



PROJECT MUSE®

Almost Kähler Forms on Rational 4-Manifolds

Tian-Jun Li, Weiyi Zhang

American Journal of Mathematics, Volume 137, Number 5, October 2015,
pp. 1209-1256 (Article)

Published by Johns Hopkins University Press

DOI: <https://doi.org/10.1353/ajm.2015.0037>



➔ *For additional information about this article*

<https://muse.jhu.edu/article/593736/summary>

ALMOST KÄHLER FORMS ON RATIONAL 4-MANIFOLDS

By TIAN-JUN LI and WEIYI ZHANG

Abstract. We study Nakai-Moishezon type question and Donaldson’s “tamed to compatible” question for almost complex structures on rational four manifolds. By extending Taubes’ subvarieties-current-form technique to J -nef genus 0 classes, we give affirmative answers of these two questions for all tamed almost complex structures on S^2 bundles over S^2 as well as for many geometrically interesting tamed almost complex structures on other rational four manifolds, including the del Pezzo ones.

Contents.

1. Introduction.
 2. Pseudo-holomorphic subvarieties and Taubes current.
 - 2.1. Pseudo-holomorphic subvarieties and their properties.
 - 2.2. The moduli space.
 - 2.3. Subvarieties through a small ball.
 - 2.4. Subvariety, Taubes current and almost Kähler form.
 3. Line class Taubes current on $\mathbb{C}P^2$.
 - 3.1. K_J -spherical classes and smooth rational curves.
 - 3.2. Moduli space of lines.
 - 3.3. Spherical Taubes current from the line class.
 4. Pencils in a big J -nef class.
 - 4.1. J -nef classes.
 - 4.2. Existence of smooth curves.
 - 4.3. Intersection properties.
 - 4.4. Vanishing locus and reducible curves.
 - 4.5. Abundance of pencils.
 5. Spherical Taubes currents from big J -nef classes.
 - 5.1. Weak Taubes currents.
 - 5.2. Tamed versus compatible.
 - 5.3. Almost Kähler cone.
 - 5.4. K -sphere cone and K -symplectic cone.
 - 5.5. Remark on the connection with Kodaira embedding.
- References.

1. Introduction. Let M be a compact, oriented, smooth manifold. An almost complex structure on M is an endomorphism of TM whose square is $-\text{Id}$. An

Manuscript received April 8, 2013; revised August 14, 2014.

Research of the first author supported in part by NSF grant 1065927; research of the second author supported in part by an AMS-Simons travel grant.

American Journal of Mathematics 137 (2015), 1209–1256. © 2015 by Johns Hopkins University Press.

almost complex structure J induces an involution on the space of 2-forms, $\Omega^2(M)$, decomposing it as $\Omega_J^+ \oplus \Omega_J^-$. J is said to be tamed if there is a symplectic form ω such that the bilinear form $\omega(\cdot, J(\cdot))$ is positive definite. In this case, ω is called a taming form of J , and we also say that J is tamed by ω . A taming form of J is said to be compatible with J if it lies in Ω_J^+ . J is said to be almost Kähler if there is a compatible form.

In [4], Donaldson raised the following question:

QUESTION 1.1. *Suppose J is an almost complex structure on a compact, oriented, smooth 4-manifold M . If J is tamed, is J almost Kähler?*

The local version of the question was known to be true in dimension 4, but false in higher dimensions [2, 12, 25, 26]. Donaldson suggested an approach via the symplectic Calabi-Yau equation, and progress on this equation has been made by Weinkove, Tosatti and Yau (cf. [30]).

It was known to hold when $M = \mathbb{C}\mathbb{P}^2$ due to deep works of Gromov [9] and Taubes [28]. In [18], we observed that this is true for any integrable J , and the same was shown for homogeneous J in [17].

In the case $b^+(M) = 1$, Taubes has recently made remarkable progress in [29]. He answers Question 1.1 affirmatively for generic tamed almost complex structures in this case.

The main purpose of this paper is to study Question 1.1 for rational 4-manifolds. Here a rational 4-manifold refers to one of the following smooth 4-manifolds: $\mathbb{C}\mathbb{P}^2$, $S^2 \times S^2$ and blow-ups of them.

Our construction is particularly successful for S^2 -bundle over S^2 :

THEOREM 1.2. *Any tamed J on $S^2 \times S^2$ or $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$ is almost Kähler.*

The remaining rational manifolds are of the form $\mathbb{C}\mathbb{P}^2 \# k \overline{\mathbb{C}\mathbb{P}^2}$ with $k \geq 2$. We settle Donaldson's question for these manifolds under a simple condition on -1 curves.

THEOREM 1.3. *Suppose $M = \mathbb{C}\mathbb{P}^2 \# k \overline{\mathbb{C}\mathbb{P}^2}$ with $k \geq 2$ and J is tamed. If there are k disjoint -1 curves, then J is almost Kähler.*

To prove his genericity result, Taubes explores close connections between pseudo-holomorphic subvarieties (see Definition 2.1) and almost Kähler forms in dimension 4. In arbitrary dimension, they have positive pairings. A special feature in dimension 4 is that they both lie in the space of closed, non-negative, J -invariant 2-currents. In particular, Taubes introduced a distributional analogue of an almost Kähler form, which we call a Taubes current (see Definition 2.11).

Let ω be a symplectic form on a compact, oriented, smooth 4-manifold with $b^+ = 1$. The basic strategy of Taubes is to first carefully pick a smooth subfamily of evenly distributed irreducible J -holomorphic subvarieties in the class of $N[\omega]$ for N large, at least when $[\omega]$ is rational, then to obtain a Taubes current via integration.

Finally, Taubes showed that such a current can be first smoothed to a J -tamed form with a dominating J -invariant part, and then further adjusted to a genuine almost Kähler form.

For a rational manifold, to avoid generic choices of J in several places of Taubes' construction, we apply the subvarieties-current-form technique to classes of genus zero smooth subvarieties with positive self-intersection. However, in general we could only hope to first construct a weaker version of Taubes current which degenerates on a finite union of subvarieties with negative self-intersection. Then we try to sum several such weak Taubes currents to get a honest Taubes current. We call Taubes current obtained this way a spherical Taubes current.

As in [29], while the construction of weak Taubes current in 5.1.1 is via integration over the irreducible subvariety part \mathcal{M}_{irr} of the moduli space, we still need a good control of the reducible subvariety part \mathcal{M}_{red} . With this in mind, we establish in [19] a clean structural picture of reducible subvarieties for a J -nef class with J -genus 0 (see Theorem 2.14). This is crucial for us to get rid of much of the "genericity" assumption of [29].

For a more detailed summary of Taubes' subvarieties-current-form technique and our adaptation, see section 2.4.3.

The subvarieties-current-form technique is also useful to further determine the almost Kähler cone

$$(1) \quad \mathcal{K}_J^c = \{[\omega] \in H^2(M; \mathbb{R}) \mid \omega \text{ is compatible with } J\}$$

in terms of the curve cone. The almost Kähler cone \mathcal{K}_J^c is a convex cohomology cone contained in the positive cone

$$\mathcal{P} = \{e \in H^2(M; \mathbb{R}) \mid e \cdot e > 0\}.$$

The curve cone of an almost complex manifold (M, J) , denoted by $A_J(M)$, is the convex cone in $H_2(M, \mathbb{R})$ generated by the set of homology classes of J -holomorphic subvarieties. Let $A_J^{\vee, >0}(M)$ be the positive dual of $A_J(M)$ under the homology-cohomology pairing, and set

$$\mathcal{P}_J = A_J^{\vee, >0}(M) \cap \mathcal{P}.$$

Clearly, $\mathcal{K}_J^c \subset A_J^{\vee, >0}(M)$ since the integral of an almost Kähler form over a J -holomorphic subvariety is positive. Motivated by the famous Nakai-Moishezon-Kleiman criterion in algebraic geometry which characterizes the ample cone in terms of the (closure of) curve cone for a projective J , and the recent Kähler version of the Nakai-Moishezon criterion (established by Buchdahl and Lamari in dimension 4, and by Demailly-Paun in arbitrary dimension), which characterizes the Kähler cone in terms of the curve cone for a Kähler J , we ask whether there is an almost Kähler version of the Nakai-Moishezon criterion.

QUESTION 1.4. *Suppose M is a compact, oriented, smooth 4-manifold with $b^+ = 1$ and J is almost Kähler. Is the almost Kähler cone dual to the curve cone, i.e., $\mathcal{K}_J^c = \mathcal{P}_J$?*

We remark that this question makes sense for any 4-manifold if we view $A_J(M)$ as a convex cone of H_+^J and define $A_J^{\vee, >0}(M)$ to be the positive dual of $A_J(M)$ under the pairing between H_+^J and H_+^+ in [18]. The question in this generality is: Is \mathcal{K}_J^c a connected component of \mathcal{P}_J ? See the survey paper [6]. On the other hand, a structural theorem of the curve cone $A_J(M)$, the cone theorem, is established for an arbitrary tamed almost complex 4-manifold in [32] by the second author.

Via a detailed analysis of spherical Taubes currents, we are able to establish the almost Kähler Nakai-Moishezon criterion in the following two cases.

THEOREM 1.5. *The almost Kähler Nakai-Moishezon criterion holds for any almost Kähler J on $S^2 \times S^2$ or $\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2$.*

THEOREM 1.6. *The almost Kähler Nakai-Moishezon criterion holds for any good almost Kähler J .*

In Theorem 1.6, a tamed J on a rational manifold is called *good* if (i) there is a smooth genus one subvariety in the anti-canonical class $-K_J$, and (ii) any irreducible genus zero subvariety of negative self-intersection is a -1 curve.

Notice that, following from Theorems 1.2 and 1.3, both Theorems 1.5 and 1.6 are still valid even if we assume J is tamed instead of almost Kähler. Recently, in [32] the second author uses J -inflation to establish the almost Kähler Nakai-Moishezon criterion for minimal ruled manifolds with a negative curve and all rational manifolds with $b^- \leq 9$.

The organization of this paper is as follows. In Section 2 we review properties of moduli space of irreducible pseudo-holomorphic subvarieties as presented in [28] and introduce Taubes current. In Section 3, to illustrate some key features of our construction, we construct spherical Taubes currents in the line class of $\mathbb{C}\mathbb{P}^2$. The situation for the line class is simpler since there are no reducible rational curves. In Section 4 we discuss various properties of J -nef spherical class and show that there are plenty of pencils in the moduli spaces. In Section 5 we combine the constructions in Sections 3 and 4 to construct weak Taubes currents from a big J -nef spherical class. An immediate consequence is the proof of Theorems 1.2 and 1.5 for S^2 -bundles over S^2 . We also study the geometry and combinatorics of K -symplectic cone and K -sphere cone of $\mathbb{C}\mathbb{P}^2 \# k \overline{\mathbb{C}\mathbb{P}^2}$ with $k \geq 2$ and prove Theorems 1.3, 1.6 and 5.19. Finally, we compare our construction with the Kodaira embedding theorem.

Acknowledgments. The authors appreciate discussions with T. Draghici, R. Gompf, Y. Ruan, C. Taubes, A. Tomassini, V. Tosatti, S. T. Yau, and K. Zhu. We are

grateful to the referee for careful reading of the manuscript and useful suggestions improving the presentation.

2. Pseudo-holomorphic subvarieties and Taubes current. Let M be a closed, oriented 4-manifold and J be a tamed almost complex structure on M . In this section we summarize properties of J -holomorphic subvarieties following [29] and introduce Taubes current.

We fix a symplectic form ω tamed by J . Such a form defines a cohomology class $[\omega]$. Moreover, the polarization of the quadratic form given by $\omega(\cdot, J(\cdot))$ defines a J -invariant metric on M . Such a metric is used implicitly in all the follows to define distances on M , integration over open sets in Cartesian products of M , norms on (complexified) tensor bundles of M .

2.1. Pseudo-holomorphic subvarieties and their properties.

Definition 2.1. A closed set $C \subset M$ with finite, nonzero 2-dimensional Hausdorff measure is said to be a J -holomorphic subvariety if it has no isolated points, and if the complement of a finite set of points in C , called the singular points, is a smooth submanifold with J -invariant tangent space.

When J is understood, we will simply call a J -holomorphic subvariety a subvariety. A subvariety is said to be smooth if it has no singular points.

A subvariety C has a canonical orientation, which is used to define integration of smooth 2-forms on its smooth part. The resulting linear functional on the space of 2-forms defines a closed, non-negative current of type $(1, 1)$. We denote the associated homology class by e_C .

2.1.1. Genus of an irreducible subvariety. A subvariety is said to be irreducible if its smooth locus is connected. Any given subvariety is a union of a finite set of irreducible subvarieties.

Suppose C is an irreducible subvariety. Then it is the image of a J -holomorphic map $\phi : C_0 \rightarrow M$ from a complex connected curve C_0 , where ϕ is an embedding off a finite set. C_0 is called the model curve and ϕ is called the tautological map. The map ϕ is uniquely determined up to automorphisms of C_0 . This understood, the homology class e_C is simply the push forward of the fundamental class of C_0 via ϕ .

The genus of an irreducible subvariety C is defined to be the genus of its model curve C_0 . There is another type of genus associated to the class e_C defined as follows.

Given a class e in $H_2(M; \mathbb{Z})$, introduce the J -genus of e ,

$$(2) \quad g_J(e) = \frac{1}{2}(e \cdot e + K_J \cdot e) + 1,$$

where K_J is the canonical class of J . Notice that $g_J(e)$ is called the virtual genus in many places in the literature.

If $e = e_C$ for some subvariety C , then $g_J(e)$ is non-negative. In fact, by the adjunction inequality in [20], $g_J(e)$ is bounded from below by the genus of the model curve C_0 of C , with equality if and only if C is smooth.

2.1.2. The normal operator D_C . Suppose C is an irreducible subvariety and C_0 is its model curve with the tautological map ϕ . Let us first introduce the normal bundle of C .

If $p \in C_0$ is not a critical point of ϕ , then $L_p = \phi_*(TC_0|_p)$ is a complex line in $TM|_{\phi(p)}$. If p_0 is a critical point of C_0 , for p in a deleted neighborhood of p_0 consisting of non-critical points, $L_p \subset TM|_{\phi(p)}$ still converge to a line L_{p_0} in $TM|_{\phi(p_0)}$. This follows from elliptic unique continuation. It also follows from the local canonical form of ϕ near p_0 (see e.g. (2.2) of [28]): there is a complex coordinate u for a disk in C_0 centered on a critical point p_0 , and a complex coordinate (z, w) centered on $\phi(p_0)$ such that ϕ has the form

$$(3) \quad u \longrightarrow (u^{n+1} + t_z, cu^{n+k+1} + t_w),$$

where $n, k \geq 1$ and $|t_z| \leq c_0|u|^{n+2}, |dt_z| \leq c_0|u|^{n+1}, |t_w| \leq c_0|u|^{n+k+2}, |dt_w| \leq c_0|u|^{n+k+2}$. Hence L extends over p_0 as the pull-back from \mathbb{C}^2 via (3) of the span of the vector field $\frac{\partial}{\partial z}$. Thus there is a complex line bundle N , whose fiber over $p \in C_0$ is the quotient complex line $TM|_{\phi(p)}/L_p$. This complex line bundle over C_0 is called the normal bundle of C .

Linearizing the equation $\bar{\partial}_J \phi = 0$ for the J -holomorphic map $\phi : C_0 \rightarrow M$, we obtain a \mathbb{R} -linear, differential operator

$$D_\phi : C^\infty(C_0; \phi^*TM) \longrightarrow C^\infty(C_0; \phi^*TM \otimes T^{0,1}C_0).$$

Choose an almost Hermitian metric on M to realize N as a subbundle of ϕ^*TM . Then it induces a canonically associated \mathbb{R} -linear, differential operator

$$(4) \quad D_C : C^\infty(C_0; N) \longrightarrow C^\infty(C_0; N \otimes T^{0,1}C_0).$$

D_C is called the normal operator of C .

Use a Hermitian metric on C_0 and the Levi-Civita connection on M to define Sobolev completions of $C^\infty(C_0; N), C^\infty(C_0; N \otimes T^{0,1}C_0)$. D_C extends to a bounded, Fredholm operator from the Hilbert space $L^2_1(C_0; N)$ to the Hilbert space $L^2(C_0; N \otimes T^{0,1}C_0)$. Denote the index of this extension by d_C . d_C is always even, and is bounded above by the even integer $2\iota_{e_C}$ defined as follows.

Definition 2.2. Given a class e , introduce its J -dimension,

$$(5) \quad \iota_e = \frac{1}{2}(e \cdot e - K_J \cdot e).$$

Note also that $d_C = 2\iota_{e_C}$ if and only if C is smooth.

2.2. The moduli space. In this section we fix a class e .

The moduli space of subvariety in the class e , \mathcal{M}_e , is defined as in [29]: Any element Θ in \mathcal{M}_e is a finite set of pairs, where each pair has the form (C, m) with $C \subset M$ an irreducible subvariety and m a positive integer. The set of pairs in an element is further constrained so that no two of its pairs have the same subvariety component, and so that $\sum m e_C = e$.

Definition 2.3. A homology class $e \in H_2(M; \mathbb{Z})$ is said to be J -effective if \mathcal{M}_e is nonempty.

2.2.1. Topology. Let $|\Theta| = \cup_{(C,m) \in \Theta} C$ denote the support of Θ . Consider the symmetric, non-negative function, ϱ , on $\mathcal{M}_e \times \mathcal{M}_e$ that is defined by the following rule:

$$(6) \quad \varrho(\Theta, \Theta') = \sup_{z \in |\Theta|} \text{dist}(z, |\Theta'|) + \sup_{z' \in |\Theta'|} \text{dist}(z', |\Theta|).$$

The function ϱ is used to measure distances on \mathcal{M}_e .

Given a smooth form ν introduce the pairing

$$(\nu, \Theta) = \sum_{(C,m) \in \Theta} m \int_C \nu.$$

The topology on \mathcal{M}_e is defined in terms of convergent sequences:

A sequence $\{\Theta_k\}$ in \mathcal{M}_e converges to a given element Θ if the following two conditions are met:

- $\lim_{k \rightarrow \infty} \varrho(\Theta, \Theta_k) = 0$.
- $\lim_{k \rightarrow \infty} (\nu, \Theta_k) = (\nu, \Theta)$ for any given smooth 2-form ν .

Here is Proposition 3.1 in [29].

PROPOSITION 2.4. *The moduli space \mathcal{M}_e is compact. In particular, only finitely many classes are of the form e_C with $(C, m) \in \Theta$ and $\Theta \in \mathcal{M}_e$.*

2.2.2. Moduli spaces of irreducible subvarieties. Fix an integer h . We define $\mathcal{M}_{h,e} \subset \mathcal{M}_e$ to be the subspace of irreducible subvarieties of genus h .

With this understood, the following is a direct consequence of the adjunction inequality.

LEMMA 2.5. *If $h = g_J(e)$ and $C \in \mathcal{M}_{h,e}$, then C is smooth.*

Let Σ be a smooth, oriented surface of genus h . Let $\mathfrak{M}_{h,e}$ be the space of somewhere injective J -holomorphic maps in the class e and originated from (Σ, j) , where j is an arbitrary complex structure on Σ . As a subset of the Fréchet space of smooth maps from Σ to M , $\mathfrak{M}_{h,e}$ has a natural topology.

Since every irreducible subvariety has a model curve, there is a surjective map Ψ from $\mathfrak{M}_{h,e}$ to $\mathcal{M}_{h,e}$. A fundamental fact established in the appendix in [29],

whose proof is rather involved, is that the topology on $\mathcal{M}_{h,e}$ is the same as the induced one from $\mathfrak{M}_{h,e}$ via Ψ . More precisely, by Lemma A.13 in [29], at any $\phi : (\Sigma, j) \rightarrow M$ in $\mathfrak{M}_{h,e}$, Ψ is a local homeomorphism from $\mathfrak{M}_{h,e}$ to $\mathcal{M}_{h,e}$ when $h \geq 2$, and in the case $h = 0, 1$, Ψ is a local homeomorphism up to automorphisms of (Σ, j) .

