A Two-stage Image Segmentation Method Using a Convex Variant of the Mumford-Shah Model and Thresholding

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Joint work with
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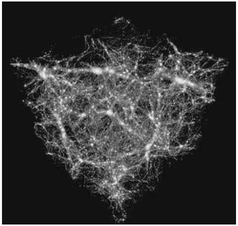
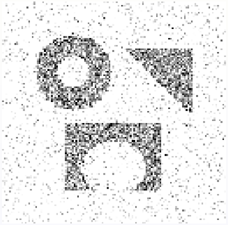
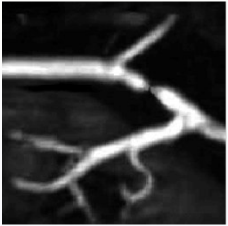
CIMI Workshop on Optimization and Statistics in Image Processing
June 24–28, 2013

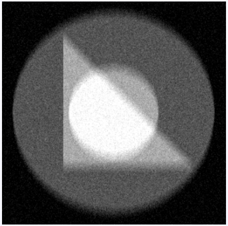
Aim: One Method to Segment Different Images

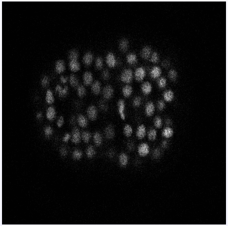

anti-mass


noisy

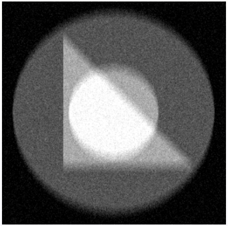

tubular


3-phase

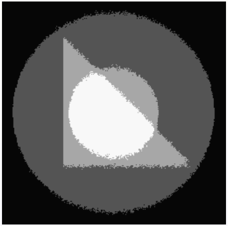

4-phase blurry


non-Gaussian

Multiphase Segmentation for Blurry Image

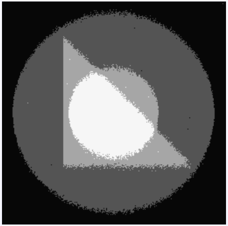


Noisy and blur image



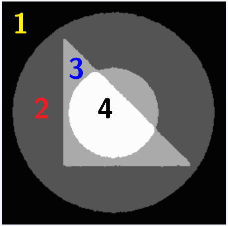
Li *et al.* (10)

Yuan *et al.* (10)



Sandberg *et al.* (10)

Steidl *et al.* (12)



Our result

Outline

Outline

Mumford-Shah Model (1989)

Mumford-Shah Energy

$$E_{\mathrm{MS}}(g,\Gamma)$$

$$\|$$

$$\frac{\lambda}{2}\int_{\Omega}(f-g)^2dx \quad + \quad \frac{\mu}{2}\int_{\Omega\setminus\Gamma}|\nabla g|^2dx \quad + \quad \mathrm{Length}(\Gamma)$$

Mumford-Shah Model (1989)

Mumford-Shah Energy

$$E_{\mathrm{MS}}(g, \Gamma)$$

$$\|$$

$$\frac{\lambda}{2}\int_{\Omega}(f-g)^2 dx \quad + \quad \frac{\mu}{2}\int_{\Omega\setminus\Gamma}|\nabla g|^2 dx \quad + \quad \mathrm{Length}(\Gamma)$$

Data fidelity: control g not far away from f.

Regularization: impose smoothness of g on $\Omega \setminus \Gamma$.

Regularization: require boundary Γ be short.

Mumford-Shah Model (1989)

Mumford-Shah Energy

$$E_{\mathrm{MS}}(g,\Gamma)$$

$$\|$$

$$\frac{\lambda}{2}\int_{\Omega}(f-g)^2dx \quad + \quad \frac{\mu}{2}\int_{\Omega\setminus\Gamma}|\nabla g|^2dx \quad + \quad \mathrm{Length}(\Gamma)$$

Nonconvex:
due to the
edge term Γ.

Finding Good Approximation of M-S Model

Mumford-Shah Energy

$E_{\mathrm{MS}}(g, \Gamma)$

Find a good approximation

$$\| $$

$$\frac{\lambda}{2}\int_{\Omega}(f-g)^2 dx \quad + \quad \frac{\mu}{2}\int_{\Omega\setminus\Gamma}|\nabla g|^2 dx \quad + \quad \mathrm{Length}(\Gamma)$$

Finding Good Approximation of M-S Model

Mumford-Shah Energy

$E_{\mathrm{MS}}(g, \Gamma)$

$$\| $$

$$\frac{\lambda}{2}\int_\Omega (f-g)^2 dx \quad + \quad \frac{\mu}{2}\int_{\Omega\setminus\Gamma} |\nabla g|^2 dx \quad + \quad \mathrm{Length}(\Gamma)$$

Find a good approximation

(1) Ambrosio and Tortorelli (1990)
(2) Gobbino (1998) and Chambolle (1999)
(3) Pock and Cremers, *et al.* (2009)

Finding Good Approximation of M-S Model

Mumford-Shah Energy

$$E_{\mathrm{MS}}(g, \Gamma)$$

Find a good approximation

$$\|$$

$$\frac{\lambda}{2}\int_{\Omega}(f-g)^2 dx \quad + \quad \frac{\mu}{2}\int_{\Omega\backslash\Gamma}|\nabla g|^2 dx \quad + \quad \mathrm{Length}(\Gamma)$$

(1) Ambrosio and Tortorelli (1990)

(2) Gobbino (1998) and Chambolle (1999)

(3) Pock and Cremers, *et al.* (2009):
A 128×128 image on a Tesla C1060 GPU machine requires 600 seconds.

