Mean Field Equation of Liouville Type with Singular Data: Topological Degree

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Abstract

We consider the following mean field equation:

$$\Delta_g v + \rho\left(\frac{h^* e^v}{\int_M h^* e^v} - 1\right) = 4\pi \sum_{j=1}^{N} \alpha_j (\delta_{q_j} - 1) \quad \text{on } M, \tag{0.1}$$

where M is a compact Riemann surface with volume 1, h^* is a positive C^1 function on M, and ρ and α_j are constants satisfying $\alpha_j > -1$. In this paper, we derive the topological-degree-counting formula for noncritical values of ρ. We also give several applications of this formula, including existence of the curvature $+1$ metric with conic singularities, doubly periodic solutions of electroweak theory, and a special case of self-gravitating strings. © 2015 Wiley Periodicals, Inc.

1 Introduction

Let (M, g) be a compact Riemann surface with volume 1, h^* be a C^1 positive function on M, and $\rho \in \mathbb{R}^+$. We consider the following mean field type equation:

$$\Delta_g v + \rho\left(\frac{h^* e^v}{\int_M h^* e^v} - 1\right) = 4\pi \sum_{j=1}^{N} \alpha(q_j)(\delta_{q_j} - 1) \quad \text{on } M, \tag{1.1}$$

where Δ is the Beltrami-Laplace operator, $\alpha(q_j) > -1$, and δ_q is the Dirac measure at $q \in M$.

Equation (1.1) arises in many different areas of mathematics. When M is the 2-sphere and $\rho = 8\pi$, (1.1) without singular sources corresponds to the Nirenberg problem of prescribing Gaussian curvature. In general, equation (1.1) is related to the existence of a positive constant curvature metric with conic singularities. In physics, equation (1.1) can be derived from the mean field limit of point vortices of the Euler flow, as studied by Caglioti et al. [13, 14], Kiessling [31], and Chanillo-Kiessling [16]. Recently it has drawn a lot of attention due to its application to

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Chern-Simons-Higgs theory (see Hong-Kim-Pac [28] and Jackiw-Weinberg [29]) and to the electroweak theory (see Ambjorn-Olesen [5]). For the recent mathematical development, we refer the readers to Nolasco-Tarantello [42–44], Spruck-Yang [48–50], Tarantello [54], and Yang [57]. See also [7, 12, 15, 21, 24–26, 30, 39, 40].

The purpose of this paper is to derive the formula for counting the Leray-Schauder degree for all solutions to equation (1.1) with noncritical parameter ρ. When $\alpha(q_j) = 0\ \forall j$, the formula was obtained in [19]. In this paper, we would like to extend our previous result to cover the case with multiple singular sources. The singular case is generally considered more important due to its application to physics, where vortices are always present in the models. On the other hand, when we compute the topological degree for a system of equations, e.g., the SU(3) Toda system, the shadow system, which is a reduction from the Toda system to a single equation, always contains singular sources even when the Toda system does not have singularity. Thus we could consider this paper to be the first step for computing the topological degree for systems.

Conventionally, equation (1.1) can be transformed to the case without singularity, but with h^* vanishing at some points. Let $G(x,q)$ be the Green function

$$\begin{cases} \Delta_x G(x,q) = -\delta_q(x) + 1, \\ \int_M G(x,q)dx = 0. \end{cases} \tag{1.2}$$

In terms of the Green function, a solution $v(x)$ of (1.1) can be written as

$$v(x) = u(x) - 4\pi \sum_{k=1}^{N} \alpha(q_k) G(x, q_k). \tag{1.3}$$

Then (1.1) is equivalent to the following:

$$\Delta u + \rho\left(\frac{he^u}{\int_M he^u} - 1\right) = 0 \quad \text{in } M, \tag{1.4}$$

where

$$h(x) = h^*(x) \exp\left\{-\sum_{k=1}^{N} 4\pi\alpha(q_k) G(x, q_k)\right\}. \tag{1.5}$$

Here $h \geq 0$ in M, and near each q_j, $h(x)$ has the form in a local coordinate:

$$h(x) = h_j(x)|x - q_j|^{2\alpha(q_j)} \quad \text{for } |x - q_j| \ll 1, \tag{1.6}$$

where $h_j(x) > 0$.

Throughout the paper, we shall consider equation (1.4) satisfying (1.6) for some $\alpha(q_j) > -1$ at each q_j. Let $\mathbb{N}^+$ denote the set of positive integers. It is well-known that if $\rho \notin \Sigma$, where

$$\Sigma := \left\{8m\pi + \Sigma_{q\in A} 8\pi(1 + \alpha(q)) \mid A \subset \{q_1, \dots, q_N\}, m \in \mathbb{N}^+ \cup \{0\}\right\} \setminus \{0\},$$

then the a priori bound holds; i.e., there exists a constant C such that for all solutions of (1.4),

$$|u(x)| \le C \quad \forall x \in M.$$

The study of a priori bounds began with the work of Brezis-Merle [11] and Li-Shafrir [33] for the case without singularities, and Bartolucci and Tarantello [10] for (1.4) with singular sources; see also [6, 8, 30, 46, 51–53]. After establishing the a priori bound, a further study naturally leads to the computation of the topological degree of (1.4).

Equation (1.4) is invariant by adding a constant to the solutions. Hence we can always normalize u to satisfy $\int_M u = 0$. Let $\mathring{H}^1$ be the space

$$\mathring{H}^1 = \left\{ u \in H^1(M) : \int_M u = 0 \right\}.$$

Set

$$T_\rho u = \rho \Delta^{-1}\left(\frac{h e^u}{\int_M h e^u} - 1 \right), \tag{1.7}$$

which maps $\mathring{H}^1(M)$ into itself. The existence of a priori bounds implies there is a large $R > 0$ such that $(I + T_\rho)u \ne 0$ for any $u \in \partial B_R$, where $B_R = \{v \in \mathring{H}^1 \mid \|v\|_{H^1} \le R\}$. Therefore the Leray-Schauder degree

$$d_\rho := \deg(I + T_\rho, B_R, 0)$$

is well-defined provide that $\rho \notin \Sigma$. Write

$$\Sigma = \{8\pi n_k \mid n_1 < n_2 < \cdots\}.$$

The homotopy invariance of the topological degree implies

- d_ρ is a constant for $\frac{\rho}{8\pi} \in (n_k, n_{k+1})$, $k = 0, 1, \ldots$. Here $n_0 = 0$.
- d_ρ is independent of h^* as long as h^* is a C^1 positive function.

To state our formula for d_ρ, we consider the following generating function g:

$$g(x) = (1 + x + x^2 + \cdots)^{-\chi(M)+N} \prod_{j=1}^{N} (1 - x^{1+\alpha(q_j)}), \tag{1.8}$$

where $\chi(M) = 2 - 2g$ is the Euler characteristic of M. We note that if $-\chi(M) + N < 0$, $(1 + x + x^2 + \cdots)^{-\chi(M)+N} = (1 - x)^{\chi(M)-N}$. Write $g(x)$ in the powers of x:

$$g(x) = 1 + b_1 x^{n_1} + b_2 x^{n_2} + \cdots + b_k x^{n_k} + \cdots .$$

The degree d_ρ can be written in terms of b_j, as shown in the following theorem:

THEOREM 1.1. *Let d_ρ be the Leray-Schauder degree for* (1.4). *Suppose $8\pi n_k < \rho < 8\pi n_{k+1}$. Then*

$$d_\rho = \sum_{j=0}^{k} b_j,$$

where $b_0 = 1$.

For the applications, it often requires that $\alpha(q_j) \in \mathbb{N}^+$. In this case, $\Sigma = \{8m\pi \mid m \in \mathbb{N}^+\}$. Suppose $\chi(M) \le 0$. Then

$$\begin{aligned} g(x) &= (1 + x + x^2 + \cdots)^{-\chi(M)} \prod_{j=1}^{N} \frac{1 - x^{1+\alpha(q_j)}}{1 - x} \\ &= (1 + x + x^2 + \cdots)^{-\chi(M)} \prod_{j=1}^{N} (1 + x + x^2 + \cdots + x^{\alpha(q_j)}) \qquad (1.9) \\ &= 1 + b_1 x + b_2 x^2 + \cdots + b_k x^k + \cdots . \end{aligned}$$

Clearly $b_j \ge 0$ for any $j \ge 1$. Thus

$$d_\rho = 1 + \sum_{j=1}^{k} b_j \ge 1 > 0$$

if $8\pi k < \rho < 8\pi(k+1)$. Hence we have the following corollary:

COROLLARY 1.2. *Suppose $\alpha(q_j) \in \mathbb{N}^+$ and $\chi(M) \le 0$. Then $d_\rho > 0$ if $\rho \ne 8\pi m$ for all $m \in \mathbb{N}^+$. Furthermore, equation* (1.4) *always has a solution if the genus of M is not zero.*

Let M be a torus. We consider the following equation:

$$\Delta u + e^u = 4\pi \sum_{j=1}^{l} \alpha_j \delta_{p_j} \quad \text{on } M, \qquad (1.10)$$

where $p_i \ne p_j$ and $\alpha_j \in \mathbb{N}$. In geometry, a solution of this equation gives rise to a positive constant curvature metric with conic singularities at $p_1, \dots, p_l$. Integrating (1.10), we can calculate the integral of e^u and show that (1.10) is equivalent to

$$\Delta u + \rho \left(\frac{e^u}{\int_M e^u} - 1 \right) = 4\pi \sum_{j=1}^{l} \alpha_j (\delta_{p_j} - 1) \quad \text{on } M,$$

where $\rho = 4\pi \sum_{j=1}^{l} \alpha_j$. Hence (1.10) is a special case of (1.1). Suppose $\sum_{j=1}^{l} \alpha_j$ is an odd integer. Then the Leray-Schauder degree of (1.10) is well-defined. Applying Theorem 1.1, we have the following result:

THEOREM 1.3. *Suppose* $\alpha_j \in \mathbb{N}$ *and* $\sum_{j=1}^{l} \alpha_j$ *is an odd integer. Then the Leray-Schauder degree for* (1.10) *is* $[\prod_{j=1}^{l}(1+\alpha_j)]/2$.

PROOF. By (1.9), the generating function $g(x)$ is

$$g(x) = \prod_{j=1}^{l} \frac{1-x^{\alpha_j+1}}{1-x} = \prod_{j=1}^{l}(1+x+x^2+\cdots+x^{\alpha_j})$$
$$= 1 + b_1 x + b_2 x^2 + \cdots + b_k x^k + \cdots + x^m,$$

where $m = \sum_{j=1}^{l} \alpha_j$. Now $\rho = 4\pi m = 8\pi \frac{m}{2}$. Since m is odd, m lies between two integers $\frac{m-1}{2}$ and $\frac{m+1}{2}$. Thus the Leray-Schauder degree δ_ρ can be computed via the formula

$$d_\rho = \sum_{k=0}^{\frac{m-1}{2}} b_k = \frac{1}{2}\sum_{k=0}^{m} b_k = \frac{g(1)}{2} = \frac{\prod_{j=1}^{l}(1+\alpha_j)}{2},$$

where we use the fact that b_k satisfies

$$b_{m-k} = b_k, \quad k = 0, 1, \ldots, m. \qquad \square$$

In [17], we can prove a stronger version of Theorem 1.3:

THEOREM A. *For any configuration* $(p_1, \ldots, p_l) \in M^l := M \times \cdots \times M$, *the number of solutions to* (1.10) $\leq [\prod_{j=1}^{l}(1+\alpha_j)]/2$.

Theorem 1.3 has important application for the Chern-Simons-Higgs equation. See Section 6. We believe that the degree formula obtained in Theorem 1.1 is useful to other problems. In Section 6 6, we will give some applications of it.

To prove Theorem 1.1, we will adopt the same strategy from [18, 19]. For some technical reasons, we first consider the case when all $\alpha(q_j)$ are not positive integers, and we prove Theorem 1.1 under this assumption. First we note that it is easy to see that $d_\rho = 1$ if $\rho \in (0, 8\pi n_1)$. Let $d_m^- = \lim_{\rho\uparrow n_m} d_\rho$ and $d_m^+ = \lim_{\rho\downarrow n_m} d_\rho$. The crucial step for Theorem 1.1 is to compute the jump $d_m^+ - d_m^-$. Obviously this jump is due to the occurrences of blowing up solutions of (1.4) with $\rho = \rho_k \to 8\pi n_m$ as $k \to \infty$. Hence we have two tasks: one is to find all blowup solutions, and the other is to compute the topological degree contributed by each blowup solution. In order to ensure that our methods are working, we assume the following assumptions throughout the paper:

(i) all $\alpha(q_j)$ are not positive integers, and
(ii) there exist some nondegenerate conditions of h.

For the precise statement of (ii), see Section 2. Under these conditions, we will first prove Theorem 1.1 for all $\alpha(q_j) \notin \mathbb{N}^+$. When $\alpha(q_j)$ is a positive integer for some q_j, we use $(1.4)_k$ to denote equation (1.4) with $\alpha(q_j)$ replaced by $\alpha_k(q_j)$. In this case, for any $\rho \notin \Sigma$, we find a sequence $\alpha_{j,k} \to \alpha(q_j)$ as $k \to \infty$ such that ρ

is not a critical parameter for equation $(1.4)_k$ with $\alpha_k(q_j) = \alpha_{j,k}$. Then by taking the limit $k \to \infty$, we can obtain the degree-counting formula for $\alpha(q_j) \in \mathbb{N}^+$.

Under assumptions (i) and (ii), our first task is contained in the following result, which was proved in [20].

THEOREM B. *Suppose $h^* \in C^3$ and assumptions* (i) *and* (ii) *hold. Then there are $\epsilon_0 > 0$ and $C > 0$ such that for any solution u of* (1.4) *with $\rho \in (8\pi n_m - \epsilon_0, 8\pi n_m + \epsilon_0)$, either $|u(x)| \le C, \forall x \in M$ or $u \in S_\rho(Q)$.*

The set $S_\rho(Q)$ describes the approximate profiles of a sequence of blowup solutions of (1.4) for which the blowup set is $Q = \{p_1^0, \ldots, p_m^0\}$. We defer the definition of $S_\rho(Q)$ to Section 2. When $p_i^0 \notin \{q_1, \ldots, q_N\}$ for all $i = 1, 2, \ldots, m$, the topological degree for all solutions in $S_\rho(Q)$ has been computed in [19]. With Theorem B, our main task of this paper is to compute the topological degree for all solutions in $S_\rho(Q)$ when $Q \cap \{q_1, \ldots, q_N\} \ne \phi$.

Our methods can also work for (1.1) with the Dirichlet boundary value problem:

$$\begin{cases} \Delta v + \rho \frac{h^* e^v}{\int_M h^* e^v} = 4\pi \sum_{j=1}^N \alpha(q_j)\delta_{q_j} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

where Ω is a bounded C^1 domain in $\mathbb{R}^2$. Let

$$\begin{aligned} g(x) &= (1 + x + x^2 + \cdots)^{-\chi(\Omega)+N} \prod_{j=1}^N (1 - x^{1+\alpha(q_j)}) \\ &= \sum_{j=0}^\infty b_j x^{n_j}, \end{aligned} \tag{1.12}$$

where $\chi(\Omega)$ denotes the number of holes bounded by Ω, i.e., the Euler characteristic of Ω, and $b_0 = 1$. Then we have a similar result.

THEOREM 1.4. *Suppose $8\pi n_k < \rho < 8\pi n_{k+1}$. Then*

$$d_\rho = \sum_{j=0}^k b_j.$$

Furthermore, if $\alpha(q_j) \in \mathbb{N}^+$ and Ω is not simply connected, then $d_\rho > 0$ if $\rho \ne 8\pi m$ for all $m \in \mathbb{N}^+$.

The paper is organized as follows. In Section 2, we review the sharp estimates to a sequence of blowup solutions of (1.1), which has been carried out in [20]. In that section, we state our assumptions (ii) explicitly and define $S_\rho(Q)$. In Section 3, we further study solutions in $S_\rho(Q)$ and obtain some sharper estimates, which is required in computing the Leray-Schauder degree. In Section 4, we state two basic lemmas, of which the proofs are postponed to the appendix, and use them to deform our original equation to some equations with simpler operators. Finally, in

Section 5, the degree-counting formula is proved, and in Section 6, we give some applications of the formula.

2 Approximation of Blowup Solutions

In this section, we construct a good approximation to a sequence of blowup solutions u_k to (1.4) with $\rho = \rho_k$ and $\lim_{k\to\infty} \rho_k = \rho_*$. For this purpose, we introduce some notation. Let $\widetilde{M} = M \setminus \{q_1, \ldots, q_N\}$, $Q_1 \subset \{q_1, \ldots, q_N\}$ (Q_1 may be empty), and $\tau = |Q_1|$. We set

$$\begin{aligned} &f_{Q_1}(x_{\tau+1}, \ldots, x_m) \\ &\quad = \sum_{j=\tau+1}^{m} (\log h(x_j) + 4\pi R(x_j; x_j)) \\ &\qquad + 8\pi \sum_{\substack{j,k=\tau+1 \\ j\neq k}}^{m} G(x_j; x_k) + \sum_{q_k \in Q_1} \sum_{j=\tau+1}^{m} 8\pi(1+\alpha_k) G(x_j; q_k), \end{aligned} \tag{2.1}$$

where $R(x,x)$ is the regular part of the Green function, $\alpha_k = \alpha(q_k)$, and $x = (x_{\tau+1}, \ldots, x_m) \in \widetilde{M} \times \cdots \times \widetilde{M}$. Also, for any $Q = (q_{i_1}, \ldots, q_{i_\tau}, p_{\tau+1}, \ldots, p_m)$, where $q_{i_j} \neq q_{i_k}$ for $j \neq k$, $p_j \in \widetilde{M}$, and $p_j \neq p_k$ for $j \neq k$, an associated function G_j^* is defined as follows:

$$G_j^*(x) = 8\pi\Big((1+\alpha_j)R(x; p_j) + \sum_{\substack{1\le i\le m \\ i\neq j}} (1+\alpha_i) G(x; p_i)\Big), \quad 1 \le j \le m, \tag{2.2}$$

where p_j denotes q_{i_j} for $j \le \tau$ and $\alpha_j = 0$ if $j > \tau$.

Let $Q_1 = (q_{i_1}, \ldots, q_{i_\tau})$. If $Q_1 \neq \varnothing$, we may assume there is $1 \le l \le \tau$ such that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_l > a_{l+1} \ge \cdots \ge \alpha_\tau.$$

Suppose $(p_{\tau+1}^0, \ldots, p_m^0)$ is a critical point of f_{Q_1}. We let $Q = (q_{i_1}, \ldots, q_{i_\tau}, p_{\tau+1}^0, \ldots, p_m^0)$ and use G_j^* associated with Q to define $l(Q)$ as follows:

(a) If $Q_1 = \phi$, then $l(Q)$ is defined as in (1.20) of [19], i.e.,

$$l(Q) = \sum_{j=\tau+1}^{m} h_j(p_j^0) e^{G_j^*(p_j^0)} \big(\Delta \log h^*\big(p_j^0\big) + 8m\pi - K\big(p_j^0\big)\big), \tag{2.3}$$

where G_j^* is defined in (2.2).

(b) If $Q_1 \neq \phi$ and $\alpha_1 > 0$, then $l(Q)$ is defined by

$$\begin{aligned} l(Q) = \sum_{j=1}^{l} \left(\frac{h_j(q_j)\rho^*}{8(1+\alpha_j)^2}\right)^{\frac{1}{1+\alpha_j}} e^{\frac{G_j^*(q_j)}{1+\alpha_1}} \\ \cdot \big(\Delta \log h^*(q_j) + \rho_* - N^* - 2K(q_j)\big), \end{aligned} \tag{2.4}$$

where $K(q_j)$ is the Gaussian curvature and $N^* = 4\pi \sum_{j=1}^{N} \alpha_j$.

(c) If $Q_1 \neq \varnothing$ and $\alpha_1 < 0$, there are two cases to be discussed.

(i) If $\alpha_1 < 0$ and $Q \neq Q_1$, then we have

$$\alpha_{\tau+1} = \cdots = \alpha_m = 0 > \alpha_j \quad \text{for } 1 \leq j \leq \tau.$$

In this case, $l(Q)$ is defined as in (1.20) of [19], i.e., as in (2.3).

(ii) If $\alpha_1 < 0$ and $Q = Q_1$, then we define $l(Q)$ as in (2.4).

From now on, we make the following assumptions.

(C1) For any subset $Q_1 = \{q_{i_1}, \dots, q_{i_\tau}\}$ of $\{q_1, \dots, q_N\}$ including the empty set, f_{Q_1} is a Morse function in $\widetilde{M}^{m-\tau} \setminus \Gamma_{m-\tau}$, where

$$\Gamma_{m-\tau} = \{(x_{\tau+1}, \dots, x_m) \mid x_i \neq x_j \text{ for } i \neq j\}.$$

(C2) For any critical point $(p^0_{\tau+1}, \dots, p^0_m)$ of f_{Q_1}, $l(Q) \neq 0$, where $Q_1 = \{q_{i_1}, \dots, q_{i_\tau}\}$ and $Q = (q_{i_1}, \dots, q_{i_\tau}, p^0_{\tau+1}, \dots, p^0_m)$.

We follow the steps in [19,20] to construct approximate solutions as follows. We will give the details of our construction for case (b) only because case (a) and (i) of case (c) are similar and their proofs were given basically in [19]. The construction for (ii) in case (c) is similar to the one we are going to give in this paper. *To simplify our description, we may also assume M has a flat metric near a neighborhood of each blowup point.* Of course, we can modify our arguments without any difficulty for the general case, as in [19].

Let $(p_1, \dots, p_m)$ satisfy $p_i = p^0_i$ for $1 \leq i \leq \tau$ and $|p_i - p^0_i| \ll 1$ for $\tau < i \leq m$, where $(p^0_{\tau+1}, \dots, p^0_m)$ is a critical point of f_{Q_1}, $\alpha_j = \alpha(q_j)$ for $j \leq \tau$, $\alpha_j = 0$ for $j > \tau$, and $0 < \alpha_1 = \cdots = \alpha_l > \alpha_j$ for $l < j \leq \tau$.