With this understood, it follows that well known topological properties of $\mathfrak{M}_{h,e}$ carry over to $\mathcal{M}_{h,e}$.

THEOREM 2.6. (Propositions 3.2 and 3.3 in [29]) *There exists a smooth map, f , from a neighborhood of 0 in $\ker D_C$ to $\text{coker } D_C$; and there exists a homeomorphism from $f^{-1}(0)$ to a neighborhood of C in $\mathcal{M}_{h,e}$ sending 0 to C .*

The subset of $\mathcal{M}_{h,e}$ where the cokernel of D_C is trivial has the structure of a smooth manifold of dimension $2\iota_e - 2(g_J(e) - h)$; and the smooth structure is such that at any point in this set, the aforementioned homeomorphism from a neighborhood of 0 in the kernel is a smooth embedding onto an open set.

For J in a residual set in the space of almost complex structures, $\text{coker } D_{(\cdot)} = 0$ for each point $\mathcal{M}_{h,e}$, and so the latter has the structure of a smooth manifold whose dimension is $2\iota_e - 2(g_J(e) - h)$.

Concerning the vanishing of $\text{coker } D_C$, we mention another fact, which is particularly useful in this paper. Note that D_C is a real Cauchy-Riemann operator on N . For such operators, there is the following automatic transversality result.

THEOREM 2.7. [10, 11] *Let (Σ, j) be a Riemann surface of genus h , and L a complex line bundle over Σ . Suppose $c_1(L) \geq 2h - 1$. Then $\text{coker } D = 0$ for any real Cauchy-Riemann operator D on L .*

2.2.3. Moduli spaces with marked points. Let $M^{[k]}$ denote the set of k tuples of pairwise distinct points in M . Given $\Omega = (z_1, \dots, z_k)$ in $M^{[k]}$, denote its support $\{z_1, \dots, z_k\}$ in M also by Ω . Let \mathcal{M}_e^Ω be the space of subvarieties in \mathcal{M}_e passing through Ω .

Notice that \mathcal{M}_e^Ω is compact since it is a closed subset of the compact space \mathcal{M}_e .

Given an integer h , let $\mathcal{M}_{h,e}^\Omega = \mathcal{M}_{h,e} \cap \mathcal{M}_e^\Omega$.

Suppose $h = g_J(e)$ and $C \in \mathcal{M}_{g_J(e),e}^\Omega$. Then C is smooth by Lemma 2.5, and so the normal bundle N is a line bundle over C itself. Consider the evaluation map at Ω , $ev^\Omega : \Gamma(N) \rightarrow \bigoplus_{p \in \Omega} N|_p$, and the operator

$$D_C \oplus ev^\Omega : \Gamma(N) \longrightarrow \Gamma(N \otimes T^{1,0}C) \oplus (\bigoplus_{p \in \Omega} N|_p).$$

The index of $D_C \oplus ev^\Omega$ is $d_C - 2k$. And the kernel of $D_C \oplus ev^\Omega$ should be thought of as giving a sort of Zariski tangent space to $\mathcal{M}_{g_J(e),e}^\Omega$ at C (as a point in the space of smooth embeddings).

The smooth subvariety C is called (J, Ω) non-degenerate if the operator $D_C \oplus ev^\Omega$ has trivial cokernel. If this is the case, $\mathcal{M}_{h,e}^\Omega$ is a smooth manifold of dimension $d_C - 2k$ around C .

It is clear that $D_C \oplus ev^\Omega$ has trivial cokernel if D_C has trivial cokernel and $ev^\Omega : \ker D_C \rightarrow \oplus_{p \in \Omega} N|_p$ is surjective. In light of Theorem 2.7, it is more useful to test the (J, Ω) non-degeneracy via the real Cauchy-Riemann operator D_C^Ω in [1], which we now describe.

The bundle N over the complex curve C has a natural holomorphic line bundle structure. Consider the holomorphic line bundle $N(\Omega)$, obtained from twisting N by the divisor $-(z_1 + \dots + z_k)$. In [1, Lemma 4], there is introduced the following operator

$$D_C^\Omega : C^\infty(C_0; N(\Omega)) \longrightarrow C^\infty(C_0; N(\Omega) \otimes T^{0,1}C_0).$$

D_C^Ω is also a real Cauchy-Riemann operator. Moreover, there is an exact sequence, (15) in [1],

$$(7) \quad 0 \longrightarrow \ker D_C^\Omega \longrightarrow \ker D_C \longrightarrow \oplus_{p \in \Omega} N|_p \longrightarrow \text{coker } D_C^\Omega \longrightarrow \text{coker } D_C \longrightarrow 0,$$

where the middle map is ev^Ω . It follows from (7) that we have

LEMMA 2.8. $D_C \oplus ev^\Omega$ has trivial cokernel if D_C^Ω has trivial cokernel.

2.3. Subvarieties through a small ball.

2.3.1. The local area bounds. The following summarizes the local area bounds of an irreducible subvariety.

LEMMA 2.9. *Let J be a tamed almost complex structure. Fix a symplectic form ω on M taming J and the induced J -invariant metric. There exists $k \geq 1$, depending only on J, ω , with the following significance: Let $C \subset M$ denote an irreducible subvariety intersecting $B_r(x)$. Fix $r > 0$. Let $a_x(2r)$ denote the area of C 's intersection with the ball of radius $2r$ in M centered at x . Then $k^{-1}r^2 < a_x(2r) < (e_C \cdot [\omega])kr^2$.*

Proof. This is based on Lemma 2.2 in [29] which states a similar bound with a constant k' when x is a point in C .

For the lower bound, notice that the intersection contains a radius r ball centered at a point in C . Take $k_1 = k'$.

For the upper bound, notice that the intersection is contained in a radius $3r$ ball centered at a point in C . Take $k_2 = 9k'$.

Thus $k = 9k'$ is as required. □

2.3.2. Local structure around a smooth subvariety. To describe the behavior of subvarieties in a neighborhood of a given point, it is useful to introduce a special sort of coordinate chart. Fix a point x in M . An adapted coordinate chart centered at x denotes complex coordinates, (z, w) defined on a radius c_0^{-1} ball centered at x with both vanishing at x , with dz and dw orthonormal at x , with $\{dz, dw\}$ spanning $T^{1,0}M$ at x , and with the norms of $|\nabla dz|$ and $|\nabla dw|$ bounded on the coordinate domain by c_0 .

Suppose C is a smooth subvariety passing through x . Fix an adapted coordinate chart centered at x so as to identify a neighborhood of x in M with a ball about the origin in \mathbb{C}^2 . There exists $R > 1$ such that C appears in the radius R^{-1} ball about the origin in \mathbb{C}^2 as the image from \mathbb{C} to \mathbb{C}^2 that has the form

$$(8) \quad u \longrightarrow \theta u + \nu$$

where $|\nu| < R|u|^2$ and $|d\nu| < R|u|$, and such that $\theta \in \mathbb{C}^2$ has norm 1.

The following is Lemma 4.2 in [29].

LEMMA 2.10. *Let C be a smooth subvariety of genus h described as above in an adapted chart centered at $x \in C$.*

Fix $\epsilon > 0$. There is a neighborhood of C in $\mathcal{M}_{e_C, h}$ where each subvariety C' intersects the ball of radius $\frac{1}{2}R^{-1}$ about the origin, and this intersection is the image of a map from \mathbb{C} to \mathbb{C}^2 of the form

$$(9) \quad u \longrightarrow x' + \theta' u + \nu'$$

with

- $|x'| < \epsilon$,
- $|\nu'| < R|u|(|x'| + |u|)$ and $|d\nu'| < R(|x'| + |u|)$,
- $\theta' \in \mathbb{C}^2$ is a unit vector with $|\theta' - \theta| < \epsilon$.

2.3.3. The exponential map \exp_C . Suppose C is smooth and coker $D_C = 0$. We describe a version of exponential map in [28, 29] to identify a ball in $\ker D_C$ with a neighborhood of C in $\mathcal{M}_{e_C, g_J(e_C)}$.

Since C is smooth, the normal bundle N can be realized as the orthogonal complement of TC in $T_{1,0}M|_C$. In this case, there exists a map, \exp_C , that is defined on a small radius disk bundle $N_1 \subset N$ and has the following properties:

- \exp_C maps the zero section to C ; and its differential along zero section is an isomorphism from $TN|_0$ to $\phi^*T_{1,0}M$.

- \exp_C embeds each fiber of N_1 as a J -holomorphic disk in M .

- $\text{dist}(\exp_C(v), C) \leq K|v|$ where K is a uniform constant independent of v and $x \in C$ for any vector $v \in N$ with small norm $|v|$.

A construction of such a map is described in Section 5d of [28]. The last item above is essentially from Lemma 5.4(3) there.

Let ζ denote a section of N_1 . Then the image in M of the map $\exp_C(\zeta(\cdot))$ is a J -holomorphic subvariety if and only if ζ obeys an equation of the form

$$(10) \quad D_C \zeta + \tau_1 \partial \zeta + \tau_0 = 0.$$

Here τ_1 and τ_0 are smooth, fiber preserving maps from N_1 to $\text{Hom}(N \otimes T^{1,0}C; N \otimes T^{0,1}C)$ and to $N \otimes T^{0,1}C$ that obey $|\tau_1(b)| \leq c_0|b|$ and $|\tau_0(b)| \leq c_0|b|^2$.

Since $\ker D_C$ is of finite dimension, all the norms on it are equivalent. Choose any norm $|\cdot|$, e.g., the L^2 norm or the sup norm, on $\ker D_C$. The map \exp_C can be used to identify a fixed radius ball of $\ker D_C$ with a neighborhood of C in $\mathcal{M}_{e_C, g_J(e_C)}$ (see Lemmas 4.6 and 4.9 in [29]): Suppose C is smooth and $\text{coker } D_C = 0$. For κ sufficiently large, there is a diffeomorphism from the radius κ^{-2} ball in $\ker D_C$ onto an open set in $\mathcal{M}_{e_C, g_J(e_C)}$ that contains the set of curves with ϱ -distance less than κ^{-3} from C . Moreover, the map in question sends a given small normed vector $\eta \in \ker D_C$ to $\exp_C(\eta + \phi_C(\eta))$, where $\phi_C : \ker D_C \rightarrow C^\infty(C; N)$ is such that $\zeta = \eta + \phi_C(\eta)$ satisfies (10) and any given C^k norm of $\phi_C(\eta)$ is bounded by a multiple of $|\eta|^2$.

2.4. Subvariety, Taubes current and almost Kähler form.

2.4.1. Non-negative currents and forms. Given an almost complex structure J on M , it acts on $\Omega^1(M)$: $J\alpha(X) = -\alpha(JX)$. The componentwise action of J on $\Omega^2(M)$ is an involution, decomposing it into $\Omega_J^+(M) \oplus \Omega_J^-(M)$, called J -invariant and J -anti-invariant parts respectively. An J -invariant 2-form is said to be non-negative if it is non-negative on any pair of tangent vectors (v, Jv) in each point, and positive if the evaluation is positive for any nonzero v . In particular, a compatible symplectic form is a positive, closed J -invariant form, and a tamed symplectic form is a closed form whose J -invariant part is positive.

Non-negative 2-forms can be constructed from 1-forms in the following way: The complexification of $\Omega^1(M)$ decomposed as $\Omega^1(M) \otimes \mathbb{C} = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$, where $\Omega^{1,0}$ is the i -eigenspace of the extended J -action. For any $\sigma \in \Omega^{1,0}(M)$, $i\sigma \wedge \bar{\sigma}$ is a non-negative form.

A 2-dimensional current is a bounded linear functional on the space of smooth 2-forms. All currents here are understood to be 2-dimensional. A current is said to be closed if it annihilates $d\Omega^1(M)$, J -invariant, or type $(1, 1)$, if it annihilates $\Omega_J^-(M)$. A J -invariant current is said to be non-negative if it is non-negative on any non-negative J -invariant form, and positive if the evaluation is positive whenever the non-negative J -invariant form is not identically zero.

If C is a J -holomorphic subvariety, using its natural orientation, it defines a closed, non-negative J -invariant current via the integration on its smooth part.

2.4.2. Taubes current and regularization. The following type of positive current was introduced in [29], which we will refer to as a Taubes current. Let

$B_t(x)$ denote the ball of radius t and center x and $f_{B_t(x)}$ denotes the characteristic function of $B_t(x)$. Remember that we choose an almost Hermitian metric on (M, J) to define the balls. When J is tamed by a symplectic form ω , we use the polarization of the quadratic form given by $\omega(\cdot, J\cdot)$.

Definition 2.11. On an almost Hermitian 4-manifold (M, J, g) , a closed positive J -invariant current T is called a Taubes current if there is a constant $k > 1$ such that for any x and small t ,

$$(11) \quad k^{-1}t^4 < T(if_{B_t(x)}\sigma \wedge \bar{\sigma}) < kt^4.$$

Here σ denotes a point-wise unit length section of $T^{1,0}M|_{B_t(x)}$.

Since M is compact, being a Taubes current is independent of the metric g . A Taubes current behaves like an almost Kähler form except it may not be smooth. The following observation is also due to Taubes (see the proof of Theorem 1 in [29]).

PROPOSITION 2.12. *Suppose M has $b^+ = 1$ and J is a tamed almost complex structure on M . Given a Taubes current T , there is an almost Kähler form α , s.t. $[\alpha] = [T]$.*

This is proved by first smoothing the Taubes current to a family of closed two forms Ω^ϵ in a standard way. The property (11) then ensures that, when ϵ is small, Ω^ϵ is non-degenerate, uniformly bounded and has dominate J -invariant part. Then the condition of $b^+ = 1$ is used to kill the anti-invariant part by L^2 method for small ϵ , keeping the two form symplectic and in the same class. The second author generalizes this regularization result to all almost complex 4-manifolds with a Taubes current [31].

2.4.3. The subvarieties-current-form technique. Let M be a compact, oriented, smooth 4-manifold with $b^+ = 1$, and J an almost complex structure on M tamed by a symplectic form ω . The basic approach of Taubes to Question 1.1 is to first carefully pick a smooth subfamily of evenly distributed irreducible J -holomorphic subvarieties in the class of $N[\omega]$ for N large, at least when $[\omega]$ is rational, then to obtain a Taubes current via integration. Finally, Proposition 2.12 provides a genuine almost Kähler form.

Given ω , let \mathcal{J}_ω be the space of almost complex structures tamed by ω . Taubes' main result in [29] affirms Question 1.1 for an open and dense subset of \mathcal{J}_ω . This generic subset is constrained by a sequence of regularity properties on the space of irreducible subvarieties in the class $N[\omega]$, as well as by similar regularity assumptions on various spaces of reducible subvarieties.

Let \mathcal{J}^t be the Fréchet space of tamed almost complex structures. Since \mathcal{J}^t is the union of \mathcal{J}_ω over all symplectic forms, Taubes' result immediately implies

that Question 1.1 has a positive answer for an open and dense subset \mathcal{J}_1 of \mathcal{J}^t . If we stratify \mathcal{J}^t according to the set of irreducible subvarieties of negative self-intersection, \mathcal{J}_1 is properly contained in the top stratum \mathcal{J}_{top} defined as follows.

Definition 2.13. A tamed J is in \mathcal{J}_{top} if any irreducible J -holomorphic subvariety with negative self-intersection is a -1 curve. Here a -1 curve refers to a smooth genus zero subvariety with self-intersection -1 .

The main purpose of this paper is to study Question 1.1 for rational manifolds by applying the subvarieties-current-form technique to classes of genus zero smooth subvarieties with positive self-intersection. The first reason to consider the space of such subvarieties is that no genericity assumption is needed, namely, we always get a smooth family for any J from the automatic transversality in [10]. This makes it possible to settle Question 1.1 completely for certain rational manifolds, where such subvarieties exist.

Given a tamed J on a rational manifold M , classes of genus zero smooth subvarieties with positive self-intersection can always be found inside S_{K_J} , the set of classes represented by smoothly embedded spheres and having J -genus zero. However, unlike the case of $N[\omega]$ in [29], such a class is often only J -nef in the sense that the pairing with an arbitrary J -holomorphic subvariety in M is non-negative but might vanish. Consequently, in general we could only hope to first construct a weak spherical Taubes current, which is a weaker version of Taubes current degenerating on a finite union of subvarieties with negative self-intersection. Then we try to find several weak spherical Taubes currents whose degeneracy loci have empty intersection and sum them to get a honest Taubes current. The Taubes current obtained this way is called a spherical Taubes current.

As in [29], while the construction of weak spherical current in 5.1.1 is via integration over the irreducible subvariety part \mathcal{M}_{irr} of the moduli space, we still need a good control of the reducible subvariety part \mathcal{M}_{red} . To achieve this, we establish in [19] the following clean structural picture of reducible subvarieties for a J -nef class in S_{K_J} :

THEOREM 2.14. *Suppose e is a J -nef class in S_{K_J} and Θ is a reducible subvariety in the class e .*

- *If Θ is connected, then each irreducible component of Θ is a smooth rational curve, and Θ is a tree configuration.*
- *If J is tamed, then Θ is connected.*

It follows that \mathcal{M}_{red} is a finite union of Cartesian products of irreducible rational curve moduli spaces, which then has expected dimension (Proposition 4.4). In particular, \mathcal{M}_{red} has codimension at least one for *any* tamed almost complex structure (Proposition 4.10). This is crucial for us to get rid of much of the “genericity” assumption of [29].

Another new feature in our construction is the existence of plenty of pencils in the moduli space (Proposition 4.9). This linearity property of the moduli space enables us to better model slices of a small neighborhood of a subvariety in terms of subspaces of the tangent space, which in turns gives rise to the desired estimate of the volume of points lying in subvarieties through a tiny ball.

Our construction of spherical Taubes currents is particularly successful for S^2 -bundle over S^2 as in Theorem 1.2 and 1.5. Theorem 1.3 is also fairly general. In particular, it applies to the entire top stratum \mathcal{J}_{top} .

Theorem 1.3 also applies to the set $\mathcal{J}_{\text{good}}$ of good almost complex structures, which are defined after Theorem 1.6. For $\mathbb{C}\mathbb{P}^2 \#_k \overline{\mathbb{C}\mathbb{P}^2}$ with $k \leq 9$, it is not hard to see that $\mathcal{J}_{\text{good}} = \mathcal{J}_{\text{top}}$. Hence, the almost Kähler Nakai-Moishezon criterion (Theorem 1.6) holds for any $J \in \mathcal{J}_{\text{top}}$ in this case. But if $k \geq 10$, $\mathcal{J}_{\text{good}}$ is a lower stratum.

3. Line class Taubes current on $\mathbb{C}\mathbb{P}^2$. In this section we begin with introducing K_J -spherical classes and discussing various automatic regularity properties of irreducible subvarieties in these classes. Then we construct a Taubes current from the space of lines in $\mathbb{C}\mathbb{P}^2$. Although in this special case, there are no reducible rational curves, it illustrates many key features of the general construction, notably, how the presence of pencils enables us to better model a small neighborhood of a subvariety in terms of the tangent space, which in turns gives rise to the order two estimate of the volume of points lying in lines through a tiny ball.

3.1. K_J -spherical classes and smooth rational curves. Let J be a tamed almost complex structure.

3.1.1. Subvarieties in a K_J -spherical classes. Let S be the set of homology classes which are represented by smoothly embedded spheres.

