MEAN FIELD EQUATIONS, HYPERELLIPTIC CURVES AND MODULAR FORMS: II

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ABSTRACT. A *pre-modular form* $Z_n(\sigma;\tau)$ of weight $\frac{1}{2}n(n+1)$ is introduced for each $n \in \mathbb{N}$, where $(\sigma,\tau) \in \mathbb{C} \times \mathbb{H}$, such that for $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, every non-trivial zero of $Z_n(\sigma;\tau)$, i.e. σ is not a 2-torsion of E_τ, corresponds to a (scaling family of) solution to the equation

$$\text{(MFE)} \qquad \triangle u + e^u = \rho\,\delta_0,$$

on the flat torus E_τ with singular strength $\rho = 8\pi n$.

In Part I [1], a hyperelliptic curve $\bar{X}_n(\tau) \subset \mathrm{Sym}^n E_\tau$, the *Lamé curve*, associated to the MFE was constructed. Our construction of $Z_n(\sigma;\tau)$ relies on a detailed study of the correspondence $\mathbb{P}^1(\mathbb{C}) \leftarrow \bar{X}_n(\tau) \to E_\tau$ induced from the hyperelliptic projection and the addition map.

As an application of the explicit form of the weight 10 pre-modular form $Z_4(\sigma;\tau)$, a counting formula for Lamé equations of degree $n=4$ with finite monodromy is given in the appendix (by Y.-C. Chou).

CONTENTS

0. INTRODUCTION

Let $E = E_\tau = \mathbb{C}/\Lambda_\tau$ be a flat torus, where $\tau \in \mathbb{H} = \{\, \tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0 \,\}$ and $\Lambda = \Lambda_\tau = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1 = 1$ and $\omega_2 = \tau$. Also $\omega_3 := \omega_1 + \omega_2$.

Convention: For $z \in \mathbb{C}$ we denote $[z] := z \pmod{\Lambda} \in E$. For a point $[z]$ in E we often write z instead of $[z]$ to simplify notations when no confusion should arise. For $N \in \mathbb{N}$, $E[N] := \{\, [z] \in E \mid Nz \in \Lambda \,\}$ is the group of N-torsion points in E. Also $E^\times := E \setminus \{[0]\}$.

Date: September 20, 2016. Revised May 15, 2017.

2010 *Mathematics Subject Classification.* 33E10, 35J08, 35J75, 14H70.

Key words and phrases. Lamé curve, Hecke function, pre-modular form.

We will use the Weierstrass elliptic function $\wp(z) = \wp(z;\Lambda)$ and its associated functions $\zeta(z;\Lambda)$ and $\sigma(z;\Lambda)$ extensively. We often write τ instead of Λ and even omit it in the notation when no confusion should arise. We take [14] as our general reference on elliptic functions.

In this paper, we continue our study, initiated in [9] and developed in Part I [1], on the singular Liouville (mean field) equation:

$$\triangle u + e^u = 8\pi n\, \delta_0 \quad \text{on } E, \tag{0.1}$$

under the flat metric. Here $\triangle = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator on E induced from $\mathbb{C}$, $n \in \mathbb{N}$, and δ_0 is the Dirac measure at $[0] \in E$.

The solvability of equation (0.1) depends on the moduli τ in a sophisticated manner. For $n = 1$, this was only recently settled in [9, 11, 2]. The aim of this paper is to develop a theory via modular forms to investigate such a dependence for all $n \in \mathbb{N}$ and to lay the foundation towards a complete resolution to equation (0.1).

We review briefly in §0.1 what had been done in earlier works (mainly in Part I) to reformulate the problem using Lamé equations and Green functions. More technical statements will be recalled in later sections whenever needed. In §0.2 we describe new results proved in this paper.

0.1. **Reduction to a Green function equation over the Lamé curve.**

0.1.1. *The Liouville curve.* It was shown in [1, Theorem 0.3 and Theorem 0.6] that if there is a solution u to equation (0.1) then it lies in a *scaling family of solutions u_λ* through the Liouville formula:

$$u_\lambda(z) = \log \frac{8e^{2\lambda}|f'(z)|^2}{(1+e^{2\lambda}|f(z)|^2)^2}, \quad \lambda \in \mathbb{R}, \tag{0.2}$$

where f is a meromorphic function on $\mathbb{C}$, known as a *developing map*, which can be normalized to satisfy the *type II constraints*:

$$f(z+\omega_j) = e^{2i\theta_j} f(z), \quad \theta_j \in \mathbb{R}, \quad j = 1, 2. \tag{0.3}$$

(i) There is a unique λ so that u_λ is *even*. Moreover, the normalized developing map f has precisely n simple zeros $[a_1], \ldots, [a_n]$ in $E^\times := E \setminus \{[0]\}$ and n simple poles $[-a_1], \ldots, [-a_n]$. They are characterized by

(ii) *The non-degenerate constraints*: $[a_i] \notin E[2]$ for all i, $[a_i] \neq \pm[a_j]$ for $i \neq j$.

(iii) The following $n-1$ *algebraic equations*:

$$\sum_{i=1}^n \wp'(a_i)\wp^r(a_i) = 0, \qquad r = 0, \ldots, n-2. \tag{0.4}$$

(iv) The *transcendental equation* on the Green function: [1]

$$\sum_{i=1}^n \nabla G(a_i) = 0. \tag{0.5}$$

[1]The Green function $G(z)$ on E is defined by $-\triangle G = \delta_0 - 1/|E|$ and $\int_E G = 0$ where $|E|$ is the area of E. Also $G(z,w) = G(z-w,0) = G(z-w)$ by the translation invariance.

The affine algebraic curve $X_n \subset \mathrm{Sym}^n E$ defined by equations (0.4) and the non-degenerate constraints is called the (n-th) *Liouville curve*.

0.1.2. *The Lamé curve.* The Liouville curve X_n has the important *hyperelliptic structure* arising from its connection with the integral *Lamé equations* on E:

$$w'' = (n(n+1)\wp + B)w, \tag{0.6}$$

where $B \in \mathbb{C}$ is usually known as the auxiliary or spectral parameter. For $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, let $w_a(z)$ be the classical *Hermite–Halphen ansatz*:

$$w_a(z) := e^{z \sum \zeta(a_i;\tau)} \prod_{i=1}^{n} \frac{\sigma(z - a_i;\tau)}{\sigma(z;\tau)}. \tag{0.7}$$

Then the following statements were proved in [1, Theorem 0.7].

(i) The point $[a] := a \pmod{\Lambda}$ lies in X_n if and only if w_a and w_{-a} are independent solutions to equation (0.6). In that case, the parameter B equals

$$B_a := (2n-1) \sum_{i=1}^{n} \wp(a_i). \tag{0.8}$$

(ii) The compactified curve

$$\bar{X}_n \subset \mathrm{Sym}^n E$$

is a hyperelliptic curve, known as the *Lamé curve*, with the added points $\bar{X}_n \setminus X_n$ being the branch points of the hyperelliptic projection

$$B : \bar{X}_n \to \mathbb{P}^1(\mathbb{C}).$$

(iii) A point $[a] \in \bar{X}_n$ is a branched point if and only if $[-a] = [a]$. In fact

$$\{[-a_i]\} \cap \{[a_i]\} \neq \emptyset \Longrightarrow [-a] = [a].$$

Also $[0] \in \{[a_i]\} \Longrightarrow [a] = 0^n$.

(iv) The unique *point at infinity* $[0]^n \in \bar{X}_n$ is a non-singular point.

(v) The finite branch points satisfy $[a] \in (E^\times)^n$, $[a_i] \neq [a_j]$ for $i \neq j$, and $[a] = [-a]$; $w_a = w_{-a}$ is still a solution to equation (0.6) with $B = B_a$. These solutions are known as the *Lamé functions*.

(vi) Let $Y_n = B^{-1}(\mathbb{C})$ be the finite part of $\bar{X}_n$. Then Y_n is parametrized by

$$Y_n \cong \{\, (B, C) \mid C^2 = \ell_n(B) \,\}$$

where $\ell_n(B)$ is the *Lamé polynomial* in B of degree $2n+1$, and $\bar{X}_n$ coincides with *the projective hyperelliptic model* of Y_n. In particular, the Lamé curve $\bar{X}_n$ is *irreducible* and is smooth if and only if $\ell_n(B)$ has no multiple roots.

Under this description, for $[a] \in X_n$, the ratio

$$f = f_a := \frac{w_a}{w_{-a}} = e^{2\sum_{i=1}^{n} \zeta(a_i) z} \prod_{i=1}^{n} \frac{\sigma(z - a_i;\tau)}{\sigma(z + a_i;\tau)}$$

gives the candidate of a developing map in (0.2). The original singular Liouville equation (0.1) is then equivalent to the Green function equation (0.5) over the unramified (Liouville) loci X_n of the map $B : \bar{X}_n \to \mathbb{P}^1(\mathbb{C})$.

0.2. **Main results: a theory of pre-modular forms.**

Recall the Weierstrass equation $\wp'^2 = 4\wp^3 - g_2\wp - g_3 = \prod_{i=1}^3(\wp - e_i)$, where $e_i(\tau) = \wp(\frac{1}{2}\omega_i;\tau)$, $i = 1,2,3$. We will also use the quasi-periods $\eta_i(\tau) := \zeta(z+\omega_i;\tau) - \zeta(z;\tau) = 2\zeta(\frac{1}{2}\omega_i;\tau)$, $i = 1,2$, extensively.

0.2.1. *The Hecke function and pre-modular forms.* For $z = x + iy = r\omega_1 + s\omega_2$, $r,s \in \mathbb{R}$, it was shown in [9, Lemma 2.3, Lemma 7.1] that

$$-4\pi\frac{\partial G}{\partial z}(z;\tau) = \zeta(z;\tau) - r\eta_1(\tau) - s\eta_2(\tau). \tag{0.9}$$

For $[z] \in E_\tau[N] \setminus \{[0]\}$, the right-hand side of equation (0.9) first appeared in [6], where Hecke showed that it is a modular form of weight one with respect to $\Gamma(N) = \{\, A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv I_2 \pmod N \,\}$. Thus we call the following function

$$Z(z;\tau) = Z_{r,s}(\tau) := \zeta(r\omega_1 + s\omega_2;\tau) - r\eta_1(\tau) - s\eta_2(\tau), \tag{0.10}$$

$(z,\tau) \in \mathbb{C} \times \mathbb{H}$, the *Hecke function*. Notice that it is holomorphic *only* in τ, and for fixed τ it depends only on $[z] = z \pmod{\Lambda_\tau} \in E_\tau$. In this paper, functions of this sort are called *pre-modular forms*.

Definition 0.1. An analytic function h in $(z,\tau) \in \mathbb{C} \times \mathbb{H}$ is *pre-modular* of weight $k \in \mathbb{N}$ if it satisfies

(1) For any fixed τ, the function $h(z)$ is analytic in z and $\bar{z}$ and it depends only on $z \pmod{\Lambda_\tau} \in E_\tau$.
(2) For any fixed *torsion type* $z \pmod{\Lambda_\tau} \in E_\tau[N]$, the function $h(\tau)$ is modular of weight k with respect to $\Gamma(N)$.

We write $h(z;\tau)$ for a pre-modular form h.

By writing $z = r + s\tau$ with $r,s \in \mathbb{R}$, it is easy to see that condition (1) is equivalent to saying that $h(z)$ is analytic and periodic in r,s with period 1. A torsion type in condition (2) is simply a choice of $r,s,\in (\frac{1}{N}\mathbb{Z})/\mathbb{Z}$. In particular, the Hecke function Z is pre-modular of weight one.

We may regard pre-modular forms as the restriction of holomorphic functions in three complex variables $(r,s,\tau) \in \mathbb{C}^2 \times \mathbb{H}$ to the $\mathbb{R}$-linear slice L defined by $r,s,\in \mathbb{R}$. Although $L \cong \mathbb{C} \times \mathbb{H}$, the embedding is not $\mathbb{C}$-linear.

The notion of pre-modular forms allows us to study *deformations* in z to relate *different* modular forms corresponding to different torsion points.

Recently this idea was applied in [2] to achieve a complete solution to equation (0.1) for $n = 1$ and for all τ. [2] In that case equations (0.4) are vacuous and the problem is equivalent to solving *non-trivial zeros* of $Z(z;\tau)$, i.e. $z \notin E_\tau[2]$. Thus, a key step towards the general cases is to generalize the pre-modular form $Z_1 = Z$ to certain "Z_n" for all $n \geq 2$.

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[2]It was stated in Part I [1, p.141–142] that such a complete solution for $n = 1$ will appear in Part II (this paper), and it was included in the first arXiv version arXiv:1502.03295v1. Later on we found its deep connection with Painlevé VI equations. Therefore we extracted that part and published it separately in [2].

0.2.2. *The main constructions.* By the anti-symmetry of ∇G, equation (0.5) holds automatically on the branch points of Y_n, hence they are referred to as *trivial solutions*. Nevertheless further investigations on the local structures of the branch points are indispensable. This is done in §1.

We proceed to construct a pre-modular form $Z_n(\sigma;\tau)$, with $\sigma \in E_\tau$, associated to the family of Lamé curves $\bar{X}_n(\tau)$, $\tau \in \mathbb{H}$. [3] It should have the property that every non-trivial solution $[a] = \{[a_1], \ldots, [a_n]\} \in X_n(\tau)$ to equation (0.5) comes from a zero of $Z_n(\sigma;\tau)$ with $\sigma = \sum_{i=1}^n [a_i] \notin E_\tau[2]$, and vice versa. The construction is stated in (0.15). Its justification consists of three steps corresponding to Theorem 0.2, 0.3 and 0.4 in the following.

Consider the meromorphic function

$$\mathbf{z}_n(a) := \zeta\Big(\sum_{i=1}^n a_i\Big) - \sum_{i=1}^n \zeta(a_i) \tag{0.11}$$

on E^n. Write $a_i = r_i\omega_1 + s_i\omega_2$. If $\sum_{i=1}^n [a_i] \ne 0$ then from equation (0.9)

$$-4\pi \sum_{i=1}^n \frac{\partial G}{\partial z}(a_i) = \sum_{i=1}^n \big(\zeta(r_i\omega_1 + s_i\omega_2) - r_i\eta_1 - s_i\eta_2\big) = Z\Big(\sum_{i=1}^n a_i\Big) - \mathbf{z}_n(a).$$

Hence the Green function equation (0.5) is equivalent to

$$\mathbf{z}_n(a) = Z\Big(\sum_{i=1}^n a_i\Big). \tag{0.12}$$

This motivates us to study the map

$$\sigma_n : \bar{X}_n \to E, \qquad [a] \mapsto \sigma_n([a]) := \sum_{i=1}^n [a_i] \tag{0.13}$$

induced from the addition map $E^n \to E$. Since the algebraic curve $\bar{X}_n$ is irreducible, σ_n is a finite morphism and $\deg \sigma_n$ is defined.

Theorem 0.2 (= Theorem 2.4)**.** *The map $\sigma_n : \bar{X}_n \to E$ has degree $\frac{1}{2}n(n+1)$.*

From Theorem 0.2, there is a polynomial

$$W_n(\mathbf{z}) \in \mathbb{Q}[g_2, g_3, \wp(\sigma), \wp'(\sigma)][\mathbf{z}]$$

of degree $\frac{1}{2}n(n+1)$ in $\mathbf{z}$ which defines the (branched) covering map σ_n.

The next task is to find a natural primitive element of this covering map, namely a rational function on $\bar{X}_n$ which has W_n as its minimal polynomial. This is achieved by the following fundamental theorem.

Theorem 0.3 (= Theorem 3.2)**.** *The rational function $\mathbf{z}_n \in K(\bar{X}_n)$ is a primitive generator for the field extension $K(\bar{X}_n)$ over $K(E)$ which is integral over the affine curve $E^\times$.*

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[3] In this paper, we often use σ as the coordinate on E whenever the map $\sigma_n : X_n \to E$ defined in (0.13) is involved. This should not be confused with the Weierstrass σ function.

