# ALMOST KÄHLER FORMS ON RATIONAL 4−MANIFOLDS

TIAN-JUN LI AND WEIYI ZHANG

ABSTRACT. We study Nakai-Moishezon type question and Donaldson's "tamed to compatible" question for almost complex structures on rational four manifolds. By extending Taubes' subvarieties–current–form technique to $J-$nef genus 0 classes, we give affirmative answers of these two questions for all tamed almost complex structures on $S^2$ bundles over $S^2$ as well as for many geometrically interesting tamed almost complex structures on other rational four manifolds, including the del Pezzo ones.

CONTENTS

## 1. Introduction

Let $M$ be a compact, oriented, smooth manifold. An almost complex structure on $M$ is an endomorphism of $TM$ whose square is $-\mathrm{Id}$. An almost complex structure $J$ induces an involution on the space of 2-forms, $\Omega^2(M)$, decomposing it as $\Omega_J^+ \oplus \Omega_J^-$. $J$ is said to be tamed if there is a symplectic form $\omega$ such that the bilinear form $\omega(\cdot, J(\cdot))$ is positive definite. In this case, $\omega$ is called a taming form of $J$, and we also say that $J$ is tamed by $\omega$. A taming form of $J$ is said to be compatible with $J$ if it lies in $\Omega_J^+$. $J$ is said to be almost Kähler if there is a compatible form.

In [4], Donaldson raised the following question:

**Question 1.1.** *Suppose $J$ is an almost complex structure on a compact, oriented, smooth $4-$manifold $M$. If $J$ is tamed, is $J$ almost Kähler?*

The local version of the question was known to be true in dimension 4, but false in higher dimensions ([26, 25, 12, 2]). Donaldson suggested an approach via the symplectic Calabi-Yau equation, and progress on this equation has been made by Weinkove, Tosatti and Yau (cf. [30]).

It was known to hold when $M = \mathbb{CP}^2$ due to deep works of Gromov [9] and Taubes [28]. In [18], we observed that this is true for any integrable $J$, and the same was shown for homogeneous $J$ in [17].

In the case $b^+(M) = 1$, Taubes has recently made remarkable progress in [29]. He answers Question 1.1 affirmatively for generic tamed almost complex structures in this case.

The main purpose of this paper is to study Question 1.1 for rational $4-$manifolds. Here a rational $4-$manifold refers to one of the following smooth 4-manifolds: $\mathbb{CP}^2$, $S^2 \times S^2$ and blow-ups of them.

Our construction is particularly successful for $S^2-$bundle over $S^2$:

**Theorem 1.2.** *Any tamed $J$ on $S^2 \times S^2$ or $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is almost Kähler.*

The remaining rational manifolds are of the form $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ with $k \geq 2$. We settle Donaldson's question for these manifolds under a simple condition on $-1$ curves.

**Theorem 1.3.** *Suppose $M = \mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ with $k \geq 2$ and $J$ is tamed. If there are $k$ disjoint $-1$ curves, then $J$ is almost Kähler.*

To prove his genericity result, Taubes explores close connections between pseudo-holomorphic subvarieties (see Definition 2.1) and almost Kähler forms in dimension 4. In arbitrary dimension, they have positive pairings. A special feature in dimension 4 is that they both lie in the space of closed, non-negative, $J-$invariant 2-currents. In particular, Taubes introduced a distributional analogue of an almost Kähler form, which we call a Taubes current (see Definition 2.11).

Let $\omega$ be a symplectic form on a compact, oriented, smooth $4-$manifold with $b^+ = 1$. The basic strategy of Taubes is to first carefully pick a smooth subfamily of evenly distributed irreducible $J-$holomorphic subvarieties in the class of $N[\omega]$ for $N$ large, at least when $[\omega]$ is rational, then to obtain a Taubes current via integration. Finally, Taubes showed that such a current can be first smoothed to a $J-$tamed form with a dominating $J-$invariant part, and then further adjusted to a genuine almost Kähler form.

For a rational manifold, to avoid generic choices of $J$ in several places of Taubes' construction, we apply the subvarieties–current–form technique to classes of genus zero smooth subvarieties with positive self-intersection. However, in general we could only hope to first construct a weaker version of Taubes current which degenerates on a finite union of subvarieties with negative self-intersection. Then we try to sum several such weak Taubes currents to get a honest Taubes current. We call Taubes current obtained this way a spherical Taubes current.

As in [29], while the construction of weak Taubes current in 5.1.1 is via integration over the irreducible subvariety part $\mathcal{M}_{irr}$ of the moduli space, we still need a good control of the reducible subvariety part $\mathcal{M}_{red}$. With this in mind, we establish in [19] a clean structural picture of reducible subvarieties for a $J-$nef class with $J-$genus 0 (see Theorem 2.14). This is crucial for us to get rid of much of the "genericity" assumption of [29].

For a more detailed summary of Taubes' subvarieties–current–form technique and our adaptation, see section 2.4.3.

The subvarieties–current–form technique is also useful to further determine the almost Kähler cone

$$\mathcal{K}_J^c = \{[\omega] \in H^2(M;\mathbb{R}) | \omega \text{ is compatible with } J\} \tag{1}$$

in terms of the curve cone. The almost Kähler cone $\mathcal{K}_J^c$ is a convex cohomology cone contained in the positive cone

$$\mathcal{P} = \{e \in H^2(M;\mathbb{R}) | e \cdot e > 0\}.$$

The curve cone of an almost complex manifold $(M, J)$, denoted by $A_J(M)$, is the convex cone in $H_2(M, \mathbb{R})$ generated by the set of homology classes of $J-$holomorphic subvarieties. Let $A_J^{\vee,>0}(M)$ be the positive dual of $A_J(M)$ under the homology-cohomology pairing, and set

$$\mathcal{P}_J = A_J^{\vee,>0}(M) \cap \mathcal{P}.$$

Clearly, $\mathcal{K}_J^c \subset A_J^{\vee,>0}(M)$ since the integral of an almost Kähler form over a $J-$holomorphic subvariety is positive. Motivated by the famous Nakai-Moishezon-Kleiman criterion in algebraic geometry which characterizes the ample cone in terms of the (closure of) curve cone for a projective $J$, and the recent Kähler version [1] of the Nakai-Moishezon criterion (in dimension 4), which characterizes the Kähler cone in terms of the curve cone for a Kähler $J$, we ask whether there is an almost Kähler version of the Nakai-Moishezon criterion.

[1]Established by Buchdahl and Lamari in dimension 4, and by Demailly-Paun in arbitrary dimension.

**Question 1.4.** [2] *Suppose $M$ is a compact, oriented, smooth $4-$manifold with $b^+=1$ and $J$ is almost Kähler. Is the almost Kähler cone dual to the curve cone, i.e. $\mathcal{K}_J^c=\mathcal{P}_J$?*

Via a detailed analysis of spherical Taubes currents, we are able to establish the almost Kähler Nakai-Moishezon criterion in the following two cases.

**Theorem 1.5.** *The almost Kähler Nakai-Moishezon criterion holds for any almost Kähler $J$ on $S^2\times S^2$ or $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$.*

**Theorem 1.6.** *The almost Kähler Nakai-Moishezon criterion holds for any good almost Kähler $J$.*

In Theorem 1.6, a tamed $J$ on a rational manifold is called *good* if (i) there is a smooth genus one subvariety in the anti-canonical class $-K_J$, and (ii) any irreducible genus zero subvariety of negative self-intersection is a $-1$ curve.

Notice that, following from Theorems 1.2 and 1.3, both Theorems 1.5 and 1.6 are still valid even if we assume $J$ is tamed instead of almost Kähler.

The organization of this paper is as follows. In section 2 we review properties of moduli space of irreducible pseudo-holomorphic subvarieties as presented in [28] and introduce Taubes current. In section 3, to illustrate some key features of our construction, we construct spherical Taubes currents in the line class of $\mathbb{CP}^2$. The situation for the line class is simpler since there are no reducible rational curves. In section 4 we discuss various properties of $J-$nef spherical class and show that there are plenty of pencils in the moduli spaces. In section 5 we combine the constructions in sections 3 and 4 to construct weak Taubes currents from a big $J-$nef spherical class. An immediate consequence is the proof of Theorems 1.2 and 1.5 for $S^2-$bundles over $S^2$. We also study the geometry and combinatorics of $K-$symplectic cone and $K-$sphere cone of $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ with $k\geq 2$ and prove Theorems 1.3, 1.6 and 5.19. Finally, we compare our construction with the Kodaira embedding theorem.

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[2] This question makes sense for any 4-manifold, if we view $A_J(M)$ as a convex cone of $H_+^J$ and define $A_J^{\vee,>0}(M)$ to be the positive dual of $A_J(M)$ under the pairing between $H_+^J$ and $H_J^+$ in [18]. The question is: Is $\mathcal{K}_J^c$ a connected component of $\mathcal{P}_J$? See the survey paper [6].

## 2. Pseudo-holomorphic subvarieties and Taubes current

Let $M$ be a closed, oriented 4−manifold and $J$ be a tamed almost complex structure on $M$. In this section we summarize properties of $J$−holomorphic subvarieties following [29] and introduce Taubes current.

We fix a symplectic form $\omega$ tamed by $J$. Such a form defines a cohomology class $[\omega]$. Moreover, the polarization of the quadratic form given by $\omega(\cdot, J(\cdot))$ defines a $J$−invariant metric on $M$. Such a metric is used implicitly in all the follows to define distances on $M$, integration over open sets in Cartesian products of $M$, norms on (complexified) tensor bundles of $M$.

### 2.1. **Pseudo-holomorphic subvarieties and their properties.**

**Definition 2.1.** *A closed set $C \subset M$ with finite, nonzero 2-dimensional Hausdorff measure is said to be a $J$−holomorphic subvariety if it has no isolated points, and if the complement of a finite set of points in $C$, called the singular points, is a smooth submanifold with $J$−invariant tangent space.*

When $J$ is understood, we will simply call a $J$−holomorphic subvariety a subvariety. A subvariety is said to be smooth if it has no singular points.

A subvariety $C$ has a canonical orientation, which is used to define integration of smooth 2−forms on its smooth part. The resulting linear functional on the space of 2−forms defines a closed, non-negative current of type $(1, 1)$. We denote the associated homology class by $e_C$.

2.1.1. *Genus of an irreducible subvariety.* A subvariety is said to be irreducible if its smooth locus is connected. Any given subvariety is a union of a finite set of irreducible subvarieties.

Suppose $C$ is an irreducible subvariety. Then it is the image of a $J$−holomorphic map $\phi : C_0 \to M$ from a complex connected curve $C_0$, where $\phi$ is an embedding off a finite set. $C_0$ is called the model curve and $\phi$ is called the tautological map. The map $\phi$ is uniquely determined up to automorphisms of $C_0$. This understood, the homology class $e_C$ is simply the push forward of the fundamental class of $C_0$ via $\phi$.

The genus of an irreducible subvariety $C$ is defined to be the genus of its model curve $C_0$. There is another type of genus associated to the class $e_C$ defined as follows.

Given a class $e$ in $H_2(M; \mathbb{Z})$, introduce the $J$−genus of $e$,

$$g_J(e) \quad = \quad \tfrac{1}{2}(e \cdot e + K_J \cdot e) + 1, \tag{2}$$

where $K_J$ is the canonical class of $J$. Notice that $g_J(e)$ is called the virtual genus in many literature.

If $e = e_C$ for some subvariety $C$, then $g_J(e)$ is non-negative. In fact, by the adjunction inequality in [21], $g_J(e)$ is bounded from below by the genus of the model curve $C_0$ of $C$, with equality if and only if $C$ is smooth.

2.1.2. *The normal operator $D_C$.* Suppose $C$ is an irreducible subvariety and $C_0$ is its model curve with the tautological map $\phi$. Let us first introduce the normal bundle of $C$.

If $p \in C_0$ is not a critical point of $\phi$, then $L_p = \phi_*(TC_0|_p)$ is a complex line in $TM|_{\phi(p)}$. If $p_0$ is a critical point of $C_0$, for $p$ in a deleted neighborhood of $p_0$ consisting of non-critical points, $L_p \subset TM|_{\phi(p)}$ still converge to a line $L_{p_0}$ in $TM|_{\phi(p_0)}$. This follows from elliptic unique continuation. It also follows from the local canonical form of $\phi$ near $p_0$ (see e.g. (2.2) of [28]): there is a complex coordinate $u$ for a disk in $C_0$ centered on a critical point $p_0$, and a complex coordinate $(z, w)$ centered on $\phi(p_0)$ such that $\phi$ has the form

$$u \to (u^{n+1} + t_z, cu^{n+k+1} + t_w), \tag{3}$$

where $n, k \geq 1$ and $|t_z| \leq c_0|u|^{n+2}$, $|dt_z| \leq c_0|u|^{n+1}$, $|t_w| \leq c_0|u|^{n+k+2}$, $|dt_w| \leq c_0|u|^{n+k+2}$. Hence $L$ extends over $p_0$ as the pull-back from $\mathbb{C}^2$ via (3) of the span of the vector field $\frac{\partial}{\partial z}$. Thus there is a complex line bundle $N$, whose fiber over $p \in C_0$ is the quotient complex line $TM|_{\phi(p)}/L_p$. This complex line bundle over $C_0$ is called the normal bundle of $C$.

Linearizing the equation $\bar{\partial}_J \phi = 0$ for the $J-$holomorphic map $\phi : C_0 \to M$, we obtain a $\mathbb{R}-$linear, differential operator

$$D_\phi : C^\infty(C_0; \phi^*TM) \to C^\infty(C_0; \phi^*TM \otimes T^{0,1}C_0).$$

Choose an almost Hermitian metric on $M$ to realize $N$ as a subbundle of $\phi^*TM$. Then it induces a canonically associated $\mathbb{R}-$linear, differential operator

$$D_C : C^\infty(C_0; N) \to C^\infty(C_0; N \otimes T^{0,1}C_0). \tag{4}$$

$D_C$ is called the normal operator of $C$.

Use a Hermitian metric on $C_0$ and the Levi-Civita connection on $M$ to define Sobolev completions of $C^\infty(C_0; N), C^\infty(C_0; N \otimes T^{0,1}C_0)$. $D_C$ extends to a bounded, Fredholm operator from the Hilbert space $L^2_1(C_0; N)$ to the Hilbert space $L^2(C_0; N \otimes T^{0,1}C_0)$. Denote the index of this extension by $d_C$. $d_C$ is always even, and is bounded above by the even integer $2\iota_{e_C}$ defined as follows.

**Definition 2.2.** *Given a class $e$, introduce its $J-$dimension,*

$$\iota_e = \frac{1}{2}(e \cdot e - K_J \cdot e). \tag{5}$$

Note also that $d_C = 2\iota_{e_C}$ if and only if $C$ is smooth.

2.2. **The moduli space.** In this subsection we fix a class $e$.

The moduli space of subvariety in the class $e$, $\mathcal{M}_e$, is defined as in [29]: Any element $\Theta$ in $\mathcal{M}_e$ is a finite set of pairs, where each pair has the form $(C, m)$ with $C \subset M$ an irreducible subvariety and $m$ a positive integer. The set of pairs in an element is further constrained so that no two of its pairs have the same subvariety component, and so that $\sum me_C = e$.

**Definition 2.3.** *A homology class $e \in H_2(M;\mathbb{Z})$ is said to be $J-$effective if $\mathcal{M}_e$ is nonempty.*

2.2.1. *Topology.* Let $|\Theta| = \cup_{(C,m)\in\Theta} C$ denote the support of $\Theta$. Consider the symmetric, non-negative function, $\varrho$, on $\mathcal{M}_e \times \mathcal{M}_e$ that is defined by the following rule:

$$\varrho(\Theta,\Theta') = \sup_{z\in|\Theta|} \operatorname{dist}(z,|\Theta'|) + \sup_{z'\in|\Theta'|} \operatorname{dist}(z',|\Theta|). \tag{6}$$

The function $\varrho$ is used to measure distances on $\mathcal{M}_e$.

Given a smooth form $\nu$ introduce the pairing

$$(\nu,\Theta) = \sum_{(C,m)\in\Theta} m\int_C \nu.$$

The topology on $\mathcal{M}_e$ is defined in terms of convergent sequences:

A sequence $\{\Theta_k\}$ in $\mathcal{M}_e$ converges to a given element $\Theta$ if the following two conditions are met:

- $\lim_{k\to\infty} \varrho(\Theta,\Theta_k) = 0$.
- $\lim_{k\to\infty} (\nu,\Theta_k) = (\nu,\Theta)$ for any given smooth 2-form $\nu$.

Here is Proposition 3.1 in [29].

**Proposition 2.4.** *The moduli space $\mathcal{M}_e$ is compact. In particular, only finitely many classes are of the form $e_C$ with $(C,m)\in\Theta$ and $\Theta\in\mathcal{M}_e$.*

2.2.2. *Moduli spaces of irreducible subvarieties.* Fix an integer $h$. We define $\mathcal{M}_{h,e} \subset \mathcal{M}_e$ to be the subspace of irreducible subvarieties of genus $h$.

With this understood, the following is a direct consequence of the adjunction inequality.

**Lemma 2.5.** *If $h = g_J(e)$ and $C \in \mathcal{M}_{h,e}$, then $C$ is smooth.*

Let $\Sigma$ be a smooth, oriented surface of genus $h$. Let $\mathfrak{M}_{h,e}$ be the space of somewhere injective $J$-holomorphic maps in the class $e$ and originated from $(\Sigma, j)$, where $j$ is an arbitrary complex structure on $\Sigma$. As a subset of the Fréchet space of smooth maps from $\Sigma$ to $M$, $\mathfrak{M}_{h,e}$ has a natural topology.

Since every irreducible subvariety has a model curve, there is a surjective map $\Psi$ from $\mathfrak{M}_{h,e}$ to $\mathcal{M}_{h,e}$. A fundamental fact established in the appendix in [29], whose proof is rather involved, is that the topology on $\mathcal{M}_{h,e}$ is the same as the induced one from $\mathfrak{M}_{h,e}$ via $\Psi$. More precisely, by Lemma A.13 in [29], at any $\phi : (\Sigma, j) \to M$ in $\mathfrak{M}_{h,e}$, $\Psi$ is a local homeomorphism from $\mathfrak{M}_{h,e}$ to $\mathcal{M}_{h,e}$ when $h \geq 2$, and in the case $h = 0, 1$, $\Psi$ is a local homeomorphism up to automorphisms of $(\Sigma, j)$.

With this understood, it follows that well known topological properties of $\mathfrak{M}_{h,e}$ carry over to $\mathcal{M}_{h,e}$.

**Theorem 2.6.** *(Propositions 3.2 and 3.3 in [29]) There exists a smooth map, $f$, from a neighborhood of* 0 *in* $\ker D_C$ *to* $\operatorname{coker} D_C$*; and there exists a* *homeomorphism from* $f^{-1}(0)$ *to a neighborhood of* $C$ *in* $\mathcal{M}_{h,e}$ *sending* $0$ *to* $C$*.*

*The subset of* $\mathcal{M}_{h,e}$ *where the cokernel of* $D_C$ *is trivial has the structure of a smooth manifold of dimension* $2\iota_e - 2(g_J(e) - h)$*; and the smooth structure is such that at any point in this set, the aforementioned homeomorphism from a neighborhood of* $0$ *in the kernel is a smooth embedding onto an open set.*

*For* $J$ *in a residual set in the space of almost complex structures,* $\mathrm{coker} D_{(\cdot)} = 0$ *for each point* $\mathcal{M}_{h,e}$*, and so the latter has the structure of a smooth manifold whose dimension is* $2\iota_e - 2(g_J(e) - h)$*.*

Concerning the vanishing of coker $D_C$, we mention another fact, which is particularly useful in this paper. Note that $D_C$ is a real Cauchy-Riemann operator on $N$. For such operators, there is the following automatic transversality result.

