Global Well-Posedness of Classical Solutions with Large Oscillations and Vacuum to the Three-Dimensional Isentropic Compressible Navier-Stokes Equations

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Abstract

We establish the global existence and uniqueness of classical solutions to the Cauchy problem for the isentropic compressible Navier-Stokes equations in three spatial dimensions with smooth initial data that are of small energy but possibly large oscillations with constant state as far field, which could be either vacuum or nonvacuum. The initial density is allowed to vanish, and the spatial measure of the set of vacuum can be arbitrarily large; in particular, the initial density can even have compact support. These results generalize previous results on classical solutions for initial densities being strictly away from vacuum and are the first for global classical solutions that may have large oscillations and can contain vacuum states. © 2012 Wiley Periodicals, Inc.

1 Introduction

The time evolution of the density and the velocity of a general viscous isentropic compressible fluid occupying a domain $\Omega \subset \mathbb{R}^3$ is governed by the compressible Navier-Stokes equations

$$
\begin{cases}\rho_t + \operatorname{div}(\rho u) = 0,\\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \nabla P(\rho) = 0,\end{cases} \tag{1.1}
$$

where $\rho \geq 0$, $u = (u^1, u^2, u^3)$, and $P(\rho) = a\rho^\gamma (a > 0, \gamma > 1)$ are the fluid density, velocity, and pressure, respectively. The constant viscosity coefficients μ and λ satisfy the physical restrictions

$$
\mu > 0, \quad \mu + \frac{3}{2}\lambda \geq 0. \tag{1.2}
$$

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Let $\Omega = \mathbb{R}^3$ and $\tilde{\rho}$ be a fixed nonnegative constant. We look for the solutions $(\rho(x,t), u(x,t))$ to the Cauchy problem for (1.1) with the far-field behavior

$$u(x,t) \to 0, \quad \rho(x,t) \to \tilde{\rho} \geq 0 \quad \text{as } |x| \to \infty, \tag{1.3}$$

and initial data

$$(\rho, u)|_{t=0} = (\rho_0, u_0), \quad x \in \mathbb{R}^3. \tag{1.4}$$

There is a huge literature on the large-time existence and behavior of solutions to (1.1). The one-dimensional problem has been studied extensively by many people (see [10, 23, 32, 33]) and the references therein. For the multidimensional case, the local existence and uniqueness of classical solutions are known in [29, 34] in the absence of vacuum and recently, for strong solutions also, in [3, 5, 6, 31] for the case that the initial density need not be positive and may vanish in open sets. The global classical solutions were first obtained by Matsumura and Nishida [28] for initial data close to a nonvacuum equilibrium in some Sobolev space H^s. In particular, the theory requires that the solution have small oscillations from a uniform nonvacuum state so that the density is strictly away from the vacuum and the gradient of the density remains bounded uniformly in time. Later, Hoff [11, 12] studied the problem for discontinuous initial data. For the existence of solutions for arbitrary data (the far-field density is vacuum, that is, $\tilde{\rho} = 0$), the major breakthrough is due to Lions [27] (see also Feireisl, Novotny, and Petzeltová [7]), where he obtains global existence of weak solutions, defined as solutions with finite energy, when the exponent γ is suitably large. The main restriction on initial data is that the initial energy is finite, so that the density vanishes at far fields or even has compact support. However, little is known on the structure of such weak solutions. Recently, under the additional assumptions that the viscosity coefficients μ and λ satisfy

$$\mu > \max\{4\lambda, -\lambda\}, \tag{1.5}$$

and for the far-field density away from vacuum ($\tilde{\rho} > 0$), Hoff and associates [14, 15, 16] obtained a new type of global weak solutions with small energy, which have extra regularity information compared with those large weak ones constructed by Lions [27] and Feireisl et al. [7]. Note that here the weak solutions may contain vacuum though the spatial measure of the set of vacuum has to be small. Moreover, under some additional conditions that prevent the appearance of vacuum states in the data, Hoff [14, 16] also obtained classical solutions.

It should be noted that in the presence of vacuum, the global well-posedness of classical solutions, and the regularity and uniqueness of those weak solutions [7, 14, 27] remains completely open. Indeed, this is a subtle issue since, in general, one would not expect such general results due to Xin's blowup results in [35], where it is shown that in the case where the initial density has compact support, any smooth solution to the Cauchy problem of the full compressible Navier-Stokes system without heat conduction blows up in finite time for any space dimension, and the same holds for the isentropic case (1.1), at least in one dimension and the symmetric two-dimensional case [20]. See also the recent generalizations to the cases for the full compressible Navier-Stokes system with heat conduction [4] and for noncompact but rapidly-decreasing-at-far-field initial densities [30].

In this paper, we will study the global existence and uniqueness of classical solutions to the Cauchy problem for the isentropic compressible Navier-Stokes equations (1.1) in three-dimensional space with smooth initial data that are of small energy but possibly large oscillations with constant state at far field which could be either vacuum ($\tilde\rho = 0$) or nonvacuum ($\tilde\rho > 0$); in particular, the initial density is allowed to vanish, and even have compact support.

Before stating the main results, we explain the notation and conventions used throughout this paper. We denote

$$\int f\,dx = \int_{\mathbb{R}^3} f\,dx.$$

For $1 \le r \le \infty$ and $\beta > 0$, we denote the standard homogeneous and inhomogeneous Sobolev spaces as follows:

$$\begin{cases} L^r = L^r(\mathbb{R}^3),\ D^{k,r} = \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^3) \mid \|\nabla^k u\|_{L^r} < \infty\},\ \|u\|_{D^{k,r}} \triangleq \|\nabla^k u\|_{L^r}, \\ W^{k,r} = L^r \cap D^{k,r},\ H^k = W^{k,2},\ D^k = D^{k,2},\ D^1 = \{u \in L^6 \mid \|\nabla u\|_{L^2} < \infty\}, \\ \dot H^\beta = \{f : \mathbb{R}^3 \to \mathbb{R} \mid \|f\|^2_{\dot H^\beta} = \int |\xi|^{2\beta} |\hat f(\xi)|^2\,d\xi < \infty\}, \end{cases}$$

where $\hat f$ is the Fourier transform of f.

The initial energy is defined as

$$C_0 = \int \left(\frac{1}{2}\rho_0|u_0|^2 + G(\rho_0)\right) dx, \tag{1.6}$$

where G denotes the potential energy density given by

$$G(\rho) \triangleq \rho \int_{\tilde\rho}^{\rho} \frac{P(s) - P(\tilde\rho)}{s^2}\,ds.$$

It is clear that

$$\begin{cases} G(\rho) = \frac{1}{\gamma - 1} P & \text{if } \tilde\rho = 0, \\ G(\rho) \ge c(\bar\rho, \tilde\rho)(\rho - \tilde\rho)^2 & \text{if } \tilde\rho > 0,\ 0 \le \rho \le \bar\rho, \end{cases}$$

for some positive constant $c(\bar\rho, \tilde\rho)$.