This means that $W_n(\mathbf{z}_n) = 0$, and conversely for generic choices of $\sigma = \sigma_0 \in E_\tau$, the roots of $W_n(\mathbf{z})(\sigma_0; \tau) = 0$ are precisely those $\frac{1}{2}n(n+1)$ values $\mathbf{z} = \mathbf{z}_n(a)$ with $\sigma_n(a) = \sigma_0$. The proof is contained in §3. Here we give a brief sketch of the idea used in the proof.

A major tool used is the *tensor product* of two Lamé equations $w'' = I_1 w$ and $w'' = I_2 w$, where $I = n(n+1)\wp(z)$, $I_1 = I + B_a$ and $I_2 = I + B_b$.

For a general point $\sigma_0 \in E$, we need to show that the $\frac{1}{2}n(n+1)$ points on the fiber of $\bar{X}_n \to E$ above σ_0 has distinct $\mathbf{z}_n$ values. From (0.11), it suffices to show that for $\sigma_n(a) = \sigma_n(b) = \sigma_0$,

$$\sum_{i=1}^n \zeta(a_i) = \sum_{i=1}^n \zeta(b_i) \Longrightarrow B_a = B_b.$$

Indeed, then we conclude $[a] = [b]$ if $\sigma_0 \notin E[2]$.

If $w_1'' = I_1 w_1$ and $w_2'' = I_2 w_2$, then the product $q = w_1 w_2$ satisfies the fourth order ODE

$$q'''' - 2(I_1 + I_2)q'' - 6I'q' + ((B_a - B_b)^2 - 2I'')q = 0. \tag{0.14}$$

We remark that if $B_a = B_b$, then $I_1 = I_2$ and q actually satisfies a third order ODE as the *second symmetric product* of a Lamé equation, which is a useful tool used in Part I in the study of the Lamé curve.

If however $B_a \neq B_b$, by the definition of w_a in (0.7) and the addition law,

$$q = w_a w_{-b} + w_{-a} w_b$$

is an *even elliptic function* solution to equation (0.14), hence a *polynomial* in $x = \wp(z)$. This leads to strong constraints on equation (0.14) in the variable x and eventually leads to a contradiction for generic choices of σ_0.

Now we set

$$Z_n(\sigma; \tau) := W_n(Z)(\sigma; \tau). \tag{0.15}$$

Then $Z_n(\sigma; \tau)$ is pre-modular of weight $\frac{1}{2}n(n+1)$. From the construction and equation (0.12) it is readily seen that $Z_n(\sigma; \tau)$ is the generalization of the Hecke function we are looking for. In fact, for $n \geq 1$, we have

Theorem 0.4. *To every scaling family $\{u_\lambda\}$ of solutions to the singular Liouville equation* (0.1) *on E_τ, the zero set $a \in X_n$ of its normalized developing map f satisfies $Z_n(\sigma_n(a); \tau) = 0$ with $\sigma_n(a) \notin E_\tau[2]$. Conversely, given $\sigma_0 \in E_\tau \setminus E_\tau[2]$ with $Z_n(\sigma_0; \tau) = 0$, there is a unique $a \in X_n$ with $\sigma_n(a) = \sigma_0$ and it determines a developing map $f = w_a / w_{-a}$ of a scaling family of solutions to equation* (0.1).

The proof is given in §4, where we also present a version of it in terms of *monodromy groups* of Lamé equations in Theorem 4.5.

For $\sigma \in E_\tau[N]$, the N-torsion points, the modular form $Z_2(\sigma; \tau)$ and $Z_3(\sigma; \tau)$ were first constructed by Dahmen [3] in his study on integral Lamé equations (0.6) with algebraic solutions (i.e. with finite monodromy group). For $n \geq 4$, the existence of a modular form $Z_n(\sigma; \tau)$ of weight $\frac{1}{2}n(n+1)$ was also conjectured in [3]. This is now settled by our results.

0.2.3. *Relation with finite gap theory.* It remains to find effective and explicit constructions of Z_n. Since σ_n is defined by the addition map, which is purely algebraic, in principle this allows us to compute the polynomial $W_n(\mathbf{z})$ for any $n \in \mathbb{N}$ by eliminating variables B and C. In practice the needed calculations are very demanding and time consuming.

In a different direction, the Lamé curve had also been studied extensively in the *finite band integration theory*. In the complex case this theory concerns the eigenvalue problem on a second order ODE

$$Lw := w'' - Iw = Bw$$

with eigenvalue B. The potential $I = I(z)$ is called a *finite-gap (band) potential* if the ODE has only logarithmic free solutions except for finitely many $B \in \mathbb{C}$. The integral Lamé equations (with $I(z) = n(n+1)\wp(z)$) provide the first non-trivial examples of them. Using this theory, Maier [12] had recently written down *an explicit map* $\pi_n : \bar{X}_n \to E$ in terms of the coordinates (B, C) on $\bar{X}_n$ (cf. Theorem 5.3). It turns out we can prove

Theorem 0.5. *The map π_n agrees with $\sigma_n : \bar{X}_n \to E$.*

This is part of Theorem 5.6 where another presentation of $\mathbf{z}_n$ in this context is also given. The main idea in the proof is to compare the Hermite–Halphen ansatz (0.7) with the *Hermite–Krichever ansatz* (given in (5.1)) of solutions to the Lamé equations (0.6).

This provides an alternative way to compute $W_n(\mathbf{z})$ by eliminating B, C. In particular the weight 10 pre-modular form $Z_4(\sigma; \tau)$ is explicitly written down in Example 5.10. The existence and effective construction of $Z_n(\sigma; \tau)$ opens the door to extend our complete results on equation (0.1) for $n = 1$ to general $n \in \mathbb{N}$.

As a related application, the explicit expression of Z_4 is used to solve Dahmen's conjecture on a counting formula for Lamé equations (0.6) with finite monodromy in the $n = 4$ case. The method works for general n once Z_n is shown to have expected asymptotic behavior at cusps. The details are given in the appendix, written by Y.-C. Chou.

Acknowledgement. The second named author would like to express his gratitude to the anonymous referee for his/her valuable suggestions.

1. Geometry of $B : \bar{X}_n \to \mathbb{P}^1(\mathbb{C})$

In this section we investigate the local structure of the branch points of the hyperelliptic projection $B : \bar{X}_n \to \mathbb{P}^1(\mathbb{C})$.

1.1. **Some useful formulas from Part I.**

We give quantitative descriptions on those results recalled in §0.1 which will be used in this paper.

Let f be a normalized developing map of a solution u to equation (0.1) with simple zeros $\{[a_1], \ldots, [a_n]\}$ and simple poles $\{[b_1], \ldots, [b_n]\}$ in E. One of the crucial properties proved in Part I (cf. §0.1.1 (i)) is the equality

$$\{[b_1], \ldots, [b_n]\} = \{[-a_1], \ldots, [-a_n]\}.$$

With this being established, the logarithmic derivative $g := (\log f)' = f'/f$ is readily seen to be an *even* elliptic function on E of the form

$$g(z) = \sum_{i=1}^{n} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)}. \tag{1.1}$$

It has simple poles at $\pm[a_i]$ and only one zero at $[0]$. Hence $\mathrm{ord}_{z=0}\, g(z) = 2n$. It leads to the $n-1$ equations in $[a_1], \ldots, [a_n]$ given in equations (0.4): under the algebraic coordinates $(w, x_i, y_i) = (\wp(z), \wp(a_i), \wp'(a_i))$,

$$\begin{aligned} g(z) &= \sum_{i=1}^{n} \frac{1}{w} \frac{y_i}{1 - x_i/w} \\ &= \sum_{i=1}^{n} \frac{y_i}{w} + \sum_{i=1}^{n} \frac{y_i x_i}{w^2} + \ldots + \sum_{i=1}^{n} \frac{y_i x_i^r}{w^{r+1}} + \ldots. \end{aligned}$$

Since $\mathrm{ord}_{z=0}\, g(z) = 2n$ and $1/w$ has a zero at $z = 0$ of order two, we get $x_i \ne x_j$ for $i \ne j$ and

$$\sum_{i=1}^{n} y_i x_i^r = 0, \qquad r = 0, \ldots, n-2. \tag{1.2}$$

Equations (1.2), together with the Weierstrass equation $y_i^2 = 4x_i^3 - g_2 x_i - g_3$ for all $i = 1, \ldots, n$, give the algebraic form of equations (0.4).

The Green function equation (0.5) is equivalent to the type II constraints (0.3) ([10, Lemma 2.4]). Indeed, by the addition law,

$$\begin{aligned} f = \exp \int g\, dz &= \exp \int \sum_{i=1}^{n} (2\zeta(a_i) - \zeta(a_i - z) - \zeta(a_i + z))\, dz \\ &= e^{2\sum_{i=1}^{n} \zeta(a_i) z} \prod_{i=1}^{n} \frac{\sigma(z - a_i)}{\sigma(z + a_i)}. \end{aligned} \tag{1.3}$$

The monodromy effect on f is then calculated from the standard formula

$$\sigma(z + \omega_j) = -e^{\eta_j (z + \frac{1}{2}\omega_j)} \sigma(z), \qquad j = 1, 2. \tag{1.4}$$

Let $a_i = r_i \omega_1 + s_i \omega_2$ for $i = 1, \ldots, n$. By way of the Legendre relation $\eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i$ we compute easily that

$$\begin{aligned} f(z + \omega_1) &= e^{-4\pi i \sum_i s_i + 2\omega_1 (\sum \zeta(a_i) - \sum r_i \eta_1 - \sum s_i \eta_2)} f(z), \\ f(z + \omega_2) &= e^{4\pi i \sum_i r_i + 2\omega_2 (\sum \zeta(a_i) - \sum r_i \eta_1 - \sum s_i \eta_2)} f(z). \end{aligned} \tag{1.5}$$

By equation (0.9) and the linear independence of ω_1 and ω_2, the equivalence of equation (0.5) and (0.3) follows immediately.

In §0.1.2 we have reviewed the hyperelliptic structure $B : \bar{X}_n \to \mathbb{P}^1(\mathbb{C})$ on the Lamé curve induced by $a \mapsto B_a = (2n-1) \sum \wp(a_i)$. Also $\bar{X}_n$ contains the Liouville curve X_n as the unramified loci. By way of Lamé equation

(0.6) with $B = B_a$ and by setting $f = w_a/w_{-a}$, where w_a is the ansatz solution (0.7), we see that solving equation (0.1) is equivalent to solving the integral Lamé equation (0.6) with *unitary projective monodromy groups*.

The finite part Y_n of $\bar{X}_n$ is defined by equation $C^2 = \ell_n(B)$ where the Lamé polynomial $\ell_n(B)$ is of degree $2n+1$ and can be effectively computed (cf. [1, Theorem 7.4]). Later we will discuss factorization properties of $\ell_n(B)$ in Proposition 2.2. Here we focus on formulas which lead to parametrization of the Lamé curve near branched points, i.e. $a \in \bar{X}_n$ with $a = -a$.

Proposition 1.1. [1, (7.5.3) and Proposition 7.5]

(1) *Let $a \in Y_n$, then (B, C) can be parameterized by $B(a) = B_a$ and*

$$C(a) = \wp'(a_i) \prod_{j \neq i} (\wp(a_i) - \wp(a_j)). \tag{1.6}$$

Here formula (1.6) *is valid for any $i \in \{1, \ldots, n\}$.*

(2) *The* limiting equations *of equations* (1.2) *at $a = 0^n \in \bar{X}_n$ are given by*

$$\sum_{i=1}^{n} t_i^{2r+1} = 0, \qquad r = 1, \ldots, n-1, \tag{1.7}$$

subject to the non-degenerate constraints $t_i \neq 0$, $t_i \neq -t_j$ for $i \neq j$.

Equations (1.7) *have a unique non-degenerate solution in $\mathbb{P}^{n-1}(\mathbb{C})$ up to permutations. It gives rise to the unique tangent direction*

$$[t] = [t_1 : \cdots : t_n] \in \mathbb{P}(T_{0^n}(\bar{X}_n)) \subset \mathbb{P}(T_{0^n}(\mathrm{Sym}^n E)).$$

Remark 1.2. Notice that (1.6) arises from (1.1) and $\mathrm{ord}_{z=0}\, g_a(z) = 2n$ in

$$g_a(z) := \sum_{i=1}^{n} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)} = \frac{\sum_{i=1}^{n} \wp'(a_i) \prod_{j \neq i} (\wp(z) - \wp(a_j))}{\prod_{i=1}^{n} (\wp(z) - \wp(a_i))},$$

where the numerator reduces to the constant $C(a)$.

1.2. **Local structures on finite and infinite branch points.**

By working on formula (1.6), we may relate various local parameters of Y_n as follows:

Lemma 1.3. *Let $a = \{a_1, \ldots, a_n\} \in Y_n \setminus X_n$ be a finite branched point and $b = \{b_1, \ldots, b_n\} \in X_n$ be a point near a.*

(i) *Let i be an index with $a_i \in E[2]$, then C can be used as a parameter for $b_i - a_i$ with $b_i'(0) \neq 0, \infty$.*

(ii) *Let i be an index with $a_i \notin E[2]$, and let i' be the corresponding index with $a_{i'} = -a_i$. Then C can be used as a parameter for $b_i + b_{i'}$ with $(b_i + b_{i'})'(0) \neq 0, \infty$.*

Proof. (i) For $a_i = -a_i$ (2-torsion) in E, we have $\wp'(a_i) = 0$ and $\wp'(b_i) = \wp''(a_i)(b_i - a_i) + o(|b_i - a_i|)$. Formula (1.6) then implies that

$$C(b) = \Big[\wp''(a_i) \prod_{j \neq i} (\wp(a_i) - \wp(a_j))\Big] (b_i - a_i) + o(|b_i - a_i|).$$

We get the inverse map $C \mapsto b_i(C) - a_i$ since $\wp''(a_i) \neq 0$ for $a_i \in E[2]$.

(ii) Similarly, for $a_i \notin E[2]$ with $a_{i'} = -a_i$, we have $\wp(b_i) - \wp(b_{i'}) = \wp'(-b_{i'})(b_i + b_{i'}) + o(|b_i + b_{i'}|)$. Since $-b_{i'}$ is close to $-a_{i'} = a_i$, we get

$$C(b) = \Big[\wp'(a_i)^2 \prod_{j \neq i, i'} (\wp(a_i) - \wp(a_j))\Big](b_i + b_{i'}) + o(|b_i + b_{i'}|).$$

Then we get the inverse map $C \mapsto (b_i + b_{i'})(C)$ since $\wp'(a_i) \neq 0$. □

Remark 1.4. From formula (1.6) (with a being substituted by b) we have $b_{i'}(C) = -b_i(-C)$, hence $b'_{i'}(C) = b'_i(-C)$. If $b'_i(0)$ and $b'_{i'}(0)$ are *finite* then they are equal and non-vanishing. If this holds for all i then $C \mapsto b(C)$ is a holomorphic map in each local branch of Y_n at a and we conclude that $a \in Y_n$ is either a smooth point or a nodal singularity ($y^2 = x^2$). We will see in Remark 5.14 that this is indeed the case. At this point we conclude only the finiteness of $(b_i + b_{i'})'(0)$ as stated in Lemma 1.3 (ii).

Now we give a precise description of the unique tangent direction at $0^n \in \bar{X}_n$. Denote by $[t] = [t_1, \ldots, t_n]$ the homogeneous coordinates satisfying the limiting equations (1.7) with non-degenerate constraints.

Let $p_j = \sum_{i=1}^n t_i^j$ be the j-th Newton symmetric polynomial and λ_j be the j-th elementary symmetric polynomial of $t_1, \ldots, t_n$. We use the convention that $p_0 = 0$, $\lambda_0 = 1$ and $\lambda_j = 0$ for $j > n$. By a Vandermonde determinant argument we have $\lambda_1 = p_1 \neq 0$.

Proposition 1.5. *The point $[t] \in \mathbb{P}(T_{0^n}(\bar{X}_n))$ is characterized by the recursions*

$$\lambda_{k+1} = 2\frac{(k-n)}{(k-2n)(k+1)}\lambda_k\lambda_1, \qquad 1 \leq k \leq n-1. \tag{1.8}$$

Proof. By Proposition 1.1 (2), namely the uniqueness of non-degenerate solutions to (1.7), we only need to verify that that the point defined by (1.8) satisfies $p_3 = p_5 = \cdots = p_{2n-1} = 0$.