**Theorem 2.7** ([10], [11])**.** *Let* $(\Sigma, j)$ *be a Riemann surface of genus* $h$*, and* $L$ *a complex line bundle over* $\Sigma$*. Suppose* $c_1(L) \geq 2h - 1$*. Then* $\mathrm{coker} D = 0$ *for any real Cauchy-Riemann operator* $D$ *on* $L$*.*

2.2.3. *Moduli spaces with marked points.* Let $M^{[k]}$ denote the set of $k$ tuples of pairwise distinct points in $M$. Given $\Omega = (z_1, \cdots, z_k)$ in $M^{[k]}$, denote its support $\{z_1, \cdots, z_k\}$ in $M$ also by $\Omega$. Let $\mathcal{M}_e^{\Omega}$ be the space of subvarieties in $\mathcal{M}_e$ passing through $\Omega$.

Notice that $\mathcal{M}_e^{\Omega}$ is compact since it is a closed subset of the compact space $\mathcal{M}_e$.

Given an integer $h$, let $\mathcal{M}_{h,e}^{\Omega} = \mathcal{M}_{h,e} \cap \mathcal{M}_e^{\Omega}$.

Suppose $h = g_J(e)$ and $C \in \mathcal{M}_{g_J(e),e}^{\Omega}$. Then $C$ is smooth by Lemma 2.5, and so the normal bundle $N$ is a line bundle over $C$ itself. Consider the evaluation map at $\Omega$, $ev^{\Omega} : \Gamma(N) \to \oplus_{p\in\Omega} N|_p$, and the operator

$$D_C \oplus ev^{\Omega} : \Gamma(N) \to \Gamma(N \otimes T^{1,0}C) \oplus (\oplus_{p\in\Omega} N|_p).$$

The index of $D_C \oplus ev^{\Omega}$ is $d_C - 2k$. And the kernel of $D_C \oplus ev^{\Omega}$ should be thought of as giving a sort of Zariski tangent space to $\mathcal{M}_{g_J(e),e}^{\Omega}$ at $C$ (as a point in the space of smooth embeddings).

The smooth subvariety $C$ is called $(J, \Omega)$ non-degenerate if the operator $D_C \oplus ev^{\Omega}$ has trivial cokernel. If this is the case, $\mathcal{M}_{h,e}^{\Omega}$ is a smooth manifold of dimension $d_C - 2k$ around $C$.

It is clear that $D_C \oplus ev^{\Omega}$ has trivial cokernel if $D_C$ has trivial cokernel and $ev^{\Omega} : \ker D_C \to \oplus_{p\in\Omega} N|_p$ is surjective. In light of Theorem 2.7, it is more useful to test the $(J, \Omega)$ non-degeneracy via the real Cauchy-Riemann operator $D_C^{\Omega}$ in [1], which we now describe.

The bundle $N$ over the complex curve $C$ has a natural holomorphic line bundle structure. Consider the holomorphic line bundle $N(\Omega)$, obtained from twisting $N$ by the divisor $-(z_1 + \cdots + z_k)$. In [1] Lemma 4, there is introduced the following operator

$$D_C^\Omega : C^\infty(C_0; N(\Omega)) \longrightarrow C^\infty(C_0; N(\Omega)\otimes T^{0,1}C_0).$$

$D_C^\Omega$ is also a real Cauchy-Riemann operator. Moreover, there is an exact sequence, (15) in [1],

$$0 \to \ker D_C^\Omega \to \ker D_C \to \oplus_{p\in\Omega} N|_p \to \mathrm{coker} D_C^\Omega \to \mathrm{coker} D_C \to 0, \tag{7}$$

where the middle map is $ev^\Omega$. It follows from (7) that we have

**Lemma 2.8.** *$D_C \oplus ev^\Omega$ has trivial cokernel if $D_C^\Omega$ has trivial cokernel.*

## 2.3. **Subvarieties through a small ball.**

2.3.1. *The local area bounds.* The following summarizes the local area bounds of an irreducible subvariety.

**Lemma 2.9.** *Let $J$ be a tamed almost complex structure. Fix a symplectic form $\omega$ on $M$ taming $J$ and the induced $J-$invariant metric. There exists $k \ge 1$, depending only on $J, \omega$, with the following significance: Let $C \subset M$ denote an irreducible subvariety intersecting $B_r(x)$. Fix $r > 0$. Let $a_x(2r)$ denote the area of $C$'s intersection with the ball of radius $2r$ in $M$ centered at $x$. Then $k^{-1}r^2 < a_x(2r) < (e_C\cdot[\omega])kr^2$.*

*Proof.* This is based on Lemma 2.2 in [29] which states a similar bound with a constant $k'$ when $x$ is a point in $C$.

For the lower bound, notice that the intersection contains a radius $r$ ball centered at a point in $C$. Take $k_1 = k'$.

For the upper bound, notice that the intersection is contained in a radius $3r$ ball centered at a point in $C$. Take $k_2 = 9k'$.

Thus $k = 9k'$ is as required. □

2.3.2. *Local structure around a smooth subvariety.* To describe the behavior of subvarieties in a neighborhood of a given point, it is useful to introduce a special sort of coordinate chart. Fix a point $x$ in $M$. An adapted coordinate chart centered at $x$ denotes complex coordinates, $(z, w)$ defined on a radius $c_0^{-1}$ ball centered at $x$ with both vanishing at $x$, with $dz$ and $dw$ orthonormal at $x$, with $\{dz, dw\}$ spanning $T^{1,0}M$ at $x$, and with the norms of $|\nabla dz|$ and $|\nabla dw|$ bounded on the coordinate domain by $c_0$.

Suppose $C$ is a smooth subvariety passing through $x$. Fix an adapted coordinate chart centered at $x$ so as to identify a neighborhood of $x$ in $M$ with a ball about the origin in $\mathbb{C}^2$. There exists $R > 1$ such that $C$ appears in the radius $R^{-1}$ ball about the origin in $\mathbb{C}^2$ as the image from $\mathbb{C}$ to $\mathbb{C}^2$ that has the form

$$u \to \theta u + \nu \tag{8}$$

where $|\nu| < R|u|^2$ and $|d\nu| < R|u|$, and such that $\theta \in \mathbb{C}^2$ has norm 1.

The following is Lemma 4.2 in [29].

**Lemma 2.10.** *Let $C$ be a smooth subvariety of genus $h$ described as above in an adapted chart centered at $x \in C$.*

*Fix $\epsilon > 0$. There is a neighborhood of $C$ in $\mathcal{M}_{e_C,h}$ where each subvariety $C'$ intersects the ball of radius $\frac{1}{2}R^{-1}$ about the origin, and this intersection is the image of a map from $\mathbb{C}$ to $\mathbb{C}^2$ of the form*

$$u \to x' + \theta' u + \nu' \tag{9}$$

*with*

- $|x'| < \epsilon$,
- $|\nu'| < R|u|(|x'| + |u|)$ *and* $|d\nu'| < R(|x'| + |u|)$,
- $\theta' \in \mathbb{C}^2$ *is a unit vector with* $|\theta' - \theta| < \epsilon$.

2.3.3. *The exponential map* $\exp_C$. Suppose $C$ is smooth and coker $D_C = 0$. We describe a version of exponential map in [28] and [29] to identify a ball in ker $D_C$ with a neighborhood of $C$ in $\mathcal{M}_{e_C,g_J(e_C)}$.

Since $C$ is smooth, the normal bundle $N$ can be realized as the orthogonal complement of $TC$ in $T_{1,0}M|_C$. In this case, there exists a map, $\exp_C$, that is defined on a small radius disk bundle $N_1 \subset N$ and has the following properties:

- $\exp_C$ maps the zero section to $C$; and its differential along zero section is an isomorphism from $TN|_0$ to $\phi^* T_{1,0}M$.
- $\exp_C$ embeds each fiber of $N_1$ as a $J$-holomorphic disk in $M$.
- $\operatorname{dist}(\exp_C(v), C) \leq K|v|$ where $K$ is a uniform constant independent of $v$ and $x \in C$ for any vector $v \in N$ with small norm $|v|$.

A construction of such a map is described in section 5d of [28]. The last item above is essentially from Lemma 5.4 (3) there.

Let $\zeta$ denote a section of $N_1$. Then the image in $M$ of the map $\exp_C(\zeta(\cdot))$ is a $J-$holomorphic subvariety if and only if $\zeta$ obeys an equation of the form

$$D_C\zeta + \tau_1\partial\zeta + \tau_0 = 0. \tag{10}$$

Here $\tau_1$ and $\tau_0$ are smooth, fiber preserving maps from $N_1$ to $\mathrm{Hom}(N \otimes T^{1,0}C; N \otimes T^{0,1}C)$ and to $N \otimes T^{0,1}C$ that obey $|\tau_1(b)| \leq c_0|b|$ and $|\tau_0(b)| \leq c_0|b|^2$.

Since $\ker D_C$ is of finite dimension, all the norms on it are equivalent. Choose any norm $|\cdot|$, e.g. the $L^2$ norm or the sup norm, on $\ker D_C$. The map $\exp_C$ can be used to identify a fixed radius ball of $\ker D_C$ with a neighborhood of $C$ in $\mathcal{M}_{e_C,g_J(e_C)}$ (see Lemmas 4.6 and 4.9 in [29]): Suppose $C$ is smooth and coker $D_C = 0$. For $\kappa$ sufficiently large, there is a diffeomorphism from the radius $\kappa^{-2}$ ball in $\ker D_C$ onto an open set in $\mathcal{M}_{e_C,g_J(e_C)}$ that contains the set of curves with $\varrho-$distance less than $\kappa^{-3}$ from $C$. Moreover, the map in question sends a given small normed vector $\eta \in \ker D_C$ to $\exp_C(\eta + \phi_C(\eta))$, where $\phi_C : \ker D_C \to C^\infty(C; N)$ is such that $\zeta = \eta + \phi_C(\eta)$ satisfies (10) and any given $C^k$ norm of $\phi_C(\eta)$ is bounded by a multiple of $|\eta|^2$.

## 2.4. **Subvariety, Taubes current and almost Kähler form.**

2.4.1. *Non-negative currents and forms.* Given an almost complex structure $J$ on $M$, it acts on $\Omega^1(M)$: $J\alpha(X) = -\alpha(JX)$. The componentwise action of $J$ on $\Omega^2(M)$ is an involution, decomposing it into $\Omega^+_J(M) \oplus \Omega^-_J(M)$, called $J-$invariant and $J-$anti-invariant parts respectively. An $J-$invariant 2-form is said to be non-negative if it is non-negative on any pair of tangent vectors $(v, Jv)$ in each point, and positive if the evaluation is positive for any nonzero $v$. In particular, a compatible symplectic form is a positive, closed $J-$invariant form, and a tamed symplectic form is a closed form whose $J-$invariant part is positive.

Non-negative 2-forms can be constructed from 1-forms in the following way: The complexification of $\Omega^1(M)$ decomposed as $\Omega^1(M)\otimes\mathbb{C} = \Omega^{1,0}(M)\oplus \Omega^{0,1}(M)$, where $\Omega^{1,0}$ is the $i-$eigenspace of the extended $J-$action. For any $\sigma \in \Omega^{1,0}(M)$, $i\sigma \wedge \bar{\sigma}$ is a non-negative form.

A 2-dimensional current is a bounded linear functional on the space of smooth 2-forms. All currents here are understood to be 2-dimensional. A current is said to be closed if it annihilates $d\Omega^1(M)$, $J-$invariant, or type (1,1), if it annihilates $\Omega^-_J(M)$. A $J-$invariant current is said to be non-negative if it is non-negative on any non-negative $J-$invariant form, and positive if the evaluation is positive whenever the non-negative $J-$invariant form is not identically zero.

If $C$ is a $J-$holomorphic subvariety, using its natural orientation, it defines a closed, non-negative $J-$invariant current via the integration on its smooth part.

2.4.2. *Taubes current and regularization.* The following type of positive current was introduced in [29], which we will refer to as a Taubes current. Let $B_t(x)$ denote the ball of radius $t$ and center $x$ and $f_{B_t(x)}$ denotes the characteristic function of $B_t(x)$. Remember that we choose an almost Hermitian metric on $(M, J)$ to define the balls. When $J$ is tamed by a symplectic form $\omega$, we use the polarization of the quadratic form given by $\omega(\cdot, J\cdot)$.

**Definition 2.11.** *On an almost Hermitian $4-$manifold $(M, J, g)$, a closed positive $J-$invariant current $T$ is called a Taubes current if there is a constant $k > 1$ such that for any $x$ and small $t$,*

$$k^{-1}t^4 < T(if_{B_t(x)}\sigma \wedge \bar{\sigma}) < kt^4. \tag{11}$$

*Here $\sigma$ denotes a point-wise unit length section of $T^{1,0}M|_{B_t(x)}$.*

Since $M$ is compact, being a Taubes current is independent of the metric $g$. A Taubes current behaves like an almost Kähler form except it may not be smooth. The following observation is also due to Taubes (see the proof of Theorem 1 in [29]).

**Proposition 2.12.** *Suppose $M$ has $b^+ = 1$ and $J$ is a tamed almost complex structure on $M$. Given a Taubes current $T$, there is an almost Kähler form $\alpha$, s.t. $[\alpha] = [T]$.*

This is proved by first smoothing the Taubes current to a family of closed two forms $\Omega^\epsilon$ in a standard way. The property (11) then ensures that, when $\epsilon$ is small, $\Omega^\epsilon$ is non-degenerate, uniformly bounded and has dominate $J-$invariant part. Then the condition of $b^+ = 1$ is used to kill the anti-invariant part by $L^2$ method for small $\epsilon$, keeping the two form symplectic and in the same class. The second author generalizes this regularization result to all almost complex $4-$manifolds with a Taubes current [32].

2.4.3. *The subvarieties–current–form technique.* Let $M$ be a compact, oriented, smooth $4-$manifold with $b^+ = 1$, and $J$ an almost complex structure on $M$ tamed by a symplectic form $\omega$. The basic approach of Taubes to Question 1.1 is to first carefully pick a smooth subfamily of evenly distributed irreducible $J-$holomorphic subvarieties in the class of $N[\omega]$ for $N$ large, at least when $[\omega]$ is rational, then to obtain a Taubes current via integration. Finally, Proposition 2.12 provides a genuine almost Kähler form.

Given $\omega$, let $\mathcal{J}_\omega$ be the space of almost complex structures tamed by $\omega$. Taubes' main result in [29] affirms Question 1.1 for an open and dense subset of $\mathcal{J}_\omega$. This generic subset is constrained by a sequence of regularity properties on the space of irreducible subvarieties in the class $N[\omega]$, as well as by similar regularity assumptions on various spaces of reducible subvarieties.

Let $\mathcal{J}^t$ be the Fréchet space of tamed almost complex structures. Since $\mathcal{J}^t$ is the union of $\mathcal{J}_\omega$ over all symplectic forms, Taubes' result immediately implies that Question 1.1 has a positive answer for an open and dense subset $\mathcal{J}_1$ of $\mathcal{J}^t$. If we stratify $\mathcal{J}^t$ according to the set of irreducible subvarieties of negative self-intersection, $\mathcal{J}_1$ is properly contained in the top stratum $\mathcal{J}_{top}$ defined as follows.

**Definition 2.13.** *A tamed $J$ is in $\mathcal{J}_{top}$ if any irreducible $J-$holomorphic subvariety with negative self-intersection is a $-1$ curve. Here a $-1$ curve refers to a smooth genus zero subvariety with self-intersection $-1$.*

The main purpose of this paper is to study Question 1.1 for rational manifolds by applying the subvarieties–current–form technique to classes of genus zero smooth subvarieties with positive self-intersection. The first reason to consider the space of such subvarieties is that no genericity assumption is needed, namely, we always get a smooth family for any $J$ from the automatic transversality in [10]. This makes it possible to settle Question 1.1 completely for certain rational manifolds, where such subvarieties exist.

Given a tamed $J$ on a rational manifold $M$, classes of genus zero smooth subvarieties with positive self-intersection can always be found inside $S_{K_J}$, the set of classes represented by smoothly embedded spheres and having $J-$genus zero. However, unlike the case of $N[\omega]$ in [29], such a class is often only $J-$nef in the sense that the pairing with an arbitrary $J-$holomorphic subvariety in $M$ is non-negative but might vanish. Consequently, in general we could only hope to first construct a weak spherical Taubes current, which is a weaker version of Taubes current degenerating on a finite union of subvarieties with negative self-intersection. Then we try to find several weak spherical Taubes currents whose degeneracy loci have empty intersection and sum them to get a honest Taubes current. The Taubes current obtained this way is called a spherical Taubes current.

As in [29], while the construction of weak spherical current in 5.1.1 is via integration over the irreducible subvariety part $\mathcal{M}_{irr}$ of the moduli space, we still need a good control of the reducible subvariety part $\mathcal{M}_{red}$. To achieve this, we establish in [19] the following clean structural picture of reducible subvarieties for a $J-$nef class in $S_{K_J}$:

**Theorem 2.14.** *Suppose $e$ is a $J-$nef class in $S_{K_J}$ and $\Theta$ is a reducible subvariety in the class $e$.*

- *If $\Theta$ is connected, then each irreducible component of $\Theta$ is a smooth rational curve, and $\Theta$ is a tree configuration.*
- *If $J$ is tamed, then $\Theta$ is connected.*

It follows that $\mathcal{M}_{red}$ is a finite union of Cartesian products of irreducible rational curve moduli spaces, which then has expected dimension (Proposition 4.4). In particular, $\mathcal{M}_{red}$ has codimension at least one for *any* tamed almost complex structure (Proposition 4.10). This is crucial for us to get rid of much of the "genericity" assumption of [29].

Another new feature in our construction is the existence of plenty of pencils in the moduli space (Proposition 4.9). This linearity property of the moduli space enables us to better model slices of a small neighborhood of a subvariety in terms of subspaces of the tangent space, which in turns gives rise to the desired estimate of the volume of points lying in subvarieties through a tiny ball.

Our construction of spherical Taubes currents is particularly successful for $S^2-$bundle over $S^2$ as in Theorem 1.2 and 1.5. Theorem 1.3 is also fairly general. In particular, it applies to the entire top stratum $\mathcal{J}_{top}$.

Theorem 1.3 also applies to the set $\mathcal{J}_{good}$ of good almost complex structures, which are defined after Theorem 1.6. For $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ with $k \leq 9$, it is not hard to see that $\mathcal{J}_{good} = \mathcal{J}_{top}$. Hence, the almost Kähler Nakai-Moishezon criterion (Theorem 1.6) holds for any $J \in \mathcal{J}_{top}$ in this case. But if $k \geq 10$, $\mathcal{J}_{good}$ is a lower stratum.

## 3. Line class Taubes current on $\mathbb{CP}^2$

In this section we begin with introducing $K_J-$spherical classes and discussing various automatic regularity properties of irreducible subvarieties in these classes. Then we construct a Taubes current from the space of lines in $\mathbb{CP}^2$. Although in this special case, there are no reducible rational curves, it illustrates many key features of the general construction, notably, how the presence of pencils enables us to better model a small neighborhood of a subvariety in terms of the tangent space, which in turns gives rise to the order two estimate of the volume of points lying in lines through a tiny ball.

3.1. $K_J-$**spherical classes and smooth rational curves.** Let $J$ be a tamed almost complex structure.

3.1.1. *Subvarieties in a $K_J-$spherical classes.* Let $S$ be the set of homology classes which are represented by smoothly embedded spheres.