Then the main results in this paper can be stated as follows:

THEOREM 1.1. *Assume that* (1.2) *holds. For given numbers* $M > 0$ (*not necessarily small*), $\beta \in (\frac{1}{2}, 1]$, *and* $\bar\rho \ge \tilde\rho + 1$, *suppose that the initial data* (ρ_0, u_0) *satisfy*

$$\begin{gathered} \rho_0|u_0|^2 + G(\rho_0) \in L^1, \quad u_0 \in \dot H^\beta \cap D^1 \cap D^3, \\ (\rho_0 - \tilde\rho, P(\rho_0) - P(\tilde\rho)) \in H^3, \end{gathered} \tag{1.7}$$

$$0 \le \inf \rho_0 \le \sup \rho_0 \le \bar\rho, \quad \|u_0\|_{\dot H^\beta} \le M, \tag{1.8}$$

and the compatibility condition

$$-\mu\triangle u_0-(\mu+\lambda)\nabla\operatorname{div}u_0+\nabla P(\rho_0)=\rho_0 g, \tag{1.9}$$

for some $g\in D^1$ *with* $\rho_0^{1/2}g\in L^2$. *Then there exists a positive constant* ε *depending on* μ, λ, $\tilde\rho$, a, γ, $\bar\rho$, β, *and* M *such that if*

$$C_0\le\varepsilon, \tag{1.10}$$

the Cauchy problem (1.1),(1.3),(1.4) *has a unique global classical solution* (ρ,u) *in* $\mathbb{R}^3\times(0,\infty)$ *satisfying, for any* $0<\tau<T<\infty$,

$$0\le\rho(x,t)\le 2\bar\rho,\quad x\in\mathbb{R}^3,\ t\ge 0, \tag{1.11}$$

$$\begin{cases}(\rho-\tilde\rho,P-P(\tilde\rho))\in C([0,T];H^3),\\ u\in C([0,T];D^1\cap D^3)\cap L^2(0,T;D^4)\cap L^\infty(\tau,T;D^4),\\ u_t\in L^\infty(0,T;D^1)\cap L^2(0,T;D^2)\cap L^\infty(\tau,T;D^2)\cap H^1(\tau,T;D^1),\\ \sqrt{\rho}u_t\in L^\infty(0,T;L^2),\end{cases} \tag{1.12}$$

and the following large-time behavior:

$$\lim_{t\to\infty}\int(|\rho-\tilde\rho|^q+\rho^{1/2}|u|^4+|\nabla u|^2)(x,t)dx=0 \tag{1.13}$$

for all

$$q\in\begin{cases}(2,\infty) & \text{for } \tilde\rho>0,\\ (\gamma,\infty) & \text{for } \tilde\rho=0.\end{cases} \tag{1.14}$$

Similar to our previous studies on the Stokes approximation equations in [26], we can obtain from (1.13) the following large-time behavior of the gradient of the density when vacuum states appear initially and the far-field density is away from vacuum, which is completely in contrast to the classical theory [16, 28].

THEOREM 1.2. *In addition to the conditions of Theorem* 1.1, *assume further that there exists some point* $x_0\in\mathbb{R}^3$ *such that* $\rho_0(x_0)=0$. *Then if* $\tilde\rho>0$, *the unique global classical solution* (ρ,u) *to the Cauchy problem* (1.1),(1.3),(1.4) *obtained in Theorem* 1.1 *has to blow up as* $t\to\infty$ *in the sense that for any* $r>3$,

$$\lim_{t\to\infty}\|\nabla\rho(\cdot,t)\|_{L^r}=\infty.$$

A few remarks are in order:

Remark 1.3. The solution obtained in Theorem 1.1 becomes a classical one for positive time. Although it has small energy, its oscillations can be arbitrarily large. In particular, both interior and far-field vacuum states are allowed.

Remark 1.4. In the case that the far-field density is away from vacuum, i.e., $\tilde\rho>0$, the conclusions in Theorem 1.1 generalize the classical theory of Matsumura and Nishida [28] to the case of large oscillations since in this case, the requirement of small energy, (1.10), is equivalent to smallness of the mean-square norm of $(\rho_0-\tilde\rho,u_0)$. However, though the large-time asymptotic behavior (1.13) is similar to that in [28], our solution may contain vacuum states whose appearance leads to the large-time blowup behavior stated in Theorem 1.2; this is in sharp contrast to the results in [16, 28], where the gradients of the density are suitably small uniformly for all time.

Remark 1.5. When the far-field density is vacuum, i.e., $\tilde{\rho} = 0$, the small energy assumption (1.10) is equivalent to both the kinetic energy and the total pressure being suitably small. There is no requirement on the size of the set of vacuum states. In particular, the initial density may have compact support. Thus, Theorem 1.1 can be regarded as the uniqueness and regularity theory of weak solutions constructed by Lions [27] and Feireisl et al. [7] when they have small initial energy.

Remark 1.6. It is worth noting that the conclusions in Theorem 1.1 for the case of $\tilde{\rho} = 0$ are somewhat surprising since for the isentropic compressible Navier-Stokes equations (1.1), any nontrivial one-dimensional smooth solution with initial compact supported density blows up in finite time [35], and the same holds true for two-dimensional smooth, spherically symmetric solutions [20]. Indeed, the blowup phenomena in both one and two dimensions are based on the fact that the support of the density will not grow in time due to the framework of the Sobolev space H^m. However, in the three-dimensional case, for compactly supported ρ_0 and smooth, spherically symmetric (ρ_0, u_0), satisfying $u_0 \in H^m$ and the conditions of Theorem 1.1, simple calculations yield that $u(\cdot, t) \in H^m$ for any finite time t. Therefore, it seems that it is the slower decay of the velocity field for large values of the spatial variable x due to the higher spatial dimensions that lead to the global existence of smooth solutions. In fact, if the smooth solution and its spatial derivatives decay fast enough for large values of the spatial variable x, it will blow up in finite time (see [30] for details).

Remark 1.7. When the far-field density is vacuum, i.e., $\tilde{\rho} = 0$, for large initial energy and large potential external forces, under the condition that $\gamma > \frac{3}{2}$, Feireisl and Petzeltová [8] proved that the density of any global weak solution converges to the steady state density in L^q-space for some q as time goes to infinity if there exists a unique steady state. Thus, if the initial energy is suitably small, for any $\gamma > 1$, we can slightly improve the large-time asymptotic behavior in [8] to (1.13) for zero external forces.

Remark 1.8. It should be emphasized that in Theorem 1.1, the viscosity coefficients are only assumed to satisfy the physical conditions (1.2), while the theory on weak small-energy solutions, developed in [14, 16], requires the additional assumption (1.5), which is crucial in establishing the time-independent upper bound for the density in the arguments in [14, 16].

Remark 1.9. For the incompressible Navier-Stokes system, a lot of results on the global well-posedness in scaling-invariant spaces are available [9, 22, 24]. In particular, Fujita and Kato [9] and Kato [22] proved that the system is globally well-posed for small initial data in the homogeneous Sobolev spaces $\dot{H}^{1/2}$ or in L^3.

In our case, since the initial energy is small, we need the boundedness assumptions on the $\dot{H}^\beta$-norm ($\beta > \frac{1}{2}$) of the initial velocity. It should be noted here that $\dot{H}^\beta \hookrightarrow L^{6/(3-2\beta)}$ and $6/(3-2\beta) > 3$ for $\beta > \frac{1}{2}$, which implies that, compared with the results in [9, 22], our conditions on the initial velocity may be optimal under the smallness conditions on the initial energy.

Remark 1.10. In this paper, we only consider the isentropic case, i.e., $P(\rho) = a\rho^\gamma$, for $a > 0$ and $\gamma > 1$. In fact, our results still hold for general barotropic flow $P(\rho)$ with $P(\rho) \in C^3([0, 2\overline{\rho}])$ satisfying

$$P(0) = 0, \quad (\rho - \widetilde{\rho})(P(\rho) - P(\widetilde{\rho})) > 0 \quad \text{for } \rho \in [0, 2\overline{\rho}],\ \rho \neq \widetilde{\rho},$$

and

$$\begin{cases} P'(\widetilde{\rho}) = 0 & \text{if } \widetilde{\rho} = 0, \\ P'(\widetilde{\rho}) > 0 & \text{if } \widetilde{\rho} > 0. \end{cases}$$

Remark 1.11. Similar ideas can be applied to study the case on the bounded domain. This will be reported in a forthcoming paper [19].

