_1 and T_2 are same-side points of $B_1B_2\cdots B_n$, we can take one of the two parts such that any B_j in that part satisfies that B_{j+1} and T_2 are on distinct sides of B_jT_2. Then we get $\omega_1+\omega_2<\angle B_{j-1}B_jB_{j+1}<\pi$, hence $\omega_1=\omega_2$.

Based on this theorem, conjugate Brocard angles are collectively denoted by ω in the following paragraphs.

4.2 Proof of Theorem 2

The theorem in this section aims to demonstrate the existence of conjugate Brocard points in a loop broken line. Besides the main theorem, several additional corollaries are given to describe other noteworthy properties of Brocard configurations.

Theorem 2 *A loop broken line $B_1B_2\cdots B_n$ has conjugate Brocard points if and only if it there exist an integer p such that $1\le p\le n-1$ and $gcd(n,p)=1$,and $B_1B_2\cdots B_n$ can be denoted as $B[n,p]$.*

Proof. Directed angles[1] are required during the proof of sufficiency.The directed angle will be designated by the symbol $\sphericalangle$. $\sphericalangle ABC$ is that the angle through which the line AB must be rotated about BC in the counterclockwise direction in order to coincide with BC.

Sufficiency:

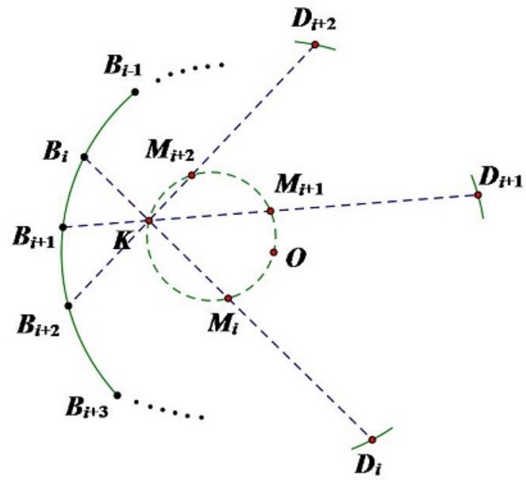

Fig.17.

For a $B[n,p]$, let B_iD_i be the harmonic diagram and D_i on the circumcircle of $B[n,p]$; and let K be the conjugate barycenter of $B[n,p]$. By Corollary 3.10, all the B_iD_i concurrent at K, where $i=1,2,\cdots$. Take M_i as the midpoint of B_iD_i, then $OM_i \perp B_iD_i$, thus $O, M_1, M_2, \cdots, M_n, K$ are concyclic; let Ψ represent the circle they lie on.

Note that for any $1 \leq i \leq n$, $D_{i+1}B_iB_{i+1}B_{i+2}$ is a harmonic quadrilateral, producing:

$$\sphericalangle D_{i+1}B_iB_{i+1} = \sphericalangle D_{i+1}B_iM_{i+1} = \sphericalangle D_{i+1}B_{i+1}B_{i+2},$$

hence the circle through M_{i+1}, B_i, B_{i+1} is tangent to $B_{i+1}B_{i+2}$.

Let $T_1^{'}$ be the intersection of the circles through M_1, B_n, B_1 and M_2, B_1, B_2. Since $\sphericalangle T_1^{'}M_1K = \sphericalangle T_1^{'}B_nB_1 = \sphericalangle T_1^{'}B_1B_2 = \sphericalangle T_1^{'}M_2K$, we get $T_1^{'} \in \Psi$.

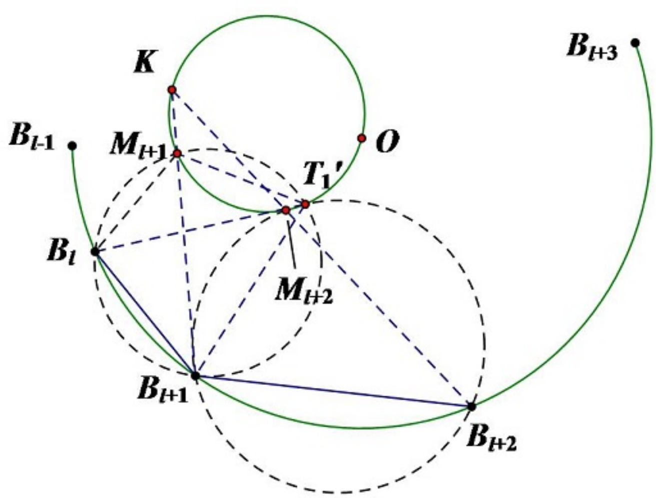

Fig.18.