The set of $K_J-$spherical classes is defined to be

$$S_{K_J} = \{e \in S | g_J(e) = 0\}.$$

The following is a consequence of Seiberg-Witten theory, see e.g. [15].

**Proposition 3.1.** *Suppose $e \in S_{K_J}$ with $e \cdot e \geq -1$. Then for any symplectic form $\omega$ taming $J$, the Gromov-Taubes invariant of $e$ is nonzero. In particular, $\mathcal{M}_e$ is nonempty, i.e. $e$ is $J-$effective.*

We use $\mathcal{M}_{irr,e}$ to denote the moduli space of irreducible subvarieties in class $e$. The following is an immediate consequence of the adjunction formula and the adjunction inequality.

**Lemma 3.2.** *For $e \in S_{K_J}$,*
- $\iota_e = e \cdot e + 1$*, where $\iota_e$ is defined in* (5).
- *every element in $\mathcal{M}_{irr,e}$ is a smooth rational curve.*

Thus for $e \in S_{K_J}$, $\mathcal{M}_{irr,e}$ is the same as $\mathcal{M}_{0,e}$. Let $\mathcal{M}_{red,e}$ denote $\mathcal{M}_e \setminus \mathcal{M}_{irr,e}$.

Given a $k(\leq \iota_e)$ tuple of distinct points $\Omega$, recall that $\mathcal{M}_e^\Omega$ is the space of subvarieties in $\mathcal{M}_e$ that contains all entries of $\Omega$. Introduce similarly $\mathcal{M}_{irr,e}^\Omega$ and $\mathcal{M}_{red,e}^\Omega$. We will often drop the subscript $e$.

Let $S^+, S^0, \mathcal{E} \subset S$ be the subsets of positive square, square 0, square $-1$ classes respectively. $S^+$ is nonempty if and only if $M$ is a rational manifold ([14]). Let $S_{K_J}^+ = S^+ \cap S_{K_J}$ and define $S_{K_J}^0$, $\mathcal{E}_{K_J}$ similarly.

Let $k$ be an integer. Denote by $\mathcal{M}_{e,k}$ the subset in $\mathcal{M}_e \times M^{[k]}$ that consists of elements of the form $(C, x_1, \cdots, x_k)$ with each $x_i \in C$. Define $\mathcal{M}_{irr,e,k}$ and $\mathcal{M}_{red,e,k}$ similarly.

Consider the restrictions of the projection map to the $M^{[k]}$ factor, $\pi_k : \mathcal{M}_{e,k} \to M^{[k]}$, $\pi_{irr,k} : \mathcal{M}_{irr,e,k} \to M^{[k]}$, $\pi_{red,k} : \mathcal{M}_{red,e,k} \to M^{[k]}$.

By Proposition 3.1, we have

**Lemma 3.3.** *Suppose $e \in S_{K_J}^+ \cup S_{K_J}^0 \cup \mathcal{E}_{K_J}$. For any $\Omega \in M^{[\iota_e]}$, $\mathcal{M}_e^\Omega$ is non-empty. In other words, $\pi_{\iota_e}$ is surjective.*

3.1.2. *Smooth rational curves.* We assume now that $e$ is a class represented by a smooth rational curve. In particular, $e \in S_{K_J}$.

Introduce

$$l_e = \max\{\iota_e, 0\}.$$

One special feature of the moduli space of smooth rational curves is the following automatic transversality, which is valid for an arbitrary tamed almost complex structure.

**Lemma 3.4.** *Let $e$ be a class represented by a smooth rational curve with $e \cdot e \geq -1$. Then $\mathcal{M}_{irr,e}$ is a smooth manifold of dimension $2l_e$.*

*Moreover, if we choose a set of $k \leq l_e$ distinct points $\Omega \subset C$, where $C \in \mathcal{M}_{irr,e}$, then $\mathcal{M}^{\Omega}_{irr,e}$ is a smooth manifold of dimension $2(l_e - k)$.*

*Proof.* This observation is essentially contained in in Corollary 2 in [27] (see also [1]).

Suppose $C \in \mathcal{M}_{irr,e}$. By Lemma 3.2, $C$ is a smooth rational curve.

Let $N$ be its normal bundle. Notice that, since $c_1(N) = e \cdot e \geq -1$ and $g = 0$, by Theorem 2.7, coker $D_C = 0$. It follows from Theorem 2.6 that $\mathcal{M}_{irr,e}$ is a smooth manifold whose dimension is $d_C$. Since $C$ is smooth, $d_C = 2\iota_e$. Since $e \cdot e \geq -1$, $\iota_e \geq 0$ and hence $l_e = \iota_e$.

The second statement is proved similarly. By Lemma 2.8, it suffices to show that $D^{\Omega}_C$ has trivial cokernel. The proof is finished by noticing that $c_1(N(\Omega)) = c_1(N) - k$ and the index of $D_C \oplus ev^{\Omega}$ is $d_C - 2k$. □

Here is another feature, specific to rational curves.

**Lemma 3.5.** *Let $e$ be a class represented by a smooth rational curve.*

- *If $e \cdot e \leq -1$, then $\mathcal{M}_{irr,e}$ consists of a single element.*
- *If $e \cdot e \geq 0$, $l = l_e$, and $(C, \Omega) \in \mathcal{M}_{irr,e,l}$, then $\mathcal{M}^{\Omega}_{irr,l}$ consists of $C$ only. In other words, $\pi_{irr,l}$ is an injective smooth map, and the image of $\pi_{irr,l}$ is disjoint from the image of $\pi_{red,l}$.*
- *If $e \cdot e \geq 1$, $l = l_e$, the same uniqueness conclusion is true when we impose constraints of $l - 1$ points and a complex direction at one of the $l - 1$ points.*

*Proof.* All the statements follow from the positivity of intersections of distinct irreducible subvarieties. For the second bullet, notice that $e \cdot e < l$. For the third bullet, notice that a tangency contributes at least 2 to the intersection number. □

**Remark 3.6.** *There is an analogous result to Propositions 3.4 involving tangency conditions. Suppose $C$ is smooth rational curve. If we choose $k \leq l$ distinct points $x_1, ..., x_k$ in $C$ and $k' < k$ with $k + k' \leq l$, then the set of smooth rational curves in $\mathcal{M}^{x_1,...,x_k}_{irr,e_C}$ having the same tangent space at the $k'$ points $x_1, ..., x_{k'}$ as $C$ is still a smooth manifold, whose dimension is $2(l-k-k')$. For this case, the Zariski tangent space is given by the subspace of $\ker D^{x_1,...,x_k}_C$, vanishing with order at least two on $x_1, ..., x_{k'}$. Thus, we are considering the line bundle $N(2x_1 + \cdots + 2x_{k'} + x_{k'+1} + \cdots + x_k)$, $N$ twisted by the divisor $-(2x_1 + \cdots + 2x_{k'} + x_{k'+1} + \cdots + x_k)$. In this case, the relevant Cauchy Riemann operator is onto if $-K_J \cdot e > k + k'$, which is automatic since $-K_J \cdot e = l + 1$.*

By Lemmas 3.4 and 3.5 we have

**Proposition 3.7.** *Let $e$ be a class represented by a smooth rational curve and $l = l_e$. Then $\mathcal{M}_{irr,e}$ has the structure of a $2l$ dimensional manifold* *and $\mathcal{M}_{irr,e,k}$ has the structure of a $2l+2k$ dimensional manifold. $\pi_{irr,k}$ : $\mathcal{M}_{irr,e,k} \to M^{[k]}$ is a smooth map from a $2l+2k$ dimensional manifold to a $4k$ dimensional manifold.*

3.2. **Moduli space of lines.** Now, we assume $M = \mathbb{CP}^2$. Let $H$ be the line class, namely, the generator of $H_2(M;\mathbb{Z})$ such that $K_J = -3H$. Note that $S_{K_J} = \{H, 2H\}$.

Let $\mathcal{M}$ denote $\mathcal{M}_H$, which we call the moduli space of lines. Notice that there are no reducible curves in $\mathcal{M}$ so $\mathcal{M}_{irr} = \mathcal{M}$. Thus by Lemma 3.2, every element in $\mathcal{M}$ is a smooth rational curve. Since $l = l_H = 2$, by Lemma 3.4, $\mathcal{M}$ is a compact, 4 dimensional smooth manifold. In fact, it is diffeomorphic to $\mathbb{CP}^2$ (see e.g. [20]). Since $\mathcal{M}$ is compact, by Lemma 2.10, we have the following (Lemma 4.12 in [29]).

**Lemma 3.8.** *There is a constant $s' < 10^{-10}$ with the following significance: Fix a point $x$ and an adapted coordinate chart centered at $x$ so as to identify a neighborhood of $x$ in $M$ with a ball about the origin in $\mathbb{C}^2$. Let $C \in \mathcal{M}$ which intersects $B_{s'}(x)$. Then $C$ intersects the ball of radius $s'^2$ centered $x$ as the image of a map from a disk in $\mathbb{C}$ about the origin to $\mathbb{C}^2$ that has the form $u \to \theta u + \mathfrak{r}(u)$, where $\theta \in \mathbb{C}^2$ has norm 1 and where $\mathfrak{r}(u)$ is such that $|\mathfrak{r}(u)| \le s'^{-1}|u|^2$ and $|d\mathfrak{r}|_v| \le s'^{-1}|u|$.*

Let $s = s'^4$.

3.2.1. *Pencils of lines.* Fix a point $x_1 \in M$. By Lemma 3.4, $\mathcal{M}^{x_1}$ is 2-dimensional manifold. In fact, $\mathcal{M}^{x_1}$ is a pencil, consisting of $\mathbb{CP}^1$ family of lines.

Choose an orthonormal basis of vectors in $T_{x_1}M$ and use it to identify the space of complex directions at $x_1$ with $\mathbb{CP}^1$. Consider the smooth map $\tau^{x_1} : \mathcal{M}^{x_1} \to \mathbb{CP}^1$ by taking the tangent line at $x_1$. The claim is that there is a unique curve through every direction at $x_1$. The uniqueness is a consequence of the positivity of intersections. The existence can be shown by taking the limit of a sequence of 'secant' curves $C_k$ through $x_1$ and $y_k$ with $y_k - x_1$ projecting to any given $\hat{\theta} \in \mathbb{CP}^1$ and $y_k \to x_1$.

3.2.2. *Norms on* $\ker_C^{x_1}$. Given a smooth curve $C$ in the pencil $\mathcal{M}^{x_1}$ with normal bundle $N$, the tangent space to $\mathcal{M}^{x_1}$ at $C$ can be identified with the vector space $\ker_C^{x_1} \subset \Gamma(N)$ that consists of the sections in the kernel of $D_C$ that vanish at $x_1$.

On this two dimensional space, besides the sup norm $\sup_C |\eta|$, there are other kinds of norms due to the fact that $\mathcal{M}^{x_1}$ is a pencil.

**Lemma 3.9.** *The following are norms on* $\ker_C^{x_1}$*:*

- *For $z \neq x_1$, the pointwise norm $|\eta(z)|$ of the vector $\eta(z)$ in $N|_z$.*
- *The pointwise norm of $\tau_*^{x_1}|_C(\eta)$ as a vector in $T_{\hat{\theta}}\mathbb{CP}^1$, where $\hat{\theta}$ denotes the direction $T_{x_1}C$ in $\mathbb{CP}^1$.*

*For a fixed curve $C$, these norms are equivalent to the sup norm* $\sup_C |\eta|$.

*Proof.* Suppose $\eta$ is a non-trivial element in $\eta \in \ker_C^{x_1}$. Both claims rely on an observation in [10] that there is a new holomorphic structure on $N$ (depending on $\eta$) with respect to which $\eta$ is a holomorphic section.

Since $c_1(N) = 1$, due to the positivity of intersection of holomorphic sections, $\eta$ only vanishes at $x_1$. Especially, $\eta(z) \neq 0$. In other words, $|\eta(z)|$ is a norm of $\ker_C^{x_1}$ if $z$ is not $x_1$.

To establish the second claim, we need the following description of $\tau_*^{x_1}|_C$ of Taubes in part g) of the Appendix. Fix an adapted coordinate chart, $(z, w)$, centered at $x_1$ so that $C$ is tangent to the $w = 0$ locus at the origin. The span of $\frac{\partial}{\partial w}$ is identified with the fiber of $N$ at $x_1$, as well as $T_{\hat{\theta}}\mathbb{CP}^1$. Choose a holomorphic coordinate, $u$, for $C$ centered at $x_1$ with $du = dz$ at $x_1$. Then $\partial\eta|_{x_1}$, when viewed as an element in $T_{\hat{\theta}}\mathbb{CP}^1$ using the identifications above, is the image of $\eta$ under the differential of $\tau^{x_1}$ at $C$.

From this description we see that $\tau_*^{x_1}|_C(\eta) = 0$ only if $\eta$ has vanishing order at least 2 at $x_1$. Since $\eta$ is a holomorphic section and $c_1(N) = 1$, this is impossible. Thus we have shown that $\tau_*^{x_1}|_C : T_C\mathcal{M}^{x_1} \to T_{\hat{\theta}}\mathbb{CP}^1$ is an isomorphism.

The last statement is clear since any two norms on a finite dimensional vector spaces are equivalent. □

Notice that, since every $C \in \mathcal{M}^{x_1}$ is a smooth curve, we have shown that $\tau^{x_1}$ is a diffeomorphism.

3.2.3. *Lines meeting a small ball and order 2 estimates.* Given $x, x_1 \in M$, we consider the subset $\mathcal{M}^{x_1;B_t(x)} \subset \mathcal{M}^{x_1}$ whose element intersects $B_t(x)$ for small $t$. Denote in what follows the line through $x_1$ and $x$ by $C$.

The following lemma is an analogue to Lemmas 4.5 and 4.9 in [29].

**Lemma 3.10.** *Let $x$, $x_1$ and $C$ be as stated as above. There is a constant $k_{3.10} > 1$, depending only on $s$, ensuring the following inequalities for $\eta \in \ker_C^{x_1}$,*

1. $\sup_C |\eta| \leq k_{3.10}|\eta(z)|$ *if $x_1$ is not in $B_s(x)$ and $z \in \overline{B_{\frac{s}{2}}(x)}$;*
2. $\sup_C |\eta| \leq k_{3.10}|\tau_*^{x_1}\eta|$ *if $x_1 \in \overline{B_s(x)}$.*

*Proof.* The constants in Lemma 3.9 can be chosen to be independent of $x$, $x_1$, $z$ since $M$, $M \setminus B_s(x)$ and $\overline{B_s(x)}$ are compact. □

Let $T^{x_1;B_t(x)}$ denote the set of points $x_2$ in $M$ that lies in a curve in $\mathcal{M}^{x_1}$ and intersecting $B_t(x)$.

**Lemma 3.11.** *Suppose $x_1$ is not in $B_s(x)$. There are constants $k$ and $\kappa$ depending on $s$ with the following significance: For $t < \kappa^{-3}$, the volume of $T^{x_1;B_t(x)}$ is bounded from above by $kt^2$.*

*Proof.* Consider the unique curve $C$ through $x_1$ and $x$. Since $\text{dist}(x_1, x) \geq s$, by Lemma 3.9, $|\eta(x)|$ is a norm on the 2−dimensional vector space $\ker_C^{x_1}$. Since $\mathcal{M}^{x_1}$ is a 2−dimensional smooth manifold, apply the implicit function theorem as in Lemma 4.7 in [29], we find there exists $\kappa > 1$ with the following property: for $t < \kappa^{-3}$, there is an embedding $\lambda_C^{x_1}$ from the $|\eta(x)| \leq \kappa t$ disk in $\ker_C^{x_1}$ onto a neighborhood in $\mathcal{M}^{x_1}$ of $C$, which brings 0 to $C$, and contains $\mathcal{M}^{x_1;B_t(x)}$ as an open set in the image. Since $\mathcal{M}^{x_1}$ and $M \setminus B_s(x)$ are compact, $\kappa$ only depends on $s$. Moreover, the image of $\eta \in \ker_C^{x_1}$ via $\lambda_C^{x_1}$ can be written as $\exp_C(\eta+\phi_C^{x_1}(\eta))$ where $\phi_C^{x_1}(\cdot)$ maps the ball $\{\eta : |\eta(x)| \leq 2\kappa^{-1}\}$ smoothly into $\ker_C^{x_1}$, and the $C^0$ norm of $\phi_C^{x_1}(\eta)$ is bounded by a multiple of $|\eta(x)|^2$.

Suppose $x_2$ in $M$ lies in a curve $C'$ in $\mathcal{M}^{x_1}$ with $C' \cap B_t(x) \neq \emptyset$. Then there is a vector $\eta \in \ker_C^{x_1}$ with norm $|\eta(x)| \leq \kappa t$, and a point $p \in C$ such that $x_2 = \lambda_C^{x_1}(\eta(p))$.

Notice that $|\eta(p) + \phi_C^{x_1}(\eta)(p)| \leq |\eta(p)| + \kappa' |\eta(p)|^2$ by the estimate of $\phi_C^{x_1}$. Hence, by the third bullet in 2.3.3,

$$\mathrm{dist}(x_2, C) = \mathrm{dist}(\lambda_C^{x_1}(\eta(p)), C) \leq K|\eta(p)|.$$

Further, by the first bullet of Lemma 3.10,

$$\mathrm{dist}(x_2, C) \leq K|\eta(p)| \leq K \sup_C |\eta| \leq K\kappa_{3.10}|\eta(x)| \leq 2K\kappa_{3.10}\kappa t.$$

This implies that $x_2$ is constrained so as to lie in a tubular neighborhood of $C$ whose radius is bounded above by $k_2' t$ (with $k_2' = 2K\kappa_{3.10}\kappa$).

The area of $C$ is bounded by $k_0 H \cdot [\omega]$. So the volume of the points in $M$ that lie on a radius $t$ tubular neighborhood of any $C$ is bounded from above by $kt^2$. □

**Lemma 3.12.** *Suppose $x_1$ is in $\overline{B_s(x)} \setminus B_{Rt}(x)$. There is a constant $k$ depending on $s$ and $R$ with the following significance:*

*The volume of points $x_2$ in $M$ lying in a curve in $\mathcal{M}^{x_1}$ and intersecting $B_t(x)$ is bounded from above by $k\frac{t^2}{d^2}$, where $d = \mathrm{dist}(x_1, x)$.*

*Proof.* Introduce $\mathcal{O}_{\hat{\theta},\delta}^{x_1} \subset \mathcal{M}^{x_1}$ to denote the set of curves that mapped under $\tau^{x_1}$ to the disk in $\mathbb{CP}^1$ of radius $\delta$ centered on the point $\hat{\theta} \in \mathbb{CP}^1$.

The following geometric consideration is (4.14) in [29]. If $d > 10^4$ and $t < \frac{s}{d}$, then the set of complex 1 dimensional lines that intersects $B_t(x)$ with $x = (d, 0)$ is contained in a disk in $\mathbb{CP}^1$ of radius less than $c_0\frac{t}{d}$ with center the image of $(1, 0)$ of $\mathbb{CP}^1$.

With this understood, by Lemma 3.8, $\mathcal{M}^{x_1;B_t(x)}$ lies in $\mathcal{O}_{\hat{\theta},\delta}^{x_1}$ with $\hat{\theta}$ here denoting the image of the point $(1, 0)$ and with $\delta < c\frac{t}{d}$.

By Lemma 3.10(2), $\mathcal{O}_{\hat{\theta},\delta}^{x_1}$ is contained in a tube of radius $k\delta$, whose volume is bounded from above by $k\delta^2$. Thus we have the desired volume estimate. □

3.3. **Spherical Taubes current from the line class.** Let $\mathcal{M}_2$ be the moduli space of lines with two distinct points,

$$\mathcal{M}_2 = \{(C, x_1, x_2) | C \in \mathcal{M}, x_i \in C\} \subset \mathcal{M} \times M^{[2]}.$$

Use $\pi_2$ to denote the projection map $\mathcal{M}_2 \to M^{[2]}$. Let $\pi_{\mathcal{M}}$ be the projection map $\mathcal{M}_2 \to \mathcal{M}$.

