A DISCRETE UNIFORMIZATION THEOREM FOR POLYHEDRAL SURFACES II

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Abstract. A discrete conformality for hyperbolic polyhedral surfaces is introduced in this paper. This discrete conformality is shown to be computable. It is proved that each hyperbolic polyhedral metric on a closed surface is discrete conformal to a unique hyperbolic polyhedral metric with a given discrete curvature satisfying Gauss-Bonnet formula. Furthermore, the hyperbolic polyhedral metric with given curvature can be obtained using a discrete Yamabe flow with surgery. In particular, each hyperbolic polyhedral metric on a closed surface with negative Euler characteristic is discrete conformal to a unique hyperbolic metric.

1. Introduction

1.1. **Statement of results.** This is a continuation of [9] in which a discrete uniformization theorem for Euclidean polyhedral metrics on closed surfaces is established. The purpose of this paper is to prove the counterpart of discrete uniformization for hyperbolic polyhedral metrics. In particular, we will introduce a discrete conformality for hyperbolic polyhedral metrics on surfaces and show the discrete conformality is algorithmic.

Recall that a *marked surface* (S, V) is a pair of a closed connected surface S together with a finite non-empty subset V. A *triangulation* of a marked surface (S, V) is a triangulation of S so that its vertex set is V. A *hyperbolic polyhedral metric* d on a marked surface (S, V) is obtained as the isometric gluing of hyperbolic triangles along pairs of edges so that its cone points are in V. It is the same as a hyperbolic cone metric on S with cone points in V. We use the terminology *polyhedral metrics* to emphasize that these metrics are determined by finite sets of data (i.e., the finite set of lengths of edges). Every hyperbolic polyhedral metric has an associated *Delaunay triangulation* which has the property that the interior of the circumcircle of each triangle contains no other vertices.

Definition 1. *(discrete conformality) Two hyperbolic polyhedral metrics d, d' on a closed marked surface (S, V) are discrete conformal if there exist a sequence of hyperbolic polyhedral metrics $d_1 = d, d_2, ..., d_m = d'$ on (S, V) and triangulations $\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_m$ of (S, V) satisfying*

(a) *each $\mathcal{T}_i$ is Delaunay in d_i,*
(b) *if $\mathcal{T}_i = \mathcal{T}_{i+1}$, there exists a function $u : V \to \mathbb{R}$, called a conformal factor, so that if e is an edge in $\mathcal{T}_i$ with end points v and v', then the lengths $x_{d_i}(e)$*

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and $x_{d_{i+1}}(e)$ *of* e *in metrics* d_i *and* d_{i+1} *are related by*

$$\sinh \frac{x_{d_{i+1}}(e)}{2} = e^{u(v)+u(v')} \sinh \frac{x_{d_i}(e)}{2},$$

(c) *if* $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, *then* (S, d_i) *is isometric to* (S, d_{i+1}) *by an isometry homotopic to the identity in* (S, V).

This definition is the hyperbolic counterpart of discrete conformality introduced in [9]. The condition (b) first appeared in [4].

Theorem 2. *Suppose* d *and* d' *are two hyperbolic (or Euclidean) polyhedral metrics given as isometric gluings of geometric triangles on a closed marked surface* (S, V). *There exists an algorithm to decide if* d *and* d' *are discrete conformal.*

The above theorem shows that discrete conformality is computable. This is in contrasts to the conformality in Riemannian geometry. Indeed, it is highly unlikely that there exist algorithms to decide if two hyperbolic (or Euclidean) polyhedral metrics on (S, V) are conformal in the Riemannian sense.

The *discrete curvature* K of a polyhedral metric d is the function defined on V sending $v \in V$ to 2π less cone angle at v. It is well known that the discrete curvature satisfies the Gauss-Bonnet identity $\sum_{v \in V} K(v) = 2\pi\chi(S) + Area(d)$ where $Area(d)$ is the area of the metric d.

Theorem 3. *Suppose* (S, V) *is a closed connected marked surface and* d *is a hyperbolic polyhedral metric on* (S, V). *Then for any* $K^* : V \to (-\infty, 2\pi)$ *with* $\sum_{v \in V} K^*(v) > 2\pi\chi(S)$, *there exists a unique hyperbolic polyhedral metric* d' *on* (S, V) *so that* d' *is discrete conformal to* d *and the discrete curvature of* d' *is* K^*. *Furthermore, the discrete Yamabe flow with surgery associated to curvature* K^* *having initial value* d *converges to* d' *exponentially fast.*

In particular, on a closed connected surface S with $\chi(S) < 0$, by choosing $K^* = 0$, we obtain,

Corollary 4. *(discrete uniformization) Let* S *be a closed connected surface of negative Euler characteristic and* $V \subset S$ *be a finite non-empty subset. Then each hyperbolic polyhedral metric* d *on* (S, V) *is discrete conformal to a unique hyperbolic metric* d^* *on the surface* S. *Furthermore, there exists a* C^1*-smooth flow on the Teichmuller space of hyperbolic polyhedral metrics on* (S, V) *which preserves discrete conformal classes and flows each polyhedral metric* d *to* d^* *as time goes to infinity.*

1.2. **Basic idea of the proof.** The basic idea of the proof is similar to that of [9]. We first introduce the Teichmüller space $T_{hp}(S, V)$ of hyperbolic polyhedral metrics on (S, V). It is shown to be a real analytic manifold which admits a cell decomposition by the work of [15] and [13]. Using the work of Kubota [14] on hyperbolic Ptolemy identity and the work of Penner [19], we show that $T_{hp}(S, V)$ is C^1 diffeomorphic to the decorated Teichmüller space so that two hyperbolic polyhedral metrics are discrete conformal if and only if their corresponding decorated metrics have the same underlying hyperbolic structure. Using this correspondence, we show Theorem 3 using a variational principle first appeared in [4].

Many arguments in this paper are similar to that of [9]. The major difference between Euclidean and hyperbolic polyhedral metrics comes from the circumcircles of triangles. Namely, the circumcircle of a hyperbolic triangle may be non-compact, i.e., a horocyle or a curve of constant distance to a geodesic. This creates many difficulties when one uses the inner angle characterization of Delaunay triangulations. To overcome this, we prove (theorem 9) that every triangle in a Delaunay triangulation of a hyperbolic polyhedral metric on a compact surface has compact circumcircle.

1.3. **Organization of the paper.** Section 2 deals with the Teichmüller space of hyperbolic polyhedral metrics, its analytic cell decomposition and Delaunay triangulations. In section 3, we show that there is a C^1 diffeomorphism between the Teichmüller space of hyperbolic polyhedral metrics and the decorated Teichmüller space. Section 4 is devoted to the proof of Theorem 3. Section 5 proves theorem 2. In the appendix, a technical lemma is proved.

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2. Teichmüller space of polyhedral metrics

2.1. **Triangulations and some conventions.** Take a finite disjoint union X of triangles and identify edges in pairs by homeomorphisms. The quotient space S is a compact surface together with a *triangulation* $\mathcal{T}$ whose simplices are the quotients of the simplices in the disjoint union X. Let $V = V(\mathcal{T})$ and $E = E(\mathcal{T})$ be the sets of vertices and edges in $\mathcal{T}$. We call $\mathcal{T}$ a *triangulation* of the marked surface (S, V). If each triangle in the disjoint union X is hyperbolic and the identification maps are isometries, then the quotient metric d on the quotient space (S, V) is a called *hyperbolic polyhedral metric*. The set of cone points of d is in V. Given a hyperbolic polyhedral metric d and a triangulation $\mathcal{T}$ on (S, V), if each triangle in $\mathcal{T}$ (in metric d) is isometric to a hyperbolic triangle, we say $\mathcal{T}$ is *geodesic* in d. If $\mathcal{T}$ is a triangulation of (S, V) isotopic to a geometric triangulation $\mathcal{T}'$ in a hyperbolic polyhedral metric d, then the *length* of an edge $e \in E(\mathcal{T})$ (or *angle* of a triangle at a vertex in $\mathcal{T}$) is defined to be the length (respectively angle) of the corresponding geodesic edge $e' \in E(\mathcal{T}')$ (triangle at the vertex) measured in metric d.

Suppose e is an edge in $\mathcal{T}$ adjacent to two distinct triangles t, t'. Then the *diagonal switch* on $\mathcal{T}$ is a new triangulation $\mathcal{T}'$ obtained from $\mathcal{T}$ by replaces e by the other diagonal in the quadrilateral $t \cup_e t'$.

For simplicity, the terms metrics and triangulations in many places will mean isotopy classes of metrics and isotopy classes of triangulations. They can be understood from the context without causing confusion.

If X is a finite set, $|X|$ denotes its cardinality and $\mathbb{R}^X$ denotes the vector space $\{f : X \to \mathbb{R}\}$. For a finite vertex set $W = \{w_1, ..., w_m\}$, we identify $\mathbb{R}^W$ with $\mathbb{R}^m$ by sending $x \in \mathbb{R}^m$ to $(x(w_1), ..., x(w_m))$.

All surfaces are assumed to be compact and connected in the rest of the paper.

2.2. **The Teichmüller space and the length coordinates.** Two hyperbolic polyhedral metrics d, d' on (S, V) are called *Teichmüller equivalent* if there is an isometry $h : (S, V, d) \to (S, V, d')$ so that h is isotopic to the identity map on (S, V). The *Teichmüller space* of all hyperbolic polyhedral metrics on (S, V), denoted by $T_{hp}(S, V)$, is the set of all Teichmüller equivalence classes of hyperbolic polyhedral metrics on (S, V).

Lemma 5. *$T_{hp}(S, V)$ is a real analytic manifold.*

Proof. Suppose $\mathcal{T}$ is a triangulation of (S,V) with the set of edges $E=E(\mathcal{T})$. Let

$$\mathbb{R}^{E(\mathcal{T})}_{\Delta}=\{x\in\mathbb{R}^E_{>0}|\forall \text{ triangle } t \text{ in } \mathcal{T} \text{ with edges } e_i,e_j,e_k, \quad x(e_i)+x(e_j)>x(e_k)\}$$

be the convex polytope in $\mathbb{R}^E$. For each $x\in\mathbb{R}^{E(\mathcal{T})}_{\Delta}$, one constructs a hyperbolic polyhedral metric d_x on (S,V) by replacing each triangle t of edges e_i,e_j,e_k by a hyperbolic triangle of edge lengths $x(e_i),x(e_j),x(e_k)$ and gluing them by isometries along the corresponding edges. This construction produces an injective map (the length coordinate associated to $\mathcal{T}$)

$$\Phi_{\mathcal{T}}:\mathbb{R}^{E(\mathcal{T})}_{\Delta}\to T_{hp}(S,V)$$

sending x to $[d_x]$. The image $P(\mathcal{T}):=\Phi_{\mathcal{T}}(\mathbb{R}^{E(\mathcal{T})}_{\Delta})$ is the space of all hyperbolic polyhedral metrics $[d]$ on (S,V) for which $\mathcal{T}$ is isotopic to a geodesic triangulation in d. We call x the *length coordinate* of d_x and $[d_x]=\Phi_{\mathcal{T}}(x)$ (with respect to $\mathcal{T}$). In general $P(\mathcal{T})\neq T_{hp}(S,V)$ (see §2.1 in [9]).