For large $\lambda_j > 0$, we set

$$U_j(x) = \lambda_j - 2\log\left(1 + \frac{\rho h_j(p_j)}{8(1+\alpha_j)^2} e^{\lambda_j} |x - p_j|^{2(1+\alpha_j)}\right), \tag{2.5}$$

which satisfies

$$\begin{cases} \Delta U_j + \rho h_j(p_j)|x - p_j|^{2\alpha_j} e^{U_j} = 0 \quad \text{in } \mathbb{R}^2, \\ U_j(p_j) = \max_{\mathbb{R}^2} U_j(x) = \lambda_j, \end{cases} \tag{2.6}$$

where h_j is defined in (1.6) if $j \leq \tau$ and $h_j = h$ if $j > \tau$.

By using $h_j(x)$, $H_j(x;t)$ is defined as follows:

$$\begin{aligned} H_j(x;t) = \exp\bigg\{ &\log\left(\frac{h_j(x+p_j)}{h_j(p_j)}\right) \\ &+ \left(G^*_j(x+p_j) - G^*_j(p_j)\right) + t \bigg\} - 1. \end{aligned} \tag{2.7}$$

By (2.1), $\nabla f_{Q_1}(p_{\tau+1}, \dots, p_m) = 0$ is equivalent to $\nabla H_j(0,0) = 0$ for $\tau < j \leq m$.

Now we construct the error terms near q_j, $1 \le j \le l$. Without loss of generality, we may assume $\nabla H_j(0;0) = (a_1, 0)$. Let $Q(x) = \frac{1}{2}(\nabla^2 \log[H_j(0;0)+1]x, x)$. Then the Taylor expansion gives

$$\log\left(\frac{h_j(x+p_j)}{h_j(p_j)}\right) + G_j^*(x+p_j) - G_j^*(p_j) = a_1 x_1 + Q(x) + \text{higher order}$$

and

$$H_j(x;t) = a_1 x_1 + t + Q(x) + \frac{1}{2}(a_1 x_1 + t)^2 + O(|x|^3 + |t|^3). \tag{2.8}$$

If $1 \le j \le l$, we let $\psi_{1,j}(y)$ and $\psi_{2,j}(y)$ be the solutions of

$$\begin{cases} \Delta\psi_{1,j}(y) + \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}(\psi_{1,j}(y) + a_1 y_1) = 0 \\ \quad \text{in } \mathbb{R}^2, \\ \psi_{1,j}(0) = 0,\ |\psi_{1,j}(y)| = O(1) \\ \quad \text{at } \infty, \end{cases} \tag{2.9}$$

and

$$\begin{cases} \Delta\psi_{2,j} + \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}(\psi_{2,j} + Q(y) \\ \quad + \frac{1}{2}(a_1 y_1 + \psi_{1,j})^2) = 0 \quad \text{in } \mathbb{R}^2, \\ \psi_{2,j}(0) = 0,\ |\psi_{2,j}(y)| = O(\log|y|) \\ \quad \text{at } \infty, \end{cases} \tag{2.10}$$

where

$$U_j = U_j(y) = -2\log\left(1 + \frac{\rho h_j(p_j)}{8(1+\alpha_j)^2}|y|^{2(1+\alpha_j)}\right).$$

The existence of $\psi_{i,j}$ has been proved in [20]. Set

$$\epsilon_j = e^{-\frac{\lambda_j}{2(1+\alpha_j)}}. \tag{2.11}$$

If $1 \le j \le l$, we define

$$\eta_j(x) = \epsilon_j \psi_{1,j}(\epsilon_j^{-1} x) + \epsilon_j^2 \psi_{2,j}(\epsilon_j^{-1} x) \quad \text{for } |x| \le 2r_0. \tag{2.12}$$

By (2.8), (2.9), and (2.10), η_j satisfies

$$\Delta\eta_j + \rho h_j(p_j)|x|^{2\alpha_j} e^{U_j(x)} \widetilde{H}_j(x;\eta_j) = 0, \tag{2.13}$$

where

$$\begin{aligned} \widetilde{H}_j(x;\eta_j) = a_1 x_1 + \eta_j + Q(x) + \frac{1}{2}(a_1 x_1 + \eta_j)^2 - \frac{1}{2}(\epsilon_j^2 \psi_{2,j}(\epsilon_j^{-1} x))^2 \\ - \epsilon_j^2 \psi_{2,j}(\epsilon_j^{-1} x)(a_1 x_1 + \epsilon_j \psi_{1,j}(\epsilon_j^{-1} x)). \end{aligned}$$

If $l+1 \le j \le m$, we set

$$\eta_j = 0.$$

For $1 \le j \le m$, we let

$$s_j = \lambda_j + 2\log\left(\frac{\rho h_j(p_j)}{8(1+\alpha_j)^2}\right) + 8\pi(1+\alpha_j)R(p_j;p_j) + \frac{d_j}{2(1+\alpha_j)}\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}, \tag{2.14}$$

where

$$d_j = \frac{\pi}{(1+\alpha_j)\sin(\frac{\pi}{1+\alpha_j})}\left(\frac{8(1+\alpha_j)^2}{\rho h_j(p_j)}\right)^{\frac{1}{1+\alpha_j}} \times (\Delta \log h^*(p_j) + \rho_* - N^* - 2K(p_j)) \tag{2.15}$$

if $1 \le j \le \tau$, and $d_j = 0$ if $\tau < j \le m$.

Let $\sigma(x)$ be a cutoff function:

$$\sigma(x) = \begin{cases} 1 & \text{if } |x| \le r_0, \\ 0 & \text{if } |x| \ge 2r_0, \end{cases}$$

and $\sigma_j(x) = \sigma(x - p_j)$. We set

$$u_{p_j}(x) = \big(U_j(x) + \eta_j(x) + 8\pi(1+\alpha_j)(R(x;p_j) - R(p_j;p_j)) + s_j\big)\sigma_j(x) + 8\pi(1+\alpha_j)G(x;p_j)(1-\sigma_j). \tag{2.16}$$

Note that

$$\eta_j(x) + \frac{d_j}{2(1+\alpha_j)}\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}} = O(e^{-\lambda_j})$$

if $j \le \tau$ and $\frac{|x|}{\epsilon_j} \ge \delta > 0$ for some $\delta > 0$.

Now we can describe our approximation solution. Let $P = (p_1, \dots, p_m)$ with $p_j = q_{i_j}$ for $1 \le j \le \tau$, $\Lambda = (\lambda_1, \dots, \lambda_m)$, and $A = (a_1, \dots, a_m)$; we set

$$u_{P,\Lambda,A} = \sum_{j=1}^{m} a_j(u_{p_j} - \overline{u}_{p_j}) \quad \text{where} \quad \overline{u}_{p_j} = \frac{1}{|M|}\int_M u_{p_j}.$$

We expect that a solution u of (1.4) with a large sup-norm can be decomposed as

$$u = u_{P,\Lambda,A} + z(x),$$

where $\|z\|_{H^1}$ is small.

To overcome the difficulty of the degeneracy of the linearized equation at u_{p_j}, we define $O_{P,\Lambda}$ and $S_\rho(Q)$ as follows: For each $P = (p_1, \dots, p_m)$ and $\Lambda =$

$(\lambda_1, \dots, \lambda_m)$, $O_{P,\Lambda}$ is defined by

$$O_{P,\Lambda} = \Big\{ w \in \mathring{H}^1(M) \,\Big|\, \int_M \nabla w \cdot \nabla u_{p_j} = \int_M \nabla w \cdot \nabla \partial_{\lambda_j} u_{p_j} = 0 \tag{2.17}$$
$$\text{for } 1 \le j \le \tau \text{ and } \int_M \nabla w \cdot \nabla u_{p_j} = \int_M \nabla w \cdot \nabla \partial_{\lambda_j} u_{p_j}$$
$$= \int_M \nabla w \cdot \nabla \partial_{p_j} u_{p_j} = 0 \text{ for } \tau + 1 \le j \le m \Big\}.$$

Also for each (P, Λ), we define

$$t_j = \lambda_j + G_j^*(p_j) + 2\log\left(\frac{\rho_* h_j(p_j)}{8(1+\alpha_j)^2}\right) \tag{2.18}$$
$$+ \frac{d_j}{2(1+\alpha_j)} \lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}} - \sum_{j=1}^m \overline{u}_{p_j} \quad \text{for } 1 \le j \le m.$$

For $Q = (q_1, \dots, q_\tau, p^0_{\tau+1}, \dots, p^0_m)$, where $(p^0_{\tau+1}, \dots, p^0_m)$ is a critical point of f_{Q_1}, we define $\lambda_1(Q, \rho)$ by the number that satisfies

$$\rho - \rho_* = \tilde{l}(Q) e^{-\frac{\lambda_1(Q,\rho)}{1+\alpha_1}}, \tag{2.19}$$

where

$$\tilde{l}(Q) = \frac{2\pi^2}{(1+\alpha_1)\sin(\frac{\pi}{1+\alpha_1})} \left(\frac{8(1+\alpha_1)^2}{\rho_* h_1(q_1)}\right)^{\frac{2}{1+\alpha_1}} e^{-\frac{G_1^*(q_1)}{1+\alpha_1}} l(Q)$$

and $l(Q)$ is defined by (2.4). Finally for some constant $c_1 > 0$, we define $S_\rho(Q)$ as follows:

$$S_\rho(Q) = \Big\{ u = u_{P,\Lambda,A} + w \,\Big|\, |p_j - p_j^0| \le c_1 e^{-\frac{\lambda_1(Q,\rho)}{1+\alpha_1}} \text{ for } \tau + 1 \le j \le m,$$
$$|\lambda_1 - \lambda_1(Q,\rho)| \le c_1 \lambda_1(Q,\rho)^{-1},\ |t_j - t_1| \le c_1 e^{-\frac{\lambda_1(Q,\rho)}{1+\alpha_1}}$$
$$\text{for } 2 \le j \le m,\ |a_j - 1| \le c_1 \lambda_1(Q,\rho)^{-\frac{1}{2}} e^{-\frac{\lambda_1(Q,\rho)}{1+\alpha_1}} \text{ for } 1 \le j \le m,$$
$$w \in O_{P,\Lambda}, \text{ and } \|w\|_{H^1} \le c_1 e^{-\frac{\lambda_1(Q,\rho)}{1+\alpha_1}} \Big\}.$$

The constant c_1 will be chosen later.

Now we can state Theorem B in Section 1, the main result proved in [20] with details.

THEOREM B. *Suppose all* $\alpha_i \notin \mathbb{N}^+$, $1 \le i \le N$, *and* h^* *is a positive* C^3 *function on* M. *If* (C1) *and* (C2) *hold, then there exist* $\epsilon_0 > 0$ *and* $C > 0$ *such that for any solution* u *of* (1.4) *with* $\rho \in (\rho_* - \epsilon_0, \rho_* + \epsilon_0)$, *either* $|u(x)| \le C$, $\forall x \in M$, *or*

$u \in S_\rho(Q)$ for some $Q = (Q_1, p^0_{\tau+1}, \dots, p^0_m)$, where $Q_1 = \{q_{j_1}, \dots, q_{j_\tau}\}$ and $(p^0_{\tau+1}, \dots, p^0_m)$ is a critical point of f_{Q_1}.

In [20], we actually prove that if u_k is a sequence of blowup solutions to (1.4) with $\rho = \rho_k$, then there exists (P_k, Λ_k, A_k) such that $u_k(x)$ can be written as

$$u_k(x) = u_{P_k,\Lambda_k,A_k} + z_k(x),$$

where $z_k(x) \in O_{P_k,\Lambda_k}$, and (P_k, Λ_k, A_k) satisfies all the conditions listed in $S_{\rho_k}(Q)$. We remark that the quantity t_j has the following meaning: For u_k, we define $w_k = u_k - \log \int_M he^{u_k}$. Let $\lambda_{k,j}$ be the maximum of u_k near p^0_j, $1 \le j \le m$. Then the $t_{k,j}$, defined in (2.18) according to $\lambda_{k,j}$, satisfies

$$t_{k,j} = \overline{w}_k(1 + o(1)), \quad 1 \le j \le m,$$

where $\overline{w}_k = \int_M he^{u_k}$.

Since $\int_M G(x; p_j)dx = 0$, it is expected that the average of u_{p_j} is also small. The lemma below shows it is true.

LEMMA 2.1.

(i) *If $1 \le j \le m$, then*

$$\int_M u_{p_j}, \quad \int_M \partial_{\lambda_j} u_{p_j} = O\big(e^{-\frac{\lambda_j}{1+\alpha_j}}\big).$$

(ii) *If $j > \tau$, then*

$$\int_M \partial_{p_j} u_{p_j} = O\big(\lambda_j e^{-\lambda_j}\big).$$

For a proof, see lemma 2.3 in [19] and lemma 4.2 in [20].

For $r_0 \le |x - p_j| \le 2r_0$, we have the following estimates:

$$\begin{aligned} U_j(x) &= \lambda_j - 2\log\left(1 + \frac{\rho h_j(p_j)}{8(1+\alpha_j)^2} e^{\lambda_j}|x - p_j|^{2(1+\alpha_j)}\right) \\ &= -\lambda_j - 2\log\left(\frac{\rho h_j(p_j)}{8(1+\alpha_j)^2}\right) - 4(1+\alpha_j)\log|x - p_j| \\ &\quad + O(e^{-\lambda_j}|x - p_j|^{-2(1+\alpha_j)}). \end{aligned} \tag{2.20}$$

Thus

$$\begin{aligned} u_{p_j}(x) - 8\pi(1+\alpha_j)G(x; p_j) &= \eta_j(x) + \frac{d_j \lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}}{2(1+\alpha_j)} + O(e^{-\lambda_j}) \\ &= O\big(e^{-\frac{\lambda_j}{1+\alpha_j}}\big) \end{aligned} \tag{2.21}$$

for $x \in B_{2r_0}(p_j) \setminus B_{r_0}(p_j)$. From [19,20], we have the following result:

LEMMA 2.2. *Let $\xi_j(x) = u_{p_j}(x) - 8\pi(1+\alpha_j)G(x; p_j)$. Then in the annulus $r_0 \le |x - p_j| \le 2r_0$ the following holds*:

(i) *For* $1 \leq j \leq m$*, the quantities* $\xi_j(x)$*,* $\partial_{\lambda_j}\xi_j(x)$*,* $\nabla_x \xi_j(x)$*, and* $\Delta_x \xi(x)$ *are all* $O(\exp(\frac{-\lambda_j}{1+\alpha_j}))$.

(ii) *For* $j > \tau$*,* $\partial_{p_j}\xi_j(x)$ *is* $O(\exp(\frac{-\lambda_j}{1+\alpha_j}))$.

3 Decomposition Lemmas

Let $u = u_{P,\Lambda,A} + w \in S_\rho(Q)$. Recall

$$t_j = s_j + 8\pi \sum_{k \neq j} (1+\alpha_k) G(p_j, p_k) - \sum_{i=1}^{m} \overline{u}_{p_i},$$

and for $x \in B(p_j, r_0)$

$$\begin{aligned} &u_{P,\Lambda,A}(x) + \log \frac{h_j(x)}{h_j(p_j)} \\ &\quad = U_j + t_j + \log(H_j(x-p_j,\eta_j)+1) + (a_j - 1)(U_j + s_j) \\ &\qquad + 8\pi \sum_{k \neq j} (1+\alpha_k)(a_k - 1) G(p_j, p_k) + O(|a_j - 1|(|y| + \eta_j|)), \end{aligned}$$

where $y = x - p_j$. The above together with (2.8) implies

$$\begin{aligned} \rho h e^{u-w} = {}& \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j + t_j} \big[1 + (a_j - 1)(U_j + s_j) \\ & + \sum_{\substack{1 \leq k \leq m \\ k \neq j}} 8\pi (1+\alpha_k)(a_k - 1) G(p_j, p_k) \\ & + \eta_j + \nabla_y H_j(0,0)\cdot y + Q_j(y) + \frac{1}{2}(\eta_j + \nabla_y H_j \cdot y)^2 \\ & + (a_j - 1)(U_j + s_j)(\nabla_y H_j(0,0)\cdot y + \eta_j) + O(\widetilde{\beta}_j)\big], \end{aligned} \tag{3.1}$$

where

$$\widetilde{\beta}_j = \lambda_1^2 \sum_{1 \leq k \leq m} |a_k - 1|^2 + |a_k - 1|(|\eta_j| + |y|) + |\eta_j|^3 + |y|^3.$$

Therefore we have on $B_{r_0}(p_j)$

$$\begin{aligned} \rho h e^{u} &= (1+w)\rho h e^{u-w} + (e^w - 1 - w)\rho h e^{u-w} \\ &= \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j + t_j} \big[1 + (a_j - 1)(U_j + s_j) \\ &\quad + \sum_{\substack{1 \leq k \leq m \\ k \neq j}} 8\pi (a_k - 1)(1+\alpha_k) G(p_j, p_k) + \end{aligned} \tag{3.2}$$

$$+\eta_j + \nabla_y H_j \cdot y + Q_j(y)$$

$$+ (a_j - 1)(U_j + s_j)(\nabla_y H_j(0,0)\cdot y + \eta_j)$$

$$+ \frac{1}{2}(\eta_j + \nabla_y H_j \cdot y)^2 + w\big] + \widetilde{E}_j,$$

where

$$\widetilde{E}_j = (e^w - 1 - w)\rho h e^{u_{P,\Lambda,A}} + \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j + t_j} O(w^2 + \widetilde{\beta}_j).$$

Let $\epsilon_2 > 0$ be small. $\widetilde{E}_j$ can be written into two parts

$$\widetilde{E}_j = \widetilde{E}_j^+ + \widetilde{E}_j^-,$$

where

$$\widetilde{E}_j^+ = \begin{cases} \widetilde{E}_j & \text{if } |w| \geq \epsilon_2, \\ 0 & \text{if } |w| < \epsilon_2, \end{cases} \qquad \widetilde{E}_j^- = \begin{cases} 0 & \text{if } |w| \geq \epsilon_2, \\ \widetilde{E}_j & \text{if } |w| < \epsilon_2. \end{cases}$$

Then

$$\widetilde{E}_j^+ = O(e^{|w| + 2\lambda_j}) \quad \text{if } |w| \geq \epsilon_2 \tag{3.3}$$

and

$$\widetilde{E}_j^- = \rho h e^{U_j + \lambda_j}|y|^{2\alpha_j} O(1)(w^2 + \widetilde{\beta}_j). \tag{3.4}$$

From now on, $O(1)$ always stands for an upper bound independent of the constants c_1 and $\lambda_1(Q,\rho)$ that appeared in $S_\rho(Q)$, provided $\lambda_1(Q,\rho)$ is large. Using the expression for $\rho h e^u$ above, we obtain the following decomposition for $\int_M \rho h e^u$:

LEMMA 3.1. *Let $\tau > 0$, $\rho_* = 8\pi(m-\tau) + \sum_{j=1}^{j=\tau} 8\pi(1+\alpha_j)$, and $u = u_{P,\Lambda,A} + w \in S_\rho(Q)$. Then as $\rho \to \rho_*$,*

$$\int_M \rho h e^u = \sum_{j=1}^m 8\pi(1+\alpha_j)e^{t_j} + \sum_{j=1}^l 2\pi d_j e^{t_j} e^{-\frac{\lambda_j}{1+\alpha_j}}$$
$$+ \sum_{j=1}^m 16\pi(1+\alpha_j)\lambda_j(a_j - 1)e^{t_j}$$
$$+ O\Big(\sum_{j=1}^m |a_j - 1| e^{\lambda_1}\Big) + O\big(e^{\lambda_1 - \frac{\lambda_1}{1+\alpha_1} - \epsilon\lambda_1}\big)$$

for some $\epsilon > 0$.

PROOF. Note that $\lambda_j = \lambda_1 + O(1)$ and $t_j = \lambda_1 + O(1)$. By (3.2),

$$\int_M \rho h e^u = \sum_j \int_{B_{r_0}(p_j)} \{\rho h_j(p_j)|y|^{2\alpha_j} e^{U_j+t_j}[1+2+\cdots] + \widetilde{E}_j\} dy + \int_{M\setminus\bigcup_j B_{r_0}(p_j)} \rho h e^u.$$

By the explicit expression of U_j, we have

$$\int_{M\setminus\bigcup_j B_{r_0}(p_j)} \rho h e^u = O(1), \tag{3.5}$$

$$\int_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j+t_j} dy = 8\pi(1+\alpha_j)e^{t_j} + O(1), \tag{3.6}$$

$$\begin{aligned} &\int_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j+t_j}(a_j-1)(U_j+s_j) dy \\ &\quad = 16\pi(1+\alpha_j)(a_j-1)\lambda_j e^{t_j} + O(|a_j-1|e^{\lambda_j}), \end{aligned} \tag{3.7}$$

where we have used

$$U_j + s_j = 2\lambda_j - 2\log\left(1 + \frac{\rho h(p_j)}{8(1+\alpha_j)^2} e^{\lambda_j}|y|^{2(1+\alpha_j)}\right) + O(1).$$

By equation (2.13) of η_j for $j \le \tau$, and the fact $\psi_{2,j}(y)y_1$ and $\psi_{2,j}(y)\psi_{1,j}(y)$ are odd functions, we have

$$\begin{aligned} &\int_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j+t_j} \\ &\qquad \times \left(\eta_j + \nabla_y H_j \cdot y + Q_j + \frac{1}{2}(\eta_j + \nabla_y H_j \cdot y)^2\right) dy \\ &\quad = -e^{t_j} \int_{B_{r_0}(p_j)} \Delta \eta_j dy + O\big(e^{t_j} e^{-\frac{2\lambda_j}{1+\alpha_j}}\big) \\ &\quad = -e^{t_j} \int_{\partial B_{r_0}(p_j)} \frac{\partial \eta_j}{\partial \nu} + O\big(e^{t_j} e^{-\frac{2\lambda_j}{1+\alpha_j}}\big) \\ &\quad = 2\pi d_j e^{-\frac{\lambda_j}{1+\alpha_j}} e^{t_j} + O\big(e^{t_j} e^{-\frac{2\lambda_j}{1+\alpha_j}}\big). \end{aligned} \tag{3.8}$$

It is not difficult to see that

$$\int_{B_{r_0}(p_j)} |y|^{2\alpha_j} e^{U_j+t_j}(a_k-1)8\pi G(p_j,p_k)dy = O\Big(\sum_{k=1}^{m}|a_k-1|e^{\lambda_j})\Big), \tag{3.9}$$

$$\begin{aligned}&\int_{B_{r_0}(p_j)} |y|^{2\alpha_j} e^{U_j+t_j}(a_j-1)[(U_j+s_j)+1+\eta_j]\nabla_y H_j(0,0)\cdot y\,dy\\ &\quad = O\big(|a_j-1|e^{\lambda_1-\frac{\lambda_1}{1+\alpha_1}}\big),\end{aligned} \tag{3.10}$$

where the cancellation occurs due to the oddness of $\nabla_y H_j(0,0)\cdot y$.