The set of K_J -spherical classes is defined to be

$$S_{K_J} = \{e \in S \mid g_J(e) = 0\}.$$

The following is a consequence of Seiberg-Witten theory, see e.g. [15].

PROPOSITION 3.1. *Suppose $e \in S_{K_J}$ with $e \cdot e \geq -1$. Then for any symplectic form ω taming J , the Gromov-Taubes invariant of e is nonzero. In particular, \mathcal{M}_e is nonempty, i.e. e is J -effective.*

We use $\mathcal{M}_{\text{irr},e}$ to denote the moduli space of irreducible subvarieties in class e . The following is an immediate consequence of the adjunction formula and the adjunction inequality.

LEMMA 3.2. *For $e \in S_{K_J}$,*

- $\iota_e = e \cdot e + 1$, where ι_e is defined in (5).
- every element in $\mathcal{M}_{\text{irr},e}$ is a smooth rational curve.

Thus for $e \in S_{K_J}$, $\mathcal{M}_{\text{irr},e}$ is the same as $\mathcal{M}_{0,e}$. Let $\mathcal{M}_{\text{red},e}$ denote $\mathcal{M}_e \setminus \mathcal{M}_{\text{irr},e}$.

Given a $k(\leq \iota_e)$ tuple of distinct points Ω , recall that \mathcal{M}_e^Ω is the space of subvarieties in \mathcal{M}_e that contains all entries of Ω . Introduce similarly $\mathcal{M}_{\text{irr},e}^\Omega$ and $\mathcal{M}_{\text{red},e}^\Omega$. We will often drop the subscript e .

Let $S^+, S^0, \mathcal{E} \subset S$ be the subsets of positive square, square 0, square -1 classes respectively. S^+ is nonempty if and only if M is a rational manifold ([14]). Let $S_{K_J}^+ = S^+ \cap S_{K_J}$ and define $S_{K_J}^0, \mathcal{E}_{K_J}$ similarly.

Let k be an integer. Denote by $\mathcal{M}_{e,k}$ the subset in $\mathcal{M}_e \times M^{[k]}$ that consists of elements of the form (C, x_1, \dots, x_k) with each $x_i \in C$. Define $\mathcal{M}_{\text{irr},e,k}$ and $\mathcal{M}_{\text{red},e,k}$ similarly.

Consider the restrictions of the projection map to the $M^{[k]}$ factor, $\pi_k : \mathcal{M}_{e,k} \rightarrow M^{[k]}$, $\pi_{\text{irr},k} : \mathcal{M}_{\text{irr},e,k} \rightarrow M^{[k]}$, $\pi_{\text{red},k} : \mathcal{M}_{\text{red},e,k} \rightarrow M^{[k]}$.

By Proposition 3.1, we have:

LEMMA 3.3. *Suppose $e \in S_{K_J}^+ \cup S_{K_J}^0 \cup \mathcal{E}_{K_J}$. For any $\Omega \in M^{[\iota_e]}$, \mathcal{M}_e^Ω is nonempty. In other words, π_{ι_e} is surjective.*

3.1.2. Smooth rational curves. We assume now that e is a class represented by a smooth rational curve. In particular, $e \in S_{K_J}$.

Introduce

$$l_e = \max\{\iota_e, 0\}.$$

One special feature of the moduli space of smooth rational curves is the following automatic transversality, which is valid for an arbitrary tamed almost complex structure.

LEMMA 3.4. *Let e be a class represented by a smooth rational curve with $e \cdot e \geq -1$. Then $\mathcal{M}_{\text{irr},e}$ is a smooth manifold of dimension $2l_e$.*

Moreover, if we choose a set of $k \leq l_e$ distinct points $\Omega \subset C$, where $C \in \mathcal{M}_{\text{irr},e}$, then $\mathcal{M}_{\text{irr},e}^\Omega$ is a smooth manifold of dimension $2(l_e - k)$.

Proof. This observation is essentially contained in Corollary 2 in [27] (see also [1]).

Suppose $C \in \mathcal{M}_{\text{irr},e}$. By Lemma 3.2, C is a smooth rational curve.

Let N be its normal bundle. Notice that, since $c_1(N) = e \cdot e \geq -1$ and $g = 0$, by Theorem 2.7, $\text{coker } D_C = 0$. It follows from Theorem 2.6 that $\mathcal{M}_{\text{irr},e}$ is a smooth manifold whose dimension is d_C . Since C is smooth, $d_C = 2\iota_e$. Since $e \cdot e \geq -1$, $\iota_e \geq 0$ and hence $l_e = \iota_e$.

The second statement is proved similarly. By Lemma 2.8, it suffices to show that D_C^Ω has trivial cokernel. The proof is finished by noticing that $c_1(N(\Omega)) = c_1(N) - k$ and the index of $D_C \oplus ev^\Omega$ is $d_C - 2k$. \square

Here is another feature, specific to rational curves.

LEMMA 3.5. *Let e be a class represented by a smooth rational curve.*

- *If $e \cdot e \leq -1$, then $\mathcal{M}_{\text{irr},e}$ consists of a single element.*
- *If $e \cdot e \geq 0$, $l = l_e$, and $(C, \Omega) \in \mathcal{M}_{\text{irr},e,l}$, then $\mathcal{M}_{\text{irr},l}^\Omega$ consists of C only. In other words, $\pi_{\text{irr},l}$ is an injective smooth map, and the image of $\pi_{\text{irr},l}$ is disjoint from the image of $\pi_{\text{red},l}$.*
- *If $e \cdot e \geq 1$, $l = l_e$, the same uniqueness conclusion is true when we impose constraints of $l - 1$ points and a complex direction at one of the $l - 1$ points.*

Proof. All the statements follow from the positivity of intersections of distinct irreducible subvarieties. For the second bullet, notice that $e \cdot e < l$. For the third bullet, notice that a tangency contributes at least 2 to the intersection number. \square

Remark 3.6. There is an analogous result to Propositions 3.4 involving tangency conditions. Suppose C is smooth rational curve. If we choose $k \leq l$ distinct points x_1, \dots, x_k in C and $k' < k$ with $k + k' \leq l$, then the set of smooth rational curves in $\mathcal{M}_{\text{irr},eC}^{x_1, \dots, x_k}$ having the same tangent space at the k' points $x_1, \dots, x_{k'}$ as C is still a smooth manifold, whose dimension is $2(l - k - k')$. For this case, the Zariski tangent space is given by the subspace of $\ker D_C^{x_1, \dots, x_k}$, vanishing with order at least two on $x_1, \dots, x_{k'}$. Thus, we are considering the line bundle $N(2x_1 + \dots + 2x_{k'} + x_{k'+1} + \dots + x_k)$, N twisted by the divisor $-(2x_1 + \dots + 2x_{k'} + x_{k'+1} + \dots + x_k)$. In this case, the relevant Cauchy Riemann operator is onto if $-K_J \cdot e > k + k'$, which is automatic since $-K_J \cdot e = l + 1$.

By Lemmas 3.4 and 3.5 we have

PROPOSITION 3.7. *Let e be a class represented by a smooth rational curve and $l = l_e$. Then $\mathcal{M}_{\text{irr},e}$ has the structure of a $2l$ dimensional manifold and $\mathcal{M}_{\text{irr},e,k}$ has the structure of a $2l + 2k$ dimensional manifold. $\pi_{\text{irr},k} : \mathcal{M}_{\text{irr},e,k} \rightarrow M^{[k]}$ is a smooth map from a $2l + 2k$ dimensional manifold to a $4k$ dimensional manifold.*

3.2. Moduli space of lines. Now, we assume $M = \mathbb{C}P^2$. Let H be the line class, namely, the generator of $H_2(M; \mathbb{Z})$ such that $K_J = -3H$. Note that $S_{K_J} = \{H, 2H\}$.

Let \mathcal{M} denote \mathcal{M}_H , which we call the moduli space of lines. Notice that there are no reducible curves in \mathcal{M} so $\mathcal{M}_{\text{irr}} = \mathcal{M}$. Thus by Lemma 3.2, every element in \mathcal{M} is a smooth rational curve. Since $l = l_H = 2$, by Lemma 3.4, \mathcal{M} is a compact, 4 dimensional smooth manifold. In fact, it is diffeomorphic to $\mathbb{C}P^2$ (see e.g. [23]). Since \mathcal{M} is compact, by Lemma 2.10, we have the following (Lemma 4.12 in [29]).

LEMMA 3.8. *There is a constant $s' < 10^{-10}$ with the following significance: Fix a point x and an adapted coordinate chart centered at x so as to identify a neighborhood of x in M with a ball about the origin in \mathbb{C}^2 . Let $C \in \mathcal{M}$ which intersects $B_{s'}(x)$. Then C intersects the ball of radius s'^2 centered x as the image of a map from a disk in \mathbb{C} about the origin to \mathbb{C}^2 that has the form $u \rightarrow \theta u + \tau(u)$,*

where $\theta \in \mathbb{C}^2$ has norm 1 and where $\tau(u)$ is such that $|\tau(u)| \leq s'^{-1}|u|^2$ and $|d\tau|_v| \leq s'^{-1}|u|$.

Let $s = s'^4$.

3.2.1. Pencils of lines. Fix a point $x_1 \in M$. By Lemma 3.4, \mathcal{M}^{x_1} is 2-dimensional manifold. In fact, \mathcal{M}^{x_1} is a pencil, consisting of $\mathbb{C}\mathbb{P}^1$ family of lines.

Choose an orthonormal basis of vectors in $T_{x_1}M$ and use it to identify the space of complex directions at x_1 with $\mathbb{C}\mathbb{P}^1$. Consider the smooth map $\tau^{x_1} : \mathcal{M}^{x_1} \rightarrow \mathbb{C}\mathbb{P}^1$ by taking the tangent line at x_1 . The claim is that there is a unique curve through every direction at x_1 . The uniqueness is a consequence of the positivity of intersections. The existence can be shown by taking the limit of a sequence of “secant” curves C_k through x_1 and y_k with $y_k - x_1$ projecting to any given $\hat{\theta} \in \mathbb{C}\mathbb{P}^1$ and $y_k \rightarrow x_1$.

3.2.2. Norms on $\ker_C^{x_1}$. Given a smooth curve C in the pencil \mathcal{M}^{x_1} with normal bundle N , the tangent space to \mathcal{M}^{x_1} at C can be identified with the vector space $\ker_C^{x_1} \subset \Gamma(N)$ that consists of the sections in the kernel of D_C that vanish at x_1 .

On this two dimensional space, besides the sup norm $\sup_C |\eta|$, there are other kinds of norms due to the fact that \mathcal{M}^{x_1} is a pencil.

LEMMA 3.9. *The following are norms on $\ker_C^{x_1}$:*

- For $z \neq x_1$, the pointwise norm $|\eta(z)|$ of the vector $\eta(z)$ in $N|_z$.
- The pointwise norm of $\tau_*^{x_1}|_C(\eta)$ as a vector in $T_{\hat{\theta}}\mathbb{C}\mathbb{P}^1$, where $\hat{\theta}$ denotes the direction $T_{x_1}C$ in $\mathbb{C}\mathbb{P}^1$.

For a fixed curve C , these norms are equivalent to the sup norm $\sup_C |\eta|$.

Proof. Suppose η is a non-trivial element in $\eta \in \ker_C^{x_1}$. Both claims rely on an observation in [10] that there is a new holomorphic structure on N (depending on η) with respect to which η is a holomorphic section.

Since $c_1(N) = 1$, due to the positivity of intersection of holomorphic sections, η only vanishes at x_1 . Especially, $\eta(z) \neq 0$. In other words, $|\eta(z)|$ is a norm of $\ker_C^{x_1}$ if z is not x_1 .

To establish the second claim, we need the following description of $\tau_*^{x_1}|_C$ of Taubes in part (g) of the appendix. Fix an adapted coordinate chart, (z, w) , centered at x_1 so that C is tangent to the $w = 0$ locus at the origin. The span of $\frac{\partial}{\partial w}$ is identified with the fiber of N at x_1 , as well as $T_{\hat{\theta}}\mathbb{C}\mathbb{P}^1$. Choose a holomorphic coordinate, u , for C centered at x_1 with $du = dz$ at x_1 . Then $\partial\eta|_{x_1}$, when viewed as an element in $T_{\hat{\theta}}\mathbb{C}\mathbb{P}^1$ using the identifications above, is the image of η under the differential of τ^{x_1} at C .

From this description we see that $\tau_*^{x_1}|_C(\eta) = 0$ only if η has vanishing order at least 2 at x_1 . Since η is a holomorphic section and $c_1(N) = 1$, this is impossible. Thus we have shown that $\tau_*^{x_1}|_C : T_C\mathcal{M}^{x_1} \rightarrow T_{\hat{\theta}}\mathbb{C}\mathbb{P}^1$ is an isomorphism.

The last statement is clear since any two norms on a finite dimensional vector spaces are equivalent. \square

Notice that, since every $C \in \mathcal{M}^{x_1}$ is a smooth curve, we have shown that τ^{x_1} is a diffeomorphism.

3.2.3. Lines meeting a small ball and order 2 estimates. Given $x, x_1 \in M$, we consider the subset $\mathcal{M}^{x_1; B_t(x)} \subset \mathcal{M}^{x_1}$ whose element intersects $B_t(x)$ for small t . Denote in what follows the line through x_1 and x by C .

The following lemma is an analogue to Lemmas 4.5 and 4.9 in [29].

LEMMA 3.10. *Let x, x_1 and C be as stated as above. There is a constant $k_{3.10} > 1$, depending only on s , ensuring the following inequalities for $\eta \in \ker_C^{x_1}$,*

- (1) $\sup_C |\eta| \leq k_{3.10} |\eta(z)|$ if x_1 is not in $B_s(x)$ and $z \in \overline{B_{\frac{s}{2}}(x)}$;
- (2) $\sup_C |\eta| \leq k_{3.10} |\tau_*^{x_1} \eta|$ if $x_1 \in \overline{B_s(x)}$.

Proof. The constants in Lemma 3.9 can be chosen to be independent of x, x_1, z since $M, M \setminus B_s(x)$ and $\overline{B_s(x)}$ are compact. \square

Let $T^{x_1; B_t(x)}$ denote the set of points x_2 in M that lies in a curve in \mathcal{M}^{x_1} and intersecting $B_t(x)$.

LEMMA 3.11. *Suppose x_1 is not in $B_s(x)$. There are constants k and κ depending on s with the following significance: For $t < \kappa^{-3}$, the volume of $T^{x_1; B_t(x)}$ is bounded from above by kt^2 .*

Proof. Consider the unique curve C through x_1 and x . Since $\text{dist}(x_1, x) \geq s$, by Lemma 3.9, $|\eta(x)|$ is a norm on the 2-dimensional vector space $\ker_C^{x_1}$. Since \mathcal{M}^{x_1} is a 2-dimensional smooth manifold, apply the implicit function theorem as in Lemma 4.7 in [29], we find there exists $\kappa > 1$ with the following property: for $t < \kappa^{-3}$, there is an embedding $\lambda_C^{x_1}$ from the $|\eta(x)| \leq \kappa t$ disk in $\ker_C^{x_1}$ onto a neighborhood in \mathcal{M}^{x_1} of C , which brings 0 to C , and contains $\mathcal{M}^{x_1; B_t(x)}$ as an open set in the image. Since \mathcal{M}^{x_1} and $M \setminus B_s(x)$ are compact, κ only depends on s . Moreover, the image of $\eta \in \ker_C^{x_1}$ via $\lambda_C^{x_1}$ can be written as $\exp_C(\eta + \phi_C^{x_1}(\eta))$ where $\phi_C^{x_1}(\cdot)$ maps the ball $\{\eta : |\eta(x)| \leq 2\kappa^{-1}\}$ smoothly into $\ker_C^{x_1}$, and the C^0 norm of $\phi_C^{x_1}(\eta)$ is bounded by a multiple of $|\eta(x)|^2$.

Suppose x_2 in M lies in a curve C' in \mathcal{M}^{x_1} with $C' \cap B_t(x) \neq \emptyset$. Then there is a vector $\eta \in \ker_C^{x_1}$ with norm $|\eta(x)| \leq \kappa t$, and a point $p \in C$ such that $x_2 = \lambda_C^{x_1}(\eta(p))$.

Notice that $|\eta(p) + \phi_C^{x_1}(\eta)(p)| \leq |\eta(p)| + \kappa' |\eta(p)|^2$ by the estimate of $\phi_C^{x_1}$. Hence, by the third bullet in 2.3.3,

$$\text{dist}(x_2, C) = \text{dist}(\lambda_C^{x_1}(\eta(p)), C) \leq K |\eta(p)|.$$

Further, by the first bullet of Lemma 3.10,

$$\text{dist}(x_2, C) \leq K |\eta(p)| \leq K \sup_C |\eta| \leq K \kappa_{3.10} |\eta(x)| \leq 2K \kappa_{3.10} \kappa t.$$

This implies that x_2 is constrained so as to lie in a tubular neighborhood of C whose radius is bounded above by $k'_2 t$ (with $k'_2 = 2K\kappa_{3,10}\kappa$).

The area of C is bounded by $k_0 H \cdot [\omega]$. So the volume of the points in M that lie on a radius t tubular neighborhood of any C is bounded from above by kt^2 . \square

LEMMA 3.12. *Suppose x_1 is in $\overline{B_s(x)} \setminus B_{Rt}(x)$. There is a constant k depending on s and R with the following significance:*

The volume of points x_2 in M lying in a curve in \mathcal{M}^{x_1} and intersecting $B_t(x)$ is bounded from above by $k \frac{t^2}{d}$, where $d = \text{dist}(x_1, x)$.

Proof. Introduce $\mathcal{O}_{\hat{\theta}, \delta}^{x_1} \subset \mathcal{M}^{x_1}$ to denote the set of curves that mapped under τ^{x_1} to the disk in $\mathbb{C}\mathbb{P}^1$ of radius δ centered on the point $\hat{\theta} \in \mathbb{C}\mathbb{P}^1$.

The following geometric consideration is (4.14) in [29]. If $d > 10^4$ and $t < \frac{s}{d}$, then the set of complex 1 dimensional lines that intersects $B_t(x)$ with $x = (d, 0)$ is contained in a disk in $\mathbb{C}\mathbb{P}^1$ of radius less than $c_0 \frac{t}{d}$ with center the image of $(1, 0)$ of $\mathbb{C}\mathbb{P}^1$.

With this understood, by Lemma 3.8, $\mathcal{M}^{x_1; B_t(x)}$ lies in $\mathcal{O}_{\hat{\theta}, \delta}^{x_1}$ with $\hat{\theta}$ here denoting the image of the point $(1, 0)$ and with $\delta < c \frac{t}{d}$.

By Lemma 3.10(2), $\mathcal{O}_{\hat{\theta}, \delta}^{x_1}$ is contained in a tube of radius $k\delta$, whose volume is bounded from above by $k\delta^2$. Thus we have the desired volume estimate. \square

3.3. Spherical Taubes current from the line class. Let \mathcal{M}_2 be the moduli space of lines with two distinct points,

$$\mathcal{M}_2 = \{(C, x_1, x_2) | C \in \mathcal{M}, x_i \in C\} \subset \mathcal{M} \times M^{[2]}.$$

Use π_2 to denote the projection map $\mathcal{M}_2 \rightarrow M^{[2]}$. Let $\pi_{\mathcal{M}}$ be the projection map $\mathcal{M}_2 \rightarrow \mathcal{M}$.

The portion of marked moduli space we need is \mathcal{M}_2^r for $0 < r < \frac{s}{10}$, subject to the constraint $d(x_1, x_2) \geq r$. Notice that the image $\pi_{\mathcal{M}}(\mathcal{M}_2^r)$ is \mathcal{M} if r is chosen that small.