We now comment on the analysis of this paper. Note that for initial data in the class satisfying (1.7)–(1.9) except $u_0 \in \dot{H}^\beta$, the local existence and uniqueness of classical solutions to the Cauchy problem, (1.1)–(1.4), have been established recently in [5]. Thus, to extend the classical solution globally in time, one needs global a priori estimates on smooth solutions to (1.1)–(1.4) in suitable higher norms. Some of the main new difficulties are due to the appearance of vacuum and that there are no other constraints on the viscosity coefficients beyond the physical conditions (1.2).

It turns out that the key issue in this paper is to derive both the time-independent upper bound for the density and the time-dependent higher norm estimates of the smooth solution (ρ, u). We start with the basic energy estimate and the initial layer analysis, and succeed in deriving an estimate on the spatial weighted L^3-norm of the velocity, and then obtain the spatial weighted mean estimates on both the gradient and the material derivatives of the velocity. This is achieved by modifying the basic elegant estimates on the material derivatives of the velocity developed by Hoff [11, 13, 14] in the theory of small-energy weak solutions with nonvacuum far fields and an interpolation argument. Then we are able to obtain the desired estimates in the $L^1(0, \min\{1, T\}; L^\infty(\mathbb{R}^3))$-norm and the time-independent ones on the $L^{8/3}(\min\{1, T\}, T; L^\infty(\mathbb{R}^3))$-norm of the effective viscous flux (see (2.5) for the definition).

It follows from these key estimates and Zlotnik's inequality (see Lemma 2.4) that the density admits a time-uniform upper bound that is the key for global estimates of classical solutions. This approach to estimate a uniform upper bound for the density is motivated by our previous analysis on the two-dimensional Stokes approximation equations in [26]. The next main step is to bound the gradients of the density and the velocity. Motivated by our recent studies [17, 18, 21] on the blowup criteria of classical (or strong) solutions to (1.1), such bounds can be obtained by solving a logarithm Gronwall inequality based on a Beal-Kato-Majda-type inequality (see Lemma 2.5) and the a priori estimates we have just derived; moreover, such a derivation yields simultaneously the bound for the $L^1(0,T;L^\infty(\mathbb{R}^3))$-norm of the gradient of the velocity; see Lemma 3.7 and its proof. It should be noted here that we do not require smallness of the gradient of the initial density, which prevents the appearance of vacuum [16, 28]. Finally, with these a priori estimates on the gradients of the density and the velocity at hand, one can estimate the higher-order derivatives by using the same arguments as in [21] to obtain the desired results.

The rest of the paper is organized as follows: In Section 2, we collect some elementary facts and inequalities that will be needed in later analysis. Section 3 is devoted to deriving the necessary a priori estimates on classical solutions that are needed to extend the local solution to all time. Then finally, the main results, Theorem 1.1 and Theorem 1.2, are proved in Section 4.

2 Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used frequently later.

We start with the local existence and uniqueness of classical solutions when the initial density may not be positive and may vanish in an open set.

LEMMA 2.1 ([5]). *For $\widetilde{\rho} \geq 0$, assume that the initial data $(\rho_0 \geq 0, u_0)$ satisfy* (1.7)–(1.9) *except $u_0 \in \dot{H}^\beta$. Then there exist a small-time $T_* > 0$ and a unique classical solution (ρ, u) to the Cauchy problem* (1.1),(1.3),(1.4) *on $\mathbb{R}^3 \times (0, T_*]$ such that*

$$\begin{cases} (\rho-\widetilde{\rho}, P-P(\widetilde{\rho})) \in C([0,T_*];H^3), \\ u \in C([0,T_*];D^1\cap D^3)\cap L^2(0,T_*;D^4), \\ u_t \in L^\infty(0,T_*;D^1)\cap L^2(0,T_*;D^2),\ \sqrt{\rho}u_{tt} \in L^2(0,T_*;L^2), \\ \sqrt{t}\,u \in L^\infty(0,T_*;D^4),\ \sqrt{t}\,u_t \in L^\infty(0,T_*;D^2),\ \sqrt{t}\,u_{tt} \in L^2(0,T_*;D^1), \\ t^{1/2}\sqrt{\rho}u_{tt} \in L^\infty(0,T_*;L^2),\ tu_t \in L^\infty(0,T_*;D^3), \\ tu_{tt} \in L^\infty(0,T_*;D^1)\cap L^2(0,T_*;D^2),\ t\sqrt{\rho}u_{ttt} \in L^2(0,T_*;L^2), \\ t^{3/2}u_{tt} \in L^\infty(0,T_*;D^2),\ t^{3/2}u_{ttt} \in L^2(0,T_*;D^1), \\ t^{3/2}\sqrt{\rho}u_{ttt} \in L^\infty(0,T_*;L^2). \end{cases} \tag{2.1}$$

Next, the following well-known Gagliardo-Nirenberg inequality will be used frequently later (see [25]).

LEMMA 2.2 (Gagliardo-Nirenberg). *For $p \in [2,6]$, $q \in (1,\infty)$, and $r \in (3,\infty)$, there exists some generic constant $C > 0$ that may depend on q and r such that for*

$f \in H^1(\mathbb{R}^3)$ *and* $g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3)$, *we have*

$$\|f\|_{L^p}^p \le C\|f\|_{L^2}^{(6-p)/2}\|\nabla f\|_{L^2}^{(3p-6)/2}, \tag{2.2}$$

$$\|g\|_{C(\overline{\mathbb{R}^3})} \le C\|g\|_{L^q}^{q(r-3)/(3r+q(r-3))}\|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \tag{2.3}$$

We now state some elementary estimates that follow from (2.2) and the standard L^p-estimate for the following elliptic system derived from the momentum equations in (1.1):

$$\triangle F = \operatorname{div}(\rho\dot{u}), \quad \mu\triangle\omega = \nabla\times(\rho\dot{u}), \tag{2.4}$$

where

$$\dot{f} \triangleq f_t + u\cdot\nabla f, \quad F \triangleq (2\mu+\lambda)\operatorname{div} u - P(\rho) + P(\widetilde{\rho}), \quad \omega \triangleq \nabla\times u, \tag{2.5}$$

are the material derivative of f, the effective viscous flux, and the vorticity, respectively.

LEMMA 2.3. *Let* (ρ, u) *be a smooth solution of* (1.1),(1.3). *Then there exists a generic positive constant* C *depending only on* μ *and* λ *such that for any* $p \in [2,6]$

$$\|\nabla F\|_{L^p} + \|\nabla\omega\|_{L^p} \le C\|\rho\dot{u}\|_{L^p}, \tag{2.6}$$

$$\begin{aligned}&\|F\|_{L^p} + \|\omega\|_{L^p} \le \\ &\quad C\|\rho\dot{u}\|_{L^2}^{(3p-6)/(2p)}(\|\nabla u\|_{L^2} + \|P - P(\widetilde{\rho})\|_{L^2})^{(6-p)/(2p)},\end{aligned} \tag{2.7}$$

$$\|\nabla u\|_{L^p} \le C(\|F\|_{L^p} + \|\omega\|_{L^p}) + C\|P - P(\widetilde{\rho})\|_{L^p}, \tag{2.8}$$

$$\|\nabla u\|_{L^p} \le C\|\nabla u\|_{L^2}^{(6-p)/(2p)}(\|\rho\dot{u}\|_{L^2} + \|P - P(\widetilde{\rho})\|_{L^6})^{(3p-6)/(2p)}. \tag{2.9}$$

PROOF. The standard L^p-estimate for elliptic system (2.4) yields (2.6) directly, which, together with (2.2) and (2.5), gives (2.7).