To estimate the terms involving w, we use (2.16) to obtain

$$\Delta u_{p_j} = \Delta U_j + \Delta\eta_j + 8\pi(1+\alpha_j) \quad \text{for } x \in B_{r_0}(p_j)$$

and

$$\Delta u_{p_j} = \Delta(u_{p_j} - 8\pi(1+\alpha_j)G(x,p_j)) + 8\pi(1+\alpha_j) \quad \text{for } x \notin B_{r_0}(p_j).$$

This together with Lemma 2.2 implies

$$\begin{aligned}&\int_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j}e^{U_j}w\\ &\quad = -\int_M w\Delta u_{p_j} + \int_{B_{r_0}(p_j)} w\Delta\eta_j\\ &\qquad + \int_{M\setminus B_{r_0}(p_j)} w\Delta(u_{p_j} - 8\pi(1+\alpha_j)G(x,p_j))\\ &\qquad + 8\pi(1+\alpha_j)\int_M w\\ &\quad = \int_M \nabla w\cdot\nabla u_{p_j} - 8\pi(1+\alpha_j)\int_M w + \int_{B_{r_0}(p_j)} w\Delta\eta_j\\ &\qquad + \int_{M\setminus B_{r_0}(p_j)} w\Delta(u_{p_j} - 8\pi(1+\alpha_j)G(x,p_j)).\end{aligned}$$

To estimate $\int_{B_{r_0}(p_j)} w\Delta\eta_j$, we choose $r_0' \in (r_0/2, r_0)$ such that

$$\int_{\partial B_{r_0'}(p_j)} |w|d\sigma \le \frac{2}{r_0}\int_{B_{r_0}(p_j)} |w|dx.$$

Hence

$$\left|\int_{\partial B_{r_0'(p_j)}} w\,\frac{\partial \eta_j}{\partial \nu}\,d\sigma\right| \le C \max_{\partial B_{r_0'(p_j)}}\left|\frac{\partial \eta_j}{\partial \nu}\right| \|w\|_{H^1} = O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big)\|w\|_{H^1},$$

where by (2.13),

$$-\int_{\partial B_{r_0'(p_j)}} \frac{\partial \eta_j}{\partial \nu}(x) = \int_{B_{r_0'(p_j)}} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}\,\widetilde{H}(x,\eta_j)dx = O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big).$$

Thus

$$\begin{aligned}\int_{B_{r_0}(p_j)} w\Delta\eta_j &= \int_{B_{r_0'}(p_j)} w\Delta\eta_j + O(e^{-\lambda_1})\int_M |w| \\ &= -\int_{B_{r_0'}(p_j)} \nabla w\nabla\eta_j + O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big)\|w\|_{H^1} \\ &\le \Big(\int_{B_{r_0'}(p_j)} |\nabla w|^2\Big)^{\frac{1}{2}}\Big(\int_{B_{r_0'}(p_j)} |\nabla \eta_j|^2\Big)^{\frac{1}{2}} + O(e^{-\lambda_1})\int_M |w| \\ &= O\big(e^{-\frac{\lambda_1}{2(1+\alpha_1)}}\big)\|w\|_{H^1}.\end{aligned}$$

By (A.2) (see Lemma A in Appendix A), we have for some $\epsilon > 0$

$$\left|\int_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}\,w\right| = O(e^{-\epsilon\lambda_1})\|w\|_{H^1}. \tag{3.11}$$

For $\widetilde{E}_j^+$, we have

$$\begin{aligned}\int_{B_{r_0}(p_j)} |\widetilde{E}_j^+(y)|dy &= O(1)\int_{B_{r_0}(p_j)\cap\{|w-\overline{w}|\ge\epsilon_2/2\}} e^{|w|+2\lambda_j}\,dy \\ &= O(1)\int_{B_{r_0}(p_j)\cap\{|w-\overline{w}|\ge\epsilon_2/2\}} e^{|w-\overline{w}|+2\lambda_j}\,dy,\end{aligned}$$

where

$$\overline{w} = \frac{\int_{B_{r_0}(p_j)} w}{\mathrm{vol}(B_{r_0}(p_j))} = O(\|w\|_{H^1}) = O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big)$$

if λ_1 is large.

Write

$$e^{|w-\overline{w}|} = e^{|w-\overline{w}|\left(1-\frac{4\pi|w-\overline{w}|}{\|w-\overline{w}\|^2}\right)} e^{\frac{4\pi|w-\overline{w}|^2}{\|w-\overline{w}\|^2}}.$$

Since $(\|w-\bar{w}\|^2)^{-1} := (\|w-\bar{w}\|^2_{H^1})^{-1} \gg 2\lambda_j$, we have

$$e^{|w-\bar{w}|\left(1-\frac{4\pi|w-\bar{w}|}{\|w-\bar{w}\|^2}\right)} \le e^{\frac{\epsilon_2}{2}\left(1-\frac{2\pi\epsilon_2}{\|w-\bar{w}\|^2}\right)} \ll e^{-2\lambda_j} \quad \text{for } |w-\bar{w}| \ge \frac{\epsilon_2}{2}.$$

Hence by the Moser-Trudinger inequality,

$$\int\limits_{B_{r_0}(p_j)\cap\{|w-\bar{w}|\ge\epsilon_2/2\}} e^{|w-\bar{w}|} \le e^{-2\lambda_j} \int\limits_{B_{r_0}(p_j)} \exp\left(\frac{4\pi|w-\bar{w}|^2}{\|w-\bar{w}\|^2}\right)dy$$
$$\le c_2 e^{-2\lambda_j},$$

which implies

$$\int\limits_{B_{r_0}(p_j)} \left|\widetilde{E}_j^+(y)\right|dy \le O(1). \tag{3.12}$$

For $\widetilde{E}_j^-$, (3.4) gives

$$\int\limits_{B_{r_0}(p_j)} |\widetilde{E}_j^-|dy \le c \int\limits_{B_{r_0}(p_j)} (|w|^2 + \widetilde{\beta}_j)\rho h_j(p_j)|y|^{2\alpha_j} e^{U_j+t_j}\, dy.$$

By Lemma A in Appendix A, the first term of the integrals can be bounded by

$$\int\limits_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j+t_j}|w|^2 dy \le c\, e^{t_j} \int\limits_M |\nabla w|^2$$
$$= O(1)c_1^2 e^{t_j} e^{-2\frac{\lambda_1}{1+\alpha_1}} = O(1)e^{\lambda_1-\frac{\lambda_1}{1+\alpha_1}-\epsilon_1\lambda_1}.$$

Since $u \in S_\rho(Q)$, the term related to $\widetilde{\beta}_j$ can be estimated as follows:

$$\int\limits_{B_{r_0}(p_j)} \widetilde{\beta}_j \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j+t_j}\, dy$$
$$= O(1)\, e^{t_j}\Bigg(\lambda_1^2 \sum_{1\le k\le m} |a_k-1|^2$$
$$+ \int\limits_{B_{r_0}(p_j)} [|\eta_j|^3 + |y|^3 + |a_k-1|(|\eta_k|+|y|)]e^{U_j}\,dy\Bigg)$$
$$= O\left(e^{t_j} e^{-\frac{\lambda_1}{1+\alpha_1}-\epsilon\lambda_1}\right)$$

for some $\epsilon > 0$ and large λ_1. So we have

$$\int\limits_{B_{r_0}(p_j)} |\widetilde{E}_j^-|dy = O(1)e^{\lambda_1-\frac{\lambda_1}{1+\alpha_1}-\epsilon\lambda_1}. \tag{3.13}$$

Therefore by (3.2) and (3.5)–(3.13), we obtain

$$\begin{aligned}\int_M \rho h e^u &= \sum_{j=1}^m \int_{B_{r_0}(p_j)} \rho h e^u dy + O(1)\\ &= \sum_{j=1}^m e^{t_j}\big(8\pi(1+\alpha_j) + 16\pi(1+\alpha_j)\lambda_j(a_j-1)\big)\\ &\quad + \sum_{j=1}^{\tau} 2\pi d_j e^{t_j} e^{-\frac{\lambda_j}{1+\alpha_j}} + O\Big(\sum_{j=1}^m |a_j-1| e^{\lambda_1}\Big) + O\big(e^{\lambda_1 - \frac{\lambda_1}{1+\alpha_1} - \epsilon\lambda_1}\big).\end{aligned}$$

□

Now, we want to express $u + T_\rho u$ in a similar formula as (3.2). For simplicity, we use Tu to denote $T_\rho u$. By Lemma 3.1, we expect that $e^{t_j}/\int_M he^u - 1$ is small. Indeed, by the definition of $S_\rho(Q)$,

$$|t_j - t_1| = O(1)e^{-\frac{\lambda_1}{(1+\alpha_1)}}.$$

By Lemma 3.1 and the Taylor expansion of the exponential function,

$$\begin{aligned}e^{-t_j}\int_M \rho h e^u &= \sum_{k=1}^m 8\pi(1+\alpha_k)e^{t_k - t_j} + \sum_{k=1}^{l} 2\pi d_k e^{t_k-t_j} e^{-\frac{\lambda_k}{1+\alpha_k}}\\ &\quad + \sum_{k=1}^m 16\pi(1+\alpha_k)\lambda_k(a_k-1)e^{t_k-t_j}\\ &\quad + O\Big(\sum_{k=1}^m |a_k - 1|\Big) + O\big(e^{-\frac{\lambda_1}{1+\alpha_1}-\epsilon\lambda_1}\big)\\ &= \rho_* + \sum_{k=1}^m 8\pi(1+\alpha_k)(t_k - t_j)\\ &\quad + \sum_{k=1}^{l} 2\pi d_k e^{-\frac{\lambda_k}{1+\alpha_k}} + \sum_{k=1}^m 16\pi(1+\alpha_k)\lambda_k(a_k-1)\\ &\quad + O(|a_k-1|) + O\big(e^{-\frac{\lambda_1}{1+\alpha_1}-\epsilon\lambda_1}\big).\end{aligned} \tag{3.14}$$

Hence

$$\begin{aligned}\frac{e^{t_j}}{\int_M he^u} - 1 &= \frac{1}{e^{-t_j}\rho\int_M he^u}\left(\rho - \frac{\rho\int_M he^u}{e^{t_j}}\right)\\ &= \theta_j + O(|a_k - 1|) + O(e^{-\frac{\lambda_1}{1+\alpha_1}-\epsilon\lambda_1}),\end{aligned} \tag{3.15}$$

where θ_j is defined by

$$\theta_j = \frac{1}{\rho_*}\Bigg[(\rho-\rho_*) - \sum_{k=1}^{l} 2\pi d_k e^{-\frac{\lambda_k}{1+\alpha_k}} - \sum_{k=1}^{m} 8\pi(1+\alpha_k)(t_k - t_j) - \sum_{k=1}^{m} 16\pi(1+\alpha_k)\lambda_k(a_k-1)\Bigg]. \tag{3.16}$$

Let

$$\beta_j = \left|\frac{e^{t_j}}{\int_M he^u} - 1\right|^2 + \widetilde{\beta}_j, \tag{3.17}$$

$$E_j = (e^w - 1 - w)\frac{\rho he^{u-w}}{\int_M he^u} + \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}(O(w^2) + O(\beta_j)). \tag{3.18}$$

Then in $B_{r_0}(p_j)$, we have by (3.2),

$$\begin{aligned}\frac{\rho he^u}{\int_M he^u} &= (1+w)\frac{\rho he^{u_{P,\Lambda,A}}}{\int_M he^u} + (e^w - 1 - w)\frac{\rho he^{u_{P,\Lambda,A}}}{\int_M he^u}\\ &= \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}\Bigg[1 + \left(\frac{e^{t_j}}{\int_M he^u} - 1\right) + (a_j-1)(U_j+s_j)\\ &\quad + \sum_{1\le k\le m, k\neq j} 8\pi(1+\alpha_k)(a_k-1)G(p_j,p_k)\\ &\quad + (a_j-1)(U_j+s_j)\nabla_y H_j(0,0)\cdot y\\ &\quad + \eta_j + \nabla_y H_j\cdot y + Q_j(y) + \frac{1}{2}(\eta_j + \nabla_y H_j\cdot y)^2 + w\Bigg] + E_j.\end{aligned} \tag{3.19}$$

Thus we have in $B_{r_0}(p_j)$,

$$\begin{aligned}&D(u+Tu)\\ &\quad = \Delta u + \frac{\rho he^u}{\int_M he^u} - \rho\\ &\quad = a_j(\Delta U_j + \Delta\eta_j) + \Delta w + \sum_{k=1}^{m} 8\pi(1+\alpha_k)a_k + \frac{\rho he^u}{\int_M he^u} - \rho\\ &\quad = -a_j\rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}\Bigg[1 + \eta_j + \nabla_y H_j\cdot y + Q_j(y)\\ &\qquad + \frac{1}{2}(\eta_j + \nabla_y H_j\cdot y)^2 - \epsilon^2\psi_{2,j}(\epsilon^{-1}y)(\epsilon\psi_{1,j}(\epsilon^{-1}y) + \nabla_y H_j\cdot y)\\ &\qquad - \frac{\epsilon^4}{2}\psi_{2,j}^2(\epsilon^{-1}y)\Bigg] + \Delta w + \Bigg[\sum_{k=1}^{m} 8\pi(1+\alpha_k) - \rho\Bigg]\\ &\qquad + \sum_{k=1}^{m} 8\pi(1+\alpha_k)(a_k-1) + \frac{\rho he^u}{\int_M he^u} =\end{aligned} \tag{3.20}$$

$$
\begin{aligned}
&= \Delta w + [\rho_* - \rho] + \sum_{k=1}^{m} 8\pi(1+\alpha_k)(a_k - 1) \\
&\quad + \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j} \Bigg[(a_j - 1)(U_j + s_j - 1) \\
&\quad + \sum_{\substack{1 \le k \le m \\ k \neq j}} 8\pi(1+\alpha_k)(a_k - 1) G(p_j, p_k) \\
&\quad + (U_j + s_j)(a_j - 1)(\nabla_y H_j(0,0) \cdot y + \eta_j) \\
&\quad + \left(\frac{e^{t_j}}{\int_M h e^u} - 1 \right) + w \Bigg] + \tilde{\tilde{E}}_j,
\end{aligned}
$$

where $\tilde{\tilde{E}}_j = E_j + a_j \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j} [\epsilon^2 \psi_{2,j}(\epsilon^{-1} y)(\eta_j + \nabla_y H_j(0,0) \cdot y) + \frac{\epsilon^4}{2} \psi_{2,j}^2(\epsilon^{-1} y)]$.

On $B_{2r_0}(p_j) \setminus B_{r_0}(p_j)$, since $u_{p_j} - \overline{u}_{p_j} - 8\pi G(x, p_j)$ is small by Lemma 2.2, we can write $\Delta(u + Tu)$ as

$$
\begin{aligned}
&\Delta(u + Tu) \\
&\quad = \Delta w + a_j \Delta(u_{p_j} - 8\pi G(x, p_j)) \\
&\quad\quad + \rho_* - \rho + \sum_{k=1}^{m} 8\pi(1+\alpha_k)(a_k - 1) \\
&\quad\quad + \frac{\rho h}{\int_M h e^u} e^{a_j[u_{p_j} - \overline{u}_{p_j} - 8\pi G(x, p_j)] + \sum_{k=1}^{m} 8\pi(1+\alpha_k) a_k G(x, p_k) + w}.
\end{aligned}
\tag{3.21}
$$

On $M \setminus \bigcup_{j=1}^{m} B_{2r_0}(p_j)$, we have

$$
\begin{aligned}
\Delta(u + Tu) = \Delta w + \rho_* - \rho &+ \sum_{k=1}^{m} 8\pi(1+\alpha_k)(a_k - 1) \\
&+ \frac{\rho h}{\int_M h e^u} e^{\sum_{k=1}^{m} 8\pi(1+\alpha_k) a_k G(x, p_k) + w}.
\end{aligned}
\tag{3.22}
$$

Based on (3.20)–(3.22), we obtain the dominant terms of $u + Tu$ on $S_\rho(Q)$ as follows:

LEMMA 3.2. *Assume none of* $\alpha_1, \alpha_2, \ldots$ *and* α_τ *are integers. Let* $\rho_* = 8\pi(m - \tau) + \sum_{k=1}^{k=\tau} 8\pi(1+\alpha_k)$ *and* $u = u_{P,\Lambda,A} + w \in S_\rho(Q)$. *Then as* $\rho \to \rho_*$:

(i) *For* $w_1 \in H^1$,

$$
\langle \nabla(u + Tu), \nabla w_1 \rangle = B(w, w_1) + O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big) \|w_1\|_{H^1},
\tag{3.23}
$$

where

$$B(w, w_1) := \int_M \left[\bigtriangledown w \bigtriangledown w_1 - \sum_{j=1}^{m} \rho h_j(p_j) |y|^{2\alpha_j} e^{U_j} \sigma_j w w_1 \right].$$

(ii) *For* $j > \tau$,

$$\begin{aligned} &\langle \bigtriangledown(u + Tu), \bigtriangledown \partial_{p_j} u_{p_j} \rangle \\ &\quad = -8\pi \bigtriangledown_y H_j(0,0) \\ &\qquad + O\left(\lambda_j |a_j - 1| + \left| \frac{e^{t_j}}{\int_M h e^u} - 1 \right| + e^{-\frac{\lambda_1}{1+\alpha_1}} \right). \end{aligned} \tag{3.24}$$

(iii) *For* $1 \leq j \leq m$,

$$\begin{aligned} &\langle \bigtriangledown(u + Tu), \bigtriangledown \partial_{\lambda_j} u_{p_j} \rangle \\ &\quad = -16\pi(1 + \alpha_j)(a_j - 1)\lambda_j \\ &\qquad - 8\pi(1 + \alpha_j)\theta_j + O\big(\max_k |a_k - 1| + e^{-\frac{\lambda_1}{1+\alpha_1} - \epsilon\lambda_1}\big). \end{aligned} \tag{3.25}$$

(iv) *For* $1 \leq j \leq m$,

$$\begin{aligned} &\langle \bigtriangledown(u + Tu), \bigtriangledown u_{p_j} \rangle \\ &= \Big(2\lambda_j - 1 + 8\pi(1 + \alpha_j) R(p_j, p_j) \\ &\qquad + 2 \log \frac{\rho h_j(p_j)}{8(1 + \alpha_j)^2} \Big) \langle \bigtriangledown u + Tu, \bigtriangledown \partial_{\lambda_j} u_{p_j} \rangle \\ &\qquad + 8\pi(1 + \alpha_j) \sum_{k \neq j} G(p_j, p_k) \langle \bigtriangledown(u + Tu), \bigtriangledown \partial_{\lambda_k} u_{p_k} \rangle \\ &\qquad + 16\pi(1 + \alpha_j)(a_j - 1)\lambda_j + O(1)\|w\|_{H^1(M)} + O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big). \end{aligned} \tag{3.26}$$

In order to know the Morse index for the solutions in $S_\rho(Q)$, we have to compute the Morse index of the bilinear form B in Lemma 3.2.

LEMMA 3.3. *Assume that all* $\alpha_j > -1$, $1 \leq j \leq \tau$, *are not integers. Let* $P = (p_1, \ldots, p_m)$ *and* $\Lambda = (\lambda_1, \ldots, \lambda_m)$, *where* $\operatorname{dist}(p_j, p_k) > 2r_0$ *for* $j \neq k$. *If* $\lambda_1(Q, \rho)$ *is large, then the symmetric bilinear form*

$$B(w, w_1) := \int_M \Big(\bigtriangledown w \bigtriangledown w_1 - \sum_{1 \leq j \leq m} \rho h_j(p_j) |y|^{2\alpha_j} e^{U_j} \sigma_j w w_1 \Big)$$

is nondegenerate and has Morse index $\sum_{i=1}^{\tau} 2(1+[\alpha_j])$ *in* $O_{P,\Lambda}$*, where* $[\alpha_j]$ *denotes the greatest integer less than or equal to* α_j*.*

We put the proofs of Lemma 3.2 and Lemma 3.3 in Appendix A and Appendix B.

4 A Deformation

Obviously, $u = u_{P,\Lambda,A} + w$ is a solution of $u + Tu = 0 \iff$ the left-hand sides of (3.23)–(3.26) vanish. To solve the system (3.23)–(3.26), we recall

$$\mathring{H}^1 = O_{P,\Lambda} \oplus \text{the linear subspace spanned by } u_{p_j},\ \partial_{\lambda_j} u_{p_j},\ \text{and } \partial_{p_j} u_{p_j}$$

and deform $u + Tu$ to a simpler operator $u + T_0 u$ by defining the operator $I + T_t, 0 \le t \le 1$ through the following relations:

$$\begin{aligned}\langle \nabla(u+T_t u), \nabla w_1\rangle &= t\langle \nabla(u+Tu), \nabla w_1\rangle + (1-t)B(w,w_1)\\ &\qquad \text{for } w_1 \in O_{P,\Lambda},\end{aligned} \tag{4.1}$$

$$\begin{aligned}\langle \nabla(u+T_t u), \nabla w_1\rangle &= t\langle \nabla(u+Tu), \nabla w_1\rangle + (1-t)B(w,w_1)\\ &\qquad \text{for } w_1 \in O_{P,\Lambda},\end{aligned} \tag{4.2}$$

$$\begin{aligned}&\langle \nabla(u+T_t u), \nabla \partial_{p_j} u_{p_j}\rangle\\ &\quad = t\langle \nabla(u+Tu), \nabla \partial_{p_j} u_{p_j}\rangle\\ &\qquad + (1-t)(-8\pi\, \nabla_y\, H_j(0,0)) \quad \text{for } \tau < j \le m,\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\langle \nabla(u+T_t u), \nabla \partial_{\lambda_j} u_{p_j}\rangle\\ &\quad = t\langle \nabla(u+Tu), \nabla \partial_{\lambda_j} u_{p_j}\rangle\\ &\qquad - (1-t)[16\pi(1+\alpha_j)(a_j-1)\lambda_j + 8\pi(1+\alpha_j)\theta_j] \quad \text{for } 1 \le j \le m,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\langle \nabla(u+T_t u), \nabla u_{p_j}\rangle = t\Big[&(2\lambda_j + O(1))\langle \nabla(u+T_t u), \nabla \partial_{\lambda_j} u_{p_j}\rangle\\ &+ \sum_{k\neq j} O(1)\langle \nabla(u+T_t u), \nabla \partial_{\lambda_k} u_{p_k}\rangle\\ &+ O(1)\|w\|_{H^1(M)} + O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big)\Big]\\ &+ 16\pi(1+\alpha_j)(a_j-1)\lambda_j \quad \text{for } 1 \le j \le m,\end{aligned} \tag{4.5}$$

where the coefficients $O(1)$ are those terms appearing in (3.26) so that $T_1 u = Tu$. From the construction above, we have $u + T_1 u = u + Tu$. When $t = 0$, the operator T_0 is simpler than T. During the deformation from T_1 to T_0, we have the following:

LEMMA 4.1. *Let* $\rho_* = 8\pi(m-\tau) + \sum_{j=1}^{j=\tau} 8\pi(1+\alpha_j)$. *Assume* $(\rho - \rho_*) \neq 0$. *Then there is* $\varepsilon_1 > 0$ *such that* $u + T_t u \neq 0$ *for* $u \in \partial S_\rho(Q)$ *and* $0 \le t \le 1$ *if* $|\rho - \rho_*| < \varepsilon_1$.

