MEAN FIELD EQUATIONS, HYPERELLIPTIC CURVES AND MODULAR FORMS: II

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ABSTRACT. A *pre-modular form* $Z_n(\sigma;\tau)$ of weight $\frac{1}{2}n(n+1)$ is introduced for each $n \in \mathbb{N}$, where $(\sigma,\tau) \in \mathbb{C} \times \mathbb{H}$, such that for $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, every non-trivial zero of $Z_n(\sigma;\tau)$, namely $\sigma \notin E_\tau[2]$, corresponds to a (scaling family of) solution to the mean field equation

$$\text{(MFE)} \qquad \triangle u + e^u = \rho\,\delta_0$$

on the flat torus E_τ with singular strength $\rho = 8\pi n$.

In Part I [3], a hyperelliptic curve $\bar{X}_n(\tau) \subset \operatorname{Sym}^n E_\tau$, the *Lamé curve*, associated to the MFE was constructed. Our construction of $Z_n(\sigma;\tau)$ relies on a detailed study on the correspondence $\mathbb{P}^1 \leftarrow \bar{X}_n(\tau) \to E_\tau$ induced from the hyperelliptic projection and the addition map.

As an application of the explicit form of the weight 10 pre-modular form $Z_4(\sigma;\tau)$, a counting formula for Lamé equations of degree $n=4$ with finite monodromy is given in the appendix (by Y.-C. Chou).

CONTENTS

0. INTRODUCTION

Let $E = E_\tau = \mathbb{C}/\Lambda_\tau$, $\tau \in \mathbb{H} = \{\, \tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0 \,\}$ and $\Lambda = \Lambda_\tau = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1 = 1$ and $\omega_2 = \tau$. In this paper, we continue our study, initiated in [10, 3], on the singular Louville (mean field) equation:

$$\triangle u + e^u = 8\pi n\,\delta_0 \quad \text{on } E, \tag{0.1}$$

under the flat metric, with δ_0 being the Dirac measure at $0 \in E$. The characteristic feature of this problem is that its solvability depends on the moduli τ in a sophisticated manner (even for $n = 1$, cf. [10]).

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It was shown in [3, §0.2.5, Theorem 0.3] that any solution to (0.1) lies in a *scaling family of solutions* u^λ through the Liouville formula:

$$u^\lambda(z) = \log \frac{8e^{2\lambda}|f'(z)|^2}{(1+e^{2\lambda}|f(z)|^2)^2}, \quad \lambda \in \mathbb{R}, \tag{0.2}$$

where the meromorphic function f on $\mathbb{C}$ is known as a *developing map* which can be chosen to be *even* and and satisfy the *type II constraints*:

$$f(z+\omega_j) = e^{2i\theta_j} f(z), \quad \theta_j \in \mathbb{R}, \quad j = 1, 2. \tag{0.3}$$

This is also known as the *unitary projective monodromy* condition.

f has precisely n simple zeros in $E^\times$ characterized by [3, Theorem 0.6]:

Thee n zeros $a_1, \ldots, a_n \in E^\times$ of f satisfy $a_i \ne \pm a_j$ for $i \ne j$, and they are completely determined by the $n-1$ algebraic equations

$$\sum_{i=1}^n \wp'(a_i)\wp^r(a_i) = 0, \qquad r = 0, \ldots, n-2, \tag{0.4}$$

together with the transcendental equation on Green function

$$\sum_{i=1}^n \nabla G(a_i) = 0. \tag{0.5}$$

Following [3], the affine algebraic curve $X_n \subset \mathrm{Sym}^n E^\times$ defined by equations (0.4) and $a_i \ne \pm a_j$ for $i \ne j$ is called the (n-th) *Liouville curve*.

We will make use of Weierstrass' elliptic function $\wp(z) = \wp(z; \Lambda)$ and its associated ζ, σ functions extensively. We use [15] as a general reference.

The Green function on E is defined by $-\triangle G = \delta_0 - 1/|E|$ and $\int_E G = 0$. For $z = x + iy = r\omega_1 + s\omega_2$, $r, s \in \mathbb{R}$, and $\eta_i = 2\zeta(\frac{1}{2}\omega_i)$, $i = 1, 2$, being the quasi-periods, it was shown in [10, Lemma 2.3, Lemma 7.1] that

$$-4\pi G_z(z;\tau) = \zeta(z;\tau) - r\eta_1(\tau) - s\eta_2(\tau). \tag{0.6}$$

For $z \in E_\tau[N]$, the N torsion points, this first appeared in [7] where Hecke showed that it is a modular form of weight one with respect to $\Gamma(N) = \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv I_2 \pmod N\}$. Thus we call

$$Z(z;\tau) = Z_{r,s}(\tau) := \zeta(r\omega_1 + s\omega_2;\tau) - r\eta_1(\tau) - s\eta_2(\tau), \tag{0.7}$$

$(z,\tau) \in \mathbb{C} \times \mathbb{H}$ the *Hecke function*, which is holomorphic *only* in τ. In this paper, analytic functions of this sort are called *pre-modular forms*.

The notion of pre-modular forms allows us to study *deformations* in σ to relate *different* modular forms. Recently this idea was successfully applied in [2] to give a complete solution to (0.1) for $n = 1$. In that case (0.4) is empty and the problem is equivalent to solving *non-trivial zeros* of $Z(z;\tau)$, i.e. $z \notin E_\tau[2]$. Thus, a key step towards the general cases is to generalize the pre-modular form $Z = Z_1$ to the corresponding Z_n for all $n \ge 2$.

Our starting point is the *hyperelliptic geometry* on X_n arising from the integral *Lamé equations* on E_τ [3, Theorem 0.7]:

$$w'' = (n(n+1)\wp + B)w. \tag{0.8}$$

For $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, let $w_a(z)$ be the classical *Hermite–Halphen ansatz*:

$$w_a(z) := e^{z \sum \zeta(a_i;\tau)} \prod_{i=1}^{n} \frac{\sigma(z - a_i;\tau)}{\sigma(z;\tau)}. \tag{0.9}$$

Denote $[a] := a \pmod \Lambda$*. Then* $[a] \in X_n$ *if and only if* w_a *and* w_{-a} *are independent solutions to* (0.8)*. In that case, the parameter B equals*

$$B_a := (2n-1) \sum_{i=1}^{n} \wp(a_i). \tag{0.10}$$

The compactified curve $\bar{X}_n \subset \mathrm{Sym}^n E$ *is a hyperelliptic curve, known as the* Lamé curve, *with the addd points* $\bar{X}_n \setminus X_n$ *being the branch points of the hyperelliptic projection* $B : \bar{X}_n \to \mathbb{P}^1$*. The* point at infinity $0^n \in \bar{X}_n$ *is always smooth. The finite branch points satisfy* $a \in (E^\times)^n$*,* $a_i \neq a_j$ *for* $i \neq j$*, and* $\{a_1, \cdots, a_n\} = \{-a_1, \cdots, -a_n\}$*;* $w_a = w_{-a}$ *is still a solution to* (0.8) *with* $B = B_a$*. These solutions are known as the* Lamé functions.

Let $Y_n = B^{-1}(\mathbb{C})$ *be the finite part of* $\bar{X}_n$*.* Y_n *can be parametrized by*

$$Y_n \cong \{(B, C) \mid C^2 = \ell_n(B)\}$$

where $\ell_n(B)$ *is the Lamé polynomial in B of degree* $2n+1$*.* $\bar{X}_n$ *is smooth if and only if* $\ell_n(B)$ *has no multiple roots.*

Further technical details needed from [3, Theorem 0.7] are summarized in Proposition 1.1 and Theorem 1.2.

By the anti-symmetry of ∇G, (0.5) holds automatically on the branch points of Y_n, hence they are referred as *trivial solutions*. We will construct a pre-modular form $Z_n(\sigma;\tau)$ with $\sigma \in E_\tau$ which is naturally associated to the family of hyperelliptic curves $\bar{X}_n(\tau)$, $\tau \in \mathbb{H}$. The goal is to show that any non-trivial solution $a = \{a_1, \cdots, a_n\} \in X_n$ to (0.5) comes from the zero of $Z_n(\sigma;\tau)$ with $\sigma = \sum_{i=1}^n a_i \notin E_\tau[2]$, and vice versa.

Consider the meromorphic function

$$\mathbf{z}_n(a) := \zeta\Big(\sum_{i=1}^{n} a_i\Big) - \sum_{i=1}^{n} \zeta(a_i) \tag{0.11}$$

on E^n. If $\sum_{i=1}^n a_i \neq 0$ then

$$-4\pi \sum \nabla G(a_i) = \sum (\zeta(r_i\omega_1 + s_i\omega_2) - r_i\eta_1 - s_i\eta_2) = Z(\textstyle\sum a_i) - \mathbf{z}_n(a).$$

Hence the Green function equation (0.5) is equivalent to

$$\mathbf{z}_n(a) = Z\Big(\sum_{i=1}^{n} a_i\Big). \tag{0.12}$$

This motivates us to study the map

$$\sigma_n : \bar{X}_n \to E, \qquad a \mapsto \sigma_n(a) := \sum_{i=1}^{n} a_i \tag{0.13}$$

induced from the addition map $E^n \to E$. The algebraic curve $\bar{X}_n(\tau)$ might be singular for some τ, but it must be irreducible (c.f. Theorem 1.2 (3)). In particular, σ_n is a finite morphism and $\deg \sigma_n$ is defined.

Recall that a *node* is a singularity of the simplest analytic type $y^2 = x^2$.

Theorem 0.1 (= Theorem 1.3 + Theorem 1.6)**.** *The Lamé curve $\bar{X}_n$ has at most nodal singularities. Moreover, the map $\sigma_n : \bar{X}_n \to E$ has degree $\frac{1}{2}n(n+1)$.*

From Theorem 0.1, there is a polynomial

$$W_n(\mathbf{z}) \in \mathbb{Q}[g_2, g_3, \wp(\sigma), \wp'(\sigma)][\mathbf{z}]$$

of degree $\frac{1}{2}n(n+1)$ in $\mathbf{z}$ which defines the (branched) covering map σ_n. Throughout the paper we use σ as the coordinate on E in $\sigma_n : \bar{X}_n \to E$ and this should not be confused with the Weierstrass σ function.

The next task is to find a natural primitive element of this covering map, namely a rational function on $\bar{X}_n$ which has W_n as its minimal polynomial. This is achieved by the following fundamental theorem:

Theorem 0.2. *The rational function $\mathbf{z}_n \in K(\bar{X}_n)$ is a primitive generator for the field extension $K(\bar{X}_n)$ over $K(E)$ which is integral over the affine curve $E^\times$.*

This means that $W_n(\mathbf{z}_n) = 0$, and conversely for general τ and $\sigma = \sigma_0 \in E_\tau$, the roots of $W_n(\mathbf{z})(\sigma_0; \tau) = 0$ are precisely those $\frac{1}{2}n(n+1)$ values $\mathbf{z} = \mathbf{z}_n(a)$ with $\sigma_n(a) = \sigma_0$. The proof is given in §2, Theorem 2.2.

A major tool used is the *tensor product* of two Lamé equations $w'' = I_1 w$ and $w' = I_2 w$, where $I = n(n+1)\wp(z)$, $I_1 = I + B_a$ and $I_2 = I + B_b$. For a general point $\sigma_0 \in E$, we need to show that the $\frac{1}{2}n(n+1)$ points on the fiber of $\bar{X}_n \to E$ above σ_0 has distinct $\mathbf{z}_n$ values. From (0.11), it is enough to show that for $\sigma_n(a) = \sigma_n(b) = \sigma_0$, $\sum \zeta(a_i) = \sum \zeta(b_i)$ implies $B_a = B_b$. Since then $a = b$ if $\sigma_0 \not\in E[2]$.

If $w_1'' = I_1 w_1$ and $w_2'' = I_2 w_2$, then the product $q = w_1 w_2$ satisfies the fourth order ODE (tensor product) given by

$$q'''' - 2(I_1 + I_2)q'' - 6I'q' + ((B_a - B_b)^2 - 2I'')q = 0. \tag{0.14}$$

We remark that if $B_a = B_b$, then $I_1 = I_2$ and q actually satisfies a third order ODE as the *second symmetric product* of a Lamé equation. This is a useful tool in Part I [3] in the study of the Lamé curve.

If however $a \neq b$, by (0.9) and addition law, $q = w_a w_{-b} + w_{-a} w_b$ is an *even elliptic function* solution to (0.14), namely a *polynomial* in $x = \wp(z)$. This leads to strong constraints on (0.14) in variable x and eventually leads to a contradiction for generic choices of σ_0.

Now we set (cf. Corollary 3.1)

$$Z_n(\sigma; \tau) := W_n(Z)(\sigma; \tau). \tag{0.15}$$

Then $Z_n(\sigma;\tau)$ is pre-modular of weight $\frac{1}{2}n(n+1)$. From the construction and (0.12) it is readily seen that $Z_n(\sigma;\tau)$ is the generalization of the Hecke function we are looking for. In fact, for $n \geq 1$, we have

Theorem 0.3. *Solutions to the singular Liouville equation* (0.1) *correspond to zeros of pre-modular form* $Z_n(\sigma;\tau)$ *in* (0.15) *with* $\sigma \notin E_\tau[2]$.

We will also present a version of Theorem 0.3 in terms of *monodromy groups* of Lamé equations (cf. Theorem 3.5).

For $\sigma \in E_\tau[N]$, the N-torsion points, the modular form $Z_2(\sigma;\tau)$ and $Z_3(\sigma;\tau)$ were first constructed by Dahmen [4] in his study on integral Lamé equations (0.8) with algebraic solutions (i.e. with finite monodromy group). For $n \geq 4$, the existence of a modular form $Z_n(\sigma;\tau)$ of weight $\frac{1}{2}n(n+1)$ was also conjectured in [4]. This is now settled by our results.

It remains to find effective and explicit constructions of Z_n. Since σ is defined by the addition map, which is purely algebraic, in principle this allows us to compute the polynomial $W_n(\mathbf{z})$ for any $n \in \mathbb{N}$ by eliminating variables B and C, though in practice the needed calculations are very demanding and time consuming.