Since each hyperbolic polyhedral metric on (S,V) admits a geometric triangulation (for instance its Delaunay triangulation), we see that $T_{hp}(S,V)=\cup_{\mathcal{T}}P(\mathcal{T})$ where the union is over all triangulations of (S,V). The space $T_{hp}(S,V)$ is a real analytic manifold with real analytic coordinate charts $\{(P(\mathcal{T}),\Phi^{-1}_{\mathcal{T}})|\mathcal{T}$ triangulations of $(S,V)\}$. To see transition functions $\Phi^{-1}_{\mathcal{T}}\Phi_{\mathcal{T}'}$ are real analytic, note that any two triangulations of (S,V) are related by a sequence of diagonal switches. Therefore, it suffices to show the result for $\mathcal{T}$ and $\mathcal{T}'$ which are related by a diagonal switch along an edge e. In this case, the transition function $\Phi^{-1}_{\mathcal{T}}\Phi_{\mathcal{T}'}$ sends $(x_0,x_1,....,x_m)$ to $(f(x_0,...,x_m),x_1,...,x_m)$ where x_0 is the length of e and f is the length of the diagonal switched edge. Let t,t' be the triangles adjacent to e so that the lengths of edges of t,t' are $\{x_0,x_1,x_2\}$ and $\{x_0,x_3,x_4\}$. Using the cosine law, we see that f is a real analytic function of $x_0,...,x_4$. □

2.3. **Delaunay triangulations and marked quadrilaterals.** Each hyperbolic triangle t in $\mathbf{H}^2$ has a circumcircle which is the curve of constant geodesic curvature containing the three vertices of t. When the circumcircle is compact, it is a hyperbolic circle. When it is not compact, it is either a horocycle or a curve of constant distance to a geodesic. We call the convex region bounded by the circumcircle the *circum-ball* of the triangle t. A *marked quadrilateral* Q is a hyperbolic quadrilateral together with a diagonal e inside Q. It is the same as a union of two hyperbolic triangles t,t' along a common edge e, i.e., $Q=t\cup_e t'$. A hyperbolic polygon is called *cyclic* if its vertices lie in a curve of constant geodesic curvature in the hyperbolic plane. A marked quadrilateral $t\cup_e t'$ is cyclic if and only if the two circumecircles for t and t' coincide.

A geodesic triangulation $\mathcal{T}$ of a hyperbolic polyhedral surface (S,V,d) is said to be *Delaunay* if for each edge e adjacent to two hyperbolic triangles t and t', the interior of the circumball of t does not contain the vertices of t' when the quadrilateral $t\cup_e t'$ is lifted to $\mathbf{H}^2$. The last condition is sometimes called the *empty ball condition*. We will call the marked quadrilateral $t\cup_e t'$ the *quadrilateral associated to the edge e*. G. Leibon [15] gave a very nice algebraic description of empty-ball condition in terms of the inner angles.

Lemma 6 (Leibon). *A geodesic triangulation $\mathcal{T}$ is Delaunay if and only if*

$$\alpha+\alpha'\leq\beta+\beta'+\gamma+\gamma' \tag{1}$$

for each edge e, where $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are angles of the two triangles in $\mathcal{T}$ having e as the common edge so that α and α' are opposite to e. Furthermore, the equality holds for e if and only if the marked quadrilateral associated to e is cyclic.

The inequality (1) can be expressed in terms of the edge lengths as follows.

Proposition 7. *A geodesic triangulation $\mathcal{T}$ is Delaunay if and only if*

$$\frac{\sinh^2(x_1/2)+\sinh^2(x_2/2)-\sinh^2(x_0/2)}{\sinh(x_1/2)\sinh(x_2/2)}+\frac{\sinh^2(x_3/2)+\sinh^2(x_4/2)-\sinh^2(x_0/2)}{\sinh(x_3/2)\sinh(x_4/2)} \geq 0 \tag{2}$$

for each edge e adjacent two triangles t, t' of edge lengths x_0, x_1, x_2 and x_0, x_3, x_4 respectively. Furthermore, the equality holds for an edge e if and only if $t \cup_e t'$ is cyclic.

Proof. We begin with

Lemma 8. *Let x_1, x_2, x_3 be side lengths of a hyperbolic triangle and a_1, a_2, a_3 be the opposite angles so that a_i is facing the edge of length x_i. Then*

$$2\sin\frac{a_2+a_3-a_1}{2}\cdot\cosh\frac{x_1}{2} = \frac{\sinh^2(x_2/2)+\sinh^2(x_3/2)-\sinh^2(x_1/2)}{\sinh(x_2/2)\sinh(x_3/2)}.$$

Proof. By the cosine law expressing x_i in terms of a_1, a_2, a_3, we have

$$\begin{aligned}
&\sinh^2(x_2/2)+\sinh^2(x_3/2)-\sinh^2(x_1/2)\\
&=\frac{1}{2}(\cosh(x_2)+\cosh(x_3)-\cosh(x_1)-1)\\
&=\frac{1}{2}[\frac{\cos a_2+\cos a_1\cos a_3}{\sin a_1\sin a_3}+\frac{\cos a_3+\cos a_1\cos a_2}{\sin a_1\sin a_2}-\frac{\cos a_1+\cos a_2\cos a_3}{\sin a_2\sin a_3}-1]\\
&=\frac{1}{2\sin a_1\sin a_2\sin a_3}(\sin(a_2+a_3)-\sin a_1)(\cos a_1+\cos(a_2-a_3))\\
&=\frac{2\sin\frac{a_2+a_3-a_1}{2}\cos\frac{a_1+a_2+a_3}{2}\cos\frac{a_1+a_2-a_3}{2}\cos\frac{a_1-a_2+a_3}{2}}{\sin a_1\sin a_2\sin a_3}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sinh^2(x_i/2) &= \frac{1}{2}(\cosh x_i - 1)\\
&= \frac{1}{2}(\frac{\cos a_i+\cos a_j\cos a_k}{\sin a_j\sin a_k}-1)\\
&= \frac{1}{2}\frac{\cos a_i+\cos(a_j+a_k)}{\sin a_j\sin a_k}\\
&= \frac{\cos\frac{a_i+a_j+a_k}{2}\cos\frac{a_i-a_j-a_k}{2}}{\sin a_j\sin a_k}.
\end{aligned}$$

Therefore

$$\frac{\sinh^2(x_2/2)+\sinh^2(x_3/2)-\sinh^2(x_1/2)}{\sinh(x_2/2)\sinh(x_3/2)}$$
$$= \frac{2\sin\frac{a_2+a_3-a_1}{2}\cos\frac{a_1+a_2+a_3}{2}\cos\frac{a_1+a_2-a_3}{2}\cos\frac{a_1-a_2+a_3}{2}}{\sin a_1\sin a_2\sin a_3\sqrt{\frac{\cos\frac{a_1+a_2+a_3}{2}\cos\frac{a_2-a_1-a_3}{2}}{\sin a_1\sin a_3}}\sqrt{\frac{\cos\frac{a_1+a_2+a_3}{2}\cos\frac{a_3-a_1-a_2}{2}}{\sin a_1\sin a_2}}}$$
$$= 2\sin\frac{a_2+a_3-a_1}{2}\cdot\sqrt{\frac{\cos\frac{a_1+a_2-a_3}{2}\cos\frac{a_1-a_2+a_3}{2}}{\sin a_2\sin a_3}}$$
$$= 2\sin\frac{a_2+a_3-a_1}{2}\cdot\cosh\frac{x_1}{2}.$$

In the last step above, we have used

$$\begin{aligned}(\cosh\frac{x_1}{2})^2 &= \frac{1}{2}(\cosh x_1+1)\\ &= \frac{1}{2}(\frac{\cos a_1+\cos a_2\cos a_3}{\sin a_2\sin a_3}+1)\\ &= \frac{1}{2}\frac{\cos a_1+\cos(a_2-a_3)}{\sin a_2\sin a_3}\\ &= \frac{\cos\frac{a_1+a_2-a_3}{2}\cos\frac{a_1-a_2+a_3}{2}}{\sin a_2\sin a_3}.\end{aligned}$$

□

Now (1) is equivalent to $\sin\frac{\beta+\gamma-\alpha}{2}+\sin\frac{\beta'+\gamma'-\alpha'}{2}\geq 0$. By Lemma 8 applied to triangles of lengths $\{x_0,x_1,x_2\}$ and $\{x_0,x_3,x_4\}$, we see that Delaunay is equivalent to (2). □

2.4. Delaunay triangulations of compact hyperbolic polyhedral surfaces.

Theorem 9. *If $\mathcal{T}$ is a Delaunay triangulation of a closed hyperbolic polyhedral surface (S,V,d), then each triangle has a compact circumcircle.*

Proof. By Proposition 7 for Delaunay triangulations inequality (2) holds. On the other hand, by lemma 4.2 of [9],

Lemma 10 ([9])**.** *Suppose $y:E(\mathcal{T})\to\mathbb{R}_{>0}$ is a function satisfying for each edge e_0 adjacent to two triangles t,t' of edges e_0,e_1,e_2 and e_0,e_3,e_4*

$$\frac{y_1^2+y_2^2-y_0^2}{y_1y_2}+\frac{y_3^2+y_4^2-y_0^2}{y_3y_4}\geq 0$$

where $y_i=y(e_i)$. Then $y(e_i)+y(e_j)>y(e_k)$ whenever e_i,e_j,e_k form edges of a triangle in $\mathcal{T}$.

Taking $y(e)=\sinh(\frac{x(e)}{2})$ in the above lemma and using (2), we obtain

$$\sinh(\frac{x(e_i)}{2})+\sinh(\frac{x(e_j)}{2})>\sinh(\frac{x(e_k)}{2}). \tag{3}$$

Now theorem 9 follows from (3) and a result in [7] page 118,

Proposition 11 (Fenchel)**.** *Let C be the circumcircle of a hyperbolic triangle of edge lengths x_i,x_j,x_k. Then C is a (compact) hyperbolic circle if and only if $\sinh(\frac{x_i}{2})+\sinh(\frac{x_j}{2})>\sinh(\frac{x_k}{2})$.*

□

Since $\sinh(a+b) > \sinh(a) + \sinh(b)$ for $a, b > 0$, by (3), we obtain

$$x(e_i) + x(e_j) > x(e_k), \tag{4}$$

whenever e_i, e_j, e_k form edges of a triangle. This implies,

Corollary 12. *Suppose $x : E(\mathcal{T}) \to \mathbb{R}_{>0}$ is a function so that (2) holds at each edge. Then x is the edge length function (in $\mathcal{T}$) of a hyperbolic polyhedral metric on (S, V).*

It is highly likely that theorem 9 still holds for hyperbolic cone metrics on high dimensional compact manifolds, i.e., empty-ball condition implies compact circumsphere. The work of [6] shows that it holds for decorated finite volume hyperbolic metrics of any dimension.