For $\eta \in \pi_2(\mathcal{M}_2^r)$, we introduce the non-negative, closed, invariant current ϕ_η . It is defined by $\phi_\eta(v) = \int_{C_\eta} v$, where C_η is the unique line $\pi_2^{-1}(\eta)$ and v is a 2-form on M . Then we have the following spherical current Φ_H in the line class given by

$$\Phi_H(v) = \int_{\eta \in \pi_2(\mathcal{M}_2^r)} \phi_\eta(v).$$

This current $\Phi_H(v)$ clearly satisfies Proposition 1.2 in [29]. Especially, it is a non-trivial, closed, non-negative J -invariant current on $\mathbb{C}\mathbb{P}^2$. In the rest of the section, we will prove that it is indeed a Taubes current.

3.3.1. Upper bound. Fix a smooth, non-increasing function $\chi : [0, \infty) \rightarrow [0, 1]$ with value 1 on $[0, \frac{1}{4}]$ and value 0 on $[\frac{1}{2}, \infty)$. Use χ_t to denote the function $\chi(t^{-1}|\cdot|)$ on \mathbb{C}^2 .

PROPOSITION 3.13. *The current Φ_H satisfies the upper bound in (11).*

Proof. Let s be as in Lemma 3.8. Fix $x \in M$ and adapted coordinates (z, w) centered at x with radius s . Let $t < 10^{-5}s$. As in [29], we only need to prove $\Phi_H(i\chi_t dz \wedge d\bar{z}) < kt^4$. Moreover, $\Phi_H(i\chi_t dz \wedge d\bar{z})$ is no greater than

$$k_0 \int_{\eta=(x_1, x_2) \in \pi_2(\mathcal{M}_2^r)} \left(\int_{C_\eta} \chi_t \omega \right).$$

Notice that $\int_{C_\eta} \chi_t \omega = 0$ if $C_\eta \cap B_t(x) = \emptyset$. If $C_\eta \cap B_t(x) \neq \emptyset$, then $C_\eta \cap B_t(x)$ is contained in a ball of radius of $2t$ centered at some point in $B_t(x)$. By Lemma 2.9, the integrand $\int_{C_\eta} \chi_t \omega$ is bounded by $H \cdot [\omega] kt^2$.

Thus it suffices to prove that the volume of the set

$$(12) \quad \{\eta = (x_1, x_2) \in \pi_2(\mathcal{M}_2^r) | C_\eta \cap B_t(x) \neq \emptyset\}$$

is $O(t^2)$.

We follow [29] to divide into three cases depending on the position of x_1 .

(I) The first case is that x_1 is away from $B_s(x)$. The upper bound for the choice of x_1 is $\text{Vol}(M)$. Now we estimate the possible choices of x_2 for a fixed x_1 . It follows from Lemma 3.11 that the volume of x_2 is $k_2 t^2$. The factors $\text{Vol}(M)$ and $k_2 t^2$ multiply to an upper bound of $O(t^2)$ for the volume of the subset in (12) with $x_1 \in M \setminus B_s(x)$. This bound depends on s .

(II) The second case is when x_1 is in $B_s(x)$ but outside $B_{Rt}(x)$, where $R = 10^5$.

Suppose C_η intersects $B_t(x)$. Since $R = 10^5$, by Lemma 3.12, the volume of this set is bounded by $k_{3.12} \delta^2$, where $\delta = c \cdot \frac{t}{\text{dist}(x_1, x)}$ and $k_{3.12}$ is the constant appeared in Lemma 3.12.

Using polar coordinates, the volume of this part of (12) is bounded from above by

$$k_3 t^2 \int_{z \in B_s(x)} \frac{1}{\text{dist}(z, x)^2} dz \leq k_2 s^2 t^2.$$

(III) The last case is when x_1 is in $B_{Rt}(x)$, this element itself would have the freedom of $\text{Vol}(B_{Rt}(x))$ which is $O(t^4)$. In this case, choices of x_1 and x_2 would multiply to contribute as the rate of $O(t^4)$.

Summing the three cases, we finish the proof. □

3.3.2. Lower bound.

PROPOSITION 3.14. *The current Φ_H satisfies the lower bound in (11).*

Proof. Let s be as in Lemma 3.8, and $t < 10^{-5}s$. Fix $x \in M$ and adapted coordinates (z, w) centered at x with radius s . As in [29], we only need to prove $\Phi_H(i\chi_t dz \wedge d\bar{z}) > k^{-1}t^4$.

The main picture to have is Lemma 2.10, applied to $B_s(x)$. Namely, inside $B_s(x)$, the curves behave as straight lines with respect to the adapted coordinates.

Fix $\epsilon > 0$. Recall that \mathcal{M}^x is a pencil, and identified with $\mathbb{C}\mathbb{P}^1$ via τ_x . Let us begin with choosing a disk $\mathcal{C}_x \subset \mathcal{M}^x$ corresponding to a disk centered at $(1, 0) \in \mathbb{C}\mathbb{P}^1$. The latter disk is chosen so that $|dz(\cdot)| \geq 2\epsilon$. By Lemma 2.10, by shrinking s if necessary, the restriction of $|dz|$ to $C \cap B_s(x)$ is greater than ϵ for $C \in \mathcal{C}_x$. Such an s can be chosen to be independent of x .

As we are estimating the lower bound, so we restrict our attention to $\eta = (x_1, x_2)$ such that x_1 is away from $B_{\frac{s}{2}}(x)$ and inside $B_s(x)$, and the line C_{x, x_1} determined by x and x_1 is contained in the disk $\mathcal{C}_x \subset \mathcal{M}$ specified above. By Lemma 2.10, the choices of x_1 constitute a compact set with volume $c_\epsilon s^4$.

Now, fix such an x_1 . Consider the set of x_2 in $B_{\frac{s}{4}}(x)$, for which $\eta = (x_1, x_2)$ contributes to $\Phi_H(ief_{B_t(x)} dz \wedge d\bar{z})$, namely, C_{x_1, x_2} intersects $B_t(x)$. Since we are estimating lower bound, we apply Lemma 2.10 to count the ones intersecting $B_{\frac{t}{2}}(x)$. Since $\text{dist}(x_1, x) \geq \frac{s}{2}$, C_{x_1, x_2} intersects $B_{\frac{t}{2}}(x)$ as long as $x_2 \in B_{\frac{s}{4}}(x)$ and $\text{dist}(x_2, C_{x_1, x}) \leq \frac{1}{4} \frac{t}{2}$. Thus the volume of x_2 is bounded by the radius $\frac{t}{8}$ tube around $C_{x_1, x} \cap B_{\frac{s}{4}}(x)$. By Lemma 2.9, or Lemma 2.10, the area of $C_{x_1, x} \cap B_{\frac{s}{4}}(x)$ is bounded from below by $a_x(\frac{s}{4})s^2$. Hence the volume of x_2 is bounded by $k_1 s^2 t^2$. Notice that, again by Lemma 2.10, when $t \leq 10^{-10}s$, the rational curve $C_{x_1, x_2} = \pi_2^{-1}(\eta)$ has the property that the restriction of $|dz|$ to $C \cap B_s(x)$ is greater than $\frac{\epsilon}{2}$. Here C denotes the line C_{x_1, x_2} .

By virtue of our choices, $\int_C(ief_{B_t(x)} dz \wedge d\bar{z})$ is bounded below by $(\frac{\epsilon}{2})^2 \cdot a_x(t)$. Thanks to Lemma 2.9, $a_x(t)$ is bounded below as $k_{2,9}^{-1}t^2$ (here $k_{2,9}$ is the constant appeared in Lemma 2.9). Given the aforementioned lower bound for $|dz|$, and given what is said in Lemma 2.9, it follows that the integral of $ief_{B_t(x)} dz \wedge d\bar{z}$ over C must be greater than $k_2^{-1}t^2$ by choosing $t (<< s)$ small enough. More precisely, this k_2^{-1} could be chosen as $(\frac{\epsilon}{2})^2 k_{2,9}^{-1}$.

Multiplying these three factors: $c = c_\epsilon s^4$, $k_1 s^2 t^2$ and $k_2^{-1}t^2$ together, we get $k^{-1}t^4 < \Phi_H(f_{B_t(x)} i\sigma \wedge \bar{\sigma})$ for some k . The constant k is further independent of x and t :

- The first constant c_ϵ is a universal constant since M is compact;
- The second constant k_1 , as in the argument, depends on the choices of x_1 (and x). But our x_1 is chosen from a compact set, so k_1 is universal as well;
- The last constant k_2 depends on x_2 (and x_1, x), and the tubular neighborhood of C_{x, x_1} we have chosen is compact, so k_2 is universal. □

We note that the following theorem due to Gromov [9] and Taubes [28] is an immediate consequence of Propositions 3.13, 3.14 and 2.12.

THEOREM 3.15. *Any tamed almost complex structure on $\mathbb{C}\mathbb{P}^2$ is almost Kähler.*

4. Pencils in a big J -nef class. Let M be a rational manifold and e a class in $S_{K_J}^+$. By Proposition 3.1, \mathcal{M}_e is nonempty. But if e is not the line class on the projective space, then \mathcal{M}_e always contains reducible subvarieties, and in fact, it could entirely consist of reducible varieties. To guarantee that there are smooth rational curves, we need to restrict to J -nef classes in $S_{K_J}^+$ (see Remark 5.11).

4.1. J -nef classes. Let J be a fixed almost complex structure on M .

Definition 4.1. A homology class $e \in H_2(M; \mathbb{Z})$ is said to be J -nef if it pairs non-negatively with any J -effective class.

4.1.1. Big J -nef and J -ample classes.

Definition 4.2. A J -nef class e is said to be big if $e \cdot e$ is positive.

The vanishing locus $Z(e)$ of a big J -nef class e is the union of irreducible subvarieties D_i such that $e \cdot e_{D_i} = 0$. Denote the complement of the vanishing locus of e by $M(e)$.

A big J -nef class e is said to be J -ample if $M(e) = M$.

The following lemma immediately follows from the positivity of intersections of distinct irreducible subvarieties.

LEMMA 4.3. *If e is represented by an irreducible J -holomorphic subvariety and $e \cdot e > 0$, then e is a big J -nef class.*

4.1.2. List of main results. We now list main properties of J -nef classes in S_{K_J} . The first one is Theorem 2.14, proved in [19]. As mentioned in the introduction, for rational manifolds, it plays a fundamental role to remove a number of the genericity assumptions in [29].

Together with Lemma 3.4, we have the following important consequence, also established in [19].

PROPOSITION 4.4. *Suppose $e \in S_{K_J}$ is a J -nef class. If $\Theta = \{(C_i, m_i), 1 \leq i \leq n\} \in \mathcal{M}_{\text{red}, e}$ is connected, then*

$$(13) \quad \sum_{(C_i, m_i) \in \Theta} m_i l_{e_{C_i}} \leq l - 1.$$

In particular,

$$(14) \quad \sum_{(C_i, m_i) \in \Theta} l_{e_{C_i}} \leq l - 1.$$

This is an analogue of Proposition 3.4 in [29], but valid for an arbitrary tamed almost complex structure. It follows that, similar to Proposition 1.1 in [29], J -nef classes have the following property:

PROPOSITION 4.5. *Suppose e is a J -nef class in $S_{K,J}$ with $e \cdot e \geq 0$. The map $\pi_l : \mathcal{M}_{\text{irr},e,l} \rightarrow M^{[l]}$ is onto the complement of a compact, measure zero subset. In particular, e is represented by a smooth rational curve.*

Fix an orthonormal frame for $T_{1,0}M|_x$ to identify the space of complex 1-dimensional subspaces with $\mathbb{C}\mathbb{P}^1$. Consider the map

$$\tau^{x,\Omega} : \mathcal{M}_e^{x,\Omega} \longrightarrow \mathbb{C}\mathbb{P}^1, \quad C \longmapsto T_x C.$$

The next result is very useful to understand this map.

PROPOSITION 4.6. *For any J -nef class e in $S_{K,J}^+$, we can choose a J -nef class H_e in $S_{K,J}^{\geq 0}$ such that $H_e \cdot e = 1$ or 2 , and $H_e \cdot e = 2$ only if H_e is proportional to e .*

For any $\Theta = \{(C_1, m_1), \dots, (C_n, m_n)\} \in \mathcal{M}_{\text{red},e}^x$ with $x \in C_1$, given $\Omega \in M^{[l-2]}$, let w_i be the cardinality of $\Omega_i = \Omega \cap C_i$.

Definition 4.7. Fix a point $x \in M(e)$. $\Omega \in M^{[l-2]}$ is called *pretty generic* with respect to e and x if

- x is distinct from any entry of Ω ;

For each $\Theta = \{(C_1, m_1), \dots, (C_n, m_n)\} \in \mathcal{M}_{\text{red},e}^x$ with $x \in C_1$;

- x is not in C_i for any $i \geq 2$;
- $\Omega_i \cap \Omega_j = \emptyset$;
- $1 + w_1 \geq m_1 e \cdot e_1 (\geq l_{e_1})$, and $w_i \geq m_i e \cdot e_i (\geq l_{e_i})$ for $i \geq 2$.

Let G_e^x be the set of pretty generic $l - 2$ tuples with respect to e and x .

PROPOSITION 4.8. *Suppose e is a big J -nef class in $S_{K,J}$ and $x \in M(e)$. Then the complement of G_e^x has complex codimension at least one in $M^{[l-2]}$.*

Further, big J -nef classes have the following property. We would like to show that if we fix x and choose Ω pretty generic, we always have smooth rational curves passing through x and Ω . Moreover, a generic complex direction in $T_x M$ would be tangent to these curves at x .

PROPOSITION 4.9. *Suppose e is big J -nef. For $x \in M(e)$ and $\Omega \in G_e^x$, The map $\tau^{x,\Omega}$ is well defined and in fact it is a homeomorphism.*

Moreover, $\tau^{x,\Omega}$ is a diffeomorphism away from the reducible curves.

Hence, for every complex direction in $T_x M = \mathbb{C}^2$, there is a (possibly reducible) rational curve tangent to it and passing through Ω and x . Moreover, except only finitely many directions, this rational curve is smooth.

Propositions 4.5, 4.6, 4.8, 4.9 will be proved in the next four subsections respectively.

4.2. Existence of smooth curves. In this section we prove Proposition 4.5. For this purpose, we need to estimate the dimension of the space of reducible curves.

4.2.1. Dimension of the moduli space of reducible curves. For

$$\Theta = \{(C_1, m_1), \dots, (C_n, m_n)\} \in \mathcal{M}_e,$$

let $\Xi_\Theta = (e_{C_1}, \dots, e_{C_n})$. Set $\chi_e = \{\Xi_\Theta | \Theta \in \mathcal{M}_{\text{red},e}\}$. Proposition 2.4 guarantees that χ_e is a finite set. Given $\Xi \in \chi_e$, the subspace of reducible curves $\mathcal{M}_\Xi \subset \mathcal{M}_{e,\text{red}}$ corresponding to Ξ is naturally identified with $\times_{e' \in \Xi} \mathcal{M}_{e',\text{irr}}$.

PROPOSITION 4.10. *Suppose e is a J -nef class in $S_{K,J}$. Then $\mathcal{M}_{\text{red},e} = \cup_{\Xi \in \chi_e} \mathcal{M}_\Xi$ is a finite union of manifolds, each with complex dimension at most $l - 1$.*

Proof. By Lemma 3.4 and Theorem 2.14, for each $\Xi \in \chi_e$, \mathcal{M}_Ξ is a manifold of complex dimension $\sum_{(C,m) \in \Theta} l_{e_C}$. Thus the assertion follows from Proposition 4.4. □

Remark 4.11. In [19] we completely determine the possible configuration with $l = 1 + \sum_{i=1}^n l_{e_i}$.

$l = 1 + \sum_{i=1}^n l_{e_i}$ if and only if each multiplicity is 1, and Θ is one of the following configurations:

- If Θ_- is empty then $n = 2$, $e_1 \cdot e_2 = 1$, $e_i \cdot e_i \geq 0$.
- If Θ_- is not empty and there is no -1 curves, then Θ_- consists of a unique element $(C_1, 1)$ with $e_1 \cdot e_1 = 1 - n \leq -2$, and Θ_+ consists of at least $n - 1 \geq 2$ elements, $e_i = \dots = e_n$ and $e_i \cdot e_i = 0$ for $i \geq 2$. Moreover, $e_1 \cdot e_2 = 1$. In short, it is a comb like configuration.

- Suppose Θ_- contains a -1 component. Then there are two cases.

(1) Θ is a successive blow-up of a smooth rational curve with non-negative self-intersection. And from the second blow-up, we only blow up at a point not lying in any component with non-negative self-intersection (there is at most one such component).

(2) Θ is a successive blow-up of a comb like configuration in the second bullet at points in C_1 and its proper transforms. Moreover, the blow-up points do not lie in C_i for each $i \geq 2$.

4.2.2. Proof of Proposition 4.5.

Proof. It follows from Proposition 3.3 that $\pi_{\text{irr},l}$ is surjective. On the other hand, by Proposition 4.10, the image of $\pi_{\text{red},l}$ is of codimension at least 2. The assertion follows. □

4.3. Intersection properties. In this section we establish Proposition 4.6.

4.3.1. Intersection properties of K_J -spherical classes.

LEMMA 4.12. *The following intersection properties hold:*

- $S_{K_J}^+$ pairs positively with $S_{K_J}^{\geq 0}$.
- $S_{K_J}^+$ pairs non-negatively with \mathcal{E}_{K_J} .
- \mathcal{E}_{K_J} pairs non-negatively with \mathcal{E}_{K_J} .

Proof. Since $b^+ = 1$, and $S_{K_J}^{\geq 0}$ pairs positively with any J -tamed symplectic form ω , $S_{K_J}^+$ pairs positively with $S_{K_J}^{\geq 0}$ by light cone lemma.

Any element in \mathcal{E}_{K_J} is J' -effective for any tamed J' with $K_{J'} = K_J$ by Proposition 3.1. For a generic tamed J' , $e \in S_{K_J}^+$ could be represented by smooth irreducible J -holomorphic curves. The second statement then follows from the positivity of intersections of distinct irreducible subvarieties.

The last item follows because any two elements e_1 and e_2 in \mathcal{E}_{K_J} have irreducible representations for a generic tamed J' with $K_{J'} = K_J$. □

LEMMA 4.13. *For any e in $S_{K_J}^+$, we can choose a class H_e in $S_{K_J}^{\geq 0}$ such that $H_e \cdot e = 1$ or 2 , and $H_e \cdot e = 2$ only if H_e is proportional to e .*

Proof. A rational manifold is either $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ or $S^2 \times S^2$.

- $S^2 \times S^2$

Denote the two factors classes by H_1 and H_2 such that $K_J = -2H_1 - 2H_2$. For the class $aH_1 + bH_2$ of an irreducible curve, the adjunction formula implies

$$(15) \quad \begin{aligned} &(aH_1 + bH_2)^2 + (-2H_1 - 2H_2)(aH_1 + bH_2) + 2 \\ &= 2ab - 2b - 2a + 2 = 2(1 - a)(1 - b) \geq 0. \end{aligned}$$

is clear that S_{K_J} is contained in the following two sequences of classes:

$$A_l = H_1 + lH_2, \quad B_l = lH_1 + H_2, \quad l \in \mathbb{Z}.$$

The sequence A_l is in $S_{K_J}^+$ if $l > 0$, and $A_l \cdot A_l = 0$ if $l = 0$. The same is true for the sequence B_l .

For $e = A_l$ with $l > 0$, choose $H_e = B_0 = H_2$, and for $e = B_l$ with $l > 0$ choose $H_e = A_0 = H_1$.

For $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ there exists a basis of spherical classes H, E_1, \dots, E_k with $H \cdot H = 1, E_i \cdot E_i = -1$ such that $K_J = -3H + E_1 + \dots + E_k$.