In a different direction, the Lamé curve had also been studied extensively in the *finite band integration theory*. In the complex case, this theory concerns about the eigenvalue problem on a second order ODE $Lw := w'' - Iw = Bw$ with eigenvalue B. The potential $I = I(z)$ is called a *finite-gap (band) potential* if the ODE has only logarithmic free solutions except for a finite number of $B \in \mathbb{C}$. The integral Lamé equations (with $I(z) = n(n+1)\wp(z)$) provide good (indeed earliest) examples of them. Using this theory, Maier [13] had recently written down *an explicit map* $\pi_n : \bar{X}_n \to E$ in terms of coordinate (B, C) on $\bar{X}_n$ (in our notations). It turns out we can prove

Theorem 0.4 (c.f. Theorem 4.5)**.** *The map* π_n *agrees with* $\sigma_n : \bar{X}_n \to E$.

This provides an alternative way to compute $W_n(\mathbf{z})$ by eliminating B, C, and §4 is devoted to this explicit construction. In particular the weight 10 pre-modular form $Z_4(\sigma;\tau)$ is explicitly written down (c.f. Example 4.9).

The existence and effective construction of $Z_n(\sigma;\tau)$ opens the door to extend our complete results on (0.1) for $n = 1$ (established in [10, 12, 2]) to general $n \in \mathbb{N}$. As a related application, the explicit expression of Z_4 is used to solve Dahmen's conjecture on a counting formula for Lamé equations (0.8) with finite monodromy for $n = 4$. The method works for general n once Z_n is shown to have expected asymptotic behavior at cusps. The details is written by Y.-C. Chou and is included in Appendix A.

1. Geometry of $\sigma_n : \bar{X}_n \to E$

The aim of this section is to prove Theorem 0.1. We first review and extend some technical details on results from [3] quoted in §0.

Proposition 1.1. [3, Theorem 6.5] *Let $a_1, \cdots, a_n$ be the zeros of a developing map f for equation* (0.1). *Then the logarithmic derivative $g = f'/f$ is given by*

$$g(z) = \sum_{i=1}^{n} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)}. \tag{1.1}$$

Moreover, $g(z)$ has $\operatorname{ord}_{z=0} g(z) = 2n$, and $a_i \notin E[2]$, $a_i \neq \pm a_j$ for $i \neq j$.

The condition $\operatorname{ord}_{z=0} g(z) = 2n$ leads to the $n-1$ equations for $a_1, \ldots, a_n$ given in (0.4): Under the notations $(w, x_j, y_j) = (\wp(z), \wp(p_j), \wp'(p_j))$,

$$\begin{aligned} g(z) &= \sum_{j=1}^{n} \frac{1}{w} \frac{y_j}{1 - x_j/w} \\ &= \sum_{j=1}^{n} \frac{y_j}{w} + \sum_{j=1}^{n} \frac{y_j x_j}{w^2} + \cdots + \sum_{j=1}^{n} \frac{y_j x_j^r}{w^{r+1}} + \cdots . \end{aligned}$$

Since $g(z)$ has a zero at $z = 0$ of order $2n$ and $1/w$ has a zero at $z = 0$ of order two, we get $x_i \neq x_j$ for $i \neq j$ and

$$\sum y_i x_i^r = 0, \qquad r = 0, \ldots, n-2. \tag{1.2}$$

This, together with the Weierstrass equation $y_i^2 = 4x_i^3 - g_2 x_i - g_3$, gives the polynomial system describing the developing maps.

The Green equation (0.5) *is equivalent to the type II condition* (0.3): The argument is essentially contained in [11, Lemma 2.4]. By the addition law,

$$\begin{aligned} f &= \exp \int g\, dz \\ &= \exp \int \sum_{i=1}^{n} (2\zeta(a_i) - \zeta(a_i - z) - \zeta(a_i + z))\, dz \\ &= e^{2\sum_{i=1}^{n} \zeta(a_i) z} \prod_{i=1}^{n} \frac{\sigma(z - a_i)}{\sigma(z + a_i)}. \end{aligned}$$

We then calculate the monodromy effect on f from

$$\sigma(z + \omega_j) = -e^{\frac{1}{2}\eta_i(z + \frac{1}{2}\omega_j)} \sigma(z), \qquad j = 1, 2. \tag{1.3}$$

Let $a_i = r_i \omega_1 + s_i \omega_2$ for $i = 1, \ldots, n$. By way of the Legendre relation $\eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i$ we compute easily that

$$\begin{aligned} f(z + \omega_1) &= e^{-4\pi i \sum_i s_i + 2\omega_1 (\sum \zeta(a_i) - \sum r_i \eta_1 - \sum s_i \eta_2)} f(z), \\ f(z + \omega_2) &= e^{4\pi i \sum_i r_i + 2\omega_2 (\sum \zeta(a_i) - \sum r_i \eta_1 - \sum s_i \eta_2)} f(z). \end{aligned} \tag{1.4}$$

By (0.6), the equivalence of (0.5) and (0.3) follows immediately.

The Liouville curve $X_n \subset \operatorname{Sym}^n E$, defined by (1.2) or (0.4), has a hyperelliptic structure under the 2 to 1 map $X_n \to \mathbb{P}^1$, $(x_i, y_i)_{i=1}^n \mapsto (2n-1) \sum_{i=1}^n x_i$ as in (0.10). This is closely related to the integral Lamé equations (0.8) since $f = w_a / w_{-a}$ where w_a is the ansatz solution (0.9).

The full information on the compactified hyperelliptic curve $\bar{X}_n \to \mathbb{P}^1$, the Lamé curve, especially on the branch points added, is described by

Theorem 1.2. [3, Theorem 0.7]

(1) *The natural compactification* $\bar{X}_n \subset \mathrm{Sym}^n E$ *coincides with the, possibly singular, projective model of the hyperelliptic curve defined by*

$$\begin{aligned} C^2 &= \ell_n(B, g_2, g_3) \\ &= 4Bs_n^2 + 4g_3 s_{n-2}s_n - g_2 s_{n-1}s_n - g_3 s_{n-1}^2, \end{aligned} \tag{1.5}$$

in (B, C), *where* $s_k = s_k(B, g_2, g_3) = r_k B^k + \cdots \in \mathbb{Q}[B, g_2, g_3]$, *is a recursively defined polynomial of homogeneous degree k with* $\deg g_2 = 2$, $\deg g_3 = 3$, *and* $B = (2n-1)s_1$.

(2) $\deg \ell_n = 2n+1$ *and* $\bar{X}_n$ *has arithmetic genus* $g = n$.

(3) *The curve* $\bar{X}_n$ *is smooth except for a finite number of* τ, *namely the discriminant loci of* $\ell_n(B, g_2, g_3)$ *so that* $\ell_n(B)$ *has multiple roots.* $\bar{X}_n$ *is an irreducible curve which is smooth at infinity.*

(4) *The* $2n+2$ *branch points* $a \in \bar{X}_n \setminus X_n$ *are characterized by* $-a = a$. *In fact* $\{-a_i\} \cap \{a_i\} \neq \emptyset \Rightarrow -a = a$. *Also* $0 \in \{a_i\} \Rightarrow a = (0, 0, \cdots, 0)$.

(5) *The limiting system of (1.2) at* $a = 0^n$ *is given by*

$$\sum_{i=1}^{n} t_i^{2r+1} = 0, \quad r = 1, \ldots, n-1 \tag{1.6}$$

under the non-degenerate constraints $t_i \neq 0$, $t_i \neq -t_j$. *Moreover,* (1.6) *has a unique non-degenerate solution in* $\mathbb{P}^{n-1}$ *up to permutations. It gives the tangent direction* $[t] \in \mathbb{P}(T_{0^n}(\bar{X}_n)) \subset \mathbb{P}(T_{0^n}(\mathrm{Sym}^n E))$.

(6) *In terms of* $a \in Y_n$, (B, C) *can be parameterized by* $B(a) = B_a$ *and*

$$C(a) = \wp'(a_i) \prod_{j \neq i} (\wp(a_i) - \wp(a_j)), \quad \text{for any } i = 1, \ldots, n. \tag{1.7}$$

The smooth point $a = 0^n \in \bar{X}_n$ is referred as *the point at infinity*. For the other $2n+1$ *finite branch points* with $a = -a$, the ansatz solution (0.9) $w_a = w_{-a}$ is still a solution to the Lamé equation. In the literature, these $2n+1$ functions are known as the *Lamé functions*.

Notice that (1.7) arises from (1.1) and $\mathrm{ord}_{z=0}\, g_a(z) = 2n$ in

$$g_a(z) := \sum_{i=1}^{n} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)} = \frac{\sum_{i=1}^{n} \wp'(a_i) \prod_{j \neq i} (\wp(z) - \wp(a_j))}{\prod_{i=1}^{n} (\wp(z) - \wp(a_i))},$$

where the numerator reduces to the constant $C(a)$. By working with (1.7), we may say a little more on the possible singularities of $\bar{X}_n(\tau)$:

Theorem 1.3. $\bar{X}_n$ *has at most nodal singularities. That is,* $\ell_n(B)$ *has at most double roots. At such a point* $a \in Y_n \setminus X_n$, *both local branches are smooth and* C *could be used as a local coordinate.*

Proof. Denote by $b = \{b_1, \cdots, b_n\} \in X_n$ a point near the branch point $a = \{a_1, \cdots, a_n\} \in Y_n \setminus X_n$. (1.7) implies that, for $a_i = -a_i$ (2-torsion) in E,

$$C(b) = \Big[\wp''(a_i) \prod\nolimits_{j \neq i} (\wp(a_i) - \wp(a_j))\Big] (b_i - a_i) + o(|b_i - a_i|).$$

Since $\wp''(a_i) \neq 0$, C can be used as a parameter and $b_i'(0) \neq 0, \infty$.

Similarly for a_i not a 2-torsion point, we denote by $a_{i'} = -a_i$ and get

$$C(b) = \Big[\wp'(a_i)^2 \prod\nolimits_{j \neq i, i'} (\wp(a_i) - \wp(a_j))\Big] (b_i + b_{i'}) + o(|b_1 + b_2|).$$

Since $\wp'(a_i) \neq 0$, C can be used as a parameter and $b_i'(0) + b_{i'}'(0) \neq 0, \infty$. Again, from (1.7) we deduce that $b_{i'}(C) = -b_i(-C)$. So $b_{i'}'(0) = b_i'(0)$ and hence they are neither 0 nor ∞.

In summary, the paramaterization $C \mapsto b(C)$ is well defined, holomorphic and non-degenerate in any chosen branch of Y_n near $a = b(0)$. Since the analytic structure at $a \in Y_n$ is of the form $C^2 = (B - \lambda)^m$, this is possible if and only if $m = 1, 2$. The singular case corresponds to $m = 2$ which leads to a double point. The two branches are all non-singular at a. □

There are four species of Lame functions, depending on the number of half periods contained in $\{a_i\}$. We call them being of type O, I, II, and III respectively. For $n = 2k$ being even, a must be of type O or II. For $n = 2k + 1$ being odd, a must be of type I or III. There are factorizations of the polynomial $\ell_n(B)$ according to the types:

Proposition 1.4. [6, 15] *In terms of $e_i = \wp(\frac{1}{2}\omega_i)$, we may write*

$$\ell_n(B) = c_n^2 l_0(B) l_1(B) l_2(B) l_3(B),$$

where $c_n \in \mathbb{Q}^+$ is a constant, $l_i(B)$'s are monic polynomials in B such that

(1) *For $n = 2k$, $l_0(B)$ consists of type O roots with $\deg l_0(B) = \frac{1}{2}n + 1 = k + 1$. For $i = 1, 2, 3$, $l_i(B)$ consists of type II roots a which does not contain $\frac{1}{2}\omega_i$. Moreover, $\deg l_i(B) = \frac{1}{2}n = k$.*

(2) *For $n = 2k + 1$, $l_0(B)$ consists of type III roots with $\deg l_0(B) = \frac{1}{2}(n - 1) = k$. For $i = 1, 2, 3$, $l_i(B)$ consists of type I roots a which contains $\frac{1}{2}\omega_i$. Moreover, $\deg l_i(B) = \frac{1}{2}(n + 1) = k + 1$.*

We remark that Proposition 1.4, Theorem 1.2 (4), (5) and Theorem 1.3 will be used in the proof of Theorem 0.1 (= Theorem 1.6 later in this section). Here are some examples to illustrate Proposition 1.4:

Example 1.5. Decomposition $\ell_n(B) = c_n^2 l_0(B) l_1(B) l_2(B) l_3(B)$ for $1 \leq n \leq 5$.

(1) $n = 1$, $k = 0$, $\bar{X}_1 \cong E$, $C^2 = \ell_1(B) = 4B^3 - g_2 B - g_3 = 4\prod_{i=1}^3 (B - e_i)$.

(2) $n = 2$, $k = 1$, (notice that $e_1 + e_2 + e_3 = 0$)

$$\begin{aligned} C^2 = \ell_2(B) &= \tfrac{4}{81} B^5 - \tfrac{7}{27} g_2 B^3 + \tfrac{1}{3} g_3 B^2 + \tfrac{1}{3} g_2^2 B - g_2 g_3 \\ &= \frac{2^2}{3^4} (B^2 - 3g_2) \prod_{i=1}^{3} (B + 3e_i). \end{aligned}$$

(3) $n = 3$, $k = 1$, $\deg l_i(B) = 2$ for $i = 1, 2, 3$,

$$\begin{aligned} C^2 = \ell_3(B) &= \frac{1}{2^2 3^4 5^4} B(16B^6 - 504g_2B^4 + 2376g_3B^3 \\ &\qquad + 4185g_2^2B^2 - 36450g_2g_3B + 91125g_3^2 - 3375g_2^3) \\ &= \frac{2^2}{3^4 5^4} B \prod_{i=1}^{3} (B^2 - 6e_iB + 15(3e_i^2 - g_2)). \end{aligned}$$

(4) $n = 4$, $k = 2$, $\deg l_0(B) = 3$,

$$C^2 = \ell_4(B) = \frac{1}{3^8 5^4 7^4} (B^3 - 52g_2B + 560g_3) \prod_{i=1}^{3} (B^2 + 10e_iB - 7(5e_i^2 + g_2)).$$

(5) $n = 5$, $k = 2$, $\deg l_i(B) = 3$ for $i = 1, 2, 3$,

$$\begin{aligned} C^2 = \ell_5(B) &= \frac{1}{3^{12} 5^4 7^4 11^2} (B^2 - 27g_2) \\ &\quad \times \prod_{i=1}^{3} (B^3 - 15e_iB^2 + (315e_i^2 - 132g_2)B + e_i(2835e_i^2 - 540g_2)). \end{aligned}$$

We are now ready to study the addition map $\sigma_n : \bar{X}_n \to E$, $a \mapsto \sigma_n(a) = \sum_{i=1}^{n} a_i$ defined in (0.13). In the rest of this section we determine $\deg \sigma_n$.