Note that $-\Delta u = -\nabla \operatorname{div} u + \nabla\times\omega$, which implies that

$$\nabla u = -\nabla(-\Delta)^{-1}\nabla\operatorname{div} u + \nabla(-\Delta)^{-1}\nabla\times\omega.$$

Thus the standard L^p-estimate shows that

$$\|\nabla u\|_{L^p} \le C(\|\operatorname{div} u\|_{L^p} + \|\omega\|_{L^p}) \quad \text{for } p \in [2,6],$$

which, together with (2.5), gives (2.8). Now (2.9) follows from (2.2), (2.8), and (2.6). □

Next, the following Zlotnik inequality will be used to get the uniform (in time) upper bound of the density ρ.

LEMMA 2.4 ([36]). *Let the function* y *satisfy*

$$y'(t) = g(y) + b'(t) \quad \text{on } [0,T], \ y(0) = y^0,$$

with $g \in C(R)$ *and* $y, b \in W^{1,1}(0,T)$. *If* $g(\infty) = -\infty$ *and*

$$b(t_2) - b(t_1) \le N_0 + N_1(t_2 - t_1) \tag{2.10}$$

for all $0 \le t_1 < t_2 \le T$ *with some* $N_0 \ge 0$ *and* $N_1 \ge 0$, *then*

$$y(t) \le \max\{y^0, \overline{\zeta}\} + N_0 < \infty \quad \text{on } [0,T],$$

where $\overline{\zeta}$ *is a constant such that*

$$g(\zeta) \le -N_1 \quad \text{for } \zeta \ge \overline{\zeta}. \tag{2.11}$$

Finally, the following Beale-Kato-Majda-type inequality, which can be found in [18] and was first proved in [1] when $\operatorname{div} u \equiv 0$, will be used later to estimate $\|\nabla u\|_{L^\infty}$ and $\|\nabla \rho\|_{L^2 \cap L^6}$.

LEMMA 2.5 ([18]). *For* $3 < q < \infty$, *there is a constant* $C(q)$ *such that the following estimate holds for all* $\nabla u \in L^2(\mathbb{R}^3) \cap D^{1,q}(\mathbb{R}^3)$:

$$\begin{aligned} \|\nabla u\|_{L^\infty(\mathbb{R}^3)} \le {} & C(\|\operatorname{div} u\|_{L^\infty(\mathbb{R}^3)} + \|\omega\|_{L^\infty(\mathbb{R}^3)}) \log(e + \|\nabla^2 u\|_{L^q(\mathbb{R}^3)}) \\ & + C\|\nabla u\|_{L^2(\mathbb{R}^3)} + C. \end{aligned} \tag{2.12}$$

3 A Priori Estimates

In this section, we will establish some necessary a priori bounds for smooth solutions to the Cauchy problem (1.1),(1.3),(1.4) to extend the local classical solution guaranteed by Lemma 2.1. Thus, let $T > 0$ be a fixed time and (ρ, u) be the smooth solution to (1.1),(1.3),(1.4) on $\mathbb{R}^3 \times (0,T]$ in the class (2.1) with smooth initial data (ρ_0, u_0) satisfying (1.7)–(1.9). To estimate this solution, we set $\sigma(t) \triangleq \min\{1, t\}$ and define

$$A_1(T) \triangleq \sup_{t \in [0,T]} \left(\sigma \|\nabla u\|_{L^2}^2\right) + \int_0^T \int \sigma \rho |\dot{u}|^2 \, dx \, dt, \tag{3.1}$$

$$A_2(T) \triangleq \sup_{t \in [0,T]} \sigma^3 \int \rho |\dot{u}|^2 \, dx + \int_0^T \int \sigma^3 |\nabla \dot{u}|^2 \, dx \, dt, \tag{3.2}$$

and

$$A_3(T) \triangleq \sup_{0 \le t \le T} \int \rho |u|^3 (x,t) dx.$$

We have the following key a priori estimates on (ρ, u).

PROPOSITION 3.1. *Under the conditions of Theorem* 1.1, *for*

$$\delta_0 \triangleq \frac{2\beta - 1}{4\beta} \in (0, \tfrac{1}{4}], \tag{3.3}$$

there exists some positive constant ε *depending on* μ, λ, $\tilde{\rho}$, a, γ, $\overline{\rho}$, β, *and* M *such that if* (ρ, u) *is a smooth solution of* (1.1),(1.3),(1.4) *on* $\mathbb{R}^3 \times (0,T]$ *satisfying*

$$\sup_{\mathbb{R}^3 \times [0,T]} \rho \le 2\overline{\rho}, \quad A_1(T) + A_2(T) \le 2C_0^{1/2}, \quad A_3(\sigma(T)) \le 2C_0^{\delta_0}, \tag{3.4}$$

the following estimates hold:

$$\sup_{\mathbb{R}^3\times[0,T]} \rho \le \frac{7\overline{\rho}}{4}, \quad A_1(T) + A_2(T) \le C_0^{1/2}, \quad A_3(\sigma(T)) \le C_0^{\delta_0}, \tag{3.5}$$

provided $C_0 \le \varepsilon$.

PROOF. Proposition 3.1 is an easy consequence of Lemmas 3.4, 3.5, and 3.6 below. □

In the following, we will use the convention that C denotes a generic positive constant depending on μ, λ, $\widetilde{\rho}$, a, γ, $\overline{\rho}$, β, and M, and we write $C(\alpha)$ to emphasize that C depends on α.

We start with the following standard energy estimate for (ρ, u) and preliminary L^2-bounds for ∇u and $\rho\dot{u}$.

LEMMA 3.2. *Let* (ρ, u) *be a smooth solution of* (1.1),(1.3),(1.4) *on* $\mathbb{R}^3 \times (0, T]$ *with* $0 \le \rho(x,t) \le 2\overline{\rho}$. *Then there is a positive constant* $C = C(\overline{\rho})$ *such that*

$$\begin{aligned} &\sup_{0\le t\le T} \int \left(\frac{1}{2}\rho|u|^2 + G(\rho)\right)dx \\ &\quad + \int_0^T \int (\mu|\nabla u|^2 + (\lambda+\mu)(\operatorname{div} u)^2)\,dx\,dt \le C_0, \end{aligned} \tag{3.6}$$

$$A_1(T) \le CC_0 + C\int_0^T\int \sigma|\nabla u|^3\,dx\,dt, \tag{3.7}$$

and

$$A_2(T) \le CC_0 + CA_1(T) + C\int_0^T\int \sigma^3|\nabla u|^4\,dx\,dt. \tag{3.8}$$

PROOF. Multiplying the first equation in (1.1) by $G'(\rho)$ and the second by u^j and integrating, then applying the far-field condition (1.3), one easily shows the energy inequality (3.6).

The proof of (3.7) and (3.8) is due to Hoff [11]. For $m \ge 0$, multiplying $(1.1)_2$ by $\sigma^m\dot{u}$ and then integrating the resulting equality over $\mathbb{R}^3$ leads to

$$\begin{aligned} &\int \sigma^m\rho|\dot{u}|^2dx \\ &\quad = \int \left(-\sigma^m\dot{u}\cdot\nabla P + \mu\sigma^m\triangle u\cdot\dot{u} + (\lambda+\mu)\sigma^m\nabla\operatorname{div} u\cdot\dot{u}\right)dx \\ &\quad \triangleq \sum_{i=1}^3 M_i. \end{aligned} \tag{3.9}$$