We then prove by induction that $\sphericalangle T_1^{'}B_{l-1}B_l = \sphericalangle T_1^{'}B_lB_{l+1}$, and $T_1^{'}, M_{l+1}, B_l, B_{l+1}$ are concyclic. Give that the proposition already holds for $l=1$, if we assume it holds for l, then $T_1^{'}, M_{l+1}, B_l, B_{l+1}$ are also concyclic. Therefore we get

$$\begin{aligned}\sphericalangle T_1^{'}B_{l+1}B_{l+2} &= \sphericalangle B_lB_{l+1}B_{l+2} - \sphericalangle B_lB_{l+1}T_1^{'} \\ &= \sphericalangle B_lM_{l+1}K - \sphericalangle B_lM_{l+1}T_1^{'} \\ &= \sphericalangle T_1^{'}M_{l+1}B_{l+1} = \sphericalangle T_1^{'}B_lB_{l+1}\end{aligned}$$

From $\sphericalangle T_1^{'}B_{l+1}B_{l+2} = \sphericalangle T_1^{'}B_lB_{l+1} = \sphericalangle T_1^{'}M_{l+1}K = \sphericalangle T_1^{'}M_{l+2}K = \sphericalangle T_1^{'}M_{l+2}B_{l+2}$ follows that $T_1^{'}, M_{l+2}, B_{l+1}, B_{l+2}$ are concyclic; the proposition holds for $l+1$.

Thus $T_1^{'}$ is the first Brocard point of $B[n,p]$. By replacing p with $n-p$ in the above statement, we can imply that $B[n,n-p]$ has the first Brocard point. The proof of sufficiency is then complete.

Necessity:

First we need to prove the $B_1B_2\cdots B_n$ is a $B\{n,p\}$.

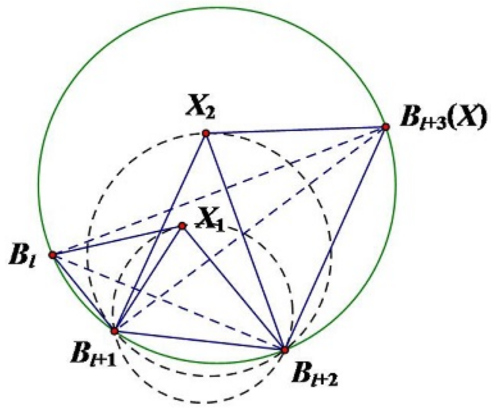

Fig.19.

Lemma 4.1 Let $B_l, B_{l+1}, B_{l+2}, B_{l+3}$ be vertices of $B(n,p)$, if X_1 and X_2 satisfies

$$\angle X_1B_lB_{l+1} = \angle X_1B_{l+1}B_{l+2} = \angle X_1B_{l+2}B_{l+3} = \omega,$$

$$\angle X_2B_{l+1}B_l = \angle X_2B_{l+2}B_{l+1} = \angle X_2B_{l+3}B_{l+2} = \omega,$$

then $B_l, B_{l+1}, B_{l+2}, B_{l+3}$ are concyclic.