For the reader's convenience we recall some definitions and facts. The function field $K(C)$ is defined for any irreducible algebraic curve C. For a finite morphism of irreducible curves $f : X \to Y$, $K(X)$ is a finite extension of $K(Y)$ and the degree of f is defined by $\deg f = [K(X) : K(Y)]$. Geometrically $\deg f$ is also the number of points for a general fiber $f^{-1}(p)$, $p \in Y$. A standard reference is [8, II.6, Proposition 6.9], where nonsingular curves are treated. The irreducible case is reduced to the nonsingular case through normalizations $\tilde{X} \to X$ and $\tilde{Y} \to Y$, since it is clear that the induced finite morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ has the same degree as f. Furthermore, the definition also extends to the case $f : X \to Y$ where $X = \bigcup_{i=1}^{k} X_i$ has a finite number of irreducible components. We require that $f|_{X_i}$ is a finite morphism for each i and then $\deg f := \sum_{i=1}^{k} \deg f|_{X_i}$. Since all curves considered here are proper (projective), it is enough to require $f|_{X_i}$ to be non-constant to ensure that it is a finite morphism.

Theorem 1.6. *The map $\sigma_n : \bar{X}_n \to E$ has degree $\frac{1}{2}n(n+1)$.*

Proof. The idea is to apply *Theorem of the Cube* [14, p.58, Corollary 2] for morphisms from an arbitrary variety V (not necessarily smooth) into abelian varieties (here the torus E): For *any* three morphisms $f, g, h : V \to E$ and a line bundle $L \in \operatorname{Pic} E$, we have

$$\begin{aligned} (f + g + h)^*L &\cong (f + g)^*L \otimes (g + h)^*L \otimes (h + f)^*L \\ &\qquad \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}. \end{aligned} \tag{1.8}$$

We will apply it to the algebraic curve $V = V_n \subset E^n$ which consists of the ordered n-tuples a's so that $V_n/S_n = \bar{X}_n$.

For any line bundle L and any *finite morphism* $f : V \to E$, we have $\deg f^* L = \deg f \deg L$. In the following we fix an L with $\deg L = 1$.

We prove inductively that for $j = 1, \ldots, n$ the morphism $f_j : V_n \to E$ defined by

$$f_j(a) := a_1 + \cdots + a_j$$

has $\deg f_j^* L = \frac{1}{2} j(j+1) n!$. The case $j = n$ then gives the result since f_n is a finite morphism which descends to σ_n under the S_n action. (Notice that the map f_j can not descend to a map on $\bar{X}_n$ for all $j < n$.)

Assuming first that it has been proved for $j = 1, 2$. To go from j to $j+1$, we let $f(a) = f_{j-1}(a)$, $g(a) = a_j$, and $h(a) = a_{j+1}$. Then by (1.8), $f_{j+1}^* L$ has degree $n!$ times

$$\tfrac{1}{2} j(j+1) + 3 + \tfrac{1}{2} j(j+1) - \tfrac{1}{2}(j-1)j - 1 - 1 = \tfrac{1}{2}(j+1)(j+2)$$

as expected.

It remains to investigate the case $j = 1$ and $j = 2$.

For $j = 1$, by Theorem 1.2 (4), the inverse image of $0 \in E$ under $f_1 : V_n \to E$ consists of a single point 0^n. By Theorem 1.2 (5), the limiting system of equations (1.6) of tangent directions, has a unique non-degenerate solution in $\mathbb{P}^{n-1}$ up to permutations. From this, we conclude that there are precisely $n!$ branches of $V_n \to E$ near 0^n. For a point $b \in E^\times$ close to 0, each branch will contribute a point a with $a_1 = b$. In particular, f_1 is a finite morphism and $\deg f_1^* L = \deg f_1 = n!$.

For $j = 2$, we consider the inverse image of $0 \in E$ under $f_2 : V_n \to E$. Namely $V_n \ni a \mapsto a_1 + a_2 = 0$.

The point $a = 0$ again contributes degree $n!$ by a similar branch argument: Indeed, over each branch near 0^n we may represent $a = (a_i(t))$ by an analytic curve in t. Then condition $t_i + t_j \neq 0$ in Theorem 1.2 (5) implies that $t \mapsto a_1(t) + a_2(t) \in E$ is still locally biholomorphic for t close to 0. As a byproduct, since every irreducible component contains a branch near 0^n, f_2 is necessarily a finite morphism and $\deg f_2^* L = \deg f_2$.

For those points $a \neq 0$ with $f_2(a) = 0$, we have $a_1 = -a_2$ and thus $a = -a$ by Theorem 1.2 (4). By Theorem 1.3 we use C as the coordinate and parameterize a (smooth) branch of V_n near a by $b(C) = (b_i(C))_{i=1}^n$ with $b(0) = a$. In the proof of Theorem 1.3 we see that $b_1'(0) = b_2'(0) \notin \{0, \infty\}$ and $b_1'(0) + b_2'(0) \neq 0, \infty$, hence f_2 is unramified at a. The degree contribution at a can thus be computed from counting points.

If $n = 2k$, by Proposition 1.4 (1) the degree contribution from type O points $a = \{\pm a_1, \cdots, \pm a_k\}$ is given by

$$(k+1) \times (k \times 2 \times (n-2)!),$$

while the degree from the type II points $\{\pm a_1, \cdots, \pm a_{k-1}, \frac{1}{2}\omega_i, \frac{1}{2}\omega_j\}$ is

$$3 \times k \times ((k-1) \times 2 \times (n-2)!).$$

The sum is $2(4k^2 - 2k)(n-2)! = 2n!$.

If $n = 2k+1$, by Proposition 1.4 (2), the degree contribution from type III points $\{\pm a_1, \cdots, \pm a_{k-1}, \frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_3\}$ is

$$k \times ((k-1) \times 2 \times (n-2)!),$$

while the type I points $\{\pm a_1, \cdots, \pm a_k, \frac{1}{2}\omega_i\}$ contribute

$$3 \times (k+1) \times (k \times 2 \times (n-2)!).$$

The sum is again $2(4k^2 + 2k)(n-2)! = 2n!$.

The counting is valid even if $\bar{X}_n$ has nodal singularities. Thus in both cases we get the total degree $n! + 2n! = 3n!$ as expected. □

To end this section, we notice that in Theorem 1.2 (5) we have $\sum_{i=1}^{n} t_i \neq 0$ by the non-vanishing of Vandermonde determinant, hence we get

Proposition 1.7. *The map σ_n is unramified at the infinity point $0^n \in \bar{X}_n$.*

2. The primitive generator $\mathbf{z}_n$

Definition 2.1 (Fundamental rational function)**.** *Consider the function on E^n:*

$$\mathbf{z}_n(a_1, \ldots, a_n) := \zeta\Big(\sum_{i=1}^{n} a_i\Big) - \sum_{i=1}^{n} \zeta(a_i).$$

$\mathbf{z}_n$ is a rational function on E^n since it is meromorphic and periodic in each a_i.

The importance of $\mathbf{z}_n$ is readily seen from investigation on the Green function equation (0.5): Let $a_i = r_i\omega_1 + s_i\omega_2$. Then

$$\begin{aligned} -4\pi \sum \nabla G(a_i) &= \sum Z(a_i) = \sum(\zeta(r_i\omega_1 + s_i\omega_2) - r_i\eta_1 - s_i\eta_2) \\ &= \zeta(\sum a_i) - (\sum r_i)\eta_1 - (\sum s_i)\eta_2 - \mathbf{z}_n(a) \\ &= Z(\sum a_i) - \mathbf{z}_n(a). \end{aligned} \tag{2.1}$$

Hence $\sum_{i=1}^{n} \nabla G(a_i) = 0 \Longleftrightarrow \mathbf{z}_n(a) = Z(\sigma_n(a))$. This links $\sigma_n(a)$ with $\mathbf{z}_n$.

When no confusion should arise, we denote the restriction $\mathbf{z}_n|_{\bar{X}_n}$ also by $\mathbf{z}_n$. Then $\mathbf{z}_n$ is a rational function on $\bar{X}_n$ with poles along the fiber $\sigma_n^{-1}(0)$. Since $\mathbf{z}_1 \equiv 0$, we assume that $n \geq 2$ to avoid trivial situation.

Theorem 2.2. *There is a (weighted homogeneous) polynomial*

$$W_n(\mathbf{z}) \in \mathbb{Q}[g_2, g_3, \wp(\sigma), \wp'(\sigma)][\mathbf{z}]$$

of $\mathbf{z}$-degree $\frac{1}{2}n(n+1)$ such that for $\sigma = \sigma_n(a) = \sum a_i$, we have

$$W_n(\mathbf{z}_n)(a) = 0.$$

Indeed, $\mathbf{z}_n(a)$ is a primitive generator of the finite extension of rational function fields $K(\bar{X}_n)$ over $K(E)$ with $W_n(\mathbf{z})$ being its minimal polynomial. [1]

[1] The coefficients lie in $\mathbb{Q}$, instead of just in $\mathbb{C}$, follows from standard elimination theory and two facts (i) The equations of $\bar{X}_n$ is defined over $\mathbb{Q}[g_2, g_3]$ (cf. (0.4)), and (ii) the addition map $E^n \to E$ is defined over $\mathbb{Q}$. In §4, we carry out the elimination procedure using resultant for another explicit presentation π_n of σ_n.

Remark 2.3. Since $\mathbf{z}_n$ has no poles over $E^\times$, it is indeed integral over the affine Weierstrass model of $E^\times$ with coordinate ring

$$R(E^\times) = \mathbb{C}[x_0, y_0]/(y_0^2 - 4x_0^3 - g_2x_0 - g_3),$$

where $x_0 = \wp(\sigma)$ and $y_0 = \wp'(\sigma)$. Thus the major statement in Theorem 0.2 is the claim that $\mathbf{z}_n$ is a primitive generator.

Proof. Since $\mathbf{z}_n \in K(\bar{X}_n)$, which is algebraic over $K(E)$ with degree $\frac{1}{2}n(n+1)$ by Theorem 1.6, its minimal polynomial $W_n(\mathbf{z}) \in K(E)[\mathbf{z}]$ exists with $d := \deg W_n$ begin a factor of $\frac{1}{2}n(n+1)$.

Notice that for $\sigma_0 \in E$ being outside the branch loci of $\sigma_n : \bar{X}_n \to E$, there are precisely $\frac{1}{2}n(n+1)$ different points $a = \{a_1, \cdots, a_n\} \in \bar{X}_n$ with $\sigma_n(a) = \sum a_i = \sigma_0$. Thus for the rational function $\mathbf{z}_n = \zeta(\sum a_i) - \sum \zeta(a_i) \in K(\bar{X}_n)$ to be a primitive generator, it is sufficient to show that $\mathbf{z}_n$ has exactly $\frac{1}{2}n(n+1)$ branches over $K(E)$. That is, $\sum \zeta(a_i)$ gives different values for different choices of those a above σ_0. Indeed, for any given $\sigma = \sigma_0$, the polynomial $W_n(\mathbf{z}) = 0$ has at most d roots. But now $\mathbf{z}_n(a)$ with $\sigma_n(a) = \sigma_0$ gives $\frac{1}{2}n(n+1)$ distinct roots of $W_n(\mathbf{z})$, hence we must conclude $d = \frac{1}{2}n(n+1)$ and $\mathbf{z}_n$ is a primitive generator.

Hence it is sufficient to show the following more precise result:

Theorem 2.4. *Let $a, b \in Y_n$ and $(a_1, \cdots, a_n)$, $(b_1, \cdots, b_n) \in \mathbb{C}^n$ be representatives of a, b such that*

$$\sum_{i=1}^n a_i = \sum_{i=1}^n b_i, \qquad \sum_{i=1}^n \zeta(a_i) = \sum_{i=1}^n \zeta(b_i). \tag{2.2}$$

Suppose that $\sum \wp(a_i) \neq \sum \wp(b_i)$. Then a, b are branch points of $Y_n \to \mathbb{P}^1$ corresponding to Lamé functions of the same type.

We emphasize that $\bar{X}_n$ is not required to be smooth.

Theorem 2.2 follows immediately by choosing σ_0 outside the branch loci of $\bar{X}_n \to E$ and $\sigma_0 \notin E[2]$. Indeed, let $a, b \in Y_n$ with $\sigma_n(a) = \sigma_n(b) = \sigma_0$ and $\mathbf{z}_n(a) = \mathbf{z}_n(b)$, or more precisely with conditions in (2.2) satisfied. By Theorem 2.4 we are left with the case $\sum \wp(a_i) = \sum \wp(b_i)$ but $a \neq b$. Then $a = -b$ by Theorem 1.2 (1), and in particular $\sigma_n(a) = -\sigma_n(b)$. Together with $\sigma_n(a) = \sigma_n(b)$ we conclude that $\sigma_0 = \sigma_n(a) = \sigma_n(b) \in E[2]$. This contradicts to the assumption $\sigma_0 \notin E[2]$. Hence we must have $a = b$. $\square$

We will give two proofs of Theorem 2.4. The first proof is longer but contains more information.