Using $(1.1)_1$ and integrating by parts gives

$$
\begin{aligned}
M_1 &= -\int \sigma^m \dot{u} \cdot \nabla P \, dx \\
&= \int \big(\sigma^m (\operatorname{div} u)_t (P - P(\widetilde{\rho})) - \sigma^m (u \cdot \nabla u) \cdot \nabla P\big) dx \\
&= \left(\int \sigma^m \operatorname{div} u (P - P(\widetilde{\rho})) dx \right)_t \\
&\quad - m\sigma^{m-1}\sigma' \int \operatorname{div} u (P - P(\widetilde{\rho})) dx \\
&\quad + \int \sigma^m \big(P' \rho (\operatorname{div} u)^2 - P(\operatorname{div} u)^2 + P \partial_i u^j \partial_j u^i\big) dx \\
&\le \left(\int \sigma^m \operatorname{div} u (P - P(\widetilde{\rho})) dx \right)_t \\
&\quad + m\sigma^{m-1}\sigma' \|P - P(\widetilde{\rho})\|_{L^2} \|\nabla u\|_{L^2} + C(\overline{\rho}) \|\nabla u\|_{L^2}^2 \\
&\le \left(\int \sigma^m \operatorname{div} u (P - P(\widetilde{\rho})) dx \right)_t + C(\overline{\rho}) \|\nabla u\|_{L^2}^2 \\
&\quad + C(\overline{\rho}) m^2 \sigma^{2(m-1)} \sigma' C_0.
\end{aligned} \tag{3.10}
$$

Integration by parts implies

$$
\begin{aligned}
M_2 &= \int \mu \sigma^m \triangle u \cdot \dot{u} dx \\
&= -\frac{\mu}{2} \big(\sigma^m \|\nabla u\|_{L^2}^2\big)_t + \frac{\mu m}{2} \sigma^{m-1} \sigma' \|\nabla u\|_{L^2}^2 \\
&\quad - \mu \sigma^m \int \partial_i u^j \partial_i (u^k \partial_k u^j) dx \\
&\le -\frac{\mu}{2} \big(\sigma^m \|\nabla u\|_{L^2}^2\big)_t + C m \sigma^{m-1} \|\nabla u\|_{L^2}^2 + C \int \sigma^m |\nabla u|^3 \, dx,
\end{aligned} \tag{3.11}
$$

and similarly,

$$
\begin{aligned}
M_3 &= -\frac{\lambda + \mu}{2} \big(\sigma^m \|\operatorname{div} u\|_{L^2}^2\big)_t + \frac{m(\lambda + \mu)}{2} \sigma^{m-1} \|\operatorname{div} u\|_{L^2}^2 \\
&\quad - (\lambda + \mu) \sigma^m \int \operatorname{div} u \operatorname{div}(u \cdot \nabla u) dx \\
&\le -\frac{\lambda + \mu}{2} \big(\sigma^m \|\operatorname{div} u\|_{L^2}^2\big)_t \\
&\quad + C m \sigma^{m-1} \|\nabla u\|_{L^2}^2 + C \int \sigma^m |\nabla u|^3 \, dx.
\end{aligned} \tag{3.12}
$$

Combining (3.9)–(3.12) leads to

$$
\begin{aligned}
&(\sigma^m B(t))' + \int \sigma^m \rho |\dot u|^2\,dx \\
&\le (Cm\sigma^{m-1} + C(\bar\rho))\|\nabla u\|_{L^2}^2 + C(\bar\rho) m^2 \sigma^{2(m-1)} \sigma' C_0 \\
&\quad + C \int \sigma^m |\nabla u|^3\,dx,
\end{aligned} \tag{3.13}
$$

where

$$
\begin{aligned}
B(t) &\triangleq \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + \frac{(\lambda+\mu)}{2}\|\operatorname{div} u\|_{L^2}^2 - \int \operatorname{div} u (P - P(\tilde\rho))dx \\
&\ge \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + \frac{(\lambda+\mu)}{2}\|\operatorname{div} u\|_{L^2}^2 - C C_0^{1/2}\|\operatorname{div} u\|_{L^2} \\
&\ge \frac{\mu}{4}\|\nabla u\|_{L^2}^2 + \frac{(\lambda+\mu)}{2}\|\operatorname{div} u\|_{L^2}^2 - C C_0.
\end{aligned} \tag{3.14}
$$

Integrating (3.13) over $(0, T)$, choosing $m = 1$, and using (3.14), one gets (3.7).

Next, for $m \ge 0$, operating $\sigma^m \dot u^j [\partial/\partial t + \operatorname{div}(u\,\cdot)]$ on $(1.1)_2^j$, summing with respect to j, and integrating the resulting equation over $\mathbb{R}^3$, one obtains after integration by parts

$$
\begin{aligned}
&\left(\frac{\sigma^m}{2}\int \rho|\dot u|^2\,dx\right)_t - \frac{m}{2}\sigma^{m-1}\sigma' \int \rho |\dot u|^2\,dx \\
&= -\int \sigma^m \dot u^j [\partial_j P_t + \operatorname{div}(\partial_j P u)]dx \\
&\quad + \mu \int \sigma^m \dot u^j [\triangle u_t^j + \operatorname{div}(u \triangle u^j)]dx \\
&\quad + (\lambda+\mu) \int \sigma^m \dot u^j [\partial_t \partial_j \operatorname{div} u + \operatorname{div}(u \partial_j \operatorname{div} u)]dx \\
&\triangleq \sum_{i=1}^3 N_i.
\end{aligned} \tag{3.15}
$$

It follows from integration by parts and using equation $(1.1)_1$ that

$$
\begin{aligned}
N_1 &= -\int \sigma^m \dot u^j [\partial_j P_t + \operatorname{div}(\partial_j P u)]dx \\
&= \int \sigma^m \big[-P'\rho \operatorname{div} u \partial_j \dot u^j + \partial_k(\partial_j \dot u^j u^k) P - P \partial_j(\partial_k \dot u^j u^k)\big]dx \\
&\le C(\bar\rho)\sigma^m \|\nabla u\|_{L^2}\|\nabla \dot u\|_{L^2} \\
&\le \delta \sigma^m \|\nabla \dot u\|_{L^2}^2 + C(\bar\rho, \delta)\sigma^m \|\nabla u\|_{L^2}^2.
\end{aligned} \tag{3.16}
$$

Integration by parts leads to

$$
\begin{aligned}
N_2 &= \mu \int \sigma^m \dot{u}^j [\triangle u_t^j + \operatorname{div}(u \triangle u^j)] dx \\
&= -\mu \int \sigma^m \big[|\nabla \dot{u}|^2 + \partial_i \dot{u}^j \partial_k u^k \partial_i u^j \\
&\qquad - \partial_i \dot{u}^j \partial_i u^k \partial_k u^j - \partial_i u^j \partial_i u^k \partial_k \dot{u}^j \big] dx \\
&\le -\frac{3\mu}{4} \int \sigma^m |\nabla \dot{u}|^2 \, dx + C \int \sigma^m |\nabla u|^4 \, dx.
\end{aligned} \tag{3.17}
$$

Similarly,

$$
N_3 \le -\frac{\mu + \lambda}{2} \int \sigma^m (\operatorname{div} \dot{u})^2 \, dx + C \int \sigma^m |\nabla u|^4 \, dx. \tag{3.18}
$$

Substituting (3.16)–(3.18) into (3.15) shows that for δ suitably small, it holds that

$$
\begin{aligned}
&\left(\sigma^m \int \rho |\dot{u}|^2 \, dx \right)_t + \mu \int \sigma^m |\nabla \dot{u}|^2 \, dx + (\mu + \lambda) \int \sigma^m (\operatorname{div} \dot{u})^2 \, dx \\
&\qquad \le m \sigma^{m-1} \sigma' \int \rho |\dot{u}|^2 \, dx + C \sigma^m \|\nabla u\|_{L^4}^4 + C(\overline{\rho}) \sigma^m \|\nabla u\|_{L^2}^2.
\end{aligned} \tag{3.19}
$$

Taking $m = 3$ in (3.19) and noticing that

$$
3 \int_0^T \sigma^2 \sigma' \int \rho |\dot{u}|^2 \, dx \, dt \le C A_1(T),
$$

we immediately obtain (3.8) after integrating (3.19) over $(0, T)$. The proof of Lemma 3.2 is completed. □

Next, the following lemma will play important roles in the estimates on both $A_i(\sigma(T))$ $(i = 1, 3)$ and the uniform upper bound of the density for small time.