Simplifying Mumford-Shah Model

Mumford-Shah Energy

$E_{\mathrm{MS}}(g, \Gamma)$

Simplify it:
$\nabla g \equiv 0$ on $\Omega \setminus \Gamma$

$$\| $$

$$\frac{\lambda}{2} \int_\Omega (f-g)^2 dx \quad + \quad \frac{\mu}{2} \int_{\Omega \setminus \Gamma} |\nabla g|^2 dx \quad + \quad \mathrm{Length}(\Gamma)$$

Restrict:

$\Omega \setminus \Gamma = \cup_i \Omega_i$
$g = c_i$ in Ω_i

Simplifying Mumford-Shah Model

Mumford-Shah Energy

$E_{\mathrm{MS}}(g,\Gamma)$

Simplify it:
$\nabla g \equiv 0$ on $\Omega \setminus \Gamma$

$$\| $$

$$\frac{\lambda}{2}\int_\Omega (f-g)^2 dx \quad + \quad \frac{\mu}{2}\int_{\Omega\setminus\Gamma} |\nabla g|^2 dx \quad + \quad \mathrm{Length}(\Gamma)$$

$= 0$

Multiphase Chan-Vese Model (02)
(g is piecewise constant function):

$$E_{\mathrm{MS}}(g,\Gamma) = \frac{\lambda}{2}\sum_i \int_{\Omega_i} (f-c_i)^2 dx + |\Gamma|$$

Restrict:

$\Omega \setminus \Gamma = \cup_i \Omega_i$
$g = c_i$ in Ω_i

Simplifying Mumford-Shah Model

Mumford-Shah Energy

$E_{\mathrm{MS}}(g, \Gamma)$

Simplify it:
$\nabla g \equiv 0$ on $\Omega \setminus \Gamma$

$$\|$$

$$\frac{\lambda}{2} \int_\Omega (f - g)^2 dx \quad + \quad \underbrace{\frac{\mu}{2} \int_{\Omega \setminus \Gamma} |\nabla g|^2 dx}_{= 0} \quad + \quad \mathrm{Length}(\Gamma)$$

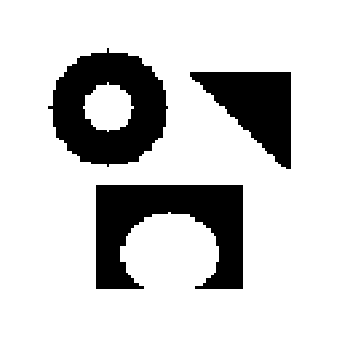
Multiphase Chan-Vese Model (02)
(g is piecewise constant function):
$E_{\mathrm{MS}}(g, \Gamma) = \frac{\lambda}{2} \sum_i \int_{\Omega_i} (f - c_i)^2 dx + |\Gamma|$

$\xrightarrow{\text{two-phase}}$

ACWE (01)
$\Omega \setminus \Gamma = \Omega_1 \cup \Omega_2$
$g = c_i$ in Ω_i

Outline

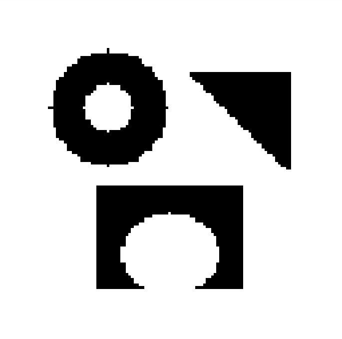
Our Motivation



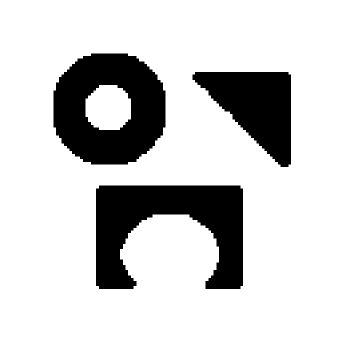
(a): True binary image

(b): Given **smooth** image

Our Motivation

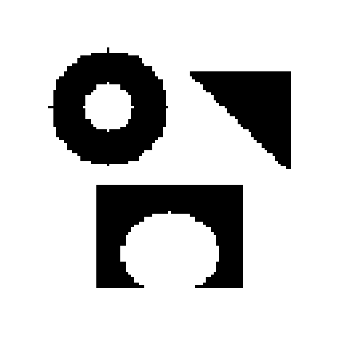


(a): True binary image

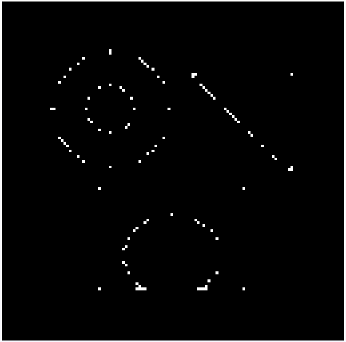


(c): Binary image from (b) with **threshold** $= 0.5$

Our Motivation



(a): True binary image



Difference of (a) and (c) (nonzero pixel values)

Aim in Segmentation

piecewise constant approximation of f

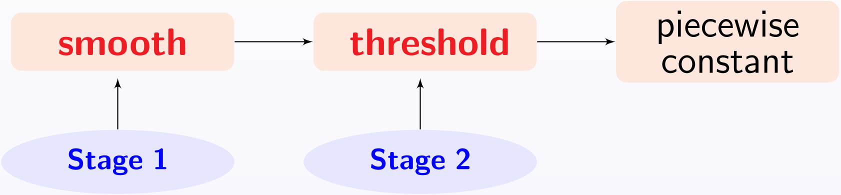
Aim in Segmentation

piecewise constant approximation of f

smooth $\longrightarrow$ **threshold** $\longrightarrow$ piecewise constant

Aim in Segmentation

piecewise constant approximation of f



Aim in Segmentation

piecewise constant approximation of f

smooth → **threshold** → piecewise constant

Stage 1 → smooth

Stage 2 → threshold

X. Cai, R. Chan, S. Morigi, and F. Sgallari, Vessel Segmentation in Medical Imaging Using a Tight-Frame Based Algorithm, *SIAM J. Imaging Sci.*, (2013)

Stage One: Convex Variant of the M-S Model

Mumford-Shah Energy

$$E_{\mathrm{MS}}(g, \Gamma)$$

Restrict:
$g \in W^{1,2}(\Omega)$

$$\|$$

$$\frac{\lambda}{2} \int_{\Omega} (f-g)^2 dx \quad + \quad \frac{\mu}{2} \int_{\Omega \setminus \Gamma} |\nabla g|^2 dx \quad + \quad \mathrm{Length}(\Gamma)$$

Stage One: Convex Variant of the M-S Model

Mumford-Shah Energy
$E_{\mathrm{MS}}(g,\Gamma)$ $\longleftarrow$ Restrict: $g \in W^{1,2}(\Omega)$