Recall that the Hermite–Halphen ansatz in (0.9)

$$w_{\pm a}(z) = e^{\pm z \sum \zeta(a_i)} \prod_{i=1}^n \frac{\sigma(z \mp a_i)}{\sigma(z)}$$

are solutions to $w'' = (n(n+1)\wp(z) + B_a)w =: I_1 w$, and

$$w_{\pm b}(z) = e^{\pm z \sum \zeta(b_i)} \prod_{i=1}^{n} \frac{\sigma(z \mp b_i)}{\sigma(z)}$$

are solutions to $w'' = (n(n+1)\wp(z) + B_b)w =: I_2 w$. Then $q_{a,-b} := w_a w_{-b}$ and $q_{-a,b} := w_{-a} w_b$ are solutions to the fourth order ODE formed by the tensor product of the two Lamé equations. By assumption.

$$q_{a,-b}(z) = \prod_{i=1}^{n} \frac{\sigma(z-a_i)\sigma(z+b_i)}{\sigma^2(z)} \tag{2.3}$$

is an elliptic function since $\sum a_i = \sum b_i$. Similarly $q_{-a,b}(z) = q_{a,-b}(-z)$ is elliptic. In particular there exists an even elliptic function solution

$$Q := \tfrac{1}{2}(q_{a,-b} + q_{-a,b}) = (-1)^n \frac{\prod_{i=1}^n \sigma(a_i)\sigma(b_i)}{z^{2n}} + \text{higher order terms.}$$

Lemma 2.5. *The fourth order ODE is given by*

$$q'''' - 2(I_1 + I_2)q'' - 6I'q' + ((B_a - B_b)^2 - 2I'')q = 0. \tag{2.4}$$

Here $I = n(n+1)\wp(z)$, $I_1 = I + B_a$ and $I_2 = I + B_b$.

Proof. This follows from a straightforward computation. Indeed,

$$\begin{aligned} q' &= w_1' w_2 + w_1 w_2', \\ q'' &= (I_1 + I_2)q + 2w_1' w_2', \\ q''' &= 2I'q + (I_1 + I_2)q' + 2(I_1 w_1 w_2' + I_2 w_1' w_2). \end{aligned}$$

Notice that if $a = b$ (or just $B_a = B_b$) then $I_1 = I_2$ and we stop here to get the third order ODE as the symmetric product of the Lamé equation.

In general, we take one more differentiation to get

$$\begin{aligned} q'''' &= 2I''q + 4I'q' + (I_1 + I_2)q'' + 2I'q' + 2(I_1 + I_2)w_1' w_2' + 4I_1 I_2 q \\ &= 2(I_1 + I_2)q'' + 6I'q' + (2I'' - (I_1 - I_2)^2)q. \end{aligned}$$

This proves the lemma. □

Now we investigate the equation in variable $x = \wp(z)$. To avoid confusion, we denote $\dot{f} = \partial f/\partial x$ and $f' = \partial f/\partial z$.

Let $y^2 = p(x) = 4x^3 - g_2 x - g_3$. Then $\wp' = y$, $\wp'' = 6\wp^2 - \frac{1}{2}g_2 = \frac{1}{2}\dot{p}(x)$. $\wp''' = 12\wp\wp' = 12xy$, $\wp'''' = 12\wp'^2 + 12\wp\wp'' = 12p(x) + 6x\dot{p}(x)$. Also

$$\begin{aligned} q' &= \dot{q}\wp' = y\dot{q}, \\ q'' &= \ddot{q}\wp'^2 + \dot{q}\wp'' = p(x)\ddot{q} + \tfrac{1}{2}\dot{p}(x)\dot{q}, \\ q''' &= \dddot{q}\wp'^3 + 3\ddot{q}\wp'\wp'' + \dot{q}\wp''', \\ q'''' &= \ddddot{q}\wp'^4 + 6\dddot{q}\wp'^2\wp'' + 3\ddot{q}(\wp'')^2 + 4\ddot{q}\wp'\wp''' + \dot{q}\wp'''' \\ &= p(x)^2\ddddot{q} + 3p(x)\dot{p}(x)\dddot{q} + \big(\tfrac{3}{4}\dot{p}(x)^2 + 48xp(x)\big)\ddot{q} + \big(12p(x) + 6x\dot{p}(x)\big)\dot{q}. \end{aligned}$$

By substituting these into (2.4) and get the ODE in x:

$$\begin{aligned} L_4\, q := {}& p^2 \ddddot{q} + 3p\dot{p}\dddot{q} + \big(\tfrac{3}{4}\dot{p}^2 - 2(2(n^2+n-12)x+\beta)p\big)\ddot{q} \\ & - \big((2(n^2+n-3)x+\beta)\dot{p} + 6(n^2+n-2)p\big)\dot{q} \\ & + \big(\alpha^2 - n(n+1)\dot{p}\big)q = 0. \end{aligned} \tag{2.5}$$

where

$$\alpha := B_a - B_b \quad \text{and} \quad \beta := B_a + B_b. \tag{2.6}$$

For the rest of the proof, we want to discuss when $L_4\, q = 0$ with $\alpha \neq 0$ has a polynomial solution. Here g_2 and g_3 could be arbitrary, not necessarily satisfy the non-degenerate condition $g_2^3 - 27g_3^2 \neq 0$.

Suppose that $q(x)$ is a polynomial in x of degree $m \geq 1$:

$$q(x) = x^m - s_1 x^{m-1} + s_2 x^{m-2} - \cdots + (-1)^m s_m, \tag{2.7}$$

which satisfies

$$\deg_x L_4\, q(x) \leq 1. \tag{2.8}$$

Then we can solve s_j recursively in terms of α^2, β and g_2, g_3.

Indeed, the top degree x^{m+2} in (2.5) has coefficient

$$\begin{aligned} & 16m(m-1)(m-2)(m-3) + 144m(m-1)(m-2) + 108m(m-1) \\ & \quad - 16(n^2+n-12)m(m-1) - 24(n^2+n-3)m \\ & \quad\quad - 24(n^2+n-2)m - 12n(n+1) \\ & = (m-n)\Big(4m^3 + (4n+68)m^2 + (8n-101)m + 3(n+1)\Big), \end{aligned}$$

which vanishes precisely when $m = n$. This we may assume that $m = n$.

The next order term x^{n+1} without the s_1 factor has coefficient

$$-8n(n-1)\beta - 12n\beta = -4n(2n+1)\beta,$$

and the coefficient of $-s_1 x^{n+1}$ is given by

$$\begin{aligned} & 16(n-1)(n-2)(n-3)(n-4) + 144(n-1)(n-2)(n-3) \\ & \quad + 108(n-1)(n-2) - 16(n^2+n-12)(n-1)(n-2) \\ & \quad - 24(n^2+n-3)(n-1) - 24(n^2+n-2)(n-1) - 12n(n+1) \\ & = -8n(2n-1)(2n+1). \end{aligned}$$

Hence

$$s_1 = \frac{\beta}{2(2n-1)}. \tag{2.9}$$

Inductively the x^{n+2-i} coefficient in (2.5) gives recursive relations to solve s_i in terms of β, α^2 and g_2, g_3 for $i = 1, \ldots, n$. It implies that

Lemma 2.6. *For $i = 1, \dots, n$, there is a polynomial expression*

$$s_i = s_i(\alpha^2, \beta, g_2, g_3) = C_i\beta^i + \cdots$$

which is homogeneous of degree i with $\deg\alpha = \deg\beta = 1$ and $\deg g_2 = 2$, $\deg g_3 = 3$. Moreover, C_i is a non-zero rational number.

A much detailed description will be given in the proof of Lemma 2.8 and the precise value of C_i can be determined (from (2.12)).

There are still two remaining terms in (2.8), that is,

$$L_4\, q = F_1(\alpha, \beta, g_2, g_3)x + F_0(\alpha, \beta, g_2, g_3). \tag{2.10}$$

The basic structure of the consistency equations is described by the following two lemmas:

Lemma 2.7. *We have*

$$\begin{aligned} F_1(\alpha, \beta) &= \alpha^2 G_1(\alpha, \beta) = \alpha^2((-1)^{n-1}s_{n-1}(\alpha^2, \beta, g_2, g_3) + \cdots), \\ F_0(\alpha, \beta) &= \alpha^2 G_0(\alpha, \beta) = \alpha^2((-1)^{n}s_{n}(\alpha^2, \beta, g_2, g_3) + \cdots). \end{aligned}$$

The remaining terms have either g_2 or g_3 as a factor, hence with lower α, β degree.

Proof. Equation (2.10) gives

$$\begin{aligned} F_1(\alpha, \beta) &= (-1)^{n-1}\alpha^2 s_{n-1} + \text{terms in } s_1, \cdots, s_{n-2}, \\ F_0(\alpha, \beta) &= (-1)^{n}\alpha^2 s_{n} + \text{terms in } s_1, \cdots, s_{n-1}. \end{aligned}$$

We note that if $\alpha = 0$, then for any β there is a solution $q(x)$ to $L_4(q) = 0$ which is a polynomial in x of degree n.

Indeed $q(x) = \prod_{i=1}^n (x - x_i)$, with $\beta = 2(2n-1)\sum_{i=1}^n x_i$, which comes from the Lamé equation (see [3, 15]). Thus $F_1(0, \beta) = 0 = F_0(0, \beta)$. Since F_i depends on α^2, we have $F_i(\alpha, \beta) = \alpha^2 G_i(\alpha, \beta)$, $i = 0, 1$, for some homogeneous polynomials G_0, G_1 in α^2, β, g_2, g_3 of degree n and $n-1$ respectively, and G_i's can be written as

$$\begin{aligned} G_1(\alpha, \beta) &= (-1)^{n-1}s_{n-1} + \cdots, \\ G_0(\alpha, \beta) &= (-1)^{n}s_{n} + \cdots. \end{aligned}$$

To see the dependence of the remaining terms on g_2 and g_3, we let $g_2 = 0 = g_3$, and then $L_4(q) \equiv \alpha^2((-1)^{n-1}s_{n-1}x + (-1)^n s_n) \pmod{x^2}$ because both $p(x) = 4x^3$ and $\dot p(x) = 12x^2$ vanish modulo x^2. Thus we have $F_1(\alpha, \beta) = (-1)^{n-1}\alpha^2 s_{n-1}$ and $F_0(\alpha, \beta) = (-1)^n\alpha^2 s_n$ whenever $g_2 = 0 = g_3$. This proves the lemma. □

Lemma 2.8. *The polynomials G_1 and G_0 have no common factors for any g_2, g_3.*

Proof. We consider first the special case $g_2 = g_3 = 0$. Then (2.8) becomes

$$\begin{aligned} &16x^6 \ddddot{q} + 144x^5 \dddot{q} + \big(108x^4 - 8x^3(2(n^2+n-12)x + \beta)\big)\ddot{q} \\ &\quad - \big(12x^2(2(n^2+n-3)x + \beta) + 24x^3(n^2+n-2)\big)\dot{q} \\ &\quad + \big(\alpha^2 - 12n(n+1)x^2\big)q \equiv 0 \pmod{\mathbb{C} \oplus \mathbb{C}x}. \end{aligned} \tag{2.11}$$

The coefficient of x^{n-k}, $k = 0, \dots, n-2$, gives recursive equation

$$(-1)^k (m_k\, s_{k+2} + n_k \beta\, s_{k+1} + \alpha^2 s_k) = 0, \tag{2.12}$$

where the constants m_k and n_k are given by

$$\begin{aligned}
m_k &= 16(n-(k+2))(n-(k+3))(n-(k+4))(n-(k+5)) \\
&\quad + 144(n-(k+2))(n-(k+3))(n-(k+4)) \\
&\quad + (108 - 16(n^2+n-12))(n-(k+2))(n-(k+3)) \\
&\quad - 24(2n^2+2n-5)(n-(k+2)) - 12n(n+1) \\
&= -4(k+2)(2n-(k+1))(2n-(2k+1))(2n-(2k+3)), \\
n_k &= (8(n-(k+1))(n-(k+2)) + 12(n-(k+1))) \\
&= 4(n-(k-1))(n-(k+1)).
\end{aligned}$$

Since $k \le n-2$, we have $m_k \ne 0$ and $n_k \ne 0$.

Let $\gamma(\alpha, \beta)$ be a non-trivial common factor of both G_1 and G_0.

In the case $g_2 = g_3 = 0$ we have $G_1 = (-1)^{n-1} s_{n-1}$ and $G_0 = (-1)^n s_n$. Then γ and α are co-prime, because if $\alpha = 0$ then $s_{n-1}(0, \beta) = c_{n-1}\beta^{n-1}$ and $s_n(0, \beta) = c_n \beta^n$ for some non-zero constants c_{n-1} and c_n. By (2.12) for $k = n-2$, we have $\gamma \mid s_{n-2}(\alpha^2, \beta, 0, 0)$ too. By induction on k for $k = n-3, \dots, 0$ in decreasing order we conclude that $\gamma \mid s_0 = 1$, which leads to a contradiction.

For $g_2, g_3 \in \mathbb{C}$, we see by Lemma 2.7 that the leading terms of G_1, G_0, as polynomials of α and β, are $(-1)^{n-1} s_{n-1}(\alpha^2, \beta, 0, 0)$ and $(-1)^n s_n(\alpha^2, \beta, 0, 0)$ respectively. Since $s_{n-1}(\alpha^2, \beta, 0, 0)$ and $s_n(\alpha^2, \beta, 0, 0)$ are co-prime, we conclude that $G_1(\alpha, \beta, g_2, g_3)$ and $G_0(\alpha, \beta, g_2, g_3)$ are also co-prime. The proof is complete. $\square$

Proposition 2.9. *The common zeros of $G_1 = 0$ and $G_0 = 0$ are precisely given by the pair of branch points (a, b) corresponding to Lame functions of the same type. If $\bar{X}_n$ is non-singular, there are exactly $n(n-1)$ such ordered pairs (a, b)'s.*

Proof. It suffices to prove the (generic) case that $\bar{X}_n$ is non-singular, namely the case that all the Lamé functions are distinct. The general case follows from the non-singular case by a limiting argument.