Proof. Construct circle Γ_1 through B_{l+1}, B_{l+2} and tangential to line $B_{l+2}B_{l+3}$; and circle Γ_2 through B_{l+1}, B_{l+2} and tangential to line B_lB_{l+1}. Let $B_{l+2}B_{l+3}$ meet the circumcircle of B_l, B_{l+1}, B_{l+2} again at X, and it is obvious that $X_1 \in \Gamma_1$, $X_2 \in \Gamma_2$. By the law of sines,

$$\frac{X_1B_{l+2}}{\sin\omega} = \frac{B_{l+1}B_{l+2}}{\sin\angle B_{l+1}X_1B_{l+2}} = \frac{B_{l+1}B_{l+2}}{\sin B_{l+2}}$$

$$\frac{X_2B_{l+1}}{\sin\omega} = \frac{B_{l+2}B_{l+1}}{\sin\angle B_{l+2}X_2B_{l+1}} = \frac{B_{l+2}B_{l+1}}{\sin B_{l+1}}$$

Hence $\dfrac{X_1B_{l+2}}{X_2B_{l+1}} = \dfrac{B_lB_{l+2}}{B_{l+1}X}$. From $\angle X_1B_{l+2}B_l = \angle XB_{l+1}X_2$ follows $\triangle X_1B_{l+2}B_l \backsim \triangle X_2B_{l+1}X$. Thus $\angle B_{l+2}XX_2 = \angle X_1B_lB_{l+1} = \omega$, then $X \equiv B_{l+3}$, thus $B_l, B_{l+1}, B_{l+2}, B_{l+3}$ are concyclic.

By Theorem 1, the conjugate Brocard angles of $B_1B_2\cdots B_n$ are equal. By Lemma 4.1 we get that the $B_1B_2\cdots B_n$ is inscribed in a circle. Therefore it can be denoted as a $B\overline{\{n,p\}}$.

Contrarily, we assume that for any p, $B_1B_2\cdots B_n$ can be noted as a $B\overline{\{n,p\}}$ but not a $B\{n,p\}$. Then for any p an integer $1 \le k \le n$ can be found that arcs $\overset{\frown}{B_{k-1}B_k}$ and $\overset{\frown}{B_kB_{k+1}}$ contains different numbers, x and y respectively, of points in $A(n)$. Without sacrificing generalizability, assume $x < y$. By corresponding the points on the arcs, there exists a B_j in $\overset{\frown}{B_kB_{k+1}}$ that satisfies $\overset{\frown}{B_{k-1}B_k} \subseteq \overset{\frown}{B_{j-1}B_j}$. For the latter case, we can see that T_1 is not a same-side point of $B(n,p)$. For the former one, we have $\angle T_1B_{j-1}B_j < \angle T_1B_{k-1}B_k$, which leads to a contradiction. Thus $B_1B_2\cdots B_n$ is a $B\{n,p\}$.

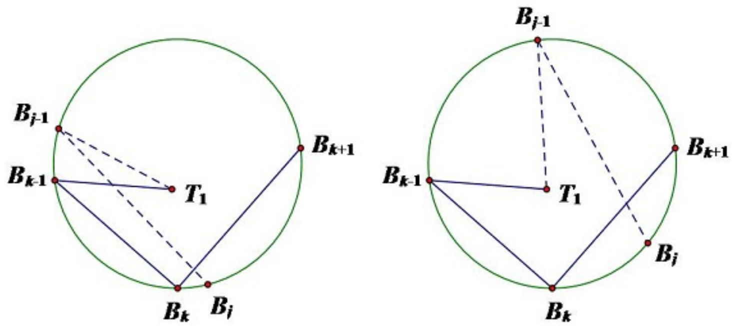

Fig.20.

By Corollary 3.12, take a proper integer $1 \le k \le n$ and point S_1 on the plane, construct $I(S_1)$: $B_i \to B_i^{'}, T_1 \to T_1^{'}, \odot O \to \odot O^{'}$ such that $B_k^{'}B_{k+1}^{'} = B_{k+1}^{'}B_{k+2}^{'} = B_{k+2}^{'}B_{k+3}^{'}$.

Construct $S^{'} = J(\odot O^{'}, T_1^{'})$, rotate $S^{'}$ by $\sphericalangle B_k^{'}O^{'}B_{k+1}^{'}$[4] around $O^{'}$, and denote the point after rotation by $S^{''}$. Next we shall prove that $S^{''}$ is actually the inversion center.