$\|$

$\frac{\lambda}{2}\int_\Omega (f-g)^2 dx \quad + \quad \frac{\mu}{2}\int_{\Omega\setminus\Gamma} |\nabla g|^2 dx \quad + \quad \mathrm{Length}(\Gamma)$

$\|$ Lemma 1 $\qquad\qquad$ $\updownarrow$ Appro. Thm. 2

$\frac{\lambda}{2}\int_\Omega (f-g)^2 dx \quad + \quad \frac{\mu}{2}\int_{\Omega} |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$

$\|$

Convex M-S Energy
$E(g)$

Stage One: Convex Variant of the M-S Model

Restrict: $g \in W^{1,2}(\Omega)$

Mumford-Shah Energy

$E_{\mathrm{MS}}(g,\Gamma)$

$\|$

$$\frac{\lambda}{2}\int_\Omega (f-g)^2\,dx \quad + \quad \frac{\mu}{2}\int_{\Omega\setminus\Gamma} |\nabla g|^2\,dx \quad + \quad \mathrm{Length}(\Gamma)$$

$\|$ **Lemma 1**

Appro. Thm. 2

$$\frac{\lambda}{2}\int_\Omega (f-g)^2\,dx \quad + \quad \frac{\mu}{2}\int_{\Omega} |\nabla g|^2\,dx \quad + \quad \int_\Omega |\nabla g|\,dx$$

$\|$

Convex M-S Energy

$E(g)$

Stage One: Lemma 1

Lemma 1

If $g \in W^{1,2}(\Omega)$ and Γ is a closed curve with Lebesgue measure $m(\Gamma) = 0$, then

$$\int_{\Gamma} |\nabla g|^2 dx = 0.$$

Proof.

Since $g \in W^{1,2}(\Omega)$, we have $\nabla g \in L^2(\Omega)$. Because of $m(\Gamma) = 0$, we get $\int_{\Gamma} |\nabla g|^2 dx = 0$ immediately. $\square$

Stage One: Convex Variant of the M-S Model

Mumford-Shah Energy

$E_{\mathrm{MS}}(g,\Gamma)$

Restrict: $g \in W^{1,2}(\Omega)$

$\|$

$\frac{\lambda}{2}\int_\Omega (f-g)^2 dx \quad + \quad \frac{\mu}{2}\int_{\Omega\setminus\Gamma} |\nabla g|^2 dx \quad + \quad \mathrm{Length}(\Gamma)$

$\|$ Lemma 1 $\qquad\qquad$ Appro. Thm. 2

$\frac{\lambda}{2}\int_\Omega (f-g)^2 dx \quad + \quad \frac{\mu}{2}\int_{\Omega} |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$

$\|$

Convex M-S Energy

$E(g)$

Stage One: Convex Variant of the M-S Model

Mumford-Shah Energy

$E_{\mathrm{MS}}(g, \Gamma)$

Restrict:
$g \in W^{1,2}(\Omega)$

$\|$

$$\frac{\lambda}{2} \int_\Omega (f-g)^2 dx \quad + \quad \frac{\mu}{2} \int_{\Omega \setminus \Gamma} |\nabla g|^2 dx \quad + \quad \mathrm{Length}(\Gamma)$$

$\|$ Lemma 1

Appro. Thm. 2

$$\frac{\lambda}{2} \int_\Omega (f-g)^2 dx \quad + \quad \frac{\mu}{2} \int_{\Omega} |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$$

$\|$

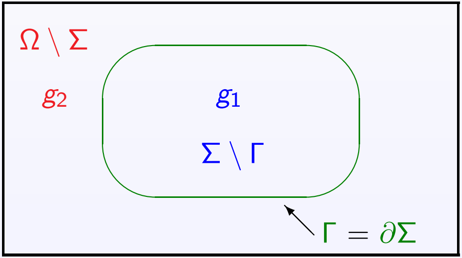
Convex M-S Energy

$E(g)$

Stage One: Theorem 2

Let $\Sigma = \overline{\text{Inside}(\Gamma)}$, $g_1 \in W^{1,2}(\Sigma \setminus \Gamma)$ and $g_2 \in W^{1,2}(\Omega \setminus \Sigma)$, rewrite the Mumford-Shah model as:

$$E^*_{\mathrm{MS}}(\Sigma, g_1, g_2) = \frac{\lambda}{2}\int_{\Sigma\setminus\Gamma}(f - g_1)^2 dx + \frac{\mu}{2}\int_{\Sigma\setminus\Gamma}|\nabla g_1|^2 dx$$
$$+\frac{\lambda}{2}\int_{\Omega\setminus\Sigma}(f - g_2)^2 dx + \frac{\mu}{2}\int_{\Omega\setminus\Sigma}|\nabla g_2|^2 dx + \mathrm{Length}(\Gamma).$$



Stage One: Theorem 2

To minimize the rewritten Mumford-Shah model:

$$E^*_{\mathrm{MS}}(\Sigma, g_1, g_2) = \frac{\lambda}{2}\int_{\Sigma\setminus\Gamma}(f-g_1)^2dx + \frac{\mu}{2}\int_{\Sigma\setminus\Gamma}|\nabla g_1|^2dx + \frac{\lambda}{2}\int_{\Omega\setminus\Sigma}(f-g_2)^2dx + \frac{\mu}{2}\int_{\Omega\setminus\Sigma}|\nabla g_2|^2dx + \boxed{\mathrm{Length}(\Gamma)}$$

let $G(g_1, g_2) = \lambda(f-g_1)^2 + \mu|\nabla g_1|^2 - \lambda(f-g_2)^2 - \mu|\nabla g_2|^2$.

Theorem 2

Given g_1 and $g_2 \in W^{1,2}(\Omega)$, a global minimizer Σ for $E^*_{\mathrm{MS}}(\Sigma; g_1, g_2)$ can be found by solving the convex minimization:

$$\min_{0\le u\le 1}\left\{\boxed{\int_\Omega |\nabla u|} + \frac{1}{2}\int_\Omega G(g_1, g_2)u(x)\right\},$$

and setting $\Sigma = \{x : u(x) \ge \rho\}$ for a.e. $\rho \in [0,1]$.

(Proof is similar to Theorem 2 of T. Chan, Esedoglu, and Nikolova (06))

Stage One: Theorem 2

Conclusion:

- $\mathrm{Length}(\Gamma) \longleftrightarrow \int_\Omega |\nabla u|$; and equal when u is binary.
- phases of g can be obtained from u by thresholding.