The portion of marked moduli space we need is $\mathcal{M}_2^r$ for $0 < r < \frac{s}{10}$, subject to the constraint $d(x_1, x_2) \geq r$. Notice that the image $\pi_{\mathcal{M}}(\mathcal{M}_2^r)$ is $\mathcal{M}$ if $r$ is chosen that small.

For $\eta \in \pi_2(\mathcal{M}_2^r)$, we introduce the non-negative, closed, invariant current $\phi_\eta$. It is defined by $\phi_\eta(v) = \int_{C_\eta} v$, where $C_\eta$ is the unique line $\pi_2^{-1}(\eta)$ and $v$ is a 2-form on $M$. Then we have the following *spherical current* $\Phi_H$ in the line class given by

$$\Phi_H(v) = \int_{\eta \in \pi_2(\mathcal{M}_2^r)} \phi_\eta(v).$$

This current $\Phi_H(v)$ clearly satisfies Proposition 1.2 in [29]. Especially, it is a non-trivial, closed, non-negative $J-$invariant current on $\mathbb{CP}^2$. In the rest of the section, we will prove that it is indeed a Taubes current.

3.3.1. *Upper bound.* Fix a smooth, non-increasing function $\chi : [0, \infty) \to [0, 1]$ with value 1 on $[0, \frac{1}{4}]$ and value 0 on $[\frac{1}{2}, \infty)$. Use $\chi_t$ to denote the function $\chi(t^{-1}|\cdot|)$ on $\mathbb{C}^2$.

**Proposition 3.13.** *The current $\Phi_H$ satisfies the upper bound in* (11).

*Proof.* Let $s$ be as in Lemma 3.8. Fix $x \in M$ and adapted coordinates $(z, w)$ centered at $x$ with radius $s$. Let $t < 10^{-5}s$. As in [29], we only need to prove $\Phi_H(i\chi_t dz \wedge d\bar{z}) < kt^4$. Moreover, $\Phi_H(i\chi_t dz \wedge d\bar{z})$ is no greater than

$$k_0 \int_{\eta=(x_1,x_2)\in\pi_2(\mathcal{M}_2^r)} (\int_{C_\eta} \chi_t \omega).$$

Notice that $\int_{C_\eta} \chi_t\omega = 0$ if $C_\eta \cap B_t(x) = \emptyset$. If $C_\eta \cap B_t(x) \neq \emptyset$, then $C_\eta \cap B_t(x)$ is contained in a ball of radius of $2t$ centered at some point in $B_t(x)$. By Lemma 2.9, the integrand $\int_{C_\eta} \chi_t\omega$ is bounded by $H \cdot [\omega]kt^2$.

Thus it suffices to prove that the volume of the set

$$\{\eta = (x_1, x_2) \in \pi_2(\mathcal{M}_2^r) | C_\eta \cap B_t(x) \neq \emptyset\} \tag{12}$$

is $O(t^2)$.

We follow [29] to divide into three cases depending on the position of $x_1$.

I. The first case is that $x_1$ is away from $B_s(x)$. The upper bound for the choice of $x_1$ is $\mathrm{Vol}(M)$. Now we estimate the possible choices of $x_2$ for a fixed $x_1$. It follows from Lemma 3.11 that the volume of $x_2$ is $k_2t^2$. The factors $\mathrm{Vol}(M)$ and $k_2t^2$ multiply to an upper bound of $O(t^2)$ for the volume of the subset in (12) with $x_1 \in M \setminus B_s(x)$. This bound depends on $s$.

II. The second case is when $x_1$ is in $B_s(x)$ but outside $B_{Rt}(x)$, where $R = 10^5$.

Suppose $C_\eta$ intersects $B_t(x)$. Since $R = 10^5$, by Lemma 3.12, the volume of this set is bounded by $k_{3.12}\delta^2$, where $\delta = c \cdot \frac{t}{\mathrm{dist}(x_1,x)}$ and $k_{3.12}$ is the constant appeared in Lemma 3.12.

Using polar coordinates, the volume of this part of (12) is bounded from above by

$$k_3 t^2 \int_{z\in B_s(x)} \frac{1}{\mathrm{dist}(z,x)^2} dz \le k_2 s^2 t^2.$$

III. The last case is when $x_1$ is in $B_{Rt}(x)$, this element itself would have the freedom of $\mathrm{Vol}(B_{Rt}(x))$ which is $O(t^4)$. In this case, choices of $x_1$ and $x_2$ would multiply to contribute as the rate of $O(t^4)$.

Summing the three cases, we finish the proof. □

3.3.2. *Lower bound.*

**Proposition 3.14.** *The current $\Phi_H$ satisfies the lower bound in* (11).

*Proof.* Let $s$ be as in Lemma 3.8, and $t < 10^{-5}s$. Fix $x \in M$ and adapted coordinates $(z, w)$ centered at $x$ with radius $s$. As in [29], we only need to prove $\Phi_H(i\chi_t dz \wedge d\bar{z}) > k^{-1}t^4$.

The main picture to have is Lemma 2.10, applied to $B_s(x)$. Namely, inside $B_s(x)$, the curves behave as straight lines with respect to the adapted coordinates.

Fix $\epsilon > 0$. Recall that $\mathcal{M}^x$ is a pencil, and identified with $\mathbb{CP}^1$ via $\tau_x$. Let us begin with choosing a disk $\mathcal{C}_x \subset \mathcal{M}^x$ corresponding to a disk centered at $(1,0) \in \mathbb{CP}^1$. The latter disk is chosen so that $|dz(\cdot)| \ge 2\epsilon$. By Lemma 2.10, by shrinking $s$ if necessary, the restriction of $|dz|$ to $C \cap B_s(x)$ is greater than $\epsilon$ for $C \in \mathcal{C}_x$. Such an $s$ can be chosen to be independent of $x$.

As we are estimating the lower bound, so we restrict our attention to $\eta = (x_1, x_2)$ such that $x_1$ is away from $B_{\frac{s}{2}}(x)$ and inside $B_s(x)$, and the line $C_{x,x_1}$ determined by $x$ and $x_1$ is contained in the disk $\mathcal{C}_x \subset \mathcal{M}$ specified above. By Lemma 2.10, the choices of $x_1$ constitute a compact set with volume $c_\epsilon s^4$.

Now, fix such an $x_1$. Consider the set of $x_2$ in $B_{\frac{s}{4}}(x)$, for which $\eta = (x_1, x_2)$ contributes to $\Phi_H(if_{B_t(x)} dz \wedge d\bar{z})$, namely, $C_{x_1,x_2}$ intersects $B_t(x)$. Since we are estimating lower bound, we apply Lemma 2.10 to count the ones intersecting $B_{\frac{t}{2}}(x)$. Since $\mathrm{dist}(x_1, x) \ge \frac{s}{2}$, $C_{x_1,x_2}$ intersects $B_{\frac{t}{2}}(x)$ as long as $x_2 \in B_{\frac{s}{4}}(x)$ and $\mathrm{dist}(x_2, C_{x_1,x}) \le \frac{1}{4}\frac{t}{2}$. Thus the volume of $x_2$ is bounded by the radius $\frac{t}{8}$ tube around $C_{x_1,x} \cap B_{\frac{s}{4}}(x)$. By Lemma 2.9, or Lemma 2.10, the area of $C_{x_1,x} \cap B_{\frac{s}{4}}(x)$ is bounded from below by $a_x(\frac{s}{4})s^2$. Hence the volume of $x_2$ is bounded by $k_1 s^2 t^2$. Notice that, again by Lemma 2.10, when $t \le 10^{-10}s$, the rational curve $C_{x_1,x_2} = \pi_2^{-1}(\eta)$ has the property that the restriction of $|dz|$ to $C \cap B_s(x)$ is greater than $\frac{\epsilon}{2}$. Here $C$ denotes the line $C_{x_1,x_2}$.

By virtue of our choices, $\int_C (if_{B_t(x)} dz \wedge d\bar{z})$ is bounded below by $(\frac{\epsilon}{2})^2 \cdot a_x(t)$. Thanks to Lemma 2.9, $a_x(t)$ is bounded below as $k_{2.9}^{-1}t^2$ (here $k_{2.9}$ is the constant appeared in Lemma 2.9). Given the aforementioned lower bound for $|dz|$ , and given what is said in Lemma 2.9, it follows that the integral of

$if_{B_t(x)}dz\wedge d\bar z$ over $C$ must be greater than $k_2^{-1}t^2$ by choosing $t(<< s)$ small enough. More precisely, this $k_2^{-1}$ could be chosen as $(\frac{\epsilon}{2})^2 k_{2.9}^{-1}$.

Multiplying these three factors: $c = c_\epsilon s^4$, $k_1 s^2 t^2$ and $k_2^{-1}t^2$ together, we get $k^{-1}t^4 < \Phi_H(f_{B_t(x)} i\sigma\wedge\bar\sigma)$ for some $k$. The constant $k$ is further independent of $x$ and $t$:

- The first constant $c_\epsilon$ is a universal constant since $M$ is compact;
- The second constant $k_1$, as in the argument, depends on the choices of $x_1$ (and $x$). But our $x_1$ is chosen from a compact set, so $k_1$ is universal as well;
- The last constant $k_2$ depends on $x_2$ (and $x_1$, $x$), and the tubular neighborhood of $C_{x,x_1}$ we have chosen is compact, so $k_2$ is universal.

□

We note that the following theorem due to Gromov [9] and Taubes [28] is an immediate consequence of Propositions 3.13, 3.14 and 2.12.

**Theorem 3.15.** *Any tamed almost complex structure on $\mathbb{CP}^2$ is almost Kähler.*

## 4. Pencils in a big $J-$nef class

Let $M$ be a rational manifold and $e$ a class in $S^+_{K_J}$. By Proposition 3.1, $\mathcal{M}_e$ is non-empty. But if $e$ is not the line class on the projective space, then $\mathcal{M}_e$ always contains reducible subvarieties, and in fact, it could entirely consist of reducible varieties. To guarantee that there are smooth rational curves, we need to restrict to $J-$nef classes in $S^+_{K_J}$ (see Remark 5.11).

### 4.1. $J-$nef classes.

Let $J$ be a fixed almost complex structure on $M$.

**Definition 4.1.** *A homology class $e\in H_2(M;\mathbb{Z})$ is said to be $J-$nef if it pairs non-negatively with any $J-$effective class.*

#### 4.1.1. *Big $J-$nef and $J-$ample classes.*

**Definition 4.2.** *A $J-$nef class $e$ is said to be big if $e\cdot e$ is positive.*

*The vanishing locus $Z(e)$ of a big $J-$nef class $e$ is the union of irreducible subvarieties $D_i$ such that $e\cdot e_{D_i}=0$. Denote the complement of the vanishing locus of $e$ by $M(e)$.*

*A big $J-$nef class $e$ is said to be $J-$ample if $M(e)=M$.*

The following lemma immediately follows from the positivity of intersections of distinct irreducible subvarieties.

**Lemma 4.3.** *If $e$ is represented by an irreducible $J-$holomorphic subvariety and $e\cdot e>0$, then $e$ is a big $J-$nef class.*

4.1.2. *List of main results.* We now list main properties of $J-$nef classes in $S_{K_J}$. The first one is Theorem 2.14, proved in [19]. As mentioned in the introduction, for rational manifolds, it plays a fundamental role to remove a number of the genericity assumptions in [29].

Together with Lemma 3.4, we have the following important consequence, also established in [19].

**Proposition 4.4.** *Suppose $e \in S_{K_J}$ is a $J-$nef class. If $\Theta = \{(C_i, m_i), 1 \leq i \leq n\} \in \mathcal{M}_{red,e}$ is connected, then*

$$\sum_{(C_i,m_i)\in\Theta} m_i l_{e_{C_i}} \leq l-1. \tag{13}$$

*In particular,*

$$\sum_{(C_i,m_i)\in\Theta} l_{e_{C_i}} \leq l-1. \tag{14}$$

This is an analogue of Proposition 3.4 in [29], but valid for an arbitrary tamed almost complex structure. It follows that, similar to Proposition 1.1 in [29], $J-$nef classes have the following property:

**Proposition 4.5.** *Suppose $e$ is a $J-$nef class in $S_{K_J}$ with $e\cdot e \geq 0$. The map $\pi_l : \mathcal{M}_{irr,e,l} \to M^{[l]}$ is onto the complement of a compact, measure zero subset. In particular, $e$ is represented by a smooth rational curve.*

Fix an orthonormal frame for $T_{1,0}M|_x$ to identify the space of complex $1-$dimensional subspaces with $\mathbb{CP}^1$. Consider the map

$$\tau^{x,\Omega} : \mathcal{M}_e^{x,\Omega} \to \mathbb{CP}^1, \quad C \mapsto T_xC.$$

The next result is very useful to understand this map.

**Proposition 4.6.** *For any $J-$nef class $e$ in $S^+_{K_J}$, we can choose a $J-$nef class $H_e$ in $S^{\geq 0}_{K_J}$ such that $H_e \cdot e = 1$ or $2$, and $H_e\cdot e = 2$ only if $H_e$ is proportional to $e$.*

For any $\Theta = \{(C_1, m_1), \cdots, (C_n, m_n)\} \in \mathcal{M}^x_{red,e}$ with $x \in C_1$, given $\Omega \in M^{[l-2]}$, let $w_i$ be the cardinality of $\Omega_i = \Omega \cap C_i$.

**Definition 4.7.** *Fix a point $x \in M(e)$. $\Omega \in M^{[l-2]}$ is called pretty generic with respect to $e$ and $x$ if*

- *$x$ is distinct from any entry of $\Omega$;*

*For each $\Theta = \{(C_1, m_1), \cdots, (C_n, m_n)\} \in \mathcal{M}^x_{red,e}$ with $x \in C_1$;*

- *$x$ is not in $C_i$ for any $i \geq 2$;*
- *$\Omega_i \cap \Omega_j = \emptyset$;*
- *$1 + w_1 \geq m_1 e\cdot e_1 (\geq l_{e_1})$, and $w_i \geq m_i e \cdot e_i (\geq l_{e_i})$ for $i \geq 2$.*

*Let $G^x_e$ be the set of pretty generic $l-2$ tuples with respect to $e$ and $x$.*

**Proposition 4.8.** *Suppose $e$ is a big $J-$nef class in $S_{K_J}$ and $x \in M(e)$. Then the complement of $G^x_e$ has complex codimension at least one in $M^{[l-2]}$.*

Further, big $J-$nef classes have the following property. We would like to show that if we fix $x$ and choose $\Omega$ pretty generic, we always have smooth rational curves passing through $x$ and $\Omega$. Moreover, a generic complex direction in $T_xM$ would be tangent to these curves at $x$.

**Proposition 4.9.** *Suppose $e$ is big $J-$nef. For $x \in M(e)$ and $\Omega \in G_e^x$, The map $\tau^{x,\Omega}$ is well defined and in fact it is a homeomorphism.*

*Moreover, $\tau^{x,\Omega}$ is a diffeomorphism away from the reducible curves.*

Hence, for every complex direction in $T_xM = \mathbb{C}^2$, there is a (possibly reducible) rational curve tangent to it and passing through $\Omega$ and $x$. Moreover, except only finitely many directions, this rational curve is smooth.

Propositions 4.5, 4.6, 4.8, 4.9 will be proved in the next four subsections respectively.

4.2. **Existence of smooth curves.** In this subsection we prove Proposition 4.5. For this purpose, we need to estimate the dimension of the space of reducible curves.

4.2.1. *Dimension of the moduli space of reducible curves.* For

$$\Theta = \{(C_1, m_1), \cdots, (C_n, m_n)\} \in \mathcal{M}_e,$$

let $\Xi_\Theta = (e_{C_1}, \cdots, e_{C_n})$. Set $\chi_e = \{\Xi_\Theta | \Theta \in \mathcal{M}_{red,e}\}$. Proposition 2.4 guarantees that $\chi_e$ is a finite set. Given $\Xi \in \chi_e$, the subspace of reducible curves $\mathcal{M}_\Xi \subset \mathcal{M}_{e,red}$ corresponding to $\Xi$ is naturally identified with $\times_{e' \in \Xi} \mathcal{M}_{e',irr}$.

**Proposition 4.10.** *Suppose $e$ is a $J-$nef class in $S_{K_J}$. Then $\mathcal{M}_{red,e} = \cup_{\Xi \in \chi_e} \mathcal{M}_\Xi$ is a finite union of manifolds, each with complex dimension at most $l-1$.*

*Proof.* By Lemma 3.4 and Theorem 2.14, for each $\Xi \in \chi_e$, $\mathcal{M}_\Xi$ is a manifold of complex dimension $\sum_{(C,m)\in\Theta} l_{e_C}$. Thus the assertion follows from Proposition 4.4. □

**Remark 4.11.** *In* [19] *we completely determine the possible configuration with $l = 1 + \sum_{i=1}^n l_{e_i}$.*

*$l = 1 + \sum_{i=1}^n l_{e_i}$ if and only if each multiplicity is 1, and*

*$\Theta$ is one of the following configurations:*

- *If $\Theta_-$ is empty then $n = 2$, $e_1 \cdot e_2 = 1$, $e_i \cdot e_i \geq 0$.*
- *If $\Theta_-$ is not empty and there is no $-1$ curves, then $\Theta_-$ consists of a unique element $(C_1, 1)$ with $e_1 \cdot e_1 = 1 - n \leq -2$, and $\Theta_+$ consists of at least $n - 1 \geq 2$ elements, $e_i = \cdots = e_n$ and $e_i \cdot e_i = 0$ for $i \geq 2$. Moreover, $e_1 \cdot e_2 = 1$. In short, it is a comb like configuration.*
- *If $\Theta_-$ contains a $-1$ curve, then either $\Theta$ is a successive infinitely near blow-up of a smooth rational curve with non-negative self-intersection, Equivalently, it means that, starting from the second blow-up, we only blow up at a point in a component with negative self-intersection.*
- *or a successive infinitely near blow-up of a comb like configuration at points in $C_1$. Here infinitely near blow up means that all the blow ups, from*

*the second one on, occur at some point not lying in the proper transform of the original configuration. Equivalently, it means that we successively blow up in the union of components of negative self-intersection.*

4.2.2. *Proof of Proposition 4.5.* .

*Proof.* It follows from Proposition 3.3 that $\pi_{irr,l}$ is surjective. On the other hand, by Proposition 4.10, the image of $\pi_{red,l}$ is of codimension at least 2. The assertion follows. □

4.3. **Intersection properties.** In this subsection we establish Proposition 4.6.

4.3.1. *Intersection properties of $K_J-$spherical classes.*

**Lemma 4.12.** *The following intersection properties hold:*

- $S^+_{K_J}$ *pairs positively with* $S^{\geq 0}_{K_J}$.
- $S^+_{K_J}$ *pairs non-negatively with* $\mathcal{E}_{K_J}$.
- $\mathcal{E}_{K_J}$ *pairs non-negatively with* $\mathcal{E}_{K_J}$.

*Proof.* Since $b^+ = 1$, and $S^{\geq 0}_{K_J}$ pairs positively with any $J-$tamed symplectic form $\omega$, $S^+_{K_J}$ pairs positively with $S^{\geq 0}_{K_J}$ by light cone lemma.