Since $\lambda_1 \neq 0$, without loss of generality we assume that $\lambda_1 = 1$. The recursions in (1.8), with $\lambda_1 = 1$, are equivalent to saying that the polynomial

$$Q(t) := \prod_{i=1}^n (1 + t_i t) = \sum_{k=0}^n \lambda_k t^k$$

coincides with the hypergeometric function $F(-n; -2n \mid 2t)$. That is, $f(t) := Q(\frac{1}{2}t)$ satisfies the hypergeometric equation

$$\big(\delta(\delta - (2n+1)) - t(\delta - n)\big)f = t\big(tf'' - (t+2n)f' + nf\big) = 0, \tag{1.9}$$

where $\delta = td/dt$. Then $g := (\log f)' = f'/f$ satisfies

$$g' + g^2 - \Big(1 + \frac{2n}{t}\Big)g + \frac{n}{t} = 0. \tag{1.10}$$

Write $g = \sum_{k=0}^{\infty} g_k\, t^k$. From

$$g = \frac{f'}{f} = \sum_{i=1}^{n} \frac{\frac{1}{2}t_i}{1+\frac{1}{2}t_i t} = \sum_{k=0}^{\infty} \frac{p_{k+1}}{2^{k+1}}\, t^k,$$

we have $g_k = p_{k+1}/2^{k+1}$ and it suffices to show $g_2 = g_4 = \cdots = g_{2n-2} = 0$.

From the series expansions

$$\begin{aligned} g' &= g_1 + \sum_{k=1}^{\infty}(k+1)g_{k+1}t^k, \\ g^2 &= \tfrac{1}{4} + \sum_{k=1}^{\infty}\Big(\sum_{j=0}^{k} g_j g_{k-j}\Big)\, t^k, \\ \Big(1+\frac{2n}{t}\Big)g &= \frac{n}{t} + \sum_{k=0}^{\infty}(g_k + 2n g_{k+1})\, t^k, \end{aligned}$$

we get by equation (1.10) that

$$\begin{aligned} &-\frac{n}{t} + \frac{n}{t} + \Big(g_1 + \tfrac{1}{4} - (\tfrac{1}{2} + 2ng_1)\Big) \\ &\quad + \sum_{k=1}^{\infty}\Big((k+1)g_{k+1} + \sum_{j=0}^{k} g_j g_{k-j} - (g_k + 2ng_{k+1})\Big)\, t^k = 0. \end{aligned}$$

Hence $g_1 = -1/(4(2n-1))$ and for all $k \ge 1$ we have recursions

$$(2n-(k+1))g_{k+1} = \sum_{j=1}^{k-1} g_j g_{k-j}. \tag{1.11}$$

(We have used the fact $g_0 = \frac{1}{2}$ to remove one g_k.)

For $k = 1$, the sum is empty and we get $g_2 = 0$. Now we conclude the proof by induction. Suppose that $g_2 = g_4 = \ldots = g_{2m} = 0$ with $m < n-1$. Then for $k = 2m+1$ we have $2n - (k+1) = 2(n-(m+1)) > 0$, and the recursions (1.11) show that $g_{2(m+1)}$ is a sum of $g_j g_{k-j}$ with either j or $k-j$ being an even number no bigger than $2m$. Hence $g_{2(m+1)} = 0$, and this completes the induction. □

2. Geometry of $\sigma_n : \bar{X}_n \to E$

The aim of this section is to prove Theorem 0.2.

2.1. **Lamé functions.** [cf. §0.1.2 (v)]

Definition 2.1. The type of a point $a \in Y_n \setminus X_n$ is defined to be the *number of half periods* contained in $a = \{a_i\}$. Hence there are four types O, I, II, III. For $n = 2k$, a must be of type O or II. For $n = 2k+1$, a must be of type I or III.

The type of a Lame functions w_a (with $[a] = [-a] \in Y_n \setminus X_n$) is defined to be the type of its zero set a accordingly.

There are factorizations of the polynomial $\ell_n(B)$ according to the types:

Proposition 2.2. [5, 14] *We may decompose $\ell_n(B; g_2, g_3)$ as*

$$\ell_n(B; g_2, g_3) = c_n^2 l_0(B) l_1(B) l_2(B) l_3(B),$$

where $c_n \in \mathbb{Q}_{>0}$ is a constant, $l_i(B)$'s are monic polynomials in B whose coefficients are polynomials in e_1, e_2, e_3 such that

(1) *For $n = 2k$, $l_0(B) = \prod(B - B_a)$ with a being of type O, and $\deg l_0(B) = \frac{1}{2}n + 1 = k + 1$. For $i = 1, 2, 3$, $l_i(B) = \prod(B - B_a)$ with a being of type II which does not contain $\frac{1}{2}\omega_i$, also $\deg l_i(B) = \frac{1}{2}n = k$.*
(2) *For $n = 2k + 1$, $l_0(B) = \prod(B - B_a)$ with a being of type III, and $\deg l_0(B) = \frac{1}{2}(n - 1) = k$. For $i = 1, 2, 3$, $l_i(B) = \prod(B - B_a)$ with a being of type I which contains $\frac{1}{2}\omega_i$, also $\deg l_i(B) = \frac{1}{2}(n + 1) = k + 1$.*

We remark that Proposition 2.2, together with Proposition 1.1 and Lemma 1.3, will be used in the proof of Theorem 0.2. Here are some examples to illustrate Proposition 2.2:

Example 2.3. Decompositions of $\ell_n(B)$ for $1 \le n \le 5$.

(1) $n = 1$, $k = 0$, $\bar{X}_1 \cong E$,

$$C^2 = \ell_1(B) = 4B^3 - g_2 B - g_3 = 4\prod_{i=1}^{3}(B - e_i).$$

(2) $n = 2$, $k = 1$, (notice that $e_1 + e_2 + e_3 = 0$)

$$\begin{aligned} C^2 = \ell_2(B) &= \tfrac{4}{81}B^5 - \tfrac{7}{27}g_2 B^3 + \tfrac{1}{3}g_3 B^2 + \tfrac{1}{3}g_2^2 B - g_2 g_3 \\ &= \frac{2^2}{3^4}(B^2 - 3g_2)\prod_{i=1}^{3}(B + 3e_i). \end{aligned}$$

(3) $n = 3$, $k = 1$, $\deg l_i(B) = 2$ for $i = 1, 2, 3$,

$$\begin{aligned} C^2 = \ell_3(B) &= \frac{1}{2^2 3^4 5^4} B(16B^6 - 504 g_2 B^4 + 2376 g_3 B^3 \\ &\qquad + 4185 g_2^2 B^2 - 36450 g_2 g_3 B + 91125 g_3^2 - 3375 g_2^3) \\ &= \frac{2^2}{3^4 5^4} B \prod_{i=1}^{3}(B^2 - 6e_i B + 15(3e_i^2 - g_2)). \end{aligned}$$

(4) $n = 4$, $k = 2$, $\deg l_0(B) = 3$,

$$C^2 = \ell_4(B) = \frac{1}{3^8 5^4 7^4}(B^3 - 52 g_2 B + 560 g_3)\prod_{i=1}^{3}(B^2 + 10 e_i B - 7(5e_i^2 + g_2)).$$

(5) $n = 5$, $k = 2$, $\deg l_i(B) = 3$ for $i = 1, 2, 3$,

$$\begin{aligned} C^2 = \ell_5(B) &= \frac{1}{3^{12} 5^4 7^4 11^2}(B^2 - 27 g_2) \\ &\quad \times \prod_{i=1}^{3}(B^3 - 15 e_i B^2 + (315 e_i^2 - 132 g_2)B + e_i(2835 e_i^2 - 540 g_2)). \end{aligned}$$

2.2. **The degree of the addition map σ_n.**

We are now ready to study the addition map $\sigma_n : \bar{X}_n \to E$, $a \mapsto \sigma_n(a) = \sum_{i=1}^n a_i$ defined in (0.13) and determine its degree $\deg \sigma_n$.

For a *finite morphism* of irreducible curves $f : X \to Y$, the function field $K(X)$ is a finite extension of $K(Y)$ and $\deg f = [K(X) : K(Y)]$. Geometrically, $\deg f$ is the number of points for a general fiber $f^{-1}(p)$, $p \in Y$. If the image curve Y is *smooth*, the degree is equal to the length of *the scheme-theoretic fiber* $f^{-1}(p)$ for any $p \in Y$. A standard reference is [7].

Theorem 2.4 (= Theorem 0.2). *The map $\sigma_n : \bar{X}_n \to E$ has degree $\frac{1}{2}n(n+1)$.*

Proof. The idea is to apply *Theorem of the Cube* [13, p.58, Corollary 2] for morphisms from an arbitrary variety V (not necessarily smooth) into abelian varieties (here the torus E). For *any* three morphisms $f, g, h : V \to E$ and a line bundle $L \in \operatorname{Pic} E$, we have

$$\begin{aligned}(f+g+h)^*L \cong{}& (f+g)^*L \otimes (g+h)^*L \otimes (h+f)^*L \\ & \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}.\end{aligned} \tag{2.1}$$

We will apply it to the algebraic curve $V = V_n \subset E^n$ which consists of the *ordered n-tuples a*'s so that $V_n/S_n = \bar{X}_n$. (Here S_n is the permutation group on n letters.)

For any line bundle L and any *finite morphism* $f : V \to E$, we have $\deg f^*L = \deg f \deg L$. In the following we fix an L with $\deg L = 1$.

We prove inductively that for $j = 1, \ldots, n$ the morphism $f_j : V_n \to E$ defined by

$$f_j(a) := a_1 + \cdots + a_j$$

has $\deg f_j^*L = \frac{1}{2}j(j+1)n!$. The case $j = n$ then gives the result since f_n is a finite morphism which descends to σ_n under the S_n action. (Notice that for $j < n$ the map f_j does not descend to a map on $\bar{X}_n$.)

Assuming first that it has been proved for $j = 1, 2$. To go from j to $j+1$, we let $f(a) = f_{j-1}(a)$, $g(a) = a_j$, and $h(a) = a_{j+1}$. Then by (2.1), f_{j+1}^*L has degree $n!$ times

$$\tfrac{1}{2}j(j+1) + 3 + \tfrac{1}{2}j(j+1) - \tfrac{1}{2}(j-1)j - 1 - 1 = \tfrac{1}{2}(j+1)(j+2)$$

as expected.

It remains to investigate the case $j = 1$ and $j = 2$.

For $j = 1$, by §0.1.2 (iii)-(iv), the inverse image of $0 \in E$ under $f_1 : V_n \to E$ consists of a single point 0^n. By Proposition 1.1 (2), the limiting system of equations (1.7) has a unique non-degenerate solution in $\mathbb{P}^{n-1}(\mathbb{C})$ up to permutations. From this, we conclude that there are precisely $n!$ branches of $V_n \to E$ near 0^n. For a point $b \in E^\times$ close to 0, each branch will contribute a point a with $a_1 = b$. In particular, f_1 is a finite morphism and $\deg f_1^*L = \deg f_1 = n!$.

For $j = 2$, we consider the (scheme-theoretic) inverse image of $0 \in E$ under $f_2 : V_n \to E$. Namely $V_n \ni a \mapsto a_1 + a_2 = 0$.

The point $a = 0$ again contributes degree $n!$ by a similar branch argument. Indeed, over each branch near 0^n we may represent $a = (a_i(z))$ by an analytic curve in z. Then condition $t_i + t_j \neq 0$ in Proposition 1.1 (2) implies that $z \mapsto a_1(z) + a_2(z) \in E$ is still locally biholomorphic for z close to 0. As a byproduct, since every irreducible component contains a branch near 0^n, f_2 is necessarily a finite morphism and $\deg f_2^* L = \deg f_2$.

For those points $a \neq 0$ with $f_2(a) = 0$, we have $a_1 = -a_2$ and thus $a = -a$ by §0.1.2 (iii), i.e. a corresponds to a branch point for the hyperelliptic projection $Y_n \to \mathbb{C}$. Let $b = (b_1, \cdots, b_n) \in V_n$ be a point near a. By Lemma 1.3, we see that $C \mapsto (b_1 + b_2)(C)$ is bi-holomorphic near $C = 0$. If $a \in V_n$ is a non-singular point then C is a local parameter and f_2 is unramified at a. In that case the degree contribution at a is one. We first treat the case that all branch points are non-singular points:

If $n = 2k$, by Proposition 2.2 (1) the degree contribution from type O points $a = \{\pm a_1, \cdots, \pm a_k\}$ is given by

$$(k+1) \times (k \times 2 \times (n-2)!),$$

while the degree from the type II points $\{\pm a_1, \cdots, \pm a_{k-1}, \frac{1}{2}\omega_i, \frac{1}{2}\omega_j\}$ is

$$3 \times k \times ((k-1) \times 2 \times (n-2)!).$$

The sum is $2(4k^2 - 2k)(n-2)! = 2n!$.

If $n = 2k+1$, by Proposition 2.2 (2), the degree contribution from type III points $\{\pm a_1, \cdots, \pm a_{k-1}, \frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_3\}$ is

$$k \times ((k-1) \times 2 \times (n-2)!),$$

while the type I points $\{\pm a_1, \cdots, \pm a_k, \frac{1}{2}\omega_i\}$ contribute

$$3 \times (k+1) \times (k \times 2 \times (n-2)!).$$

The sum is again $2(4k^2 + 2k)(n-2)! = 2n!$.

Thus in both cases we get the total degree $n! + 2n! = 3n!$ as expected.

If $Y_n = Y_n(\tau_0)$ is singular, let $a \in Y_n \setminus X_n$ be a singular (branch) point with $C^2 = h(B)(B - B_a)^m$, $m \geq 2$ and $h(B_a) \neq 0$. The curve Y_n arises from flat degenerations of smooth curves $Y_n(\tau)$ where m linear factors become the same. By Lemma 1.3, f_2 leads to an analytic equivalence between C and $b_1 + b_2$ near a. That is, the equation $b_1 + b_2 = 0$ is simply $C = 0$. Let $\tilde{B} = B - B_a$. Then the analytic structure sheaf of $f_2^{-1}(0)$ at a is given by

$$\mathbb{C}[\![\tilde{B}, C]\!]/\langle C^2 - h(B_a + \tilde{B})\tilde{B}^m, C\rangle \cong \mathbb{C}[\![\tilde{B}]\!]/\langle \tilde{B}^m \rangle,$$

which also has length m. This shows that f_2 is compatible with the degeneration and the degree counting then follows from the smooth case. $\square$

3. THE PRIMITIVE GENERATOR $\mathbf{z}_n$

We prove Theorem 0.3, in a more precise form, in Theorem 3.2.

3.1. **Setup of the proof.**

Definition 3.1 (The fundamental rational function $\mathbf{z}_n$ on $\bar{X}_n$). The function

$$\mathbf{z}_n(a_1, \ldots, a_n) := \zeta\Big(\sum_{i=1}^{n} a_i\Big) - \sum_{i=1}^{n} \zeta(a_i), \quad a_i \in \mathbb{C},$$

is meromorphic and periodic in each a_i, hence it defines a rational function on E^n. By symmetry, it descends to a rational function on $\mathrm{Sym}^n E$. We denote the restriction $\mathbf{z}_n|_{\bar{X}_n}$ also by $\mathbf{z}_n$, which is a rational function on $\bar{X}_n$ with poles along the fiber $\sigma_n^{-1}(0)$.

The importance of $\mathbf{z}_n$ is readily seen from investigation on the Green function equation (0.5): Let $a_i = r_i\omega_1 + s_i\omega_2$. Then by (0.9),

$$\begin{aligned} -4\pi \sum \frac{\partial G}{\partial z}(a_i) &= \sum(\zeta(r_i\omega_1 + s_i\omega_2) - r_i\eta_1 - s_i\eta_2) \\ &= Z(\textstyle\sum a_i) - \mathbf{z}_n(a). \end{aligned} \tag{3.1}$$

Hence $\sum_{i=1}^n \nabla G(a_i) = 0 \Longleftrightarrow \mathbf{z}_n(a) = Z(\sigma_n(a))$. This links $\sigma_n(a)$ with $\mathbf{z}_n$.