- $\mathbb{C}P^2$.

H and $2H$ are the only classes in S_{K_J} .

- $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

For $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and class $\alpha H + \beta E$, the adjunction formula is of the form

$$(16) \quad (\alpha - 1)(\alpha - 2) - \beta(\beta + 1) \geq 0$$

S_{K_J} is a subset of the following sequence

$$D_s = sH + (1 - s)E, \quad s \in \mathbb{Z}.$$

Choose $H_e = H - E$.

- $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ with $k \geq 2$.

When $k \geq 2$, it is hard to explicitly describe the classes in $S_{K_J}^{\geq 0}$. We invoke the classification up to Cremona equivalence (see [13]). As noted in [13], any class in $S_{K_J}^{\geq 0}$ is Cremona equivalent to one of the following classes:

- (1) $2H, H$,
- (2) $(n + 1)H - nE_1, n \geq 1$,
- (3) $(n + 1)H - nE_1 - E_2, n \geq 1$.

Case (1). If e is equivalent to $2H$ or H , we choose $H_e = \frac{e}{\sqrt{e \cdot e}}$. The class H_e is J -nef since it is proportional to the J -nef class e .

Case (2). If e is equivalent to $(n + 1)H - nE_1, n \geq 1$, we choose H_e to be $H - E_1$ under the same equivalence. Notice that $H - E_1 \in S_{K_J}^0$ and $H_e \cdot e = 1$.

Case (3). When the class e is equivalent to $2H - E_1 - E_2$, then we could also choose $H_e = e$.

Case (4). If e is equivalent to $(n + 1)H - nE_1 - E_2, n \geq 2$, we again choose H_e to be $H - E_1$ under the same equivalence. Notice that $H_e \cdot e = 1$. □

4.3.2. J -nef classes on S^2 -bundles over S^2 . For $S^2 \times S^2$, the negative self-intersection classes must be of the form $aH_1 + bH_2$ with $ab < 0$.

It follows from (15) that class of any irreducible curve satisfies $(1 - a)(1 - b) \geq 0$. Thus the only possible negative square irreducible J -curves are in the classes A_p with $p < 0$ or B_p with $p < 0$.

Moreover, given J , there is at most one such curve by the positivity of intersections.

Case (i). There are irreducible J -curves with negative self-intersection.

Case (ii). A_p is J -effective for some $p < 0$.

Case (iii). B_p is J -effective for some $p < 0$.

The negative self-intersection classes must be of the form $aH + bE$ with $|a| < |b|$. Then the only possible negative square irreducible J -curves are in the classes

$$(1 - s)H + sE_1, \quad s > 0.$$

These classes are in S_{K_J} . Moreover, there is at most one such curve due to positivity of intersections.

In summary,

LEMMA 4.14. For $S^2 \times S^2$,

In case (i), both A_l and B_l are J -nef if $l \geq 0$, and J -ample if $l > 0$.

In case (ii), A_l is J -nef when $l \geq -p$, and J -ample if $l \geq -p + 1$. B_0 is J -nef.

In case (iii), B_l is J -nef when $l \geq -p$, and ample if $l \geq -p + 1$. A_0 is J -nef.

For $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, $H - E$ is J -nef for any J . If $sH + (1 - s)E_1$ is J -effective for some $s \leq 0$, then D_l is J -nef for $l \geq 1 - s$, and J -ample for $l \geq 2 - s$.

Notice that for an S^2 -bundle over S^2 , there is always a J -ample class, and there is always a J -nef class with self-intersection 0.

4.3.3. A criterion for $H - E_i$ to be J -nef. For $\mathbb{C}P^2 \#_k \overline{\mathbb{C}P^2}$ we need the following observation.

LEMMA 4.15. *Suppose there is an irreducible curve class pairing negatively with $H - E_1$, say $e_C = aH - b_1E_1 - \dots - b_nE_n$. Then $a \leq 0$. The same is true for $H - E_i$.*

Proof. To see this, observe that $(H - E_1) \cdot e_C < 0$ means $a < b_1$. If $a > 0$ then $b_1 \geq 2$. Now the K_J -adjunction number

$$e_C \cdot e_C + K_J \cdot e_C \leq a^2 - b_1^2 - 3a + b_1 \leq (b_1 - 1)^2 - 3(b_1 - 1) - b_1^2 + b_1 = -4b_1 + 4$$

is less than -2 , which is impossible. □

4.3.4. Proof of Proposition 4.6.

Proof. Suppose e is J -nef. It suffices to show that we can further choose H_e in Lemma 4.13 to be J -nef.

For $S^2 \times S^2$, if $e = A_l$ with $l > 0$, $H_e = B_0 = H_2$ is J -nef. The second case is similar.

For $\mathbb{C}P^2$, H and $2H$ are J -nef for any tamed J . In both cases, we choose H_e to be H .

For $\mathbb{C}P^2 \#_k \overline{\mathbb{C}P^2}$, cases (1) and (3) are clear. For cases (2) and (4) we will show that $H_e = H - E_1$ is J -nef.

Suppose there is an irreducible curve class pairing negatively with $H - E_1$, say $e_C = aH - b_1E_1 - \dots - b_nE_n$. By the lemma above, $a \leq 0$.

If $e = (n + 1)H - nE_1$, $n \geq 1$, is J -nef, then $H \cdot e_C \geq -n(H - E_1) \cdot e_C > 0$. This implies that $a > 0$.

If $e = (n + 1)H - nE_1 - E_2$, $n \geq 2$ is J -nef, then

$$(H - E_2) \cdot e_C \geq -n(H - E_1) \cdot e_C \geq n.$$

This means $a \geq b_2 + n$. Since $a \leq 0$, we have $b_2 < a \leq 0$. Thus the K_J -adjunction number

$$e_C \cdot e_C + K_J \cdot e_C \leq a^2 - b_2^2 - 3a + b_2 \leq -2n \leq -4,$$

which is impossible. □

4.4. Vanishing locus and reducible curves. Suppose e is a big J -nef class in $S_{K,J}$, especially $l = e^2 + 1 \geq 2$. Recall the complement of the vanishing locus of e is denoted by $M(e)$. Fix $x \in M(e)$.

4.4.1. Reducible rational curves through a point in a big J -nef class.

Consider $\Theta = \{(C_1, m_1), \dots, (C_n, m_n)\} \in \mathcal{M}_{\text{red},e}^x$ and assume $x \in C_1$. Denote e_{C_i} by e_i .

Set $\chi_e^x = \{\Xi \mid \Theta \in \mathcal{M}_{\text{red},e}^x\}$. Given $\Xi \in \chi_e^x$, the subspace of reducible curves $\mathcal{M}_{\Xi}^x \subset \mathcal{M}_{\text{red},e}^x$ corresponding to Ξ is naturally identified with $\mathcal{M}_{\text{irr},e_1}^x \times \times_{i \geq 2} \mathcal{M}_{\text{irr},e_i}$.

We estimate the dimension of $\mathcal{M}_{\text{red},e}^x$.

PROPOSITION 4.16. *Suppose e is a big J -nef class in S_{K_J} . For any $x \in M(e)$, $\mathcal{M}_{\text{red},e}^x$ is a union of manifolds with complex dimension at most $l - 2$.*

Proof. Given $\Theta = \{(C_1, m_1), \dots, (C_n, m_n)\} \in \mathcal{M}_{\text{red},e}^x$, let $\Xi = \Xi_{\Theta}$.

By Lemma 3.4 and Theorem 2.14, \mathcal{M}_{Ξ}^x has complex dimension

$$\dim_{\mathbb{C}} \mathcal{M}_{\text{irr},e_1}^x + \sum_{i=2}^n l_{e_i}.$$

First suppose that $e_1 \cdot e_1 \geq 0$. By Lemma 3.4, $\mathcal{M}_{\text{irr},e_1}^x$ has complex dimension $l_{e_1} - 1$. Thus we need to show

$$l_{e_1} - 1 + \sum_{i=2}^n l_{e_i} \leq l - 2,$$

which is the same as (14).

Now suppose that $e_1 \cdot e_1 < 0$. Then we need to show $\sum_{i=2}^n l_{e_i} \leq l - 2$.

We assume $l_{e_j} = 0$ when $j \leq k$, and $l_{e_j} > 0$ when $j \geq k + 1$.

By Lemma 2.7 in [19], Θ is connected, so $e_j \cdot (e - m_j e_j) \geq 1$ for each $j \geq 2$.

Therefore l can be estimated as follows:

$$\begin{aligned} l - 1 &= e \cdot e \\ &= \left(\sum_{j=2}^n m_j e_j + m_1 e_1 \right) \cdot e \\ &\geq \sum_{j=k+1}^n m_j e_j \cdot (m_j e_j + (e - m_j e_j)) + m_1 e_1 \cdot e \\ (17) \quad &= \sum_{j=k+1}^n (m_j^2 e_j \cdot e_j + m_j e_j \cdot (e - m_j e_j)) + m_1 e_1 \cdot e \\ &\geq \sum_{j=k+1}^n m_j l_{e_j} + m_1 e_1 \cdot e \\ &= \sum_{j=1}^n m_j l_{e_j} + m_1 e_1 \cdot e. \end{aligned}$$

Recall that we assume $x \in C_1$ and $x \in M(e)$, therefore $e \cdot e_1 > 0$. Hence in this case, we have also shown that \mathcal{M}_{Ξ}^x has complex dimension at most $l - 2$. \square

This can be viewed as a version of Lemma 4.7 in [29].

Now consider the subset $\mathcal{M}_{\text{red},e}^x$ (x nodal) of $\mathcal{M}_{\text{red},e}$ where $x \in C_1$ and $x \in C_i$ for some $i \geq 2$.

LEMMA 4.17. *Suppose e is a big J -nef class in $S_{K,J}$. For any $x \in M(e)$, $\mathcal{M}_{\text{red},e}^x$ (x nodal) is a union of manifolds with complex dimension at most $l - 3$.*

Proof. The proof is similar to that of Proposition 4.16.

Given $\Theta = \{(C_1, m_1), \dots, (C_n, m_n)\} \in \mathcal{M}_{\text{red},e}^x$ (x nodal), let $\Xi = \Xi_{\Theta}$. Assume without loss of generality that $x \in C_2$.

By Lemma 3.4 and Theorem 2.14, \mathcal{M}_{Ξ}^x ($x \in C_2$) has complex dimension

$$\dim_{\mathbb{C}} \mathcal{M}_{\text{irr},e_1}^x + \dim_{\mathbb{C}} \mathcal{M}_{\text{irr},e_2}^x + \sum_{i=3}^n l_{e_i}.$$

We need to divide into four cases: (I) $e_1 \cdot e_1 \geq 0, e_2 \cdot e_2 \geq 0$. (II) $e_1 \cdot e_1 < 0, e_2 \cdot e_2 < 0$. (III) $e_1 \cdot e_1 \geq 0, e_2 \cdot e_2 < 0$. (IV) $e_1 \cdot e_1 < 0, e_2 \cdot e_2 \geq 0$.

In Case (I),

$$\dim \mathcal{M}_{\Xi}^x(x \in C_2) = l_{e_1} - 1 + l_{e_2} - 1 + \sum_{i=3}^n l_{e_i} \leq l - 3$$

by Lemma 3.4 and (14).

In Case II, we also assume $l_{e_j} = 0$ when $j \leq k$, and $l_{e_j} > 0$ when $j \geq k + 1$. By Lemma 2.7 in [19], Θ is connected, so $e_j \cdot (e - m_j e_j) \geq 1$ for each $j \geq 3$. Therefore l can be estimated as follows:

$$\begin{aligned} l - 1 &= e \cdot e \\ &= \left(\sum_{j=3}^n m_j e_j + m_1 e_1 + m_2 e_2 \right) \cdot e \\ &\geq \sum_{j=k+1}^n m_j e_j \cdot (m_j e_j + (e - m_j e_j)) + m_1 e_1 \cdot e + m_2 e_2 \cdot e \\ &= \sum_{j=k+1}^n (m_j^2 e_j \cdot e_j + m_j e_j \cdot (e - m_j e_j)) + m_1 e_1 \cdot e + m_2 e_2 \cdot e \\ &\geq \sum_{j=k+1}^n m_j l_{e_j} + m_1 e_1 \cdot e + m_2 e_2 \cdot e \\ &= \sum_{j=1}^n m_j l_{e_j} + m_1 e_1 \cdot e + m_2 e_2 \cdot e. \end{aligned}$$

Recall that we assume $x \in C_1 \cap C_2$ and $x \in M(e)$, therefore $e \cdot e_1 > 0, e \cdot e_2 > 0$. Hence in this case, we have also shown that \mathcal{M}_{Ξ}^x has complex dimension at most $l - 3$.

Cases (III) and (IV) are similar, we only prove Case (IV). In this case, we need to show $\sum_{i=2}^n l_{e_i} \leq l - 2$, which is exactly the second case of Proposition 4.16. \square

4.4.2. Pretty generic $l - 2$ tuple.

LEMMA 4.18. *For any $x \in M(e)$ and $\Omega \in G_e^x$, any two elements in $\mathcal{M}_e^{x,\Omega}$ intersect only at the $l - 1$ points x, Ω . So it is like a pencil in algebraic geometry. Moreover,*

- *Given z distinct from x and Ω , $\mathcal{M}^{z,x,\Omega}$ consists of unique curve.*
- *There is a unique curve in e passing through the $l - 1$ points x, Ω and a given direction at one of these points.*
- *$\mathcal{M}_{\text{red}}^{x,\Omega}$ is a finite set. Moreover, these reducible curves cannot be tangent to each others at x .*
- *We can define the map $\tau^{x,\Omega} : \mathcal{M}^{x,\Omega} \rightarrow \mathbb{C}\mathbb{P}^1, C \mapsto T_x C$.*

Proof. Suppose Θ, Θ' are two elements in $\mathcal{M}^{z,x,\Omega}$. If Θ, Θ' have no common components, then the first bullet follows from local positivity of intersection. Hence we assume they have common components.

We rewrite two subvarieties $\Theta, \Theta' \in \mathcal{M}_e$, allowing $m_i = 0$ in the notation, such that they have the same set of irreducible components formally, i.e. $\Theta = \{(C_i, m_i)\}$ and $\Theta' = \{(C_i, m'_i)\}$. Then for each C_i , if $m_i \leq m'_i$, we change the components to $(C_i, 0)$ and $(C_i, m'_i - m_i)$. Apply this process to all i and discard finally all components with multiplicity 0 and denote them by Θ_0, Θ'_0 and still use (C_i, m_i) and (C_i, m'_i) to denote their components. Notice they are homologous, formally have homology class $e - \sum m_{k_i} [C_{k_i}] - \sum m'_{l_j} [C_{l_j}]$ with pairwise different C_{k_i} and C_{l_j} . For each C_{k_i} , we know that there should be originally at least $m'_{k_i} e \cdot e_{k_i}$ points on it. Notice $\sum m_i e \cdot e_i = \sum m'_i e \cdot e_i = e^2 = l - 1$, and none of x, Ω are nodal points of Θ and Θ' . Hence either $e \cdot e_{k_i} = 0$, or $e \cdot e_{k_i} = 1$ and $m'_{k_i} - m_{k_i} = 1$. And the latter case could happen at most once. Similarly for C'_{l_j} . Therefore we know there are at least $e \cdot e_{\Theta_0}$ intersections (among x, Ω) of Θ_0, Θ'_0 .

Notice $e \cdot e_{\Theta_0} > e_{\Theta_0}^2$ since original Θ, Θ' have at least one common component. Hence there are more intersections than the homology intersection number $e_{\Theta_0}^2$ of our new subvariety Θ_0 and Θ'_0 . Then our claim follows from the local positivity of intersection.

The second bullet follows by the same argument.

For the third bullet, first notice $\chi_{\text{red},e}^{x,\Omega}$ is a finite set by Proposition 2.4. Hence we could fix the type $\Xi = (e_i) \in \chi_{\text{red},e}^{x,\Omega}$. Apply the same process, the removed components should have the same multiplicities since $l - 1 = \sum m_i e \cdot e_{C_i}$. And if $\Theta \neq \Theta_0$, the above argument claims contradiction. Hence, it implies there are no common

components of Θ, Θ' . Especially, it implies there are no spheres of negative self-intersection as components. Hence, it is the situation that Corollary 2 in [27] applies, which says that $\mathcal{M}_{\text{red}}^{x,\Omega}$ is isolated in the compact space $\mathcal{M}^{x,\Omega}$. Hence $\mathcal{M}_{\text{red}}^{x,\Omega}$ is a finite set.

Since x is not a nodal point of any Θ , and thanks to the second bullet, the map $\tau^{x,\Omega}$ is well-defined. □

4.4.3. Proof of Proposition 4.8.

Proof. The complement of G_e^x is the union of the four sets $V_i, i = 1, 2, 3, 4$: V_i is the set of points in $M^{[l-2]}$ violating the i -th item of Definition 4.7, but not the previous items.

It is easy to see that V_1 has complex codimension 2.

To estimate the dimensions of V_2, V_3, V_4 , consider the map

$$\pi_{\text{red},l-2}^x : \mathcal{M}_{\text{red},e,l-2}^x \longrightarrow M^{[l-2]}.$$

We first deal with V_2 . For each $\Xi = (e_i) \in \chi_e^x$, let $V_2(\Xi)$ be the image of the map $\pi_{\text{red},l-2}^x$ restricted to $\mathcal{M}_{\Xi,l-2}^x(x \text{ nodal})$. Here $\mathcal{M}_{\Xi,l-2}^x(x \text{ nodal}) \subset \mathcal{M}_{\Xi,l-2}^x$ consists of $\Theta = \{(C_i, m_i)\} \times \Omega \in \mathcal{M}_{\Xi}^x \times M^{[l-2]}$ with $x \in C_1$, and $x \notin \Omega$.

Clearly, V_2 is the union of $V_2(\Xi)$ over $\Xi = (e_i) \in \chi_e^x$. By Lemma 4.17,

$$\dim_{\mathbb{C}} \mathcal{M}_{\Xi,l-2}^x(x \text{ nodal}) \leq (l-3) + (l-2) = 2(l-2) - 1.$$

Thus V_2 has complex codimension at least 1 in $M^{[l-2]}$.

For the set V_3 , it is similarly the union of $V_3(\Xi)$ over $\Xi = (e_i) \in \chi_e^x$, where $V_3(\Xi)$ is the image of the map $\pi_{\text{red},l-2}^x$ restricted to $\mathcal{M}_{\Xi,l-2}^x(\Omega \text{ nodal})$. Here $\mathcal{M}_{\Xi,l-2}^x(\Omega \text{ nodal}) \subset \mathcal{M}_{\Xi,l-2}^x$ consists of $\Theta = \{(C_i, m_i)\} \times \Omega \in \mathcal{M}_{\Xi}^x \times M^{[l-2]}$ with $x \in C_1$, and

- x is not in C_i for any $i \geq 2$,
- there exists $i \neq j$ and $y \in M$ such that $y \in \Omega \cap C_i$ and $y \in \Omega \cap C_j$, i.e., $y \in \Omega$

is a nodal point of Θ .

By Proposition 4.16, $\dim_{\mathbb{C}} \mathcal{M}_{\Xi}^x \leq l-2$. Given

$$\{(C_i, m_i)\} \times \Omega \in \mathcal{M}_{\Xi,l-2}^x(\Omega \text{ nodal}),$$

there is a nodal point y in Ω . Observe that such a nodal point is in the intersection of C_i and C_j , so it has 0 dimensional freedom. Hence

$$\dim_{\mathbb{C}} \mathcal{M}_{\Xi,l-2}^x(\Omega \text{ nodal}) \leq (l-2) + (l-3) = 2(l-2) - 1.$$

Thus V_3 has complex codimension at least 1.