For any two Lamé functions w_a, w_b of the same type, it is easy to see that we may arrange the representatives of a and b so that (2.2) holds. It follows that $q := q_{a,-b} = q_{-a,b}$ (see (2.3)) is an even elliptic function solution to (2.4), or equivalently $q(x)$ is a polynomial solution to $L_4\, q(x) = 0$.

From the above discussion, (α, β) must be a common root of G_1 and G_0 (where $\alpha = B_a - B_b$, $\beta = B_a + B_b$). By Lemma 2.6 and 2.7, we have $\deg G_1 = n-1$ and $\deg G_0 = n$ and G_1, G_0 are co-prime to each other by Lemma 2.8. Hence by Bezout theorem there are at most $n(n-1)$ common roots.

On the other hand, the number of such ordered pairs can be determined by Proposition 1.4. Indeed, if $n = 2k$ is even, then we have

$$(k+1)k + 3k(k-1) = 4k^2 - 2k = n(n-1)$$

such pairs. If $n = 2k+1$ is odd, the number of pairs is given by

$$k(k-1) + 3(k+1)k = 4k^2 + 2k = n(n-1).$$

Hence in all cases the number of ordered pairs coming from the Lamé functions of the same type agrees with the Bezout degree of the polynomial system defined by $G_1 = 0 = G_0$. Thus these $n(n-1)$ pairs form the zero locus as expected (and there is no infinity contribution). $\square$

The above discussions from Lemma 2.5 to Proposition 2.9 constitute a complete proof of Theorem 2.4. Here is a summary: We already know that Q is an even elliptic function with singularity only at $0 \in E$. Thus

$$Q(x) = c\prod_{i=1}^{n}(\wp(z) - \wp(c_i)) =: c\prod_{i=1}^{n}(x - x_i)$$

is a polynomial solution to the ODE (2.5) with $\alpha = B_a - B_b$, $\beta = B_a + B_b$.

Since $\alpha = B_a - B_b \neq 0$, by Lemma 2.7 (α, β) must be a common root of $G_1(\alpha, \beta) = 0 = G_0(\alpha, \beta)$. Then Proposition 2.9 says that (α, β) is pair of Lamé functions of the same type. This proves Theorem 2.4.

For future reference, we combine Theorem 2.4 and Proposition 2.9 into the following statement on a fourth order ODE which arises from the *tensor product of two different (integral) Lamé equations* with the same parameter n.

Due to its importance, we will give a second (shorter and more direct) proof of the part corresponding to Theorem 2.4.

Theorem 2.10. *Let $I(z) = n(n+1)\wp(z)$. The fourth order ODE*

$$q''''(z) - 2(I+\beta)q''(z) - 6I'q'(z) + (\alpha^2 - 2I'')q(z) = 0 \tag{2.13}$$

with $\alpha \neq 0$ has an elliptic function solution if and only if (α, β) is a pair of common root to $G_0(\alpha, \beta) = 0$ and $G_1(\alpha, \beta) = 0$. Moreover, this solution must be even.

Second Proof to Theorem 2.4. Following the definition of $q_{a,-b}(z)$ in (2.3), we now consider the odd elliptic solution to (2.13) (= (2.4)) instead:

$$q(z) = \tfrac{1}{2}(q_{a,-b}(z) - q_{-a,b}(z)),$$

which has a pole of order $3 + 2l$ at $0 \in E$ with $l \leq n-2$. Thus $q(z)/\wp'(z)$ is an even elliptic function with the only pole at 0 since $q(\frac{1}{2}\omega_i) = 0$ for $1 \leq i \leq 3$. If $q(z)$ does not vanish completely, then

$$q(z) = c\wp'(z)\prod_{i=1}^{l}(\wp(z) - \wp(c_i)) =: c\wp'(z)f(\wp(z)),$$

where $f(x) = \prod_{i=1}^{l}(x - \wp(c_i)) = x^l - s_1x^{l-1} + \cdots + (-1)^l s_l$.

By Lemma 2.5, $q(z)$ satisfies

$$\begin{aligned} &q''''(z) - 2(\beta + 2n(n+1)\wp(z))q''(z) \\ &\quad - 6n(n+1)\wp'(z)q'(z) + (\alpha^2 - 2n(n+1)\wp''(z))q(z) = 0. \end{aligned} \tag{2.14}$$

By straightforward calculations, we can compute all derivatives of q in terms of derivatives of $\wp(z)$ and $f'(x)$. For example,

$$
\begin{aligned}
q'(z) &= \wp''(z)f(x) + \wp'(z)^2 f'(x),\\
q''(z) &= \wp'''(z)f(x) + 3\wp''(z)\wp'(z)f'(x) + \wp'(z)^3 f''(x), \quad \text{etc.}
\end{aligned}
$$

Then (2.14) is equivalent to

$$
\begin{aligned}
&f(x)\Big((360 - 96n(n+1))x^2 - 24\beta x + (4n(n+1) - 18)g_2 + \alpha^2\Big)\\
&+f'(x)\Big((1320 - 96n(n+1))x^3 - 36\beta x^2\\
&\qquad\qquad + (12n(n+1) - 150)g_2 x + (6n(n+1) - 60)g_3 + 3\beta g_2\Big)\\
&+f''(x)\Big((1020 - 16n(n+1))x^4 - 8\beta x^3 + (4n(n+1) - 210)g_2 x^2\\
&\qquad\qquad + (2\beta g_2 + (4n(n+1) - 120)g_3)x + 2\beta g_3 + \tfrac{15}{4}g_2^2\Big)\\
&+f'''(x)(60x^2 - 30g_2)(4x^3 - g_2 x - g_3)\\
&+f''''(x)(4x^3 - g_2 x - g_3)^2 = 0.
\end{aligned}
$$

By comparing the coefficients of x^{l+2}, we obtain

$$
\begin{aligned}
&(360 - 96n(n+1)) + l(1320 - 96n(n+1)) + l(l-1)(1020 - 16n(n+1))\\
&\quad + 240l(l-1)(l-2) + 16l(l-1)(l-2)(l-3) = 0.
\end{aligned}
$$

After simplification, this is reduced to

$$4n(n+1) = (2l+3)(2l+5),$$

which obviously leads to a contradiction since the RHS is odd. Therefore we must have $q \equiv 0$ from the beginning. That is, $\{a_i, -b_i\} = \{-a_i, b_i\}$.

If one of a, b does not correspond to a Lamé function, say $a \in X_n$, then $\{a_1, \cdots, a_n\} \cap \{-a_1, \cdots, -a_n\} = \emptyset$ and we conclude that $\{a_i\} = \{b_i\}$. Otherwise a and b correspond to Lamé functions of the same type. $\square$

Example 2.11. For $n = 2$, $\beta = B_a + B_b$, $\alpha = B_a - B_b$, we have

$$s_1 = \tfrac{1}{6}\beta, \qquad s_2 = \tfrac{1}{36}\beta^2 + \tfrac{1}{72}\alpha^2 - \tfrac{1}{4}g_2.$$

The first compatibility equation from x^1 is

$$s_1(\alpha^2 + 36g_2) - 6\beta g_2 = 0.$$

After substituting s_1 we get

$$\tfrac{1}{6}\alpha^2\beta = 0. \tag{2.15}$$

The second compatibility equation from x^0 is

$$s_2(\alpha^2 + 6g_2) - s_1(\beta g_2 + 24g_3) + 4\beta g_3 + \tfrac{3}{2}g_2^2 = 0.$$

By substituting s_1, s_2 and noticing the (expected) cancellations we get

$$\alpha^2(\tfrac{1}{36}\beta^2 + \tfrac{1}{72}\alpha^2 - \tfrac{1}{6}g_2) = 0. \tag{2.16}$$

If $B_a \neq B_b$ then (2.15) implies that $B_b = -B_a$ and then (2.16) leads to

$$B_a^2 = 3g_2 \Longrightarrow \wp(a_1) + \wp(a_2) = \pm\sqrt{g_2/3}.$$

By Example 1.5 (2), such $a \in \bar{X}_2$ lies in the branch loci of the hyperelliptic (Lamé) curve. In particular, $a, b \in \sigma^{-1}(0)$ and they are excluded by the assumption in Proposition 2.4. Denote by $\wp(\pm q_\pm) = \pm\sqrt{g_2/12}$. Then $a := \{q_+, -q_+\} \neq b := \{q_-, -q_-\}$ unless $g_2 = 0$. When $g_2 \neq 0$, $\mathbf{z}_2$ fails to distinguish the two points a and b. When $g_2 = 0$ (equivalently $\tau = e^{\pi i/3}$), $a = b$ becomes a (singular) branch point for $\sigma : \bar{X}_2 \to E_\tau$.

Example 2.12. For $n = 3$, $\beta = B_a + B_b$, $\alpha = B_a - B_b$. Then

$$\begin{aligned} s_1 &= \tfrac{1}{10}\beta, \\ s_2 &= \tfrac{1}{600}(4\beta^2 + \alpha^2 - 150g_2), \\ s_3 &= \tfrac{1}{3600}(2\beta^3 + 3\alpha^2\beta - 120\beta g_2 + 900g_3). \end{aligned}$$

The two compatibility equations from x^1 and x^0 are

$$\begin{aligned} 0 &= \tfrac{1}{600}\alpha^2(4\beta^2 + \alpha^2 + 60g_2), \\ 0 &= \tfrac{1}{3600}\alpha^2(2\beta^3 + 3\alpha^2\beta - 90\beta g_2 + 540g_3). \end{aligned}$$

If $\alpha \neq 0$ then $\alpha^2 = -4\beta^2 - 60g_2$ and the second equation becomes

$$\beta^3 + 27g_2\beta - 54g_3 = 0.$$

It is clear that there are only finite solutions (B_a, B_b)'s to this, though it may not be so straightforward to see that these 6 solution pairs (for generic tori) come from the branch loci as proved in Proposition 2.9.

3. Pre-modular forms $Z_n(\sigma; \tau)$

We call a real analytic function in $(\sigma, \tau) \in \mathbb{C} \times \mathbb{H}$ *pre-modular* if it is (holomorphic and) modular in τ for $\Gamma(N)$ whenever we fix $\sigma \pmod{\Lambda_\tau} \in E_\tau[N]$. Theorem 2.2 and Hecke's theorem on Z [7] (cf. (0.7)) then imply

Corollary 3.1. *$Z_n(\sigma; \tau) := W_n(Z)(\sigma; \tau)$ is pre-modular of weight $\frac{1}{2}n(n+1)$, with Z, $\wp(\sigma)$, $\wp'(\sigma)$, g_2, g_3 being of weight 1, 2, 3, 4, 6 respectively.*

Now we prove Theorem 0.3.

We call the $2n+1$ branch points $a \in Y_n \setminus X_n$ *trivial critical points* since $a = -a$ and the Green equation (0.5) holds trivially. They satisfy a nice compatibility condition with the case $n = 1$ under the addition map:

Lemma 3.2. *Let $a = \{a_1, \cdots, a_n\} \in Y_n$ be a solution to the Green equation $\sum_{i=1}^n \nabla G(a_i) = 0$. Then a is trivial, i.e. $a = -a$, if and only if $\sigma_n(a) \in E[2]$.*

Proof. If a is trivial, then $\sigma_n(a) \in E[2]$ clearly. If a is non-trivial, i.e. $a \in X_n$, by (1.4), it gives rise to a type II developing map f with

$$f(z+\omega_1) = e^{-4\pi i \sum_i s_i} f(z), \qquad f(z+\omega_2) = e^{4\pi i \sum_i r_i} f(z).$$

Here $a_i = r_i\omega_1 + s_i\omega_2$ for $i = 1, \ldots, n$.

If $\sigma_n(a) \in E[2]$, then both exponential factors reduce to one and we conclude that $f(z)$ is an elliptic function on E. Notice that the only zero of $f'(z)$ is at $z = 0$ which has order $2n$, and the only poles of $f'(z)$ are at $-a_i$ of order 2, $i = 1, \ldots, n$. This forces that $\sigma_n(a) \equiv 0 \pmod{\Lambda}$ and

$$f'(z) = \textstyle\sum_{j=1}^n E_j \wp(z + a_j) + C_1$$

for some constants $E_1, \ldots, E_n$ and C_1, since f' is residue free. Then

$$f(z) = -\textstyle\sum_{j=1}^n E_j \zeta(z + a_i) + C_1 z + C_2$$

for some constant C_2. But $f(z)$ is elliptic, which implies that $C_1 = 0$ and $\sum_{j=1}^n E_j = 0$. Now $f^{2k-1}(0) = 0$ for $k = 1, \ldots, n$ leads to a system of linear equations in E_j's (c.f. [3, Lemma 2.5]):

$$\textstyle\sum_{j=1}^n \wp^k(a_j) E_j = 0, \qquad k = 1, \ldots, n.$$

But then $\wp(a_i) \neq \wp(a_j)$ for $i \neq j$ forces that $E_j = 0$ for all j. This is a contradiction and so we must have $\sigma_n(a) \notin E[2]$. $\square$

The following theorem completes the proof of Theorem 0.3:

Theorem 3.3 (Extra critical points vs zeros of pre-modular forms)**.**

(i) *Given $\sigma_0 \in E_\tau \setminus E_\tau[2]$ with $Z_n(\sigma_0; \tau) = 0$, there is a unique $a \in X_n$ such that $\sigma_n(a) = \sigma_0$ and $\mathbf{z}_n(a) = Z(\sigma_0)$.*

(ii) *Conversely, if $a \in X_n$ and $\mathbf{z}_n(a) = Z(\sigma(a))$, then $Z_n(\sigma(a); \tau) = 0$ and $\sigma_n(a) \notin E_\tau[2]$.*

Proof. (i) For any given σ_0, by substituting σ by σ_0 in $W_n(\mathbf{z})$, we get a polynomial $W_{n,\sigma_0}(\mathbf{z})$ of degree $\frac{1}{2}n(n+1)$. Since $W_n(\mathbf{z})$ is the minimal polynomial of the rational function $\mathbf{z}_n \in K(\bar{X}_n)$ over $K(E)$, those $\mathbf{z}_n(a)$ with $a \in \bar{X}_n$ and $\sigma_n(a) = \sigma_0$ give precisely all the roots of $W_{n,\sigma_0}(a)$, counted with multiplicities.