Theorem 3.2 (= Theorem 0.3)**.** *There is a weighted homogeneous polynomial*

$$W_n(\mathbf{z}) \in \mathbb{Q}[g_2, g_3, \wp(\sigma), \wp'(\sigma)][\mathbf{z}]$$

of $\mathbf{z}$*-degree $d_n = \deg \sigma_n$ such that for $\sigma = \sigma_n(a) = \sum a_i$, we have*

$$W_n(\mathbf{z}_n)(a) = 0.$$

Here the weights of $\mathbf{z}$, $\wp(\sigma)$, $\wp'(\sigma)$, g_2, g_3 are 1, 2, 3, 4, 6 respectively.

Indeed, $\mathbf{z}_n(a)$ is a primitive generator of the finite extension of rational function field $K(\bar{X}_n)$ over $K(E)$ with $W_n(\mathbf{z})$ being its minimal polynomial. [4]

Moreover, the extension is integral over the affine curve $E^\times$.

Proof. There is nothing to prove for $n = 1$, so we assume that $n \geq 2$.

Since $\mathbf{z}_n \in K(\bar{X}_n)$, which is algebraic over $K(E)$ with degree d_n, its minimal polynomial $W_n(\mathbf{z}) \in K(E)[\mathbf{z}]$ exists with $d := \deg W_n \mid d_n$.

Notice that for $\sigma_0 \in E$ being outside the branch loci of $\sigma_n : \bar{X}_n \to E$, there are precisely d_n different points $a = \{a_1, \cdots, a_n\} \in \bar{X}_n$ with $\sigma_n(a) = \sum a_i = \sigma_0$. Thus for the rational function $\mathbf{z}_n = \zeta(\sum a_i) - \sum \zeta(a_i) \in K(\bar{X}_n)$ to be a primitive generator, it is sufficient to show that $\mathbf{z}_n$ has exactly d_n branches over $K(E)$. That is, $\sum \zeta(a_i)$ gives different values for different choices of those a above σ_0. Indeed, for any given $\sigma = \sigma_0$, the polynomial $W_n(\mathbf{z}) = 0$ has at most d roots. But now $\mathbf{z}_n(a)$ with $\sigma_n(a) = \sigma_0$ gives d_n distinct roots of $W_n(\mathbf{z})$, hence we must conclude $d = d_n$ and $\mathbf{z}_n$ is a primitive generator.

Hence it is sufficient to show the following more precise result:

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[4]The fact that the coefficients lie in $\mathbb{Q}$, instead of just in $\mathbb{C}$, follows from standard elimination theory and two facts (i) The equations of $\bar{X}_n$ are defined over $\mathbb{Q}[g_2, g_3]$ (cf. equations (1.2)), and (ii) the addition map $E^n \to E$ is defined over $\mathbb{Q}$. In §5, we carry out the elimination procedure using the resultant for another explicit presentation π_n of σ_n.

Theorem 3.3. *Let $a, b \in Y_n$ and $(a_1, \cdots, a_n)$, $(b_1, \cdots, b_n) \in \mathbb{C}^n$ be representatives of a, b such that*

$$\sum_{i=1}^n a_i = \sum_{i=1}^n b_i, \qquad \sum_{i=1}^n \zeta(a_i) = \sum_{i=1}^n \zeta(b_i). \tag{3.2}$$

Suppose that $\sum \wp(a_i) \ne \sum \wp(b_i)$. Then a, b are branch points of $Y_n \to \mathbb{C}$ which contains the same number of half periods. Equivalently, the Lamé functions w_a and w_b are of the same type.

We emphasize that $\bar{X}_n$ is not required to be smooth.

Theorem 3.2 follows immediately by choosing σ_0 outside the branch loci of $\bar{X}_n \to E$ and $\sigma_0 \notin E[2]$. Indeed, let $a, b \in Y_n$ with $\sigma_n(a) = \sigma_n(b) = \sigma_0$ and $\mathbf{z}_n(a) = \mathbf{z}_n(b)$, or more precisely with conditions in (3.2) satisfied. By Theorem 3.3 we are left with the case $\sum \wp(a_i) = \sum \wp(b_i)$ but $a \ne b$. Then $a = -b$ by Proposition 1.1 (1), and in particular $\sigma_n(a) = -\sigma_n(b)$. Together with $\sigma_n(a) = \sigma_n(b)$ we conclude that $\sigma_0 = \sigma_n(a) = \sigma_n(b) \in E[2]$. This contradicts to the assumption $\sigma_0 \notin E[2]$. Hence we must have $a = b$.

Since $\mathbf{z}_n$ has no poles over $E^\times$, it is indeed integral over the affine Weierstrass model of $E^\times$ with coordinate ring (let $x_0 = \wp(\sigma)$, $y_0 = \wp'(\sigma)$)

$$R(E^\times) = \mathbb{C}[x_0, y_0]/(y_0^2 - 4x_0^3 - g_2x_0 - g_3).$$

The homogeneity of $W_n(\mathbf{z})$ also follows from this. □

Remark 3.4. By Theorem 0.2 we have $d_n = \frac{1}{2}n(n+1)$. We do not use this result in the formulation nor in the proof of Theorem 3.2.

We will give two proofs to Theorem 3.3 in §3.2 and §3.3. The first proof is longer but contains more information. Both proofs are based on the following basic lemma.

Lemma 3.5 (Tensor product). *Let $I = n(n+1)\wp(z)$, $I_1 = I + B_a$ and $I_2 = I + B_b$. Suppose that $w_1'' = I_1 w_1$ and $w_2'' = I_2 w_2$. Then the product $q := w_1 w_2$ satisfies the following fourth order ODE:*

$$q'''' - 2(I_1 + I_2)q'' - 6I'q' + ((B_a - B_b)^2 - 2I'')q = 0. \tag{3.3}$$

Proof. This follows from a straightforward computation. Indeed,

$$\begin{aligned} q' &= w_1'w_2 + w_1w_2', \\ q'' &= (I_1 + I_2)q + 2w_1'w_2', \\ q''' &= 2I'q + (I_1 + I_2)q' + 2(I_1w_1w_2' + I_2w_1'w_2). \end{aligned}$$

Notice that if $a = b$ (or just $B_a = B_b$) then $I_1 = I_2$ and we stop here to get the third order ODE as the symmetric product of the Lamé equation.

In general, we take one more differentiation to get

$$\begin{aligned} q'''' &= 2I''q + 4I'q' + (I_1 + I_2)q'' + 2I'q' + 2(I_1 + I_2)w_1'w_2' + 4I_1I_2q \\ &= 2(I_1 + I_2)q'' + 6I'q' + (2I'' - (I_1 - I_2)^2)q. \end{aligned}$$

This proves the lemma. □

Recall from the Hermite–Halphen ansatz in (0.7) that

$$w_{\pm a}(z) = e^{\pm z \sum \zeta(a_i)} \prod_{i=1}^{n} \frac{\sigma(z \mp a_i)}{\sigma(z)}$$

are solutions to $w'' = (n(n+1)\wp(z) + B_a)w =: I_1 w$, and

$$w_{\pm b}(z) = e^{\pm z \sum \zeta(b_i)} \prod_{i=1}^{n} \frac{\sigma(z \mp b_i)}{\sigma(z)}$$

are solutions to $w'' = (n(n+1)\wp(z) + B_b)w =: I_2 w$. Then $q_{a,-b} := w_a w_{-b}$ and $q_{-a,b} := w_{-a} w_b$ are solutions to equation (3.3).

3.2. **The proof to Theorem 3.3.**

By assumption we have $\sum a_i = \sum b_i$, hence

$$q_{a,-b}(z) = \prod_{i=1}^{n} \frac{\sigma(z - a_i)\sigma(z + b_i)}{\sigma^2(z)} \tag{3.4}$$

is an elliptic function. Similarly $q_{-a,b}(z) = q_{a,-b}(-z)$ is elliptic. In particular there exists an *even* elliptic function solution to equation (3.3), namely

$$Q := \tfrac{1}{2}(q_{a,-b} + q_{-a,b}) = (-1)^n \frac{\prod_{i=1}^{n} \sigma(a_i)\sigma(b_i)}{z^{2n}} (1 + O(|z|)). \tag{3.5}$$

Let q be an *even elliptic solution* to equation (3.3). Then we may investigate it in variable $x = \wp(z)$. To avoid confusion, we denote

$$\dot{f} = \partial f / \partial x \quad \text{and} \quad f' = \partial f / \partial z.$$

Let $y^2 = p(x) = 4x^3 - g_2 x - g_3$. Then $\wp' = y$, $\wp'' = 6\wp^2 - \frac{1}{2} g_2 = \frac{1}{2}\dot{p}(x)$. $\wp''' = 12\wp\wp' = 12xy$, $\wp'''' = 12\wp'^2 + 12\wp\wp'' = 12p(x) + 6x\dot{p}(x)$. Also

$$\begin{aligned}
q' &= \dot{q}\wp' = y\dot{q}, \\
q'' &= \ddot{q}\wp'^2 + \dot{q}\wp'' = p(x)\ddot{q} + \tfrac{1}{2}\dot{p}(x)\dot{q}, \\
q''' &= \dddot{q}\,\wp'^3 + 3\ddot{q}\wp'\wp'' + \dot{q}\wp''', \\
q'''' &= \ddddot{q}\,\wp'^4 + 6\dddot{q}\,\wp'^2\wp'' + 3\ddot{q}(\wp'')^2 + 4\ddot{q}\wp'\wp''' + \dot{q}\wp'''' \\
&= p(x)^2\ddddot{q} + 3p(x)\dot{p}(x)\dddot{q} + \big(\tfrac{3}{4}\dot{p}(x)^2 + 48xp(x)\big)\ddot{q} + \big(12p(x) + 6x\dot{p}(x)\big)\dot{q}.
\end{aligned}$$

By substituting these into equation (3.3) we get the ODE in x:

$$\begin{aligned}
L_4\, q := p^2\ddddot{q} + 3p\dot{p}\dddot{q} &+ \big(\tfrac{3}{4}\dot{p}^2 - 2(2(n^2+n-12)x + \beta)p\big)\ddot{q} \\
&- \big((2(n^2+n-3)x + \beta)\dot{p} + 6(n^2+n-2)p\big)\dot{q} \\
&+ \big(\alpha^2 - n(n+1)\dot{p}\big)q = 0,
\end{aligned} \tag{3.6}$$

where

$$\alpha := B_a - B_b \quad \text{and} \quad \beta := B_a + B_b. \tag{3.7}$$

We would like to find constraints for equation $L_4\, q = 0$ with $\alpha \neq 0$ to have a polynomial solution $q(x)$. Here g_2 and g_3 could be arbitrary, not necessarily satisfying the non-degeneracy condition $g_2^3 - 27g_3^2 \neq 0$.

Suppose that $q(x)$ is a polynomial in x of degree $m \geq 1$:

$$q(x) = x^m - s_1 x^{m-1} + s_2 x^{m-2} - \cdots + (-1)^m s_m, \tag{3.8}$$

which satisfies

$$\deg_x L_4\, q(x) \leq 1. \tag{3.9}$$

Then we can solve s_j recursively in terms of α^2, β and g_2, g_3.

Indeed, the top degree term x^{m+2} in equation (3.6) has coefficient

$$\begin{aligned}
&16m(m-1)(m-2)(m-3) + 144m(m-1)(m-2) + 108m(m-1) \\
&\quad - 16(n^2+n-12)m(m-1) - 24(n^2+n-3)m \\
&\quad\quad - 24(n^2+n-2)m - 12n(n+1) \\
&= (m-n)\Big(4m^3 + (4n+68)m^2 + (8n-101)m + 3(n+1)\Big),
\end{aligned}$$

which vanishes precisely when $m = n$. Thus we may assume $m = n$.

The next order term x^{n+1} without the s_1 factor has coefficient

$$-8n(n-1)\beta - 12n\beta = -4n(2n+1)\beta,$$

and the coefficient of $-s_1 x^{n+1}$ is given by

$$\begin{aligned}
&16(n-1)(n-2)(n-3)(n-4) + 144(n-1)(n-2)(n-3) \\
&\quad + 108(n-1)(n-2) - 16(n^2+n-12)(n-1)(n-2) \\
&\quad - 24(n^2+n-3)(n-1) - 24(n^2+n-2)(n-1) - 12n(n+1) \\
&= -8n(2n-1)(2n+1).
\end{aligned}$$

Hence

$$s_1 = \frac{\beta}{2(2n-1)}. \tag{3.10}$$

Inductively the coefficients of x^{n+2-i} in equation (3.6) for $i = 1, \ldots, n$ give rise to recursions to solve s_i in terms of β, α^2 and g_2, g_3. Hence we get

Lemma 3.6. *For $i = 1, \ldots, n$, there is a polynomial expression*

$$s_i = s_i(\alpha^2, \beta, g_2, g_3) = C_i \beta^i + \cdots$$

which is homogeneous of degree i with $\deg \alpha = \deg \beta = 1$ *and* $\deg g_2 = 2$, $\deg g_3 = 3$. [5] *Moreover, C_i is a non-zero rational number.*

There are still two remaining terms in (3.9). That is,

$$L_4\, q = F_1(\alpha, \beta, g_2, g_3)x + F_0(\alpha, \beta, g_2, g_3). \tag{3.11}$$

The basic structure on the consistency equations is described by the following two lemmas:

[5] Notice that the weight, as assigned in Theorem 3.2, is twice the degree.

Lemma 3.7. *We have*

$$
\begin{aligned}
F_1(\alpha,\beta,g_2,g_3) &= \alpha^2 G_1(\alpha,\beta,g_2,g_3) = \alpha^2((-1)^{n-1}s_{n-1}(\alpha^2,\beta,g_2,g_3)+\cdots),\\
F_0(\alpha,\beta,g_2,g_3) &= \alpha^2 G_0(\alpha,\beta,g_2,g_3) = \alpha^2((-1)^{n}s_{n}(\alpha^2,\beta,g_2,g_3)+\cdots).
\end{aligned}
$$

For the remaining terms, each term of them has either g_2 or g_3 as a factor, hence it has lower degree in α, β.

Proof. Equation (3.11) gives

$$
\begin{aligned}
F_1(\alpha,\beta,g_2,g_3) &= (-1)^{n-1}\alpha^2 s_{n-1} + \text{terms in } s_1,\cdots,s_{n-2},\\
F_0(\alpha,\beta,g_2,g_3) &= (-1)^{n}\alpha^2 s_{n} + \text{terms in } s_1,\cdots,s_{n-1}.
\end{aligned}
$$

If $\alpha = 0$ then for any $\beta \in \mathbb{C}$ there is an $a \in Y_n$ with $B_a = \beta/2$ and a polynomial solution $q_a(x) = w_a w_{-a} = \prod_{i=1}^n (x - \wp(a_i))$ to the symmetric product of the Lamé equation, hence a polynomial solution to $L_4(q) = 0$.