Stage One: Theorem 2

Conclusion:

- $\mathrm{Length}(\Gamma) \longleftrightarrow \int_\Omega |\nabla u|$; and equal when u is binary.
- phases of g can be obtained from u by thresholding.

Observation: (especially when g is nearly binary)

- jump set of $g \approx$ jump set of u.
- $\int_\Omega |\nabla g| \approx \int_\Omega |\nabla u| \longleftrightarrow \mathrm{Length}(\Gamma)$.

Stage One: Convex Variant of the M-S Model

Convex M-S Energy

$E(g)$

$\|$

$\frac{\lambda}{2}\int_\Omega (f-g)^2 dx \quad + \quad \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$

Stage One

Stage One: Convex Variant of the M-S Model

Stage One

Convex M-S Energy

$$E(g)$$

$$\|$$

$$\frac{\lambda}{2}\int_\Omega (f-g)^2 dx \quad + \quad \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$$

extendable to blurred images

$$\frac{\lambda}{2}\int_\Omega (f-\mathcal{A}g)^2 dx \quad + \quad \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$$

Stage One: Unique Minimizer

Our *convex* variant of the Mumford-Shah model is:

$$\boxed{E(g) = \frac{\lambda}{2}\int_\Omega (f - \mathcal{A}g)^2 dx + \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx + \int_\Omega |\nabla g| dx}$$

Its discrete version:

$$\boxed{\frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|\nabla g\|_1}$$

Theorem 3

Let Ω be a bounded connected open subset of $\mathbb{R}^2$ with a Lipschitz boundary. Let $\mathrm{Ker}(\mathcal{A})\bigcap\mathrm{Ker}(\nabla) = \{0\}$ and $f \in L^2(\Omega)$, where $\mathcal{A}$ is a bounded linear operator from $L^2(\Omega)$ to itself. Then $E(g)$ has a unique minimizer $g \in W^{1,2}(\Omega)$.

Stage Two: Thresholding

Convex M-S Energy

$$E(g)$$

Stage One

$$E(g) = \frac{\lambda}{2}\int_\Omega (f-\mathcal{A}g)^2 dx + \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx + \int_\Omega |\nabla g| dx$$

Stage Two: Thresholding

Convex M-S Energy

$E(g)$ ← Stage One

$$\| $$

$$\frac{\lambda}{2}\int_\Omega (f - \mathcal{A}g)^2 dx \quad + \quad \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$$

$$\Downarrow$$

Stage Two → **Threshold g to piecewise constant**

Our Two-stage Segmentation Algorithm

Given f

↓

Stage One: solve

$$\min_g \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|\nabla g\|_1 \right\}$$

↓

Stage Two: determine thresholds from g

$$\rho_1 \leq \cdots \leq \rho_{K-1}$$

↓

K segments

Numerical Aspects: Stage One

Given f

↓

Stage One: solve

$$\min_g \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|\nabla g\|_1 \right\}$$

↓

Stage Two: determine thresholds from g

$$\rho_1 \le \cdots \le \rho_{K-1}$$

↓

K segments

Split-Bregman (ADMM) (Goldstein and Osher, 09); **Augmented Lagrangian** (Tai, *et. al.*, 09); **Chambolle-Pock method** (Chambolle and Pock, 10); etc.

Numerical Aspects: Stage One

Given f

Stage One: solve

$$\min_g \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|\nabla g\|_1 \right\}$$

Stage Two: determine thresholds from g

$$\rho_1 \leq \cdots \leq \rho_{K-1}$$

K segments

Split-Bregman (ADMM) (Goldstein and Osher, 09);
Augmented Lagrangian (Tai, *et. al.*, 09);
Chambolle-Pock method (Chambolle and Pock, 10);
etc.

Numerical Aspects: Stage One

Split-Bregman method for solving our model

$$\boxed{\min_g \left\{ \underbrace{\frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2}_{F(g)} + \|\nabla g\|_1 \right\}.}$$

Note that $F(g)$ is quadratic in g.

Idea is to separate the g's in $F(g)$ and $\|\nabla g\|_1$. Set

$$\begin{cases} d_x = \nabla_x g, \\ d_y = \nabla_y g. \end{cases}$$

Numerical Aspects: Stage One

Solve:

$$\boxed{\min_{g,d_x,d_y} \left\{ F(g) + \|(d_x, d_y)\|_1 \right\} \quad \text{s.t.} \quad d_x = \nabla_x g, d_y = \nabla_y g}$$

Split-Bregman iteration:

$$\begin{aligned}(g^{k+1}, d_x^{k+1}, d_y^{k+1}) &= \operatorname{argmin}_{g,d_x,d_y} \Big\{ F(g) + \|(d_x, d_y)\|_1 \\ &\quad + \frac{\sigma}{2}\|d_x - \nabla_x g - b_x^k\|_2^2 + \frac{\sigma}{2}\|d_y - \nabla_y g - b_y^k\|_2^2 \Big\},\end{aligned}$$

$$b_x^{k+1} = b_x^k + (\nabla_x g^{k+1} - d_x^{k+1}), \quad b_y^{k+1} = b_y^k + (\nabla_y g^{k+1} - d_y^{k+1}).$$

Numerical Aspects: Stage One

Given f

↓

Stage One: solve

$$\min_g \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|\nabla g\|_1 \right\}$$

← **Solved!**

↓

Stage Two: determine thresholds from g

$$\rho_1 \leq \cdots \leq \rho_{K-1}$$

↓

K segments

Numerical Aspects: Stage Two

Given f

↓

Stage One: solve

$\min_g \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|\nabla g\|_1 \right\}$

↓

Stage Two: determine thresholds from g

$\rho_1 \leq \cdots \leq \rho_{K-1}$

↓

K segments

Automatic way to determine thresholds by K-means

1. Linear-stretch g to $[0, 1]$:
$$\bar{g} = (g - g_{\min})/(g_{\max} - g_{\min}).$$
2. Segment the histogram of $\bar{g}$ into K clusters, and compute the mean value of each cluster:
$$m_1 \leq m_2 \leq \cdots \leq m_K.$$
3. Define $(K-1)$ thresholds:
$$\rho_i = (m_i + m_{i+1})/2, \;\; i = 1, \ldots, K-1.$$