Now $Z(\sigma_0)$ is a root of $W_{n,\sigma_0}(\mathbf{z})$ with $\sigma_0 \notin E[2]$, hence there is a point $a \in X_n$ corresponds to it, i.e. $Z(\sigma_0) = \mathbf{z}_n(a)$ with $\sigma_n(a) = \sigma_0$, which is unique by Theorem 2.4. Notice that if $a \in \bar{X}_n \setminus X_n$ then $a = -a$ and then $\sigma_n(a) \in E[2]$. So in fact we must have $a \in X_n$.

(ii) It is clear that $Z_n(\sigma(a)) \equiv W_n(Z(\sigma(a)) = W_n(\mathbf{z}_n(a)) = 0$. Since $a \in X_n$, by (2.1) we have $\sum_{i=1}^n \nabla G(a_i) = 0$. But since a is non-trivial ($a \in X_n$ by assumption), Lemma 3.2 implies that $\sigma_n(a) \notin E[2]$. $\square$

We present below an extended version of Theorem 0.3 in terms of *monodromy groups of Lamé equations*. The original case of mean field equations corresponds to the case with *unitary monodromy* (cf. [3]).

Let $a = \{a_1, \cdots, a_n\} \in X_n$, $B_a = (2n-1)\sum_{i=1}^{n} \wp(a_i)$ and w_a, w_{-a} be the independent ansatz solutions (0.9) to $w'' = (n(n+1)\wp(z) + B_a)w$. From (1.3), one calculate easily that the monodromy matrices are given by

$$\begin{aligned} \begin{pmatrix} w_a \\ w_{-a} \end{pmatrix} (z+\omega_1) &= \begin{pmatrix} e^{-2\pi i r} & 0 \\ 0 & e^{2\pi i r} \end{pmatrix} \begin{pmatrix} w_a \\ w_{-a} \end{pmatrix} (z), \\ \begin{pmatrix} w_a \\ w_{-a} \end{pmatrix} (z+\omega_2) &= \begin{pmatrix} e^{2\pi i s} & 0 \\ 0 & e^{-2\pi i s} \end{pmatrix} \begin{pmatrix} w_a \\ w_{-a} \end{pmatrix} (z), \end{aligned} \tag{3.1}$$

where the two *complex numbers* $r, s \in \mathbb{C}$ are uniquely determined by

$$r\omega_1 + s\omega_2 = \sigma(a) = \sum_{i=1}^{n} a_i, \qquad r\eta_1 + s\eta_2 = \sum_{i=1}^{n} \zeta(a_i). \tag{3.2}$$

The system is non-singular by the Legendre relation $\omega_1\eta_2 - \omega_2\eta_1 = -2\pi i$.

The next lemma extends Lemma 3.2:

Lemma 3.4. *Let $a \in X_n$ with (r, s) given by* (3.2). *Then $(r, s) \notin \frac{1}{2}\mathbb{Z}^2$.*

Proof. If $(r, s) \in \frac{1}{2}\mathbb{Z}^2$ then $f := w_a/w_{-a}$ is elliptic by (3.1). Since

$$f' = \frac{w_a' w_{-a} - w_a w_{-a}'}{w_a^2} = \frac{C}{w_a^2},$$

we find that $z = 0$ is the only zero of $f'(z)$, which has order $2n$. The proof of Lemma 3.2 for this f goes through and leads to a contradiction. ☐

Now we consider $Z_{r,s}(\tau)$ in (0.7) but with $r, s, \in \mathbb{C}$, and define

$$Z_{n;r,s}(\tau) := W_n(Z_{r,s})(r + s\tau; \tau), \qquad r, s \in \mathbb{C}. \tag{3.3}$$

It reduces to $Z_n(\sigma; \tau)$ for $\sigma = r + s\tau$ when $r, s \in \mathbb{R}$ (see [2] for its role in the isomonodromy problems and Painleve VI equations).

By substituting $Z_n(\sigma; \tau)$ with $Z_{n;,r,s}(\tau)$ and using Lemma 3.4 in place of Lemma 3.2, the proof of Theorem 3.3 also leads to:

Theorem 3.5. *Let $r, s \in \mathbb{C}$. Then any non-trivial solution τ to $Z_{n;r,s}(\tau) = 0$, i.e. with $r + s\tau \pmod{\Lambda_\tau} \notin E_\tau[2]$, corresponds to an $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ such that $a \pmod{\Lambda_\tau} \in X_n(\tau)$ and*

$$\sum_{i=1}^{n} a_i = r + s\tau, \qquad \sum_{i=1}^{n} \zeta(a_i; \tau) = r\eta_1(\tau) + s\eta_2(\tau).$$

Equivalently, by (3.2), *the Lame equation $w'' = (n(n+1)\wp(z; \Lambda_\tau) + B_a)w$ has its monodromy representation given by* (3.1).

We leave the straightforward justifications to the interested reader.

4. AN EXPLICIT DETERMINATION OF Z_n

From the equations of $\bar{X}_n \subset \mathrm{Sym}^n E$ (cf. (0.4)) and the recursively defined algebraic formula of the addition map $E^n \to E$, in principle it is possible to compute W_n and hence Z_n by *elimination theory* (cf. [9]). However we shall present a more direct approach on this to reveal more structures inside it.

Besides the Hermite–Halphen ansatz (0.9), there is another ansatz, the *Hermite–Krichever ansatz*, which can also be used to construct solutions to the integral Lamé equation (0.8). It takes the form

$$\psi(z) := \Big(U(\wp(z)) + V(\wp(z))\frac{\wp'(z)+\wp'(a_0)}{\wp(z)-\wp(a_0)}\Big)\frac{\sigma(z-a_0)}{\sigma(z)}e^{(\zeta(a_0)+\kappa)z}, \tag{4.1}$$

where $U(x)$ and $V(x)$ are polynomials in x, $a_0 \in E^\times$, and $\kappa \in \mathbb{C}$ is a constant. As usual, we set $(x, y) = (\wp(z), \wp'(z))$ and $(x_0, y_0) = (\wp(a_0), \wp'(a_0))$ to be the corresponding algebraic coordinates.

Notice that (4.1) makes sense since ψ only has poles at $z = 0$ (the one at $z = a_0$ from $(\wp(z) - \wp(a_0))^{-1}$ cancels with the zero from $\sigma(z - a_0)$). Moreover, in order for $\mathrm{ord}_{z=0}\,\psi(z) = -n$, we must have

Lemma 4.1 (Degree constraints)**.**

(i) *If* $n = 2m$ *with* $m \in \mathbb{N}$ *then* $\deg U \le m-1$ *and* $\deg V = m-1$.
(ii) *If* $n = 2m+1$ *with* $m \in \mathbb{N}\cup\{0\}$ *then* $\deg U = m$ *and* $\deg V \le m-1$.

By an obvious normalization, in case (i) we may assume that $U(x) = \sum_{i=0}^{m-1} u_i x^i$, $V(x) = \sum_{i=0}^{m-1} v_i x^i$ with $v_{m-1} = 1$, and in case (ii) $U(x) = \sum_{i=0}^{m} u_i x^i$ with $u_m = 1$ and $V(x) = \sum_{i=0}^{m-1} v_i x^i$. In both cases, the requirement that $\psi(z)$ satisfies (0.8) leads to recursive relations on u_i's and v_i's. In doing so, it is more convenient to work on the algebraic coordinates. This had been carried out by Maier in [13, §4]. The following is a summary:

In case (i) the recursion determines v_i ($v_{m-1} = 1$) and then u_i for $i = m-1, m-2, \cdots$ in decreasing order. In case (ii) it starts with $u_m = 1$ and determines v_i and then u_i for $i = m-1, m-2, \cdots$. There are two compatibility equations coming from $u_{-1}(B, \kappa, x_0, y_0) = 0$ and $v_{-1}(B, \kappa, x_0, y_0) = 0$. The two parameters x_0, y_0 satisfy $y_0^2 = 4x_0^3 - g_2x_0 - g_3$. Hence there are four variables $(B, \kappa, x_0, y_0) \in \mathbb{C}^4$ which are subject to three polynomial equations. By taking in to account the limiting cases with $(x_0, y_0) = (\infty, \infty)$, this recovers the Lame curve $\bar{Y}_n$, which was denoted by Γ_ℓ in [13] with $\ell = n$.

There are four natural coordinate projections (rational functions) $\bar{Y}_n \to \mathbb{P}^1$, namely B, κ, x_0 and y_0 respectively. The first one $B : \bar{Y}_n \to \mathbb{P}^1$ is simply the hyperelliptic structure map. The main result in [13] is an explicit description of the other 3 maps in terms of the coordinates (B, C) on $\bar{Y}_n$:

Theorem 4.2 ([13, Theorem 4.1]). *For all $n \in \mathbb{N}$ and $i \in \{1, 2, 3\}$,*

$$\begin{aligned} x_0(B) &= e_i + \frac{4}{n^2(n+1)^2} \frac{l_i(B) lt_i(B)^2}{l_0(B) lt_0(B)^2}, \\ y_0(B, C) &= \frac{16}{n^3(n+1)^3} \frac{C}{c_n} \frac{lt_1(B) lt_2(B) lt_3(B)}{l_0(B)^2 lt_0(B)^3}, \\ \kappa(B, C) &= -\frac{(n-1)(n+2)}{n(n+1)} \frac{C}{c_n} \frac{l_\theta(B)}{l_0(B) lt_0(B)}. \end{aligned} \tag{4.2}$$

The formula for $x_0(B)$ is independent of the choices of i.

All the factors lie in $\mathbb{Q}[e_1, e_2, e_3, g_2, g_3, B]$ and are monic *in B. They are homogeneous with weights of B, e_i, g_2, g_3 being $1, 1, 2, 3$ respectively.*

As a simple consistency check, we have $C^2 = \ell_n(B)$ by Proposition 1.4.

In (4.2), $lt_j(B)$, $j = 0, 1, 2, 3$, are the *twisted Lamé polynomials* whose zeros correspond to solutions to (0.8) given by the Hermite–Krichever ansatz with $\kappa \neq 0$ and $a_0 = 0$, $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_2$, $\frac{1}{2}\omega_3$ respectively, i.e. $(x_0, y_0) = (\infty, \infty)$, $(e_1, 0)$, $(e_2, 0)$, $(e_3, 0)$ respectively.

The polynomial $l_\theta(B)$ is the *theta-twisted polynomial* whose roots correspond to the case $\kappa = 0$ and $a_0 \notin E[2]$. (For $\kappa = 0$ and $a_0 \in E[2]$ they correspond to the *ordinary* Lamé polynomials $l_i(B)$'s.)

Remark 4.3. In [13] $\nu = C/c_n$ is used instead. Also $l_0(B), l_i(B), lt_0(B), lt_i(B)$, and $l_\theta(B)$ $(i = 1, 2, 3)$ are written there as $L^I_\ell(B; g_2, g_3)$, $L^{II}_\ell(B; e_i, g_2, g_3)$, $Lt^I_\ell(B; g_2, g_3)$, $Lt^{II}_\ell(B; e_i, g_2, g_3)$, and $L\theta_\ell(B; g_2, g_3)$ respectively, where $\ell = n$.

The compatibility equations from the recursive formulas for these special cases give rise to explicit formulas for $lt_j(B)$'s and $l_\theta(B)$'s. Tables for $lt_0(B)$, $l_\theta(B)$ up to $n = 8$, and for $lt_i(B)$ up to $n = 6$, are given in [13, Table 5, 6].

Example 4.4. We recall Maier's formulas for $lt_j(B)$ and $l_\theta(B)$ for $n \le 4$.

(1) First of all, $l_\theta(B) = 1$ for $n \le 3$. For $n = 4$,

$$l_\theta(B) = B^2 - \tfrac{193}{3} g_2.$$

Also for $n = 1$, $lt_j(B) = 1$ for all j.

(2) $n = 2$: $lt_0(B) = 1$, $lt_i(B) = B - 6e_i$ for $i = 1, 2, 3$.

(3) $n = 3$: $lt_0(B) = B^2 - \frac{75}{4} g_2$, and for $i = 1, 2, 3$,

$$lt_i(B) = B^2 - 15 e_i B + \tfrac{75}{4} g_2 - 225 e_i^2.$$

(4) $n = 4$: $lt_0(B) = B^3 - \frac{343}{4} g_2 B - \frac{1715}{2} g_3$. For $i = 1, 2, 3$,

$$\begin{aligned} lt_i(B) = B^4 &- 55 e_i B^3 + (\tfrac{539}{4} g_2 - 945 e_i^2) B^2 \\ &+ (1960 e_i g_2 + 2450 g_3) B + 61740 e_i^2 g_2 - 68600 e_i g_3 - 9261 g_2^2. \end{aligned}$$

To apply Theorem 4.2, we need to compare the projection map

$$\pi_n : \bar{Y}_n \to E, \qquad a \mapsto \pi_n(a) := a_0. \tag{4.3}$$

with the addition map $\sigma_n : \bar{Y}_n \to E$. They turn out to be the same!

Theorem 4.5. *$\pi_n(a) = \sigma_n(a)$. Moreover, $\kappa(a) = -\mathbf{z}_n(a)$.*

Proof. During the proof we view $a_i \in \mathbb{C}$ instead of its image $[a_i] \in E$.