Any element in $\mathcal{E}_{K_J}$ is $J'-$effective for any tamed $J'$ with $K_{J'} = K_J$ by Proposition 3.1. For a generic tamed $J'$, $e \in S^+_{K_J}$ could be represented by smooth irreducible $J-$holomorphic curves. The second statement then follows from the positivity of intersections of distinct irreducible subvarieties.

The last item follows because any two elements $e_1$ and $e_2$ in $\mathcal{E}_{K_J}$ have irreducible representations for a generic tamed $J'$ with $K_{J'} = K_J$. □

**Lemma 4.13.** *For any $e$ in $S^+_{K_J}$, we can choose a class $H_e$ in $S^{\geq 0}_{K_J}$ such that $H_e \cdot e = 1$ or 2, and $H_e \cdot e = 2$ only if $H_e$ is proportional to $e$.*

*Proof.* A rational manifold is either $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ or $S^2 \times S^2$.

• $S^2 \times S^2$

Denote the two factors classes by $H_1$ and $H_2$ such that $K_J = -2H_1 - 2H_2$. For the class $aH_1 + bH_2$ of an irreducible curve, the adjunction formula implies

$$\begin{aligned} & (aH_1 + bH_2)^2 + (-2H_1 - 2H_2)(aH_1 + bH_2) + 2 \\ = \; & 2ab - 2b - 2a + 2 = 2(1-a)(1-b) \geq 0. \end{aligned} \tag{15}$$

is clear that $S_{K_J}$ is contained in the following two sequences of classes:

$$A_l = H_1 + lH_2, \quad B_l = lH_1 + H_2, \quad l \in \mathbb{Z}.$$

The sequence $A_l$ is in $S^+_{K_J}$ if $l > 0$, and $A_l \cdot A_l = 0$ if $l = 0$. The same is true for the sequence $B_l$.

For $e = A_l$ with $l > 0$, choose $H_e = B_0 = H_2$, and for $e = B_l$ with $l > 0$ choose $H_e = A_0 = H_1$.

For $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ there exists a basis of spherical classes $H, E_1, ..., E_k$ with $H \cdot H = 1, E_i \cdot E_i = -1$ such that $K_J = -3H + E_1 + \cdots + E_k$.

• $\mathbb{CP}^2$

$H$ and $2H$ are the only classes in $S_{K_J}$.

• $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$

For $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ and class $\alpha H+\beta E$, the adjunction formula is of the form

$$(\alpha-1)(\alpha-2)-\beta(\beta+1)\geq 0 \tag{16}$$

$S_{K_J}$ is a subset of the following sequence

$$D_s=sH+(1-s)E,\quad s\in\mathbb{Z}.$$

Choose $H_e=H-E$.

• $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ with $k\geq 2$

When $k\geq 2$, it is hard to explicitly describe the classes in $S^{\geq 0}_{K_J}$. We invoke the classification up to Cremona equivalence (see [13]). As noted in [13], any class is in $S^{\geq 0}_{K_J}$ is Cremona equivalent to one of the following classes

(1) $2H$, $H$,
(2) $(n+1)H-nE_1, n\geq 1$,
(3) $(n+1)H-nE_1-E_2, n\geq 1$.

Case (1). If $e$ is equivalent to $2H$ or $H$, we choose $H_e=\frac{e}{\sqrt{e\cdot e}}$. The class $H_e$ is $J-$nef since it is proportional to the $J-$nef class $e$.

Case (2). If $e$ is equivalent to $(n+1)H-nE_1, n\geq 1$, we choose $H_e$ to be $H-E_1$ under the same equivalence. Notice that $H-E_1\in S^0_{K_J}$ and $H_e\cdot e=1$.

Case (3). When the class $e$ is equivalent to $2H-E_1-E_2$, then we could also choose $H_e=e$.

Case (4) If $e$ is equivalent to $(n+1)H-nE_1-E_2, n\geq 2$, we again choose $H_e$ to be $H-E_1$ under the same equivalence. Notice that $H_e\cdot e=1$. □

4.3.2. *$J-$nef classes on $S^2-$bundles over $S^2$.* For $S^2\times S^2$, the negative self-intersection classes must be of the form $aH_1+bH_2$ with $ab<0$.

It follows from (15) that class of any irreducible curve satisfies $(1-a)(1-b)\geq 0$. Thus the only possible negative square irreducible $J-$curves are in the classes $A_p$ with $p<0$ or $B_p$ with $p<0$.

Moreover, given $J$, there is at most one such curve by the positivity of intersections.

Case (i). There are irreducible $J-$curves with negative self-intersection.

Case (ii). $A_p$ is $J-$effective for some $p<0$.

Case (iii). $B_p$ is $J-$effective for some $p<0$.

The negative self-intersection classes must be of the form $aH+bE$ with $|a|<|b|$. Then the only possible negative square irreducible $J-$curves are in the classes

$$(1-s)H+sE_1,\quad s>0.$$

These classes are in $S_{K_J}$. Moreover, there is at most one such curve due to positivity of intersections.

In summary,

**Lemma 4.14.** *For $S^2 \times S^2$,*

*In case (i), both $A_l$ and $B_l$ are $J-$nef if $l \geq 0$, and $J-$ample if $l > 0$.*

*In case (ii), $A_l$ is $J-$nef when $l \geq -p$, and $J-$ample if $l \geq -p+1$. $B_0$ is $J-$nef.*

*In case (iii), $B_l$ is $J-$nef when $l \geq -p$, and ample if $l \geq -p+1$. $A_0$ is $J-$nef.*

*For $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, $H-E$ is $J-$nef for any $J$. If $sH+(1-s)E_1$ is $J-$effective for some $s \leq 0$, then $D_l$ is $J-$nef for $l \geq 1-s$, and $J-$ample for $l \geq 2-s$.*

Notice that for an $S^2-$bundle over $S^2$, there is always a $J-$ample class, and there is always a $J-$nef class with self-intersection 0.

4.3.3. *A criterion for $H-E_i$ to be $J-$nef.* For $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ we need the following observation.

**Lemma 4.15.** *Suppose there is an irreducible curve class pairing negatively with $H-E_1$, say $e_C = aH - b_1E_1 - ... - b_nE_n$. Then $a \leq 0$. The same is true for $H-E_i$.*

*Proof.* To see this, observe that $(H-E_1)\cdot e_C < 0$ means $a < b_1$. If $a > 0$ then $b_1 \geq 2$. Now the $K_J-$adjunction number

$$e_C\cdot e_C + K_J\cdot e_C \leq a^2 - b_1^2 - 3a + b_1 \leq (b_1-1)^2 - 3(b_1-1) - b_1^2 + b_1 = -4b_1 + 4$$

is less than $-2$, which is impossible. □

4.3.4. *Proof of Proposition 4.6.*

*Proof.* Suppose $e$ is $J-$nef. It suffices to show that we can further choose $H_e$ in Lemma 4.13 to be $J-$nef.

For $S^2\times S^2$, if $e = A_l$ with $l > 0$, $H_e = B_0 = H_2$ is $J-$nef. The second case is similar.

For $\mathbb{CP}^2$, $H$ and $2H$ are $J-$nef for any tamed $J$. In both cases, we choose $H_e$ to be $H$.

For $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, cases (1) and (3) are clear. For cases (2) and (4) we will show that $H_e = H - E_1$ is $J-$nef.

Suppose there is an irreducible curve class pairing negatively with $H-E_1$, say $e_C = aH - b_1E_1 - ... - b_nE_n$. By the lemma above, $a \leq 0$.

If $e = (n+1)H - nE_1, n \geq 1$, is $J-$nef, then $H\cdot e_C \geq -n(H-E_1)\cdot e_C > 0$. This implies that $a > 0$.

If $e = (n+1)H - nE_1 - E_2, n \geq 2$ is $J-$nef, then

$$(H-E_2)\cdot e_C \geq -n(H-E_1)\cdot e_C \geq n.$$

This means $a \geq b_2 + n$. Since $a \leq 0$, we have $b_2 < a \leq 0$. Thus the $K_J-$adjunction number

$$e_C\cdot e_C + K_J\cdot e_C \leq a^2 - b_2^2 - 3a + b_2 \leq -2n \leq -4,$$

which is impossible. □

4.4. **Vanishing locus and reducible curves.** Suppose $e$ is a big $J-$nef class in $S_{K_J}$, especially $l = e^2 + 1 \geq 2$. Recall the complement of the vanishing locus of $e$ is denoted by $M(e)$. Fix $x \in M(e)$.

4.4.1. *Reducible rational curves through a point in a big $J-$nef class.* Consider $\Theta = \{(C_1, m_1), \cdots, (C_n, m_n)\} \in \mathcal{M}^x_{red,e}$ and assume $x \in C_1$. Denote $e_{C_i}$ by $e_i$.

Set $\chi^x_e = \{\Xi_\Theta | \Theta \in \mathcal{M}^x_{red,e}\}$. Given $\Xi \in \chi^x_e$, the subspace of reducible curves $\mathcal{M}^x_\Xi \subset \mathcal{M}^x_{red,e}$ corresponding to $\Xi$ is naturally identified with $\mathcal{M}^x_{irr,e_1} \times \times_{i\geq 2} \mathcal{M}_{irr,e_i}$.

We estimate the dimension of $\mathcal{M}^x_{red,e}$.

**Proposition 4.16.** *Suppose $e$ is a big $J-$nef class in $S_{K_J}$. For any $x \in M(e)$, $\mathcal{M}^x_{red,e}$ is a union of manifolds with complex dimension at most $l-2$.*

*Proof.* Given $\Theta = \{(C_1, m_1), \cdots, (C_n, m_n)\} \in \mathcal{M}^x_{red,e}$, let $\Xi = \Xi_\Theta$.

By Lemma 3.4 and Theorem 2.14, $\mathcal{M}^x_\Xi$ has complex dimension

$$\dim_{\mathbb{C}} \mathcal{M}^x_{irr,e_1} + \sum_{i=2}^n l_{e_i}.$$

First suppose that $e_1 \cdot e_1 \geq 0$. By Lemma 3.4, $\mathcal{M}^x_{irr,e_1}$ has complex dimension $l_{e_1} - 1$. Thus we need to show

$$l_{e_1} - 1 + \sum_{i=2}^n l_{e_i} \leq l - 2,$$

which is the same as (14).

Now suppose that $e_1 \cdot e_1 < 0$. Then we need to show $\sum_{i=2}^n l_{e_i} \leq l - 2$.

We assume $l_{e_j} = 0$ when $j \leq k$, and $l_{e_j} > 0$ when $j \geq k+1$.

By Lemma 2.7 in [19], $\Theta$ is connected, so $e_j \cdot (e - m_j e_j) \geq 1$ for each $j \geq 2$. Therefore $l$ can be estimated as follows:

$$\begin{aligned}
l - 1 &= e \cdot e \\
&= (\textstyle\sum_{j=2}^n m_j e_j + m_1 e_1) \cdot e \\
&\geq \textstyle\sum_{j=k+1}^n (m_j e_j \cdot (m_j e_j + (e - m_j e_j)) + m_1 e_1 \cdot e \\
&= \textstyle\sum_{j=k+1}^n (m_j^2 e_j \cdot e_j + m_j e_j \cdot (e - m_j e_j)) + m_1 e_1 \cdot e \\
&\geq \textstyle\sum_{j=k+1}^n m_j l_{e_j} + m_1 e_1 \cdot e \\
&= \textstyle\sum_{j=1}^n m_j l_{e_j} + m_1 e_1 \cdot e.
\end{aligned} \tag{17}$$

Recall that we assume $x \in C_1$ and $x \in M(e)$, therefore $e \cdot e_1 > 0$. Hence in this case, we have also shown that $\mathcal{M}^x_\Xi$ has complex dimension at most $l-2$.

□

This can be viewed as a version of Lemma 4.7 in [29].

Now consider the subset $\mathcal{M}^x_{red,e}(x \text{ nodal})$ of $\mathcal{M}_{red,e}$ where $x \in C_1$ and $x \in C_i$ for some $i \geq 2$.

**Lemma 4.17.** *Suppose $e$ is a big $J-$nef class in $S_{K_J}$. For any $x \in M(e)$, $\mathcal{M}^x_{red,e}(x \text{ nodal})$ is a union of manifolds with complex dimension at most $l-3$.*

*Proof.* The proof is similar to that of Proposition 4.16.

Given $\Theta = \{(C_1, m_1), \cdots, (C_n, m_n)\} \in \mathcal{M}^x_{red,e}(x \text{ nodal})$, let $\Xi = \Xi_\Theta$. Assume without loss of generality that $x \in C_2$.

By Lemma 3.4 and Theorem 2.14, $\mathcal{M}^x_\Xi(x \in C_2)$ has complex dimension

$$\dim_{\mathbb{C}} \mathcal{M}^x_{irr,e_1} + \dim_{\mathbb{C}} \mathcal{M}^x_{irr,e_2} + \sum_{i=3}^n l_{e_i}.$$

We need to divide into four cases: I. $e_1 \cdot e_1 \geq 0, e_2 \cdot e_2 \geq 0$. II. $e_1 \cdot e_1 < 0, e_2 \cdot e_2 < 0$. III. $e_1 \cdot e_1 \geq 0, e_2 \cdot e_2 < 0$. IV. $e_1 \cdot e_1 < 0, e_2 \cdot e_2 \geq 0$.

In Case I,

$$\dim \mathcal{M}^x_\Xi(x \in C_2) = l_{e_1} - 1 + l_{e_2} - 1 + \sum_{i=3}^n l_{e_i} \leq l - 3$$

by Lemma 3.4 and (14).

In Case II, we also assume $l_{e_j} = 0$ when $j \leq k$, and $l_{e_j} > 0$ when $j \geq k+1$. By Lemma 2.7 in [19], $\Theta$ is connected, so $e_j \cdot (e - m_j e_j) \geq 1$ for each $j \geq 3$. Therefore $l$ can be estimated as follows:

$$\begin{aligned}
l - 1 &= e \cdot e \\
&= (\textstyle\sum_{j=3}^n m_j e_j + m_1 e_1 + m_2 e_2) \cdot e \\
&\geq \textstyle\sum_{j=k+1}^n (m_j e_j \cdot (m_j e_j + (e - m_j e_j)) + m_1 e_1 \cdot e + m_2 e_2 \cdot e \\
&= \textstyle\sum_{j=k+1}^n (m_j^2 e_j \cdot e_j + m_j e_j \cdot (e - m_j e_j)) + m_1 e_1 \cdot e + m_2 e_2 \cdot e \\
&\geq \textstyle\sum_{j=k+1}^n m_j l_{e_j} + m_1 e_1 \cdot e + m_2 e_2 \cdot e \\
&= \textstyle\sum_{j=1}^n m_j l_{e_j} + m_1 e_1 \cdot e + m_2 e_2 \cdot e.
\end{aligned}$$

Recall that we assume $x \in C_1 \cap C_2$ and $x \in M(e)$, therefore $e \cdot e_1 > 0$, $e \cdot e_2 > 0$. Hence in this case, we have also shown that $\mathcal{M}^x_\Xi$ has complex dimension at most $l - 3$.

Cases III and IV are similar, we only prove Case IV. In this case, we need to show $\sum_{i=2}^n l_{e_i} \leq l - 2$, which is exactly the second case of Proposition 4.16. □

4.4.2. *Pretty generic $l-2$ tuple.*

**Lemma 4.18.** *For any $x \in M(e)$ and $\Omega \in G^x_e$, any two elements in $\mathcal{M}^{x,\Omega}_e$ intersect only at the $l-1$ points $x, \Omega$. So it is like a pencil in algebraic geometry. Moreover,*

- *Given $z$ distinct from $x$ and $\Omega$, $\mathcal{M}^{z,x,\Omega}$ consists of unique curve.*
- *There is a unique curve in $e$ passing through the $l-1$ points $x, \Omega$ and a given direction at one of these points.*
- *$\mathcal{M}^{x,\Omega}_{red}$ is a finite set. Moreover, these reducible curves cannot be tangent to each others at $x$.*
- *We can define the map $\tau^{x,\Omega} : \mathcal{M}^{x,\Omega} \to \mathbb{CP}^1, \quad C \mapsto T_xC$.*

*Proof.* Suppose $\Theta, \Theta'$ are two elements in $\mathcal{M}^{z,x,\Omega}$. If $\Theta, \Theta'$ have no common components, then the the first bullet follows from local positivity of intersection. Hence we assume they have common components.

We rewrite two subvarieties $\Theta, \Theta' \in \mathcal{M}_e$, allowing $m_i = 0$ in the notation, such that they have the same set of irreducible components formally, i.e. $\Theta = \{(C_i, m_i)\}$ and $\Theta' = \{(C_i, m'_i)\}$. Then for each $C_i$, if $m_i \le m'_i$, we change the components to $(C_i, 0)$ and $(C_i, m'_i - m_i)$. Apply this process to all $i$ and discard finally all components with multiplicity 0 and denote them by $\Theta_0, \Theta'_0$ and still use $(C_i, m_i)$ and $(C_i, m'_i)$ to denote their components. Notice they are homologous, formally have homology class $e - \sum m_{k_i}[C_{k_i}] - \sum m'_{l_j}[C_{l_j}]$ with pairwise different $C_{k_i}$ and $C_{l_j}$. For each $C_{k_i}$, we know that there should be originally at least $m'_{k_i} e \cdot e_{k_i}$ points on it. Notice $\sum m_i e \cdot e_i = \sum m'_i e \cdot e_i = e^2 = l-1$, and none of $x, \Omega$ are nodal points of $\Theta$ and $\Theta'$. Hence either $e \cdot e_{k_i} = 0$, or $e \cdot e_{k_i} = 1$ and $m'_{k_i} - m_{k_i} = 1$. And the latter case could happen at most once. Similarly for $C'_{l_j}$. Therefore we know there are at least $e \cdot e_{\Theta_0}$ intersections (among $x, \Omega$) of $\Theta_0, \Theta'_0$.

Notice $e \cdot e_{\Theta_0} > e^2_{\Theta_0}$ since original $\Theta, \Theta'$ have at least one common component. Hence there are more intersections than the homology intersection number $e^2_{\Theta_0}$ of our new subvariety $\Theta_0$ and $\Theta'_0$. Then our claim follows from the local positivity of intersection.

The second bullet follows by the same argument.

For the third bullet, first notice $\chi^{x,\Omega}_{red,e}$ is a finite set by Proposition 2.4. Hence we could fix the type $\Xi = (e_i) \in \chi^{x,\Omega}_{red,e}$. Apply the same process, the removed components should have the same multiplicities since $l - 1 = \sum m_i e \cdot e_{C_i}$. And if $\Theta \neq \Theta_0$, the above argument claims contradiction. Hence, it implies there are no common components of $\Theta, \Theta'$. Especially, it implies there are no spheres of negative self-intersection as components. Hence, it is the situation that Corollary 2 in [27] applies, which says that $\mathcal{M}^{x,\Omega}_{red}$ is isolated in the compact space $\mathcal{M}^{x,\Omega}$. Hence $\mathcal{M}^{x,\Omega}_{red}$ is a finite set.

Since $x$ is not a nodal point of any $\Theta$, and thanks to the second bullet, the map $\tau^{x,\Omega}$ is well-defined. □

4.4.3. *Proof of Proposition 4.8.*

*Proof.* The complement of $G_e^x$ is the union of the four sets $V_i, i = 1, 2, 3, 4$: $V_i$ is the set of points in $M^{[l-2]}$ violating the $i-$th item of Definition 4.7, but not the previous items.

It is easy to see that $V_1$ has complex codimension 2.