Finally, we deal with V_4 . For each $\Xi = (e_i) \in \chi_e^x$, define $\mathcal{M}_{\Xi,l-2}^x(\Omega_i \text{ non-generic}) \subset \mathcal{M}_{\Xi,l-2}^x$ consisting of $\Theta = \{(C_i, m_i)\} \times \{\Omega_i\} \in \mathcal{M}_{\Xi}^x \times M^{[l-2]}$ with $x \in C_1$, and

- x is not in C_i for any $i \geq 2$,

- $\Omega_i \subset C_i$,
- $\Omega_i \in M^{[|\Omega_i|]}$, $\Omega_i \cap \Omega_j = \emptyset$, and $\sum |\Omega_i| = l - 2$,
- $1 + |\Omega_1| < l_{e_1}$ or $|\Omega_i| < l_{e_i}$ for $i \geq 2$.

Clearly, under $\pi_{\text{red},l-2}^x$, the union of the image of $\mathcal{M}_{\Xi,l-2}^x(\Omega_i \text{ non-generic})$ over $\Xi = (e_i) \in \chi_e^x$ is the rest part of the complement of G_e^x .

By the estimate (17), $\dim_{\mathbb{C}} \mathcal{M}_{\Xi}^x \leq l - 2$ if $x \in M(e)$. If $\dim_{\mathbb{C}} \mathcal{M}_{\Xi}^x < l - 2$, then already

$$\dim_{\mathbb{C}} \mathcal{M}_{\Xi,l-2}^x \leq (l - 3) + (l - 2) = 2(l - 2) - 1.$$

So we assume that $\dim_{\mathbb{C}} \mathcal{M}_{\Xi}^x = l - 2$. In this case, $\sum_i l_{e_i} = l - 1$. And for any $\{(C_i, m_i)\} \times \{\Omega_i\} \in \mathcal{M}_{\Xi,l-2}^x(\Omega_i \text{ non-generic})$, either $1 + |\Omega_1| < l_{e_1}$, or $|\Omega_j| < l_{e_j}$ for $j \geq 2$.

Observe that $\pi_{\text{red},l-2}^x$ restricted to $\mathcal{M}_{\Xi,l-2}^x$ is of the form:

$$\times \pi_{e_i,|\Omega_i|} : \mathcal{M}_{e_1}^{x,\Omega_1} \times_{i \geq 2} \mathcal{M}_{e_i}^{\Omega_i} \longrightarrow \times_i M^{[|\Omega_i|]}.$$

The source $\mathcal{M}_{\Xi,l-2}^x$ has total complex dimension $2(l - 2)$. But when some $|\Omega_i| < l_{e_i}$ (or $1 + |\Omega_1| < l_{e_1}$), $\pi_{e_i,|\Omega_i|}$ drops dimension since

$$\dim_{\mathbb{C}} \mathcal{M}_{e_i,l_{e_i}-p} = l_{e_i} + l_{e_i} - p > \dim_{\mathbb{C}} M^{l_{e_i}-p} = 2(l_{e_i} - p). \quad \square$$

4.5. Abundance of pencils. In this section we establish Proposition 4.9.

4.5.1. $\mathcal{M}_e^{x,\Omega}$ is homeomorphic to S^2 .

PROPOSITION 4.19. *Let e be a big J -nef class in S_{K_J} . Fix a point $x \in M(e)$ and choose $\Omega \in G_e^x$. Then $\mathcal{M}_e^{x,\Omega}$ is homeomorphic to S^2 .*

Proof. By Lemma 4.6, there is another J -nef class H_e in S_{K_J} such that $H_e \cdot e = 1$ or 2. We prove that $\mathcal{M}_e^{x,\Omega}$ is homeomorphic to a smooth representative of H_e .

Let us first assume that $H_e \cdot e = 1$.

By the first item of Proposition 4.5, we can choose a smooth rational curve S representative of H_e such that it does not pass through any entry of Ω and x . This is possible since H_e is J -nef and the space of reducible H_e -curves is of codimension at least 1 by Proposition 4.10. Moreover, the space of irreducible H_e -curves containing x or any entry of Ω is of codimension 1 by Proposition 3.4.

Given any $z \in S$, z is distinct from x or any entry of Ω . By the first bullet of Lemma 4.18, there is a unique (although possibly reducible) rational curve $C_{x,z,\Omega}$ in class e passing through x , z and Ω . Thus we obtain a map $h : z \mapsto C_{x,z,\Omega}$ from S to $\mathcal{M}_e^{x,\Omega}$.

The map h is surjective since $H_e \cdot e \neq 0$. Since S is also J -holomorphic and $H_e \cdot e = 1$ any curve in $\mathcal{M}_e^{x,\Omega}$ intersects with S at a unique point by the positivity of intersection. Therefore h is also one-to-one.

Now let us show that h is a homeomorphism, namely both h and h^{-1} are continuous. Since $S = S^2$ is Hausdorff and $\mathcal{M}_e^{x,\Omega}$ is compact, if we can show that $h^{-1} : \mathcal{M}_e^{x,\Omega} \rightarrow S$ is continuous, it follows that h is also continuous. To show h^{-1} is continuous, consider a sequence $C_i \in \mathcal{M}_e^{x,\Omega}$ approaching to its Gromov-Hausdorff limit C . Let the intersection of C_i (resp. C) with S be p_i (resp. p). Then p_i has to approach p by the first item of the definition of topology on \mathcal{M}_e . Therefore h is a homeomorphism.

The case that $H_e \cdot e = 2$ is similar. We choose the smooth rational curve S representative of H_e such that it passes x but not any entry of Ω . This is achieved by Proposition 4.16 and Proposition 3.4 applied to the J -nef class H_e . Here we also need to use the fact that in this case H_e is proportional to e and hence x is also in $M(H_e)$.

Then we vary z in S . If $z \neq x$, we choose the rational curve $C_{x,z,\Omega}$ in class e passing through x, z and Ω . If $z = x$, we choose the rational curve $C_{x,x,\Omega}$ in class e passing through x, Ω and tangent to S at x . The sphere $C_{x,z,\Omega} \in \mathcal{M}_e^{x,\Omega}$ is unique by the second bullet of Lemma 4.18. We thus again obtain a map $h : z \mapsto C_{x,z,\Omega}$ from S^2 to $\mathcal{M}_e^{x,\Omega}$. This map is clearly surjective. Since S is J -holomorphic with $x \in S$ and $H_e \cdot e = 2$, any curve in $\mathcal{M}_e^{x,\Omega}$ either intersects with S at a unique point other than x or is tangent to S at x by the positivity of intersection. Therefore h is also one-to-one. Now we show that this map is a homeomorphism. As before, we only need to show that $h^{-1} : \mathcal{M}_e^{x,\Omega} \rightarrow S$ is continuous. Again, consider a sequence $C_i \in \mathcal{M}_e^{x,\Omega}$ approaching to its Gromov-Hausdorff limit C . Let the intersection of C_i (resp. C) with S be p_i and x (resp. p and x). If C_i (or C) tangent to S , let p_i (or p) be x . Then p_i has to approach p by the first item of the definition of topology on \mathcal{M}_e . Therefore h is a homeomorphism. □

4.5.2. Proof of Proposition 4.9.

Proof. Fix an orthonormal frame for $T_{1,0}M|_x$ to identify the space of complex 1-dimensional subspaces with $\mathbb{C}\mathbb{P}^1$. Consider the map

$$\tau^{x,\Omega} : \mathcal{M}_e^{x,\Omega} \longrightarrow \mathbb{C}\mathbb{P}^1, \quad C \longmapsto T_x C.$$

By Proposition 4.19, $\tau^{x,\Omega}$ is a map from S^2 to S^2 . An injective continuous map from S^2 to S^2 has to be a homeomorphism.

By the second assertion of Lemma 3.5, these curves cannot be tangent to each other at x if one of them is irreducible. Moreover, by the third bullet of Lemma 4.18 there are finitely many reducible curves in this family. In fact, we have shown that $|\mathcal{M}_{\text{red}}^{x,\Omega}|$ is bounded by $|\chi_{\text{red},e}|$, which only depends on e and J . We also know that these reducible curves cannot be tangent to each others. To summarize, the map is injective. □

5. Spherical Taubes currents from big J -nef classes.

5.1. Weak Taubes currents. We begin with introducing the notion of spherical current from a big J -nef class.

5.1.1. Spherical currents. We continue to assume J is a tamed almost complex structure. Suppose e is a big J -nef sphere class in S_{KJ} . Then $l_e = \iota_e \geq 2$. Fix $x \in M$. Let us first choose a constant r_0 . Let \mathcal{A} be the measure zero set of non-prettly-generic points with respect to x when $x \in M$ if $M(e) = M$ or x belongs to a compact subset $K \subset M(e)$ if $M(e) \subsetneq M$. Choose a small open neighborhood $\mathcal{OB}(\mathcal{A})$ with $\text{Vol}(\mathcal{OB}(\mathcal{A})) < 10^{-5l} \text{Vol}(M)$. By Proposition 4.9 and the third item of Lemma 4.18, $\mathcal{M}_{\text{red}}^{x,\Omega}$ are finite points in $\mathcal{M}^{x,\Omega} = S^2$ if $\Omega \in G_e^x$. Then we choose r_0 small enough such that $\mathcal{M}_{\text{irr}}^{x,\Omega} \cap \mathcal{M}_{\text{irr}}^{r_0} \neq \emptyset$ if Ω is chosen from the complement of $\mathcal{OB}(\mathcal{A})$.

Lemma 3.8 is still valid in this situation with \mathcal{M} replaced by $\mathcal{M}_{\text{irr}}^{r_0}$, which is Lemma 4.12 in [29]. We choose the constant k_{r_0} as in Lemma 4.12 of [29] (or $s'_{r_0} = k_{r_0}^{-1}$ as in Lemma 3.8) and $s = k_{r_0}^{-4}$. Let $B_s(x)$ be the ball of radius s centered at x .

We define a current Φ_e in the following manner. Recall

$$\mathcal{M}_{\text{irr},l} = \{(C, x_1, \dots, x_l) \mid C \in \mathcal{M}_{\text{irr}}, x_i \in M\} \subset \mathcal{M} \times M^{[l]}.$$

Use π_l to denote the projection map $\mathcal{M}_{\text{irr},l} \rightarrow M^{[l]}$. The portion of marked moduli space we choose is $\mathcal{M}_{\text{irr},l}^{r_0,r}$, consisting of the set of marked curves with distance at least r_0 to \mathcal{M}_{red} and $d(x_i, x_j) \geq r$ for any $i \neq j$. Here we suppose $r < \frac{s}{10}$.

We first define $\phi_\eta(v) = \int_C v$. Here $\eta \in \pi_l(\mathcal{M}_{\text{irr},l}^{r_0,r})$, C is the unique rational curve in $\pi_l^{-1}(\eta)$ and v is a 2-form on M . Then we have the following *spherical current*

$$\Phi_e(v) = \int_{\eta \in \pi_l(\mathcal{M}_{\text{irr},l}^{r_0,r})} \phi_\eta(v).$$

The spherical current Φ_e defined clearly satisfies Proposition 1.2 in [29]. Especially, it is a non-trivial, closed, non-negative J -invariant current on M .

5.1.2. Estimates of the pencil $\mathcal{M}^{x_1,\Omega}$. Fix x_1 and $\Omega \in G_e^{x_1}$. We write $\Omega = (x_3, \dots, x_l)$. By Proposition 4.9, $\mathcal{M}^{x_1,\Omega}$ is a pencil. Moreover, by removing an open neighborhood of these finite directions corresponding to reducible curves, we can suppose the remaining directions correspond to the curves with distance at least r_0 from \mathcal{M}_{red} .

We now assume x_1 and any entry of Ω are chosen from $M \setminus B_s(x)$ (but Ω not necessarily belongs to G_e^x). When the curve $C_{x,x_1,\Omega}$ is in $\mathcal{M}^{x_1,\Omega,r_0}$, with x_1 and Ω chosen from the compact set above (i.e., $d(x_i, x_j) \geq r$, $d(x_i, x) \geq s$ and $C_{x,x_1,\Omega} \in$

$\mathcal{M}^{x_1, \Omega, r_0}$, there is a number $T > 0$ such that for any $z \in \overline{B_T(x)}$, the sphere $C_{z, x_1, \Omega}$ is smooth and in $\mathcal{M}_{\text{irr}}^{\frac{r_0}{2}}$. In other words, the part of $\mathcal{M}^{x_1, \Omega}$ intersecting $\overline{B_T(x)}$ is a pencil of smooth curves. Clearly, this is also true for any $t \leq T$. Let us denote this set by $\mathcal{M}^{x_1, \Omega; B_t(x)}$. Notice the first defining condition of $\mathcal{M}_{\text{irr}, l}^{r_0, r}$ guarantees $\text{dist}(x, M(e)) \geq r_0$ if $M(e) \neq \emptyset$.

Given a smooth curve C in this pencil with normal bundle N , the tangent space to $\mathcal{M}_{\text{irr}}^{x_1, \Omega}$ at C can be identified with the vector space $\ker_{C, x_1, \Omega} \subset \Gamma(N)$ that consists of the sections in the kernel of D_C that vanish at x_1 and Ω .

On this two dimensional space, there are several norms. Let $\nu \in \ker_{C, x_1, \Omega}$.

- The L^2 norm $\|\nu\|_2$;
- The sup norm $\sup_C |\nu|$;
- For $z \neq x_1$ or any entry of Ω , the pointwise norm $|\nu(z)|$;
- By choosing x_1, Ω as above, we could still define $\tau^{x_1, \Omega}$ or τ^{Ω, x_1} by taking the complex direction $T_{x_1}C$ (or $T_{x_3}C$). Let u denote the direction $T_{x_1}C$ in $\mathbb{C}\mathbb{P}^1$, $\tau_*^{x_1, \Omega} : T_C \mathcal{M}^{x_1, \Omega} \rightarrow T_u \mathbb{C}\mathbb{P}^1$ is an isomorphism. We thus could speak of the pointwise norm of $\tau_*^{x_1, \Omega} \nu$ as a vector in $T_u \mathbb{C}\mathbb{P}^1$.

For a fixed curve C , these norms are equivalent. Since $\mathcal{M}_{\text{irr}, l}^{r_0, r}$ is compact, if we have compact families of choices of x_1, Ω, z , we have uniform constants as in Lemma 3.10.

LEMMA 5.1. *Let x, x_1, Ω and C be as stated as above. There is a constant $k_{5.1} > 1$, depending only on r_0, r, s and T , ensuring the following inequalities for $\nu \in \ker_{C, x_1, \Omega}$:*

- (1) $\sup_C |\nu| \leq k_{5.1} |\nu(z)|$ if x_1 and any entry of Ω are not in $B_s(x)$ and $z \in \overline{B_{\frac{s}{2}}(x)}$;
- (2) $\sup_C |\nu| \leq k_{5.1} |\tau_*^{x_1, \Omega} \nu|$ if $x_1 \in \overline{B_s(x)}$, and $\sup_C |\nu| \leq k_{5.1} |\tau_*^{\Omega, x_1} \nu|$ if $x_3 \in \overline{B_s(x)}$.

Proof. The constant in (1) can be chosen to be independent of x, x_1, Ω, z , and C since $M, M \setminus B_s(x), M \setminus B_r(x_i), \overline{B_{\frac{s}{2}}(x)}$ and $\mathcal{M}^{x_1, \Omega; B_T(x)}$ are compact.

The constant in (2) is uniform because (x_1, Ω) is chosen from a compact set in $M^{[l-1]}$. □

Similarly, Lemmas 3.11 and 3.12 are also valid with apparent modification in the statement.

Let $T^{x_1, \Omega; B_t(x)}$ denote the set of points x_2 in M that lies in a curve in $\mathcal{M}^{x_1, \Omega}$ and intersecting $\overline{B_t(x)}$.

LEMMA 5.2. *Suppose x_1 and each entry of Ω are not in $B_s(x)$. There are constants k and κ depending on s with the following significance: For $t < \kappa^{-3}$, the volume of $T^{x_1, \Omega; B_t(x)}$ is bounded from above by kt^2 .*

Proof. The proof is similar to that of Lemma 3.11. Let $C = C_{x, x_1, \Omega}$. Since $\text{dist}(x_1, x) \geq s$ and $\text{dist}(\Omega, x) \geq s$, $|\nu(x)|$ is a norm on the 2-dimensional vector

space $\ker_{C,x_1,\Omega}$. Now $\mathcal{M}^{x_1,\Omega;B_t(x)}$ is a 2-dimensional smooth compact manifold. As argued in Lemma 3.11,

$$\text{dist}(x_2, C) \leq 2K\kappa_{5,1}\kappa t.$$

Then the volume of $T^{x_1,\Omega;B_t(x)}$ is bounded from above by kt^2 . □

LEMMA 5.3. *Suppose $w = x_3$ is in $\overline{B_s(x)} \setminus B_{Rt}(x)$. There is a constant k depending on s and R with the following significance:*

The volume of points x_2 in M lying in a curve in $\mathcal{M}^{x_1,\Omega,r_0}$ and intersecting $B_t(x)$ is bounded from above by $k\frac{t^2}{d^2}$, where $d = \text{dist}(w, x)$.

Proof. The proof is identical to that of Lemma 3.12, with the discussion on the map τ^{Ω,x_1} (notice $w = x_3$ is the first entry of the superscript) and Lemma 5.1(2) in place of Lemma 3.10(2). □

5.1.3. Upper bound for a big J -nef class. Now we denote the center of B in definition 2.11 by x where x is any point in M and denote the ball by $B_t(x)$.

PROPOSITION 5.4. *Let e be a big J -nef class in $S_{K,J}$. The current Φ_e satisfies the upper bound in (11).*

Proof. Choose s as in the beginning of this section.

Fix $x \in M$ and adapted coordinates (z, w) centered at x with radius s .

Let $0 < t < 10^{-5}s$.

As in [29], we only need to prove $\Phi_e(i\chi_t dz \wedge d\bar{z}) < kt^4$. Let us denote and group the l points by x_1 and x_2 and $\Omega = (x_3, \dots, x_l)$. Moreover, $\Phi_e(i\chi_t dz \wedge d\bar{z})$ is no greater than

$$k_0 \int_{\eta=(x_1,x_2,\Omega) \in \pi_l(\mathcal{M}_{\text{irr},l}^{r_0,r})} \left(\int_{C_\eta} \chi_t \omega \right).$$

Notice that $\int_{C_\eta} \chi_t \omega = 0$ if $C_\eta \cap B_t(x) = \emptyset$. If $C_\eta \cap B_t(x) \neq \emptyset$, then $C_\eta \cap B_t(x)$ is contained in a ball of radius of $2t$ centered at some point in $B_t(x)$. By Lemma 2.9, the integrand $\int_{C_\eta} \chi_t \omega$ is bounded by $H \cdot [\omega]kt^2$.

Thus it suffices to prove that the volume of the set

$$(18) \quad \{\eta = (x_1, x_2, \Omega) \in \pi_l(\mathcal{M}_{\text{irr},l}^{r_0,r}), C_\eta \cap B_t(x) \neq \emptyset\}$$

is $O(t^2)$.

We choose Ω, x_1 and x_2 in turns.

We have three cases depending on the positions of Ω and x_1 .

(I) The first case is that x_1 and each element of Ω are all away from $B_s(x)$.

Since our moduli space for integration is $\mathcal{M}_{\text{irr},l}^{r_0,r}$, and $x_1 \in M \setminus B_s(x)$, $\Omega \in \{(x_3, \dots, x_l) | x_i \in M \setminus B_s(x)\}$. The corresponding upper bounds for these two factors are $\text{Vol}(M)$ and $\text{Vol}(M)^{l-2}$ respectively.