Let $a \in Y_n$. The two expressions (0.9) and (4.1), which correspond to the same solution to the Lamé equation (0.8), must be proportional to each other by a constant. Hence we get

$$\kappa(a) = \sum_{i=1}^{n} \zeta(a_i) - \zeta(a_0).$$

Recall that $\mathbf{z}_n(a) = \zeta(\sigma_n(a)) - \sum_{i=1}^{n} \zeta(a_i)$. Then

$$\mathbf{z}_n(a) + \kappa(a) = \zeta(\sigma_n(a)) - \zeta(a_0). \tag{4.4}$$

As a well defined meromorphic function on $\bar{Y}_n$, we conclude that

$$a_0(a) = \sigma_n(a) + c$$

for some constant $c \in \mathbb{C}$. Consider a point $a \in Y_n \setminus X_n$ with $\sigma_n(a) = \frac{1}{2}\omega_1$, i.e. $l_1(B_a) = 0$. Such a exists by Proposition 1.4. Then $\mathbf{z}_n(a) = 0$ trivially. We also have $\kappa(a) = 0$ by Theorem 4.2 since

$$C_a^2 = c_n^2 l_0(B_a) l_1(B_a) l_2(B_a) l_3(B_a) = 0$$

(again by Proposition 1.4). So (4.4) implies $0 = \frac{1}{2}\eta_1 - \zeta(\frac{1}{2}\omega_1 + c)$, and hence $c = 0$. This proves $\sigma_n(a) = a_0$, which represents $\pi_n(a)$ in E, and also $\kappa(a) = -\mathbf{z}_n(a)$. The proof is complete. $\square$

Now we may describe the explicit construction of the polynomial $W_n(\mathbf{z})$ in Theorem 2.2 based on Theorem 4.2. It is indeed merely an application of the elimination theory using resultant.

By Theorem 4.2 and 4.5, we may eliminate C to get

$$\frac{y_0}{\mathbf{z}_n} = \frac{16}{n^2(n+1)^2(n-1)(n+2)} \frac{lt_1(B) lt_2(B) lt_3(B)}{l_0(B) lt_0(B)^2 l_\theta(B)}, \tag{4.5}$$

which leads to a polynomial equation $g = 0$ for

$$g := \mathbf{z} \prod_{i=1}^{3} lt_i(B) - y_0 \frac{n^2(n+1)^2(n-1)(n+2)}{16} l_0(B) lt_0(B)^2 l_\theta(B). \tag{4.6}$$

On the other hand, the three rational expressions of x_0 lead to $f = 0$ for

$$\begin{aligned} f &:= l_i(B) lt_i(B)^2 - (x_0 - e_i) \frac{n^2(n+1)^2}{4} l_0(B) lt_0(B)^2 \\ &= \frac{1}{3} \sum_{i=1}^{3} l_i(B) lt_i(B)^2 - x_0 \frac{n^2(n+1)^2}{4} l_0(B) lt_0(B)^2. \end{aligned} \tag{4.7}$$

Notice that f, g are polynomials in g_2, g_3 (and B, x_0, y_0) instead of e_i's.

Let $R(f, g; B)$ be the *resultant of the two polynomials* f and g arising from the elimination of the variable B. Standard elimination theory (see e.g [9, Chapter 5]) implies that $R(f, g; B)$ gives *the equation* defining the branched covering map $\sigma_n : \bar{Y}_n \to E$ outside the loci $C = 0$:

Proposition 4.6. *$R(f,g;B)(\mathbf{z}) = \lambda_n W_n(\mathbf{z}) \in \mathbb{Q}[g_2,g_3,x_0,y_0][\mathbf{z}]$, where $\lambda_n = \lambda_n(g_2,g_3,x_0,y_0)$ is independent of $\mathbf{z}$.*

In particular, the pre-modular form $Z_n(\sigma;\tau) = W_n(Z)(\sigma;\tau)$ can be explicitly computed for any $n \in \mathbb{N}$ by way of the resultant $R(f,g;B)$.

In practice, such a computation is time consuming even using computer. In the following, we apply it to the initial cases up to $n = 4$. As before we denote $x_0 = \wp(\sigma) =: \wp$ and $y_0 = \wp'(\sigma) =: \wp'$.

Example 4.7. For $n = 2$, it is easy to see that

$$\begin{aligned} f &= B^3 - 9\wp B^2 + 27(g_2\wp + g_3), \\ g &= \mathbf{z}B^3 - 9\wp' B^2 - 9\mathbf{z}g_2 B + 27(g_2\wp' - 2\mathbf{z}g_3). \end{aligned}$$

The resultant $R(f,g;B)$ is calculated by the 6×6 *Sylvester determinant*:

$$\begin{vmatrix} 1 & -9\wp & 0 & 27(g_2\wp + g_3) & 0 & 0 \\ 0 & 1 & -9\wp & 0 & 27(g_2\wp+g_3) & 0 \\ 0 & 0 & 1 & -9\wp & 0 & 27(g_2\wp+g_3) \\ \mathbf{z} & -9\wp' & -9\mathbf{z}g_2 & 27(g_2\wp' - 2\mathbf{z}g_3) & 0 & 0 \\ 0 & \mathbf{z} & -9\wp' & -9\mathbf{z}g_2 & 27(g_2\wp' - 2\mathbf{z}g_3) & 0 \\ 0 & 0 & \mathbf{z} & -9\wp' & -9\mathbf{z}g_2 & 27(g_2\wp' - 2\mathbf{z}g_3) \end{vmatrix}.$$

A direct evaluation gives

$$R(f,g;B)(\mathbf{z}) = -3^9\Delta(\wp')^2(\mathbf{z}^3 - 3\wp\mathbf{z} - \wp').$$

Here $\Delta = g_2^3 - 27g_3^2$ is the discriminant. This gives $W_2(\mathbf{z}) = \mathbf{z}^3 - 3\wp\mathbf{z} - \wp'$ and $Z_2(\sigma;\tau) = W_2(Z) = Z^3 - 3\wp Z - \wp'$.

Example 4.8. For $n = 3$, we have

$$\begin{aligned} f &= 16B^6 - 576B^5\wp + 360B^4 g_2 + 5400B^3(5g_3 + 4g_2\wp) \\ &\quad - 3375B^2g_2^2 - 84375\Delta - 101250Bg_2(3g_3 + 2g_2\wp), \\ g &= 16B^6\mathbf{z} - 1440B^5\wp' - 1800B^4g_2\mathbf{z} + 54000B^3(g_2\wp' - g_3\mathbf{z}) \\ &\quad - 16875B^2g_2^2\mathbf{z} - 506250Bg_2^2\wp' + 421875\Delta\mathbf{z}. \end{aligned}$$

It takes a couple seconds to evaluate the corresponding 12×12 Sylvester determinant (e.g. using *Mathematica*) to get

$$R(f,g;B)(\mathbf{z}) = 2^{36}3^{27}5^{30}\Delta^5(\wp')^4 W_3(\mathbf{z}),$$

where $W_3(\mathbf{z})$ is given by

$$W_3(\mathbf{z}) = \mathbf{z}^6 - 15\wp\mathbf{z}^4 - 20\wp'\mathbf{z}^3 + (\tfrac{27}{4}g_2 - 45\wp^2)\mathbf{z}^2 - 12\wp\wp'\mathbf{z} - \tfrac{5}{4}\wp'^2.$$

It seems impractical to evaluate this resultant by hand. .

Both Z_2 and Z_3 are known to Dahmen [4]. Here is a new example:

Example 4.9. For $n = 4$, the expansions of the polynomials f and g, as given in (4.7) and (4.6) by a direct substitution, are already too complicate to put here. Nevertheless, a couple hours *Mathematica* calculation gives

$$R(f,g;B)(\mathbf{z}) = -2^{80}3^{63}5^{60}7^{63}\Delta^{18}(\wp')^8 W_4(\mathbf{z}),$$

where $W_4(\mathbf{z})$ is the degree 10 polynomial:

$$
\begin{aligned}
W_4(\mathbf{z}) &= \mathbf{z}^{10} - 45\wp\mathbf{z}^8 - 120\wp'\mathbf{z}^7 + (\tfrac{399}{4}g_2 - 630\wp^2)\mathbf{z}^6 - 504\wp\wp'\mathbf{z}^5 \\
&\quad - \tfrac{15}{4}(280\wp^3 - 49g_2\wp - 115g_3)\mathbf{z}^4 + 15(11g_2 - 24\wp^2)\wp'\mathbf{z}^3 \\
&\quad - \tfrac{9}{4}(140\wp^4 - 245g_2\wp^2 + 190g_3\wp + 21g_2^2)\mathbf{z}^2 \\
&\quad - (40\wp^3 - 163g_2\wp + 125g_3)\wp'\mathbf{z} + \tfrac{3}{4}(25g_2 - 3\wp^2)(\wp')^2.
\end{aligned}
\tag{4.8}
$$

The weight 10 pre-modular form $Z_4(\sigma;\tau)$ is then obtained.

We end this section with a brief discussion on the *rationality property*. We have constructed two affine curves from $\bar{X}_n$. One is the hyperelliptic model $Y_n = \{(B, C) \mid C^2 = \ell_n(B)\}$, another one is $Y'_n := \{(x_0, y_0, \mathbf{z}) \mid y_0^2 = 4x_0^2 - g_2x_0 - g_3,\ W_n(x_0, y_0; \mathbf{z}) = 0\}$ which is understood as a degree $\frac{1}{2}n(n+1)$ branched cover of the original curve $E = \{(x_0, y_0) \mid y_0^2 = 4x_0^3 - g_2x_0 - g_3\}$ under the projection $\sigma'_n : Y'_n \to E$ with defining equation $W_n(\mathbf{z}) = 0$.

Y_n is birational to Y'_n over E, namely the addition map $\sigma_n : Y_n \to E$ is compatible with $\sigma'_n : Y'_n \to E$. Notice that both ℓ_n and W_n have coefficients in $\mathbb{Q}[g_2, g_3]$. The explicit birational map $\phi : (B, C) \dashrightarrow (x_0, y_0, \mathbf{z})$ (given in Theorem 4.2 and 4.5 via $\mathbf{z}_n = -\kappa$) also has coefficients in $\mathbb{Q}[g_2, g_3]$. This implies that ϕ is defined over $\mathbb{Q}$. Moreover ϕ extends to a *birational morphism*

$$
\begin{tikzcd}
\bar{Y}_n \cong \bar{X}_n \arrow[rr, "\phi"] \arrow[dr, "\sigma_n"'] & & \bar{Y}'_n \arrow[dl, "\sigma'_n"] \\
& E &
\end{tikzcd}
$$

by identifying $\sigma_n^{-1}(0_E)$ with $\mathbf{z}_n^{-1}(\infty)$. The morphism ϕ is an isomorphism outside those branch points for $Y_n \to \mathbb{P}^1$ (i.e. $C = 0$). In particular, the non-isomorphic loci lie in $\mathbf{z}_n = 0$ by (4.2) and Theorem 4.5.

Remark 4.10. In contrast to the smoothness of $Y_n(\tau)$ for general τ, for all $n \geq 3$ the model $Y'_n(\tau)$ is singular at points $\mathbf{z} = 0 = y_0$ (and hence $x_0 = e_i$ for some i). Indeed from (4.2) this is equivalent to $C = 0$ and $l_i(B)lt_i(B)^2 = 0$ for some $1 \leq i \leq 3$. For $n = 2$, there is only one solution B for each fixed i (c.f. Example 4.4). However, for $n \geq 3$ there are more than one solutions B. These points $(B, 0) \in Y_n$ are collapsed to the same point $(x_0, y_0, \mathbf{z}) = (e_i, 0, 0) \in Y'_n$ under ϕ, thus $(e_i, 0, 0)$ is a singular point of Y'_n.

For $n = 3, 4$ this is easily seen from the equation $W_n(\mathbf{z}) = 0$ given above since it contains a quadratic polynomial in $(\mathbf{z}, \wp')$ as its lowest degree terms.

In particular, the birational map ϕ^{-1} is also represented by rational functions $B = B(x_0, y_0, \mathbf{z})$ and $C = C(x_0, y_0, \mathbf{z})$ with coefficients in $\mathbb{Q}[g_2, g_3]$ and with at most poles along $\mathbf{z} = 0$. In principle such an explicit inverse can be obtained by a Groebner basis calculation associated to the ideal of the graph Γ_ϕ. The following statement is clear from the above description:

Proposition 4.11. *Let E be defined over $\mathbb{Q}$, i.e. $g_2, g_3 \in \mathbb{Q}$. Then the Lamé curve $\bar{Y}_n$ is also defined over $\mathbb{Q}$ for all $n \in \mathbb{N}$. Moreover, $\bar{Y}'_n$ and all the morphisms $\sigma_n, \sigma'_n, \phi$ are also defined over $\mathbb{Q}$.*

A rational point $(B, C) \in \bar{Y}_n$ is mapped to a rational point $(x_0, y_0, \mathbf{z}) \in \bar{Y}'_n$ by ϕ. For the converse, given $(x_0, y_0) \in E(\mathbb{Q})$, a point $(x_0, y_0, \mathbf{z})$ in the σ'_n fiber gives a unique $(B, C) \in \bar{Y}_n(\mathbb{Q})$ if $\mathbf{z} \in \mathbb{Q}$ and $(x_0, y_0, \mathbf{z}) \neq (e_i, 0, 0)$ for any i.

Remark 4.12. It is well known that there are only few (i.e. at most finite) rational points on a *non-elliptic* hyperelliptic curve. This phenomenon is consistent with the irreducibility of the polynomial $W_n(\mathbf{z})$ over $K(E)$ in light of Hilbert's irreducibility theorem that there is a infinite (Zariski dense) set of $(g_2, g_3, x_0, y_0) \in \mathbb{Q}^4$ so that the specialization of $W_n(\mathbf{z})$ is still irreducible. Nevertheless, it might be interesting to see if $\mathbf{z}_n$ plays any role in the study of rational points.

Appendix A. A counting formula for Lamé equations

By You-Cheng Chou [2]

Using the pre-modular forms constructed in §3 and §4, we verify the $n = 4$ case of Dahmen's conjectural counting formula (Conjecture 73 in [4]) for integral Lamé equations with finite monodromy. It is known that the finite monodromy group is necessarily a dihedral group.