For any $1 \le i \le n$, by the basic properties of inversion,

$$\angle T_1 B_i B_{i+1} = \angle T_1 B_{i+1} B_{i+2}$$

$$\Leftrightarrow B_{i+1}B_{i+2} \text{ is tangent to the circle through } T_1, B_i, B_{i+1}$$

$$\Leftrightarrow \text{the circle through } S_1, B_{i+1}^{'}, B_{i+2}^{'} \text{ is tangent to the one through } T_1^{'}, B_i^{'}, B_{i+1}^{'}.$$

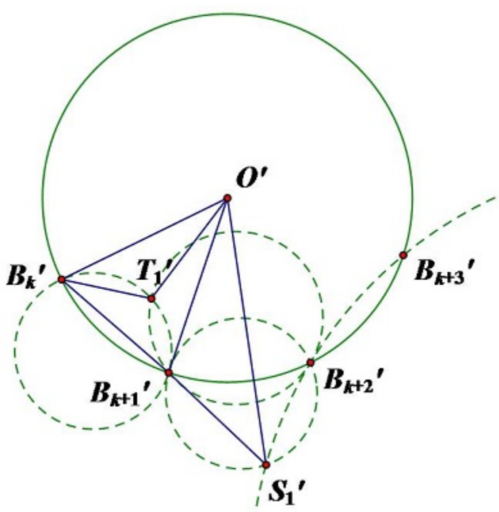

Fig.21.

The circle that passes through $B_{i+1}^{'}$,$B_{i+2}^{'}$ and is tangent to the circle through $T_1^{'}, B_i^{'}, B_{i+1}^{'}$ is denoted by $\sum_i$.It is easy to know that $\sum_i$ is unique. Then S_1 is one of the intersections of $\sum_k$ and $\sum_{k+1}$ which is apart from $B_{k+1}^{'}$ and can be uniquely determined.

Note a conclusion. Let A, B, C lie on $\odot O$ and B be the midpoint of $\widehat{AC}$. Construct $\triangle OAM$ inversely similar[5]to $\triangle ONB$, then the circle through A, M, B is tangent to the circle through C, N, B.

[4]The angle size and rotation direction are required to be the same as the directed angle.

[5]Two figures are said to be inversely similar if all the corresponding angles are equal but are described in opposite senses.

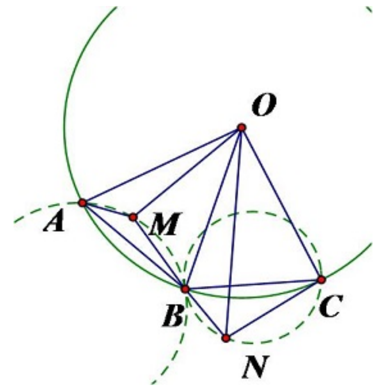

Fig.22.

We can see that $\triangle O^{'}B_{k+1}^{'}T_1^{'}$ and $\triangle O^{'}S^{''}B_{k+2}^{'}$ are inversely similar, as are $\triangle O^{'}B_{k+2}^{'}T_1^{'}$ and $\triangle O^{'}S^{''}B_{k+3}^{'}$. From the conclusion above it follows that $S^{''} \in \sum_k, S^{''} \in \sum_{k+1}$. Hence $S^{''} \equiv S_1$.

We claim that the $B^{'}\{n,p\}$ is a $B^{'}\langle n,p\rangle$. Take C_{k+4} on the arc $\widehat{B_{k+3}^{'}B_{k+2}^{'}}$ such that $B_{k+3}^{'}C_{k+4} = B_{k+2}^{'}B_{k+3}^{'}$, from the conclusion above we imply that the circles through $T_1^{'}, B_{k+2}^{'}, B_{k+3}^{'}$ and $S^{''}, B_{k+3}^{'}, C_{k+4}$, respectively, are tangent. Note that $\sum_{k+3}$ and $\odot O^{'}$ have a unique intersection other than $B_{l+3}^{'}$, thus $C_{k+4} \equiv B_{k+4}^{'}$, then $B_{k+3}^{'}B_{k+4}^{'} = B_{k+2}^{'}B_{k+3}^{'}$.

By similar argument we can complete the proof by induction. Theorem 3.4 demonstrates that the $B\{n,p\}$ is a $B[n,p]$ and we complete the prove of necessity.