The classical way of constructing many Delaunay triangulations of a polyhedral metric d on (S, V) is as follows. See for instance [3]. Define the *Voronoi decomposition* of (S, V, d) to be the collection of 2-cells $\{R(v)|v \in V\}$ where $R(v) = \{x \in S|d(x, v) \leq d(x, v') \text{ for all } v' \in V\}$. Its dual is called a *Delaunay tessellation* $\mathcal{C}(d)$ of (S, V, d). It is a cell decomposition of (S, V) with vertices V and two vertices v, v' jointed by an edge if and only if $R(v) \cap R(v')$ is 1-dimensional. By definition, each 2-cell in the Delaunay tessellation is a convex polygon inscribed to a compact circle in $\mathbf{H}^2$ whose center is a vertex of the Voronoi decomposition. By further triangulating all non-triangular 2-dimensional cells (without introducing extra vertices) in $\mathcal{C}(d)$, one obtains a Delaunay triangulation of (S, V, d). This Delaunay triangulation has the property that the circumcircles of triangles are hyperbolic circles (i.e., compact). Indeed, the centers of the circumcircles are the vertices in the Voronoi cell decomposition. Conversely, if $\mathcal{T}$ is a Delaunay triangulation with compact circumcircles for all triangles, then it is a triangulation of the Delaunay tessellation. Combining theorem 9, we obtain part (a) of the following,

Proposition 13. *(a) Suppose $\mathcal{T}$ is a geodesic triangulation of a compact hyperbolic polyhedral surface (S, V, d). Then $\mathcal{T}$ satisfies the empty-ball condition if and only if it is a geodesic triangulation of the Delaunay tessellation.*

(b) If $\mathcal{T}$ and $\mathcal{T}'$ are Delaunay triangulations of a hyperbolic polyhedral metric d on a closed marked surface (S, V), then there exists a sequence of Delaunay triangulations $\mathcal{T}_1 = \mathcal{T}, \mathcal{T}_2, ..., \mathcal{T}_k = \mathcal{T}'$ of d so that $\mathcal{T}_{i+1}$ is obtained from $\mathcal{T}_i$ by a diagonal switch.

(c) Suppose $\mathcal{T}$ is a Delaunay triangulation of a compact hyperbolic polyhedral surface (S, V, d) whose diameter is D. Then the length of each edge e in $\mathcal{T}$ is at most $2D$. In particular, there exists an algorithm to find all Delaunay triangulations of a hyperbolic polyhedral surface.

Proof. Part(b) of the proposition follows from part(a) and the well known fact that any two geodesic triangulations of the Delaunay tessellation are related by a sequence of diagonal switches. Indeed, any two geodesic triangulations of a convex cyclic polygon are related by a sequence of (geodesic) diagonal switches. See for instance [3] for a proof.

To see part (c), if e is an edge dual to two Voronoi cells $R(v)$ and $R(v')$, then the length of e is at most the sum of the diameters of $R(v)$ and $R(v')$. However, the diameters of $R(v)$ and $R(v')$ are bounded by the diameter of the surface S. Thus, the length of e is at most $2D$. It is well known that for any constant C, there exists an algorithm to list all geodesic paths in (S,V,d) of lengths at most C joining V to V. Therefore, we can list algorithmically all Delaunay triangulations of a given polyhedral metric on (S,V).

□

Note that if we remove the compactness of the space S, then there are examples of geodesic triangulations with empty-ball condition which does not come from dual of Voronoi cell. See [6].

For a triangulation $\mathcal{T}$ of (S,V), the associated Delaunay cell in $T_{hp}(S,V)$ is defined to be

$$D_c(\mathcal{T}) \quad = \{[d] \in T_{hp}(S,V) | \mathcal{T} \text{ is isotopic to a Delaunay triangulation of } d\}.$$

Theorem 9 and corollary 12 show that $D_c(\mathcal{T})$ is defined by a finite set of real analytic inequalities (i.e., (2)). On the other hand, Leibon showed in [15] that $D_c(\mathcal{T})$ is a cell. Putting these together, one obtains

Theorem 14 (Hazel[13], Leibon[15])**.** *There is a real analytic cell decomposition*

$$T_{hp}(S,V) = \cup_{[\mathcal{T}]} D_c(\mathcal{T})$$

invariant under the action of the mapping class group where the union is over all isotopy classes $[\mathcal{T}]$ of triangulations of (S,V).

3. Diffeomorphism between two Teichmüller spaces

One of the main tools used in our proof is the decorated Teichmüller space theory developed by R. Penner [19]. See also [2], [10] and [9] for a discussion of Delaunay triangulations of decorated metrics.

Recall that S is a closed connected surface and $V = \{v_1, ..., v_n\} \subset S$ and let $\Sigma = S - V$. We assume $n \geq 1$ and the Euler characteristic $\chi(\Sigma) < 0$. A *decorated hyperbolic metric* is a complete hyperbolic metric d of finite area on Σ together with a horoball H_i at the i-th cusp for each v_i. The decorated metric will be written as a pair (d,w) where $w = (w_1, ..., w_n) \in \mathbb{R}^n_{>0}$ so that w_i is the length of the horocycle ∂H_i. The decorated Teichmüller space, denoted by $T_D(\Sigma)$, is the space of all decorated metrics on Σ modulo isometries homotopic to the identity and preserving decorations. For a given triangulation $\mathcal{T}$ of (S,V), let $\Psi_{\mathcal{T}} : \mathbb{R}^E_{>0} \to T_D(\Sigma)$ be the λ-length coordinate (see [19]) and let $D(\mathcal{T})$ be the set of all decorated hyperbolic metrics (d,w) in $T_D(\Sigma)$ so that $\mathcal{T}$ is isotopic to a Delaunay triangulation of (d,w). See [19] or [9] for details.

Fix a triangulation $\mathcal{T}$ of (S,V), we have two coordinate maps $\Phi^{-1}_{\mathcal{T}} : P(\mathcal{T}) \to \mathbb{R}^{E(\mathcal{T})}$ and $\Psi_{\mathcal{T}} : \mathbb{R}^{E(\mathcal{T})} \to T_D(S,V)$. Consider the smooth embedding $A_{\mathcal{T}} : P(\mathcal{T}) \to T_D(\Sigma)$ defined by $\Psi_{\mathcal{T}} \circ \Theta \circ \Phi^{-1}_{\mathcal{T}}$, where $\Theta : \mathbb{R}^{E(\mathcal{T})} \to \mathbb{R}^{E(\mathcal{T})}$ sends $(x_0, x_1, x_2, ...)$ to $(\sinh(x_0/2), \sinh(x_1/2), \sinh(x_2/2), ...)$, i.e., $\Theta(x)(e) = \sinh(x(e)/2)$.

Theorem 15. *For each triangulation $\mathcal{T}$ of (S,V), $A_{\mathcal{T}}|_{D_c(\mathcal{T})}$ is a real analytic diffeomorphism from $D_c(\mathcal{T})$ onto $D(\mathcal{T})$.*

Proof. To see that $A_{\mathcal{T}}$ maps $D_c(\mathcal{T})$ bijectively onto $D(\mathcal{T})$, it suffices to show that $\Theta \circ \Phi^{-1}_{\mathcal{T}}(D_c(\mathcal{T})) = \Psi^{-1}_{\mathcal{T}}(D(\mathcal{T}))$.

The space $\Psi^{-1}_{\mathcal{T}}(D(\mathcal{T}))$ can be characterized as follows. For each edge e in $(S,\mathcal{T})$ with a decorated hyperbolic metric (d,w), let a, a' be the two angles facing e and

b, b', c, c' be the angles adjacent to the edge e. Then $\mathcal{T}$ is Delaunay in the metric (d, w) if and only if for each edge $e \in E(\mathcal{T})$ (see [19], or [10]),

$$a + a' \leq b + b' + c + c'. \tag{5}$$

Let t and t' be the triangle adjacent to e and e, e_1, e_2 be edges of t and e, e_3, e_4 be the edges of t'. Let the λ-length of e be λ_0 and the λ-length of e_i be λ_i. Recall the cosine law for decorated ideal triangles [19] states that $\alpha = \frac{x}{yz}$ where α is the angle (i.e., the length of the horocyclic arc) and x, y, z are the λ-lengths so that x faces α. Using it, one sees that (5) is equivalent to

$$\frac{\lambda_0}{\lambda_1\lambda_2} + \frac{\lambda_0}{\lambda_3\lambda_4} \leq \frac{\lambda_1}{\lambda_0\lambda_2} + \frac{\lambda_2}{\lambda_0\lambda_1} + \frac{\lambda_3}{\lambda_0\lambda_4} + \frac{\lambda_4}{\lambda_0\lambda_3}, \tag{6}$$

for each $e \in E(\mathcal{T})$.

Rearranging terms, we see (6) is equivalent to

$$0 \leq \frac{\lambda_1^2 + \lambda_2^2 - \lambda_0^2}{\lambda_1\lambda_2} + \frac{\lambda_3^2 + \lambda_4^2 - \lambda_0^2}{\lambda_3\lambda_4}, \tag{7}$$

for each $e \in E(\mathcal{T})$.

Therefore,

$$\Psi_{\mathcal{T}}^{-1}(D(\mathcal{T})) = \{(\lambda_0, \lambda_1, ..., \lambda_{|E|}) \in \mathbb{R}^E_{>0} | \text{ (7) holds at each edge } e \in E(\mathcal{T})\}.$$

By Theorem 9 and proposition 7, the characterization of a hyperbolic polyhedral metric d which is Delaunay in $\mathcal{T}$ in terms of the length coordinate $x = \Phi_{\mathcal{T}}^{-1}(d)$ is as follows. Take an edge $e \in E(\mathcal{T})$ and let t and t' be the triangles adjacent to e so that e, e_1, e_2 are edges of t and e, e_3, e_4 are the edge of t'. Suppose the length of e (in d) is x_0 and the length of e_i is x_i, $i = 1, ..., 4$. Then, by Proposition 7,

$$0 \leq \frac{\sinh^2(x_1/2) + \sinh^2(x_2/2) - \sinh^2(x_0/2)}{\sinh(x_1/2)\sinh(x_2/2)} + \frac{\sinh^2(x_3/2) + \sinh^2(x_4/2) - \sinh^2(x_0/2)}{\sinh(x_3/2)\sinh(x_4/2)} \tag{8}$$

holds for each edge $e \in E(\mathcal{T})$.