PROOF. Assume $u = u_{P,\Lambda,A} + w \in \overline{S}_\rho(Q)$, the closure of $S_\rho(Q)$, and $u + T_t u = 0$ for some $0 \le t \le 1$. We will show that $u \notin \partial S_\rho(Q)$.

From $\langle \nabla(u + T_t u), \nabla w \rangle = 0$, we have by Lemma 3.2

$$\|w\|_{H^1}^2 \le O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big)\|w\|_{H^1}.$$

This implies

$$\|w\|_{H^1} = O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big) \le c_2\, e^{-\frac{\lambda_1}{1+\alpha_1}} \tag{4.6}$$

for some constant c_2 independent of c_1.

Using $\langle \nabla(u+T_t u), \nabla \partial_{\lambda_j} u_{p_j} \rangle = 0$ and $\langle \nabla(u+T_t u), \nabla u_{p_j} \rangle = 0$, we conclude from (4.5) and (4.6) that

$$16\pi\lambda_j(1+\alpha_j)(a_j - 1) = O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big) \tag{4.7}$$

for $j = 1, \dots, m$. Hence if $|\rho - \rho_*|$ is sufficiently small, then

$$|a_j - 1| = O\big(\lambda_1^{-1} e^{-\frac{\lambda_1}{1+\alpha_1}}\big) < c_1 \lambda_1^{-1/2}(Q,\rho) e^{-\frac{\lambda_1(Q,\rho)}{1+\alpha_1}} \quad \text{for } 1 \le j \le m. \tag{4.8}$$

Next we estimate the term $e^{t_j} (\int_M he^u)^{-1} - 1$. By $\langle \nabla(u + T_t u), \nabla \partial_{\lambda_j} u \rangle = 0$, we conclude from (iii) of Lemma 3.2 and (4.8) that

$$\theta_j + 2\lambda_j(a_j - 1) = O\big(\max|a_k - 1| + e^{-\frac{\lambda_1}{1+\alpha_1} - \epsilon\lambda_1}\big) = O\big(\lambda_1^{-1} e^{-\frac{\lambda_1}{1+\alpha_1}}\big) \tag{4.9}$$

and

$$e^{t_j}\bigg(\int_M he^u\bigg)^{-1} - 1 = \theta_j + O\big(\max|a_k - 1| + e^{-\frac{\lambda_1}{1+\alpha_1} - \epsilon\lambda_1}\big) = O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big). \tag{4.10}$$

Together with $\langle \Delta(u + T_t u), \nabla \partial_{p_j} u_{p_j} \rangle = 0$ for $j > \tau$ and (4.8), (4.10), and (ii) of Lemma 3.2, we have

$$\begin{aligned} |\nabla_y H_j(0,0)| &= O\bigg(\lambda_j|a_j - 1| + \bigg|\frac{e^{t_j}}{\int_M he^u} - 1\bigg| + e^{-\frac{\lambda_1}{1+\alpha_1}}\bigg) \\ &\le O(1) e^{-\frac{\lambda_1}{1+\alpha_1}}. \end{aligned} \tag{4.11}$$

for $j > \tau$. Since f_{Q_1} is a Morse function at Q, we have

$$\big|p_j - p_j^0\big| \le c\, |\nabla_y H_j(0,0)| \le c_3 e^{-\frac{\lambda_1(Q,\rho)}{1+\alpha_1}} \tag{4.12}$$

for $j > \tau$.

It remains to estimate $t_j - t_1$ and $\lambda_1 - \lambda_1(Q,\rho)$. Recall that

$$\theta_j = \frac{1}{\rho_*}\bigg((\rho-\rho_*) - \sum_{k=1}^{l} 2\pi d_k e^{-\frac{\lambda_k}{1+\alpha_k}} - \sum_{k=1}^{m} 8\pi(1+\alpha_k)(t_k - t_j) - \sum_{k=1}^{m} 16\pi(1+\alpha_k)\lambda_k(a_k - 1)\bigg).$$

From (4.9), we obtain

$$\begin{aligned} O(1)\lambda_1^{-1}e^{-\frac{\lambda_1}{1+\alpha_1}} &= 8\pi\sum_j (1+\alpha_j)[\theta_j + 2\lambda_j(a_j-1)] \\ &= \frac{1}{8\pi}\bigg(\rho - \rho_* - \sum_{k=1}^{l} 2\pi d_k e^{-\frac{\lambda_k}{1+\alpha_k}}\bigg), \end{aligned} \tag{4.13}$$

where all t_j cancel out with each other. Here $\rho_* = 8\pi\sum_{j=1}^{m}(1+\alpha_j)$ is used. By the definition of d_k in (2.15) and the assumption

$$|t_j - t_1| \le c_1 e^{-\frac{\lambda_1(Q,\rho)}{1+\alpha_1}},$$

we have

$$\sum_{k=1}^{l} 2\pi d_k e^{-\frac{\lambda_k}{1+\alpha_k}} = \tilde{l}(Q)e^{-\frac{\lambda_1}{1+\alpha_1}} + O\big((c_1+1)e^{-\frac{\lambda_1}{1+\alpha_1} - \epsilon\lambda_1}\big). \tag{4.14}$$

Also, we have

$$\rho - \rho_* = \tilde{l}(Q)e^{-\frac{\lambda_1(Q,\rho)}{1+\alpha_1}}.$$

Therefore (4.13) implies

$$O(1)\lambda_1^{-1}e^{-\frac{\lambda_1}{1+\alpha_1}} = \tilde{l}(Q)\big(e^{-\frac{\lambda_1(Q,\rho)}{1+\alpha_1}} - e^{-\frac{\lambda_1}{1+\alpha_1}}\big).$$

This in turn gives

$$|\lambda_1 - \lambda_1(Q,\rho)| = c_4\lambda_1(Q,\rho)^{-1} \tag{4.15}$$

for some c_4 independent of c_1.

To obtain estimates for $t_j - t_1,\ j \ge 2$, we note $\theta_j = O(e^{-\lambda_1/(1+\alpha_1)})$. Then (3.16) implies

$$\bigg|t_j - \frac{1}{\rho_*}\sum_k 8\pi(1+\alpha_k)t_k\bigg| = O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big)$$

and

$$\begin{aligned} |t_j - t_1| &\le \bigg|t_j - \frac{1}{\rho_*}\sum_k 8\pi(1+\alpha_k)t_k\bigg| + \bigg|\frac{1}{\rho_*}\sum_k 8\pi(1+\alpha_k)t_k - t_1\bigg| \\ &= O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big) \le c_5 e^{-\frac{\lambda_1}{1+\alpha_1}} \end{aligned} \tag{4.16}$$

for $j \geq 2$, where c_5 is independent of c_1.

From (4.6), (4.8), (4.12), (4.15), and (4.16), we obtain $u \notin \partial S_\rho(Q)$ if we choose $c_1 > \max\{c_2, c_3, c_4, c_5\}$. The proof is complete. □

5 Degree-Counting Formula

In this section, we will apply Lemma 4.1 to derive the degree-counting formula for all solutions in $S_\rho(Q)$. To compute $\deg(u + Tu; S_\rho(Q), 0)$, we set $S^*(Q) = \{(P, \Lambda, A) : u_{P,\Lambda,A} + w \in S_\rho(Q), w \in O_{P,\Lambda}\}$ and define $\Phi_Q = (\Phi_{Q,1}, \Phi_{Q,2}, \Phi_{Q,3})$:

$$\Phi_{Q,1}^{(j)} = \langle \nabla(u + T_0 u), \nabla \partial_{p_j} u_{p_j} \rangle = -8\pi \nabla_y H_j(0,0) \qquad \text{for } \tau < j \leq m,$$

$$\Phi_{Q,2}^{(j)} = \langle \nabla(u + T_0 u), \nabla \partial_{\lambda_j} u_{p_j} \rangle \qquad \text{for } 1 \leq j \leq m,$$

$$\Phi_{Q,3}^{(j)} = \langle \nabla(u + T_0 u), \nabla u_{p_j} \rangle = 16\pi(1 + \alpha_j)(a_j - 1)\lambda_j \quad \text{for } 1 \leq j \leq m.$$

Clearly, by Lemma 3.3,

$$\deg(u + Tu; S_\rho(Q), 0) = \deg(\Phi_Q; S^*(Q), 0), \tag{5.1}$$

because the bilinear form B has an even Morse index. Now we state the degree-counting formula for (5.1).

LEMMA 5.1. *Assume* (C1) *and* (C2) *defined in Section* 2 *hold. Denote* $Q = (Q_1, Q_2)$. *Then*

$$\deg(\Phi_Q; S^*(Q), 0) = \operatorname{sgn}(\rho - \rho_*)(-1)^{m + \operatorname{ind}_{f_{Q_1}}(Q_2)},$$

where $\operatorname{sgn}(\rho - \rho_*) = 1$ *if* $\rho > \rho_*$ *and* -1 *if* $\rho < \rho_*$, *and* $\operatorname{ind}_{f_{Q_1}}(Q_2)$ *is the Morse index of* f_{Q_1} *at* Q_2, *which is defined to be zero if* Q_2 *is empty.*

Note that the sign for $\rho - \rho_*$ at Q is completely determined by the quantity $\tilde{l}(Q)$; that is, for each Q, $(\rho - \rho_*)\tilde{l}(Q)$ is always positive under assumptions (C1) and (C2).

PROOF. To compute the degree, we can simplify the problem further by replacing Φ_Q by a new map $\hat{\Phi}_Q$ defined as follows: $\hat{\Phi}_{Q,1} = \Phi_{Q,1}$, $\hat{\Phi}_{Q,3} = \Phi_{Q,3}$,

$$\begin{aligned}\hat{\Phi}_{Q,2}^{(j)} &= \Phi_{Q,2}^{(j)} - \frac{8\pi(1+\alpha_j)}{\rho_*} \sum_{k=1}^{m} \Phi_{Q,3}^{(k)} + \Phi_{Q,3}^{(j)} \\ &= -\frac{8\pi(1+\alpha_j)}{\rho_*}\Bigg[\rho - \rho_* - 8\pi \sum_{k=1}^{m} [(1+\alpha_k)(t_k - t_j)] \\ &\qquad - 2\pi \sum_{1 \leq k \leq l} d_k e^{-\frac{\lambda_k}{1+\alpha_k}}\Bigg].\end{aligned} \tag{5.2}$$

Clearly, we have

$$\frac{\partial \widehat{\Phi}_{Q,1}}{\partial \Lambda} = \frac{\partial \widehat{\Phi}_{Q,1}}{\partial A} = \frac{\partial \widehat{\Phi}_{Q,2}}{\partial A} = 0, \tag{5.3}$$

$$\Phi_Q(P, \Lambda, A) = 0 \quad \text{if and only if} \quad \widehat{\Phi}_Q(P, \Lambda, A) = 0, \tag{5.4}$$

$$\deg(\Phi_Q; S^*(Q), 0) = \deg(\widehat{\Phi}_Q; S^*(Q), 0). \tag{5.5}$$

Moreover, $\widehat{\Phi}_{Q,1} = 0$ and $\widehat{\Phi}_{Q,3} = 0$ if and only if

$$P = Q, \quad A = (1, 1, \dots, 1), \tag{5.6}$$

and $\widehat{\Phi}_{Q,2} = 0$ if and only if

$$\begin{cases} t_1 = t_2 = \cdots = t_m, \\ \rho - \rho_* = 2\pi \sum_{k=1}^{l} d_k\, e^{-\frac{\lambda_k}{1+\alpha_k}}. \end{cases} \tag{5.7}$$

It is not difficult to see that if $|\rho - \rho_*|$ is sufficiently small, equation (5.7) possesses a unique solution $\Lambda(Q, \rho) = (\lambda_1, \dots, \lambda_m)$ up to permutation. Hence $(Q, \Lambda(Q, \rho), A)$ is the unique solution of $\widehat{\Phi}_Q = 0$, where $A = (1, \dots, 1)$. By (5.3), the degree of $\widehat{\Phi}_Q$ at $(Q, \Lambda(Q, \rho), A)$ is the sign of the product of

$$\det\left(\frac{\partial \Phi_{Q,1}}{\partial(p_{\tau+1}, \dots, p_m)}\right) \cdot \det\left(\frac{\partial \widehat{\Phi}_{Q,2}}{\partial \Lambda}\right) \cdot \det\left(\frac{\partial \widehat{\Phi}_{Q,3}}{\partial A}\right),$$

where $\det\big(\frac{\partial \Phi_{Q,1}}{\partial(p_{\tau+1},\dots,p_m)}\big)$ is defined to be 1 if Q_2 is empty. Thus

$$\deg(\Phi_Q; S^*(Q), 0) = (-1)^{\operatorname{ind}_{f_{Q_1}}(Q_2)} \operatorname{sgn} \det\left(\frac{\partial \widehat{\Phi}_{Q,2}}{\partial \Lambda}\right). \tag{5.8}$$

To compute $\det\big(\frac{\partial \widehat{\Phi}_{Q,2}}{\partial \Lambda}\big)$, we recall that

$$t_j = \lambda_j + \frac{d_j}{2(1+\alpha_j)} \lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}} - \sum_{j=1}^{m} \overline{u}_{p_j} + \text{const}.$$

Thus

$$\frac{\partial t_j}{\partial \lambda_k} = \left[1 + \left(\frac{d_j}{2(1+\alpha_j)} - \frac{d_j}{2(1+\alpha_j)^2}\lambda_j\right) e^{-\frac{\lambda_j}{1+\alpha_j}}\right]\delta_{jk} - \frac{\partial \overline{u}_{p_k}}{\partial \lambda_k}.$$

By (5.2), we have

$$\begin{aligned} \frac{\partial \widehat{\Phi}^{(j)}_{Q,2}}{\partial \lambda_j} &= -\Big[\sum_{k \neq j} (1+\alpha_k)\Big] \frac{\partial t_j}{\partial \lambda_j} + O\big(e^{-\frac{\lambda_j}{1+\alpha_j}}\big) \\ &= -\Big[\sum_{k \neq j} (1+\alpha_k)\Big]\left[1 - \frac{d_j}{2(1+\alpha_j)^2}\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\right] + O\big(e^{-\frac{\lambda_j}{1+\alpha_j}}\big) \end{aligned}$$

and

$$\frac{\partial\widehat{\Phi}^{(j)}_{Q,2}}{\partial\lambda_k} = (1+\alpha_k)\frac{\partial t_k}{\partial\lambda_k} - \frac{d_k}{4(1+\alpha_k)}e^{-\frac{\lambda_k}{1+\alpha_k}}$$
$$= (1+\alpha_k)\left[1 - \frac{d_k}{2(1+\alpha_k)^2}\lambda_k e^{-\frac{\lambda_k}{1+\alpha_k}}\right] + O\left(e^{-\frac{\lambda_k}{1+\alpha_k}}\right)$$

for $k \neq j$. Here we replace $\widehat{\Phi}_{Q,2}$ by $\frac{\rho^*}{64\pi^2(1+\alpha_j)}\widehat{\Phi}_{Q,2}$ (still denoted by $\widehat{\Phi}_{Q,2}$) for simplicity of notation. Denote

$$B = \sum_{j=1}^{m}(1+\alpha_j), \quad \gamma_k = 1 - \frac{d_k}{2(1+\alpha_k)^2}\lambda_k e^{-\frac{\lambda_k}{1+\alpha_k}}, \quad \text{and} \quad \delta_j = \sum_{k=1}^{m}\frac{\partial\widehat{\Phi}^{(j)}_{Q,2}}{\partial\lambda_k}. \tag{5.9}$$

Thus we have

$$\det\left[\frac{\partial}{\partial\Lambda}\widehat{\Phi}^{(j)}_{Q,2}\right]$$
$$= \det\begin{pmatrix} (-B+(1+\alpha_1))\gamma_1+(*) & (1+\alpha_2)\gamma_2+(*) & \cdots & (1+\alpha_m)\gamma_m+(*) \\ (1+\alpha_1)\gamma_1+(*) & (-B+(1+\alpha_2))\gamma_2+(*) & \cdots & (1+\alpha_m)\gamma_m+(*) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ (1+\alpha_1)\gamma_1+(*) & -(1+\alpha_2)\gamma_2+(*) & \cdots & (-B+(1+\alpha_m))\gamma_m+(*) \end{pmatrix}$$
$$= \det\begin{pmatrix} \delta_1 & (1+\alpha_2)\gamma_2+(*) & \cdots & (1+\alpha_m)\gamma_m+(*) \\ \delta_2 & (-B+(1+\alpha_2))\gamma_2+(*) & \cdots & (1+\alpha_m)\gamma_m+(*) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \delta_m & (1+\alpha_2)\gamma_2+(*) & \cdots & (-B+(1+\alpha_m))\gamma_m+(*) \end{pmatrix}$$
$$= \det\begin{pmatrix} \delta_1 & (1+\alpha_2)\gamma_2+(*) & (1+\alpha_3)\gamma_3+(*) & \cdots & (1+\alpha_m)\gamma_m+(*) \\ \delta_2-\delta_1 & -B\gamma_2+(*) & (*) & \cdots & (*) \\ \delta_3-\delta_1 & (*) & -B\gamma_3+(*) & \cdots & (*) \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ \delta_m-\delta_1 & (*) & (*) & \cdots & -B\gamma_m+(*) \end{pmatrix}$$
$$= \det\begin{pmatrix} \sum_j \frac{(1+\alpha_j)}{B}\delta_j & (*) & (*) & \cdots & (*) \\ \delta_2-\delta_1 & -B\gamma_2+(*) & (*) & \cdots & (*) \\ \delta_3-\delta_1 & (*) & -B\gamma_3+(*) & \cdots & (*) \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ \delta_m-\delta_1 & (*) & (*) & \cdots & -B\gamma_m+(*) \end{pmatrix}$$

where all the terms $(*)$ are bounded by $O(e^{-\lambda_1/(1+\alpha_1)})$. Now using $\alpha_1 = \cdots = \alpha_l$, we have

$$\begin{aligned}
&\sum_j (1+\alpha_j)\delta_j \\
&= \sum_j (1+\alpha_j)\Bigg[-\Big[\sum_{k\neq j}(1+\alpha_k)\Big]\Bigg(1-\Bigg[\frac{d_j}{2(1+\alpha_j)^2}\lambda_j - \frac{d_j}{2(1+\alpha_j)}\Bigg]e^{-\frac{\lambda_j}{1+\alpha_j}} - \frac{\partial \bar{u}_{p_j}}{\partial \lambda_j}\Bigg) \\
&\quad - \frac{d_j}{4(1+\alpha_j)}e^{-\frac{\lambda_j}{1+\alpha_j}} \\
&\quad + \Bigg(\sum_{k\neq j}(1+\alpha_k)\Bigg(1-\Bigg[\frac{d_k\lambda_k}{2(1+\alpha_k)^2} - \frac{d_k}{2(1+\alpha_k)}\Bigg]e^{-\frac{\lambda_k}{1+\alpha_k}} - \frac{\partial \bar{u}_{p_k}}{\partial \lambda_k}\Bigg) - \frac{d_k e^{-\frac{\lambda_k}{1+\alpha_k}}}{4(1+\alpha_k)}\Bigg)\Bigg] \\
&= -\sum_j (1+\alpha_j)\cdot \sum_{1\le j\le m}\frac{d_j}{4(1+\alpha_j)}e^{-\frac{\lambda_j}{1+\alpha_j}} \\
&= -\frac{B}{4(1+\alpha_1)}\sum_{1\le j\le l} d_j e^{-\frac{\lambda_j}{1+\alpha_j}} + O\big(e^{-\frac{\lambda_1}{1+\alpha_1}-\epsilon\lambda_1}\big) \\
&= -\frac{B}{4(1+\alpha_1)}(\rho-\rho_*) + O\big(e^{-\frac{\lambda_1}{1+\alpha_1}-\epsilon\lambda_1}\big)
\end{aligned}$$

for some $\epsilon > 0$. Thus

$$\det\Bigg[\frac{\partial}{\partial\Lambda}\hat{\Phi}^{(j)}_{Q,2}\Bigg] = (-1)^m\frac{B^{m-1}}{4(1+\alpha_1)}(\rho-\rho_*) + O(e^{-\frac{\lambda_1}{1+\alpha_1}-\epsilon\lambda_1})$$

and

$$\deg(\Phi_Q; S^*(Q), 0) = \operatorname{sgn}(\rho-\rho_*)(-1)^{m+\operatorname{ind}_{f_{Q_1}}(Q_2)}. \qquad \square$$

Now we are in a position to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Let B_R and d_ρ denote

$$B_R = \{u \in \overset{\circ}{H}{}^1(M) : \|u\|_{H^1(M)} < R\}$$

and

$$d_\rho = \deg(I + T_\rho; B_R, 0).$$

Recall

$$\begin{aligned}
\Sigma &= \Big\{8\pi\Big(m + \sum_{q\in A}\alpha(q)\Big)\ \Big|\ m = 1,2,\ldots \text{ and } A \subset \{q_1,\ldots,q_N\}\Big\} \\
&= \{8\pi n_k \mid k = 1,2,\ldots\},
\end{aligned}$$

where $d_0 = 1 < d_1 < d_2 < \cdots$.