Corollary 4.2 Rotate $J(\odot O^{'}, T_1^{'})$ by $\dfrac{2p\pi}{n}$ around $O^{'}$ in counterclockwise direction,it will be coinciding with S_1.

Corollary 4.3 $\omega = \dfrac{n-2p}{2n}\pi - \angle O^{'}S_1T_1^{'} = \dfrac{n-2p}{2n}\pi - \angle OS_1T_1$. Herein $p < \dfrac{n}{2}$.

Corollary 4.4 $\angle T_1OT_2 = 2\omega$, $OT_1 = OT_2$. By means of directed angle, we get $\sphericalangle T_1OT_2 = 2\sphericalangle T_1B_iB_{i+1}$.

Combining this with Theorem 3.4, we have the corollary below, linking harmonic $n-$point range to loop broken lines which has conjugate Brocard points. We also demonstrate geometric methods to construct them.

Corollary 4.5 By constructing an inversion on $A^{'}\langle n\rangle$, we can get all possible loop broken line which has conjugate Brocard points.

Corollary 4.6 Let T be an arbitrary point inside $\odot O$ and B_k be any point on $\odot O$. For integer p of $1 \le p \le n-1$ and $gcd(n,p) = 1$, there exist a unique $B[n,p]$ that lies on $\odot O$ and contains B_k, with T being its first Brocard point.

4.3 Proof of Theorem 3

In this section we simply investigate common conjugate Brocard points of several loop broken lines, which are defined as Brocard points of q-period loop broken lines.

Definition 4.7 If there exists a point T such that T is a same-side point of each of the q loop broken lines $B_1(n,p), B_2(n,p), \cdots, B_q(n,p)$, then these q loop broken lines are collectively called a *q-period loop broken line*, denoted by $B(n,p,q)$. T is called a *common same-side point* of $B(n,p,q)$. Each of the broken lines is referred to as a *period* of $B(n,p,q)$.

Definition 4.8 For a $B(n,p,q)$, one of its common same-side points T_1 is called the *first Brocard point* if $\angle T_1B_iB_{i+1} = \omega_1$ is valid for all $1 \le i \le n$ in each period; ω_1 is called the *first Brocard angle.*

Similarly, one of its common same-side points T_2 is called the *second Brocard point* if $\angle T_2A_iA_{i-1} = \omega_2$ is valid for all $1 \le i \le n$ in each period; ω_2 is called the *second Brocard angle.*

If a $B(n,p,q)$ has both the first and the second Brocard points T_1 and T_2, then they are collectively called the conjugate Brocard points of $B(n,p,q)$; ω_1 and ω_2 are collectively called the *conjugate Brocard angles.*

Theorem 3 *For an integer p such that $1 \le p \le n-1$ and $gcd(n,p) = 1$, if each period of a $B(n,p,q)$ has conjugate Brocard points and can be denoted as a $B_i[n,p]$, then $B(n,p,q)$ has conjugate Brocard points if and only if the conjugate Brocard points and circumcircle of each period coincide.*

Proof.

Sufficiency:

Suppose all the period of $B(n,p,q)$ are inscribed in $\odot O$, all the conjugate Brocard points coincide at T_1 and T_2. By Corollary 4.4 we know that the Brocard angle of each period is $\frac{1}{2}\angle T_1OT_2$. T_1 and T_2 are then the conjugate Brocard points of $B(n,p,q)$.

Necessity:

We may assume that $p < \frac{n}{2}$. Note that the conjugate Brocard point of $B(n,p,q)$ is also the conjugate Brocard point of any of its periods.Hence the conjugate Brocard points of each period coincide at T_1 and T_2. By Corollary 4.4, considering directed angles, we then establish that the circumcenter of each period coincides.

From Corollary 4.2 we know that for the first isodynamic point in each period $S_1T_1^{'} = S_1T_2^{'}$ is valid. By Corollary 4.3 ,we have

$$\angle OS_1T_1 = \frac{n-2p}{2n}\pi - \omega_1.$$

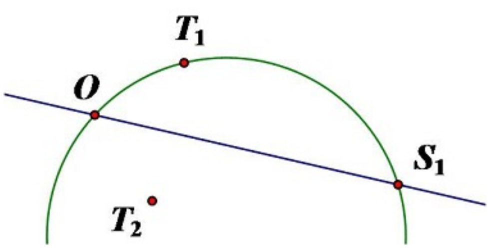

Fig.23.