This shows that

$$\Phi_{\mathcal{T}}^{-1}(D_c(\mathcal{T})) = \{x \in \mathbb{R}^E_{>0} | \text{ (8) holds for } e \in E, \text{ and (9) holds for each triangle}\}$$

where

$$x(e_i) + x(e_j) > x(e_k), \quad e_i, e_j, e_k \text{ form edges of a triangle in } \mathcal{T}. \tag{9}$$

Now inequality (7) is the same as (8) by taking λ_i to be $\sinh(x_i/2)$ for each i. This shows $\Theta \circ \Phi_{\mathcal{T}}^{-1}(D_c(\mathcal{T})) \subset \Psi_{\mathcal{T}}^{-1}(D(\mathcal{T}))$. On the other hand, corollary 12 implies that for each $\lambda \in \Psi_{\mathcal{T}}^{-1}(D(\mathcal{T}))$ and a triangle of edges e_i, e_j, e_k, we have $x(e_i) + x(e_j) > x(e_k)$ where $x(e) = 2\sinh^{-1}(\lambda(e))$, i.e., condition (9) is a consequence of (8). Therefore $\Theta \circ \Phi_{\mathcal{T}}^{-1}(D_c(\mathcal{T})) = \Psi_{\mathcal{T}}^{-1}(D(\mathcal{T}))$.

Finally, since $\Phi_{\mathcal{T}}$, $\Psi_{\mathcal{T}}$ and Θ are real analytic diffeomorphisms and $A_{\mathcal{T}} = \Psi_{\mathcal{T}} \circ \Theta \circ \Phi_{\mathcal{T}}^{-1}$ and $A_{\mathcal{T}}^{-1} = \Phi_{\mathcal{T}} \circ \Theta^{-1} \circ \Psi_{\mathcal{T}}^{-1}$, we see that $A_{\mathcal{T}}$ is a real analytic diffeomorphism. □

3.1. **The Ptolemy identity and diagonal switch.** Let Q be a convex quadrilateral Q in the Euclidean plane $\mathbf{E}^2$, or the hyperbolic plane $\mathbf{H}^2$ or the 2-sphere $\mathbf{S}^2$ so that its edges are a, b, a', b' counted cyclically and its diagonals are c, c'. We say Q is *cyclic* if it is circumscribed to a circle in $\mathbf{E}^2$, or $\mathbf{S}^2$, or a curve of constant geodesic curvature in $\mathbf{H}^2$. Let $l(e)$ to be the length of an edge e.

The classical Ptolemy theorem states that a Euclidean quadrilateral Q is cyclic if and only if the following holds

$$l(a)l(a') + l(b)l(b') = l(c)l(c').$$

In the 19-th century, Jean Darboux and Ferdinand Frobenius proved that a spherical quadrilateral Q is cyclic if and only if

$$\sin(\frac{l(a)}{2})\sin(\frac{l(a')}{2}) + \sin(\frac{l(b)}{2})\sin(\frac{l(b')}{2}) = \sin(\frac{l(c)}{2})\sin(\frac{l(c')}{2}).$$

The hyperbolic case was established by T. Kubota in 1912 [14]. He proved,

Proposition 16 (Kubota)**.** *A hyperbolic quadrilateral Q is inscribed to a curve of constant geodesic curvature in $\mathbf{H}^2$ if and only if*

$$\sinh(\frac{l(a)}{2})\sinh(\frac{l(a')}{2}) + \sinh(\frac{l(b)}{2})\sinh(\frac{l(b')}{2}) = \sinh(\frac{l(c)}{2})\sinh(\frac{l(c')}{2}). \tag{10}$$

Penner's Ptolemy identity [19] also takes the same form. Namely, if Q is a decorated ideal quadrilateral in $\mathbf{H}^2$ so that the λ-lengths of the its edges are A, B, A', B' counted cyclically and its diagonal are C, C', then

$$AA' + BB' = CC'. \tag{11}$$

The most remarkable feature of these theorems is that all equations take the same form as $xx' + yy' = zz'$ which we will call the Ptolemy identity. The Ptolemy identity also plays the key role for cluster algebras associated to surfaces [8].

The relationship between the Ptolemy identity and the diagonal switch operation on Delaunay triangulations is the following. If $\mathcal{T}$ and $\mathcal{T}'$ are two Delaunay triangulations of a Euclidean (or hyperbolic or spherical) polyhedral surface (S, V, d) so that they are related by a diagonal switch from edge e to edge e', then the change of the lengths from $l(e)$ and $l(e')$ is governed by one of the Ptolemy identities listed above.

Casey's generalization of Ptolemy's theorem is another direction where Ptolemy identity plays a key role. Furthermore, Casey's theorem is known to be true for Euclidean, hyperbolic, spherical and even Minkowski planes. In [11], we will exam the related discrete conformality in the new setting.

3.2. **A globally defined diffeomorphism.**

Theorem 17. *Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two triangulations of (S, V) so that $D_c(\mathcal{T}) \cap D_c(\mathcal{T}') \neq \emptyset$. Then*

$$A_{\mathcal{T}}|_{D_c(\mathcal{T})\cap D_c(\mathcal{T}')} = A_{\mathcal{T}'}|_{D_c(\mathcal{T})\cap D_c(\mathcal{T}')}. \tag{12}$$

In particular, the gluing of these $A_{\mathcal{T}}|_{D_c(\mathcal{T})}$ mappings produces a homeomorphism $A = \cup_{\mathcal{T}} A_{\mathcal{T}}|_{D_c(\mathcal{T})} : T_{hp}(S, V) \to T_D(\Sigma)$ such that $A(d)$ and $A(d')$ have the same underlying hyperbolic structure if and only if d and d' are discrete conformal.

Proof. Suppose $d \in D_c(\mathcal{T}) \cap D_c(\mathcal{T}')$, i.e., $\mathcal{T}$ and $\mathcal{T}'$ are both Delaunay in the hyperbolic polyhedral metric d. Then by proposition 13 there exists a sequence of triangulations $\mathcal{T}_1 = \mathcal{T}, \mathcal{T}_2, ..., T_k = \mathcal{T}'$ on (S, V) so that each $\mathcal{T}_i$ is Delaunay in d and $\mathcal{T}_{i+1}$ is obtained from $\mathcal{T}_i$ by a diagonal switch. In particular, $A_{\mathcal{T}}(d) = A_{\mathcal{T}'}(d)$ follows from $A_{\mathcal{T}_i}(d) = A_{\mathcal{T}_{i+1}}(d)$ for $i = 1, 2, ..., k-1$. Thus, it suffices to show $A_{\mathcal{T}}(d) = A_{\mathcal{T}'}(d)$ when $\mathcal{T}'$ is obtained from $\mathcal{T}$ by a diagonal switch along an edge e. This is the same as showing $\Psi_{\mathcal{T}}^{-1}\Psi_{\mathcal{T}'} = \Theta\Phi_{\mathcal{T}}^{-1}\Phi_{\mathcal{T}'}\Theta^{-1}$ at the point $x = \Psi_{\mathcal{T}'}^{-1}(d)$. On the other hand, $\Psi_{\mathcal{T}}^{-1}\Psi_{\mathcal{T}'}(x)$ and $\Theta\Phi_{\mathcal{T}}^{-1}\Phi_{\mathcal{T}'}\Theta^{-1}(x)$ have the same coordinate except at the e edge of diagonal switch. For the edge e, the two coordinates are the same due to the Penner's Ptolemy identity (11) (for $\Psi_{\mathcal{T}}^{-1}\Psi_{\mathcal{T}'}$) and Kubota's Ptolemy identity (10) (for $\Phi_{\mathcal{T}}^{-1}\Phi_{\mathcal{T}'}$). These two identities differ by a change of variable $t \to \sinh(\frac{t}{2})$ which corresponds to Θ. Therefore, $A_{\mathcal{T}}(d) = A_{\mathcal{T}'}(d)$.

Taking the inverse, we obtain

$$A_{\mathcal{T}}^{-1}|_{D(\mathcal{T}) \cap D(\mathcal{T}')} = A_{\mathcal{T}'}^{-1}|_{D(\mathcal{T}) \cap D(\mathcal{T}')}. \tag{13}$$

Lemma 18. (a) *$D_c(\mathcal{T}) \cap D_c(\mathcal{T}') \neq \emptyset$ if and only if $D(\mathcal{T}) \cap D(\mathcal{T}') \neq \emptyset$.*

(b) *The gluing map $A = \cup_{\mathcal{T}} A_{\mathcal{T}}|_{D_c(\mathcal{T})} : T_c \to T_D$ is a homeomorphism invariant under the action of the mapping class group.*

Proof. By (12) and (13), the maps $A = \cup_{\mathcal{T}} A_{\mathcal{T}}|_{D_c(\mathcal{T})} : T_c \to T_D$ and $B = \cup_{\mathcal{T}} A_{\mathcal{T}}^{-1}|_{D(\mathcal{T})} : T_D \to T_c$ are well defined and continuous. Since $A(D_c(\mathcal{T}) \cap D_c(\mathcal{T}')) \subset D(\mathcal{T}) \cap D(\mathcal{T}')$ and $B(D(\mathcal{T}) \cap D(\mathcal{T}')) \subset D_c(\mathcal{T}) \cap D_c(\mathcal{T}')$, part (a) follows. To see part (b), by Penner's result [19] that $T_D = \cup_{\mathcal{T}} D(\mathcal{T})$, the map A is onto. To see A is injective, suppose $x_1 \in D_c(\mathcal{T}_1), x_2 \in D_c(\mathcal{T}_2)$ so that $A(x_1) = A(x_2) \in D(\mathcal{T}_1) \cap D(\mathcal{T}_2)$. Apply (13) to $A_{\mathcal{T}_1}^{-1}|, A_{\mathcal{T}_2}^{-1}|$ on the set $D(\mathcal{T}_1) \cap D(\mathcal{T}_2)$ at the point $A(x_1)$, we conclude that $x_1 = x_2$. This shows that A is a bijection with inverse B. Since both A and B are continuous, A is a homeomorphism. □ □

Now if d and d' are two discrete conformally equivalent hyperbolic polyhedral metrics, then $A(d)$ and $A(d')$ are of the form (p, w) and (p, w') due to the definitions. Indeed, if d and d' are related by condition (b) in definition 1, then the discrete conformality translates to the change of decoration without changing the hyperbolic metric. (This is the same proof as in [9], lemma 3.1). If d and d' are related by condition (c) in definition 1, then the two triangulations $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$ are both Delaunay in $[d]$. Therefore, in this case, $A(d) = A(d')$.