For those Ω and x_1 contributing to the integration, $\mathcal{M}^{\Omega, x_1, x'} \subset \mathcal{M}_{\text{irr}}^{r_0}$ for some $x' \in \overline{B_t(x)}$. Thus by choosing $t < T$, for any $z \in \overline{B_t(x)}$, the unique smooth rational curve $C_{\Omega, x_1, z} \in \mathcal{M}_{\text{irr}}^{\frac{r_0}{2}}$.

Now we estimate the possible choices of x_2 . By the above picture, the part of $\mathcal{M}^{\Omega, x_1; B_t(x)}$ is a pencil. By Lemma 5.2, the volume of x_2 is bounded from above by $k_1 t^2$. This constant k_1 could be chosen uniformly since the possible set of Ω and x_1 is a closed, then compact, subset of $M^{[l-1]}$. The factors $\text{Vol}(M)^{l-2}$, $\text{Vol}(M)$ and $k_1 t^2$, multiply to an upper bound of $O(t^2)$ for the volume of the subset in (18) with x_1 and each element of Ω in $M \setminus B_s(x)$.

(II) The second case is when x_1 or some entry of Ω , say w , satisfies $Rt < \text{dist}(w, x) < s$, where $R = 10^5$. The proof is almost identical to part II of Proposition 3.13 but invoking Lemma 5.3 instead.

(III) The last case is when any entry of Ω or x_1 is in $B_{Rt}(x)$. This is exactly the last case of Proposition 3.13.

Summing the three cases, we finish the proof. □

5.1.4. Taubes current from a J -ample class.

PROPOSITION 5.5. *Let e be a J -ample class e in $S_{K,J}$. Then the current Φ_e is a Taubes current, i.e., it satisfies (11). Consequently, there is an almost Kähler form in the same class.*

Proof. Thanks to Proposition 5.4, we only prove the lower bound $k^{-1}t^4 < \Phi_e(f_{B_t(x)} i\sigma \wedge \bar{\sigma})$ here. As in [29], we prove $k^{-1}t^4 < \Phi_e(ief_{B_t(x)} dz \wedge d\bar{z})$. Let us denote the l points by x_1, x_2 and $\Omega = (x_3, \dots, x_l)$. If $l = 2$, we only have x_1 and x_2 . Let $0 < t < 10^{-5}s$.

The main picture to have is Lemma 2.10, applied to $B_s(x)$. Namely, inside $B_s(x)$, the curves behave as straight lines with respect to the adapted coordinates.

Since we are estimating the lower bound, in addition to choosing Ω outside a small open neighborhood $\mathcal{OB}(\mathcal{A})$ of the measure zero set \mathcal{A} , we also know that each entry of Ω is away from $B_{\frac{s}{2}}(x)$ and each entry of Ω is at least of distance r from each others. This set of Ω is compact in $M^{[l-2]}$. To summarize, by choosing s small, all such Ω constitute a compact set of volume no smaller than

$$\left(\text{Vol}(M) - (l-2) \max_{x \in M} \text{Vol}(B_{\frac{s}{2}}(x)) \right)^{l-2} - \text{Vol}(\mathcal{OB}(\mathcal{A})) > \left(\frac{\text{Vol}(M)}{2} \right)^{l-2}.$$

Before making choices of x_1 and x_2 , we digress to choose a compact submanifold $\mathcal{C}_{x, \Omega} \subset \mathcal{M}_{\text{irr}}^{x, \Omega}$.

By Proposition 4.9 and Lemma 4.18, for a pretty generic Ω , except for finitely many complex directions in $T_x M$, there is a smooth rational curve passing through this direction and Ω . Recall that $\mathcal{M}^{x, \Omega}$ is a pencil, and identified with $\mathbb{C}P^1$ via $\tau^{x, \Omega}$. Then the set $\mathcal{C}_{x, \Omega}$ is characterized by the following two properties:

- Its image under $\tau^{x,\Omega}$ is contained in a disk $|dz|(\cdot) \geq 2\epsilon$ in $\mathbb{C}\mathbb{P}^1$;
- Any curve $C \in \mathcal{C}_{x,\Omega}$ has distance at least r from $\mathcal{M}_{\text{red},e}$.

It is a (nonempty) compact submanifold of $\mathcal{M}_{\text{irr}}^{x,\Omega,r_0}$ (of real dimension two). Recall we choose r_0 small enough such that $\mathcal{M}_{\text{irr}}^{x,\Omega,r_0} = \mathcal{M}_{\text{irr}}^{x,\Omega} \cap \mathcal{M}_{\text{irr}}^{r_0} \neq \emptyset$. Moreover, for C chosen from $\mathcal{C}_{x,\Omega}$, the restriction of $|dz|$ to $C \cap B_s(x)$ is greater than ϵ when s is chosen sufficiently small.

Now, let us choose x_1 . Again, we choose $x_1 \in B_s(x)$ away from $B_{\frac{s}{2}}(x)$ and $B_{\frac{s}{2}}(x_i)$ where x_i 's are entries of Ω . Additionally, we choose x_1 such that the rational curve $C_{x_1,x,\Omega}$ determined by x, Ω and x_1 is contained in the compact submanifold $\mathcal{C}_{x,\Omega} \subset \mathcal{M}_{\text{irr}}^{x,\Omega}$ specified above. By Lemma 4.18, the choices of x_1 constitute a compact set with nonzero volume, say $c_\Omega s^4$.

Now, with x_1 fixed, we consider the set of x_2 in $B_{\frac{s}{4}}(x)$, for which $\eta = (x_1, x_2, \Omega)$ contributes to $\Phi_e(\text{if}_{B_t(x)} dz \wedge d\bar{z})$, namely $C_{x_1,x_2,\Omega}$ intersects $B_t(x)$. This part of argument is identical to the corresponding part in Proposition 3.14 with $C_{x_1,x_2,\Omega}$ and π_l appearing in place of C_{x_1,x_2} and π_2 . We have the lower bound $k_1 s^2 t^2$.

Let $C = C_{x_1,x_2,\Omega}$, then $\int_C(\text{if}_{B_t(x)} dz \wedge d\bar{z}) \geq k_2^{-1} t^2$ by Lemma 2.9 as in Proposition 3.14.

Multiplying these four factors: $(\frac{\text{Vol}(M)}{2})^{l-2}$, $c_\Omega s^4$, $k_1 s^2 t^2$ and $k_2^{-1} t^2$ together, we get $k^{-1} t^4 < \Phi_e(\text{if}_{B_t(x)} dz \wedge d\bar{z})$. These constants c_Ω, k_1, k_2 are independent of t as one can check from the proof. The constant k could be chosen universal by the same reasoning in Proposition 3.14.

The last statement follows from Proposition 2.12. □

5.1.5. Weak Taubes current from a big J -nef class. If e is big J -nef but not J -ample, Φ_e is not a Taubes current, since no irreducible curves in class e pass through points in the vanishing locus $Z(e)$. Nonetheless, the following observation will be very useful.

PROPOSITION 5.6. *Let e be a big J -nef class. Then the current Φ_e is non-negative, and over any (4-dimensional) compact submanifold K of the complement $M(e)$, it satisfies (11) for a constant $k > 1$ depending only on K .*

Proof. Φ_e is a non-negative current by definition, and the upper bound is from Proposition 5.4. We only need to prove that it is bounded from below by $k^{-1} t^4$ on any compact submanifold $K \subset M(e)$. The proof is almost identical to the proof of Proposition 5.5. Notice that all the relevant results in Section 4 are established for $x \in M(e)$. Hence the proof goes almost verbatim as that of Proposition 5.5 when x is chosen from a compact submanifold $K \subset M(e)$, and Ω, x_1 chosen from $K'^{[l-2]}$, K' (instead of from $M^{[l-2]}$ and M) respectively. Here K' is another (4-dimensional) compact submanifold of $M(e)$ such that $K \subsetneq K'$, which satisfies $B_s(p) \subset K'$ if $p \in K$. □

We call a current in Proposition 5.6 a *weak Taubes current*. By summing up weak Taubes currents with disjoint zero locus, we obtain Taubes currents.

PROPOSITION 5.7. *Let e_i be big J -nef classes in S_{K_J} and denote Z_i the zero locus of e_i . If $\cap Z_i = \emptyset$, then there is a Taubes current in the class $e = \sum_i a_i e_i$, with $a_i > 0$. In turn, we obtain an almost Kähler form in the class e .*

5.2. Tamed versus compatible.

5.2.1. Proof of Theorem 1.2.

Proof. Given any J on S^2 -bundle over S^2 it follows from Lemma 4.14 that there is always a J -ample class. Now the conclusion follows from Proposition 5.5. □

5.2.2. Tameness and foliation. Now we have shown that every tamed J on an S^2 -bundle over S^2 is almost Kähler, a related question is when J is tamed.

Suppose M is an S^2 -bundle over S^2 and J is an almost complex structure on M , not necessarily tamed. Then there is a J -nef class in $S_{K_J}^0$. If J is tamed, by Proposition 3.3, e is J -effective, and moreover, there is a J -holomorphic foliation by smooth rational curves. We would like to know whether the converse is also true.

QUESTION 5.8. *For an S^2 -bundle over S^2 , suppose there is a J -holomorphic foliation by smooth rational curves. Is J tamed?*

Note that if we assume further that J is fibred, Gompf’s construction in [8] produces a tamed symplectic form.

Here is an analogous question for $\mathbb{C}P^2$.

QUESTION 5.9. *For $\mathbb{C}P^2$, J is tamed if and only if there is a pencil of smooth rational curves?*

5.2.3. $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ with $k \geq 2$. From now on we sometimes denote $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ by M_k . For M_k with $k \geq 2$, there are no J -ample classes in S_{K_J} . So we apply Proposition 5.7 to construct Taubes currents.

Let H, E_i be an orthogonal basis of $H_2(M_k; \mathbb{Z})$ with $H^2 = 1, E_i^2 = -1$. Such a basis is called a standard basis.

Given any tamed almost complex structure J , there is a standard basis such that $K_J = -3H + \sum E_i$, called a standard basis adapted to J . This follows from the uniqueness of symplectic canonical classes, up to diffeomorphisms. Notice that $E_i \in \mathcal{E}_{K_J}$ and $H \in S_{K_J}^+$. Hence H is J -effective. And it pairs positively with any J -tamed symplectic form.

Suppose J is tamed and there is a configuration of k disjoint -1 curves C_i . Notice that we have a standard basis (H, E_1, \dots, E_i) adapted to J , with $E_i = [C_i]$ and H the unique square 1 class with $H \cdot K_J < 0$ and $H \cdot E_i = 0, i = 1, \dots, k$.

LEMMA 5.10. *Suppose J is tamed and there is a configuration of k disjoint -1 curves C_i . Then the classes $H, 2H, nH - (n - 1)E_i, nH - (n - 1)E_i - E_j$ and $H - E_i$ in S_{K_J} are J -nef.*

Proof. Let C be an irreducible curve distinct from any C_i , and suppose $[C] = aH - \sum_i b_i E_i$. Since C_i is irreducible, $b_i \geq 0$ by the positivity of intersection. Since any J -tamed ω is positive on H , $a > 0$.

Clearly $H, 2H$ are J -nef since $a > 0$.

Since C is an irreducible curve, $g_J(e_C) \geq 0$, so we have

$$\begin{aligned}
 (19) \quad & -2 \leq 2g_J(e_C) - 2 \\
 & = K_J \cdot C + C^2 \\
 & = -3a + \sum_i b_i + a^2 - \sum_i b_i^2 \\
 & = (a - b_1)(a + b_1 - 1) - 2a - \sum_{i \geq 2} (b_i^2 - b_i).
 \end{aligned}$$

The second term is at most -2 . The third term is non-positive. Therefore $a \geq b_1$. In fact, the same argument shows that $a \geq b_i$ for any i .

It follows that C cannot pair negatively with $nH - (n - 1)E_i, H - E_i$.

If C pairs negatively with $nH - (n - 1)E_i - E_j$, then

$$na - (n - 1)b_i - b_j < 0,$$

which implies that either $a < b_i$ or $a < b_j$. But this is impossible so $nH - (n - 1)E_i - E_j$ is also J -nef. □

Remark 5.11. Under the assumption of Lemma 5.10, it is not true that any class in $S_{K_J}^{\geq 0}$ is J -nef. Consider the holomorphic blow up of 3 points on a line l in $\mathbb{C}P^2$ and let C_1, C_2, C_3 be the exceptional curves. Then the class $2H - E_1 - E_2 - E_3$ in S_{K_J} is not J -nef since it pairs negatively with the class of the proper transform l' of the line l , which is $H - E_1 - E_2 - E_3$. In particular, there are no smooth rational curves in the class $2H - E_1 - E_2 - E_3$.

Remark 5.12. Notice that, under the assumption of Lemma 5.10, there is at least one J -nef class in $S_{K_J}^0$. It is natural to wonder whether this is true for any tamed almost complex structure. If so, by Proposition 4.5, every tamed rational 4-manifold has a rational curve “foliation”, with only finitely many reducible leaves.

We are ready to prove Theorem 1.3.

Proof. Observe that, for each i , $2H - E_i$ is a K_J -spherical class with square 3, and it is J -nef by Lemma 5.10.

Let $Z_i = Z(2H - E_i)$ be the zero locus of $2H - E_i$. If $C \neq C_i$ is an irreducible curve in Z_i , then $b_1 = 2a > a$. This is impossible by the proof of Lemma 5.10. Therefore

$$Z_i = \{C_1, \dots, C_{i-1}, C_{i+1}, \dots\}.$$

Clearly, $\cap_{1 \leq i \leq k} Z_i = \emptyset$.

Therefore there is an almost Kähler form in the class $\sum_{i=1}^k (2H - E_i)$ by Proposition 5.7. In particular, J is almost Kähler. \square

COROLLARY 5.13. *If J is in \mathcal{J}_{top} or $\mathcal{J}_{\text{good}}$, then J is almost Kähler.*

Proof. For any tamed J , by Proposition 3.3, each E in \mathcal{E}_{K_J} is represented by a J -holomorphic subvariety Θ . We claim that E pairs non-negatively with any irreducible subvariety C whose class is not E . This is clear if C has non-negative self-intersection. Suppose C has negative self-intersection. Then by our assumption on J , C is either a -1 curve or the anti-canonical curve. The -1 curve case follows from Lemma 4.12 (3), and the anti-canonical curve case follows from the adjunction formula.

The following result is proved in [19] as Proposition 4.25:

PROPOSITION 5.14. *Suppose J is tamed, $e \in S_{K_J}$ and $\Theta = \{(C_i, m_i)\} \in \mathcal{M}_e$. If $e \cdot e_{C_i} \geq 0$, then Θ is connected and each component C_i is a smooth rational curve.*

Our Θ satisfies the condition, so Θ is connected and each component is a smooth rational curve. Since Θ is connected, if it is reducible, it must contain an irreducible component F with self-intersection at most -2 . But by our assumption on J there are no smooth rational curves of self-intersection less than -1 .

We have shown that each E in \mathcal{E}_{K_J} is represented by a -1 curve. In particular, given a standard basis $\{E_i\}$ adapted to J , there are k disjoint -1 curves in the k classes E_i . Now apply Theorem 1.3. \square

COROLLARY 5.15. *Suppose $h_l = lH - \sum_{i=1}^k E_i$ with $l^2 > \max\{k, 9\}$. If J is an almost complex structure on M_k tamed by a symplectic form in the class h_l , then J is almost Kähler. Moreover, there is an almost Kähler form in the class h_l .*

Proof. Since the h_l area of E_i is 1, a subvariety representing E_i is irreducible, and hence it is smooth. Thus J is almost Kähler by Theorem 1.3.

Since h_l is in the J -tamed cone \mathcal{K}_J^t and J is almost Kähler, the last claim follows from the equality (23) between \mathcal{K}_J^t and the almost Kähler cone. \square

We call a tamed almost complex structure J del Pezzo if it is tamed by a symplectic form in the class $-K_J$. By Corollary 5.15, del Pezzo J is almost Kähler.

Remark 5.16. It is observed by Pinsonnault [24] that for any tamed J on M_k , there exists at least one (smooth) -1 curve.

5.3. Almost Kähler cone. We first introduce an open convex cone associated to J .

Definition 5.17. For a tamed almost complex structure J on M_k (resp. $S^2 \times S^2$), the open convex cone \mathcal{S}_J is defined to be the interior of the convex cone generated by big J -nef classes in S_{K_J} if it is of dimension $k + 1$ (resp. 2). Otherwise, it is defined as \emptyset .

For an almost Kähler J , we have the following

LEMMA 5.18. *If J is almost Kähler, then \mathcal{S}_J is contained in $\mathcal{K}_J^c(M)$.*

Proof. Let $\mathcal{S}_J \neq \emptyset$. Suppose u is the class of an almost Kähler form. Observe that given any class $e \in \mathcal{S}_J$, $e - tu$ is in \mathcal{S}_J for t small. Thus $e - tu = \sum a_i e_i$ with $a_i > 0$ and e_i big J -nef. Hence $e = (e - tu) + tu \in \mathcal{K}_J^c(M)$ by Propositions 5.6 and 2.12. □

5.3.1. S^2 -bundles over S^2 .

Proof of Theorem 1.5. By Lemma 5.18 it suffices to show that

$$\mathcal{S}_J = \mathcal{P}_J.$$

We establish this by describing explicitly the curve cone A_J . We have mentioned that there is always a foliation by smooth rational curves. One boundary of the curve cone is generated by the class of such a foliation. The other boundary is generated by a transversal class.

We start with $S^2 \times S^2$.

Given any J , denote the class of a foliation by H_2 . The curve cone A_J is generated by H_2 and a transversal class $H_1 - lH_2$ for some $l \geq 0$.

The ≥ 0 -dual of A_J is generated by $A = H_1 + lH_2$ and $B = H_2$.

If $l = 0$, then $A = H_1$ and it is approximated by the sequence of big J -nef classes in S_{K_J} , $pH_1 + H_2$. B is approximated by the sequence of big J -nef classes in S_{K_J} , $H_1 + qH_2$. Therefore

$$\mathcal{S}_J = \mathcal{P}_J = \{aA + bB \mid a > 0, b > 0\} = \{xH_1 + yH_2 \mid x > 0, y > 0\}.$$

If $l > 0$, then A itself is a big J -nef class in S_{K_J} , and B is approximated by the sequence of big J -nef classes in S_{K_J} , $H_1 + qH_2$, $q \geq l$. Therefore

$$\mathcal{S}_J = \mathcal{P}_J = \{aA + bB \mid a > 0, b > 0\} = \{xH_1 + yH_2 \mid y > lx > 0\}.$$

For $M = \mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$, the proof is similar.

Given any J , the unique class of foliation is $H - E$. The curve cone A_J is generated by $H - E$ and a transversal class $D_{-l} = -lH + (l + 1)E$ for some l .

The ≥ 0 -dual of A_J is generated by $B = H - E$ and $A = (l + 1)H - lE$.

For each l , the class A is a big J -nef class in S_{K_J} , and B is approximated by the sequence of big J -nef classes in S_{K_J} , $(p + 1)H - pE, p \geq l$. Thus

$$\mathcal{S}_J = \mathcal{P}_J = \{aA + bB \mid a > 0, b > 0\} = \left\{ xH - yE \mid \frac{l+1}{l}y > x > y > 0 \right\}. \quad \square$$

5.3.2. $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$. In this case we will again apply Lemma 5.18 to probe the almost Kähler cone.