LEMMA 3.3. *Let* (ρ, u) *be a smooth solution of* (1.1),(1.3),(1.4) *on* $\mathbb{R}^3 \times (0, T]$ *satisfying* (3.4). *Then there exist positive constants* K *and* ε_0, *both depending only on* μ, λ, $\widetilde{\rho}$, a, γ, $\overline{\rho}$, β, *and* M, *such that*

$$
\sup_{0 \le t \le \sigma(T)} t^{1-\beta} \|\nabla u\|_{L^2}^2 + \int_0^{\sigma(T)} t^{1-\beta} \int \rho |\dot{u}|^2 \, dx \, dt \le K(\overline{\rho}, M), \tag{3.20}
$$

$$
\sup_{0 \le t \le \sigma(T)} t^{2-\beta} \int \rho |\dot{u}|^2 \, dx + \int_0^{\sigma(T)} t^{2-\beta} \int |\nabla \dot{u}|^2 \, dx \, dt \le K(\overline{\rho}, M), \tag{3.21}
$$

provided $C_0 \le \varepsilon_0$.

PROOF. As in [13], we define w_1 and w_2 to be the solution to

$$
L w_1 = 0, \quad w_1(x, 0) = w_{10}(x), \tag{3.22}
$$

and

$$Lw_2 = -\nabla P(\rho), \quad w_2(x,0) = 0, \tag{3.23}$$

respectively, with L being the linear differential operator defined by

$$\begin{aligned}(Lw)^j &\triangleq \rho w_t^j + \rho u\cdot\nabla w^j - (\mu\Delta w^j + (\mu+\lambda)\operatorname{div} w_{x_j})\\ &= \rho\dot{w}^j - (\mu\Delta w^j + (\mu+\lambda)\operatorname{div} w_{x_j}), \quad j = 1,2,3.\end{aligned}$$

Straightforward energy estimates show that

$$\sup_{0\le t\le\sigma(T)}\int \rho|w_1|^2\,dx + \int_0^{\sigma(T)}\int |\nabla w_1|^2\,dx\,dt \le C(\overline{\rho})\int |w_{10}|^2\,dx, \tag{3.24}$$

and

$$\sup_{0\le t\le\sigma(T)}\int \rho|w_2|^2\,dx + \int_0^{\sigma(T)}\int |\nabla w_2|^2\,dx\,dt \le C(\overline{\rho})C_0. \tag{3.25}$$

It follows from (3.22) and a standard L^2-estimate for elliptic systems that

$$\|\nabla w_1\|_{L^6} \le C\|\nabla^2 w_1\|_{L^2} \le C\|\rho\dot{w}_1\|_{L^2}. \tag{3.26}$$

Multiplying (3.22) by w_{1t} and integrating the resulting equality over $\mathbb{R}^3$, we get by (3.26) and $(3.4)_3$ that

$$\begin{aligned}&\frac{1}{2}\left(\mu\|\nabla w_1\|_{L^2}^2 + (\mu+\lambda)\|\operatorname{div} w_1\|_{L^2}^2\right)_t + \int \rho|\dot{w}_1|^2\,dx\\ &= \int \rho\dot{w}_1(u\cdot\nabla w_1)dx\\ &\le C(\overline{\rho})\left(\int \rho|\dot{w}_1|^2\,dx\right)^{1/2}\left(\int \rho|u|^3\,dx\right)^{1/3}\|\nabla w_1\|_{L^6}\\ &\le C(\overline{\rho})C_0^{\delta_0/3}\int \rho|\dot{w}_1|^2\,dx,\end{aligned}$$

which, together with Gronwall's inequality and (3.24), gives

$$\sup_{0\le t\le\sigma(T)}\|\nabla w_1\|_{L^2}^2 + \int_0^{\sigma(T)}\int \rho|\dot{w}_1|^2\,dx\,dt \le C\|\nabla w_{10}\|_{L^2}^2, \tag{3.27}$$

$$\sup_{0\le t\le\sigma(T)} t\|\nabla w_1\|_{L^2}^2 + \int_0^{\sigma(T)} t\int \rho|\dot{w}_1|^2\,dx\,dt \le C\|w_{10}\|_{L^2}^2, \tag{3.28}$$

provided $C_0 \le \varepsilon_{01} \triangleq (2C(\overline{\rho}))^{-3/\delta_0}$.

Since the solution operator $w_{10} \mapsto w_1(\cdot,t)$ is linear, by the standard Stein-Weiss interpolation argument [2], one can deduce from (3.27) and (3.28) that for any $\theta\in[\beta,1]$,

$$\sup_{0\le t\le\sigma(T)} t^{1-\theta}\|\nabla w_1\|_{L^2}^2 + \int_0^{\sigma(T)} t^{1-\theta}\int \rho|\dot{w}_1|^2\,dx\,dt \le C\|w_{10}\|_{\dot{H}^\theta}^2, \tag{3.29}$$

with a uniform constant C independent of θ.

Next, we estimate w_2. It follows in a similar way to (2.6) and (2.8) that

$$\begin{cases}\|\nabla((2\mu+\lambda)\operatorname{div} w_2-(P-P(\widetilde{\rho})))\|_{L^2}\le C\|\rho\dot{w}_2\|_{L^2},\\ \|\nabla w_2\|_{L^6}\le C(\|\rho\dot{w}_2\|_{L^2}+\|P-P(\widetilde{\rho})\|_{L^6}).\end{cases}\tag{3.30}$$

Multiplying (3.23) by w_{2t}, integrating the resultant equation over $\mathbb{R}^3$, and using (3.30), one has

$$\begin{aligned}
&\frac{1}{2}\left(\mu\|\nabla w_2\|_{L^2}^2+(\mu+\lambda)\|\operatorname{div} w_2\|_{L^2}^2-2\int(P-P(\widetilde{\rho}))\operatorname{div} w_2\,dx\right)_t\\
&\quad+\int\rho|\dot{w}_2|^2\,dx\\
&\qquad=\int\rho\dot{w}_2(u\cdot\nabla w_2)dx-\int P_t\operatorname{div} w_2\,dx\\
&\qquad\le C(\overline{\rho})\left(\int\rho|\dot{w}_2|^2\,dx\right)^{1/2}\left(\int\rho|u|^3\,dx\right)^{1/3}\|\nabla w_2\|_{L^6}\\
&\qquad\quad+\int\operatorname{div} w_2\operatorname{div}((P-P(\widetilde{\rho}))u)dx\\
&\qquad\quad+\int(P(\widetilde{\rho})+(\gamma-1)P)\operatorname{div} u\operatorname{div} w_2\,dx\\
&\qquad\le C(\overline{\rho})C_0^{\delta_0/3}\left(\int\rho|\dot{w}_2|^2\,dx\right)^{1/2}(\|\rho^{1/2}\dot{w}_2\|_{L^2}+\|P-P(\widetilde{\rho})\|_{L^6})\\
&\qquad\quad-\int(P-P(\widetilde{\rho}))u\cdot\nabla\left(\operatorname{div} w_2-\frac{P-P(\widetilde{\rho})}{2\mu+\lambda}\right)dx\\
&\qquad\quad+\frac{1}{2(2\mu+\lambda)}\int(P-P(\widetilde{\rho}))^2\operatorname{div} u\,dx+C\|\nabla u\|_{L^2}^2+C\|\nabla w_2\|_{L^2}^2\\
&\qquad\le C(\overline{\rho})C_0^{\delta_0/3}\int\rho|\dot{w}_2|^2\,dx+CC_0^{1/3}\\
&\qquad\quad+C\|P-P(\widetilde{\rho})\|_{L^3}\|u\|_{L^6}\|\rho^{1/2}\dot{w}_2\|_{L^2}\\
&\qquad\quad+C\|P-P(\widetilde{\rho})\|_{L^4}^4+C\|\nabla u\|_{L^2}^2+C\|\nabla w_2\|_{L^2}^2\\
&\qquad\le C(\overline{\rho})C_0^{\delta_0/3}\int\rho|\dot{w}_2|^2\,dx+CC_0^{1/3}+C\|\nabla u\|_{L^2}^2+C\|\nabla w_2\|_{L^2}^2,
\end{aligned}$$