To estimate the dimensions of $V_2, V_3, V_4$, consider the map

$$\pi^x_{red,l-2} : \mathcal{M}^x_{red,e,l-2} \to M^{[l-2]}.$$

We first deal with $V_2$. For each $\Xi = (e_i) \in \chi^x_e$, let $V_2(\Xi)$ be the image of the map $\pi^x_{red,l-2}$ restricted to $\mathcal{M}^x_{\Xi,l-2}(x \text{ nodal})$. Here $\mathcal{M}^x_{\Xi,l-2}(x \text{ nodal}) \subset \mathcal{M}^x_{\Xi,l-2}$ consists of $\Theta = \{(C_i, m_i)\} \times \Omega \in \mathcal{M}^x_\Xi \times M^{[l-2]}$ with $x \in C_1$, and $x \notin \Omega$.

Clearly, $V_2$ is the union of $V_2(\Xi)$ over $\Xi = (e_i) \in \chi^x_e$. By Lemma 4.17,

$$\dim_{\mathbb{C}} \mathcal{M}^x_{\Xi,l-2}(x \text{ nodal}) \le (l-3) + (l-2) = 2(l-2) - 1.$$

Thus $V_2$ has complex codimension at least 1 in $M^{[l-2]}$.

For the set $V_3$, it is similarly the union of $V_3(\Xi)$ over $\Xi = (e_i) \in \chi^x_e$, where $V_3(\Xi)$ is the image of the map $\pi^x_{red,l-2}$ restricted to $\mathcal{M}^x_{\Xi,l-2}(\Omega \text{ nodal})$. Here $\mathcal{M}^x_{\Xi,l-2}(\Omega \text{ nodal}) \subset \mathcal{M}^x_{\Xi,l-2}$ consists of $\Theta = \{(C_i, m_i)\} \times \Omega \in \mathcal{M}^x_\Xi \times M^{[l-2]}$ with $x \in C_1$, and

- $x$ is not in $C_i$ for any $i \ge 2$,
- there exists $i \ne j$ and $y \in M$ such that $y \in \Omega \cap C_i$ and $y \in \Omega \cap C_j$, i.e. $y \in \Omega$ is a nodal point of $\Theta$.

By Proposition 4.16, $\dim_{\mathbb{C}} \mathcal{M}^x_\Xi \le l-2$. Given $\{(C_i, m_i)\} \times \Omega \in \mathcal{M}^x_{\Xi,l-2}(\Omega \text{ nodal})$, there is a nodal point $y$ in $\Omega$. Observe that such a nodal point is in the intersection of $C_i$ and $C_j$, so it has 0 dimensional freedom. Hence

$$\dim_{\mathbb{C}} \mathcal{M}^x_{\Xi,l-2}(\Omega \text{ nodal}) \le (l-2) + (l-3) = 2(l-2) - 1.$$

Thus $V_3$ has complex codimension at least 1.

Finally, we deal with $V_4$. For each $\Xi = (e_i) \in \chi^x_e$, define $\mathcal{M}^x_{\Xi,l-2}(\Omega_i \text{ non-generic}) \subset \mathcal{M}^x_{\Xi,l-2}$ consisting of $\Theta = \{(C_i, m_i)\} \times \{\Omega_i\} \in \mathcal{M}^x_\Xi \times M^{[l-2]}$ with $x \in C_1$, and

- $x$ is not in $C_i$ for any $i \ge 2$,
- $\Omega_i \subset C_i$,
- $\Omega_i \in M^{[|\Omega_i|]}$, $\Omega_i \cap \Omega_j = \emptyset$, and $\sum |\Omega_i| = l - 2$,
- $1 + |\Omega_1| < l_{e_1}$ or $|\Omega_i| < l_{e_i}$ for $i \ge 2$.

Clearly, under $\pi^x_{red,l-2}$, the union of the image of $\mathcal{M}^x_{\Xi,l-2}(\Omega_i \text{ non-generic})$ over $\Xi = (e_i) \in \chi^x_e$ is the rest part of the complement of $G^x_e$.

By the estimate (17), $\dim_{\mathbb{C}} \mathcal{M}^x_\Xi \le l-2$ if $x \in M(e)$. If $\dim_{\mathbb{C}} \mathcal{M}^x_\Xi < l-2$, then already

$$\dim_{\mathbb{C}} \mathcal{M}^x_{\Xi,l-2} \le (l-3) + (l-2) = 2(l-2) - 1.$$

So we assume that $\dim_{\mathbb{C}} \mathcal{M}^x_\Xi = l-2$. In this case, $\sum_i l_{e_i} = l-1$. And for any $\{(C_i, m_i)\} \times \{\Omega_i\} \in \mathcal{M}^x_{\Xi,l-2}(\Omega_i \text{ non-generic})$, either $1 + |\Omega_1| < l_{e_1}$, or $|\Omega_j| < l_{e_j}$ for $j \ge 2$.

Observe that $\pi^x_{red,l-2}$ restricted to $\mathcal{M}^x_{\Xi,l-2}$ is of the form:

$$\times\pi_{e_i,|\Omega_i|}:\mathcal{M}^{x,\Omega_1}_{e_1}\times_{i\geq 2}\mathcal{M}^{\Omega_i}_{e_i}\to\times_i M^{[|\Omega_i|]}.$$

The source $\mathcal{M}^x_{\Xi,l-2}$ has total complex dimension $2(l-2)$. But when some $|\Omega_i|<l_{e_i}$ (or $1+|\Omega_1|<l_{e_1}$), $\pi_{e_i,|\Omega_i|}$ drops dimension since

$$\dim_{\mathbb{C}}\mathcal{M}_{e_i,l_{e_i}-p}=l_{e_i}+l_{e_i}-p>\dim_{\mathbb{C}}M^{l_{e_i}-p}=2(l_{e_i}-p).$$

□

4.5. **Abundance of pencils.** In this subsection we establish Proposition 4.9.

4.5.1. $\mathcal{M}^{x,\Omega}_e$ *is homeomorphic to* $S^2$.

**Proposition 4.19.** *Let $e$ be a big $J-$nef class in $S_{K_J}$. Fix a point $x\in M(e)$ and choose $\Omega\in G^x_e$. Then $\mathcal{M}^{x,\Omega}_e$ is homeomorphic to $S^2$.*

*Proof.* By Lemma 4.6, there is another $J-$nef class $H_e$ in $S_{K_J}$ such that $H_e\cdot e=1$ or 2. We prove that $\mathcal{M}^{x,\Omega}_e$ is homeomorphic to a smooth representative of $H_e$.

Let us first assume that $H_e\cdot e=1$.

By the first item of Proposition 4.5, we can choose a smooth rational curve $S$ representative of $H_e$ such that it does not pass through any entry of $\Omega$ and $x$. This is possible since $H_e$ is $J-$nef and the space of reducible $H_e-$curves is of codimension at least 1 by Proposition 4.10. Moreover, the space of irreducible $H_e-$curves containing $x$ or any entry of $\Omega$ is of codimension 1 by Proposition 3.4.

Given any $z\in S$, $z$ is distinct from $x$ or any entry of $\Omega$. By the first bullet of Lemma 4.18, there is a unique (although possibly reducible) rational curve $C_{x,z,\Omega}$ in class $e$ passing through $x$, $z$ and $\Omega$. Thus we obtain a map $h:z\mapsto C_{x,z,\Omega}$ from $S$ to $\mathcal{M}^{x,\Omega}_e$.

The map $h$ is surjective since $H_e\cdot e\neq 0$. Since $S$ is also $J-$holomorphic and $H_e\cdot e=1$ any curve in $\mathcal{M}^{x,\Omega}_e$ intersects with $S$ at a unique point by the positivity of intersection. Therefore $h$ is also one-to-one.

Now let us show that $h$ is a homeomorphism, namely both $h$ and $h^{-1}$ are continuous. Since $S=S^2$ is Hausdorff and $\mathcal{M}^{x,\Omega}_e$ is compact, if we can show that $h^{-1}:\mathcal{M}^{x,\Omega}_e\to S$ is continuous, it follows that $h$ is also continuous. To show $h^{-1}$ is continuous, consider a sequence $C_i\in\mathcal{M}^{x,\Omega}_e$ approaching to its Gromov-Hausdorff limit $C$. Let the intersection of $C_i$ (resp. $C$) with $S$ be $p_i$ (resp. $p$). Then $p_i$ has to approach $p$ by the first item of the definition of topology on $\mathcal{M}_e$. Therefore $h$ is a homeomorphism.

The case that $H_e\cdot e=2$ is similar. We choose the smooth rational curve $S$ representative of $H_e$ such that it passes $x$ but not any entry of $\Omega$. This is achieved by Proposition 4.16 and Proposition 3.4 applied to the $J-$nef class $H_e$. Here we also need to use the fact that in this case $H_e$ is proportional to $e$ and hence $x$ is also in $M(H_e)$.

Then we vary $z$ in $S$. If $z \neq x$, we choose the rational curve $C_{x,z,\Omega}$ in class $e$ passing through $x$, $z$ and $\Omega$. If $z = x$, we choose the rational curve $C_{x,x,\Omega}$ in class $e$ passing through $x$, $\Omega$ and tangent to $S$ at $x$. The sphere $C_{x,z,\Omega} \in \mathcal{M}_e^{x,\Omega}$ is unique by the second bullet of Lemma 4.18. We thus again obtain a map $h : z \mapsto C_{x,z,\Omega}$ from $S^2$ to $\mathcal{M}_e^{x,\Omega}$. This map is clearly surjective. Since $S$ is $J-$holomorphic with $x \in S$ and $H_e \cdot e = 2$, any curve in $\mathcal{M}_e^{x,\Omega}$ either intersects with $S$ at a unique point other than $x$ or is tangent to $S$ at $x$ by the positivity of intersection. Therefore $h$ is also one-to-one. Now we show that this map is a homeomorphism. As before, we only need to show that $h^{-1} : \mathcal{M}_e^{x,\Omega} \to S$ is continuous. Again, consider a sequence $C_i \in \mathcal{M}_e^{x,\Omega}$ approaching to its Gromov-Hausdorff limit $C$. Let the intersection of $C_i$ (resp. $C$) with $S$ be $p_i$ and $x$ (resp. $p$ and $x$). If $C_i$ (or $C$) tangent to $S$, let $p_i$ (or $p$) be $x$. Then $p_i$ has to approach $p$ by the first item of the definition of topology on $\mathcal{M}_e$. Therefore $h$ is a homeomorphism. □

4.5.2. *Proof of Proposition 4.9.*

*Proof.* Fix an orthonormal frame for $T_{1,0}M|_x$ to identify the space of complex $1-$dimensional subspaces with $\mathbb{CP}^1$. Consider the map

$$\tau^{x,\Omega} : \mathcal{M}_e^{x,\Omega} \to \mathbb{CP}^1, \quad C \mapsto T_xC.$$

By Proposition 4.19, $\tau^{x,\Omega}$ is a map from $S^2$ to $S^2$. An injective continuous map from $S^2$ to $S^2$ has to be a homeomorphism.

By the second assertion of Lemma 3.5, these curves cannot be tangent to each other at $x$ if one of them is irreducible. Moreover, by the third bullet of Lemma 4.18 there are finitely many reducible curves in this family. In fact, we have shown that $|\mathcal{M}_{red}^{x,\Omega}|$ is bounded by $|\chi_{red,e}|$, which only depends on $e$ and $J$. We also know that these reducible curves cannot be tangent to each others. To summarize, the map is injective. □

## 5. Spherical Taubes currents from big $J-$nef classes

### 5.1. **Weak Taubes currents.**

We begin with introducing the notion of spherical current from a big $J-$nef class.

5.1.1. *Spherical currents.* We continue to assume $J$ is a tamed almost complex structure. Suppose $e$ is a big $J-$nef sphere class in $S_{K_J}$. Then $l_e = \iota_e \geq 2$. Fix $x \in M$. Let us first choose a constant $r_0$. Let $\mathcal{A}$ be the measure zero set of non-pretty-generic points with respect to $x$ when $x \in M$ if $M(e) = M$ or $x$ belongs to a compact subset $K \subset M(e)$ if $M(e) \subsetneq M$. Choose a small open neighborhood $\mathcal{OB}(\mathcal{A})$ with $\mathrm{Vol}(\mathcal{OB}(\mathcal{A})) < 10^{-5l}\mathrm{Vol}(M)$. By Proposition 4.9 and the third item of Lemma 4.18, $\mathcal{M}_{red}^{x,\Omega}$ are finite points in $\mathcal{M}^{x,\Omega} = S^2$ if $\Omega \in G_e^x$. Then we choose $r_0$ small enough such that $\mathcal{M}_{irr}^{x,\Omega} \cap \mathcal{M}_{irr}^{r_0} \neq \emptyset$ if $\Omega$ is chosen from the complement of $\mathcal{OB}(\mathcal{A})$.

Lemma 3.8 is still valid in this situation with $\mathcal{M}$ replaced by $\mathcal{M}_{irr}^{r_0}$, which is Lemma 4.12 in [29]. We choose the constant $k_{r_0}$ as in Lemma 4.12 of [29] (or $s'_{r_0} = k_{r_0}^{-1}$ as in Lemma 3.8) and $s = k_{r_0}^{-4}$. Let $B_s(x)$ be the ball of radius $s$ centered at $x$.

We define a current $\Phi_e$ in the following manner. Recall

$$\mathcal{M}_{irr,l} = \{(C, x_1, \cdots, x_l) | C \in \mathcal{M}_{irr}, x_i \in M\} \subset \mathcal{M} \times M^{[l]}.$$

Use $\pi_l$ to denote the projection map $\mathcal{M}_{irr,l} \to M^{[l]}$. The portion of marked moduli space we choose is $\mathcal{M}_{irr,l}^{r_0,r}$, consisting of the set of marked curves with distance at least $r_0$ to $\mathcal{M}_{red}$ and $d(x_i, x_j) \geq r$ for any $i \neq j$. Here we suppose $r < \frac{s}{10}$.

We first define $\phi_\eta(v) = \int_C v$. Here $\eta \in \pi_l(\mathcal{M}_{irr,l}^{r_0,r})$, $C$ is the unique rational curve in $\pi_l^{-1}(\eta)$ and $v$ is a 2-form on $M$. Then we have the following *spherical current*

$$\Phi_e(v) = \int_{\eta \in \pi_l(\mathcal{M}_{irr,l}^{r_0,r})} \phi_\eta(v).$$

The spherical current $\Phi_e$ defined clearly satisfies Proposition 1.2 in [29]. Especially, it is a non-trivial, closed, non-negative $J-$invariant current on $M$.

5.1.2. *Estimates of the pencil $\mathcal{M}^{x_1,\Omega}$.* Fix $x_1$ and $\Omega \in G_e^{x_1}$. We write $\Omega = (x_3, \cdots, x_l)$. By Proposition 4.9, $\mathcal{M}^{x_1,\Omega}$ is a pencil. Moreover, by removing an open neighborhood of these finite directions corresponding to reducible curves, we can suppose the remaining directions correspond to the the curves with distance at least $r_0$ from $\mathcal{M}_{red}$.

We now assume $x_1$ and any entry of $\Omega$ are chosen from $M \setminus B_s(x)$ (but $\Omega$ not necessarily belongs to $G_e^x$). When the curve $C_{x,x_1,\Omega}$ is in $\mathcal{M}^{x_1,\Omega,r_0}$, with $x_1$ and $\Omega$ chosen from the compact set above (i.e. $d(x_i, x_j) \geq r$, $d(x_i, x) \geq s$ and $C_{x,x_1,\Omega} \in \mathcal{M}^{x_1,\Omega,r_0}$), there is a number $T > 0$ such that for any $z \in \overline{B_T(x)}$, the sphere $C_{z,x_1,\Omega}$ is smooth and in $\mathcal{M}_{irr}^{\frac{r_0}{2}}$. In other words, the part of $\mathcal{M}^{x_1,\Omega}$ intersecting $\overline{B_T(x)}$ is a pencil of smooth curves. Clearly, this is also true for any $t \leq T$. Let us denote this set by $\mathcal{M}^{x_1,\Omega;B_t(x)}$. Notice the first defining condition of $\mathcal{M}_{irr,l}^{r_0,r}$ guarantees $\text{dist}(x, M(e)) \geq r_0$ if $M(e) \neq \emptyset$.

Given a smooth curve $C$ in this pencil with normal bundle $N$, the tangent space to $\mathcal{M}_{irr}^{x_1,\Omega}$ at $C$ can be identified with the vector space $\ker_{C,x_1,\Omega} \subset \Gamma(N)$ that consists of the sections in the kernel of $D_C$ that vanish at $x_1$ and $\Omega$.

On this two dimensional space, there are several norms. Let $\nu \in \ker_{C,x_1,\Omega}$.

- The $L^2$ norm $||\nu||_2$;
- The sup norm $\sup_C |\nu|$;
- For $z \neq x_1$ or any entry of $\Omega$, the pointwise norm $|\nu(z)|$;
- By choosing $x_1$, $\Omega$ as above, we could still define $\tau^{x_1,\Omega}$ or $\tau^{\Omega,x_1}$ by taking the complex direction $T_{x_1}C$ (or $T_{x_3}C$). Let $u$ denote the direction $T_{x_1}C$ in $\mathbb{CP}^1$, $\tau_*^{x_1,\Omega} : T_C\mathcal{M}^{x_1,\Omega} \to T_u\mathbb{CP}^1$ is an isomorphism. We thus could speak of the pointwise norm of $\tau_*^{x_1,\Omega}\nu$ as a vector in $T_u\mathbb{CP}^1$.

For a fixed curve $C$, these norms are equivalent. Since $\mathcal{M}^{r_0,r}_{irr,l}$ is compact, if we have compact families of choices of $x_1, \Omega, z$, we have uniform constants as in Lemma 3.10.

**Lemma 5.1.** *Let $x$, $x_1$, $\Omega$ and $C$ be as stated as above. There is a constant $k_{5.1} > 1$, depending only on $r_0$, $r$, $s$ and $T$, ensuring the following inequalities for $\nu \in \ker_{C,x_1,\Omega}$:*

1. $\sup_C |\nu| \leq k_{5.1}|\nu(z)|$ *if $x_1$ and any entry of $\Omega$ are not in $B_s(x)$ and $z \in \overline{B_{\frac{s}{2}}(x)}$;*
2. $\sup_C |\nu| \leq k_{5.1}|\tau^{x_1,\Omega}_* \nu|$ *if $x_1 \in \overline{B_s(x)}$, and* $\sup_C |\nu| \leq k_{5.1}|\tau^{\Omega,x_1}_* \nu|$ *if $x_3 \in \overline{B_s(x)}$.*

*Proof.* The constant in (1) can be chosen to be independent of $x$, $x_1$, $\Omega$, $z$, and $C$ since $M$, $M \setminus B_s(x)$, $M \setminus B_r(x_i)$, $\overline{B_{\frac{s}{2}}(x)}$ and $\mathcal{M}^{x_1,\Omega;B_T(x)}$ are compact.

The constant in (2) is uniform because $(x_1, \Omega)$ is chosen from a compact set in $M^{[l-1]}$. □

Similarly, Lemmas 3.11 and 3.12 are also valid with apparent modification in the statement.

Let $T^{x_1,\Omega;B_t(x)}$ denote the set of points $x_2$ in $M$ that lies in a curve in $\mathcal{M}^{x_1,\Omega}$ and intersecting $\overline{B_t(x)}$.