On the other hand, if two hyperbolic cone metrics d, d' satisfy that $A(d)$ and $A(d')$ are of the form (p, w) and (p, w'), consider a generic smooth path $\gamma(t) = (p, w(t)), t \in [0, 1]$, in $T_D(\Sigma)$ from (p, w) to (p, w') so that $\gamma(t)$ intersects the cells $D(\mathcal{T})$'s transversely. This implies that γ passes through a finite set of cells $D(\mathcal{T}_i)$ and $\mathcal{T}_j$ and $\mathcal{T}_{j+1}$ are related by a diagonal switch. Let $t_0 = 0 < ... < t_m = 1$ be a partition of $[0, 1]$ so that $\gamma([t_i, t_{i+1}]) \subset D(\mathcal{T}_i)$. Say d_i is the hyperbolic polyhedral metric so that $A(d_i) = \gamma(t_i) \in D(\mathcal{T}_i) \cap D(\mathcal{T}_{i+1})$, $d_1 = d$ and $d_m = d'$. Then by definition, the sequences $\{d_1, ..., d_m\}$ and the associated Delaunay triangulations $\{\mathcal{T}_1, ..., T_m\}$ satisfy the definition of discrete conformality for d, d'.

□

Theorem 19. *The homeomorphism $A : T_{hp}(S, V) \to T_D(\Sigma)$ is a C^1 diffeomorphism.*

Proof. It suffices to show that for a point $d \in D_c(\mathcal{T}) \cap D_c(\mathcal{T}')$, the derivatives $DA_{\mathcal{T}}(d)$ and $DA_{\mathcal{T}'}(d)$ are the same. Since both $\mathcal{T}$ and $\mathcal{T}'$ are Delaunay in d and are related by a sequence of Delaunay triangulations (in d) $\mathcal{T}_1 = \mathcal{T}, \mathcal{T}_2, ..., \mathcal{T}_k = \mathcal{T}'$, $DA_{\mathcal{T}}(d) = DA_{\mathcal{T}'}(d)$ follows from $DA_{\mathcal{T}_i}(d) = DA_{\mathcal{T}_{i+1}}(d)$ for $i = 1, 2, ..., k-1$. Therefore, it suffices to show $DA_{\mathcal{T}}(d) = DA_{\mathcal{T}'}(d)$ when $\mathcal{T}$ and $\mathcal{T}'$ are related by a diagonal switch at an edge e. In the coordinates $\Phi_{\mathcal{T}}$ and $\Psi_{\mathcal{T}}$, the fact that $DA_{\mathcal{T}}(d) = DA_{\mathcal{T}'}(d)$ is equivalent to the following smoothness question on the diagonal lengths.

Lemma 20. *Suppose Q is a convex hyperbolic quadrilateral whose four edges are of lengths x, y, z, w (counted cyclically) and the length of a diagonal is a. Suppose $A(x,y,z,w,a)$ is the length of the other diagonal and $B(x,y,z,w,a) = 2\sinh^{-1}(\frac{s(x)s(z)+s(y)s(w)}{s(a)})$ where $s(t) = \sinh(\frac{t}{2})$. If a point (x,y,z,w,a) satisfies $A(x,y,z,w,a) = B(x,y,z,w,a)$, i.e., Q is inscribed in a curve of constant geodesic curvature, then $DA(x,y,z,w,a) = DB(x,y,z,w,a)$ where DA is the derivative of A.*

Due to the lengthy proof of this lemma, we defer it to the appendix.

□

Corollary 21. *For a given hyperbolic polyhedral metric d on (S,V), the set of all Teichmüller equivalence classes of hyperbolic metrics on (S,V) which are discrete conformal to d is C^1-diffeomorphic to $\mathbb{R}^{|V|}$.*

4. Discrete Uniformization for Hyperbolic Polyhedral Metrics

This section proves theorem 3 which is the main result of this paper.

By Corollary 21, Theorem 3 is equivalent to a statement about the composition map of the discrete curvature map K and $(A|)^{-1}$ defined on $\{p\} \times \mathbb{R}^n_{>0} \subset T_D(\Sigma)$ for any $p \in T(\Sigma)$. Here $K : T_{hp}(S,V) \to (-\infty, 2\pi)^n$ is the map sending a metric d to its discrete curvature K_d. Let us make a change of variables from $w = (w_1, ..., w_n) \in \mathbb{R}^n_{>0}$ to $u = (u_1, ..., u_n) \in \mathbb{R}^n$ where $u_i = \ln(w_i)$. We write $w = w(u)$. For a given $p \in T(\Sigma)$, define F to be the composition of K and $(A|)^{-1}$ from $\mathbb{R}^n$ to $(-\infty, 2\pi)^n$ by

$$F(u) = K_{A^{-1}(p,w(u))}. \tag{14}$$

By the Gauss-Bonnet theorem, the image $F(u)$ lies in the open subset $\mathbf{P} = \{x \in (-\infty, 2\pi)^n | \sum_{i=1}^n x_i > 2\pi\chi(S)\}$ of $\mathbb{R}^n$. Theorem 3 is equivalent to that $F : \mathbb{R}^n \to \mathbf{P}$ is a bijection. We will show a stronger statement that F is a homeomorphism.

For simplicity, we use $s(t)$ to denote the function $\sinh(\frac{t}{2})$.

4.1. Injectivity of F.

Since A is a C^1 diffeomorphism and the discrete curvature $K : \mathcal{T}_{hp}(S,V) \to \mathbb{R}^V$ is real analytic, hence the map F is C^1 smooth.

On the other hand, we have,

Theorem 22 (Akiyoshi [1])**.** *For any finite area complete hyperbolic metric p on Σ, there are only finitely many isotopy classes of triangulations $\mathcal{T}$ so that $([p] \times \mathbb{R}^n_{>0}) \cap D(\mathcal{T}) \neq \emptyset$.*

Let $\mathcal{T}_i$, $i = 1, ..., k$, be the set of all triangulations so that $(\{p\} \times \mathbb{R}^n) \cap D(\mathcal{T}_i) \neq \emptyset$ and $\{p\} \times \mathbb{R}^n \subset \cup_{i=1}^k D(\mathcal{T}_i)$.

Lemma 23. *Let $\phi : \mathbb{R}^n \to \{p\} \times \mathbb{R}^n$ be $\phi(x) = (p, x)$ and $U_i = \phi^{-1}((\{p\} \times \mathbb{R}^n) \cap D(\mathcal{T}_i)) \subset \mathbb{R}^n$ and $J = \{i| \, int(U_i) \neq \emptyset\}$. Then $\mathbb{R}^n = \cup_{i\in J} U_i$ and U_i is real analytic diffeomorphic to a convex polytope in $\mathbb{R}^n$.*

Proof. By definition, both $\{p\} \times \mathbb{R}^n$ and $D(\mathcal{T}_i)$ are closed and semi algebraic in $T_D(\Sigma)$. Therefore U_i is closed in $\mathbb{R}^n$ and is diffeomorphic under $w = w(u)$ to a semi-algebraic set. Now by definition, $Y := \cup_{i\in J} U_i$ is a closed subset of $\mathbb{R}^n$ since U_i is closed. If $Y \neq \mathbb{R}^n$, then the complement $\mathbb{R}^n - Y$ is a non-empty open set which is diffeomorphic under $w = w(u)$ to a finite union of real algebraic sets of dimension less than n. This is impossible.

Finally, we will show that for any triangulation $\mathcal{T}$ of (S, V) and $p \in T(\Sigma)$, the intersection $U = \phi^{-1}((\{p\} \times R^n) \cap D(\mathcal{T}))$ is real analytically diffeomorphic to a convex polytope in a Euclidean space. In fact $\Psi_{\mathcal{T}}^{-1}(U) \subset \mathbb{R}^{E(\mathcal{T})}$ is real analytically diffeomorphic to a convex polytope. To this end, let $b = \Psi_{\mathcal{T}}(p, (1, 1, ..., 1))$. By definition, $\Psi_{\mathcal{T}}^{-1}(U)$ is give by

$$\{x \in \mathbb{R}_{>0}^{E(\mathcal{T})} | \exists \lambda \in \mathbb{R}_{>0}^V, \sinh(x(e)/2) = b(e)\lambda(v_1)\lambda(v_2), \partial e = \{v_1, v_2\}, \text{ Delaunay condition (2) holds for } x\}.$$

We claim that the Delaunay condition (2) consists of linear inequalities in the variable $\delta : V \to \mathbb{R}_{>0}$ where $\delta(v) = \lambda(v)^{-2}$. Indeed, suppose the two triangles adjacent to the edge $e = (v_1, v_2)$ have vertices v_1, v_2, v_3 and v_1, v_2, v_4. Let x_{ij} (respectively b_{ij}) be the value of x (respectively b) at the edge joining v_i, v_j, and $\lambda_i = \lambda(v_i)$ and let $s(t)$ be the function $\sinh(\frac{t}{2})$. By definition, $s(x_{ij}) = b_{ij}\lambda_i\lambda_j$. The Delaunay condition (2) at the edge $e = (v_1 v_2)$ says that

$$\frac{s(x_{12})^2}{s(x_{31})s(x_{32})} + \frac{s(x_{12})^2}{s(x_{41})s(x_{42})} \leq \frac{s(x_{31})}{s(x_{32})} + \frac{s(x_{32})}{s(x_{31})} + \frac{s(x_{41})}{s(x_{42})} + \frac{s(x_{42})}{s(x_{41})}$$

It is the same as, using $s(x_{ij}) = b_{ij}\lambda_i\lambda_j$,

$$c_3\frac{\lambda_1\lambda_2}{\lambda_3^2} + c_4\frac{\lambda_1\lambda_2}{\lambda_4^2} \leq c_1\frac{\lambda_2}{\lambda_1} + c_2\frac{\lambda_1}{\lambda_2},$$

where c_i is some constant depending only on b_{jk}'s. Dividing above inequality by $\lambda_1\lambda_2$ and using $\delta_i = \lambda_i^{-2}$, we obtain

$$c_3\delta_3 + c_4\delta_4 \leq c_1\delta_1 + c_2\delta_2 \tag{15}$$

at each edge $e \in E(\mathcal{T})$. This shows for b fixed, the set of all possible values of δ form a convex polytope $\mathbf{Q}$ defined by (15) at all edges and $\delta(v) > 0$ at all $v \in V$. On the other hand, by definition, the map from $\mathbf{Q}$ to $\Psi_{\mathcal{T}}^{-1}(U)$ sending δ to $x = x(\delta)$ given by $x(vv') = 2\sinh^{-1}(\frac{b(vv')}{\sqrt{\delta(v)\delta(v')}})$ is a real analytic diffeomorphism. Thus the result follows. □

Write $F = (F_1, ..., F_n)$ which is C^1 smooth. The work of Bobenko-Pinkall-Springborn ([4], proposition 5.1.5) shows that

(a) $F_j|_{U_h}$ is real analytic so that $\frac{\partial F_i}{\partial u_j} = \frac{\partial F_j}{\partial u_i}$ in U_h for all $h \in J$,

(b) the Hessian matrix $[\frac{\partial F_i}{\partial u_j}]$ is positive definite on each U_h.