Numerical Aspects: Stage Two

Given f

↓

Stage One: solve

$$\min_g \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|\nabla g\|_1 \right\}$$

↓

Stage Two: determine thresholds from g

$$\rho_1 \le \cdots \le \rho_{K-1}$$

↓

K segments

Other Ways

1. Choose by the user: ρ^U.
2. Two-phase: $\rho^M = \mathrm{mean}(\bar{g})$.

Numerical Aspects: Stage Two

Given f

Stage One: solve

$$\min_g \left\{ \frac{\lambda}{2}\|f - \mathcal{A}g\|_2^2 + \frac{\mu}{2}\|\nabla g\|_2^2 + \|\nabla g\|_1 \right\}$$

Stage Two: determine thresholds from g

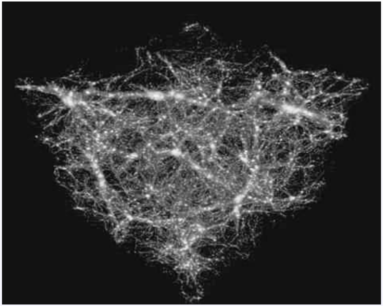
$$\rho_1 \leq \cdots \leq \rho_{K-1}$$

K segments

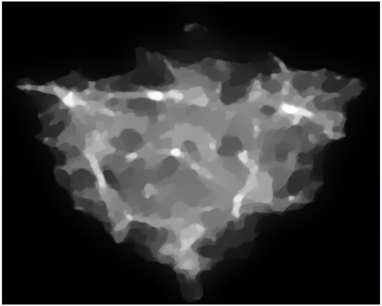
1. No need to recompute g when using different thresholds!
2. No need to fix the number of phases K at the beginning!

Outline

Anti-mass Image: Stage One Solution

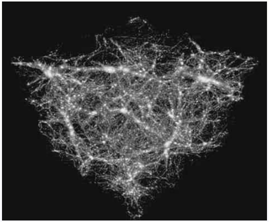


Given image



Our solution g

Anti-mass Image: Results Comparison



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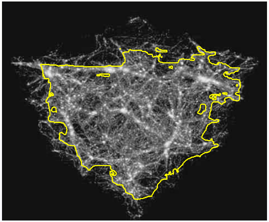
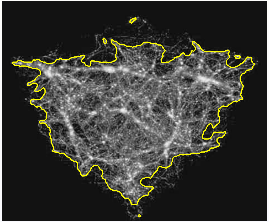
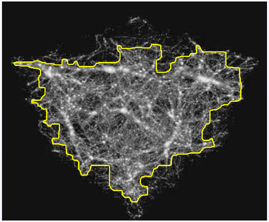


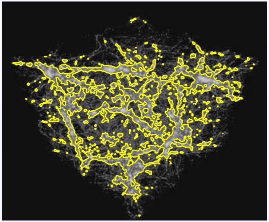
Chan-Vese (01)



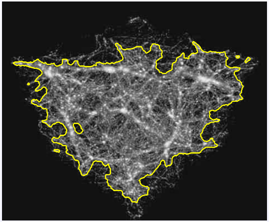
Our: $\rho^M = 0.1898$



Dong *et al.* (10)

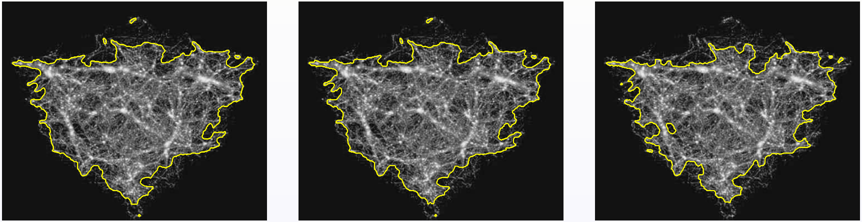


Yuan *et al.* (10)



Our: $\rho_1 = 0.2669$

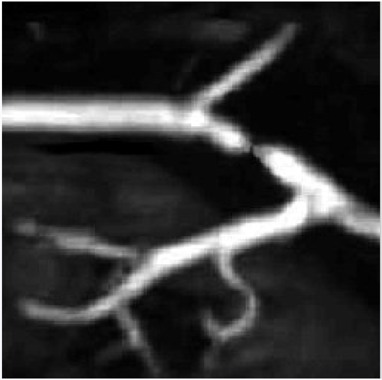
Anti-mass Image: Our Result



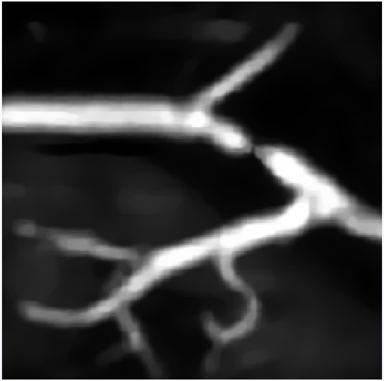
$\rho^M = 0.1898$ $\rho^U = 0.2$ $\rho_1 = 0.2669$

Different thresholds give different meaningful segmentation results. No need to solve the convex model again when thresholds changed.

Tubular Image: Stage One Solution

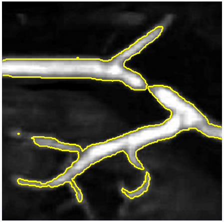


Given image

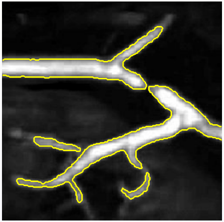


Our solution g

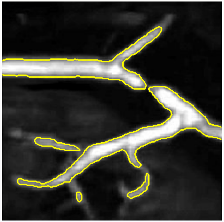
Tubular Image: Results Comparison



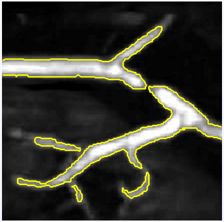
Chan-Vese (01)



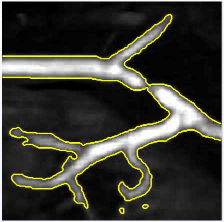
Yuan *et al.* (10)