**Lemma 5.2.** *Suppose $x_1$ and each entry of $\Omega$ are not in $B_s(x)$. There are constants $k$ and $\kappa$ depending on $s$ with the following significance: For $t < \kappa^{-3}$, the volume of $T^{x_1,\Omega;B_t(x)}$ is bounded from above by $kt^2$.*

*Proof.* The proof is similar to that of Lemma 3.11. Let $C = C_{x,x_1,\Omega}$. Since $\text{dist}(x_1, x) \geq s$ and $\text{dist}(\Omega, x) \geq s$, $|\nu(x)|$ is a norm on the 2−dimensional vector space $\ker_{C,x_1,\Omega}$. Now $\mathcal{M}^{x_1,\Omega;B_t(x)}$ is a 2−dimensional smooth compact manifold. As argued in Lemma 3.11,

$$\text{dist}(x_2, C) \leq 2K\kappa_{5.1}\kappa t.$$

Then the volume of $T^{x_1,\Omega;B_t(x)}$ is bounded from above by $kt^2$. □

**Lemma 5.3.** *Suppose $w = x_3$ is in $\overline{B_s(x)} \setminus B_{Rt}(x)$. There is a constant $k$ depending on $s$ and $R$ with the following significance:*

*The volume of points $x_2$ in $M$ lying in a curve in $\mathcal{M}^{x_1,\Omega,r_0}$ and intersecting $B_t(x)$ is bounded from above by $k\frac{t^2}{d^2}$, where $d = dist(w, x)$.*

*Proof.* The proof is identical to that of Lemma 3.12, with the discussion on the map $\tau^{\Omega,x_1}$ (notice $w = x_3$ is the first entry of the superscript) and Lemma 5.1(2) in place of Lemma 3.10(2). □

5.1.3. *Upper bound for a big $J-$nef class.* Now we denote the center of $B$ in definition 2.11 by $x$ where $x$ is any point in $M$ and denote the ball by $B_t(x)$.

**Proposition 5.4.** *Let $e$ be a big $J-$nef class in $S_{K_J}$. The current $\Phi_e$ satisfies the upper bound in* (11).

*Proof.* Choose $s$ as in the beginning of this section.

Fix $x \in M$ and adapted coordinates $(z, w)$ centered at $x$ with radius $s$.

Let $0 < t < 10^{-5}s$.

As in [29], we only need to prove $\Phi_e(i\chi_t dz \wedge d\bar{z}) < kt^4$. Let us denote and group the $l$ points by $x_1$ and $x_2$ and $\Omega = (x_3, \cdots, x_l)$. Moreover, $\Phi_e(i\chi_t dz \wedge d\bar{z})$ is no greater than

$$k_0 \int_{\eta=(x_1,x_2,\Omega)\in\pi_l(\mathcal{M}^{r_0,r}_{irr,l})} (\int_{C_\eta} \chi_t \omega)$$

Notice that $\int_{C_\eta} \chi_t\omega = 0$ if $C_\eta \cap B_t(x) = \emptyset$. If $C_\eta \cap B_t(x) \neq \emptyset$, then $C_\eta \cap B_t(x)$ is contained in a ball of radius of $2t$ centered at some point in $B_t(x)$. By Lemma 2.9, the integrand $\int_{C_\eta} \chi_t\omega$ is bounded by $H \cdot [\omega]kt^2$.

Thus it suffices to prove that the volume of the set

$$\{\eta = (x_1, x_2, \Omega) \in \pi_l(\mathcal{M}^{r_0,r}_{irr,l}), C_\eta \cap B_t(x) \neq \emptyset\} \tag{18}$$

is $O(t^2)$.

We choose $\Omega$, $x_1$ and $x_2$ in turns.

We have three cases depending on the positions of $\Omega$ and $x_1$.

I. The first case is that $x_1$ and each element of $\Omega$ are all away from $B_s(x)$.

Since our moduli space for integration is $\mathcal{M}^{r_0,r}_{irr,l}$, and $x_1 \in M \setminus B_s(x)$, $\Omega \in \{(x_3, \cdots, x_l)|x_i \in M \setminus B_s(x)\}$. The corresponding upper bounds for these two factors are $\mathrm{Vol}(M)$ and $\mathrm{Vol}(M)^{l-2}$ respectively.

For those $\Omega$ and $x_1$ contributing to the integration, $\mathcal{M}^{\Omega,x_1,x'} \subset \mathcal{M}^{r_0}_{irr}$ for some $x' \in \overline{B_t(x)}$. Thus by choosing $t < T$, for any $z \in \overline{B_t(x)}$, the unique smooth rational curve $C_{\Omega,x_1,z} \in \mathcal{M}^{\frac{r_0}{2}}_{irr}$.

Now we estimate the possible choices of $x_2$. By the above picture, the part of $\mathcal{M}^{\Omega,x_1;B_t(x)}$ is a pencil. By Lemma 5.2, the volume of $x_2$ is bounded from above by $k_1t^2$. This constant $k_1$ could be chosen uniformly since the possible set of $\Omega$ and $x_1$ is a closed, then compact, subset of $M^{[l-1]}$. The factors $\mathrm{Vol}(M)^{l-2}$, $\mathrm{Vol}(M)$ and $k_1t^2$, multiply to an upper bound of $O(t^2)$ for the volume of the subset in (18) with $x_1$ and each element of $\Omega$ in $M \setminus B_s(x)$.

II. The second case is when $x_1$ or some entry of $\Omega$, say $w$, satisfies $Rt < \mathrm{dist}(w, x) < s$, where $R = 10^5$. The proof is almost identical to part II of Proposition 3.13 but invoking Lemma 5.3 instead.

III. The last case is when any entry of $\Omega$ or $x_1$ is in $B_{Rt}(x)$. This is exactly the last case of Proposition 3.13.

Summing the three cases, we finish the proof. □

5.1.4. *Taubes current from a J−ample class.*

**Proposition 5.5.** *Let e be a J−ample class e in $S_{K_J}$. Then the current $\Phi_e$ is a Taubes current, i.e. it satisfies* (11). *Consequently, there is an almost Kähler form in the same class.*

*Proof.* Thanks to Proposition 5.4, we only prove the lower bound $k^{-1}t^4 < \Phi_e(f_{B_t(x)} i\sigma \wedge \bar{\sigma})$ here. As in [29], we prove $k^{-1}t^4 < \Phi_e(if_{B_t(x)}dz \wedge d\bar{z})$. Let us denote the $l$ points by $x_1$, $x_2$ and $\Omega = (x_3, \cdots, x_l)$. If $l = 2$, we only have $x_1$ and $x_2$. Let $0 < t < 10^{-5}s$.

The main picture to have is Lemma 2.10, applied to $B_s(x)$. Namely, inside $B_s(x)$, the curves behave as straight lines with respect to the adapted coordinates.

Since we are estimating the lower bound, in addition to choosing $\Omega$ outside a small open neighborhood $\mathcal{OB}(\mathcal{A})$ of the measure zero set $\mathcal{A}$, we also know that each entry of $\Omega$ is away from $B_{\frac{s}{2}}(x)$ and each entry of $\Omega$ is at least of distance $r$ from each others. This set of $\Omega$ is compact in $M^{[l-2]}$. To summarize, by choosing $s$ small, all such $\Omega$ constitute a compact set of volume no smaller than

$$(\mathrm{Vol}(M) - (l-2)\max_{x\in M} \mathrm{Vol}(B_{\frac{s}{2}}(x)))^{l-2} - \mathrm{Vol}(\mathcal{OB}(\mathcal{A})) > (\frac{\mathrm{Vol}(M)}{2})^{l-2}.$$

Before making choices of $x_1$ and $x_2$, we digress to choose a compact submanifold $\mathcal{C}_{x,\Omega} \subset \mathcal{M}_{irr}^{x,\Omega}$.

By Proposition 4.9 and Lemma 4.18, for a pretty generic $\Omega$, except for finitely many complex directions in $T_xM$, there is a smooth rational curve passing through this direction and $\Omega$. Recall that $\mathcal{M}^{x,\Omega}$ is a pencil, and identified with $\mathbb{CP}^1$ via $\tau^{x,\Omega}$. Then the set $\mathcal{C}_{x,\Omega}$ is characterized by the following two properties:

- Its image under $\tau^{x,\Omega}$ is contained in a disk $|dz|(\cdot) \geq 2\epsilon$ in $\mathbb{CP}^1$;
- Any curve $C \in \mathcal{C}_{x,\Omega}$ has distance at least $r$ from $\mathcal{M}_{red,e}$.

It is a (nonempty) compact submanifold of $\mathcal{M}_{irr}^{x,\Omega,r_0}$ (of real dimension two). Recall we choose $r_0$ small enough such that $\mathcal{M}_{irr}^{x,\Omega,r_0} = \mathcal{M}_{irr}^{x,\Omega} \cap \mathcal{M}_{irr}^{r_0} \neq \emptyset$. Moreover, for $C$ chosen from $\mathcal{C}_{x,\Omega}$, the restriction of $|dz|$ to $C \cap B_s(x)$ is greater than $\epsilon$ when $s$ is chosen sufficiently small.

Now, let us choose $x_1$. Again, we choose $x_1 \in B_s(x)$ away from $B_{\frac{s}{2}}(x)$ and $B_{\frac{s}{2}}(x_i)$ where $x_i$'s are entries of $\Omega$. Additionally, we choose $x_1$ such that the rational curve $C_{x_1,x,\Omega}$ determined by $x$, $\Omega$ and $x_1$ is contained in the compact submanifold $\mathcal{C}_{x,\Omega} \subset \mathcal{M}_{irr}^{x,\Omega}$ specified above. By Lemma 4.18, the choices of $x_1$ constitute a compact set with nonzero volume, say $c_\Omega s^4$.

Now, with $x_1$ fixed, we consider the set of $x_2$ in $B_{\frac{s}{4}}(x)$, for which $\eta = (x_1, x_2, \Omega)$ contributes to $\Phi_e(if_{B_t(x)}dz\wedge d\bar{z})$, namely $C_{x_1,x_2,\Omega}$ intersects $B_t(x)$. This part of argument is identical to the corresponding part in Proposition 3.14 with $C_{x_1,x_2,\Omega}$ and $\pi_l$ appearing in place of $C_{x_1,x_2}$ and $\pi_2$. We have the lower bound $k_1s^2t^2$.

Let $C = C_{x_1,x_2,\Omega}$, then $\int_C (if_{B_t(x)}dz \wedge d\bar{z}) \geq k_2^{-1}t^2$ by Lemma 2.9 as in Proposition 3.14.

Multiplying these four factors: $(\frac{\mathrm{Vol}(M)}{2})^{l-2}$, $c_\Omega s^4$, $k_1s^2t^2$ and $k_2^{-1}t^2$ together, we get $k^{-1}t^4 < \Phi_e(if_{B_t(x)}dz \wedge d\bar{z})$. These constants $c_\Omega$, $k_1, k_2$ are independent of $t$ as one can check from the proof. The constant $k$ could be chosen universal by the same reasoning in Proposition 3.14.

The last statement follows from Proposition 2.12. □

5.1.5. *Weak Taubes current from a big $J-$nef class.* If $e$ is big $J-$nef but not $J-$ample, $\Phi_e$ is not a Taubes current, since no irreducible curves in class $e$ pass through points in the vanishing locus $Z(e)$. Nonetheless, the following observation will be very useful.

**Proposition 5.6.** *Let $e$ be a big $J-$nef class. Then the current $\Phi_e$ is non-negative, and over any (4−dimensional) compact submanifold $K$ of the complement $M(e)$, it satisfies* (11) *for a constant $k > 1$ depending only on $K$.*

*Proof.* $\Phi_e$ is a non-negative current by definition, and the upper bound is from Proposition 5.4. We only need to prove that it is bounded from below by $k^{-1}t^4$ on any compact submanifold $K \subset M(e)$. The proof is almost identical to the proof of Proposition 5.5. Notice that all the relevant results in Section 4 are established for $x \in M(e)$. Hence the proof goes almost verbatim as that of Proposition 5.5 when $x$ is chosen from a compact submanifold $K \subset M(e)$, and $\Omega$, $x_1$ chosen from $K'^{[l-2]}$, $K'$ (instead of from $M^{[l-2]}$ and $M$) respectively. Here $K'$ is another (4−dimensional) compact submanifold of $M(e)$ such that $K \subsetneq K'$, which satisfies $B_s(p) \subset K'$ if $p \in K$. □

We call a current in Proposition 5.6 a *weak Taubes current*. By summing up weak Taubes currents with disjoint zero locus, we obtain Taubes currents.

**Proposition 5.7.** *Let $e_i$ be big $J-$nef classes in $S_{K_J}$ and denote $Z_i$ the zero locus of $e_i$. If $\cap Z_i = \emptyset$, then there is a Taubes current in the class $e = \sum_i a_i e_i$, with $a_i > 0$. In turn, we obtain an almost Kähler form in the class $e$.*

## 5.2. **Tamed versus compatible.**

5.2.1. *Proof of Theorem 1.2.*

*Proof.* Given any $J$ on $S^2-$bundle over $S^2$ it follows from Lemma 4.14 that there is always a $J-$ample class. Now the conclusion follows from Proposition 5.5. □

5.2.2. *Tameness and foliation.* Now we have shown that every tamed $J$ on an $S^2-$bundle over $S^2$ is almost Kähler, a related question is when $J$ is tamed.

Suppose $M$ is an $S^2-$bundle over $S^2$ and $J$ is an almost complex structure on $M$, not necessarily tamed. Then there is a $J-$nef class in $S^0_{K_J}$. If $J$ is tamed, by Proposition 3.3, $e$ is $J-$effective, and moreover, there is a $J-$holomorphic foliation by smooth rational curves. We would like to know whether the converse is also true.

**Question 5.8.** *For an $S^2-$bundle over $S^2$, suppose there is a $J-$holomorphic foliation by smooth rational curves. Is $J$ tamed?*

Note that if we assume further that $J$ is fibred, Gompf's construction in [8] produces a tamed symplectic form.

Here is an analogous question for $\mathbb{CP}^2$.

**Question 5.9.** *For $\mathbb{CP}^2$, $J$ is tamed if and only if there is a pencil of smooth rational curves?*

5.2.3. $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ *with* $k\geq 2$. From now on we sometimes denote $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ by $M_k$. For $M_k$ with $k\geq 2$, there are no $J-$ample classes in $S_{K_J}$. So we apply Proposition 5.7 to construct Taubes currents.

Let $H, E_i$ be an orthogonal basis of $H_2(M_k;\mathbb{Z})$ with $H^2=1, E_i^2=-1$. Such a basis is called a standard basis.

Given any tamed almost complex structure $J$, there is a standard basis such that $K_J=-3H+\sum E_i$, called a standard basis adapted to $J$. This follows from the uniqueness of symplectic canonical classes, up to diffeomorphisms. Notice that $E_i\in\mathcal{E}_{K_J}$ and $H\in S^+_{K_J}$. Hence $H$ is $J-$effective. And it pairs positively with any $J-$tamed symplectic form.

Suppose $J$ is tamed and there is a configuration of $k$ disjoint $-1$ curves $C_i$. Notice that we have a standard basis $(H, E_1,\cdots,E_i)$ adapted to $J$, with $E_i=[C_i]$ and $H$ the unique square 1 class with $H\cdot K_J<0$ and $H\cdot E_i=0, i=1,\cdots,k$.

**Lemma 5.10.** *Suppose $J$ is tamed and there is a configuration of $k$ disjoint $-1$ curves $C_i$. Then the classes $H$, $2H$, $nH-(n-1)E_i$, $nH-(n-1)E_i-E_j$ and $H-E_i$ in $S_{K_J}$ are $J-$nef.*

*Proof.* Let $C$ be an irreducible curve distinct from any $C_i$, and suppose $[C]=aH-\sum_i b_iE_i$. Since $C_i$ is irreducible, $b_i\geq 0$ by the positivity of intersection. Since any $J-$tamed $\omega$ is positive on $H$, $a>0$.

Clearly $H, 2H$ are $J-$nef since $a>0$.

Since $C$ is an irreducible curve, $g_J(e_C)\geq 0$, so we have

$$
\begin{array}{rcl}
-2 & \leq & 2g_J(e_C)-2\\
 & = & K_J\cdot C+C^2\\
 & = & -3a+\sum_i b_i+a^2-\sum_i b_i^2\\
 & = & (a-b_1)(a+b_1-1)\\
 & & -2a\\
 & & -\sum_{i\geq 2}(b_i^2-b_i).
\end{array}
\tag{19}
$$

The second term is at most $-2$. The third term is non-positive. Therefore $a\geq b_1$. In fact, the same argument shows that $a\geq b_i$ for any $i$.

It follows that $C$ cannot pair negatively with $nH-(n-1)E_i$, $H-E_i$.

If $C$ pairs negatively with $nH-(n-1)E_i-E_j$, then

$$na-(n-1)b_i-b_j<0,$$

which implies that either $a<b_i$ or $a<b_j$. But this is impossible so $nH-(n-1)E_i-E_j$ is also $J-$nef. $\square$

**Remark 5.11.** *Under the assumption of Lemma 5.10, it is not true that any class in $S^{\geq 0}_{K_J}$ is $J-$nef. Consider the holomorphic blow up of 3 points on a line $l$ in $\mathbb{CP}^2$ and let $C_1, C_2, C_3$ be the exceptional curves. Then the class $2H-E_1-E_2-E_3$ in $S_{K_J}$ is not $J-$nef since it pairs negatively with the* *class of the proper transform $l'$ of the line $l$, which is $H - E_1 - E_2 - E_3$. In particular, there are no smooth rational curves in the class $2H-E_1-E_2-E_3$.*

**Remark 5.12.** *Notice that, under the assumption of Lemma 5.10, there is at least one $J-$nef class in $S^0_{K_J}$. It is natural to wonder whether this is true for any tamed almost complex structure. If so, by Proposition 4.5, every tamed rational $4-$manifold has a rational curve 'foliation', with only finitely many reducible leafs.*

We are ready to prove Theorem 1.3.

*Proof.* Observe that, for each $i$, $2H - E_i$ is a $K_J-$spherical class with square 3, and it is $J-$nef by Lemma 5.10.

Let $Z_i = Z(2H - E_i)$ be the zero locus of $2H - E_i$. If $C \neq C_i$ is an irreducible curve in $Z_1$, then $b_1 = 2a > a$. This is impossible by the proof of Lemma 5.10. Therefore

$$Z_i = \{C_1, \cdots, C_{i-1}, C_{i+1}, \cdots\}.$$

Clearly, $\cap_{1\leq i\leq k} Z_i = \emptyset$.

Therefore there is an almost Kähler form in the class $\sum_{i=1}^k (2H - E_i)$ by Proposition 5.7. In particular, $J$ is almost Kähler. □

**Corollary 5.13.** *If $J$ is in $\mathcal{J}_{top}$ or $\mathcal{J}_{good}$, then $J$ is almost Kähler.*

*Proof.* For any tamed $J$, by Proposition 3.3, each $E$ in $\mathcal{E}_{K_J}$ is represented by a $J-$holomorphic subvariety $\Theta$. We claim that $E$ pairs non-negatively with any irreducible subvariety $C$ whose class is not $E$. This is clear if $C$ has non-negative self-intersection. Suppose $C$ has negative self-intersection. Then by our assumption on $J$, $C$ is either a $-1$ curve or or the anti-canonical curve. The $-1$ curve case follows from Lemma 4.12 (3), and the anti-canonical curve case follows from the adjunction formula.

The following result is proved in [19] as Proposition 4.25:

**Proposition 5.14.** *Suppose $J$ is tamed, $e \in S_{K_J}$ and $\Theta = \{(C_i, m_i)\} \in \mathcal{M}_e$. If $e\cdot e_{C_i} \geq 0$, then $\Theta$ is connected and each component $C_i$ is a smooth rational curve.*

Our $\Theta$ satisfies the condition, so $\Theta$ is connected and each component is a smooth rational curve. Since $\Theta$ is connected, if it is reducible, it must contain an irreducible component $F$ with self-intersection at most $-2$. But by our assumption on $J$ there are no smooth rational curves of self-intersection less than $-1$.