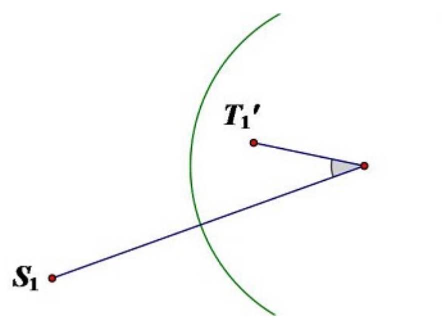

Fig.24.

Hence for each period, its S_1 can be determined by the intersection of the perpendicular bisector of T_1T_2 and a circle through O, T_1. Thus all the periods of $B(n,p,q)$ share a common isodynamic point.

Construct $I(S_1) : T_1 \to T_1^{'}, \odot O \to \odot O^{'}$. By Corollary 4.2, $\angle S_1O^{'}T_1^{'} = \dfrac{2p\pi}{n}$.

By Corollary 4.2, rotating $J(\odot O^{'}, T_1^{'})$ by $\dfrac{2p\pi}{n}$ around $O^{'}$ in an anticlockwise direction will be coinciding with S_1, the position of $O^{'}$ and the radius of $\odot O^{'}$ are determined by S_1 and $T_1^{'}$. Therefore we know that the radius of the circumcircle for each period are equal. The proof is complete.

To end this chapter, we provide a kind of loop broken line with a first Brocard point that does not have conjugate Brocard points, in order to demonstrate the value of our results.

Construction Let O be the centre of an $A\langle n\rangle$. Take a point X_1 on line A_1A_2 such that $A_1X_1 > A_1A_2$. Draw a series of triangles $\triangle OA_1X_1, \triangle OX_1X_2, \triangle OX_2X_3, \cdots, \triangle OX_{n-3}X_{n-2}$ directly similar to each other. Next we hope to construct a loop broken line $A_1X_1X_2\cdots X_{n-1}$ with a first Brocard point. Let $\omega_1 = \angle OX_1A_1$, let c be the circle such that for a point B on c, $\angle OBX_{n-2} = \omega_1$. Let A_1D be the ray such that $\angle OA_1D = \omega_1$. If c and A_1D intersects then we can take X_{n-1}as one of

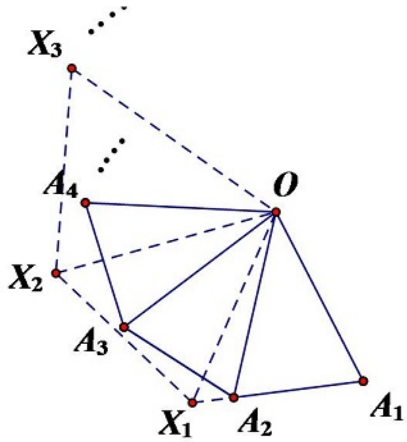

Fig.25.

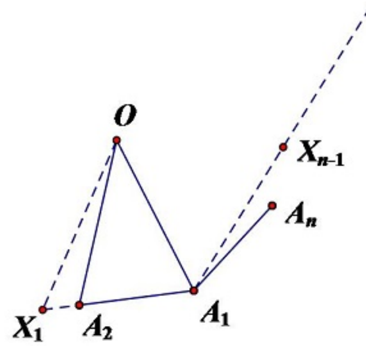

Fig.26.

the intersections. Actually, by making the distance between A_2, X_1 short enough we can let them intersects. Then $O$is the first Brocard point of $A_1X_1X_2\cdots X_{n-1}$.

Therefore an $A_1X_1X_2\cdots X_{n-1}$ with a first Brocard point can be found. On the other hand, by considering A_1, X_1, X_2, X_3, we know that it is not inscribed in a circle. By Theorem 2 we demonstrate that it does not have feature conjugate Brocard points.

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