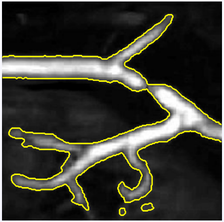
$\rho_1 = 0.4019$



Dong *et al.* (10)

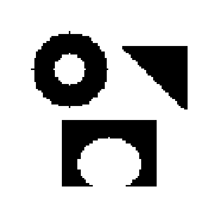


Cai *et al.* (13)

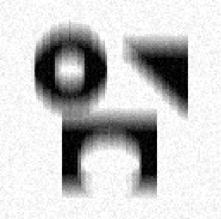


$\rho^M = 0.1760$

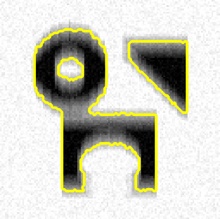
Motion Blurred and Noisy Image



Clean image



Given blurred image



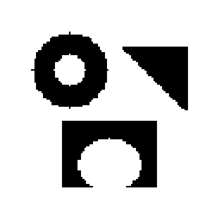
Chan-Vese (01)

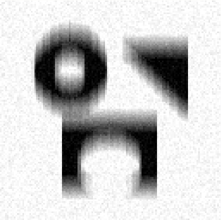
Dong *et al.* (10)

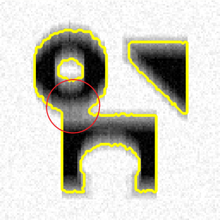
Yuan *et al.* (10)

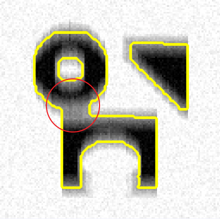
$\rho_1 = 0.5048$

Motion Blurred and Noisy Image

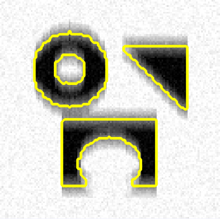

Clean image


Given blurred image


Chan-Vese (01)

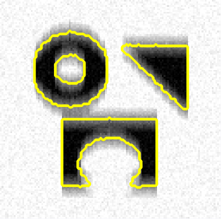

Dong *et al.* (10)


Yuan *et al.* (10)

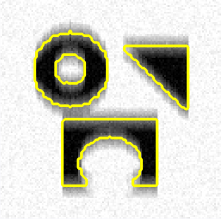

$\rho_1 = 0.5048$

Motion Blurred and Noisy Image

$\rho^M = 0.7661$



$\rho^U = 0.6$



$\rho_1 = 0.5048$

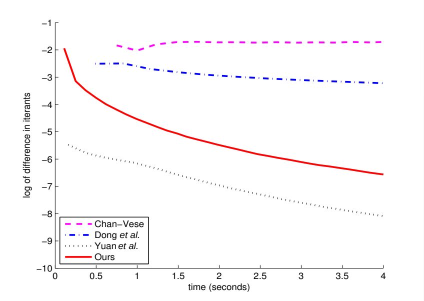
Robust with respect to the thresholds chosen.

Segmentation Changes with Threshold

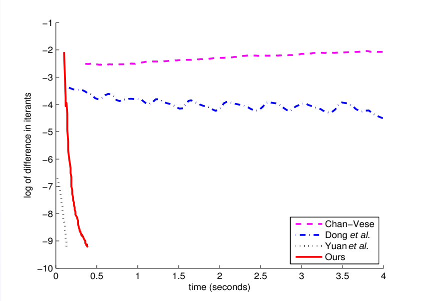
(Loading Video...)

Convergence History

Log of difference in iterates versus time



Anti-mass image



Motion blurred and noisy image

Our method is very stable.

CPU Time

Two-phase: iteration numbers and CPU time in second

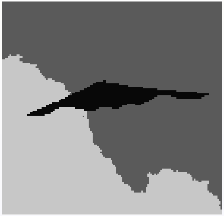
	C-V (01)		Dong (10)		Yuan (10)		Our method	
Example	iter.	time	iter.	time	iter.	time	iter.	time
Anti-mass	1000	263.73	300	83.82	64	6.01	172	18.38
Tubular	1000	76.62	300	32.17	18	0.37	115	3.03
Motion	1300	28.19	300	10.18	20	0.09	52	1.13

Our method is faster than others except Yuan's, but our segmentation results are better.

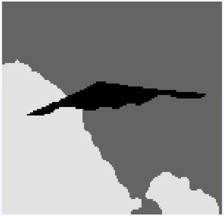
Three-phase Segmentation

Given image

Li *et al.* (10)

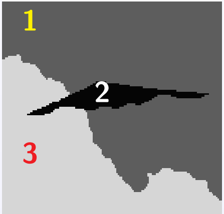


Sandberg *et al.* (10)



Yuan *et al.* (10)

Our solution g



Our 3 phases from g

Three-phase Segmentation

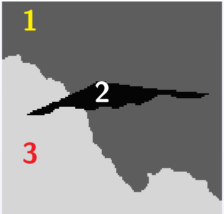

Given image


Li *et al.* (10)


Sandberg *et al.* (10)

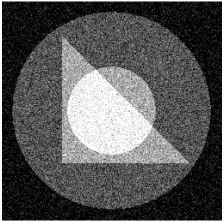

Yuan *et al.* (10)


Our solution g



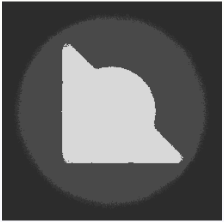
Our 3 phases from g

Four-phase Segmentation: Noisy image

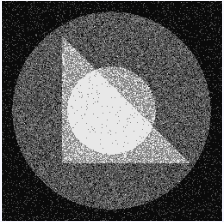


Given noisy image

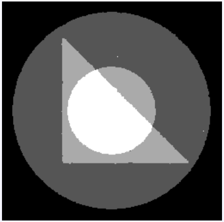
Yuan *et al.* (10)



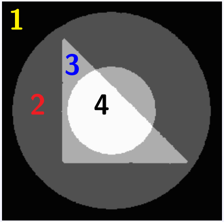
Li *et al.* (10)



Sandberg *et al.* (10)

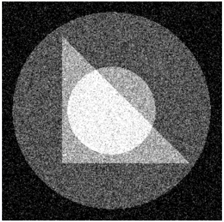


Steidl *et al.* (12)