We first consider the case when $\alpha(q) \notin \mathbb{N}$, $j = 1,2,\ldots,N$. Let $\rho_* = 8\pi(m + \sum_{q\in A}\alpha(q)) \in \Sigma$, where $|A| = \tau \ge 0$, $d^+_{\rho_*} = \lim_{\rho\to\rho_*^+} d_\rho$, and

$d^-_{\rho_*} = \lim_{\rho\to\rho_*^-} d_\rho$. By Theorem B, we know that for $\rho > \rho_*$ close to ρ_*, the degree d_ρ of the nonlinear map $u + T_\rho u$ can be counted by

$$
\begin{aligned}
d^+_{\rho_*} &= d_\rho \\
&= \sum_m \frac{1}{m!} \sum_{l(Q)>0}^{(m)} \deg(I + T_\rho; S_\rho(Q), 0) + \deg(I + T_\rho; B_C, 0) \\
&= \sum_m \frac{1}{m!} \sum_{l(Q)>0}^{(m)} \deg(I + T_\rho; S_\rho(Q), 0) + d_{\rho_*},
\end{aligned}
\tag{5.10}
$$

where C is the number in Theorem B and the summation $\sum_{l(Q)>0}^{(m)}$ is taken over all $Q = (q_{j_1}, \dots, q_{j_\tau}, p_1, \dots, p_m) = (Q_1, p_1, \dots, p_m)$ with $(p_1, \dots, p_m)$ a critical point of f_{Q_1} such that $l(Q) > 0$. Here, $m!$ appears in the denominator because any permutation of $(p_1, \dots, p_m)$ is considered to be the same solution, and similarly for $\rho < \rho_*$ close to ρ_*,

$$
d^-_{\rho_*} = \sum_m \frac{1}{m!} \sum_{l(Q)<0}^{(m)} \deg(I + T_\rho; S_\rho(Q), 0) + d_{\rho_*}.
\tag{5.11}
$$

By Lemma 5.1, (5.10) and (5.11) imply

$$
\begin{aligned}
d^+_{\rho_*} &- \sum_m \frac{(-1)^{m+\tau}}{m!} \sum_{l(Q)>0}^{(m)} (-1)^{\mathrm{ind}_{f_{Q_1}}(Q_2)} \\
&= d_{\rho_*} = d^-_{\rho_*} + \sum_m \frac{(-1)^{m+\tau}}{m!} \sum_{l(Q)<0}^{(m)} (-1)^{\mathrm{ind}_{f_{Q_1}}(Q_2)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d^+_{\rho_*} - d^-_{\rho_*} &= \frac{1}{m!} \sum_Q^{(m)} (-1)^{m+\tau} (-1)^{\mathrm{ind}_{f_{Q_1}}(Q_2)} \\
&= \sum \frac{(-1)^{m+\tau}}{m!} \chi(M^m_* \setminus \Gamma_m),
\end{aligned}
\tag{5.12}
$$

where the last summation is taken for all m and τ such that

$$
\rho_* = 8\pi \sum_{\substack{q\in Q_1 \\ |Q_1|=\tau|}} (1 + \alpha(q)) + 8\pi m.
$$

We note that the last equality is due to the Hopf theorem, where

$$
M_* = M \setminus \{q_1, \dots, q_N\}
$$

and $\chi(M^m_* \setminus \Gamma_m)$ is the Euler characteristic of $M^m_* \setminus \Gamma_m$, which is defined to be 1 if $m = 0$.

To compute $\chi(M_*^m \backslash \Gamma_m)$, we consider the following fiberation: $\mu : M_*^m \backslash \Gamma_m \to M_*^{m-1} \setminus \Gamma_{m-1}$, defined by

$$\mu(x_1, \ldots, x_{m-1}, x_m) = (x_1, \ldots, x_{m-1}).$$

Clearly, each fiber $\mu^{-1}((x_1, \ldots, x_{m-1}))$ is $M_* \setminus \{x_1, \ldots, x_{m-1}\}$, which is homomorphic to each other. Therefore $\chi(M_*^m \setminus \Gamma_m)$ can be computed through the fiber $M_* \setminus \{x_1, \ldots, x_{m-1}\}$ and $M_*^{m-1} \setminus \Gamma_{m-1}$, that is,

$$\begin{aligned} \chi(M_*^m \setminus \Gamma_m) &= \chi(M_* \setminus \{(x_1, \ldots, x_{m-1})\}) \cdot \chi(M_*^{m-1} \setminus \Gamma_{m-1}) \\ &= (\chi(M_*) - m + 1)\chi(M_*^{m-1} \setminus \Gamma_{m-1}). \end{aligned}$$

See [47] for the proof of the identity above. By induction, we then have for $m \geq 1$

$$\begin{aligned} \chi(M_*^m \setminus \Gamma_m) &= \chi(M_*)(\chi(M_*) - 1) \cdots (\chi(M_*) - m + 1) \\ &= (\chi(M) - N)(\chi(M) - N - 1) \cdots (\chi(M) - N - m + 1). \end{aligned}$$

Thus we have by (5.12)

$$\begin{aligned} d_{\rho_*}^+ - d_{\rho_*}^- = &\sum_{m>0} \frac{(-1)^\tau}{m!}(-\chi(M) + N) \cdots (-\chi(M) + N - m + 1) \\ &+ \sum_{m=0} (-1)^\tau. \end{aligned} \tag{5.13}$$

On the other hand, we let

$$\begin{aligned} g(x) &= (1 + x + x^2 + x^3 + \ldots)^{-\chi(M)+N} \prod_{j=1}^{N} (1 - x^{1+\alpha_j}) \\ &= b_0 + b_1 x^{n_1} + b_2 x^{n_2} + \cdots + b_k x^{n_k} + \cdots. \end{aligned}$$

Suppose $-\chi(M) + N > 0$ first. The exponent n_k has the form $n_k = m + \sum_{i=1}^{\tau'}(1 + \alpha_{j_i})$ for some $(m, \{j_1, \ldots, j_{\tau'}\})$. The monomial x^{n_k} comes from the products of the terms in $g(x)$,

$$(-1)^{\tau'} x^{n_k} = \prod_{s=1}^{s=-\chi(M)+N} x^{\gamma_s} \prod_{i=1}^{\tau'} (-x^{1+\alpha_{j_i}}),$$

where $\sum_{s=1}^{s=-\chi(M)+N} \gamma_s = m$. The number of different $(\gamma_1, \ldots, \gamma_{-\chi(M)+N})$ that satisfies $\sum_{s=1}^{s=-\chi(M)+N} \gamma_s = m$ can be calculated by the formula

$$\binom{-\chi(M) + N + m - 1}{m}.$$

Therefore the coefficient b_k of x^{n_k} equals

$$b_k = \sum (-1)^{\tau'} \binom{-\chi(M) + N + m - 1}{m},$$

where the summation is taken over the set

$$\left\{(m,\{j_1,\dots,j_{\tau'}\}) : m+\sum_{i=1}^{\tau'}(1+\alpha_j)=n_k\right\},$$

i.e., by (5.13)

$$d^+_{d_k}-d^-_{d_k}=b_k.$$

It is easy to see that

$$d_\rho=\sum_{k=0}^{s} b_k \tag{5.14}$$

if $8\pi n_s<\rho<8\pi n_{s+1}$. Theorem 1.1 is proved in this case.

If $-\chi(M)+N=0$, then

$$\binom{-\chi(M)+N+m-1}{m}=0$$

for $m>0$. Hence $b_k=\sum(-1)^{\tau'}$ and (5.14) holds again.

For $-\chi(M)+N<0$,

$$(1+x+x^2+\cdots)^{-\chi(M)+N}=(1-x)^{\chi(M)-N}.$$

Since in this case

$$\binom{-\chi(M)+N+m-1}{m}=\begin{cases}0 & \text{if } m>\chi(M)-N,\\ (-1)^m\binom{\chi(M)-N}{m} & \text{if } m\le\chi(M)-N,\end{cases} \tag{5.15}$$

are the coefficients in the expansion of $(1-x)^{\chi(M)-N}$, we can show that (5.14) holds by a similar argument as above. Therefore Theorem 1.1 is proved if all $\alpha\notin\mathbb{N}$. For $\alpha(q)\in\mathbb{N}$, as mentioned in the Introduction, one can see that Theorem 1.1 holds by taking $\alpha_k(q)\to\alpha(q)$ with $\alpha_k(q)\notin\mathbb{N}$. □

6 Applications

In this section, we shall give some examples to apply our degree-counting formula. The first example is the equation studied recently by Poliakovsky and Tarantello [45] concerning self-gravitating strings for some physics model:

$$\Delta v+\lambda e^{av}+\prod_{j=1}^{N}|x-p_j|^2e^v=0\quad\text{in }\mathbb{R}^2. \tag{6.1}$$

The details of the derivation of (6.1) can be found in [57]. In [45], they studied the structure of the radial solution v of (6.1), where all vortex points p_j coincide with 0, particularly for the case $a=1$:

$$\begin{cases}\Delta u+K(x)e^{u(x)}=0 & \text{in }\mathbb{R}^2,\\ u(x)=-(2\beta+4)\log|x|+O(1) & \text{at }\infty,\end{cases} \tag{6.2}$$

where $K(x) = 1 + |x|^{2N}$. Equation (6.2) has been studied for more than three decades. See [34, 36] for radial solutions and [22] for nonradial solutions. For nonradial solutions most of the previous works have focused on the situation when $N < 0$ (N is defined in the following Theorem 6.1). Poliakovsky and Tarantello showed that (6.2) has a solution if and only if $\beta \in (2N-2, 2N)$ (see corollary 1.2 in [45]). We want to extend their result to nonradial $K(x)$ of equation (6.2), including (6.1) with $a = 1$.

THEOREM 6.1. *Suppose $K(x)$ is a positive C^1 function in $\mathbb{R}^2$ and satisfies the condition that $K(1/|x|)|x|^{2N}$ is a positive C^1 function in a small neighborhood of* 0. *Then for any $\beta \in (2\max\{1, N\} - 2, 2N)$, equation* (6.2) *has a solution.*

PROOF. We can write equation (6.2) on $\mathbb{S}^2$ via the stereographic projection of $\mathbb{S}^2 \to \mathbb{R}^2$:

$$\Delta w + \rho\left(\frac{h(x)e^w}{\int h(x)e^w} - \frac{1}{4\pi}\right) = 4\pi(\beta - N)\left(\delta_N - \frac{1}{4\pi}\right),$$

where $\rho = 4\pi(\beta + 2)$, and $h(x)$ is a positive C^1 function on $\mathbb{S}^2$. Then the generating function $g(x)$ for the degree d_ρ is

$$g(x) = (1-x)(1-x^{\alpha+1}) = 1 - x - x^{\alpha+1} + x^{\alpha+2},$$

where $\alpha = \beta - N$. Clearly $\alpha > -1$ if and only if $K(x)e^u \in L^1(\mathbb{R}^2)$. If $8\pi(\alpha + 2) > \rho > 8\pi\max\{1, \alpha+1\}$, then the degree d_ρ is -1. This condition is equivalent to $\beta \in (2\max\{1, N\} - 2, 2N)$. □

Our second application is the existence of a constant curvature metric on $\mathbb{S}^2$ with conic singularities. This problem has been completely solved if the number of singularities is less than four; see [27, 55]. When the number of singularities is greater than or equal to four, this problem becomes very delicate and few results are known. Here we shall apply the degree-counting formula to prove some existence of such a metric if the number of singularities is four.

We consider

$$\Delta u + e^u - 2 = 4\pi\beta\delta_{p_0} + 4\pi\alpha\sum_{i=1}^{3}\delta_{p_i} \quad \text{on } \mathbb{S}^2, \tag{6.3}$$

where $p_0 \notin \{p_1, p_2, p_3\}$. Let

$$\Sigma = \{(2j-3)(\alpha+1) + 2m + 1 \mid j = 1, 2, 3,\ m \text{ is a positive integer}\}.$$

THEOREM 6.2. *Suppose $-\frac{2}{3} \le \alpha \le -\frac{1}{2}$ and $\beta > 0$. Then equation* (6.3) *has a solution if $\beta \notin \Sigma$.*

PROOF. We note that equation (6.3) is equivalent to

$$\Delta u + \rho\left(\frac{e^u}{\int e^u} - \frac{1}{4\pi}\right) = 4\pi\beta\left(\delta_{p_0} - \frac{1}{4\pi}\right) + 4\pi\alpha\sum_{j=1}^{3}\left(\delta_{p_j} - \frac{1}{4\pi}\right),$$

where $\rho = 4\pi(\beta + 3\alpha + 2)$. Set

$$\rho_* = \frac{1}{2}(\beta + 3\alpha + 2).$$

Obviously, $\rho_* < \beta + 1$ if $\beta > 0, \alpha < 0$.

On the other hand, the generating function

$$\begin{aligned} g(x) &= (1 - x^{\alpha+1})^3(1 + x + x^2 + \cdots)^2(1 - x^{\beta+1}) \\ &= (1 - 3x^{1+\alpha} + 3x^{2(1+\alpha)} - x^{3(1+\alpha)})\left(\sum_{k=0}^{\infty} a_k x^k\right)(1 - x^{\beta+1}), \end{aligned}$$

where a_k is strictly increasing in k. Since $\beta + 1 > \rho_*$, we only have to consider the function

$$\tilde{g}(x) = (1 - 3x^{1+\alpha} + 3x^{2(1+\alpha)} - x^{3(1+\alpha)})\left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{l=0}^{\infty} b_l x^{n_l},$$

where $n_l = j(1 + \alpha) + m$, $m \in \mathbb{N}^+$, $j = 0, 1, 2, 3$. Therefore $\rho_* \notin \{n_l\}_{l=0}^{\infty} \Leftrightarrow \beta \notin \Sigma$.

Obviously we have the following four cases.

Case A. $k_0 < \rho_* < k_0 - 1 + 3(1 + \alpha)$, $(k_0 \geq 1)$.
We only have to consider the terms from $(1 - 3x^{1+\alpha} + 3x^{2(1+\alpha)} - x^{3(1+\alpha)})x^l$, $l = k_0 - 1, k_0 : a_{k_0-1}(x^{k_0-1} - 3x^{k_0+\alpha} + 3x^{k_0-1+2(1+\alpha)}) + a_{k_0}x^{k_0}$. Thus

$$\sum_{k \leq k_0} b_k = a_{k_0-1} + a_{k_0} > 0.$$

Case B. $k_0 - 1 + 3(1 + \alpha) < \rho_* < k_0 + (1 + \alpha)$.
We only have to consider the terms from $(1 - 3x^{1+\alpha} + 3x^{2(1+\alpha)} - x^{3(1+\alpha)})x^l$, $l = k_0 : a_{k_0}x^{k_0}$. Thus

$$\sum_{k \leq k_0 - 1 + 3(1+\alpha)} b_k = a_{k_0} > 0.$$

Case C. $k_0 + (1 + \alpha) < \rho_* < k_0 + 2(1 + \alpha)$.
We only have to consider the terms from $(1 - 3x^{1+\alpha} + 3x^{2(1+\alpha)} - x^{3(1+\alpha)})x^l$, $l = k_0 : a_{k_0}(x^{k_0} - 3x^{k_0+(1+\alpha)})$. Thus

$$\sum_{k \leq k_0 + (1+\alpha)} b_k = -2a_{k_0} < 0.$$

Case D. $k_0 + 2(1 + \alpha) < \rho_* < k_0 + 1$.
We only have to consider $(1-3x^{1+\alpha}+3x^{2(1+\alpha)}-x^{3(1+\alpha)})x^l$, $l = k_0 : a_{k_0}(x^{k_0} -$

$3x^{k_0+(1+\alpha)} + 3x^{k_0+2(1\alpha)})$. Thus

$$\sum_{k \le k_0+2(1+\alpha)} b_k = a_{k_0} > 0. \qquad \square$$

Remark 1. If $1+\alpha < \frac{1}{3}$ and $\rho_* \in (k_0 + 3(1+\alpha), k_0+1)$, then the degree d_ρ is zero.

Remark 2. When $\beta \in \Sigma$, the existence of solutions to (6.3) is a difficult question. In fact, it will depend on the configuration $\{p_0, p_1, p_2, p_3\}$. For $\alpha = -\frac{1}{2}$, it is closely related to the Liouville equation (1.10) on a torus. See [37].

Our third application is the existence of multiparities in the electroweak theory of Glasgow-Salam-Weinberg, which has been actively studied in recent years. Abrikosov [1] first predicted the appearance of spatially periodic vortex line in superconductive materials. Ambjorn and Olesen [2–5] found that in the electroweak theory, periodic vortices could be realized as solutions of self-dual Bogomol′nyĭ type equations, which can further be reduced to

$$\begin{cases} \Delta u + 4g^2 e^u + g^2 e^w = 4\pi \sum_{l=1}^m n_l \delta_{p_l} & \text{in } T, \\ \Delta w - 2g^2 e^u - \dfrac{g^2}{2\cos^2\theta}(e^w - \varphi_0^2) = 0, \end{cases} \tag{6.4}$$

where φ_0, θ, and g are constants. For the derivation of (6.4), we refer the readers to [9, 54, 57]. By integration, we have

$$4g^2 \int e^u = \frac{4\pi N - g^2\varphi_0^2|T|}{\sin^2\theta}, \tag{6.5}$$

$$g^2 \int e^w = \frac{g^2\varphi_0^2|T| - 4\pi\cos^2\theta N}{\sin^2\theta}, \tag{6.6}$$

where $N = \sum_{l=1}^m n_l$. Thus the necessary condition for solvability of (6.4) is that N must satisfy

$$g^2\varphi_0^2 < \frac{4\pi N}{|T|} < \frac{g^2\varphi_0^2}{\cos^2\theta}. \tag{6.7}$$

The conjecture proposed in [57] is to ask whether (6.7) is also sufficient for the solvability of (6.4). We use the degree-counting formula to prove that this conjecture is true at least for most cases.

THEOREM 6.3. *Assume*

$$\frac{4\pi N - g^2\varphi_0^2|T|}{\sin^2\theta} \notin 8\pi\mathbb{N}.$$

Then (6.7) *is a necessary and sufficient condition for the existence of a self-dual vortex solution of* (6.4).

PROOF. By (6.5) and (6.6), equation (6.4) can be written in the form of mean field equations:

$$(6.8)\qquad \begin{cases}\Delta u_1 + \lambda\left[\frac{e^{u_1}}{\int_M e^{u_1}} - \frac{1}{|T|}\right] + \mu\left[\frac{e^{u_2}}{\int_M e^{u_2}} - \frac{1}{|T|}\right] \\ \quad = 4\pi\sum_{j=1}^N n_j\left[\delta_{p_j} - \frac{1}{|T|}\right] \\ \Delta u_2 - \frac{\mu}{2\cos^2\theta}\left[\frac{e^{u_2}}{\int_M e^{u_2}} - \frac{1}{|T|}\right] - \frac{1}{2}\lambda\left[\frac{e^{u_1}}{\int_M e^{u_1}} - \frac{1}{|T|}\right] = 0,\end{cases}$$

where

$$\lambda = \frac{4\pi N - g^2\varphi_0^2|T|}{\sin^2\theta}, \quad \mu = \frac{g^2\varphi_0^2|T| - 4\pi\cos^2\theta N}{\sin^2\theta}.$$

By (6.7), both λ and μ are positive constants.

To solve (6.8), we deform it into a decoupled system:

$$(6.9)\qquad \begin{cases}\Delta u_1^t + \lambda\left[\frac{e^{u_1^t}}{\int_M e^{u_1^t}} - \frac{1}{|T|}\right] + \mu t\left[\frac{e^{u_2^t}}{\int_M e^{u_2^t}} - \frac{1}{|T|}\right] \\ \quad = 4\pi\sum_{j=1}^N n_j\left[\delta_{p_j} - \frac{1}{|T|}\right], \\ \Delta u_2^t - \frac{\mu t}{2\cos^2\theta}\left[\frac{e^{u_2^t}}{\int_M e^{u_2^t}} - \frac{1}{|T|}\right] - \frac{1}{2}\lambda t\left[\frac{e^{u_1^t}}{\int_M e^{u_1^t}} - \frac{1}{|T|}\right] = 0, \\ \int u_i^t = 0,\ i = 1, 2.\end{cases}$$

Since $\lambda \notin 8\pi\mathbb{N}$ and u_2^t cannot blow up due to the negative sign of the coefficients of the second equation, (u_1^t, u_2^t) remains bounded for $t \in [0, 1]$. For $t = 0$, the system is decoupled as

$$\begin{cases}\Delta u_1 + \lambda\left(\frac{e^{u_1}}{\int_M e^{u_1}} - \frac{1}{|T|}\right) = 4\pi\sum_{j=1}^N n_j\left(\delta_{p_j} - \frac{1}{|T|}\right), \\ \Delta u_2 = 0, \\ \int u_i = 0,\ i = 1, 2,\end{cases}$$

and $u_2 = 0$. Thus the Leray-Schauder degree of (6.4) equals the degree to the system for $t = 0$, which is nonzero by Theorem 1.1. □

Our last application is the existence of solutions of the Chern-Simons equation

$$(6.10)\qquad \Delta u + \frac{1}{\epsilon^2}e^u(1 - e^u) = 4\pi\sum_{j=1}^N n_j\delta_{p_j},$$

where $n_j \in \mathbb{N}^+$ and p_j are distinct points in a torus $\mathbb{T}$. In [23], Choe and Kim are able to classify the behavior of solutions u_ϵ as $\epsilon \to 0$. A sequence of solutions u_ϵ is called bounded if $u_\epsilon - 2\log\epsilon$ converges to a solution of

$$\Delta w + \rho\frac{e^w}{\int e^w} = 4\pi\sum_{j=1}^N n_j\delta_{p_j},$$

where $\rho = 4\pi\sum_{j=1}^N n_j = 4\pi N^*$ as $\epsilon \to 0$.

THEOREM 6.4. *Suppose $N^* = \sum_{j=1}^N n_j$ is odd. Then there are solutions u_ϵ of* (6.10) *such that $u_\epsilon - 2\log\epsilon$ converges to w in $C^2_{\mathrm{loc}}(T \setminus \{p_1, \dots, p_N\})$, where w satisfies*

$$\Delta w + \rho\left(\frac{e^w}{\int e^w} - \frac{1}{|T|}\right) = 4\pi \sum_{j=1}^{N} n_j \left(\delta_{p_j} - \frac{1}{|T|}\right), \tag{6.11}$$

with $\rho = 4\pi N^*$.