A.1. **Dahmen's conjecture.** Let $L_n(N)$ be the number of Lamé equations $w'' = (n(n+1)\wp(z) + B)w$ up to linear equivalence which has finite monodromy isomorphic to the dihedral group D_N. Using the Hermite–Halphen ansatz (0.9) and the theory in §3, the problem is reduced to the zero counting of the $\mathrm{SL}(2, \mathbb{Z})$ modular form

$$M_n(N) := \prod_{\substack{0 \le k_1, k_2 < N \\ \gcd(k_1, k_2, N) = 1}} Z_n\Big(\frac{k_1 + k_2\tau}{N}; \tau\Big).$$

Using this, by repeating Dahmen's argument in [4], Lemma 65, 74, we get

Proposition A.1. *Suppose that for all $N \in \mathbb{Z}_{\ge 3}$ and $n \in \mathbb{N}$ we have that*

$$\nu_\infty(M_n(N)) = a_n \phi(N) + b_n \phi\Big(\frac{N}{2}\Big),$$

where $a_{2m} = a_{2m+1} = m(m+1)/2$, $b_{2m} = b_{2m-1} = m^2$. Then

$$L_n(N) = \tfrac{1}{2}\left(\frac{n(n+1)\Psi(N)}{24} - \left(a_n\phi(N) + b_n\phi\Big(\frac{N}{2}\Big)\right)\right) + \tfrac{2}{3}\epsilon_n(N),$$

where $\epsilon_n(N) = 1$ if $N = 3$ and $n \equiv 1 \pmod 3$, and $\epsilon_n(N) = 0$ otherwise.

Furthermore, $Z_n(\sigma; \tau)$ with σ a torsion point has only simple zeros in $\tau \in \mathbb{H}$.

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Proof. Recall the formula for $\mathrm{SL}(2,\mathbb{Z})$ modular forms of weight k:

$$\sum_{P\neq\infty,i,\rho}\nu_P(f)+\nu_\infty(f)+\frac{\nu_i(f)}{2}+\frac{\nu_\rho(f)}{3}=\frac{k}{12}.$$

For $f=M_n(N)$, the weight $k=\frac{1}{2}n(n+1)\Psi(N)$. Notice that the counting is always doubled under the symmetry $(k_1,k_2)\to(N-k_1,N-k_2)$, thus by [4], Lemma 65, an upper bound for $L_n(N)$ is given by

$$U_n(N):=\tfrac{1}{2}\left(\frac{n(n+1)\Psi(N)}{24}-\left(a_n\phi(N)+b_n\phi\Big(\frac{N}{2}\Big)\right)\right)+\tfrac{2}{3}\epsilon_n(N).$$

That is, $L_n(N)\le U_n(N)$. Moreover, the equality holds if and only if each factor $Z_n((k_1+k_2\tau)/N;\tau)$ has only simple zeros.

We will show the equality holds by comparing it with the counting formula for the projective monodormy group $PL_n(N)$ (c.f. [4], Lemma 74).

We recall the relation between $L_n(N)$ and $PL_n(N)$:

$$PL_n(N)=\begin{cases}L_n(N)+L_n(2N) & \text{if } N \text{ is odd},\\ L_n(2N) & \text{if } N \text{ is even.}\end{cases}$$

If n is even and N is odd, we have

$$\begin{aligned}PL_n(N)&=L_n(N)+L_n(2N)\\ &\le\tfrac{1}{2}\left(\frac{n(n+1)\Psi(N)}{24}-\left(\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}\phi(N)+\frac{n^2}{4}\phi\Big(\frac{N}{2}\Big)\right)\right)+\tfrac{2}{3}\epsilon_n(N)\\ &\quad+\tfrac{1}{2}\left(\frac{n(n+1)\Psi(2N)}{24}-\left(\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}\phi(2N)+\frac{n^2}{4}\phi(N)\right)\right)+\tfrac{2}{3}\epsilon_n(2N)\\ &=\frac{n(n+1)}{12}\left(\Psi(N)-3\phi(N)\right)+\tfrac{2}{3}\epsilon_n(N)\end{aligned}$$

For the last equality, we use $\epsilon_n(2N)=0$, $\Psi(2N)=3\Psi(N)$ and $\phi(2N)=\phi(N)$. (If N is even, the relations are $\epsilon_n(N)=0$, $\Psi(2N)=4\Psi(N)$ and $\phi(2N)=\phi(N)$.) For the other three cases with (n,N) being (even, even), (odd, odd) or (odd, even), the computations are similar, and all lead to

$$PL_n(N)\le\frac{n(n+1)}{12}\left(\Psi(N)-3\phi(N)\right)+\tfrac{2}{3}\epsilon_n(N).$$

On the other hand, using the method of *dessin d'enfants*, Dahmen showed directly that the equality holds [5]. Thus all the intermediate inequalities are indeed equalities, and in particular $L_n(N)=U_n(N)$ holds. $\square$

A.2. **q-expansions for some modular forms.** Recall that

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \frac{1}{(m+z)^k} &= \frac{1}{(k-1)!}(-2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n z},\\
\sum_{n\in\mathbb{Z}} \frac{1}{(x+n)^2} &= \pi^2 \cot^2(\pi x) + \pi^2,\\
\sum_{n\in\mathbb{Z}} \frac{1}{(x+n)^3} &= \pi^3 \cot^3(\pi x) + \pi^3 \cot(\pi x).
\end{aligned}$$

We compute the q-expansions for $g_2, g_3, \wp, \wp', Z$, where $q = e^{2\pi i \tau}$:

$$g_2 = 60 \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^4} = 60\Big(2\zeta(4) + 2\frac{(-2\pi i)^4}{3!} \sum_{n=1}^{\infty} \sigma_3(n) q^n\Big),$$

where $\sigma_k(n) := \sum_{d|n} d^k$. Similarly,

$$g_3 = 140 \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^6} = 140\Big(2\zeta(6) + 2\frac{(-2\pi i)^6}{5!} \sum_{n=1}^{\infty} \sigma_5(n) q^n\Big).$$

Let $z = t + s\tau$. For $s = 0$, we have

$$\begin{aligned}
\wp'(t;\tau) &= -2 \sum_{n,m\in\mathbb{Z}} \frac{1}{(t+n+m\tau)^3}\\
&= -2 \sum_{n\in\mathbb{Z}} \frac{1}{(t+n)^3} - 2 \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \left(\frac{1}{(m\tau+n+t)^3} - \frac{1}{(m\tau+n-t)^3} \right)\\
&= -2 \sum_{n\in\mathbb{Z}} \frac{1}{(t+n)^3} - 2 \sum_{m=1}^{\infty} \frac{(-2\pi i)^3}{2!} \sum_{n=1}^{\infty} n^2 \left(e^{2\pi i n(m\tau+t)} - e^{2\pi i n(m\tau-t)} \right)\\
&= -2\pi^3 \cot(\pi t) - 2\pi^3 \cot^3(\pi t) + 16\pi^3 \sum_{n,m=1}^{\infty} n^2 \sin(2\pi n t)\, q^{nm}.
\end{aligned}$$

$$\begin{aligned}
\wp(t;\tau) &= \frac{1}{t^2} + \sum_{(n,m)\neq(0,0)} \left(\frac{1}{(t+n+m\tau)^2} - \frac{1}{(n+m\tau)^2} \right)\\
&= \sum_{n\in\mathbb{Z}} \frac{1}{(t+n)^2} - \sum_{n=1}^{\infty} \frac{2}{n^2} + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \left(\frac{1}{(m\tau+t+n)^2} + \frac{1}{(m\tau-t+n)^2} - \frac{2}{(m\tau+n)^2} \right)\\
&= \pi^2 \cot^2(\pi t) + \tfrac{2}{3}\pi^2 + \sum_{m=1}^{\infty} (-2\pi i)^2 \sum_{n=1}^{\infty} \left(e^{2\pi i n(m\tau+t)} + e^{2\pi i n(m\tau-t)} - 2e^{2\pi i n m\tau} \right)\\
&= \pi^2 \cot^2(\pi t) + \tfrac{2}{3}\pi^2 + 8\pi^2 \sum_{n,m=1}^{\infty} (1 - \cos 2n\pi t) q^{nm}.
\end{aligned}$$

Also, the Hecke function Z (cf. (0.7)):

$$Z(t;\tau) = \pi \cot(\pi t) + 4\pi \sum_{n,m=1}^{\infty} (\sin 2n\pi t) q^{nm}.$$

For $s = \frac{1}{2}$, we have

$$\begin{aligned}
\wp'(t+\tfrac{1}{2}\tau;\tau) &= -2 \sum_{(n,m)\neq(0,0)} \frac{1}{(t+n+(\frac{1}{2}+m)\tau)^3} \\
&= -2 \sum_{m=1}^{\infty} \left(\sum_{m\in\mathbb{Z}} \frac{1}{(n+t+(m-\frac{1}{2})\tau)^3} - \sum_{n\in\mathbb{Z}} \frac{1}{(n-t+(m-\frac{1}{2})\tau)^3} \right) \\
&= -2\frac{(-2\pi i)^3}{2!} \sum_{n,m=1}^{\infty} n^2 \left(e^{2\pi i n(t+(m-\frac{1}{2})\tau)} - e^{2\pi i n(-t)+(m-\frac{1}{2})\tau} \right) \\
&= 16\pi^3 \sum_{n,m=1}^{\infty} n^2 (\sin 2\pi n t) q^{n(m-\frac{1}{2})}.
\end{aligned}$$

Similarly,

$$\wp(t+\tfrac{1}{2}\tau;\tau) = -\tfrac{1}{3}\pi^2 + 8\pi^2 \sum_{n,m=1}^{\infty} n q^{nm} - 8\pi^2 \sum_{n,m=1}^{\infty} n (\cos 2\pi n t) q^{n(m-\frac{1}{2})},$$

and $Z(t+\frac{1}{2}\tau;\tau) = 4\pi \sum_{n,m=1}^{\infty} (\sin 2\pi n t) q^{n(m-\frac{1}{2})}$.

A.3. **The counting formula for $n = 4$.** Now we give the computations for $n = 4$ and prove the formula $L_4(N) = U_4(N)$ from Proposition A.1.

Theorem A.2. *For $n = 4$ and $N \in \mathbb{Z}_{\geq 3}$, we have*

$$L_4(N) = \tfrac{1}{2} \left(\tfrac{5}{6}\Psi(N) - \left(3\phi(N) + 4\phi\left(\frac{N}{2}\right) \right) \right).$$

Moreover, $Z_4(\sigma;\tau)$ with $\sigma \in E_\tau[N]$ has only simple zeros in $\tau \in \mathbb{H}$.

Proof. For $n = 4$, the pre-modular form $Z_4 = W_4(Z)$ is given in (4.8):

$$\begin{aligned}
W_4(Z) = Z^{10} &- 45\wp Z^8 - 120\wp' Z^7 + (\tfrac{399}{4}g_2 - 630\wp^2)Z^6 - (504\wp\wp')Z^5 \\
&- \tfrac{15}{4}(280\wp^3 - 49g_2\wp - 115g_3)Z^4 + 15(11g_2 - 24\wp^2)\wp' Z^3 \\
&- \tfrac{9}{4}(140\wp^4 - 245g_2\wp^2 + 190g_3\wp + 21g_2^2)Z^2 \\
&- (40\wp^3 - 163g_2\wp + 125g_3)\wp Z + \tfrac{3}{4}(25g_2 - 3\wp^2)\wp'^2,
\end{aligned}$$

where Z is the Hecke function. We compute the asymptotic behavior of $W_4(Z)$ when $\tau \to \infty$. Let $z = t + s\tau$. We divide the problem into two cases

(1) $s \equiv 0 \pmod 1$: According to the q-expansion given in §A.2, we have

$$g_2 \to \tfrac{3}{4}\pi^4, \qquad g_3 \to \tfrac{8}{27}\pi^6, \qquad Z(z) \to \pi\cot(\pi t),$$
$$\wp'(z) \to -2\pi^3\cot(\pi t) - 2\pi^3\cot^3(\pi t), \qquad \wp(z) \to \pi^2\cot^2(\pi t) + \tfrac{2}{3}\pi^2.$$

A direct computation shows that $W_4(Z)$ has zeros at ∞ when $s = 0$.

By replacing all the modular forms $g_2, g_3, \wp, \wp'$ and Z in $W_4(Z)$ with their q-expansions, we have (e.g. using *Mathematica*)

$$W_4(Z) = 2^{14}3^3 5^2 7\,\pi^{10}\cos^2(\pi t)\sin^2(\pi t) q^3 + O(q^4)$$

(2) $s \not\equiv 0 \pmod 1$: In this case we have

$$Z \to 2\pi i\left(s - \tfrac{1}{2}\right), \qquad \wp(z) \to -\tfrac{1}{3}\pi^2,$$

$$\wp'(z) \to 0, \qquad g_2 \to \tfrac{4}{3}\pi^4, \qquad g_3 \to \tfrac{8}{27}\pi^6.$$

Hence the constant term of $W_4(Z)$ is given by

$$\begin{aligned} W_4(z) = & -64\pi^{10}(-2+s)(-1+s)^2 s^2 (1+s) \\ & \times (-3+2s)(-1+2s)^2(1+2s) + O(q). \end{aligned}$$

If $s \not\equiv 0 \pmod 1$ then $W_4(Z)$ has zero at $\tau = \infty \iff s \equiv \frac{1}{2} \pmod 1$.

Now we fix $s = \frac{1}{2}$ and replace the modular forms $g_2, g_3, \wp, \wp'$ and Z in $W_4(Z)$ with their q-expansions. We get

$$W_4(Z) = 2^{10}3^3 5^2 7\, \pi^{10} \cos(\pi t)^2 \sin(\pi t)^2 q^2 + O(q^3).$$

These computations for the q-expansions imply that

$$\begin{aligned} \nu_\infty(M_4(N)) &= 3\,\#\{\, 1 \le k_1 \le N \mid \gcd(N, k_1) = 1 \,\} \\ &\qquad + 2\,\#\{\, 0 \le k_1 \le N \mid \gcd(N/2, k_1) = 1 \,\} \\ &= 3\phi(N) + 4\phi(N/2). \end{aligned}$$

Since the value of $\nu_\infty(M_4(N))$ coincides with the assumption in Proposition A.1 for $n = 4$, the theorem follows from it accordingly. □

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