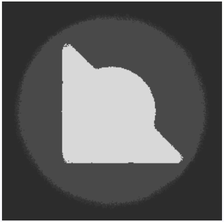
Our 4 phases from g

Four-phase Segmentation: Noisy image

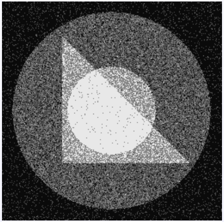


Given noisy image

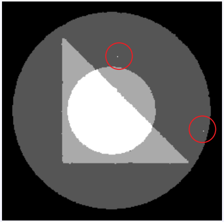
Yuan *et al.* (10)



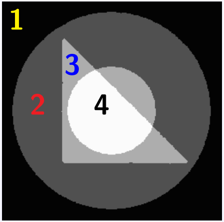
Li *et al.* (10)



Sandberg *et al.* (10)

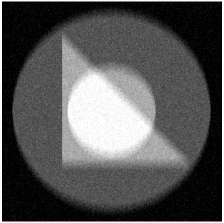


Steidl *et al.* (12)

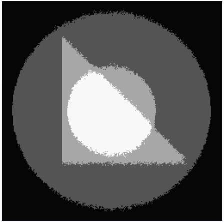


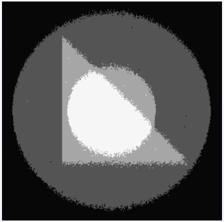
Our 4 phases from *g*

Four-phase Segmentation: Noisy and blurry image

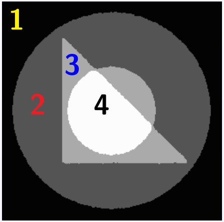

Noisy & blurry


Yuan *et al.* (10)


Li *et al.* (10)


Sandberg *et al.* (10)

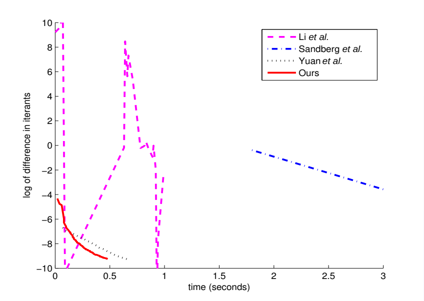

Steidl *et al.* (12)



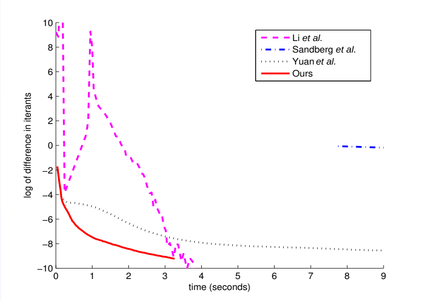
Our 4 phases from g

Convergence History

Log of difference in iterates versus time



Three-phase image



Four-phase noisy image

Our method is very stable.

CPU Time

Multiphase: iteration numbers and CPU time in second

	Li (10)		Sandberg (10)		Yuan (10)		Our method	
Example	iter.	time	iter.	time	iter.	time	iter.	time
3-phase	100	1.56	2	3.15	32	0.58	62	0.57
4-phase	100	7.64	12	90.59	134	14.51	112	3.04
4-phase-blur	100	7.26	13	93.79	57	5.82	78	2.90

Our method is the best and fastest for multiphase segmentation.

Cai, C., and Zeng, *A Two-stage Image Segmentation Method using a Convex Variant of the Mumford-Shah Model and Thresholding*, SIAM J. Imag. Sci. (2013).

Outline

1. Mumford-Shah Model
2. Our Two-stage Image Segmentation Method
3. Experimental Results
4. **Extensions to Other Noise Models**
5. Conclusions

Poisson and Multiplicative Gamma Noises

- Poisson noise: observed image $f(x)$ follows

$$p_{f(x)}(n; g(x)) = \frac{(g(x))^n e^{-g(x)}}{n!}$$

with mean $g(x)$.

- Multiplicative Gamma noise: $f = g \cdot \eta$ where $\eta(x)$ follows:

$$p_{\eta(x)}(y; \theta, K) = \frac{1}{\theta^K \Gamma(K)} y^{K-1} e^{-\frac{y}{\theta}} \text{ for } y \geq 0.$$

with mean 1 and variance of $\frac{1}{K}$.

2-stage Method:

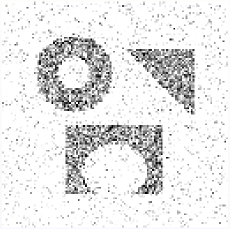
First stage: solve for

$$\min_g \left\{ \lambda \int_\Omega (\mathcal{A}g - f \log \mathcal{A}g) dx + \frac{\mu}{2} \int_\Omega |\nabla g|^2 dx + \int_\Omega |\nabla g| dx \right\}.$$

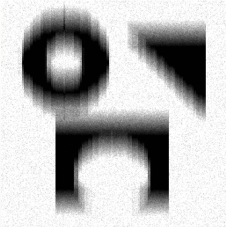
- data fitting term good for Poisson noise from MAP analysis
- also suitable for Gamma noise (Steidl and Teuber (10)).
- objective functional is convex (solved by Chambolle-Pock)
- admits unique solution if $\mathrm{Ker}(\mathcal{A}) \cap \mathrm{Ker}(\nabla) = 0$.

Second stage: threshold the solution to get the phases.

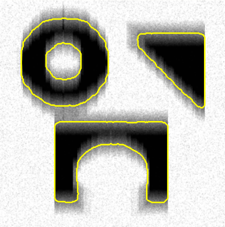
Example 1: Poisson noise with motion blur



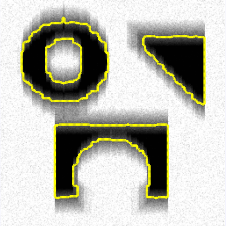
Noisy image



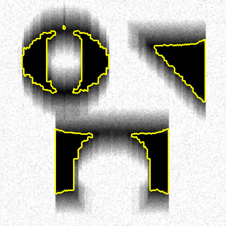
Noisy & blurred



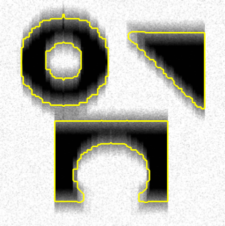
Yuan *et al.* (10)



Dong *et al.* (10)