Obviously Theorem 6.4 is a consequence of Corollary 1.2 or Theorem 1.3 because the topological degree for (6.11) is nonvanishing. Theorem 1.3 tells us that equation (6.11) might have $(\prod_{j=1}^N (1+n_j))/2$ solutions for generic $(p_1, \dots, p_N)$. For (6.10), we have the following conjecture:

CONJECTURE. *For generic $(p_1, \dots, p_N)$, if ϵ is small, equation* (6.10) *has at least $\prod_{j=1}^N (1+n_j)$ solutions.*

Finally, we want to make a remark about Theorem 1.1, where the coefficient $h(x)$ of (1.4) can be extended to a general class of functions including the one satisfying (1.6). Let $h(x)$ be C^1 and positive except at a finite set $\{q_1, \dots, q_N\}$. In a neighborhood of each q_j, we assume

$$h(x) = h_j(x) Q_j(x - q_j), \quad h_j \in C^1, \quad \text{and} \quad h_j(q_j) > 0, \tag{6.12}$$

where $Q_j(y)$ is a C^1 homogeneous function of degree $2\alpha_j$, $\alpha_j > -1$, for $|y| > 0$ such that

$$c_0 |y|^{2\alpha_j} \le Q_j(y) \le c_1 |y|^{2\alpha_j}.$$

As in Section 1, we can prove for $\rho \notin \Sigma$, solutions of (1.4) with the assumption that (6.12) is uniformly bounded and the Leray-Schauder degree d_ρ is the same as Theorem 1.1.

To prove this, we deform h by h_t such that $h_t > 0$ except at q_j, $j = 1, 2, \dots, N$, and

$$h_t(y) = h_j(y)(t Q_j(y) + (1-t)|y|^{2\alpha_j})$$

in a neighbor of q_j and is equal to h outside the union of those neighborhoods. Let $u_k = u_{k,t_k}$ be a sequence of blowup solutions of (1.4) with $h = h_t$. Suppose p is a blowup point. Write $w_k = u_k - \log \int h_t e^{u_k}$. Then the Pohozaev identity implies

$$\int_{\partial B_{r_0}(p)} (x-p)\cdot \nabla w_k \frac{\partial w_k}{\partial \nu}\, d\sigma - \frac{1}{2} \int_{\partial B_{r_0}(p)} (x-p)\cdot \nu \left| \frac{\nabla w_k}{\partial \nu} \right|^2 d\sigma =$$

$$\int_{B_k(p)} (2h_{t_k} + (x-p)\cdot \nabla h_{t_k})(y) e^{w_k(y)}\, dy - \int_{\partial B_k(p)} h_{t_k}(x) \frac{\partial}{\partial \nu} e^{w_k(y)}\, d\sigma.$$

Since $u_k(y) = w_k + \log \int h_{t_k} e^{u_k} \to -\frac{m}{2\pi} \log|y-p|$ + a harmonic function for $y \in B_{r_0}(p) \setminus \{p\}$, it is easy to compute from the Potholed identity

$$m = \lim_{k\to\infty} \int_{B_k(p)} h_{t_k}(y) e^{w_k(y)} = 8\pi(1+\alpha_j).$$

Hence

$$\rho = \lim_{k\to\infty} \rho_k = 8\pi \sum_{\alpha_j \in A} (1+\alpha_j) \in \Sigma.$$

By this deformation, we see that d_ρ is independent of t. Hence the conclusion is proved.

Appendix A Proof of Lemma 3.2

This section is devoted to the proof of Lemma 3.2. Let

$$\begin{aligned}\overline{u}_\alpha &:= \int_{B_{r_0}(0)} \frac{|y|^{2\alpha} e^\lambda}{(1+e^\lambda |y|^{2+2\alpha})^2} u(y)dy \Big/ \int_{\mathbb{R}^2} \frac{|y|^{2\alpha} e^\lambda}{(1+e^\lambda |y|^{2+2\alpha})^2} dy \\ &= \frac{1+\alpha}{\pi} \int_{B_{r_0}(0)} \frac{|y|^{2\alpha} e^\lambda}{(1+e^\lambda |y|^{2+2\alpha})^2} u(y)dy.\end{aligned}$$

Then we have the following Poincaré-type inequality:

$$\text{(A.1)} \qquad \int_{B_{r_0}(0)} \frac{|y|^{2\alpha} e^\lambda}{(1+e^\lambda |y|^{2+2\alpha})^2} w^2 dy \le c\left(\|w\|^2_{H^1(B_{r_0}(0))} + \overline{w}_\alpha^2\right)$$

for some constant c independent of λ (see [20] for example). Using (A.1), we can prove the following result:

LEMMA A. *Let $P = (p_1, \ldots, p_m)$ and $\Lambda = (\lambda_1, \ldots, \lambda_m)$. Assume $w \in O_{P,\Lambda}$. Then there is a constant c and $\epsilon > 0$ such that for large λ_j*

$$\text{(A.2)} \qquad \int_{B_{r_0}(q_j)} |y|^{2\alpha_j} e^{U_j} w = O(e^{-\epsilon\lambda_j})\|w\|_{H^1}$$

and

$$\text{(A.3)} \qquad \int_{B_{r_0}(q_j)} |y|^{2\alpha_j} e^{U_j} w^2 dy = O(1)\|w\|^2_{H^1(M)}.$$

For a proof, see [20].

PROOF OF LEMMA 3.2. We start with part (i). Let $w_1 \in O_{P,\Lambda}$. Recall $u = u_{P,\Lambda,A} + w$ and $w \in O_{P,\Lambda}$. We can decompose

$$\langle \nabla(u+Tu), \nabla w_1\rangle = -\langle \Delta(u+Tu), w_1\rangle$$

into several parts according to (3.20)–(3.22) as follows:

$$\begin{aligned}
&\langle \nabla(u+Tu), \nabla w_1\rangle \\
&\quad = \int \nabla w \cdot \nabla w_1 - \sum_{1\le j\le m} \int_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j} \sigma_j w w_1 \\
&\qquad + \text{remainder terms.} \\
&\quad := B(w, w_1) + \text{remainder terms.}
\end{aligned}$$

Clearly, B is a symmetric bilinear form in $O_{P,\Lambda}$.

Since $w_1 \in O_{P,\Lambda}$, by (A.2) and (3.15), we have for large λ_1,

$$\left|\left[\frac{e^{t_j}}{\int_M he^u} - 1\right] \int_{B_{r_0}(0)} w_1 \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}\, dy\right| = O\big(e^{-(\frac{1}{1+\alpha_1}+\epsilon)\lambda_1}\big)\|w_1\|_{H^1(M)}, \tag{A.4}$$

$$\begin{aligned}
&2\lambda_j(a_j-1)\int_{B_{r_0}(p_j)} \nabla H_j \cdot y\rho h(p_j)|y|^{2\alpha_j} e^{U_j} w_1\, dy \\
&\quad \le O(1)\lambda_j|a_j-1|\left[\int_{B_{r_0}(p_j)} |y|^{2+2\alpha_j} e^{U_j}\, dy\right]^{\frac{1}{2}} \|w_1\|_{H^1(M)} \\
&\quad \le O(1)\lambda_j|a_j-1|e^{-\frac{\lambda_1}{2(1+\alpha_1)}}\|w_1\|_{H^1(M)} = O(1)e^{-\frac{\lambda_1}{1+\alpha_1}}\|w_1\|_{H^1(M)}.
\end{aligned} \tag{A.5}$$

Similarly, we have

$$\lambda_j(a_j-1)\int_{B_{r_0}(p_j)} |y|^{2\alpha_j} e^{U_j} \eta_j w_1\, dy = O(1)e^{-\frac{\lambda_1}{1+\alpha_1}}\|w_1\|_{H^1(M)}. \tag{A.6}$$

By Lemma A, we have for large λ_1

$$\begin{aligned}
&\int_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}(a_j-1)(U_j+s_j-1)w_1\, dy \\
&\quad = 2\lambda_j \int_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}(a_j-1)w_1\, dy \\
&\qquad + \int_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}(a_j-1)(U_j-\lambda_j+O(1))w_1\, dy =
\end{aligned} \tag{A.7}$$

$$
\begin{aligned}
&= 2\lambda_j(a_j-1)O(e^{-\epsilon\lambda_j})\|w_1\|_{H^1(M)}\\
&\quad + O(a_j-1)\Bigg(\int\limits_{B_{r_0}(p_j)}|y|^{2\alpha_j}e^{U_j}(U_j-\lambda_j+O(1))^2dy\Bigg)^{\frac{1}{2}}\\
&\qquad \times\Bigg(\int\limits_{B_{r_0}(p_j)}|y|^{2\alpha_j}e^{U_j}w_1^2dy\Bigg)^{\frac{1}{2}}\\
&= O(1)|a_j-1|\|w_1\|_{H^1(M)} = O(1)e^{-\frac{\lambda_1}{1+\alpha_1}}\|w_1\|_{H^1(M)}.
\end{aligned}
$$

As for $\tilde{\tilde{E}}_j$, we define E^+ and E^- as before:

$$
E^+ = \begin{cases}\tilde{\tilde{E}}_j & \text{if } |w|\ge\varepsilon_2,\\ 0 & \text{if } |w|<\varepsilon_2,\end{cases}\qquad E^- = \begin{cases}0 & \text{if } |w|\ge\varepsilon_2,\\ \tilde{\tilde{E}}_j & \text{if } |w|<\varepsilon_2,\end{cases}
$$

where ε_2 is a small number. Then we use (3.12) and a similar argument there to obtain

$$
\begin{aligned}
\int\limits_{B_{r_0}(p_j)}|E^+w_1|dy &\le \Bigg(\int\limits_{B_{r_0}(p_j)}|E^+|^2dy\Bigg)^{\frac{1}{2}}\Bigg(\int\limits_{B_{r_0}(p_j)}w_1^2dy\Bigg)^{\frac{1}{2}}\\
&= O(e^{-b\lambda_j})\|w_1\|_{H^1(M)}
\end{aligned}
\tag{A.8}
$$

for any fix $b>0$. For E^-, we use (3.4) and Lemma A to obtain

$$
\begin{aligned}
&\int\limits_{B_{r_0}(p_j)}|E^-w_1|dy\\
&\quad\le O(1)\int\limits_{B_{r_0}(p_j)}h_j(p_j)|y|^{2\alpha_j}e^{U_j}(O(w^2)+O(\beta_j))w_1\,dy\\
&\quad= O(\varepsilon_2)\Bigg(\int\limits_{B_{r_0}}|y|^{2\alpha_j}e^{U_j}w^2dy\Bigg)^{\frac{1}{2}}\Bigg(\int\limits_{B_{r_0}}|y|^{2\alpha_j}e^{U_j}w_1^2dy\Bigg)^{\frac{1}{2}}\\
&\qquad + O\big(e^{-2\frac{\lambda_1}{1+\alpha_1}}\big)\Bigg(\int\limits_{B_{r_0}}|y|^{2\alpha_j}e^{U_j}w_1^2dy\Bigg)^{\frac{1}{2}} + \int\limits_{B_{r_0}}|y|^{2\alpha_j}e^{U_j}|y|^3|w_1| =
\end{aligned}
$$

$$
\begin{aligned}
&= O(\varepsilon_2)\|w\|_{H^1(M)}\|w_1\|_{H^1(M)} + O\big(e^{-2\frac{\lambda_1}{1+\alpha_1}}\big)\|w_1\|_{H^1(M)} \\
&\quad + \bigg(\int_{B_{r_0}} |y|^{2b\alpha_j} e^{bU_j}|y|^6\bigg)^{\frac{1}{2}}\bigg(\int_{B_{r_0}} e^{(2-b)U_j}|y|^{2(2-b)\alpha_j} w_1^2 dy\bigg)^{\frac{1}{2}} \\
&= O(\varepsilon_2 c_1 e^{-\frac{\lambda_1}{1+\alpha_1}})\|w_1\|_{H^1(M)} + O(e^{-(2-\frac{b}{2})\frac{\lambda_1}{1+\alpha_1}})\|w_1\|_{H^1(M)}
\end{aligned}
$$

for $2 > b > \frac{4}{2+\alpha_j}$, and

$$
\int_{B_{r_0}(p_j)} |E^- w_1| dy = O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big)\|w_1\|_{H^1(M)} \tag{A.9}
$$

provided that $\varepsilon_2 c_1 < 1$.

By Lemma 2.2,

$$
\int_{B_{2r_0}(p_j)\setminus B_{r_0}(p_j)} \Delta(u_{p_j} - 8\pi(1+\alpha_j)G(x,p_j))w_1 = O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big)\|w_1\|_{H^1(M)}. \tag{A.10}
$$

For the nonlinear term appearing in ΔTu on $M \setminus \bigcup_{j=1}^m B_{r_0}(p_j)$, we first note that

$$
\begin{aligned}
\int_{M\setminus\bigcup_{j=1}^m B_{r_0}(p_j)} |e^w w_1| &= O\bigg(\int_{|w|\ge\varepsilon_2} |e^w w_1| + \int_{|w|<\varepsilon_2} |e^{\varepsilon_2} w_1|\bigg) \\
&= O(1)\|w_1\|_{H^1(M)}
\end{aligned}
$$

by (3.12). Using $\int_M he^u = O(e^{\lambda_1})$, we have

$$
\begin{aligned}
\int_{M\setminus\bigcup_{j=1}^m B_{r_0}(p_j)} \frac{\rho h e^u}{\int_M he^u}|w_1| &= O(e^{-\lambda_1}) \int_{M\setminus\bigcup_{j=1}^m B_{r_0}(p_j)} e^w |w_1| \\
&= O(e^{-\lambda_1})\|w_1\|_{H^1(M)}.
\end{aligned} \tag{A.11}
$$

Putting (A.4)–(A.11) together, we obtain

$$
\langle \nabla(u+Tu), \nabla w_1\rangle = B(w, w_1) + O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big)\|w_1\|_{H^1(M)}.
$$

This proves part (i).

Next, we prove part (iii). We have by (2.5) and (2.14),

$$\begin{aligned}\partial_{\lambda_j} u_{p_j} &= \left(2 - \frac{\frac{\rho h_j(p_j)}{4(1+\alpha_j)^2} e^{\lambda_j} |y|^{2(1+\alpha_j)}}{1 + \frac{\rho h_j(p_j)}{8(1+\alpha_j)^2} e^{\lambda_j} |y|^{2(1+\alpha_j)}} + O\big(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\big)\right)\sigma_j \\ &= \big[(1 + \partial_{\lambda_j} U_j) + O\big(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\big)\big]\sigma_j .\end{aligned} \tag{A.12}$$

Since $w \in O_{P,\Lambda}$, we have

$$\int_M \nabla w \cdot \nabla \partial_{\lambda_j} u_{p_j} = 0.$$

By (A.12) and scaling,

$$\begin{aligned}\int_{B_{r_0}(p_j)} \partial_{\lambda_j} u_{p_j} \, dy &= \int_{B_{r_0}(p_j)} \frac{2}{1 + \frac{\rho h_j(p_j)}{8(1+\alpha_j)^2} e^{\lambda_j} |y|^{2(1+\alpha_j)}} \, dy + O\big(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\big) \\ &= O\big(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\big).\end{aligned}$$

Hence

$$(|\rho_* - \rho| + |a_k - 1|) \int_{B_{r_0}(p_j)} \partial_{\lambda_j} u_{p_j} \, dy = O\big(e^{-\frac{\lambda_1}{(1+\alpha_1)} - \epsilon\lambda_1}\big) \tag{A.13}$$

for some $\epsilon > 0$. By (A.12), we also have

$$\begin{aligned}&\int_{B_{r_0}(p_j)} \rho h_j(p_j) |y|^{2\alpha_j} e^{U_j} \partial_{\lambda_j} u_{p_j} \, dy \\ &= \int_{B_{r_0}(p_j)} \rho h_j(p_j) |y|^{2\alpha_j} e^{U_j} \big(1 + \partial_{\lambda_j} U_j + O\big(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\big)\big) dy \\ &= \int_{\mathbb{R}^2} \rho h_j(p_j) |y|^{2\alpha_j} e^{U_j} (1 + \partial_{\lambda_j} U_j) dz + O\big(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\big) \\ &= 8\pi(1 + \alpha_j) + O\big(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\big),\end{aligned} \tag{A.14}$$

and

$$
\begin{aligned}
&\int_{B_{r_0}} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j} \\
&\qquad \times \left[-2\log\left(1+\frac{\rho h_j(p_j)e^{\lambda_j}}{8(1+\alpha_j)^2}|y|^{2(1+\alpha_j)}\right)\right]\partial_{\lambda_j} u_{p_j}\, dy \\
&= \int_{\mathbb{R}^2} \frac{8(1+\alpha_j)^2 r^{2\alpha_j}}{(1+r^{2(1+\alpha_j)})^2}[-2\log(1+r^{2(1+\alpha_j)})] \\
&\qquad \times \left(\frac{2}{1+r^{2(1+\alpha_j)}} + O\big(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\big)\right) dz \\
&\quad + O(e^{-\lambda_j}) \\
&= -8\pi(1+\alpha_j) + O\big(e^{-\frac{\lambda_j}{1+\alpha_j}}\big).
\end{aligned}
\tag{A.15}
$$

(A.14) and (A.15) together yield

$$
\begin{aligned}
&\int_{B_{r_0}(p_j)} \left(\rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}\partial_{\lambda_j} u_{p_j}\left[(a_j-1)(U_j+s_j-1)\right.\right. \\
&\qquad \left.\left. + \sum 8\pi(1+\alpha_k)(a_k-1)G(p_j,p_k) + \left(\frac{e^{t_j}}{\int he^u}-1\right)\right]\right) dy \\
&= 8\pi(1+\alpha_j)\left(2\lambda_j(a_j-1) + \frac{e^{t_j}}{\int he^u} - 1\right) + O(1)(\max_k |a_k-1|).
\end{aligned}
\tag{A.16}
$$

To estimate the term with $w\,\partial_{\lambda_j} u_{p_j}$, note that $w \in O_{P,\Lambda}$ implies

$$
\begin{aligned}
0 &= \int_M \nabla w \cdot \nabla \partial_{\lambda_j} u_{p_j} = -\int_M w\Delta(\partial_{\lambda_j} u_{p_j}) \\
&= -\int_{B_{r_0}(p_j)} w\Delta(\partial_{\lambda_j} U_j) dy + O\Big(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\|w\|_{H^1}\Big) \\
&= \int_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j} w \partial_{\lambda_j} U_j\, dy + O\Big(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\|w\|_{H^1}\Big).
\end{aligned}
$$

Together with Lemma A, we conclude from the above the following:

$$\begin{aligned}(A.17)\qquad &\int_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j} w\partial_{\lambda_j} u_{p_j}\\ &= \int_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j} w\big(1+\partial_{\lambda_j} U_j + O\big(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\big)\big)dy\\ &= O\big(e^{-\frac{\epsilon\lambda_1}{1+\alpha_1}}\big)\|w\|_{H^1} = O\big(e^{-\frac{\lambda_1}{(1+\alpha_1)}-\epsilon\lambda_1}\big).\end{aligned}$$

The other integrals of $\partial_{\lambda_j} u_{p_j}$ with other terms in (3.20) would be smaller than

$$O\big(\max|a_k-1| + e^{-\frac{\lambda_1}{(1+\alpha_1)}-\epsilon\lambda_1}\big).$$

Since the computations are straightforward, we omit the details here.

Now we go to integrate over $M\setminus B_{r_0}(p_j)$. Since $\partial_{\lambda_j} u_{p_j} = 0$ on $M\setminus B_{2r_0}(p_j)$, we only need to consider the integrals on $B_{2r_0}(p_j)$ on $B_{2r_0}(p_j)\setminus B_{r_0}(p_j)$, $e^u = O(e^w)$, $\frac{\rho h e^u}{\int_M he^u} = O(e^{-\lambda_1})e^w$, and $\partial_{\lambda_j} u_{p_j} = O(\lambda_j e^{-\lambda_1/(1+\alpha_1)})$ by using the Moser-Trudinger inequality as in the proof of (3.12),

$$\begin{aligned}(A.18)\qquad \int_{B_{2r_0}(p_j)\cap\{|w|\ge\epsilon_2\}} e^{|w|} &\le e^{-2\lambda_j}\int_{B_{r_0}(p_j)} \exp\Big(\frac{4\pi|w-\bar{w}|^2}{\|w-\bar{w}\|^2}\Big)dy\\ &\le c_2 e^{-2\lambda_j}.\end{aligned}$$

This implies

$$(A.19)\qquad \int_{B_{2r_0}(p_j)} e^{|w|} = O(1)$$

and

$$(A.20)\qquad \int_{B_{2r_0}(p_j)\setminus B_{r_0}(p_j)} \frac{\rho h e^u}{\int_M he^u}\partial_{\lambda_j} u_{p_j} = O\big(e^{-\frac{\lambda_1}{(1+\alpha_1)}-\epsilon\lambda_1}\big).$$

By Lemma 2.2, we have

$$\begin{aligned}(A.21)\qquad &\int_{B_{2r_0}(p_j)\setminus B_{r_0}(p_j)} \Delta\big(u_{p_j} - \bar{u}_{p_j} - 8\pi(1+\alpha_j)G(x,p_j)\big)\cdot\partial_{\lambda_j} u_{p_j}\\ &\qquad = O\big(e^{-2\frac{\lambda_1}{(1+\alpha_1)}}\big).\end{aligned}$$

By (3.15), (A.13), and (A.16)–(A.21), we obtain

$$\langle \nabla(u+Tu), \nabla \partial_{\lambda_j} u_{p_j} \rangle = -16\pi(1+\alpha_j)(a_j-1)\lambda_j - 8\pi(1+\alpha_j)\theta_j + O\big(\max|a_k-1| + e^{-\frac{\lambda_1}{1+\alpha_1}-\epsilon\lambda_1}\big).$$

This proves part (iii).