Thus $F_1(0,\beta,g_2,g_3) = 0 = F_0(0,\beta,g_2,g_3)$. Since F_i depends on α^2, we have $F_i = \alpha^2 G_i$, $i = 0, 1$, for some homogeneous polynomials G_0, G_1 in α^2, β, g_2, g_3 of degree n and $n-1$ respectively, and G_i's can be written as

$$
\begin{aligned}
G_1 &= (-1)^{n-1}s_{n-1} + \cdots,\\
G_0 &= (-1)^{n}s_{n} + \cdots.
\end{aligned}
$$

To see the dependence of the remaining terms on g_2 and g_3, we let $g_2 = 0 = g_3$, and then $L_4(q) \equiv \alpha^2((-1)^{n-1}s_{n-1}x + (-1)^n s_n) \pmod{x^2}$ because both $p(x) = 4x^3$ and $\dot p(x) = 12x^2$ vanish modulo x^2. Thus we have $F_1(\alpha,\beta) = (-1)^{n-1}\alpha^2 s_{n-1}$ and $F_0(\alpha,\beta) = (-1)^n\alpha^2 s_n$ whenever $g_2 = 0 = g_3$. This proves the lemma. ☐

Lemma 3.8. *The polynomials G_1 and G_0 have no common factors for any g_2, g_3.*

Proof. We consider first the special case $g_2 = g_3 = 0$. Then (3.9) becomes

$$
\begin{aligned}
&16x^6\ddddot{q} + 144x^5\dddot{q} + \big(108x^4 - 8x^3(2(n^2+n-12)x+\beta)\big)\ddot{q}\\
&\quad - \big(12x^2(2(n^2+n-3)x+\beta) + 24x^3(n^2+n-2)\big)\dot{q}\\
&\qquad + \big(\alpha^2 - 12n(n+1)x^2\big)q \equiv 0 \pmod{\mathbb{C}\oplus\mathbb{C}x}.
\end{aligned}
\tag{3.12}
$$

The coefficients of x^{n-k}, $k = 0, \ldots, n-2$, lead to recursive equations

$$
(-1)^k(m_k\, s_{k+2} + n_k\beta\, s_{k+1} + \alpha^2 s_k) = 0, \tag{3.13}
$$

where the constants m_k and n_k are given by

$$\begin{aligned}
m_k &= 16(n-(k+2))(n-(k+3))(n-(k+4))(n-(k+5)) \\
&\quad + 144(n-(k+2))(n-(k+3))(n-(k+4)) \\
&\qquad + (108-16(n^2+n-12))(n-(k+2))(n-(k+3)) \\
&\qquad\quad - 24(2n^2+2n-5)(n-(k+2)) - 12n(n+1) \\
&= -4(k+2)(2n-(k+1))(2n-(2k+1))(2n-(2k+3)), \\
n_k &= (8(n-(k+1))(n-(k+2)) + 12(n-(k+1))) \\
&= 4(n-(k-1))(n-(k+1)).
\end{aligned}$$

Since $k \le n-2$, we have $m_k \neq 0$ and $n_k \neq 0$.

Let γ be a non-trivial common factor of G_1 and G_0. Under the assumption $g_2 = g_3 = 0$ we have $G_1 = (-1)^{n-1} s_{n-1}$ and $G_0 = (-1)^n s_n$. Then γ and α are co-prime, because if $\alpha = 0$ then $s_{n-1}(0,\beta) = c_{n-1}\beta^{n-1}$ and $s_n(0,\beta) = c_n\beta^n$ for some non-zero constants c_{n-1} and c_n. By the recursive equation (3.13) for $k = n-2$, we have $\gamma \mid s_{n-2}(\alpha^2,\beta,0,0)$ too. By induction on k for $k = n-3,\ldots,0$ in decreasing order we conclude that $\gamma \mid s_0 = 1$, which leads to a contradiction.

For $g_2, g_3 \in \mathbb{C}$, we see by Lemma 3.7 that the *leading terms* of G_1, G_0, as polynomials in α and β, are $(-1)^{n-1}s_{n-1}(\alpha^2,\beta,0,0)$ and $(-1)^n s_n(\alpha^2,\beta,0,0)$ respectively. Since $s_{n-1}(\alpha^2,\beta,0,0)$ and $s_n(\alpha^2,\beta,0,0)$ are co-prime, as we have just seen, we conclude that $G_1(\alpha,\beta,g_2,g_3)$ and $G_0(\alpha,\beta,g_2,g_3)$ are also co-prime. The proof is complete. $\square$

Proposition 3.9. *The common zeros of $G_1 = 0$ and $G_0 = 0$ consist of pairs of branch points (a,b) corresponding to Lame functions of the same type. If $\bar{X}_n$ is non-singular, there are exactly $n(n-1)$ such ordered pairs (a,b)'s.*

Proof. It suffices to prove the (generic) case that $\bar{X}_n$ is non-singular, namely the case that all the Lamé functions are distinct. The general case follows from the non-singular case by a limiting argument.

For any two Lamé functions w_a, w_b of the same type (cf. §2.1), it is easy to see that we may arrange the representatives of a and b so that equations (3.2) holds. It follows that $q := q_{a,-b} = q_{-a,b}$ (cf. (3.4)) is an even elliptic function solution to equation (3.3), or equivalently $q(x)$ is a polynomial solution to $L_4\, q(x) = 0$ in variable $x = \wp(z)$.

From the above discussion, (α,β) must be a common root of G_1 and G_0 (where $\alpha = B_a - B_b$, $\beta = B_a + B_b$). By Lemma 3.6 and 3.7, we have $\deg G_1 = n-1$ and $\deg G_0 = n$ and G_1, G_0 are co-prime to each other by Lemma 3.8. Hence by Bezout theorem there are at most $n(n-1)$ common zeros.

On the other hand, the number of such ordered pairs can be determined by Proposition 2.2. Indeed, if $n = 2k$ then we have

$$(k+1)k + 3k(k-1) = 4k^2 - 2k = n(n-1)$$

such pairs. If $n = 2k+1$, the number of pairs is given by

$$k(k-1) + 3(k+1)k = 4k^2 + 2k = n(n-1).$$

Hence in all cases the number of ordered pairs coming from the Lamé functions of the same type agrees with the Bezout degree of the polynomial system defined by $G_1 = 0 = G_0$. Thus these $n(n-1)$ pairs form the zero locus as expected (and there is no infinity contribution). $\square$

The above discussions from Lemma 3.5 to Proposition 3.9 constitute a complete proof to Theorem 3.3. Here is a summary: we already know that Q in (3.5) is an even elliptic function with singularity only at $0 \in E$. Thus

$$Q(x) = c\prod_{i=1}^{n}(\wp(z) - \wp(c_i)) =: c\prod_{i=1}^{n}(x - x_i)$$

is a polynomial solution to the ODE (3.6) with $\alpha = B_a - B_b$, $\beta = B_a + B_b$.

Since $\alpha = B_a - B_b \neq 0$, by Lemma 3.7 (α, β) must be a common root of $G_1(\alpha, \beta) = 0 = G_0(\alpha, \beta)$. Then Proposition 3.9 says that (α, β) is pair of Lamé functions of the same type. This proves Theorem 3.3.

For future reference, we summarize the results into the following statement on a fourth order ODE which arises from the *tensor product of two different Lamé equations* with the same parameter $n \in \mathbb{N}$.

Theorem 3.10. *Let $I(z) = n(n+1)\wp(z)$. The fourth order ODE*

$$q''''(z) - 2(I+\beta)q''(z) - 6I'q'(z) + (\alpha^2 - 2I'')q(z) = 0 \tag{3.14}$$

with $\alpha \neq 0$ has an elliptic function solution q if and only if (α, β) is common zero to $G_0(\alpha, \beta) = 0$ and $G_1(\alpha, \beta) = 0$. Moreover, this solution q must be even.

Example 3.11. For $n = 2$, $\beta = B_a + B_b$, $\alpha = B_a - B_b$, we have

$$s_1 = \tfrac{1}{6}\beta, \qquad s_2 = \tfrac{1}{36}\beta^2 + \tfrac{1}{72}\alpha^2 - \tfrac{1}{4}g_2.$$

The first compatibility equation from x^1 is

$$s_1(\alpha^2 + 36g_2) - 6\beta g_2 = 0.$$

After substituting s_1 we get

$$\tfrac{1}{6}\alpha^2\beta = 0. \tag{3.15}$$

The second compatibility equation from x^0 is

$$s_2(\alpha^2 + 6g_2) - s_1(\beta g_2 + 24g_3) + 4\beta g_3 + \tfrac{3}{2}g_2^2 = 0.$$

By substituting s_1, s_2 and noticing the (expected) cancellations we get

$$\alpha^2\left(\tfrac{1}{36}\beta^2 + \tfrac{1}{72}\alpha^2 - \tfrac{1}{6}g_2\right) = 0. \tag{3.16}$$

If $B_a \neq B_b$ then (3.15) implies that $B_b = -B_a$ and then (3.16) leads to

$$B_a^2 = 3g_2 \Longrightarrow \wp(a_1) + \wp(a_2) = \pm\sqrt{g_2/3}.$$

By Example 2.3 (2), such $a \in \bar{X}_2$ lies in the branch loci of the Lamé curve. In particular, $a, b \in \sigma_2^{-1}(0)$. Denote by $\wp(\pm q_\pm) = \pm\sqrt{g_2/12}$. Then $a := \{q_+, -q_+\} \neq b := \{q_-, -q_-\}$ unless $g_2 = 0$. When $g_2 \neq 0$, $\mathbf{z}_2$ fails to distinguish the two points a and b. When $g_2 = 0$ (equivalently $\tau = e^{\pi i/3}$), $a = b$ becomes a singular branch point for $\sigma_2 : \bar{X}_2 \to E_\tau$.

Example 3.12. For $n = 3$, $\beta = B_a + B_b$, $\alpha = B_a - B_b$. Then

$$
\begin{aligned}
s_1 &= \tfrac{1}{10}\beta,\\
s_2 &= \tfrac{1}{600}(4\beta^2 + \alpha^2 - 150 g_2),\\
s_3 &= \tfrac{1}{3600}(2\beta^3 + 3\alpha^2\beta - 120\beta g_2 + 900 g_3).
\end{aligned}
$$

The two compatibility equations from x^1 and x^0 are

$$
\begin{aligned}
0 &= \tfrac{1}{600}\alpha^2(4\beta^2 + \alpha^2 + 60 g_2),\\
0 &= \tfrac{1}{3600}\alpha^2(2\beta^3 + 3\alpha^2\beta - 90\beta g_2 + 540 g_3).
\end{aligned}
$$

If $\alpha \neq 0$ then $\alpha^2 = -4\beta^2 - 60 g_2$ and the second equation becomes

$$\beta^3 + 27 g_2 \beta - 54 g_3 = 0.$$

It is clear that there are only finite solutions (B_a, B_b)'s to this, though it may not be so straightforward to see that these 6 solution pairs (for generic tori) come from the branch loci as proved in Proposition 3.9.

3.3. **Second proof to Theorem 3.3.**

Proof. Following the definition of $q_{a,-b}(z)$ in (3.4), we now consider the *odd elliptic solution* to equation (3.3) (= equation (3.14)) instead:

$$q(z) := \tfrac{1}{2}(q_{a,-b}(z) - q_{-a,b}(z)).$$

The function $q(z)$ has a pole of order $3 + 2l$ at $0 \in E$ with $l \leq n - 2$. Thus $q(z)/\wp'(z)$ is an even elliptic function with the only pole at 0 since $q(\frac{1}{2}\omega_i) = 0$ for $1 \leq i \leq 3$. If $q(z)$ does not vanish completely, then

$$q(z) = c\wp'(z)\prod_{i=1}^{l}(\wp(z) - \wp(c_i)) =: c\wp'(z) f(\wp(z)),$$

where $f(x) = \prod_{i=1}^{l}(x - \wp(c_i)) = x^l - s_1 x^{l-1} + \cdots + (-1)^l s_l$.

Equation (3.14) now reads as

$$
\begin{aligned}
&q''''(z) - 2(\beta + 2n(n+1)\wp(z))q''(z)\\
&\qquad - 6n(n+1)\wp'(z)q'(z) + (\alpha^2 - 2n(n+1)\wp''(z))q(z) = 0.
\end{aligned}
\tag{3.17}
$$

By straightforward calculations, we get derivatives of q in terms of derivatives of $\wp(z)$ and $f'(x)$. For example,

$$
\begin{aligned}
q'(z) &= \wp''(z) f(x) + \wp'(z)^2 f'(x),\\
q''(z) &= \wp'''(z) f(x) + 3\wp''(z)\wp'(z) f'(x) + \wp'(z)^3 f''(x), \quad \text{etc.}
\end{aligned}
$$

Then (3.17) is equivalent to

$$\begin{aligned}
&f(x)\Big((360-96n(n+1))x^2-24\beta x+(4n(n+1)-18)g_2+\alpha^2\Big)\\
&+f'(x)\Big((1320-96n(n+1))x^3-36\beta x^2\\
&\qquad\qquad+(12n(n+1)-150)g_2x+(6n(n+1)-60)g_3+3\beta g_2\Big)\\
&+f''(x)\Big((1020-16n(n+1))x^4-8\beta x^3+(4n(n+1)-210)g_2x^2\\
&\qquad\qquad+(2\beta g_2+(4n(n+1)-120)g_3)x+2\beta g_3+\tfrac{15}{4}g_2^2\Big)\\
&+f'''(x)(60x^2-30g_2)(4x^3-g_2x-g_3)\\
&+f''''(x)(4x^3-g_2x-g_3)^2=0.
\end{aligned}$$

By comparing the coefficients of x^{l+2}, we obtain

$$\begin{aligned}
&(360-96n(n+1))+l(1320-96n(n+1))+l(l-1)(1020-16n(n+1))\\
&\quad+240l(l-1)(l-2)+16l(l-1)(l-2)(l-3)=0.
\end{aligned}$$

After simplification, this is reduced to

$$4n(n+1)=(2l+3)(2l+5),$$

which obviously leads to a contradiction since the number in the right-hand side is odd while the number in the left-hand side is even. Therefore we must have $q\equiv 0$ from the beginning. That is, $\{a_i,-b_i\}=\{-a_i,b_i\}$.

If one of a,b does not correspond to a Lamé function, say $a\in X_n$, then $\{a_i\}\cap\{-a_i\}=\emptyset$ by §0.1.2 (iii) and we conclude that $\{a_i\}=\{b_i\}$. Otherwise a and b correspond to Lamé functions of the same type. □

3.4. The degree of the rational function $\mathbf{z}_n$.

Theorem 3.13. *The structure of the map $\mathbf{z}_n:\bar{X}_n\to\mathbb{P}^1(\mathbb{C})$ over $\infty\in\mathbb{P}^1(\mathbb{C})$ is analytically equivalent to $\sigma_n:\bar{X}_n\to E$ over 0. In particular it has the same degree as the one for σ_n, namely* $\deg\mathbf{z}_n=\deg\sigma_n=\frac{1}{2}n(n+1)$.

Proof. By definition, $\mathbf{z}_n^{-1}(\infty)=\sigma_n^{-1}(0)$ as sets. So the crucial point is to compare the ramification structures of $\bar{X}_n\to E$ at $0\in E$ and $\bar{X}_n\to\mathbb{P}^1(\mathbb{C})$ at $\infty\in\mathbb{P}^1(\mathbb{C})$. Let $a\in\bar{X}_n$ with $\sigma_n(a)=0$. Then for $b=\{b_i\}_{i=1}^n\in\bar{X}_n$ in a small analytic neighborhood of a we have $b_i\neq 0$ all i.

If $a\neq 0^n$, then every b_i is away from 0 and

$$(\mathbf{z}_n(b))^{-1}=\Big(\zeta(\sigma_n(b))-\sum_{i=1}^n\zeta(b_i)\Big)^{-1}=\sigma_n(b)+o(\sigma_n(b)).$$

In terms of the coordinate of $\mathbb{P}^1(\mathbb{C})$ at ∞, the map $\mathbf{z}_n$ near $a\neq 0^n$ is seen to be analytically equivalent to σ_n.

At $a = 0^n$, we compute the expansion of $(\mathbf{z}_n(b))^{-1}$ as

$$\frac{\sigma_n(b)}{\sigma_n(b)\zeta(\sigma_n(b)) - \sigma_n(b)\sum_{i=1}^n \zeta(b_i)} = \sigma_n(b)\Big(1 + \sigma_n(b)\sum_{i=1}^n \zeta(b_i) + O(\sigma_n^4(b))\Big).$$

The tangent direction (t_i) at 0^n is related to (b_i) through the asymptotic

$$t_i|B|^{1/2} \sim -1/b_i$$

(cf. [1, Proposition 7.5] and the proof therein). Hence

$$\lim_{b\to 0} \sigma_n(b)\sum_{i=1}^n \zeta(b_i) = \sum_{i=1}^n t_i^{-1} \sum_{i=1}^n t_i =: \Lambda_n.$$

The precise value of Λ_n follows from Proposition 1.5:

$$\Lambda_n = \frac{e_{n-1}e_1}{e_n} = \tfrac{1}{2}n(n+1) \neq -1.$$

Hence $(\mathbf{z}_n(b))^{-1} = (1+\Lambda_n)\sigma_n(b) + o(\sigma_n(b))$ and we again have the analytic equivalence (up to a constant multiple).