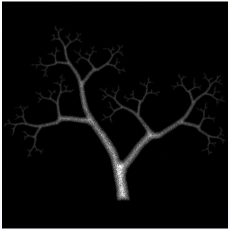


Sawatzky *et al.* (13)

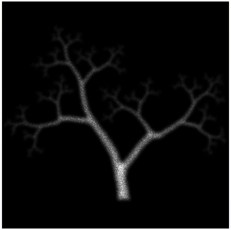


Our method

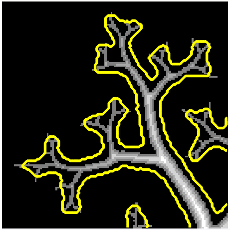
Example 2: Gamma noise with Gaussian blur



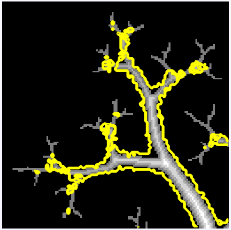
Original image



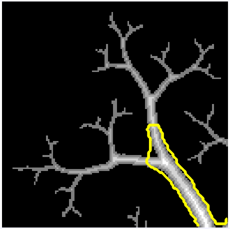
Noisy & blurred



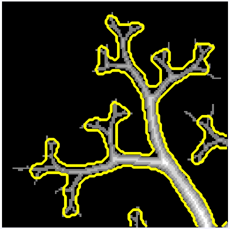
Yuan *et al.* (10)



Dong *et al.* (10)



Sawatzky *et al.* (13)



Our method

Segmentation Changes with Threshold

(Loading Video...)

Real cell image from an automated cell tracking system.

CPU Time

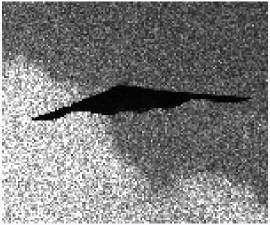
2-phase: iteration numbers and CPU time in second

	Yuan*		Dong*		Sawatzky		Our method	
Test	iter.	time	iter.	time	iter.	time	iter.	time
Example 1	25	0.2	66	2.0	19	152.0	325	4.1
Example 2	31	3.8	295	37.1	25	1220.3	263	18.9
Boat	78	3.3	187	12.7	13	324.5	61	1.5
Anti-mass	74	5.3	239	19.5	25	562.8	80	3.2
Cells	41	7.0	300	50.5	25	2478.5	101	6.3
Bacteria	51	6.3	300	36.1	25	1435.7	74	3.9

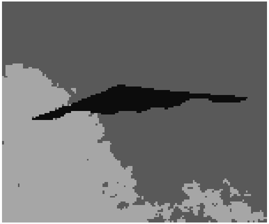
Yuan's and Dong's algorithms were applied on images after Anscombe transformation.

Example 3: Gamma noise

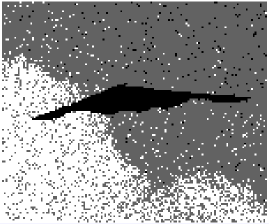
Original image



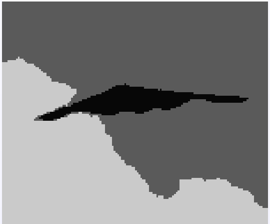
Noisy image



Yuan *et al.* (10)

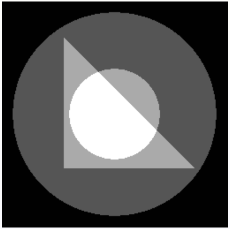


Li *et al.* (10)

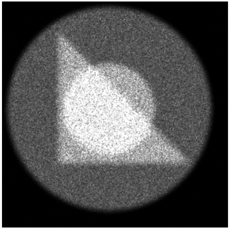


Our method

Example 4: Gamma noise with Gaussian blur

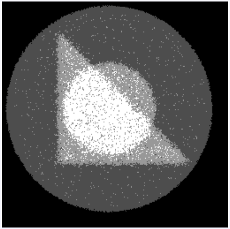


Original image

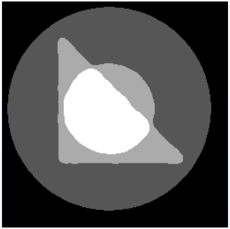


Noisy & blurred

Yuan *et al.* (10)



Li *et al.* (10)



Our method

CPU Time

Multi-phase: iteration numbers and CPU time in second

	Yuan		Li		Our Method	
Test	iter.	time	iter.	time	iter.	time
Example 3	105	0.9	62	0.6	58	0.2
Example 4	97	4.7	60	1.9	225	3.3
4-phase Poisson	130	6.3	53	2.3	35	0.5
MRI	95	4.7	144	4.4	60	2.0

C., Yang, and Zeng, *A Two-stage Image Segmentation Method for Blurry Images with Poisson or Multiplicative Gamma Noise*, Submitted.

Outline

Relationship with Image Restoration

Convex M-S Energy

$$E(g)$$

$$\|$$

$$\frac{\lambda}{2}\int_\Omega (f-\mathcal{A}g)^2 dx \quad + \quad \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$$

Relationship with Image Restoration

Convex M-S Energy

$$E(g)$$

$$\|$$

$$\frac{\lambda}{2}\int_\Omega (f - \mathcal{A}g)^2 dx \quad + \quad \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$$

$\mathcal{A} = \mathcal{I}$

omit

ROF Model (1992)

Rudin, Osher and Fatemi

Relationship with Image Restoration

Convex M-S Energy

$E(g)$

$\|$

$$\frac{\lambda}{2}\int_\Omega (f-\mathcal{A}g)^2 dx \quad + \quad \frac{\mu}{2}\int_\Omega |\nabla g|^2 dx \quad + \quad \int_\Omega |\nabla g| dx$$

$\mathcal{A}=\mathcal{I}$

omit

ROF Model (1992)

Rudin, Osher and Fatemi

staircase

remove by introducing higher-order derivative:

Chan (00), Lysaker (03), Steidl (08), Bredies (10), Hintermüller (10), etc.

Conclusions

- Convex segmentation model with unique solution. Can be solved easily and fast.
- No need to recompute the whole model when thresholds or number of phases changes.
- Easily extendable to e.g. blurry images and non-Gaussian noise.
- Link image segmentation and image restoration.

Conclusions

THANK YOU!