We have shown that each $E$ in $\mathcal{E}_{K_J}$ is represented by a $-1$ curve. In particular, given a standard basis $\{E_i\}$ adapted to $J$, there are $k$ disjoint $-1$ curves in the $k$ classes $E_i$. Now apply Theorem 1.3. □

**Corollary 5.15.** *Suppose $h_l = lH - \sum_{i=1}^k E_i$ with $l^2 > \max\{k, 9\}$. If $J$ is an almost complex structure on $M_k$ tamed by a symplectic form in the class $h_l$, then $J$ is almost Kähler. Moreover, there is an almost Kähler form in the class $h_l$.*

*Proof.* Since the $h_l$ area of $E_i$ is 1, a subvariety representing $E_i$ is irreducible, and hence it is smooth. Thus $J$ is almost Kähler by Theorem 1.3.

Since $h_l$ is in the $J-$tamed cone $\mathcal{K}_J^t$ and $J$ is almost Kähler, the last claim follows from the equality (23) between $\mathcal{K}_J^t$ and the almost Kähler cone. □

We call a tamed almost complex structure $J$ del Pezzo if it is tamed by a symplectic form in the class $-K_J$. By Corollary 5.15, del Pezzo $J$ is almost Kähler.

**Remark 5.16.** *It is observed by Pinsonnault* [24] *that for any tamed $J$ on $M_k$, there exists at least one (smooth) $-1$ curve.*

5.3. **Almost Kähler cone.** We first introduce an open convex cone associated to $J$.

**Definition 5.17.** *For a tamed almost complex structure $J$ on $M_k$ (resp. $S^2 \times S^2$), the open convex cone $\mathcal{S}_J$ is defined to be the interior of the convex cone generated by big $J-$nef classes in $S_{K_J}$ if it is of dimension $k+1$ (resp. 2). Otherwise, it is defined as $\emptyset$.*

For an almost Kähler $J$, we have the following

**Lemma 5.18.** *If $J$ is almost Kähler, then $\mathcal{S}_J$ is contained in $\mathcal{K}_J^c(M)$.*

*Proof.* Let $\mathcal{S}_J \neq \emptyset$. Suppose $u$ is the class of an almost Kähler form. Observe that given any class $e \in \mathcal{S}_J$, $e - tu$ is in $\mathcal{S}_J$ for $t$ small. Thus $e - tu = \sum a_i e_i$ with $a_i > 0$ and $e_i$ big $J-$nef . Hence $e = (e - tu) + tu \in \mathcal{K}_J^c(M)$ by Propositions 5.6 and 2.12. □

5.3.1. *$S^2-$bundles over $S^2$.*

*Proof of Theorem 1.5.* By Lemma 5.18 it suffices to show that

$$\mathcal{S}_J = \mathcal{P}_J$$

We establish this by describing explicitly the curve cone $A_J$. We have mentioned that there is always a foliation by smooth rational curves. One boundary of the curve cone is generated by the class of such a foliation. The other boundary is generated by a transversal class.

We start with $S^2 \times S^2$.

Given any $J$, denote the class of a foliation by $H_2$. The curve cone $A_J$ is generated by $H_2$ and a transversal class $H_1 - lH_2$ for some $l \geq 0$.

The $\geq 0-$dual of $A_J$ is generated by $A = H_1 + lH_2$ and $B = H_2$.

If $l = 0$, then $A = H_1$ and it is approximated by the sequence of big $J-$nef classes in $S_{K_J}$, $pH_1 + H_2$. $B$ is approximated by the sequence of big $J-$nef classes in $S_{K_J}$, $H_1 + qH_2$. Therefore

$$\mathcal{S}_J = \mathcal{P}_J = \{aA + bB | a > 0, b > 0\} = \{xH_1 + yH_2 | x > 0, y > 0\}.$$

If $l > 0$, then $A$ itself is a big $J-$nef class in $S_{K_J}$, and $B$ is approximated by the sequence of big $J-$nef classes in $S_{K_J}$, $H_1 + qH_2$, $q \geq l$. Therefore

$$\mathcal{S}_J = \mathcal{P}_J = \{aA + bB | a > 0, b > 0\} = \{xH_1 + yH_2 | y > lx > 0\}.$$

For $M = \mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, the proof is similar.

Given any $J$, the unique class of foliation is $H - E$. The curve cone $A_J$ is generated by $H - E$ and a transversal class $D_{-l} = -lH + (l+1)E$ for some $l$.

The $\geq 0-$dual of $A_J$ is generated by $B = H - E$ and $A = (l+1)H - lE$.

For each $l$, the class $A$ is a big $J-$nef class in $S_{K_J}$, and $B$ is approximated by the sequence of big $J-$nef classes in $S_{K_J}$, $(p+1)H - pE, p \geq l$. Thus

$$\mathcal{S}_J = \mathcal{P}_J = \{aA + bB | a > 0, b > 0\} = \{xH - yE | \frac{l+1}{l} y > x > y > 0\}.$$

□

5.3.2. $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$. In this case we will again apply Lemma 5.18 to probe the almost Kähler cone.

In fact, for $J \in \mathcal{J}_{top}$, Question 1.4 takes a particular simple form, which we now explain. Among all $J \in \mathcal{J}^t$, $J$ in the top stratum $\mathcal{J}_{top}$ has the maximal $\mathcal{P}_J$. In fact, when $b^+(M) = 1$ and $J \in \mathcal{J}_{top}$, $\mathcal{P}_J$ is equal to the $K_J-$symplectic cone $\mathcal{C}_{K_J}$ introduced in [16]:

$$\mathcal{C}_{K_J} = \{e \in H^2(M; \mathbb{R}) | e = [\omega] \text{ for some } \omega \text{ with } K_\omega = K_J\}. \tag{20}$$

Here $K_\omega$ is the symplectic canonical class of $\omega$. Thus the almost Kähler Nakai-Moishezon criterion for $J \in \mathcal{J}_{top}$ is the same as

$$\mathcal{K}_J^c(M) = \mathcal{C}_{K_J}.$$

Although we cannot verify the almost Kähler Nakai-Moishezon criterion for $J \in \mathcal{J}_{top}$ when $k \geq 10$, we have the following partial result.

**Theorem 5.19.** *Suppose $M$ is a rational manifold. If $J \in \mathcal{J}_{top}$, then $\mathcal{K}_J^c(M) \supset \mathcal{C}_{K_J} \cap \{e | e \cdot K_J < 0\}$.*

To establish Theorem 5.19 we introduce several open convex cones associated to $K_J$.

**Definition 5.20.** *For a tamed almost complex structure $J$, the open convex cone $\mathcal{S}^+_{K_J}$ is the interior of the convex cone generated by classes in $S^+_{K_J}$. It is called the positive $K_J-$sphere cone.*

Notice that $\mathcal{S}^+_{K_J}$ contains $\mathcal{S}_{J'}$ for any tamed $J'$ with $K_{J'} = K_J$.

According to [16], the $K_J-$symplectic cone $\mathcal{C}_{K_J}$ has the following characterization,

$$\mathcal{C}_{K_J} = \{e \in \mathcal{P} | e \cdot E > 0 \text{ for any } E \in \mathcal{E}_{M,K_J}\}. \tag{21}$$

Finally, introduce the open subcone of $\mathcal{C}_{K_J}$,

$$P_{K_J} := \{e \in \mathcal{C}_{K_J} | e \cdot (-K_J) > 0\} \tag{22}$$

**Proposition 5.21.** *For any $J$ on $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, the positive $K-$sphere cone $\mathcal{S}^+_{K_J}$ coincides with $P_{K_J}$.*

We will defer the proof of Proposition 5.21 to the next subsection, where we need to review $P-$cell in [7] and $K-$symplectic cone in [16].

**Lemma 5.22.** *If $J$ is in $\mathcal{J}_{top}$ or $\mathcal{J}_{good}$, then $\mathcal{S}_J = \mathcal{S}^+_{K_J}$.*

*Proof.* When $J$ is in $\mathcal{J}_{top}$ or $\mathcal{J}_{good}$, by Lemma 5.10 and the proof of Corollary 5.13, any class in $S^+_{K_J}$ is $J-$nef.

□

*Proof of Theorem 5.19 and Theorem 1.6.* Both claims follow from Lemmas 5.18, 5.22, Proposition 5.21, and (22). □

It can be easily shown that if $J$ is del Pezzo then $J \in \mathcal{J}_{top}$, thus we have

**Corollary 5.23.** *If $J$ is del Pezzo, then $\mathcal{K}^c_J(M) = \mathcal{C}_{K_J}$. In other words, the almost Kähler cone either does not contain $-K_J$, or is equal to $\mathcal{C}_{K_J}$.*

This result also follows from the $J-$inflation approach in [31]. The $J-$inflation along a smooth $J-$holomorphic subvariety, which is due to McDuff [22] and extended by Buse [3], can be effectively applied to probe the $J-$tamed cone of a tamed almost complex structure $J$,

$$\mathcal{K}^t_J = \{[\omega] \in H^2(M;\mathbb{R}) | \omega \text{ tames } J\}.$$

Clearly, $\mathcal{K}^c_J \subset \mathcal{K}^t_J$. When $b^+ = 1$, it was shown in [18] that if $J$ is almost Kähler, then

$$\mathcal{K}^c_J = \mathcal{K}^t_J. \tag{23}$$

The equality (23) combined with the calculation of the tamed cone in [31] gives an alternative proof of Corollary 5.23. In fact, in [31] we use this approach to establish the almost Kähler Nakai-Moishezon criterion for minimal ruled manifolds and all rational manifolds with $b^- \leq 8$.

5.4. $K-$**sphere cone and** $K-$**symplectic cone.** In this subsection we establish Proposition 5.21.

5.4.1. *$P-$cells.* Suppose $M$ is an oriented closed manifold with odd intersection form, $b^+ = 1$, $b^- = n$ and no torsion in $H^2(M;\mathbb{Z})$. A basis $(x, \alpha_1, \cdots, \alpha_n)$ for $H^2(M;\mathbb{Z})$ is called standard if $x^2 = 1$, and $\alpha_i^2 = -1$ for each $i = 1, \cdots, n$. Let

$$\begin{aligned} \mathcal{P} &= \{e \in H^2(M;\mathbb{R}) | e \cdot e > 0\} \\ \mathcal{B} &= \{e \in H^2(M;\mathbb{R}) | e \cdot e = 0\} \\ \overline{\mathcal{P}} &= \{e \in H^2(M;\mathbb{R}) | e \cdot e \geq 0\}. \end{aligned}$$

For each class $x \in H^2(M;\mathbb{Z})$ with $x^2 < 0$, we define $x^\perp \in H^2(M;\mathbb{R})$ to be the orthogonal subspace to $x$ with respect to the cup product, and we call $(x^\perp) \cap \mathcal{P}$ the wall in $\mathcal{P}$ defined by $x$. Let $\mathcal{W}_1$ be the set of walls in $\mathcal{P}$ defined by integral classes with square $-1$. A chamber for $\mathcal{W}_1$ is the closure in $\mathcal{P}$ of a connected component of $\mathcal{P} - \cup_{W \in \mathcal{W}_1} W$.

Any point $x \in \mathcal{P}$ with square 1 at which $n$ mutually perpendicular walls of $\mathcal{W}_1$ meet is called a corner. Any corner is an integral class (see Lemma

2.2 in [7]). Suppose $C$ is a chamber for $\mathcal{W}_1$. If $x$ is a corner in $C$, a standard basis $(x, \alpha_1, \cdots, \alpha_n)$ for $H^2(M;\mathbb{Z})$ is called a standard basis adapted to $C$ if $\alpha_i \cdot C \geq 0$ for each $i$. The canonical class of the pair $(x, C)$ is defined to be $\kappa(x, C) = 3x - \sum_i \alpha_i$. Suppose $C$ is a chamber for $\mathcal{W}_1$ and $x$ is a corner in $C$, we define

$$P(x, C) = C \cap \{e \in \mathcal{P} | \kappa(x, C) \cdot e \geq 0\}.$$

Any subset of $\mathcal{P}$ of the form $P(x, C)$ is called a $P-$cell.

5.4.2. *$\mathcal{C}_K$ and $P_K$.* We are back to the situation that $M = M_k$ and $J$ is a tamed almost complex structure. Denote $K_J$ by $K$.

By Lemma 2.4 in [13], $\overline{P}_K$, the closure of $P_K$, is a $P-$cell and $\kappa(P_K) = -K$.

**Lemma 5.24.** *1. $P_K$ is an open convex polytope in $\mathcal{P}$. Each wall of $P_K$ is either a wall of a class in $\mathcal{E}_{M,K}$, or the wall of $K$ if $k \geq 9$.*

*2. The face $F_{E_k}$ of $P_{M_k,K}$ corresponding to $E_k$ is naturally identified with $P_{M_{k-1},K}$.*

*Proof.* Statement 1 is due to Friedman and Morgan. It states that $P_K$ does not have round boundary, i.e. boundary contributed by $\mathcal{B}$, although $\mathcal{C}_K$ always has a round boundary when $k > 9$. It is easy to see that when $k \leq 8$, $P_K$ is just a chamber. In other words, each wall of it is a wall of a class in $\mathcal{E}_{M,K}$. When $k > 9$, the class $K$ does contribute a wall to $P_K$.

For 2, $E_k$ is orthogonal to all classes in $\mathcal{E}_{M_{k-1},K}$. In fact

$$\mathcal{E}_{M_{k-1},K} = \{e \in \mathcal{E}_{M_k,K} | e \cdot E_k = 0\}.$$

Suppose $u$ is in the interior of the face $F_{E_k}$. Then $u$ is positive on $\mathcal{E}_{M_{k-1},K} \subset \mathcal{E}_{M_k,K}$. If we consider the expansion of $u$ with respect to the standard basis, then $u$ has no $E_k$ coefficient. Hence $F_{E_k} \subset P_{M_{k-1},K}$.

Conversely, if $u \in P_{M_{k-1},K}$, then $u$ is orthogonal to $E_k$, and $u$ is positive on $\mathcal{E}_{M_{k-1},K}$. Moreover, for any class $e \in \mathcal{E}_{M_k,K}$ with nonzero $E_k$ coefficient, $u \cdot e = u(e + (e \cdot E_k)E_k)$. Notice that $u^2 > 0$ and

$$(e + (e \cdot E_k)E_k)^2 = -1 + (e \cdot E_k)^2 \geq 0.$$

Further, since $e \cdot E_k > 0$, $u$ and $e + (e \cdot E_k)E_k$ pairs positively with any symplectic form. By the light cone lemma, we have $u \cdot e > 0$. This proves that $P_{M_{k-1},K} \subset F_{E_k}$. □

5.4.3. *Proof of Proposition 5.21.*

*Proof.* First of all,

$$\mathcal{S}_K^+ \subset P_K.$$

This follows from (21), (22), the positive pairing between $S_K^+$ and $\mathcal{E}_K$ by Lemma 4.12 (2), and the positive pairing between $S_K^+$ and $K$ by the adjunction formula.

So now we start to prove $P_K \subset \mathcal{S}_K^+$.

For $k = 1$, it is clear and is essentially contained in the proof of Theorem 1.5.

For $2 \le k \le 8$, we do induction. Suppose we have done the case when $k < l \le 8$, we want to argue that for $M_l = \mathbb{CP}^2 \# l\overline{\mathbb{CP}^2}$, $\mathcal{S}_K^+ = P_K$.

In this case, by Lemma 5.24 (1), $P_K$ is an open polytope with each face of the boundary a wall of a class in $\mathcal{E}_{M_l,K}$. Hence $e$ could be written as a finite combination $\sum_{i=1}^q a_i e_i$ with $e_i$ in a boundary face and $a_i > 0$. Notice each boundary face $F_{E_i'}$ (with $E_i' \in \mathcal{E}_{M_l,K}$) of $P_{M_l,K}$ corresponds to $P_{M_{l-1},K}$ by Lemma 5.24 (2). Then by induction assumption, each $e_i \in \mathcal{S}^+_{M_{l-1},K}$. Hence $e \in \mathcal{S}_K^+$ as well by definition.

When $k \ge 9$, we still do induction. However, in this situation, $P_K$ does have a wall contributed by the class $-K$.

Consider a class of the form $V_a = aH - \sum_{i=1}^l E_i$. Given $e \in P_K$, we can find $a < 3$ such that $e \cdot (aH - \sum E_i) = 0$. This is because $V_3 = -K$ pairs positively with $e$, and when $a = 0$, $V_0$ pairs negatively with $e$.

Notice that $V_a \cdot V_3 < V_3 \cdot V_3 \le 0$, so the hypersurface of $V_a$ does not intersect the wall of $-K$. Choose a generic line $L$ in this hypersurface such that $e \in L$ and $L$ intersects the polytope inside the interior of the boundary faces $F_1, F_2$ at $e_1$ and $e_2$.

Then $e = a_1 e_1 + a_2 e_2$ with $a_i > 0$. Each class $e_i$ lies in the interior of the face $F_i$, which corresponds to $P_{M_{l-1},K}$ by Lemma 5.24 (2). By induction assumption, $e_i \in \mathcal{S}_K^+$. This finishes the proof. □

5.5. **Remark on the connection with Kodaira embedding.** Finally we would like to provide a heuristic comparison of the genus zero subvariety-current-form construction with the Kodaira embedding theorem.

Unlike linear systems in algebraic geometry, the moduli spaces of pseudo-holomorphic subvarieties generally have no natural linear structure. In algebraic geometry, we obtain a linear system of divisors from the vector space of sections of a holomorphic line bundle. For almost complex structures, there is no such direct global correspondence. In general, only locally there are some traces of linearity. Via the implicit function theorem, Taubes in [29] uses the linear structure on the kernel of the normal operator $D_C$ to linearize the moduli space near a subvariety $C$. Another instance is that, inside a very small ball, subvarieties behave like lines in the standard $\mathbb{C}^2$ (Lemma 2.10).

However, it strikes us again and again that the moduli spaces of genus zero pseudo-holomorphic subvarieties possess local as well as semi-global linearity properties. Notably, in Proposition 4.9 we show that there are plenty of pencils of genus zero subvarieties in a big $J-$nef class $e$. To find a pencil, we usually fix $e \cdot e$ points suitably, then look at the local or global moduli space passing through these points. We are thus led to the following global question.

**Question 5.25.** *Suppose $J$ is a tamed almost complex structure on a rational manifold, and $e$ is represented by a smooth $J-$holomorphic genus zero subvariety. Does $\mathcal{M}_e$ have the structure of a complex projective space of dimension $e\cdot e+1$?*

In fact, this has a positive answer when $e\cdot e\leq 0$, and when $e$ is the line class of $\mathbb{CP}^2$. If we can answer positively Question 5.25 in general, it is tempting to further link the construction of an almost Kähler form via genus zero subvarieties on a rational manifold to the construction of a Kähler form on a Hodge manifold via Kodaira embedding. Given a $J-$ample class $e$ in $S_{K_J}$, when $J$ is integrable and hence Kähler, holomorphic sections of $L$ provide a holomorphic embedding $\tau_L: M\to\mathbb{CP}^N$, where $L$ is line bundle corresponding to the class $e$ and $N=h^0(L)-1=e\cdot e+1$. The pullback $\tau_L^*\omega_{FS}$ of the standard Fubini-Study form gives us a Kähler form in the class $e$. When $J$ is not integrable, one could heuristically think that the family of $J-$tamed forms with dominating $J-$invariant part in the construction come from a family of embeddings $\tau_e(\epsilon): M\to\mathcal{M}_e=\mathbb{CP}^N$, which are close to being holomorphic.

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School of Mathematics, University of Minnesota, Minneapolis, MN 55455
*E-mail address*: `tjli@math.umn.edu`

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, England
*E-mail address*: `weiyi.zhang@warwick.ac.uk`