For the proof of part (iv), we have

$$\langle \nabla(u+Tu), \nabla u_{p_j} \rangle = \langle \nabla(u+Tu), \nabla(u_{p_j} - \overline{u}_{p_j}) \rangle = -\langle \Delta(u+Tu), u_{p_j} - \overline{u}_{p_j} \rangle.$$

First, we have $\langle 1, u_{p_j} - \overline{u}_{p_j} \rangle = 0$ and $\langle \Delta w, u_{p_j} - \overline{u}_{p_j} \rangle = 0$ on M because $w \in O_{P,\Lambda}$. To estimate the other terms, we note that

$$\begin{aligned} u_{p_j} - \overline{u}_{p_j} &= 2\lambda_j - 2\log\Big(1 + \frac{\rho h_j(p_j)}{8(1+\alpha_j)^2} e^{\lambda_j}|y|^{2(1+\alpha_j)}\Big) \\ &\quad + 8\pi(1+\alpha_j)R(p_j,p_j) + 2\log\frac{\rho h_j(p_j)}{8(1+\alpha_j)^2} \\ &\quad + O(|y|) + O\big(\lambda_j e^{-\frac{\lambda_j}{1+\alpha_j}}\big). \end{aligned} \tag{A.22}$$

By scaling, we have

$$\begin{aligned} &-4(a_j-1)\lambda_j \int_{\mathbb{R}^2} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j} \log\Big(1 + \frac{\rho h_j(p_j)}{8(1+\alpha_j)^2} e^{\lambda_j}|y|^{2(1+\alpha_j)}\Big) dy \\ &\quad + O(|a_j-1|) \\ &= -16\pi(1+\alpha_j)(a_j-1)\lambda_j + O(|a_j-1|). \end{aligned} \tag{A.23}$$

Let A represent any constant term in

$$(a_j-1)(U_j+s_j-1), \quad \sum_{k\neq j}(a_k-1)(1+\alpha_k)G(p_j,p_k), \quad \text{and} \quad \frac{e^{t_j}}{\int he^u} - 1.$$

For simplicity of notation, we set $W_j(x) = \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}$. By comparing (A.12) and (A.22), we have

$$\begin{aligned} &\int_{B_{r_0}(p_j)} W_j A(u_{p_j} - \overline{u}_{p_j}) dy \\ &\quad = \Big(2\lambda_j - 1 + 8\pi(1+\alpha_j)R(p_j,p_j) + 2\log\frac{\rho h_j(p_j)}{8(1+\alpha_j)^2}\Big) \\ &\qquad \times \int_{B_{r_0}(p_j)} W_j A\, \partial_{\lambda_j} u_{p_j} + O\big(e^{-\frac{\lambda_j}{1+\alpha_j}}\big), \end{aligned} \tag{A.24}$$

where (A.16) is used. It is also easy to see that the integral of $u_{p_j} - \overline{u}_{p_j}$ with all remainder terms in (3.20) are smaller than $O(e^{-\lambda_j/1+\alpha_j} + \|w\|_{H^1})$. For example, by (A.2) of Lemma A,

$$\begin{aligned}
&\int_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j} w(u_{p_j} - \overline{u}_{p_j})dy \\
&= \int_{B_{r_0}} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j} w \\
&\qquad \times \left[\lambda_j + s_j - 2\log\left(1 + \frac{\rho h_j(p_j) e^{\lambda_j}|y|^{2(1+\alpha_j)}}{8(1+\alpha_j)^2}\right) + O(|y|)\right]dy \\
&= O(\lambda_j e^{-\epsilon\lambda_1})\|w\|_{H^1(M)} + \left(\int_{B_{r_0}(p_j)} \rho h_j(p_j) e^{U_j}|y|^{2\alpha_j} w^2 dy\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{B_{r_0}} \rho h_j(p_j) e^{U_j}|y|^{2\alpha_j} \right. \\
&\qquad \left. \times \left[\log\left(1 + \frac{\rho h_j(p_j) e^{\lambda_j}|y|^{2(1+\alpha_j)}}{8(1+\alpha_j)^2}\right) + O(|y|)\right]^2 dy\right)^{\frac{1}{2}} \\
&= O(1)\|w\|_{H^1(M)}.
\end{aligned} \tag{A.25}$$

Since $u_{p_j} - \overline{u}_{p_j} = O(1)$ on $M \setminus B_{r_0}(p_j)$, by Lemma 2.1,

$$\int_{B_{2r_0}(p_j)\setminus B_{r_0}(p_j)} \Delta(u_{p_j} - 8\pi(1+\alpha_j)G(x, p_j))(u_{p_j} - \overline{u}_{p_j}) = O\left(e^{-\frac{\lambda_1}{1+\alpha_1}}\right). \tag{A.26}$$

For the integration on $B_{r_0}(p_k), k \neq j$, the dominant term can be estimated by

$$\begin{aligned}
&\int_{B_{r_0}(p_k)} \rho h_k(p_k)|y|^{2\alpha_k} e^{U_k} \\
&\qquad \times \left[(a_k - 1)(U_k - s_k - 1) + \left(\frac{e^{t_k}}{\int_M he^u} - 1\right)\right](u_{p_j} - \overline{u}_{p_j})dy \\
&= (1+\alpha_j)(1+\alpha_k)\Big[128\pi^2 G(p_k, p_j)(a_k - 1)\lambda_k \\
&\qquad + 64\pi^2 G(p_k, p_j)\left(\frac{e^{t_k}}{\int_M he^u} - 1\right)\Big] + O(|a_k - 1|) =
\end{aligned} \tag{A.27}$$

$$= 64\pi^2(1+\alpha_j)(1+\alpha_k)G(p_k, p_j)\left[\left(\frac{e^{t_k}}{\int_M he^u} - 1\right) + 2(a_k - 1)\lambda_k\right]$$
$$+\ O(|a_k - 1| + e^{-\frac{\lambda_1}{1+\alpha_1} - \epsilon\lambda_1})$$
$$= -8\pi(1+\alpha_j)G(p_k, p_j)\langle \nabla(u + Tu), \nabla\partial_{\lambda_k} u_{p_k}\rangle$$
$$+\ O\big(|a_k - 1| + e^{-\frac{\lambda_1}{1+\alpha_1} - \epsilon\lambda_1}\big).$$

where we use (iii) of Lemma 3.2 proved earlier. It is easy to see that the other terms on $B_{r_0}(p_k)$ are bounded by $O(e^{-\lambda_1/(1+\alpha_1)-\epsilon\lambda_1})$.

For the integration outside $\bigcup_{j=1}^m B_{r_0}(p_j)$, we have $(\int_M he^u)^{-1} = O(e^{-\lambda_j})$ and

$$\text{(A.28)} \qquad \int\limits_{B_{2r_0}(p_j)\setminus B_{r_0}(p_j)} \frac{\rho he^u(u_{p_j} - \overline{u}_{p_j})}{\int_M he^u} = \int\limits_{B_{2r_0}(p_j)\setminus B_{r_0}(p_j)} O(e^{-\lambda_j})e^w = O(e^{-\lambda_j}).$$

Similarly,

$$\text{(A.29)} \qquad \int\limits_{M\setminus\bigcup_j B_{2r_0}(p_j)} \frac{\rho h}{\int_M he^u} e^u(u_{p_j} - \overline{u}_{p_j}) = O(e^{-\lambda_j}).$$

Therefore, by (A.23)–(A.29), we have

$$\langle \nabla u + Tu, \nabla(u_{p_j} - \overline{u}_{p_j})\rangle$$
$$= \left(2\lambda_j - 1 + 8\pi(1+\alpha_j)R(p_j, p_j) + 2\log\frac{\rho h_j(p_j)}{8(1+\alpha_j)^2}\right)\langle \nabla u + Tu, \nabla\partial_{\lambda_j} u_{p_j}\rangle$$
$$+\ 8\pi(1+\alpha_j)\sum_{k\neq j} G(p_j, p_k)\langle \nabla(u + Tu), \nabla\partial_{\lambda_k} u_k\rangle + 16\pi(a_j - 1)\lambda_j(1+\alpha_j)$$
$$+\ O(1)\|w\|_{H^1(M)} + O\big(e^{-\frac{\lambda_1}{1+\alpha_1}}\big).$$

The proof of part (iv) is complete.

To prove part (ii), we note that

$$\langle \nabla(u + Tu), \nabla\partial_{p_j} u_{p_j}\rangle = \langle \nabla(u + Tu), \nabla\partial_{p_j}(u_{p_j} - \overline{u}_{p_j})\rangle.$$

From $w \in O_{P,\Lambda}$, we have

$$\text{(A.30)} \qquad \int\limits_M \nabla w \cdot \nabla\partial_{p_j}(u_{p_j} - \overline{u}_{p_j}) = 0.$$

Since $\int_M (u_{p_j} - \overline{u}_{p_j}) = 0$, we have $\int_M \partial_{p_j}(u_{p_j} - \overline{u}_{p_j}) = 0$ and

$$\int_M \Big[8\pi m + \sum_{k=1}^{\tau} 8\pi\alpha_k - \rho + \sum_{k=1}^{m} 8\pi(1+\alpha_k)(a_k - 1)\Big](u_{p_j} - \overline{u}_{p_j}) = 0. \tag{A.31}$$

On $B_{r_0}(p_j)$, by Lemma 2.1,

$$\partial_{p_j} u_{p_j} = -\nabla_y U_j + \frac{\partial_{p_j} h(p_j)}{h(p_j)}(\partial_{\lambda_j} U_j - 1) + 2\partial_{p_j} \log h(p_j) + 8\pi \partial_{p_j} R_j(x)\big|_{x=p_j} + O(|y|). \tag{A.32}$$

Since $\nabla_y U_j$ is an odd function, we have

$$\int_{B_{r_0}(p_j)} \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}(U_j - s_j - 1) \nabla_y U_j \, dy = O(1). \tag{A.33}$$

Hence by Lemma 2.2 and the fact that $\partial_{\lambda_j} U_j$ is bounded,

$$\int_{B_{r_0}(p_j)} \rho h_j(p_j) e^{U_j}(U_j + s_j - 1)\partial_{p_j}(u_{p_j} - \overline{u}_{p_j}) = O(\lambda_j).$$

For the other terms in (3.20), we have the following estimates:

$$\int_{B_{r_0}(p_j)} \rho h_j(p_j) e^{U_j} \partial_{p_j}(u_{p_j} - \overline{u}_{p_j}) dy = O(1), \tag{A.34}$$

$$\int_{B_{r_0}(p_j)} \rho h_j(p_j) e^{U_j} O(|y|)\partial_{p_j}(u_{p_j} - \overline{u}_{p_j}) dy = O(1), \tag{A.35}$$

$$\int_{B_{r_0}(p_j)} \rho h_j(p_j) e^{U_j} \nabla H_j(p_j) \cdot y \nabla_y U_j \, dy = (8\pi + O(e^{-\lambda_j})) \nabla H_j(p_j), \tag{A.36}$$

and

$$\int_{B_{r_0}(p_j)} \rho h_j(p_j) e^{U_j} \nabla H_j(p_j) \cdot y \partial_{p_j}(u_{p_j} - \overline{u}_{p_j}) dy = 8\pi \nabla H_j(p_j) + O\big(\lambda_j e^{-\frac{3}{2}\lambda_j}\big), \tag{A.37}$$

where we have used $\nabla H_j(p_j) = O(\lambda_j e^{-\lambda_j})$ for $u \in S_\rho(Q)$.

By Lemma 2.2 and (A.32),

$$\int_{B_{r_0}(p_j)} \rho h_j(p_j) e^{U_j} w \partial_{p_j}(u_{p_j} - \overline{u}_{p_j}) dy = O\big(e^{-\frac{3}{10}\lambda_j}\big)\|w\|_{H^1(M)}.$$

Since $\partial_{p_j}(u_{p_j} - \overline{u}_{p_j}) = O(e^{\frac{1}{2}\lambda_j})$, as in the proof of part (iii), we have

$$\int_{B_{r_0}(p_j)} E\partial_{p_j}(u_{p_j} - \overline{u}_{p_j}) dy = O\big(\lambda_j^3 e^{-\frac{3}{2}\lambda_j}\big).$$

On $M \setminus B_{r_0}(p_j)$, $\partial_{p_j}(u_{p_j} - \overline{u}_{p_j}) = O(1)$. Hence by Lemma 2.1,

$$\int_{B_{2r_0}(p_j)\setminus B_{r_0}(p_j)} \Delta(u_{p_j} - 8\pi G(x, p_j)) \cdot \partial_{p_j}(u_{p_j} - \overline{u}_{p_j}) = O\left(\frac{\lambda_j}{e^{\lambda_j}}\right).$$

Since $(\int_M he^u)^{-1} = O(e^{-\lambda_j})$, the integral of the products of $\partial_{p_j}(u_{p_j} - \overline{u}_{p_j})$ and the nonlinear terms in (3.21) and (3.22) are of order

$$O(e^{-\lambda_j}) \int_M e^w = O(e^{-\lambda_j}).$$

The estimates above imply

$$\begin{aligned}\langle \nabla Tu, \nabla\partial_{p_j}(u_{p_j} - \overline{u}_{p_j})\rangle &= -8\pi \nabla H_j(p_j)\\ &\quad + O\left(|a_j - 1|\lambda_j + \left|\frac{e^{t_j}}{\int_M he^u} - 1\right| + e^{-\frac{\lambda_1}{1+\alpha_1}}\right).\end{aligned}$$

This proves part (ii) and hence the proof of Lemma 3.2 is complete. □

Appendix B Proof of Lemma 3.3

We first note that

(B.1) $$\|u_{p_j}\|^2 \sim \lambda_j,$$

(B.2) $$\|\partial_{p_j} u_{p_j}\|^2 \sim e^{\frac{\lambda_1}{1+\alpha_1}},$$

(B.3) $$\|\partial_{\lambda_j} u_{p_j}\|^2 \sim 1,$$

where $\|w\|$ denotes $\|w\|_{H^1}$ and the notation "$A \sim B$" means that both ratios A/B and B/A are bounded by a positive constant independent of (P, Λ) if λ_1 is large enough. We also know that

(B.4) $$\langle \nabla u_{p_j}, \nabla\partial_{\lambda_j} u_{p_j}\rangle = O(1) \quad \text{for } 1 \le j \le m,$$

(B.5) $$\langle \nabla u_{p_j}, \nabla\partial_{p_j} u_{p_j}\rangle = O(1) \quad \text{for } \tau < j \le m,$$

(B.6) $$\langle \nabla\partial_{\lambda_j} u_{p_j}, \nabla\partial_{p_j} u_{p_j}\rangle = O(1) \quad \text{for } \tau < j \le m.$$

Therefore $u_{p_j}/\|u_{p_j}\|$, $\partial_{p_j} u_{p_j}/\|\partial_{p_j} u_{p_j}\|$, and $\partial_{\lambda_j} u_{p_j}/\|\partial_{\lambda_j} u_{p_j}\|$ are almost orthogonal to each other.

Let

$$V_\alpha(y) = \frac{|y|^{2\alpha}}{(1+|y|^{2+2\alpha})^2}.$$

Consider the eigenvalue problem with eigenvalue γ:

$$\Delta\psi + (8(1+\alpha)^2 + \gamma)|y|^{2\alpha} e^{V_\alpha}\psi = 0 \quad \text{on } R^2. \tag{B.7}$$

According to the results in [20], there are $2(1+[\alpha])+1$ negative eigenvalues and 0 is a simple eigenvalue in the space of bounded functions. The eigenfunction corresponding to $\gamma = 0$ can be written down as

$$\widetilde{\psi}_0(y) = \frac{1-|y|^{2+2\alpha}}{1+|y|^{2+2\alpha}}.$$

One of the eigenfunctions corresponding to those $2(1+[\alpha])+1$ negative eigenvalues is radially symmetric (in fact, it is the constant function with $\gamma = -8(1+\alpha)^2$). The others have the forms: $g_1(r)\cos k\theta$ or $g_2(r)\sin k\theta$, $1 \le k \le 1+[\alpha]$, where $g_j(r)$, $j = 1, 2$, does not change sign and $g_j(r) \to 0$ as $r \to \infty$. For $1 \le n \le 2(1+[\alpha])$, let $\widetilde{\psi}_n$ denote those nonradially symmetric eigenfunctions corresponding to the negative eigenvalue $\gamma_{\alpha,n}$. It is easy to see that $\|\nabla\widetilde{\psi}_n\|^2$ is finite. Hence we can always assume $\|\nabla\widetilde{\psi}_n\|^2 = 1$. These eigenfunctions can be used to construct approximate eigenfunctions corresponding to the bilinear form in the lemma as follows.

In $B_{2r_0}(p_j)$, we let

$$\widetilde{\psi}_{p_j,n}(y) = c_j\widetilde{\sigma}_j\widetilde{\psi}_n(R_j(y-p_j)),\ R_j = e^{\frac{\lambda_j}{2(1+\alpha_j)}}, \tag{B.8}$$

where $\widetilde{\sigma}_j$ is a symmetric cutoff function satisfying $\operatorname{supp}\widetilde{\sigma}_j \subset B_{r_0}(p_j)$, and a constant c_j is added so that $\|\widetilde{\psi}_{p_j,n}\| = 1$. Clearly, $\widetilde{\psi}_{p_j,n} \in \overset{\circ}{H}{}^1$.

To project $\widetilde{\psi}_{p_j,n}$ onto the space $O_{P,\Lambda}$, we need to calculate the inner products of $\widetilde{\psi}_{p_j,n}(y)$ and the functions u_{p_k} and $\partial_{\lambda_k} u_{p_k}$ for $1 \le k \le m$, and $\partial_{p_k} u_{p_k}$ for $k > \tau$.

We consider the case $k \ne j$ first. Since $u_{p_k} = 8(1+\alpha_k)\pi G(x, p_k)$ on $B_{2r_0}(p_j)$, we have

$$\begin{aligned}\int_M \nabla\widetilde{\psi}_{p_j,n}\nabla u_{p_k} &= -\int_M \widetilde{\psi}_{p_j,n}\Delta u_{p_k}\\ &= -8\pi(1+\alpha_k)\int_{B_{2r_0}(p_j)} \widetilde{\psi}_{p_j,n} = 0.\end{aligned} \tag{B.9}$$

Similarly, we have

$$\int_M \nabla \widetilde{\psi}_{p_j,n} \nabla \frac{\partial_{p_k} u_{p_k}}{\|\partial_{p_k} u_{p_k}\|_{H^1}} = - \frac{1}{\|\partial_{p_k} u_{p_k}\|_{H^1}} \int_{B_{2r_0}(p_j)} \widetilde{\psi}_{p_j,n} \Delta \partial_{p_k} u_{p_k} = o\big(e^{-\frac{1}{2}\frac{\lambda_1}{1+\alpha_1}}\big)$$

by (B.2) for $k > \tau$, and

$$\int_M \nabla \widetilde{\psi}_{p_j,n} \nabla \frac{\partial_{\lambda_k} u_{p_k}}{\|\partial_{\lambda_k} u_{p_k}\|_{H^1}} = 0$$

for $1 \le k \le m$ because they have different support.

For $k = j$,

$$\text{(B.10)} \qquad \begin{aligned} \int_M \nabla \widetilde{\psi}_{p_j,n} \nabla u_{p_j} &= - \int_{B_{r_0}(p_j)} \widetilde{\psi}_{p_j,n} [\Delta U_j + 8\pi(1+\alpha_j) + \Delta \eta_j] \\ &= - \int_{B_{2r_0}(p_j)} \widetilde{\psi}_{p_j,n} |y|^{2\alpha_j} e^{U_j} + o(1) = o(1), \end{aligned}$$

where a cancellation due to the presence of $\cos k\theta$ is used. By (A.12) and the fact $\partial_{\lambda_j} u_{p_j}$ is small on $B_{2r_0} \setminus B_{r_0}$,

$$\begin{aligned} \int_M \nabla \widetilde{\psi}_{p_j,n} \cdot \nabla \partial_{\lambda_j} u_{p_j} &= - \int_M \widetilde{\psi}_{p_j,n} \Delta(\partial_{\lambda_j} u_{p_j}) \\ &= - \int_{B_{r_0}} \widetilde{\psi}_{p_j,n} \Delta(\partial_{\lambda_j} U_j) + O(e^{-\lambda_1}) \\ &= \int_{|y|<r_0} \widetilde{\psi}_{p_j,n} \rho h_j(p_j) |y|^{2\alpha_j} e^{U_j} \partial_{\lambda_j} U_j \, dy + o(1) \\ &= o(1). \end{aligned}$$

Now let $\psi_{p_j,n}$ be the projection of $\widetilde{\psi}_{p_j,n}$ onto the space $O_{P,\Lambda}$. From the estimates above,

$$\text{(B.11)} \qquad \psi_{p_j,n} = \widetilde{\psi}_{p_j,n} + o(1)$$

and

$$\text{(B.12)} \qquad \int_M \nabla \psi_{p_j,n} \nabla \psi_{p_{j'},n'} = \int_M \nabla \widetilde{\psi}_{p_j,n} \nabla \widetilde{\psi}_{p_{j'},n'} + o(1) = o(1)$$

if $j \neq j'$ or $n \neq n'$.

Let

$$W = \operatorname{span}\{\psi_{p_j,n} : 1 \le j \le \tau,\ 1 \le n \le 2(1+[\alpha_j])\},$$
$$w_1 = \sum_{1\le j\le\tau} \sum_{1\le n\le 2(1+[\alpha_j])} b_{j,n}\psi_{p_j,n} \in W, \quad \text{and} \quad \sum_{j,n} b_{j,n}^2 = 1.$$

The value of the bilinear form at (w_1, w_1) is equal to

$$\begin{aligned} &B(w_1, w_1)\\ &= \sum_{1\le j\le\tau} \sum_{1\le k\le 2(1+[\alpha_j])} \int_{B_{r_0}(p_j)} b_{j,k}^2[|\nabla\psi_{p_j,n}|^2 - \rho h_j(p_j)|y|^{2\alpha_j}e^{U_j}\psi_{p_j,n}^2]\\ &\qquad + o(1)\\ &= \sum_{1\le j\le\tau} \sum_{1\le k\le 2(1+[\alpha_j])} \int_{B_{r_0}(p_j)} b_{j,k}^2\gamma_{\alpha_j,k}\rho h_j(p_j)|y|^{2\alpha_j}e^{U_j}\psi_{p_j,n}^2 + o(1) \qquad \text{(B.13)}\\ &\le -d_0\sum_{j,k} b_{j,k}^2 + o(1) < -\frac{d_0}{2}, \end{aligned}$$

where $0 < d_0 \le \min\{-\gamma_{\alpha_j,k}\}$. Hence $B(w,w)$ is negative definite on W. By an argument of linear algebra, this implies that the Morse index of $B(w,w)$ in $\mathring{H}^1$ is no less than $\sum_{1\le j\le\tau} 2(1+[\alpha_j])$.