which, together with (3.25) and Gronwall's inequality, gives

$$\sup_{0\le t\le\sigma(T)}\|\nabla w_2\|_{L^2}^2+\int_0^{\sigma(T)}\int\rho|\dot{w}_2|^2\,dx\,dt\le CC_0^{1/3},\tag{3.31}$$

provided $C_0 \le \varepsilon_{02} \triangleq (2C(\bar\rho))^{-3/\delta_0}$. Taking $w_{10} = u_0$ so that $w_1 + w_2 = u$, we then conclude from (3.29) and (3.31) that for any $\theta \in [\beta, 1]$,

$$\sup_{0\le t\le\sigma(T)} t^{1-\theta}\|\nabla u\|_{L^2}^2 + \int_0^{\sigma(T)} t^{1-\theta}\int \rho|\dot u|^2\,dx\,dt \le C\|u_0\|_{\dot H^\theta}^2 + CC_0^{1/3}, \tag{3.32}$$

provided $C_0 \le \varepsilon_0 \triangleq \min\{\varepsilon_{01}, \varepsilon_{02}\}$. Thus, (3.20) follows from (3.32) directly.

To prove (3.21), we take $m = 2 - \beta$ in (3.19) to obtain, after integrating (3.19) over $(0, \sigma(T))$ and using (3.32) and (2.9), that

$$\begin{aligned}
&\sup_{0\le t\le\sigma(T)} t^{2-\beta}\int \rho|\dot u|^2\,dx + \int_0^{\sigma(T)} t^{2-\beta}\int |\dot u|^2\,dx\,dt \\
&\le C\int_0^{\sigma(T)} t^{2-\beta}\|u\|_{L^4}^4\,dt + C(\bar\rho, M) \\
&\le C\int_0^{\sigma(T)} t^{2-\beta}\|u\|_{L^2}\left(\|\rho\dot u\|_{L^2}^3 + \|P - P(\tilde\rho)\|_{L^6}^3\right)dt + C(\bar\rho, M) \\
&\le C\int_0^{\sigma(T)} t^{(2\beta-1)/2}\big(t^{1-\beta}\|u\|_{L^2}^2\big)^{1/2}\big(t^{2-\beta}\|\rho^{1/2}\dot u\|_{L^2}^2\big)^{1/2}\big(t^{1-\beta}\|\rho^{1/2}\dot u\|_{L^2}^2\big)dt \\
&\quad + C(\bar\rho, M) \\
&\le C(\bar\rho, M)\left(\sup_{0\le t\le\sigma(T)} t^{2-\beta}\int \rho|\dot u|^2\,dx\right)^{1/2} + C(\bar\rho, M),
\end{aligned}$$

which implies (3.21). Thus, we finish the proof of Lemma 3.3. □

The following lemma will give an estimate of $A_3(\sigma(T))$.

LEMMA 3.4. *If (ρ, u) is a smooth solution of* (1.1),(1.3),(1.4) *on $\mathbb{R}^3 \times (0, T]$ satisfying* (3.4)*, there exists a positive constant ε_1 depending on μ, λ, $\tilde\rho$, a, γ, $\bar\rho$, β, and M such that the following estimate holds for δ_0 defined by* (3.3):

$$\sup_{0\le t\le\sigma(T)} \int \rho|u|^3(x,t)\,dx \le C_0^{\delta_0}, \tag{3.33}$$

provided $C_0 \le \varepsilon_1$.

PROOF. Multiplying $(1.1)_2$ by $3|u|u$ and integrating the resulting equation over $\mathbb{R}^3$, we obtain by (2.9) that

$$\begin{aligned}
&\frac{d}{dt}\int \rho|u|^3\,dx \\
&\le C\int |u|\,|\nabla u|^2\,dx + C\int |P - P(\tilde\rho)|\,|u|\,|u|dx \\
&\le C\|u\|_{L^6}\,\|u\|_{L^2}^{3/2}\,\|u\|_{L^6}^{1/2} + C\|P - P(\tilde\rho)\|_{L^3}\,\|u\|_{L^6}\,\|u\|_{L^2} \le
\end{aligned}$$

$$
\begin{aligned}
&\le C\|u\|_{L^2}^{5/2}(\|\rho\dot u\|_{L^2}+\|P-P(\tilde\rho)\|_{L^6})^{1/2}+CC_0^{1/6}\|u\|_{L^2}^2\\
&\le C\|u\|_{L^2}^{5/2}\big(\|\rho\dot u\|_{L^2}+C_0^{1/6}\big)^{1/2}+CC_0^{1/6}\|u\|_{L^2}^2\\
&\le Ct^{(2\delta_0-3/2)(1-\beta)}\big(t^{1-\beta}\|u\|_{L^2}^2\big)^{-2\delta_0+5/4}\big(t^{1-\beta}\|\rho^{1/2}\dot u\|_{L^2}^2\big)^{1/4}\|u\|_{L^2}^{4\delta_0}\\
&\quad+CC_0^{1/12}t^{-3(1-\beta)/4}\big(t^{1-\beta}\|u\|_{L^2}^2\big)^{3/4}\|u\|_{L^2}+CC_0^{1/6}\|u\|_{L^2}^2,
\end{aligned}
$$

which together with (3.20) and (3.6) gives

$$
\begin{aligned}
&\sup_{0\le t\le\sigma(T)}\int\rho|u|^3\,dx\\
&\le C(\bar\rho,M)\Big(\int_0^{\sigma(T)}t^{-\frac{2(3-4\delta_0)(1-\beta)}{3-8\delta_0}}dt\Big)^{(3-8\delta_0)/4}\Big(\int_0^{\sigma(T)}\|\nabla u\|_{L^2}^2\,dt\Big)^{2\delta_0}\\
&\quad+C(\bar\rho,M)C_0^{1/12}\Big(\int_0^{\sigma(T)}t^{-3(1-\beta)/2}dt\Big)^{1/2}\Big(\int_0^{\sigma(T)}\|\nabla u\|_{L^2}^2\,dt\Big)^{1/2}\\
&\quad+\int\rho_0|u_0|^3\,dx+CC_0\\
&\le C(\bar\rho,M)C_0^{2\delta_0},
\end{aligned}
\tag{3.34}
$$

provided $C_0\le\varepsilon_0$, where in the last inequality we have used the following simple facts:

$$
\begin{aligned}
\int\rho_0|u_0|^3\,dx&\le C\left(\int\rho_0|u_0|^2\,dx\right)^{3(2\beta-1)/(4\beta)}\|u_0\|_{\dot H^\beta}^{3/(2\beta)}\\
&\le C(\bar\rho,M)C_0^{2\delta_0}
\end{aligned}
\tag{3.35}
$$

and

$$
\frac{2(3-4\delta_0)(1-\beta)}{3-8\delta_0}=1-\frac{\beta(2\beta-1)}{2-\beta}<1
$$

due to (3.3) and $\beta\in(\frac12,1]$. Thus, it follows from (3.34) that (3.33) holds provided $C_0\le\varepsilon_1$, where

$$
\varepsilon_1\triangleq\min\{\varepsilon_0,(C(\bar\rho,M))^{-1/\delta_0}\}=\min\{\varepsilon_0,(C(\bar\rho,M))^{-4\beta/(2\beta-1)}\}.
$$

The proof of Lemma 3.4 is completed. □

LEMMA 3.5. *There exists a positive constant* $\varepsilon_2(\mu,\lambda,\tilde\rho,a,\gamma,\bar\rho,\beta,M)\le\varepsilon_1$ *such that, if* (ρ,u) *is a smooth solution of* (1.1),(1.3),(1.4) *on* $\mathbb{R}^3\times(0,T]$ *satisfying* (3.4)*, then*

$$
A_1(T)+A_2(T)\le C_0^{1/2},
\tag{3.36}
$$

provided $C_0\le\varepsilon_2$.