In particular, $\deg \mathbf{z}_n = \deg \sigma_n = \frac{1}{2}n(n+1)$ by Theorem 0.2. □

4. Pre-modular forms $Z_n(\sigma;\tau)$

Pre-modular forms are defined in Definition 0.1. Since the Hecke function is pre-modular of weight one, Theorem 3.2 then implies

Corollary 4.1. *$Z_n(\sigma;\tau) := W_n(Z)(\sigma;\tau)$ is pre-modular of weight $\frac{1}{2}n(n+1)$, with Z, $\wp(\sigma)$, $\wp'(\sigma)$, g_2, g_3 being of weight 1, 2, 3, 4, 6 respectively.*

4.1. **The completion of the proof to Theorem 0.4.**

We call the $2n+1$ branch points $a \in Y_n \setminus X_n$ *trivial critical points* since $a = -a$ and the Green equation (0.5) holds trivially. They satisfy a nice compatibility condition with the case $n = 1$ under the addition map:

Lemma 4.2. *Let $a = \{a_1, \cdots, a_n\} \in Y_n$ be a solution to the Green equation $\sum_{i=1}^n \nabla G(a_i) = 0$. Then a is trivial, i.e. $a = -a$, if and only if $\sigma_n(a) \in E[2]$.*

Proof. If a is trivial, then $\sigma_n(a) \in E[2]$ clearly. If a is non-trivial, i.e. $a \in X_n$, by equations (1.5), it gives rise to a type II developing map f with

$$f(z+\omega_1) = e^{-4\pi i \sum_i s_i} f(z), \qquad f(z+\omega_2) = e^{4\pi i \sum_i r_i} f(z).$$

Here $a_i = r_i\omega_1 + s_i\omega_2$ for $i = 1, \ldots, n$.

If $\sigma_n(a) \in E[2]$, then both exponential factors reduce to one and we conclude that $f(z)$ is an elliptic function on E. Notice that the only zero of $f'(z)$ is at $z = 0$ which has order $2n$, and the only poles of $f'(z)$ are at $-a_i$ of order 2, $i = 1, \ldots, n$. This forces that $\sigma_n(a) = 0$ and

$$f'(z) = \textstyle\sum_{j=1}^n E_j \wp(z+a_j) + C_1$$

for some constants $E_1, \ldots, E_n$ and C_1, since f' is residue free. Then

$$f(z) = -\sum_{j=1}^n E_j \zeta(z + a_i) + C_1 z + C_2$$

for some constant C_2. But $f(z)$ is elliptic, which implies that $C_1 = 0$ and $\sum_{j=1}^n E_j = 0$. Now $f^{2k-1}(0) = 0$ for $k = 1, \ldots, n$ leads to a system of linear equations in E_j's (c.f. [1, Lemma 2.3.1]):

$$\sum_{j=1}^n \wp^k(a_j) E_j = 0, \qquad k = 1, \ldots, n.$$

Then $\wp(a_i) \neq \wp(a_j)$ for $i \neq j$ forces that $E_j = 0$ for all j. This is a contradiction and we conclude that $\sigma_n(a) \notin E[2]$. □

The following theorem completes the proof to Theorem 0.4:

Theorem 4.3 (Extra critical points versus non-trivial zeros of $Z_n(\sigma; \tau)$)**.**

(i) *Given $\sigma_0 \in E_\tau \setminus E_\tau[2]$ with $Z_n(\sigma_0; \tau) = 0$, there is a unique $a \in X_n$ such that $\sigma_n(a) = \sigma_0$ and $\mathbf{z}_n(a) = Z(\sigma_0)$.*

(ii) *Conversely, if $a \in X_n$ and $\mathbf{z}_n(a) = Z(\sigma(a))$ then $Z_n(\sigma(a); \tau) = 0$ and $\sigma_n(a) \notin E_\tau[2]$.*

Proof. (i) For any given σ_0, by substituting σ with σ_0 in $W_n(\mathbf{z})$, we get a polynomial $W_{n,\sigma_0}(\mathbf{z})$ of degree $\frac{1}{2}n(n+1)$. Since $W_n(\mathbf{z})$ is the minimal polynomial of the rational function $\mathbf{z}_n \in K(\bar{X}_n)$ over $K(E)$, those $\mathbf{z}_n(a)$ with $a \in \bar{X}_n$ and $\sigma_n(a) = \sigma_0$ give rise to all the roots, counted with multiplicities, of $W_{n,\sigma_0}(a) = 0$.

Now $Z(\sigma_0)$ is a root of $W_{n,\sigma_0}(\mathbf{z}) = 0$ with $\sigma_0 \notin E[2]$, hence there is a point $a \in X_n$ which corresponds to it. That is, $Z(\sigma_0) = \mathbf{z}_n(a)$ with $\sigma_n(a) = \sigma_0$, and a is unique by Theorem 3.3. Notice that if $a \in \bar{X}_n \setminus X_n$ then $a = -a$ and then $\sigma_n(a) \in E[2]$. So we must have $a \in X_n$.

(ii) It is clear that $Z_n(\sigma(a)) = W_n(Z(\sigma(a)) = W_n(\mathbf{z}_n(a)) = 0$. Since $a \in X_n$, by equation (3.1) we have $\sum_{i=1}^n \nabla G(a_i) = 0$. But since a is non-trivial, Lemma 4.2 implies that $\sigma_n(a) \notin E[2]$. □

4.2. **Monodromy aspects.**

We present below an extended version of Theorem 0.4 in terms of *monodromy groups of Lamé equations*. The original case of mean field equations corresponds to the case with *unitary monodromy*.

Let $a = \{a_1, \cdots, a_n\} \in X_n$, $B_a = (2n-1)\sum_{i=1}^n \wp(a_i)$ and w_a, w_{-a} be the independent ansatz solutions (0.7) to $w'' = (n(n+1)\wp(z) + B_a)w$. From equations (1.4), one calculates that the monodromy matrices are given by

$$\begin{aligned} \begin{pmatrix} w_a \\ w_{-a} \end{pmatrix}(z + \omega_1) &= \begin{pmatrix} e^{-2\pi i s} & 0 \\ 0 & e^{2\pi i s} \end{pmatrix} \begin{pmatrix} w_a \\ w_{-a} \end{pmatrix}(z), \\ \begin{pmatrix} w_a \\ w_{-a} \end{pmatrix}(z + \omega_2) &= \begin{pmatrix} e^{2\pi i r} & 0 \\ 0 & e^{-2\pi i r} \end{pmatrix} \begin{pmatrix} w_a \\ w_{-a} \end{pmatrix}(z), \end{aligned} \tag{4.1}$$

where the two numbers $r, s \in \mathbb{C} \pmod{\mathbb{Z}}$ are uniquely determined by

$$r\omega_1 + s\omega_2 = \sigma(a) = \sum_{i=1}^n a_i, \qquad r\eta_1 + s\eta_2 = \sum_{i=1}^n \zeta(a_i). \tag{4.2}$$

The system is non-singular by the Legendre relation $\omega_1\eta_2 - \omega_2\eta_1 = -2\pi i$.

The next lemma extends Lemma 4.2:

Lemma 4.4. *Let $a \in X_n$ with (r, s) given by* (4.2)*. Then $(r, s) \notin \frac{1}{2}\mathbb{Z}^2$.*

Proof. If $(r, s) \in \frac{1}{2}\mathbb{Z}^2$ then $f := w_a/w_{-a}$ is elliptic by equations (4.1). Since

$$f' = \frac{w_a' w_{-a} - w_a w_{-a}'}{w_{-a}^2} = \frac{C}{w_{-a}^2},$$

we find that $z = 0$ is the only zero of $f'(z)$ in E, which has order $2n$. The proof of Lemma 4.2 for this f goes through and leads to a contradiction. □

Now we consider $Z_{r,s}(\tau)$ in (0.10) but with $r, s, \in \mathbb{C}$, and define

$$Z_{n;r,s}(\tau) := W_n(Z_{r,s})(r + s\tau; \tau), \qquad r, s \in \mathbb{C}. \tag{4.3}$$

It reduces to $Z_n(\sigma; \tau)$ for $\sigma = r + s\tau$ when $r, s \in \mathbb{R}$ (see [2] for its role in the isomonodromy problems and Painleve VI equations).

By substituting $Z_n(\sigma; \tau)$ with $Z_{n;,r,s}(\tau)$ and using Lemma 4.4 in place of Lemma 4.2, the proof of Theorem 4.3 also leads to:

Theorem 4.5. *Let $r, s \in \mathbb{C}$. Then any non-trivial solution τ to $Z_{n;r,s}(\tau) = 0$, i.e. with $r + s\tau \pmod{\Lambda_\tau} \notin E_\tau[2]$, corresponds to an $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ such that $a \pmod{\Lambda_\tau} \in X_n(\tau)$ and*

$$\sum_{i=1}^n a_i = r + s\tau, \qquad \sum_{i=1}^n \zeta(a_i; \tau) = r\eta_1(\tau) + s\eta_2(\tau).$$

Equivalently, by equations (4.2)*, the Lame equation $w'' = (n(n+1)\wp(z; \Lambda_\tau) + B_a)w$ has its monodromy representation given by equations* (4.1)*.*

We leave the straightforward justifications to the interested readers.

5. AN EXPLICIT DETERMINATION OF Z_n

From the equations of $\bar{X}_n \subset \mathrm{Sym}^n E$ (cf. (1.2)) and the recursively defined algebraic formula for the addition map $E^n \to E$, in principle it is possible to compute W_n and hence Z_n by *elimination theory* (cf. [8]). However we shall present a more direct approach on this to reveal more structures inside it.

5.1. **Comparisons with the Hermite–Krichever ansatz.**

Besides the Hermite–Halphen ansatz (0.7), there is another ansatz, the *Hermite–Krichever ansatz*, which can also be used to construct solutions to the integral Lamé equation (0.6). It takes the form

$$\psi(z) := \Big(U(\wp(z)) + V(\wp(z))\frac{\wp'(z) + \wp'(a_0)}{\wp(z) - \wp(a_0)}\Big)\frac{\sigma(z - a_0)}{\sigma(z)} e^{(\zeta(a_0)+\kappa)z}, \tag{5.1}$$

where $U(x)$ and $V(x)$ are polynomials in x, the point $a_0 \in \mathbb{C} \setminus \Lambda$, and $\kappa \in \mathbb{C}$ is a constant. As usual, we set $(x, y) = (\wp(z), \wp'(z))$ and $(x_0, y_0) = (\wp(a_0), \wp'(a_0))$ to be the corresponding algebraic coordinates.

The ansatz (5.1) makes sense since ψ only has poles at $z = 0$. The two poles at $z = \pm a_0$ from $(\wp(z) - \wp(a_0))^{-1}$ cancel with the zeros from $\sigma(z - a_0)$ and $\wp'(z) + \wp'(a_0)$. In order for $\operatorname{ord}_{z=0} \psi(z) = -n$, we must have

Lemma 5.1 (Degree constraints)**.**

(i) *If* $n = 2m$ *with* $m \in \mathbb{N}$ *then* $\deg U \le m - 1$ *and* $\deg V = m - 1$.
(ii) *If* $n = 2m + 1$ *with* $m \in \mathbb{N} \cup \{0\}$ *then* $\deg U = m$ *and* $\deg V \le m - 1$.

By an obvious normalization, in case (i) we may assume that

$$U(x) = \sum_{i=0}^{m-1} u_i x^i, \qquad V(x) = \sum_{i=0}^{m-1} v_i x^i \quad \text{with } v_{m-1} = 1,$$

and in case (ii) we assume that $U(x) = \sum_{i=0}^{m} u_i x^i$ with $u_m = 1$ and $V(x) = \sum_{i=0}^{m-1} v_i x^i$. In both cases, the requirement that $\psi(z)$ satisfies equation (0.6) leads to recursions on u_i's and v_i's. It turns out to be convenient to work with coordinates (B, κ, x_0, y_0) to parametrize u_i's and v_i's, and this was carried out by Maier in [12, §4]. The following is a summary:

In case (i) the recursion determines v_i ($v_{m-1} = 1$) and then u_i for $i = m-1, m-2, \ldots$ in descending order. In case (ii) it starts with $u_m = 1$ and determines v_i and then u_i for $i = m-1, m-2, \ldots$ also in descending order.

There are two compatibility equations coming from

$$u_{-1}(B, \kappa, x_0, y_0) = 0 \quad \text{and} \quad v_{-1}(B, \kappa, x_0, y_0) = 0.$$

The two parameters x_0, y_0 satisfy $y_0^2 = 4x_0^3 - g_2 x_0 - g_3$. Hence there are four variables $(B, \kappa, x_0, y_0) \in \mathbb{C}^4$ which are subject to three polynomial equations. By taking into account the limiting cases with $(x_0, y_0) = (\infty, \infty)$, this recovers the Lame curve $\bar{Y}_n$ (which is denoted by Γ_ℓ in [12] with $\ell = n$).

There are four natural coordinate projections (rational functions) $\bar{Y}_n \to \mathbb{P}^1(\mathbb{C})$, namely B, κ, x_0 and y_0 respectively. The first one $B : \bar{Y}_n \to \mathbb{P}^1(\mathbb{C})$ is simply the hyperelliptic structure map. The main result in [12] is an explicit description of the other 3 maps in terms of the coordinates (B, C) on $\bar{Y}_n$. To state it we need to first recall some variants of Lamé polynomials.

Definition 5.2. [12, Definition 3.2, 3.4, 3.6]

(1) The *twisted Lamé polynomials* $lt_j(B)$, $j = 0, 1, 2, 3$ are monic polynomials whose zeros correspond to solutions to (0.6) given by the Hermite–Krichever ansatz with $\kappa \ne 0$ and $a_0 = 0$, $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_2$, $\frac{1}{2}\omega_3$ respectively, i.e. $(x_0, y_0) = (\infty, \infty)$, $(e_1, 0)$, $(e_2, 0)$, $(e_3, 0)$ respectively.
(2) The *theta-twisted polynomial* $l_\theta(B)$ is the monic polynomial whose roots correspond to the case $\kappa = 0$ and $a_0 \notin E[2]$. (For $\kappa = 0$ and $a_0 \in E[2]$ they correspond to the *ordinary* Lamé polynomials $l_i(B)$'s.)

Theorem 5.3 ([12, Theorem 4.1]). *For all $n \in \mathbb{N}$ and $i \in \{1,2,3\}$,*

$$\begin{aligned} x_0(B) &= e_i + \frac{4}{n^2(n+1)^2} \frac{l_i(B) lt_i(B)^2}{l_0(B) lt_0(B)^2}, \\ y_0(B,C) &= \frac{16}{n^3(n+1)^3} \frac{C}{c_n} \frac{lt_1(B) lt_2(B) lt_3(B)}{l_0(B)^2 lt_0(B)^3}, \\ \kappa(B,C) &= -\frac{(n-1)(n+2)}{n(n+1)} \frac{C}{c_n} \frac{l_\theta(B)}{l_0(B) lt_0(B)}. \end{aligned} \tag{5.2}$$

The formula for $x_0(B)$ is valid for all three choices of i.

All the factors lie in $\mathbb{Q}[e_1, e_2, e_3, g_2, g_3, B]$ and are monic *in B. They are homogeneous with degrees of B, e_i, g_2, g_3 being* $1,1,2,3$ *respectively.*

As a simple consistency check, we have $C^2 = \ell_n(B)$ by Proposition 2.2.

Remark 5.4. In [12] $\nu = C/c_n$ is used instead.

The polynomials $l_0(B), l_i(B)$ ($i = 1,2,3$) are written there as $L^I_\ell(B; g_2, g_3)$, $L^{II}_\ell(B; e_i, g_2, g_3)$, called the Lamé spectral polynomials, where $\ell = n$.