Let $W^\perp$ be the orthogonal complement of the space generated by W in $\mathring{H}^1$. We claim that there exists $d_1 > 0$ such that

$$B(w,w) \ge d_1\|w\|_{H^1(M)}^2 \quad \text{for } w \in W^\perp \cap O_{P,\Lambda} \tag{B.14}$$

or, equivalently, there exists some $\tilde{d}_1 > 0$ such that

$$J(w) = B(w,w) \ge d_1 > 0$$

for $w \in W^\perp \cap O_{P,\Lambda}$ satisfying

$$\int_M \sum_j \rho h_j(p_j)|y|^{2\alpha_j}e^{U_j}\sigma_j w^2 = 1.$$

After the claim, it is obvious to see that the Morse index of $B(w,w)$ is equal to $\sum_{1\le j\le\tau} 2(1+[\alpha_j])$ and $|B(w,w)| \ge d_2\|w\|^2$ for all $w \in W^\perp \cap O_{P,\Lambda}$ with some $d_2 > 0$.

Let $\beta_{P,\Lambda} = \min_{w\in O_{P,\Lambda}\bigcap W^\perp, w\ne 0} J(w)$ and $w_{P,\Lambda}$ be a minimizer satisfying $J(w_{P,\Lambda}) = \beta_{P,\Lambda}$ and

$$\int_M \sum_j \rho h_j(p_j)|y|^{2\alpha_j}e^{U_j}\sigma_j w^2 = 1. \tag{B.15}$$

Then $w_{P,\Lambda}$ satisfies

$$\Delta w_{P,\Lambda} + (1+\beta_{P,\Lambda})\Big(\sum_j \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}\sigma_j\Big) w_{P,\Lambda} = \psi_{P,\Lambda} \tag{B.16}$$

for some $\psi_{P,\Lambda} \in \widetilde{W} = (W^{\perp} \cap O_{P,\Lambda})^{\perp}$, where $\perp$ denotes the orthogonal complement in $\overset{\circ}{H}{}^1$. We will show that there is $c > 0$ such that $\beta_{P,\Lambda} \geq c$ if all λ_j's are large. Suppose on the contrary that there are $P_k = (p_{1,k}, \dots, p_{m,k})$ with $p_{j,k} = q_j$ for $1 \leq j \leq \tau$ and $\Lambda_k = (\lambda_{1,k}, \dots, \lambda_{m,k})$ such that $\operatorname{dist}(p_{i,k}, p_{j,k}) \geq 2r_0$ for $i \neq j$, $\min_j \lambda_{j,k} \to \infty$, and $\beta_{P_k,\Lambda_k} \to \beta_\infty \leq 0$ as $k \to \infty$. We will show that a contradiction follows from this.

Let $\beta_k = \beta_{P_k,\Lambda_k}$, $w_k = w_{P_k,\Lambda_k}$. For simplicity of notation, let us write ψ_β^k, $\beta = 1, 2, \dots, L$, to stand for $u_{p_j}/\|u_{p_j}\|$, $\partial_{p_j} u_{p_j}/\|\partial_{p_j} u_{p_j}\|$, $\partial_{\lambda_j} u_{p_j}/\|\partial_{\lambda_j} u_{p_j}\|$, and $\psi_{p_j,n}$. Then the multiple $\psi_{P,\Lambda}$ can be written as

$$\psi_{P_k,\Lambda_k} = b_0^k + \sum_{\beta=1}^{L} b_\beta^k \Delta \psi_\beta^k,$$

where $b_\beta^k \in \mathbb{R}$, $\beta = 1, \dots, L$. Due to the almost orthogonality of ψ_β^k and their closeness to eigenfunctions, we can show

$$b_\beta^k \to 0 \quad \text{as } k \to \infty,\ 0 \leq \beta \leq L.$$

First, since $w_k \in O_{P,\Lambda}$, Lemma A implies

$$\int \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}\sigma_j w_k \, dy = O(e^{-\epsilon_0\lambda_1}) = o(1) \quad \forall j.$$

Thus

$$b_0^k = (1+\beta_k)\sum_j \int \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}\sigma_j w_k \, dy = o(1). \tag{B.17}$$

Now we compute $b_{\beta_0}^k$. If $\psi_{\beta_0}^k = u_{p_j}/\|u_{p_j}\|$, then

$$u_{p_j} = -2\log\Big(1 + \frac{\rho h_j(p_j)}{8(1+\alpha_j)^2} e^{\lambda_j}|y|^{2\alpha_j+2}\Big) + \eta_j + \text{const} + O(|y|).$$

By Lemma A, we have

$$\begin{aligned}
&\int \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}\sigma_j w_k \psi_{\beta_0}^k \, dy \\
&= \frac{1}{\|u_{p_j}\|}\Big[\int \rho h_j(p_j)|y|^{2\alpha_j} e^{U_j}\Big[-2\log\Big(1 + \frac{\rho h_j(p_j)}{8(1+\alpha_j)^2} e^{\lambda_j}|y|^{2\alpha_j+2}\Big)\Big] w_k \\
&\qquad + O(\lambda_j e^{-\epsilon_1\lambda_1})\|w_k\|_{H^1}\Big] \\
&= \frac{O(1)}{\|u_{p_j}\|}\|w_k\|_{H^1} = O(1)\|w_k\|_{H^1}\lambda_1^{-\frac{1}{2}}.
\end{aligned}$$

Also by (B.4) and (B.5), we have $\langle \psi^k_{\beta_0}, \psi^k_\beta \rangle_{H^1} = O(\lambda_1^{-1/2})$ if $\beta_0 \neq \beta$. Thus

$$\begin{aligned} b_{\beta_0} &+ O\left(\frac{1}{\lambda_j^{1/2}}\right) \max_\beta |b_\beta| \\ &= \int \psi^k_{\beta_0} \Delta w_k + (1+\beta_k) \int \Big(\sum_j \rho h_j(p_j) |y|^{2\alpha_j} e^{U_j} \sigma_j \Big) \psi^k_{\beta_0} w_k \\ &= O(1) \|w_k\|_{H^1} \lambda_1^{-\frac{1}{2}}, \end{aligned} \tag{B.18}$$

where $0 = \langle w_k, \psi^k_\beta \rangle_{H^1} = -\langle w_k, \Delta \psi^k_\beta \rangle$ is used. For $\psi^k_\beta = \partial_{\lambda_j} u_{p_j}, \partial_{p_j} u_{p_j}$, or $\psi_{j,n}$, we have $\langle \bigtriangledown w_k, \bigtriangledown \psi^k_\beta \rangle = 0$. Then

$$\begin{aligned} 0 = \langle \bigtriangledown w_k, \bigtriangledown \psi^k_\beta \rangle &= -\int w_k \Delta \psi^k_\beta \\ &= -\int_{B_{r_0}(p_j)} \rho h_j(p_j) |y|^{2\alpha_j} e^{U_j} w_k \psi^k_\beta + o(1) \|w_k\|_{H^1}, \end{aligned}$$

which implies

$$\int \rho h_j(p_j) |y|^{2\alpha_j} e^{U_j} w_k \psi^k_\beta = o(1) \|w_k\|_{H^1}. \tag{B.19}$$

By equation (B.16), we have

$$b_\beta + o(1) \max_l |b_l| = (1+\beta_k) \int \sum_{j=1}^m \rho h_j(p_j) |y|^{2\alpha_j} e^{U_j} w_k \psi^k_\beta = o(1) \|w_k\|_{H^1}.$$

Thus

$$b_\beta = o(1) \|w_k\|, \quad \beta = 1, \ldots, L.$$

On the other hand, from (B.16)

$$\|w_k\|^2 = (1+\beta_k) \int \Big(\sum_{j=1}^m \rho h_j(p_j) |y|^{2\alpha_j} e^{U_j} \sigma_j \Big) w_k^2 = 1 + \beta_k \leq 2.$$

We have proved the smallness of each b_β.

Now set $e_k = \max_M |w_k|$. By (B.15), $e_k \geq e_0 > 0$ for some constant e_0. Let $\mathrm{dist}(x, p_k)$ denote the distance from x to the set $\{p_{1,k}, \ldots, p_{m,k}\}$.

Step 1. We show that for any fixed $r_1 > 0$

$$\limsup_k \max_{\mathrm{dist}(x,p_k) \geq r_1} \left(\frac{|w_k(x)|}{e_k} \right) = 0. \tag{B.20}$$

Suppose this is not true. Then after passing to a subsequence, there are $c_1 > 0$ and $\{x_k\} \subset M$ satisfying $\operatorname{dist}(x_k, p_k) \ge r_1$ and $|w_k(x_k)| \ge c_1 e_k$. Let $g_k(x) = w_k(x)e_k^{-1}$. Then after passing to a subsequence again, $\lim_{k\to\infty} x_k = x_\infty$ for some x_∞, $\lim_{k\to\infty} p_{j,k} = p_j$, g_k tends to some g in $C^{2,\delta}_{\mathrm{loc}}(M \setminus \{p_1, \dots, p_m\})$, and g satisfies

$$\begin{cases} \Delta g = 0,\ |g| \le 1 & \text{on } M \setminus \{p_1, \dots, p_m\}, \\ \int_M g = 0,\ |g(x_\infty)| \ge c_1. \end{cases}$$

Here $b^k_\beta \to 0$ is used. By the removability of singularities for bounded harmonic functions, we have $\Delta g = 0$ on M. Hence $g = 0$ on M. This contradicts $|g(x_\infty)| \ge c_1 > 0$. Therefore we may assume the maximum of $|w_k|$ is attained in $B_{r_0}(p_{j,k})$ for large k.

Step 2. If the maximum of $|w_k|$ is attained in $B_{r_0}(p_{j,k})$ for some $j > \tau$, let $g_k(y) = w_k e_k^{-1}$ be defined as before and $\widetilde{g}_k(z) = g_k(z/R_k)$. Then elliptic estimates imply there is a subsequence of $\widetilde{g}_k$ (still denoted by $\widetilde{g}_k$) such that $\widetilde{g}_k$ converges uniformly in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$.

We claim that $\widetilde{g}_k$ converges to 0 uniformly on any fixed ball. Suppose this is not true. Then after passing to a subsequence, there are $r > 0$ and $\{z_k\}$ such that $|z_k| \le r$, $|\widetilde{g}_k(z_k)| = \max_{|z|\le r} |\widetilde{g}_k(z)|$, $\lim_{k\to\infty} z_k = z_\infty$, and $\widetilde{g}_k$ converges to some $\widetilde{g}$ in $C^2_{loc}(\mathbb{R}^2)$ with

$$\begin{cases} \Delta \widetilde{g} + (1+\beta_\infty)\frac{8(1+\alpha_j)^2|z|^{2\alpha_j}}{(1+|z|^{2(1+\alpha_j)})^2}\widetilde{g} = 0,\ |\widetilde{g}| \le 1 \quad \text{on } \mathbb{R}^2 \\ \widetilde{g}(z_\infty) \ne 0, \end{cases}$$

where we use $b^k_\beta \to 0$ as $k \to \infty$.

Then, by multiplying $\partial_{\lambda_j} u_{p_\beta}$ or $\psi_{p_j,n}$, we have by (B.19),

$$\text{(B.21)}\qquad \int_{\mathbb{R}^2} \frac{|z|^{2\alpha_j}\widetilde{g}(z)}{(1+|z|^{2(1+\alpha_j)})^2}\,dz = \int_{\mathbb{R}^2} \frac{|z|^{2(1+\alpha_j)}-1}{|z|^{2(1+\alpha_j)}+1}\frac{|z|^{2\alpha_j}\widetilde{g}(z)}{(1+|z|^{2(1+\alpha_j)})^2} = 0$$

and

$$\text{(B.22)}\qquad \int_{\mathbb{R}^2} \widetilde{\psi}_n \frac{\widetilde{g}(z)|z|^{2\alpha_j}}{(1+|z|^{2(1+\alpha_j)})^2}\,dz = 0 \quad \text{for } 1 \le n \le 2(1+[\alpha_j]).$$

But

$$\left\{1, \widetilde{\psi}_n, \frac{|z|^{2(1+\alpha_j)}-1}{|z|^{2(1+\alpha_j)}+1}\right\}$$

are the only eigenfunctions of (B.7) with oppositive eigenvalues under the weight $|z|^{2\alpha_j}(|z|^{2(1+\alpha_j)}+1)^{-2}$ if $j \le \tau$. Thus we may assume $j > \tau$. Then (B.19) for

$\psi^k_\beta = \partial_{p_j} u_{p_j}$ gives

$$\int\limits_{\mathbb{R}^2} \frac{1}{(1+|z|^2)^2} \frac{z}{1+|z|^2} \widetilde{g}\, dz = 0,$$

which, together with (B.21) and (B.22), again leads to $\widetilde{g} \equiv 0$, which of course yields a contradiction. Hence we have proved $\widetilde{g} \equiv 0$, the claim.

Let $A_k(t) = \frac{1}{2\pi t} \int_{|z|=t} \widetilde{g}_k(z)$ be the average of $\widetilde{g}_k$ on the sphere $|z| = t$. Then A_k satisfies $|A_k(t)| \leq 1$ and

$$\text{(B.23)} \qquad \Delta A_k + (1+\beta_k) \frac{8(1+\alpha_j)^2 |z|^{2\alpha_j}}{(1+|z|^{2(1+\alpha_j)})^2} A_k = e_k^{-1} R_j^{-2} \int\limits_{|z|=t} \psi_{P_k,\Lambda_k}\, d\sigma.$$

The right-hand side

$$\psi_{P_k,\Lambda_k} = b^k_0 + b^k_{\beta_0} \psi^k_{b_o} + \sum_{\beta \neq \beta_0} b_\beta \psi^k_\beta,$$

where $\psi^k_{\beta_0} = \Delta \frac{u_{p_j}}{\|u_{p_j}\|}$). In $B_{r_0}(p_j)$, the nonzero terms in $\sum_{\beta\neq\beta_0} b_\beta \psi^k_\beta$ consist only of $\Delta\partial_{\lambda_j} u_{p_j}$, $\psi_{j,n}$, $\Delta \frac{u_{p_k}}{\|u_{p_k}\|}$ $(= \lambda_1^{-1/2} 8\pi(1+\alpha_k))$, $k \neq j$, if $j \leq \tau$, and an additional term $\Delta\partial_{p_j} u_{p_j}$ if $j > \tau$.

Thus for $r \leq r_0$,

$$\begin{aligned} e_k^{-1} R_j^{-2} \|u_{p_j}\| \overline{\psi}_{P_k,\Lambda_k}(t) &= \Delta V_{\alpha_j}(z) + \Delta\partial_{\lambda_j} V_{\alpha_j}(z) + O(1)\lambda_1^{-\frac{1}{2}} R_j^{-2} \\ &= O(1)\left\{ \frac{1}{t^{2+4\alpha_j}} + \lambda_1^{-\frac{1}{2}} R_j^{-2} \right\}, \end{aligned}$$

where $\overline{\psi}_{P_k,\Lambda_k}(t) = \int_{|z|=t} \psi_{P_k,\Lambda_k}(z) d\sigma$.

Let $\varphi_0(z) = \frac{|z|^{2\alpha_j}-1}{|z|^{2\alpha_j}+1}$. Then φ_0 satisfies

$$\Delta\varphi_0 + \frac{8(1+\alpha_j)^2 |z|^{2\alpha_j}}{(1+|z|^{2(1+\alpha_j)})^2} \varphi_0 = 0 \quad \text{in } \mathbb{R}^2.$$

Step 3. Assume $\max_{t \leq r_0 R_j} |A_k(t)|$ does not tends to 0. Then since $\widetilde{g}_k$ tends to 0 on any compact sets and on $|z| = r_0 R_j$, after passing to a subsequence, there is t_k satisfying

$$\frac{|A_k(t_k)|}{\varphi_0(t_k)} = \max_{2 \leq t \leq r_0 R_k} \frac{|A_k(t)|}{\varphi_0(t)} \quad \text{and} \quad \lim_{k\to\infty} t_k = \infty$$

due to Step 2. For $2 \le t \le t_k$, we have

$$\begin{aligned}
&\left| t\varphi_0^2(t)\frac{d}{dt}\left(\frac{A_k(t)}{\varphi_0(t)}\right)\right| \\
&\quad = \left|\int_t^{t_k}\left[-\frac{8\beta_k(1+\alpha_j)^2 s^{2\alpha_j}}{(1+s^{2(1+\alpha_j)})^2}A_k\varphi_0 + e_k^{-1}R_j^{-2}\overline{\psi}_{P_k,\Lambda_k}(s)\varphi_0\right]s\,ds\right| \\
&\quad \le C_2\left(\frac{1}{t^{2(1+\alpha_j)}} + \frac{t_k^2}{R_j^2}\lambda_1^{-1}\right) \\
&\quad = C_2\left(\frac{1}{t^{2(1+\alpha_j)}} + o(1)\lambda_1^{-1}\right)
\end{aligned}$$

for some $C_2 > 0$ because $\frac{t_k^2}{R_j^2} \to 0$ as $k \to \infty$ by Step 1. Hence for $2 \le t \le t_k$,

$$\begin{aligned}
\text{(B.24)}\qquad \left|\frac{A_k(t_k)}{\varphi_0(t_k)} - \frac{A_k(t)}{\varphi_0(t)}\right| &\le \int_t^{t_k} C_3\left(\frac{1}{s^{3+2\alpha_j}} + \frac{o(1)}{\lambda_1 s}\right)ds \\
&\le C_3\left(\frac{1}{t^{2(1+\alpha_j)}} + \frac{o(1)}{\lambda_1}\log t_k\right).
\end{aligned}$$

Since $\widetilde{g}_k$ converges to 0 on any compact set uniformly, there are $\overline{t}_k \le t_k$ such that $A_k(\overline{t}_k) \to 0$ and $\overline{t}_k \to \infty$ as $k \to \infty$. Let $t = \overline{t}_k$ in (B.24). Then

$$\left|\frac{A_k(t_k)}{\varphi_0(t_k)}\right| \le \left|\frac{A_k(\overline{t}_k)}{\varphi_0(\overline{t}_k)}\right| + C_3\left(\frac{1}{\overline{t}_k^{2(1+\alpha_j)}} + \frac{\log R_k}{R_k^{2(1+\alpha_j)}}\right) \to 0.$$

This implies $\max_{t \le r_0 R_k} |A_k(t)| \to 0$ as $k \to \infty$.

Since we assume w_k attains its maximum in $B_{r_0}(p_j)$, there is $\{z_k\}$ satisfying $|z_k| \le r_0 R_k$, $|\widetilde{g}_k(z_k)| = 1$, and $|z_k| \to \infty$ as $k \to \infty$. We may assume $\widetilde{g}_k(z_k) = 1$. Since $\max_{t \le r_0 R_k} |A_k| \to 0$, there is y_k satisfying $|z_k| = |y_k|$ and $|\widetilde{g}_k(y_k)| \le \frac{1}{4}$. After taking a rotation of the coordinates, we may assume $z_k = (z_k^1, z_k^2)$ and $y_k = (-z_k^1, z_k^2)$. For $z = (z^1, z^2)$, define $D_k(z) = \widetilde{g}_k(z^1, z^2) - \widetilde{g}_k(-z^1, z^2)$. Then D_k satisfies $D_k(z_k) \ge \frac{3}{4}$ and

$$\Delta D_k + (1+\beta_k)\frac{8(1+\alpha_j)^2|z|^{2\alpha_j}}{(1+|z|^{2(1+\alpha_j)})^2}D_k + h_k = 0,$$

where

$$h_k(z) = \sum b_\beta^k\big(\psi_\beta^k(z) - \psi_\beta^k(-z_1, z_2)\big).$$

Here $\psi_\beta^k(z)$ are those terms that come from $\Delta\partial_{p_j}u_{p_j}/\|\Delta\partial_{p_j}u_{p_j}\|$ and $\Delta\psi_n$. Note that

$$\begin{aligned}
R_j^{-2}\Delta\partial_{p_j}u_{p_j}\big(R_j^{-\frac{1}{2}}y\big) &= \Delta\left(\frac{z}{1+|z|^2}\right) = O\left(\frac{1}{(1+|z|)^5}\right), \\
R_j^{-2}\Delta\psi_{n,j}\big(R_j^{-\frac{1}{2}}y\big) &= \Delta\widetilde{\psi}_n(z) = O\left(\frac{1}{(1+|z|)^5}\right)
\end{aligned}$$

if $\widetilde{\psi}_{n,j} = g(r)\cos k\theta$ or $g(r)\sin k\theta$. Thus

$$h_k(z) = o(1)(1+|z|)^{-5}$$

for $|z| \le r_0 R_j$ and $z^1 \ge 0$. By the Green representation

$$\begin{aligned}\frac{3}{4} &\le D_k(z_k)\\ &= \int_{|z|\le r_0R_j, z^1\ge 0} G^+(z,z_k)\left(\frac{8(1+\beta_k)(1+\alpha_j)^2|z|^{2\alpha_j}}{(1+|z|^{2(1+\alpha_j)})^2}(D_k+h_k(z))\right)dz \\ &\quad + \int_{\{\text{boundary}\}} \frac{\partial}{\partial\nu} G^+(z,z_k) D_k\, ds,\end{aligned} \tag{B.25}$$

where $G^+(y,z) \le \frac{1}{2\pi}\log\frac{|y-z|}{|y-z^*|}$ with $z^* = (-z^1, z^2)$ is the Green function of $-\Delta$ on the half $\{|z|\le r_0R_k,\ z^1\ge 0\}$. Since $D_k(z)=0$ on $z^1=0$ and $\widetilde{g}_k \to 0$ on $|z| = r_0R_k$, the boundary integral in (B.25) tends to 0. Clearly, the integral with $h_k(z) = o(1)$. Using the fact $|D_k|\le 1$, the first term in the right-hand side of (B.25) equals

$$\int_{\substack{|z|\le\sqrt{|z_k|}\\ z^1\ge 0}} + \int_{\substack{|z-z_k|\le\frac{1}{2}z_k\\ z^1\ge 0}} + \int_{\substack{|z|\ge 2|z_k|\\ z^1\ge 0}} + \int_{\substack{|z-z_k|\ge\frac{1}{2}|z_k|\\ \sqrt{|z_k|}\le|z|\le 2|z_k|\\ z^1\ge 0}} =$$

$$O\left(\frac{1}{\sqrt{|z_k|}} + \frac{1}{|z_k|^{2(1+\alpha_j)}} + \frac{1}{|z_k|^{2(1+\alpha_j)}} + \frac{1}{|z_k|^{1+\alpha_j}}\right) \to 0$$

as $k\to\infty$ since $\lim_{k\to\infty}|z_k| = \infty$, where we have used $G^+(z,z_k) = O(\frac{1}{\sqrt{|z_k|}})$ for $|z|\le\sqrt{|z_k|}$. Therefore

$$\frac{3}{4} \le D_k(z_k) \le o(1) \quad \text{as } k\to\infty.$$

We obtain a contradiction. The proof of Lemma 3.3 is complete.

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