Therefore, the 1-form $\eta = \sum_i F_i(u)du_i$ is a C^1 smooth 1-form on $\mathbb{R}^n$ so that $d\eta = 0$ on each $U_h, h \in J$. This implies that $d\eta = 0$ in $\mathbb{R}^n$. Hence the integral

$$W(u) = \int_0^u \eta \tag{16}$$

is a well defined C^2 smooth function on $\mathbb{R}^n$ so that its Hessian matrix is positive definite. Therefore, W is convex in $\mathbb{R}^n$ so that its gradient $\nabla W = F$. Now F is injective due to the following well known lemma,

Lemma 24. *If $W : \Omega \to \mathbb{R}$ is a C^1-smooth strictly convex function on an open convex set $\Omega \subset \mathbb{R}^m$, then its gradient $\nabla W : \Omega \to \mathbb{R}^m$ is an embedding.*

4.2. **The map F is onto.** Since both $\mathbb{R}^n$ and $\mathbf{P} = \{x \in (-\infty, 2\pi)^n | \sum_{i=1}^n x_i > 2\pi\chi(S)\}$ are connected manifolds of dimension n and F is injective and continuous, it follows that $F(\mathbb{R}^n)$ is open in $\mathbf{P}$. To show that F is onto, it suffices to prove that $F(\mathbb{R}^n)$ is closed in $\mathbf{P}$.

To this end, take a sequence $\{u^{(m)}\}$ in $\mathbb{R}^n$ which leaves every compact set in $\mathbb{R}^n$. We will show that $\{F(u^{(m)})\}$ leaves each compact set in $\mathbf{P}$. By taking subsequences, we may assume that for each index $i = 1, 2, ..., n$, the limit $\lim_m u_i^{(m)} = t_i$ exists in $[-\infty, \infty]$. Furthermore, by Akiyoshi's theorem that the space $p \times \mathbb{R}^n$ is in the union of a finite number of Delaunay cells $D(\mathcal{T})$, we may assume, after taking another subsequence, that the corresponding hyperbolic polyhedral metrics $d_m = A^{-1}(p, w(u^{(m)}))$ are in $D(\mathcal{T})$ for one triangulation $\mathcal{T}$. We will calculate in the length coordinate $\Phi_{\mathcal{T}}$ below.

Since $u^{(m)}$ does not converge to any vector in $\mathbb{R}^n$, there exists $t_i = \infty$ or $-\infty$. Let us label vertices $v \in V$ by *black* and *white* as follows. The vertex v_i is black if and only if $t_i = -\infty$ and all other vertices are white.

Lemma 25. *(a) There does not exist a triangle $\tau \in \mathcal{T}$ with exactly two white vertices.*

(b) If $\Delta v_1v_2v_3$ is a triangle with exactly one white vertex at v_1, then the inner angle of the triangle at v_1 converges to 0 as $m \to \infty$ in the metrics d_m.

Proof. To see (a), suppose otherwise, using the $\Phi_{\mathcal{T}}$ length coordinate, we see the given assumption is equivalent to following. There exists a hyperbolic triangle of lengths $l_1^{(m)}, l_2^{(m)}, l_3^{(m)}$ such that $s(l_i^{(m)}) = s(a_i)e^{u_j^{(m)}+u_k^{(m)}}$, $\{i, j, k\} = \{1, 2, 3\}$, where $\lim_m u_i^{(m)} > -\infty$ for $i = 2, 3$ and $\lim_m u_1^{(m)} = -\infty$. By applying $\sinh(t/2)$ to the triangle inequality $l_2^{(m)} + l_3^{(m)} > l_1^{(m)}$ and using angle sum formula for sinh, we obtain

$$s(l_2^{(m)})\sqrt{1 + s(l_3^{(m)})^2} + s(l_3^{(m)})\sqrt{1 + s(l_2^{(m)})^2} > s(l_1^{(m)}).$$

Thus

$$s(a_2)e^{u_1^{(m)}+u_3^{(m)}}\sqrt{1 + s(a_3)^2e^{2u_1^{(m)}+2u_2^{(m)}}} + s(a_3)e^{u_1^{(m)}+u_2^{(m)}}\sqrt{1 + s(a_2)^2e^{2u_1^{(m)}+2u_3^{(m)}}}$$
$$> s(a_1)e^{u_2^{(m)}+u_3^{(m)}}.$$

This is the same as

$$s(a_2)\sqrt{e^{-2u_2^{(m)}} + s(a_3)^2e^{2u_1^{(m)}}} + s(a_3)\sqrt{e^{-2u_3^{(m)}} + s(a_3)^2e^{2u_1^{(m)}}} > s(a_1)e^{-u_1^{(m)}}.$$

However, by the assumption, the right-hand-side tends to ∞ and the left-hand-side is bounded. The contradiction shows that (a) holds.

To see (b), we use the same notation as in the proof of (a). Let $\alpha_1^{(m)}$ be the inner angle at v_1 of the triangle $\Delta v_1v_2v_3$ in d_m metric. Our goal is to show $\lim_m \alpha_1^{(m)} = 0$.

Since the sequence of hyperbolic polyhedral metrics $\{d_m\}$ are Delaunay in the same triangulation $\mathcal{T}$, by proposition 11, the three numbers $s(l_1^{(m)}), s(l_2^{(m)}), s(l_3^{(m)})$ satisfy the triangle inequality. Therefore, for each m, there is a Euclidean triangle whose sides have lengths $s(l_1^{(m)}), s(l_2^{(m)}), s(l_3^{(m)})$. Since $s(l_i^{(m)}) = s(a_i)e^{u_j^{(m)}+u_k^{(m)}}$, this triangle is similar to the Euclidean triangle Δ whose sides have lengths $s(a_1)e^{-u_1^{(m)}}$, $s(a_1)e^{-u_2^{(m)}}$ and $s(a_1)e^{-u_3^{(m)}}$. By the assumption that $\lim_m u_1^{(m)} > -\infty$ and $\lim_m u_2^{(m)} = -\infty$ and $\lim_m u_3^{(m)} = -\infty$, the three edge lengths $s(a_1)e^{-u_1^{(m)}}, s(a_1)e^{-u_2^{(m)}}, s(a_1)e^{-u_3^{(m)}}$ tend to $t \in \mathbb{R}$, ∞ and ∞ respectively. Therefore the angle in the Euclidean triangle Δ opposite to the edge of length $s(a_1)e^{-u_1^{(m)}}$ approaches 0. By the cosine law for Euclidean triangle, we obtain

$$\lim_m \frac{s(l_2^{(m)})^2 + s(l_3^{(m)})^2 - s(l_1^{(m)})^2}{2s(l_2^{(m)})s(l_3^{(m)})} = 1.$$

On the other hand, from Lemma 8, we have

$$\sin\frac{\alpha_2^{(m)} + \alpha_3^{(m)} - \alpha_1^{(m)}}{2} \cdot \cosh\frac{l_1^{(m)}}{2} = \frac{s(l_2^{(m)})^2 + s(l_3^{(m)})^2 - s(l_1^{(m)})^2}{2s(l_2^{(m)})s(l_3^{(m)})}.$$

Also we have $\lim_m l_1^{(m)} = 0$ due to $\lim_m u_2^{(m)} = -\infty$ and $\lim_m u_3^{(m)} = -\infty$. Hence

$$\lim_m \sin\frac{\alpha_2^{(m)} + \alpha_3^{(m)} - \alpha_1^{(m)}}{2} = 1.$$

It is equivalent to

$$\lim_m(\alpha_2^{(m)} + \alpha_3^{(m)} - \alpha_1^{(m)}) = \pi \geq \lim_m(\alpha_2^{(m)} + \alpha_3^{(m)} + \alpha_1^{(m)}).$$

Thus

$$\lim_m \alpha_1^{(m)} \leq 0.$$

Hence

$$\lim_m \alpha_1^{(m)} = 0.$$

□

We now finish the proof of $F(\mathbb{R}^n) = \mathbf{P}$ as follows.

Case 1. All vertices are white. There exists $t_i = \infty$. Let $\triangle v_iv_jv_k$ be a triangle at vertex v_i. There exists a hyperbolic triangle of lengths $l_i^{(m)}, l_j^{(m)}, l_k^{(m)}$ such that $s(l_i^{(m)}) = s(a_i)e^{u_j^{(m)}+u_k^{(m)}}$ (similar formulas hold for $l_j^{(m)}$ and $l_k^{(m)}$). Then

$\lim_m l_j^{(m)} = \lim_m l_k^{(m)} = \infty$. Let $\alpha_i^{(m)}$ be the inner angle at v_i. By the cosine rule,

$$\begin{aligned}
\lim_m \cos\alpha_i^{(m)} &= \lim_m \frac{-\cosh l_i^{(m)} + \cosh l_j^{(m)} \cosh l_k^{(m)}}{\sinh l_j^{(m)} \sinh l_k^{(m)}} \\
&= \lim_m \frac{-\cosh l_i^{(m)} + \cosh l_j^{(m)} \cosh l_k^{(m)}}{\cosh l_j^{(m)} \cosh l_k^{(m)}} \cdot \lim_m \frac{\cosh l_j^{(m)} \cosh l_k^{(m)}}{\sinh l_j^{(m)} \sinh l_k^{(m)}} \\
&= \lim_m \frac{-\cosh l_i^{(m)} + \cosh l_j^{(m)} \cosh l_k^{(m)}}{\cosh l_j^{(m)} \cosh l_k^{(m)}} \\
&= -\lim_m \frac{\cosh l_i^{(m)}}{\cosh l_j^{(m)} \cosh l_k^{(m)}} + 1 \\
&= -\lim_m \frac{2s(l_i^{(m)})^2 + 1}{(2s(l_j^{(m)})^2 + 1)(2s(l_k^{(m)})^2 + 1)} + 1 \\
&= -\lim_m \frac{2s(l_i^{(m)})^2}{(2s(l_j^{(m)})^2 + 1)(2s(l_k^{(m)})^2 + 1)} + 1 \\
&= -\lim_m \frac{2s(a_i)^2 e^{2u_j^{(m)} + 2u_k^{(m)}}}{(2s(a_j)^2 e^{2u_i^{(m)} + 2u_k^{(m)}} + 1)(2s(a_k)^2 e^{2u_i^{(m)} + 2u_j^{(m)}} + 1)} + 1 \\
&= -\lim_m \frac{2s(a_i)^2}{(2s(a_j)^2 e^{2u_i^{(m)}} + e^{-2u_k^{(m)}})(2s(a_k)^2 e^{2u_i^{(m)}} + e^{-2u_j^{(m)}})} + 1 \\
&= 1.
\end{aligned}$$

Therefore each inner angle at v_i approaches 0. The curvature of d_m at v_i approaches 2π. This shows that $F(u^{(m)})$ tends to infinity of $\mathbf{P}$.