The polynomials $lt_0(B), lt_i(B)$ ($i = 1,2,3$) are written there as $Lt^I_\ell(B; g_2, g_3)$, $Lt^{II}_\ell(B; e_i, g_2, g_3)$, called the twisted Lamé polynomials.

Also $l_\theta(B)$ is written there as $L\theta_\ell(B; g_2, g_3)$.

The compatibility equations from the recursive formulas for these special cases give rise to explicit formulas for $lt_j(B)$'s and $l_\theta(B)$'s. Tables for $lt_0(B)$, $l_\theta(B)$ up to $n = 8$, and for $lt_i(B)$ up to $n = 6$, are given in [12, Table 5, 6].

Example 5.5. We recall Maier's formulas for $lt_j(B)$ and $l_\theta(B)$ for $n \le 4$.

(1) First of all, $l_\theta(B) = 1$ for $n \le 3$. For $n = 4$,

$$l_\theta(B) = B^2 - \tfrac{193}{3} g_2.$$

Also for $n = 1$, $lt_j(B) = 1$ for all j.

(2) $n = 2$: $lt_0(B) = 1$, $lt_i(B) = B - 6e_i$ for $i = 1,2,3$.

(3) $n = 3$: $lt_0(B) = B^2 - \frac{75}{4} g_2$, and for $i = 1,2,3$,

$$lt_i(B) = B^2 - 15 e_i B + \tfrac{75}{4} g_2 - 225 e_i^2.$$

(4) $n = 4$: $lt_0(B) = B^3 - \frac{343}{4} g_2 B - \frac{1715}{2} g_3$. For $i = 1,2,3$,

$$\begin{aligned} lt_i(B) = B^4 &- 55 e_i B^3 + (\tfrac{539}{4} g_2 - 945 e_i^2) B^2 \\ &+ (1960 e_i g_2 + 2450 g_3) B + 61740 e_i^2 g_2 - 68600 e_i g_3 - 9261 g_2^2. \end{aligned}$$

To apply Theorem 5.2, we need to compare the projection map

$$\pi_n : \bar{Y}_n \to E, \qquad a \mapsto \pi_n(a) := a_0. \tag{5.3}$$

with the addition map $\sigma_n : \bar{Y}_n \to E$. They turn out to be the same!

Theorem 5.6. *$\pi_n(a) = \sigma_n(a)$. Moreover, $\kappa(a) = -\mathbf{z}_n(a)$.*

Proof. During the proof we view $a_i \in \mathbb{C}$ instead of its image $[a_i] \in E$.

Let $a \in Y_n$. The two expressions (0.7) and (5.1), which correspond to the same solution to the Lamé equation (0.6), must be proportional to each other by a constant. Hence we get

$$\kappa(a) = \sum_{i=1}^{n} \zeta(a_i) - \zeta(a_0).$$

Recall that $\mathbf{z}_n(a) = \zeta(\sigma_n(a)) - \sum_{i=1}^{n} \zeta(a_i)$. Then

$$\mathbf{z}_n(a) + \kappa(a) = \zeta(\sigma_n(a)) - \zeta(a_0). \tag{5.4}$$

As a well-defined meromorphic function on $\bar{Y}_n$, we conclude that

$$a_0(a) = \sigma_n(a) + c$$

for some constant $c \in \mathbb{C}$. Consider a point $a \in Y_n \setminus X_n$ with $\sigma_n(a) = \frac{1}{2}\omega_1$, i.e. $l_1(B_a) = 0$. Such an a exists by Proposition 2.2. Then $\mathbf{z}_n(a) = 0$ trivially. We also have $\kappa(a) = 0$ by Theorem 5.2 since

$$C_a^2 = c_n^2 l_0(B_a) l_1(B_a) l_2(B_a) l_3(B_a) = 0$$

(again by Proposition 2.2). So equation (5.4) implies $0 = \frac{1}{2}\eta_1 - \zeta(\frac{1}{2}\omega_1 + c)$, and hence $c = 0$. This proves $\sigma_n(a) = a_0$, which represents $\pi_n(a)$ in E, and also $\kappa(a) = -\mathbf{z}_n(a)$. The proof is complete. □

5.2. **Effective construction and explicit formulas for Z_n, $n \le 4$.**

Now we describe an explicit construction, based on Theorem 5.3, of the polynomial $W_n(\mathbf{z})$ in Theorem 3.2. It is an application of elimination theory using resultants.

By Theorem 5.3 and Theorem 5.6, we may eliminate C to get

$$\frac{y_0}{\mathbf{z}_n} = \frac{16}{n^2(n+1)^2(n-1)(n+2)} \frac{lt_1(B) lt_2(B) lt_3(B)}{l_0(B) lt_0(B)^2 l_\theta(B)}, \tag{5.5}$$

which leads to a polynomial equation $g = 0$ for

$$g := \mathbf{z} \prod_{i=1}^{3} lt_i(B) - y_0 \frac{n^2(n+1)^2(n-1)(n+2)}{16} l_0(B) lt_0(B)^2 l_\theta(B). \tag{5.6}$$

On the other hand, the three rational expressions of x_0 lead to $f = 0$ for

$$\begin{aligned} f &:= \frac{1}{3} \sum_{i=1}^{3} l_i(B) lt_i(B)^2 - x_0 \frac{n^2(n+1)^2}{4} l_0(B) lt_0(B)^2 \\ &= l_i(B) lt_i(B)^2 - (x_0 - e_i) \frac{n^2(n+1)^2}{4} l_0(B) lt_0(B)^2, \quad i \in \{1,2,3\}. \end{aligned} \tag{5.7}$$

Notice that f, g are polynomials in g_2, g_3 (and B, x_0, y_0) instead of in e_i's.

Let $R(f, g; B)$ be the *resultant of the two polynomials* f and g arising from the elimination of the variable B. Standard elimination theory (see e.g [8, Chapter 5]) implies that $R(f, g; B)$ gives *the equation* defining the branched covering map $\sigma_n : \bar{Y}_n \to E$ outside the loci $C = 0$:

Proposition 5.7. *$R(f,g;B)(\mathbf{z}) = \lambda_n W_n(\mathbf{z}) \in \mathbb{Q}[g_2,g_3,x_0,y_0][\mathbf{z}]$, where $\lambda_n = \lambda_n(g_2,g_3,x_0,y_0)$ is independent of $\mathbf{z}$.*

In particular, the pre-modular form $Z_n(\sigma;\tau) = W_n(Z)(\sigma;\tau)$ can be explicitly computed for any $n \in \mathbb{N}$ by way of $R(f,g;B)$.

In practice, such a computation is time consuming even using computer. In the following, we apply it to the initial cases up to $n = 4$. As before we denote $x_0 = \wp(\sigma) =: \wp$ and $y_0 = \wp'(\sigma) =: \wp'$.

Example 5.8. For $n = 2$, it is easy to see that

$$\begin{aligned} f &= B^3 - 9\wp B^2 + 27(g_2\wp + g_3), \\ g &= \mathbf{z}B^3 - 9\wp' B^2 - 9\mathbf{z}g_2 B + 27(g_2\wp' - 2\mathbf{z}g_3). \end{aligned}$$

The resultant $R(f,g;B)$ is calculated by the 6×6 *Sylvester determinant*:

$$\begin{vmatrix} 1 & -9\wp & 0 & 27(g_2\wp+g_3) & 0 & 0 \\ 0 & 1 & -9\wp & 0 & 27(g_2\wp+g_3) & 0 \\ 0 & 0 & 1 & -9\wp & 0 & 27(g_2\wp+g_3) \\ \mathbf{z} & -9\wp' & -9\mathbf{z}g_2 & 27(g_2\wp'-2\mathbf{z}g_3) & 0 & 0 \\ 0 & \mathbf{z} & -9\wp' & -9\mathbf{z}g_2 & 27(g_2\wp'-2\mathbf{z}g_3) & 0 \\ 0 & 0 & \mathbf{z} & -9\wp' & -9\mathbf{z}g_2 & 27(g_2\wp'-2\mathbf{z}g_3) \end{vmatrix}.$$

A direct evaluation gives

$$R(f,g;B)(\mathbf{z}) = -3^9\Delta(\wp')^2(\mathbf{z}^3 - 3\wp\mathbf{z} - \wp').$$

Here $\Delta = g_2^3 - 27g_3^2$ is the discriminant. This gives $W_2(\mathbf{z}) = \mathbf{z}^3 - 3\wp\mathbf{z} - \wp'$ and $Z_2(\sigma;\tau) = W_2(Z) = Z^3 - 3\wp Z - \wp'$.

Example 5.9. For $n = 3$, we have

$$\begin{aligned} f &= 16B^6 - 576B^5\wp + 360B^4 g_2 + 5400B^3(5g_3 + 4g_2\wp) \\ &\quad - 3375B^2g_2^2 - 84375\Delta - 101250Bg_2(3g_3 + 2g_2\wp), \\ g &= 16B^6\mathbf{z} - 1440B^5\wp' - 1800B^4g_2\mathbf{z} + 54000B^3(g_2\wp' - g_3\mathbf{z}) \\ &\quad - 16875B^2g_2^2\mathbf{z} - 506250Bg_2^2\wp' + 421875\Delta\mathbf{z}. \end{aligned}$$

It takes a couple seconds to evaluate the corresponding 12×12 Sylvester determinant (in *Mathematica*) to get

$$R(f,g;B)(\mathbf{z}) = 2^{36}3^{27}5^{30}\Delta^5(\wp')^4 W_3(\mathbf{z}),$$

where $W_3(\mathbf{z})$ is given by

$$W_3(\mathbf{z}) = \mathbf{z}^6 - 15\wp\mathbf{z}^4 - 20\wp'\mathbf{z}^3 + (\tfrac{27}{4}g_2 - 45\wp^2)\mathbf{z}^2 - 12\wp\wp'\mathbf{z} - \tfrac{5}{4}\wp'^2.$$

It seems impractical to compute this resultant by hand.

Both Z_2 and Z_3 are known to Dahmen [3]. Here is a new example:

Example 5.10. For $n = 4$, the expansions of the polynomials f and g, as given in (5.7) and (5.6) by a direct substitution, are already too long to put here. Nevertheless, a couple hours *Mathematica* calculation gives

$$R(f,g;B)(\mathbf{z}) = -2^{80}3^{63}5^{60}7^{63}\Delta^{18}(\wp')^8 W_4(\mathbf{z}),$$

where $W_4(\mathbf{z})$ is the degree 10 polynomial:

$$\begin{aligned} W_4(\mathbf{z}) = {}& \mathbf{z}^{10} - 45\wp\mathbf{z}^8 - 120\wp'\mathbf{z}^7 + (\tfrac{399}{4}g_2 - 630\wp^2)\mathbf{z}^6 - 504\wp\wp'\mathbf{z}^5 \\ & - \tfrac{15}{4}(280\wp^3 - 49g_2\wp - 115g_3)\mathbf{z}^4 + 15(11g_2 - 24\wp^2)\wp'\mathbf{z}^3 \\ & - \tfrac{9}{4}(140\wp^4 - 245g_2\wp^2 + 190g_3\wp + 21g_2^2)\mathbf{z}^2 \\ & - (40\wp^3 - 163g_2\wp + 125g_3)\wp'\mathbf{z} + \tfrac{3}{4}(25g_2 - 3\wp^2)(\wp')^2. \end{aligned} \tag{5.8}$$

The weight 10 pre-modular form $Z_4(\sigma;\tau)$ is then obtained.

5.3. **Remarks on rationality and singularities of the Lamé curve.**

We have constructed two affine curves from $\bar{X}_n$. One is the hyperelliptic model $Y_n = \{(B,C) \mid C^2 = \ell_n(B)\}$, another one is $Y'_n := \{(x_0,y_0,\mathbf{z}) \mid y_0^2 = 4x_0^2 - g_2x_0 - g_3,\ W_n(x_0,y_0;\mathbf{z}) = 0\}$ which is a degree $\frac{1}{2}n(n+1)$ branched cover of the original curve $E = \{(x_0,y_0) \mid y_0^2 = 4x_0^3 - g_2x_0 - g_3\}$ under the projection $\sigma'_n : Y'_n \to E$ with defining equation $W_n(\mathbf{z}) = 0$.

The curve Y_n is birational to Y'_n over E, namely the addition map $\sigma_n : Y_n \to E$ is compatible with $\sigma'_n : Y'_n \to E$. Notice that both ℓ_n and W_n have coefficients in $\mathbb{Q}[g_2,g_3]$. The explicit birational map $\phi : (B,C) \dashrightarrow (x_0,y_0,\mathbf{z})$ (given in Theorem 5.3 and 5.6 via $\mathbf{z}_n = -\kappa$) also has coefficients in $\mathbb{Q}[g_2,g_3]$. This implies that ϕ is defined over $\mathbb{Q}$. Moreover ϕ extends to a *birational morphism*

$$\begin{tikzcd} \bar{Y}_n \cong \bar{X}_n \arrow[rr, "\phi"] \arrow[dr, "\sigma_n"'] & & \bar{Y}'_n \arrow[dl, "\sigma'_n"] \\ & E & \end{tikzcd}$$

by identifying $\sigma_n^{-1}(0_E)$ with $\mathbf{z}_n^{-1}(\infty)$. The morphism ϕ is an isomorphism outside those branch points for $Y_n \to \mathbb{P}^1(\mathbb{C})$ (i.e. $C = 0$). In particular, the non-isomorphic loci lie in $\mathbf{z}_n = 0$ by formulas (5.2) and Theorem 5.6.

Remark 5.11. In contrast to the smoothness of $Y_n(\tau)$ for general τ, for all $n \geq 3$ the model $Y'_n(\tau)$ is singular at points $\mathbf{z} = 0 = y_0$ (and hence $x_0 = e_i$ for some i). Indeed from (5.2) this is equivalent to $C = 0$ and $l_i(B)lt_i(B)^2 = 0$ for some $1 \leq i \leq 3$. For $n = 2$, there is only one solution B for each fixed i (cf. Example 5.5). However, for $n \geq 3$ there are more than one such solutions B. These points $(B,0) \in Y_n$ are collapsed to the same point $(x_0,y_0,\mathbf{z}) = (e_i,0,0) \in Y'_n$ under ϕ, thus $(e_i,0,0)$ is a singular point of Y'_n.

For $n = 3,4$ this is easily seen from the equation $W_n(\mathbf{z}) = 0$ given above since it contains a quadratic polynomial in $(\mathbf{z},\wp')$ as its lowest degree terms.

In particular, the birational map ϕ^{-1} is also represented by rational functions $B = B(x_0,y_0,\mathbf{z})$ and $C = C(x_0,y_0,\mathbf{z})$ with coefficients in $\mathbb{Q}[g_2,g_3]$ and with at most poles along $\mathbf{z} = 0$. In principle such an explicit inverse can be obtained by a Groebner basis calculation associated to the ideal of the graph Γ_ϕ. The following statement is clear from the above description:

Proposition 5.12. *Let E be defined over $\mathbb{Q}$, i.e. $g_2, g_3 \in \mathbb{Q}$. Then the Lamé curve $\bar{Y}_n$ is also defined over $\mathbb{Q}$ for all $n \in \mathbb{N}$. Moreover, $\bar{Y}'_n$ and all the morphisms $\sigma_n, \sigma'_n, \phi$ are also defined over $\mathbb{Q}$.*

A rational point $(B, C) \in \bar{Y}_n$ is mapped to a rational point $(x_0, y_0, \mathbf{z}) \in \bar{Y}'_n$ by ϕ. For the converse, given $(x_0, y_0) \in E(\mathbb{Q})$, a point $(x_0, y_0, \mathbf{z})$ in the σ'_n-fiber gives a unique $(B, C) \in \bar{Y}_n(\mathbb{Q})$ if $\mathbf{z} \in \mathbb{Q}$ and $(x_0, y_0, \mathbf{z}) \neq (e_i, 0, 0)$ for any i.