In fact, for $J \in \mathcal{J}_{\text{top}}$, Question 1.4 takes a particular simple form, which we now explain. Among all $J \in \mathcal{J}^t$, J in the top stratum \mathcal{J}_{top} has the maximal \mathcal{P}_J . In fact, when $b^+(M) = 1$ and $J \in \mathcal{J}_{\text{top}}$, \mathcal{P}_J is equal to the K_J -symplectic cone \mathcal{C}_{K_J} introduced in [16]:

$$(20) \quad \mathcal{C}_{K_J} = \{e \in H^2(M; \mathbb{R}) \mid e = [\omega] \text{ for some } \omega \text{ with } K_\omega = K_J\}.$$

Here K_ω is the symplectic canonical class of ω . Thus the almost Kähler Nakai-Moishezon criterion for $J \in \mathcal{J}_{\text{top}}$ is the same as

$$\mathcal{K}_J^c(M) = \mathcal{C}_{K_J}.$$

Although we cannot verify the almost Kähler Nakai-Moishezon criterion for $J \in \mathcal{J}_{\text{top}}$ when $k \geq 10$, we have the following partial result.

THEOREM 5.19. *Suppose M is a rational manifold. If $J \in \mathcal{J}_{\text{top}}$, then $\mathcal{K}_J^c(M) \supset \mathcal{C}_{K_J} \cap \{e \mid e \cdot K_J < 0\}$.*

To establish Theorem 5.19 we introduce several open convex cones associated to K_J .

Definition 5.20. For a tamed almost complex structure J , the open convex cone $\mathcal{S}_{K_J}^+$ is the interior of the convex cone generated by classes in $S_{K_J}^+$. It is called the positive K_J -sphere cone.

Notice that $\mathcal{S}_{K_J}^+$ contains $\mathcal{S}_{J'}$ for any tamed J' with $K_{J'} = K_J$.

According to [16], the K_J -symplectic cone \mathcal{C}_{K_J} has the following characterization,

$$(21) \quad \mathcal{C}_{K_J} = \{e \in \mathcal{P} \mid e \cdot E > 0 \text{ for any } E \in \mathcal{E}_{M, K_J}\}.$$

Finally, introduce the open subcone of \mathcal{C}_{K_J} ,

$$(22) \quad P_{K_J} := \{e \in \mathcal{C}_{K_J} \mid e \cdot (-K_J) > 0\}.$$

PROPOSITION 5.21. *For any J on $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$, the positive K -sphere cone $\mathcal{S}_{K,J}^+$ coincides with $P_{K,J}$.*

We will defer the proof of Proposition 5.21 to the next subsection, where we need to review P -cell in [7] and K -symplectic cone in [16].

LEMMA 5.22. *If J is in \mathcal{J}_{top} or $\mathcal{J}_{\text{good}}$, then $\mathcal{S}_J = \mathcal{S}_{K,J}^+$.*

Proof. When J is in \mathcal{J}_{top} or $\mathcal{J}_{\text{good}}$, by Lemma 5.10 and the proof of Corollary 5.13, any class in $\mathcal{S}_{K,J}^+$ is J -nef. □

Proof of Theorem 5.19 and Theorem 1.6. Both claims follow from Lemmas 5.18, 5.22, Proposition 5.21, and (22). □

It can be easily shown that if J is del Pezzo then $J \in \mathcal{J}_{\text{top}}$, thus we have

COROLLARY 5.23. *If J is del Pezzo, then $\mathcal{K}_J^c(M) = \mathcal{C}_{K,J}$. In other words, the almost Kähler cone either does not contain $-K_J$, or is equal to $\mathcal{C}_{K,J}$.*

This result also follows from the J -inflation approach in [32]. The J -inflation along a smooth J -holomorphic subvariety, which is due to McDuff [21] and extended by Buse [3], can be effectively applied to probe the J -tamed cone of a tamed almost complex structure J ,

$$\mathcal{K}_J^t = \{[\omega] \in H^2(M; \mathbb{R}) \mid \omega \text{ tames } J\}.$$

Clearly, $\mathcal{K}_J^c \subset \mathcal{K}_J^t$. When $b^+ = 1$, it was shown in [18] that if J is almost Kähler, then

$$(23) \quad \mathcal{K}_J^c = \mathcal{K}_J^t.$$

The equality (23) combined with the calculation of the tamed cone in [32] gives an alternative proof of Corollary 5.23. In fact, in [32] we use this approach to establish the almost Kähler Nakai-Moishezon criterion for minimal ruled manifolds and all rational manifolds with $b^- \leq 9$.

5.4. K -sphere cone and K -symplectic cone. In this section we establish Proposition 5.21.

5.4.1. P -cells. Suppose M is an oriented closed manifold with odd intersection form, $b^+ = 1$, $b^- = n$ and no torsion in $H^2(M; \mathbb{Z})$. A basis $(x, \alpha_1, \dots, \alpha_n)$ for $H^2(M; \mathbb{Z})$ is called standard if $x^2 = 1$, and $\alpha_i^2 = -1$ for each $i = 1, \dots, n$. Let

$$\begin{aligned} \mathcal{P} &= \{e \in H^2(M; \mathbb{R}) \mid e \cdot e > 0\} \\ \mathcal{B} &= \{e \in H^2(M; \mathbb{R}) \mid e \cdot e = 0\} \\ \overline{\mathcal{P}} &= \{e \in H^2(M; \mathbb{R}) \mid e \cdot e \geq 0\}. \end{aligned}$$

For each class $x \in H^2(M; \mathbb{Z})$ with $x^2 < 0$, we define $x^\perp \in H^2(M; \mathbb{R})$ to be the orthogonal subspace to x with respect to the cup product, and we call $(x^\perp) \cap \mathcal{P}$ the wall in \mathcal{P} defined by x . Let \mathcal{W}_1 be the set of walls in \mathcal{P} defined by integral classes with square -1 . A chamber for \mathcal{W}_1 is the closure in \mathcal{P} of a connected component of $\mathcal{P} - \cup_{W \in \mathcal{W}_1} W$.

Any point $x \in \mathcal{P}$ with square 1 at which n mutually perpendicular walls of \mathcal{W}_1 meet is called a corner. Any corner is an integral class (see Lemma 2.2 in [7]). Suppose C is a chamber for \mathcal{W}_1 . If x is a corner in C , a standard basis $(x, \alpha_1, \dots, \alpha_n)$ for $H^2(M; \mathbb{Z})$ is called a standard basis adapted to C if $\alpha_i \cdot C \geq 0$ for each i . The canonical class of the pair (x, C) is defined to be $\kappa(x, C) = 3x - \sum_i \alpha_i$. Suppose C is a chamber for \mathcal{W}_1 and x is a corner in C , we define

$$P(x, C) = C \cap \{e \in \mathcal{P} \mid \kappa(x, C) \cdot e \geq 0\}.$$

Any subset of \mathcal{P} of the form $P(x, C)$ is called a P -cell.

5.4.2. C_K and P_K . We are back to the situation that $M = M_k$ and J is a tamed almost complex structure. Denote K_J by K .

By Lemma 2.4 in [13], \overline{P}_K , the closure of P_K , is a P -cell and $\kappa(P_K) = -K$.

LEMMA 5.24. (1) P_K is an open convex polytope in \mathcal{P} . Each wall of P_K is either a wall of a class in $\mathcal{E}_{M,K}$, or the wall of K if $k \geq 9$.

(2) The face F_{E_k} of $P_{M_k,K}$ corresponding to E_k is naturally identified with $P_{M_{k-1},K}$.

Proof. Statement (1) is due to Friedman and Morgan. It states that P_K does not have round boundary, i.e. boundary contributed by \mathcal{B} , although C_K always has a round boundary when $k > 9$. It is easy to see that when $k \leq 8$, P_K is just a chamber. In other words, each wall of it is a wall of a class in $\mathcal{E}_{M,K}$. When $k > 9$, the class K does contribute a wall to P_K .

For (2), E_k is orthogonal to all classes in $\mathcal{E}_{M_{k-1},K}$. In fact

$$\mathcal{E}_{M_{k-1},K} = \{e \in \mathcal{E}_{M_k,K} \mid e \cdot E_k = 0\}.$$

Suppose u is in the interior of the face F_{E_k} . Then u is positive on $\mathcal{E}_{M_{k-1},K} \subset \mathcal{E}_{M_k,K}$. If we consider the expansion of u with respect to the standard basis, then u has no E_k coefficient. Hence $F_{E_k} \subset P_{M_{k-1},K}$.

Conversely, if $u \in P_{M_{k-1},K}$, then u is orthogonal to E_k , and u is positive on $\mathcal{E}_{M_{k-1},K}$. Moreover, for any class $e \in \mathcal{E}_{M_k,K}$ with nonzero E_k coefficient, $u \cdot e = u(e + (e \cdot E_k)E_k)$. Notice that $u^2 > 0$ and

$$(e + (e \cdot E_k)E_k)^2 = -1 + (e \cdot E_k)^2 \geq 0.$$

Further, since $e \cdot E_k > 0$, u and $e + (e \cdot E_k)E_k$ pairs positively with any symplectic form. By the light cone lemma, we have $u \cdot e > 0$. This proves that $P_{M_{k-1},K} \subset F_{E_k}$. □

5.4.3. Proof of Proposition 5.21.

Proof. First of all,

$$S_K^+ \subset P_K.$$

This follows from (21), (22), the positive pairing between S_K^+ and \mathcal{E}_K by Lemma 4.12(2), and the positive pairing between S_K^+ and K by the adjunction formula.

So now we start to prove $P_K \subset S_K^+$.

For $k = 1$, it is clear and is essentially contained in the proof of Theorem 1.5.

For $2 \leq k \leq 8$, we do induction. Suppose we have done the case when $k < l \leq 8$, we want to argue that for $M_l = \mathbb{C}\mathbb{P}^2 \# l\mathbb{C}\mathbb{P}^2$, $S_K^+ = P_K$.

In this case, by Lemma 5.24(1), P_K is an open polytope with each face of the boundary a wall of a class in $\mathcal{E}_{M_l,K}$. Hence e could be written as a finite combination $\sum_{i=1}^q a_i e_i$ with e_i in a boundary face and $a_i > 0$. Notice each boundary face $F_{E'_i}$ (with $E'_i \in \mathcal{E}_{M_l,K}$) of $P_{M_l,K}$ corresponds to $P_{M_{l-1},K}$ by Lemma 5.24(2). Then by induction assumption, each $e_i \in S_{M_{l-1},K}^+$. Hence $e \in S_K^+$ as well by definition.

When $k \geq 9$, we still do induction. However, in this situation, P_K does have a wall contributed by the class $-K$.

Consider a class of the form $V_a = aH - \sum_{i=1}^l E_i$. Given $e \in P_K$, we can find $a < 3$ such that $e \cdot (aH - \sum E_i) = 0$. This is because $V_3 = -K$ pairs positively with e , and when $a = 0$, V_0 pairs negatively with e .

Notice that $V_a \cdot V_3 < V_3 \cdot V_3 \leq 0$, so the hypersurface of V_a does not intersect the wall of $-K$. Choose a generic line L in this hypersurface such that $e \in L$ and L intersects the polytope inside the interior of the boundary faces F_1, F_2 at e_1 and e_2 .

Then $e = a_1 e_1 + a_2 e_2$ with $a_i > 0$. Each class e_i lies in the interior of the face F_i , which corresponds to $P_{M_{l-1},K}$ by Lemma 5.24(2). By induction assumption, $e_i \in S_K^+$. This finishes the proof. □

5.5. Remark on the connection with Kodaira embedding. Finally we would like to provide a heuristic comparison of the genus zero subvariety-current-form construction with the Kodaira embedding theorem.

Unlike linear systems in algebraic geometry, the moduli spaces of pseudo-holomorphic subvarieties generally have no natural linear structure. In algebraic geometry, we obtain a linear system of divisors from the vector space of sections of a holomorphic line bundle. For almost complex structures, there is no such direct global correspondence. In general, only locally there are some traces of linearity. Via the implicit function theorem, Taubes in [29] uses the linear structure on the kernel of the normal operator D_C to linearize the moduli space near a subvariety

C. Another instance is that, inside a very small ball, subvarieties behave like lines in the standard \mathbb{C}^2 (Lemma 2.10).

However, it strikes us again and again that the moduli spaces of genus zero pseudo-holomorphic subvarieties possess local as well as semi-global linearity properties. Notably, in Proposition 4.9 we show that there are plenty of pencils of genus zero subvarieties in a big J -nef class e . To find a pencil, we usually fix $e \cdot e$ points suitably, then look at the local or global moduli space passing through these points. We are thus led to the following global question.

QUESTION 5.25. *Suppose J is a tamed almost complex structure on a rational manifold, and e is represented by a smooth J -holomorphic genus zero subvariety. Does \mathcal{M}_e have the structure of a complex projective space of dimension $e \cdot e + 1$?*

In fact, this has a positive answer when $e \cdot e \leq 0$, and when e is the line class of $\mathbb{C}\mathbb{P}^2$. If we can answer positively Question 5.25 in general, it is tempting to further link the construction of an almost Kähler form via genus zero subvarieties on a rational manifold to the construction of a Kähler form on a Hodge manifold via Kodaira embedding. Given a J -ample class e in S_{K_J} , when J is integrable and hence Kähler, holomorphic sections of L provide a holomorphic embedding $\tau_L : M \rightarrow \mathbb{C}\mathbb{P}^N$, where L is line bundle corresponding to the class e and $N = h^0(L) - 1 = e \cdot e + 1$. The pullback $\tau_L^* \omega_{FS}$ of the standard Fubini-Study form gives us a Kähler form in the class e . When J is not integrable, one could heuristically think that the family of J -tamed forms with dominating J -invariant part in the construction come from a family of embeddings $\tau_e(\epsilon) : M \rightarrow \mathcal{M}_e = \mathbb{C}\mathbb{P}^N$, which are close to being holomorphic.

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455
E-mail: tjli@math.umn.edu

MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL,
 ENGLAND
E-mail: weiyi.zhang@warwick.ac.uk

REFERENCES

- [1] J.-F. Barraud, Nodal symplectic spheres in $\mathbb{C}\mathbb{P}^2$ with positive self-intersection, *Int. Math. Res. Not. IMRN* **1999** (1999), no. 9, 495–508.
- [2] R. L. Bryant, Submanifolds and special structures on the octonians, *J. Differential Geom.* **17** (1982), no. 2, 185–232.
- [3] O. Buse, Negative inflation and stability in symplectomorphism groups of ruled surfaces, *J. Symplectic Geom.* **9** (2011), no. 2, 147–160.

- [4] S. K. Donaldson, Two-forms on four-manifolds and elliptic equations, *Inspired by S. S. Chern, Nankai Tracts Math.*, vol. 11, World Sci. Publ., Hackensack, NJ, 2006, pp. 153–172.
- [5] T. Draghici, T.-J. Li, and W. Zhang, Symplectic forms and cohomology decomposition of almost complex four-manifolds, *Int. Math. Res. Not. IMRN* **2010** (2010), no. 1, 1–17.
- [6] ———, On the J -anti-invariant cohomology of almost complex 4-manifolds, *Q. J. Math.* **64** (2013), no. 1, 83–111.
- [7] R. Friedman and J. W. Morgan, On the diffeomorphism types of certain algebraic surfaces. I, *J. Differential Geom.* **27** (1988), no. 2, 297–369.
- [8] R. E. Gompf, Locally holomorphic maps yield symplectic structures, *Comm. Anal. Geom.* **13** (2005), no. 3, 511–525.
- [9] M. Gromov, Pseudoholomorphic curves in symplectic manifolds, *Invent. Math.* **82** (1985), no. 2, 307–347.
- [10] H. Hofer, V. Lizan, and J.-C. Sikorav, On genericity for holomorphic curves in four-dimensional almost-complex manifolds, *J. Geom. Anal.* **7** (1997), no. 1, 149–159.
- [11] S. Ivashkovich and V. Shevchishin, Structure of the moduli space in a neighborhood of a cusp-curve and meromorphic hulls, *Invent. Math.* **136** (1999), no. 3, 571–602.
- [12] M. Lejmi, Strictly nearly Kähler 6-manifolds are not compatible with symplectic forms, *C. R. Math. Acad. Sci. Paris* **343** (2006), no. 11-12, 759–762.
- [13] B.-H. Li and T.-J. Li, Symplectic genus, minimal genus and diffeomorphisms, *Asian J. Math.* **6** (2002), no. 1, 123–144.
- [14] T.-J. Li, Smoothly embedded spheres in symplectic 4-manifolds, *Proc. Amer. Math. Soc.* **127** (1999), no. 2, 609–613.
- [15] T.-J. Li and A.-K. Liu, General wall crossing formula, *Math. Res. Lett.* **2** (1995), no. 6, 797–810.
- [16] ———, Uniqueness of symplectic canonical class, surface cone and symplectic cone of 4-manifolds with $B^+ = 1$, *J. Differential Geom.* **58** (2001), no. 2, 331–370.
- [17] T.-J. Li and A. Tomassini, Almost Kähler structures on four dimensional unimodular Lie algebras, *J. Geom. Phys.* **62** (2012), no. 7, 1714–1731.
- [18] T.-J. Li and W. Zhang, Comparing tamed and compatible symplectic cones and cohomological properties of almost complex manifolds, *Comm. Anal. Geom.* **17** (2009), no. 4, 651–683.
- [19] ———, J -holomorphic curves in a nef class, preprint, <http://arxiv.org/abs/1210.3337>, *Int. Math. Res. Not. IMRN* (to appear), 2012.
- [20] D. McDuff, The structure of rational and ruled symplectic 4-manifolds, *J. Amer. Math. Soc.* **3** (1990), no. 3, 679–712.
- [21] ———, Symplectomorphism groups and almost complex structures, *Essays on Geometry and Related Topics, Vol. 1, 2, Monogr. Enseign. Math.*, vol. 38, Enseignement Math., Geneva, 2001, pp. 527–556.
- [22] D. McDuff and D. Salamon, *J-Holomorphic Curves and Symplectic Topology*, *Amer. Math. Soc. Colloq. Publ.*, vol. 52, Amer. Math. Soc., Providence, RI, 2004.
- [23] B. McKay, Almost complex rigidity of the complex projective plane, *Proc. Amer. Math. Soc.* **135** (2007), no. 2, 597–603.
- [24] M. Pinsonnault, Maximal compact tori in the Hamiltonian group of 4-dimensional symplectic manifolds, *J. Mod. Dyn.* **2** (2008), no. 3, 431–455.
- [25] T. Rivière and G. Tian, The singular set of J -holomorphic maps into projective algebraic varieties, *J. Reine Angew. Math.* **570** (2004), 47–87.
- [26] J.-C. Sikorav, Some properties of holomorphic curves in almost complex manifolds, *Holomorphic Curves in Symplectic Geometry, Progr. Math.*, vol. 117, Birkhäuser, Basel, 1994, pp. 165–189.
- [27] ———, The gluing construction for normally generic J -holomorphic curves, *Symplectic and Contact Topology: Interactions and Perspectives (Toronto, ON/Montreal, QC, 2001)*, *Fields Inst. Commun.*, vol. 35, Amer. Math. Soc., Providence, RI, 2003, pp. 175–199.
- [28] C. H. Taubes, $SW \Rightarrow Gr$: from the Seiberg-Witten equations to pseudo-holomorphic curves, *J. Amer. Math. Soc.* **9** (1996), no. 3, 845–918.
- [29] ———, Tamed to compatible: symplectic forms via moduli space integration, *J. Symplectic Geom.* **9** (2011), no. 2, 161–250.
- [30] V. Tosatti and B. Weinkove, The Calabi-Yau equation, symplectic forms and almost complex structures, *Geometry and Analysis. No. 1, Adv. Lect. Math. (ALM)*, vol. 17, Int. Press, Somerville, MA, 2011, pp. 475–493.
- [31] W. Zhang, From Taubes currents to almost Kähler forms, *Math. Ann.* **356** (2013), no. 3, 969–978.
- [32] ———, The curve cone of almost complex 4-manifolds, preprint, <http://arxiv.org/abs/1501.06744>, 2015.