Case 2. All vertices are black. Then the length of each edge approaches 0. Each hyperbolic triangle approaches a Euclidean triangle. The sum of the curvatures at all vertices approaches $2\pi\chi(S)$. This shows that $F(u^{(m)})$ tends to infinity of $\mathbf{P}$.

Case 3. There exist both white and black vertices. Since the surface S is connected, there exists an edge e whose end points v, v_1 have different colors. Assume v is white and v_1 is black. Let $v_1, ..., v_k$ be the set of all vertices adjacent to v so that v, v_i, v_{i+1} form vertices of a triangle and let $v_{k+1} = v_1$. Now applying part (a) of Lemma 25 to triangle $\Delta v v_1 v_2$ with v white and v_1 black, we conclude that v_2 must be black. Repeating this to $\Delta v v_2 v_3$ with v white and v_2 black, we conclude v_3 is black. Inductively, we conclude that all v_i's, for $i = 1, 2, ..., k$, are black. By part (b) of Lemma 25 , we conclude that the curvature of d_m at v tends to 2π. This shows that $F(u^{(m)})$ tends to infinity of $\mathbf{P}$.

Cases 1,2,3 show that $F(\mathbb{R}^n)$ is closed in $\mathbf{P}$. Therefore $F(\mathbb{R}^n) = \mathbf{P}$.

4.3. **Discrete Yamabe flow.** Given $K^* \in (-\infty, 2\pi)^V$ so that $\sum_{v\in V} K^*(v) > 2\pi\chi(S)$, by the proof above, there exists $u^* \in \mathbb{R}^n$ so that $F(u^*) = K^*$. Furthermore, the function F is the gradient $\bigtriangledown W$ of a strictly convex function $W(u)$ defined on (16) on $\mathbb{R}^n$.

The discrete Yamabe flow with surgery is defined to be the gradient flow of the strictly convex function $W^*(u) = W(u) - \sum_{i=1} K_i^* u_i$. This flow is a generalization of the discrete Yamabe flow introduced in [16]. Since $F(u^*) = K^*$, we see

$\bigtriangledown W^*(u^*) = 0$, i.e., W^* has a unique minimal point u^* in $\mathbb{R}^n$. It follows that the gradient flow of W^* converges to the minimal point u^* as time approaches infinity.

In the formal notation, the flow takes the form $\frac{du_i(t)}{dt} = K_i - K_i^*$ and $u(0) = 0$. The exponential convergence of the flow can be established using exactly the same method used for Theorem 1.4 of [16].

5. Algorithmic aspect of discrete conformality

We will prove theorem 2 in this section.

Suppose α and α' are two hyperbolic (or Euclidean) polyhedral metrics on (S, V) given in terms of edge lengths in two geodesic triangulations $\mathcal{T}$ and $\mathcal{T}'$, i.e., $l = \Phi_{\mathcal{T}}^{-1}(\alpha)$ and $l' = \Phi_{\mathcal{T}'}^{-1}(\alpha')$ are two vectors in $\mathbb{R}^{E(\mathcal{T})}$ and $\mathbb{R}^{E(\mathcal{T}')}$. We will produce an algorithm to decide if d and d' are discrete conformal using the data $(\mathcal{T}, l)$ and $(\mathcal{T}', l')$.

There are two steps involved in the algorithm.

In the first step, using proposition 13(c), we may assume that both $\mathcal{T}$ and $\mathcal{T}'$ are Delaunay in metrics α and α' respectively. (The same also holds for Euclidean polyhedral metrics. This is a well known fact from computational geometry. See for instance [3]). Next, consider two decorated hyperbolic metrics $(d, w) = A_{\mathcal{T}}(\alpha)$ and $(d', w') = A_{\mathcal{T}'}(\alpha')$ with their respective Penner's λ-coordinates $y = \Psi_{\mathcal{T}}^{-1}(d, w)$ and $y' = \Psi_{\mathcal{T}'}^{-1}(d', w')$. By theorem 17, we see Theorem 2 follows from,

Proposition 26. *Suppose two decorated hyperbolic metrics (d, w) and (d', w') in $T_D(\Sigma)$ are given in terms of λ-lengths in two triangulations. There exists an algorithm to decide if $d = d'$.*

Proof. By the construction $y = \Psi_{\mathcal{T}}^{-1}(d, w)$ and $y' = \Psi_{\mathcal{T}'}^{-1}(d', w')$ are the two λ-lengths. Our goal is to use y and y' to decide if $d = d'$. There are two cases according to $\mathcal{T}$ and $\mathcal{T}'$ are isotopic or not.

In the first case, $\mathcal{T}$ and $\mathcal{T}'$ are isotopic. Then it is known by the work of Penner [19] that $d = d'$ if and only if the associated Thurston's shear coordinates of y and y' are the same. Here the shear coordinate z of y is defined to be $z(e) = \frac{y(e_1)y(e_3)}{y(e_2)y(e_4)}$ with e_1, e_2, e_3, e_4 being a (fixed) cyclically ordered edges of the quadrilateral associated to e. Thus one can check algorithmically if $d = d'$ using y and y'.

In the second case that $\mathcal{T}$ and $\mathcal{T}'$ are not isotopic, we can algorithmically produce $y'' = \Psi_{\mathcal{T}}^{-1}(d', w')$ from y' and $\mathcal{T}'$. Indeed, a well known theorem of L. Mosher [18] says that there exists an algorithm to produce a finite set of triangulations $\mathcal{T}_1 = \mathcal{T}', \mathcal{T}_2, ..., \mathcal{T}_k = \mathcal{T}$ so that $\mathcal{T}_{i+1}$ is obtained from $\mathcal{T}_i$ by a diagonal switch. Penner's Ptolemy identity shows that one can compute algorithmically $\Psi_{\mathcal{T}_{i+1}}^{-1}(d', w')$ from $\Psi_{\mathcal{T}_i}^{-1}(d', w')$. Thus we can algorithmically compute the new λ-length coordinate $y'' = \Psi_{\mathcal{T}}^{-1}(d', w')$ from $y' = \Psi_{\mathcal{T}'}^{-1}(d', w')$. This reduces the problem to the first case. □

6. Appendix

In the appendix we prove Lemma 20. Let $s(x) = \sinh \frac{x}{2}$.

Lemma 27 (Fenchel [7] page 118)**.** *Given a hyperbolic triangle with side lengths a, b, c, then*

$$\frac{(s(a)s(b)s(c))^2}{(s(a)+s(b)+s(c))(s(a)+s(b)-s(c))(s(b)+s(c)-s(a))(s(c)+s(b)-s(a))}$$

equals

- $\frac{1}{4}\sinh^2 r$ *if the triangle has a compact circumcircle of radius* r*,*
- ∞ *if the circumcircle is a horocycle,*
- $-\frac{1}{4}\cosh^2 D$ *if the circumcircle is of constant distance* D *to a geodesic.*

As a corollary we have,

Lemma 28. *Denote by* α, β, γ *the angles opposite to the sides with lengths* a, b, c*. Then*

$$\frac{\sinh a}{\sin \alpha} = 2\zeta \cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2},$$

where ζ *equals*

- $\tanh r$ *if the triangle has a compact circumcircle of radius* r*,*
- 1 *if the circumcircle is a horocycle,*
- $\coth D$ *if the circumcircle is of constant distance* D *to a geodesic.*

Proof. Assume that the triangle has a circumscribed circle of radius r. By using the cosine rule and Lemma 27,

$$\begin{aligned}
\sin\alpha &= (1-\cos^2\alpha)^{\frac{1}{2}} \\
&= \frac{(-\cosh^2 a - \cosh^2 b - \cosh^2 c + 1 + 2\cosh a\cosh b\cosh c)^{\frac{1}{2}}}{\sinh b \sinh c} \\
&= \frac{2}{\sinh b\sinh c}\cdot\{4s(a)^2s(b)^2s(c)^2+ \\
&\quad 2s(a)^2s(b)^2+2s(b)^2s(c)^2+2s(c)^2s(a)^2-s(a)^4-s(b)^4-s(c)^4\}^{\frac{1}{2}} \\
&= \frac{2}{\sinh b\sinh c}\cdot\{4s(a)^2s(b)^2s(c)^2+ \\
&\quad (s(a)+s(b)+s(c))(s(a)+s(b)-s(c))(s(b)+s(c)-s(a))(s(c)+s(b)-s(a))\}^{\frac{1}{2}} \\
&= \frac{2}{\sinh b\sinh c}\cdot\{4s(a)^2s(b)^2s(c)^2+\frac{4s(a)^2s(b)^2s(c)^2}{\sinh^2 r}\}^{\frac{1}{2}} \\
&= \frac{4}{\sinh b\sinh c}\cdot s(a)s(b)s(c)\frac{\cosh r}{\sinh r}.
\end{aligned}$$

By taking limit with $r\to\infty$, we can prove the lemma for the case that the triangle has a horocyclic circumcircle.

Similar calculation can be used to prove the lemma for the case that the triangle has a circumscribed equidistant curve. □

Lemma 29. *Let* a, b, c, d *be the side lengths of a hyperbolic quadrilateral and* e, f *the diagonal lengths so that* a, b, c, d *are cyclically ordered edge lengths and edges of lengths* a, b, e *form a triangle.*

(i) *The vertices of this quadrilateral lie on a curve of constant geodesic curvature.*

(ii) *Ptolemy's formula holds:*

$$s(e)s(f) = s(a)s(c) + s(b)s(d).$$

(iii)

$$s(e)^2 = (s(a)s(c)+s(b)s(d))\frac{s(a)s(d)+s(b)s(c)}{s(a)s(b)+s(c)s(d)}, \tag{17}$$

and

$$s(f)^2 = (s(a)s(c) + s(b)s(d))\frac{s(a)s(b) + s(c)s(d)}{s(a)s(d) + s(b)s(c)}.$$

Proof. (i)$\Longrightarrow$(ii). It was proved by T. Kubota [14].
(ii)$\Longrightarrow$(i). It was proved by Joseph E. Valentine [21], Theorem 3.4.
(iii)$\Longrightarrow$(ii). The product of the two equations in (iii) produces the equation in (ii).
(i)$\Longrightarrow$(iii).
Case 1. When the vertices lie on a circle, it was proved in [12] (theorem 1, page 4).