Remark 5.13. It is well known that there are only few (i.e. at most finite) rational points on a *non-elliptic* hyperelliptic curve. This phenomenon is consistent with the irreducibility of the polynomial $W_n(\mathbf{z})$ over $K(E)$ in light of Hilbert's irreducibility theorem that there is an infinite (Zariski dense) set of $(g_2, g_3, x_0, y_0) \in \mathbb{Q}^4$ so that the specialization of $W_n(\mathbf{z})$ over there are all irreducible. Nevertheless, it might be interesting to see if $\mathbf{z}_n$ plays any role in the study of rational points.

Remark 5.14. It was proved in [12, Proposition 3.2] that *a Lamé curve is either smooth or nodal, and there is at most one node*. The proof relies on the degree formula $\deg \pi_n = \frac{1}{2}n(n+1) = \deg \kappa$ which was quoted as *a significant formula from finite-band integration theory* without explicit references in [12, p.1139] . While this might be well-known to experts in this field, we want to point out that it also follows from Theorem 5.6 and our degree formula $\deg \sigma_n = \frac{1}{2}n(n+1) = \deg \mathbf{z}_n$ in Theorem 2.4 and Theorem 3.13.

Appendix A. A counting formula for Lamé equations

By You-Cheng Chou [6]

Using the pre-modular forms constructed in §4 and §5, we verify the $n = 4$ case of Dahmen's conjectural counting formula [3, Conjecture 73] for integral Lamé equations with finite monodromy. It is known that the finite monodromy group is necessarily a dihedral group.

For $N \in \mathbb{N}$ we denote by $\phi(N) := \#\{k \in \mathbb{Z} \mid \gcd(k, N) = 1,\ 0 \le k < N\}$ the Euler function and we set $\phi(N) = 0$ if $N \notin \mathbb{N}$. Similarly we define $\Psi(N) := \#\{(k_1, k_2) \mid \gcd(k_1, k_2, N) = 1,\ 0 \le k_i < N\}$.

A.1. **Dahmen's conjecture.**

Let $L_n(N)$ be the number of Lamé equations $w'' = (n(n+1)\wp(z) + B)w$, up to linear equivalence, which has finite monodromy group isomorphic to the dihedral group D_N. Using the Hermite–Halphen ansatz (0.7) and the theory in §4, the problem is reduced to the zero counting of the following $\mathrm{SL}(2, \mathbb{Z})$ modular form

$$M_n(N) := \prod_{\substack{0 \le k_1, k_2 < N \\ \gcd(k_1, k_2, N) = 1}} Z_n\Big(\frac{k_1 + k_2\tau}{N}; \tau\Big).$$

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Using this, by repeating Dahmen's argument in [3, Lemma 65, 74], we get

Proposition A.1. *Suppose that for all $N \in \mathbb{Z}_{\geq 3}$ and $n \in \mathbb{N}$ we have that*

$$\nu_\infty(M_n(N)) = a_n\phi(N) + b_n\phi\Big(\frac{N}{2}\Big),$$

where $a_{2m} = a_{2m+1} = m(m+1)/2$, $b_{2m} = b_{2m-1} = m^2$. Then

$$L_n(N) = \tfrac{1}{2}\left(\frac{n(n+1)\Psi(N)}{24} - \left(a_n\phi(N) + b_n\phi\Big(\frac{N}{2}\Big)\right)\right) + \tfrac{2}{3}\epsilon_n(N),$$

where $\epsilon_n(N) = 1$ if $N = 3$ and $n \equiv 1 \pmod 3$, and $\epsilon_n(N) = 0$ otherwise.

Furthermore, $Z_n(\sigma;\tau)$ with σ a torsion point has only simple zeros in $\tau \in \mathbb{H}$.

Proof. Recall the formula for $\mathrm{SL}(2,\mathbb{Z})$ modular forms of weight k:

$$\sum_{P\neq\infty, i, \rho} \nu_P(f) + \nu_\infty(f) + \frac{\nu_i(f)}{2} + \frac{\nu_\rho(f)}{3} = \frac{k}{12}.$$

The modular form $f = M_n(N)$ has weight $k = \frac{1}{2}n(n+1)\Psi(N)$. Notice that the counting is always doubled under the symmetry $(k_1, k_2) \to (N - k_1, N - k_2)$, thus by [3, Lemma 65] an upper bound for $L_n(N)$ is given by

$$U_n(N) := \tfrac{1}{2}\left(\frac{n(n+1)\Psi(N)}{24} - \left(a_n\phi(N) + b_n\phi\Big(\frac{N}{2}\Big)\right)\right) + \tfrac{2}{3}\epsilon_n(N).$$

That is, $L_n(N) \leq U_n(N)$. Moreover, the equality holds if and only if each factor $Z_n((k_1 + k_2\tau)/N; \tau)$ has only simple zeros.

We will show the equality holds by comparing it with the counting formula for the projective monodormy group $PL_n(N)$ (cf. [3, Lemma 74]).

We recall the relation between $L_n(N)$ and $PL_n(N)$:

$$PL_n(N) = \begin{cases} L_n(N) + L_n(2N) & \text{if } N \text{ is odd,} \\ L_n(2N) & \text{if } N \text{ is even.} \end{cases}$$

If n is even and N is odd, we have

$$\begin{aligned} PL_n(N) &= L_n(N) + L_n(2N) \\ &\leq \tfrac{1}{2}\left(\frac{n(n+1)\Psi(N)}{24} - \left(\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}\phi(N) + \frac{n^2}{4}\phi\Big(\frac{N}{2}\Big)\right)\right) + \tfrac{2}{3}\epsilon_n(N) \\ &\quad + \tfrac{1}{2}\left(\frac{n(n+1)\Psi(2N)}{24} - \left(\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}\phi(2N) + \frac{n^2}{4}\phi(N)\right)\right) + \tfrac{2}{3}\epsilon_n(2N) \\ &= \frac{n(n+1)}{12}\left(\Psi(N) - 3\phi(N)\right) + \tfrac{2}{3}\epsilon_n(N) \end{aligned}$$

For the last equality, we use $\epsilon_n(2N) = 0$, $\Psi(2N) = 3\Psi(N)$ and $\phi(2N) = \phi(N)$. (If N is even, the relations are $\epsilon_n(N) = 0$, $\Psi(2N) = 4\Psi(N)$ and $\phi(2N) = \phi(N)$.) For the other three cases with (n, N) being (even, even), (odd, odd) or (odd, even), the computations are similar. They lead to

$$PL_n(N) \leq \frac{n(n+1)}{12}\left(\Psi(N) - 3\phi(N)\right) + \tfrac{2}{3}\epsilon_n(N).$$

On the other hand, using the method of *dessin d'enfants*, Dahmen showed directly that the equality holds [4]. Thus all the intermediate inequalities are indeed equalities, and in particular $L_n(N) = U_n(N)$ holds. $\square$

A.2. **q-expansions for some modular forms.**

Recall that

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \frac{1}{(m+z)^k} &= \frac{1}{(k-1)!}(-2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n z},\\
\sum_{n\in\mathbb{Z}} \frac{1}{(x+n)^2} &= \pi^2 \cot^2(\pi x) + \pi^2,\\
\sum_{n\in\mathbb{Z}} \frac{1}{(x+n)^3} &= \pi^3 \cot^3(\pi x) + \pi^3 \cot(\pi x).
\end{aligned}$$

We compute the q-expansions for $g_2, g_3, \wp, \wp', Z$, where $q = e^{2\pi i \tau}$:

$$g_2 = 60 \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^4} = 60\Big(2\zeta(4) + 2\frac{(-2\pi i)^4}{3!} \sum_{n=1}^{\infty} \sigma_3(n) q^n\Big),$$

where $\sigma_k(n) := \sum_{d|n} d^k$. Similarly,

$$g_3 = 140 \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^6} = 140\Big(2\zeta(6) + 2\frac{(-2\pi i)^6}{5!} \sum_{n=1}^{\infty} \sigma_5(n) q^n\Big).$$

Let $z = r + s\tau$. For $s = 0$, we have

$$\begin{aligned}
\wp'(r;\tau) &= -2 \sum_{n,m\in\mathbb{Z}} \frac{1}{(r+n+m\tau)^3}\\
&= -2 \sum_{n\in\mathbb{Z}} \frac{1}{(r+n)^3} - 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \left(\frac{1}{(m\tau+n+r)^3} - \frac{1}{(m\tau+n-r)^3}\right)\\
&= -2 \sum_{n\in\mathbb{Z}} \frac{1}{(r+n)^3} - 2\sum_{m=1}^{\infty} \frac{(-2\pi i)^3}{2!} \sum_{n=1}^{\infty} n^2 \left(e^{2\pi i n(m\tau+r)} - e^{2\pi i n(m\tau-r)}\right)\\
&= -2\pi^3 \cot(\pi r) - 2\pi^3 \cot^3(\pi r) + 16\pi^3 \sum_{n,m=1}^{\infty} n^2 \sin(2\pi n r)\, q^{nm}.
\end{aligned}$$

$$\begin{aligned}
\wp(r;\tau) &= \frac{1}{r^2} + \sum_{(n,m)\neq(0,0)} \left(\frac{1}{(r+n+m\tau)^2} - \frac{1}{(n+m\tau)^2}\right)\\
&= \sum_{n\in\mathbb{Z}} \frac{1}{(r+n)^2} - \sum_{n=1}^{\infty} \frac{2}{n^2} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \left(\frac{1}{(m\tau+r+n)^2} + \frac{1}{(m\tau-r+n)^2} - \frac{2}{(m\tau+n)^2}\right)\\
&= \pi^2 \cot^2(\pi r) + \tfrac{2}{3}\pi^2 + \sum_{m=1}^{\infty} (-2\pi i)^2 \sum_{n=1}^{\infty} \left(e^{2\pi i n(m\tau+r)} + e^{2\pi i n(m\tau-r)} - 2e^{2\pi i n m\tau}\right)\\
&= \pi^2 \cot^2(\pi r) + \tfrac{2}{3}\pi^2 + 8\pi^2 \sum_{n,m=1}^{\infty} (1 - \cos 2n\pi r) q^{nm}.
\end{aligned}$$

Also, for the Hecke function Z (cf. (0.10)) we have

$$Z(r;\tau) = \pi\cot(\pi r) + 4\pi \sum_{n,m=1}^{\infty} (\sin 2n\pi r) q^{nm}.$$

For $s = \frac{1}{2}$, we have

$$\begin{aligned}
\wp'(r+\tfrac{1}{2}\tau;\tau) &= -2 \sum_{(n,m)\neq(0,0)} \frac{1}{(r+n+(\frac{1}{2}+m)\tau)^3} \\
&= -2\sum_{m=1}^{\infty}\left(\sum_{m\in\mathbb{Z}} \frac{1}{(n+r+(m-\frac{1}{2})\tau)^3} - \sum_{n\in\mathbb{Z}} \frac{1}{(n-r+(m-\frac{1}{2})\tau)^3}\right) \\
&= -2\frac{(-2\pi i)^3}{2!} \sum_{n,m=1}^{\infty} n^2 \left(e^{2\pi i n(r+(m-\frac{1}{2})\tau)} - e^{2\pi i n(-r)+(m-\frac{1}{2})\tau}\right) \\
&= 16\pi^3 \sum_{n,m=1}^{\infty} n^2 (\sin 2\pi n r) q^{n(m-\frac{1}{2})}.
\end{aligned}$$

Similarly,

$$\wp(r+\tfrac{1}{2}\tau;\tau) = -\tfrac{1}{3}\pi^2 + 8\pi^2 \sum_{n,m=1}^{\infty} n q^{nm} - 8\pi^2 \sum_{n,m=1}^{\infty} n(\cos 2\pi n r) q^{n(m-\frac{1}{2})},$$

and $Z(r+\frac{1}{2}\tau;\tau) = 4\pi\sum_{n,m=1}^{\infty}(\sin 2\pi n r) q^{n(m-\frac{1}{2})}$.

A.3. **The counting formula for $n = 4$.**

Now we give the computations for $n = 4$ and prove the formula $L_4(N) = U_4(N)$ from Proposition A.1.

Theorem A.2. *For $n = 4$ and $N \in \mathbb{Z}_{\geq 3}$, we have*

$$L_4(N) = \tfrac{1}{2}\left(\tfrac{5}{6}\Psi(N) - \left(3\phi(N) + 4\phi\left(\frac{N}{2}\right)\right)\right).$$

Moreover, $Z_4(\sigma;\tau)$ with $\sigma \in E_\tau[N]$ has only simple zeros in $\tau \in \mathbb{H}$.

Proof. For $n = 4$, the pre-modular form $Z_4 = W_4(Z)$ is given in (5.8):

$$\begin{aligned}
W_4(Z) = Z^{10} &- 45\wp Z^8 - 120\wp' Z^7 + (\tfrac{399}{4}g_2 - 630\wp^2)Z^6 - (504\wp\wp')Z^5 \\
&- \tfrac{15}{4}(280\wp^3 - 49g_2\wp - 115g_3)Z^4 + 15(11g_2 - 24\wp^2)\wp' Z^3 \\
&- \tfrac{9}{4}(140\wp^4 - 245g_2\wp^2 + 190g_3\wp + 21g_2^2)Z^2 \\
&- (40\wp^3 - 163g_2\wp + 125g_3)\wp Z + \tfrac{3}{4}(25g_2 - 3\wp^2)\wp'^2,
\end{aligned}$$

where Z is the Hecke function. We compute the asymptotic behavior of $W_4(Z)$ when $\tau \to \infty$. Let $z = r + s\tau$. We divide the problem into two cases

(1) $s \equiv 0 \pmod 1$: according to the q-expansion given in §A.2, we have

$$g_2 \to \tfrac{3}{4}\pi^4, \qquad g_3 \to \tfrac{8}{27}\pi^6, \qquad Z(z) \to \pi\cot(\pi r),$$

$$\wp'(z) \to -2\pi^3\cot(\pi r) - 2\pi^3\cot^3(\pi r), \qquad \wp(z) \to \pi^2\cot^2(\pi r) + \tfrac{2}{3}\pi^2.$$

A direct computation shows that $W_4(Z)$ has a zero at ∞ when $s = 0$.

By replacing all the modular forms $g_2, g_3, \wp, \wp'$ and Z appeared in $W_4(Z)$ with their q-expansions, we have (e.g. using *Mathematica*)

$$W_4(Z) = 2^{14}3^3 5^2 7\, \pi^{10} \cos^2(\pi r) \sin^2(\pi r) q^3 + O(q^4)$$

(2) $s \not\equiv 0 \pmod 1$: in this case we have

$$Z \to 2\pi i \left(s - \tfrac{1}{2}\right), \qquad \wp(z) \to -\tfrac{1}{3}\pi^2,$$
$$\wp'(z) \to 0, \qquad g_2 \to \tfrac{4}{3}\pi^4, \qquad g_3 \to \tfrac{8}{27}\pi^6.$$

Hence the constant term of $W_4(Z)$ is given by

$$\begin{aligned} W_4(z) = &-64\pi^{10}(-2+s)(-1+s)^2 s^2(1+s) \\ &\times (-3+2s)(-1+2s)^2(1+2s) + O(q). \end{aligned}$$

If $s \not\equiv 0 \pmod 1$ then $W_4(Z)$ has a zero at $\tau = \infty \iff s \equiv \frac{1}{2} \pmod 1$.

Now we fix $s = \frac{1}{2}$ and replace the modular forms $g_2, g_3, \wp, \wp'$ and Z appeared in $W_4(Z)$ with their q-expansions. We get

$$W_4(Z) = 2^{10}3^3 5^2 7\, \pi^{10} \cos^2(\pi r) \sin^2(\pi r) q^2 + O(q^3).$$

These computations for the q-expansions imply that

$$\begin{aligned} \nu_\infty(M_4(N)) &= 3\,\#\{\, 1 \le k_1 \le N \mid \gcd(N, k_1) = 1 \,\} \\ &\quad + 2\,\#\{\, 0 \le k_1 \le N \mid \gcd(N/2, k_1) = 1 \,\} \\ &= 3\phi(N) + 4\phi(N/2). \end{aligned}$$

Since the value of $\nu_\infty(M_4(N))$ coincides with the assumption in Proposition A.1 for $n = 4$, the theorem follows from it accordingly. $\square$

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