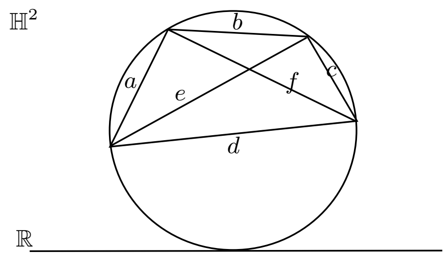

FIGURE 1.

Case 2. When the vertices lie on a horocycle, for example as in Figure 1, we have

$$\begin{aligned} s(e) &= s(a) + s(b), \\ s(f) &= s(b) + s(c), \\ s(d) &= s(a) + s(b) + s(c). \end{aligned}$$

Then the equations in (iii) hold.
Case 3. When the vertices lie on a geodesic, without loss of generality, we may assume

$$\begin{aligned} e &= a + b, \\ f &= b + c, \\ d &= a + b + c. \end{aligned}$$

Direct calculation shows that

$$s(a)s(c) + s(b)s(d) = s(a)s(c) + s(b)s(a + b + c) = s(a + b)s(c + b).$$

Similarly,

$$\begin{aligned} s(a)s(d) + s(b)s(c) &= s(a + b)s(a + c), \\ s(a)s(b) + s(c)s(d) &= s(c + a)s(c + b). \end{aligned}$$

Therefore the right hand side of (17) equals

$$s(a + b)s(c + b)\frac{s(a + b)s(a + c)}{s(c + a)s(c + b)} = s(a + b)^2 = s(e)^2.$$

Similar argument proves the equation involving $s(f)$.
Case 4. When the vertices lie on an equidistant curve with distance D to its geodesic axis, project the vertices to the geodesic axis. The corresponding distance between those projection of vertices are denoted by $\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}$.

By Case 3, we have

$$s(\overline{e})^2 = (s(\overline{a})s(\overline{c}) + s(\overline{b})s(\overline{d}))\frac{s(\overline{a})s(\overline{d}) + s(\overline{b})s(\overline{c})}{s(\overline{a})s(\overline{b}) + s(\overline{c})s(\overline{d})}.$$

Since

$$s(x) = s(\overline{x})\cosh D$$

for $x = a, b, c, d, e, f$, we have

$$s(e)^2 = (s(a)s(c) + s(b)s(d))\frac{s(a)s(d) + s(b)s(c)}{s(a)s(b) + s(c)s(d)}.$$

□

6.1. **Proof of Lemma 20.** First, we verify that

$$\frac{\partial A}{\partial x}|_{A=B} = \frac{\partial B}{\partial x}.$$

The role of x, y, z, w are the same with respect to a. It is enough to verify the case of variable x.

Now let $\alpha, \alpha', \beta, \beta'$ be the angles formed by the pairs of edges $\{a, y\}$, $\{a, x\}$, $\{a, z\}$, $\{a, w\}$ as Figure 2.

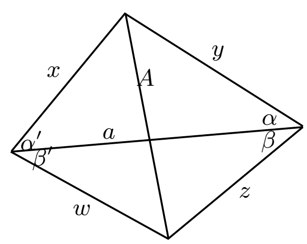

FIGURE 2.

In the triangle of lengths y, z, A, by the cosine rule,

$$\cosh A = \cosh y \cosh z - \sinh y \sinh z \cos(\alpha + \beta).$$

Taking derivative of both sides with respect to x, we have

$$\frac{\partial A}{\partial x} = \frac{\sinh y \sinh z \sin(\alpha + \beta)}{\sinh A} \cdot \frac{\partial \alpha}{\partial x}.$$

In the triangle of lengths x, y, a, by the derivative of cosine rule [17], we have

$$\frac{\partial \alpha}{\partial x} = \frac{\sinh x}{\sinh y \sinh a \sin \alpha}.$$

Therefore,

$$\frac{\partial A}{\partial x} = \frac{\sinh z}{\sinh a} \cdot \frac{\sin(\alpha + \beta)}{\sinh A} \cdot \frac{\sinh x}{\sin \alpha}.$$

In the triangle of lengths y, z, A, Lemma 28 implies that

$$\frac{\sinh A}{\sin(\alpha + \beta)} = 2\zeta_1 \cosh \frac{A}{2} \cosh \frac{y}{2} \cosh \frac{z}{2}. \tag{18}$$

In the triangle of lengths x, y, a, Lemma 28 implies that

$$\frac{\sinh x}{\sin \alpha} = 2\zeta_2 \cosh \frac{x}{2} \cosh \frac{y}{2} \cosh \frac{a}{2}. \tag{19}$$

Therefore,

$$\frac{\partial A}{\partial x} = \frac{\sinh z}{\sinh a} \cdot \frac{2\zeta_2 \cosh \frac{x}{2} \cosh \frac{y}{2} \cosh \frac{a}{2}}{2\zeta_1 \cosh \frac{A}{2} \cosh \frac{y}{2} \cosh \frac{z}{2}} = \frac{\sinh \frac{z}{2} \cosh \frac{x}{2} \zeta_2}{\sinh \frac{a}{2} \cosh \frac{A}{2} \zeta_1}.$$

When $A = B$, by Lemma 29, the vertices of the hyperbolic quadrilateral lie on a circle, a horocycle or an equidistant curve. Thus $\zeta_1 = \zeta_2$.

Therefore

$$\frac{\partial A}{\partial x}|_{A=B} = \frac{\sinh \frac{z}{2} \cosh \frac{x}{2}}{\sinh \frac{a}{2} \cosh \frac{B}{2}} = \frac{\partial B}{\partial x}.$$

Second, we verify that

$$\frac{\partial A}{\partial a}|_{A=B} = \frac{\partial B}{\partial a}.$$

In the triangle of lengths y, z, A, by the cosine rule,

$$\cosh A = \cosh y \cosh z - \sinh y \sinh z \cos(\alpha + \beta).$$

Taking derivative of both sides with respect to a, we have

$$\frac{\partial A}{\partial a} = \frac{\sinh y \sinh z \sin(\alpha + \beta)}{\sinh A} \cdot (\frac{\partial \alpha}{\partial a} + \frac{\partial \beta}{\partial a}).$$

In the triangle of length x, y, a, by the derivative of cosine rule [17], we have

$$\frac{\partial \alpha}{\partial a} = -\frac{\sinh x}{\sinh y \sinh a \sin \alpha} \cos \alpha'.$$

In the triangle of length w, z, a, by the derivative of cosine rule [17], we have

$$\frac{\partial \beta}{\partial a} = -\frac{\sinh w}{\sinh z \sinh a \sin \beta} \cos \beta'.$$

Therefore

$$\frac{\partial A}{\partial a} = -\frac{\sin(\alpha + \beta)}{\sinh A \sinh a}(\frac{\sinh z \sinh x \cos \alpha'}{\sin \alpha} + \frac{\sinh y \sinh w \cos \beta'}{\sin \beta}).$$

By the equations (18) and (19), we have

$$\frac{\sin(\alpha + \beta)}{\sin \alpha} = \frac{\zeta_2 \cosh \frac{a}{2} \sinh \frac{A}{2}}{\zeta_1 \cosh \frac{z}{2} \sinh \frac{x}{2}}.$$

By the similar calculation, we have

$$\frac{\sin(\alpha + \beta)}{\sin \beta} = \frac{\zeta_3 \cosh \frac{a}{2} \sinh \frac{A}{2}}{\zeta_1 \cosh \frac{y}{2} \sinh \frac{w}{2}},$$

there ζ_3 is the corresponding quantity of the triangle of lengths w, z, a.

Therefore

$$\frac{\partial A}{\partial a} = -\frac{1}{\cosh \frac{A}{2} \sinh \frac{a}{2}}(\frac{\zeta_2}{\zeta_1} \sinh \frac{z}{2} \cosh \frac{x}{2} \cos \alpha' + \frac{\zeta_3}{\zeta_1} \sinh \frac{y}{2} \cosh \frac{w}{2} \cos \beta').$$

When $A = B$, by Lemma 29, the vertices of the hyperbolic quadrilateral lie on a circle, a horocycle or an equidistant curve. Thus $\zeta_1 = \zeta_2 = \zeta_3$.

Therefore

$$\frac{\partial A}{\partial a}|_{A=B} = -\frac{1}{\cosh \frac{B}{2} \sinh \frac{a}{2}}(\sinh \frac{z}{2} \cosh \frac{x}{2} \cos \alpha' + \sinh \frac{y}{2} \cosh \frac{w}{2} \cos \beta').$$

On the other hand

$$\frac{\partial B}{\partial a} = -\frac{\sinh\frac{B}{2}\cosh\frac{a}{2}}{\cosh\frac{B}{2}\sinh\frac{a}{2}}.$$

To prove $\frac{\partial A}{\partial a}|_{A=B} = \frac{\partial B}{\partial a}$, it remains to show that

$$\sinh\frac{z}{2}\cosh\frac{x}{2}\cos\alpha' + \sinh\frac{y}{2}\cosh\frac{w}{2}\cos\beta' = \sinh\frac{B}{2}\cosh\frac{a}{2}. \tag{20}$$

In the triangle of length x, y, a, by the cosine rule,

$$\cos\alpha' = \frac{-\cosh y + \cosh x \cosh a}{\sinh x \sinh a}.$$

In the triangle of length w, z, a, by the cosine rule,

$$\cos\beta' = \frac{-\cosh z + \cosh w \cosh a}{\sinh w \sinh a}.$$

Therefore the equation (20) is equivalent to

$$\begin{aligned} &\frac{\sinh\frac{z}{2}}{2\sinh\frac{x}{2}}(-\cosh y + \cosh x\cosh a) + \frac{\sinh\frac{y}{2}}{2\sinh\frac{w}{2}}(-\cosh z + \cosh w \cosh a) \\ &= \sinh\frac{B}{2}\cosh\frac{a}{2}\sinh a. \end{aligned} \tag{21}$$

Using the notation $s(t) = \sinh\frac{t}{2}$, we have $\cosh t = 2s(t)^2 + 1$. Therefore the equation (21) is equivalent to

$$\begin{aligned} &\frac{s(z)}{s(x)}(2s(a)^2 s(x)^2 + s(a)^2 + s(x)^2 - s(y)^2) \\ &+ \frac{s(y)}{s(w)}(2s(a)^2 s(w)^2 + s(a)^2 + s(w)^2 - s(z)^2) \\ &= 2s(B)s(a)(s(a)^2+1) \\ &= 2(s(x)s(z) + s(y)s(w))(s(a)^2+1), \end{aligned}$$

the second equality is due to Ptolemy's formula.

After simplify we obtain

$$s(a)^2 = (s(x)s(z) + s(y)s(w))\frac{s(x)s(w) + s(y)s(z)}{s(x)s(y) + s(z)s(w)}.$$

This is exactly the result of Lemma 29.

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