PROOF. Lemma 3.2 shows that

$$A_1(T) + A_2(T) \le C(\widetilde{\rho})C_0 + C(\widetilde{\rho})\int_0^T \sigma^3\|\nabla u\|_{L^4}^4\,ds + C(\widetilde{\rho})\int_0^T \sigma\|\nabla u\|_{L^3}^3\,ds. \tag{3.37}$$

Due to (2.8),

$$\int_0^T \sigma^3\|\nabla u\|_{L^4}^4\,ds \le C\int_0^T \sigma^3\big(\|F\|_{L^4}^4 + \|\omega\|_{L^4}^4\big)ds + C\int_0^T \sigma^3\|P - P(\widetilde{\rho})\|_{L^4}^4\,ds. \tag{3.38}$$

It follows from (2.7) that

$$\begin{aligned}
&\int_0^T \sigma^3\left(\|F\|_{L^4}^4 + \|\omega\|_{L^4}^4\right)ds\\
&\le C\int_0^T \sigma^3\,(\|\nabla u\|_{L^2} + \|P - P(\widetilde{\rho})\|_{L^2})\,\|\rho\dot{u}\|_{L^2}^3 ds\\
&\le C(\widetilde{\rho})\sup_{t\in(0,T]}\big(\sigma^{3/2}\|\sqrt{\rho}\dot{u}\|_{L^2}\big(\sigma^{1/2}\|\nabla u\|_{L^2} + C_0^{1/2}\big)\big)\int_0^T\int \sigma\rho|\dot{u}|^2\,dx\,ds\\
&\le C(\widetilde{\rho})\big(A_1^{1/2}(T) + C_0^{1/2}\big)A_2^{1/2}(T)A_1(T)\\
&\le C(\widetilde{\rho})C_0.
\end{aligned} \tag{3.39}$$

To estimate the second term on the right-hand side of (3.38), one deduces from $(1.1)_1$ that $P - P(\widetilde{\rho})$ satisfies

$$(P - P(\widetilde{\rho}))_t + u\cdot\nabla(P - P(\widetilde{\rho})) + \gamma(P - P(\widetilde{\rho}))\operatorname{div} u + \gamma P(\widetilde{\rho})\operatorname{div} u = 0. \tag{3.40}$$

Multiplying (3.40) by $3(P - P(\widetilde{\rho}))^2$ and integrating the resulting equality over $\mathbb{R}^3$, one gets after using $\operatorname{div} u = \frac{1}{2\mu+\lambda}(F + P - P(\widetilde{\rho}))$ that

$$\begin{aligned}
&\frac{3\gamma - 1}{2\mu + \lambda}\|P - P(\widetilde{\rho})\|_{L^4}^4\\
&= -\left(\int (P - P(\widetilde{\rho}))^3\,dx\right)_t - \frac{3\gamma - 1}{2\mu + \lambda}\int (P - P(\widetilde{\rho}))^3 F\,dx\\
&\quad - 3\gamma P(\widetilde{\rho})\int (P - P(\widetilde{\rho}))^2 \operatorname{div} u\,dx\\
&\le -\left(\int (P - P(\widetilde{\rho}))^3\,dx\right)_t + \eta\|P - P(\widetilde{\rho})\|_{L^4}^4 + C_\eta\|F\|_{L^4}^4 + C_\eta\|\nabla u\|_{L^2}^2.
\end{aligned} \tag{3.41}$$

Multiplying (3.41) by σ^3, integrating the resulting inequality over $(0,T)$, and choosing η suitably small, one arrives at

$$\begin{aligned}&\int_0^T \sigma^3\|P-P(\widetilde{\rho})\|_{L^4}^4\,dt\\ &\quad\le C\sup_{0\le t\le T}\|P-P(\widetilde{\rho})\|_{L^3}^3+C\int_0^{\sigma(T)}\|P-P(\widetilde{\rho})\|_{L^3}^3\,dt\\ &\qquad+C(\overline{\rho})\int_0^T\sigma^3\|F\|_{L^4}^4\,ds+C(\overline{\rho})C_0\\ &\quad\le C(\overline{\rho})C_0,\end{aligned}\tag{3.42}$$

where (3.39) has been used. Therefore, collecting (3.38), (3.39), and (3.42) shows that

$$\int_0^T\sigma^3\big(\|\nabla u\|_{L^4}^4+\|P-P(\widetilde{\rho})\|_{L^4}^4\big)ds\le C(\overline{\rho})C_0.\tag{3.43}$$

Finally, we estimate the last term on the right-hand side of (3.37). First, (3.43) implies that

$$\int_{\sigma(T)}^T\int\sigma|\nabla u|^3\,dx\,ds\le\int_{\sigma(T)}^T\int(|\nabla u|^4+|\nabla u|^2)dx\,ds\le CC_0.\tag{3.44}$$

Next, one deduces from (2.9), (3.20), and (3.4) that

$$\begin{aligned}&\int_0^{\sigma(T)}\sigma\|\nabla u\|_{L^3}^3\,dt\\ &\le C(\overline{\rho})\int_0^{\sigma(T)}t\|\nabla u\|_{L^2}^{3/2}\big(\|\rho\dot{u}\|_{L^2}^{3/2}+C_0^{1/4}\big)dt\\ &\le C(\overline{\rho})\int_0^{\sigma(T)}(t^{(1-\beta)/2}\|\nabla u\|_{L^2})\|\nabla u\|_{L^2}^{1/2}\left(t\int\rho|\dot{u}|^2\,dx\right)^{3/4}dt+C(\overline{\rho})C_0\\ &\le C(\overline{\rho})\sup_{t\in(0,\sigma(T)]}(t^{(1-\beta)/2}\|\nabla u\|_{L^2})\int_0^{\sigma(T)}\|\nabla u\|_{L^2}^{1/2}\left(t\int\rho|\dot{u}|^2\,dx\right)^{3/4}dt\\ &\quad+C(\overline{\rho})C_0\\ &\le C(\overline{\rho},M)A_1^{3/4}C_0^{1/4}+C(\overline{\rho})C_0\\ &\le C(\overline{\rho},M)C_0^{5/8},\end{aligned}\tag{3.45}$$

provided $C_0\le\varepsilon_1$. It thus follows from (3.37) and (3.43)–(3.45) that the left-hand side of (3.36) is bounded by

$$C(\overline{\rho},M)C_0^{5/8}\le C_0^{1/2}$$

provided

$$C_0\le\varepsilon_2\triangleq\min\{\varepsilon_1,(C(\overline{\rho},M))^{-8}\}.$$

The proof of Lemma 3.5 is completed. □

We now proceed to derive a uniform (in time) upper bound for the density, which turns out to be the key to obtain all the higher-order estimates and thus to extend the classical solution globally. We will use an approach motivated by our previous study on the two-dimensional Stokes approximation equations [26].

LEMMA 3.6. *There exists a positive constant $\varepsilon = \varepsilon(\overline{\rho}, M)$ as described in Theorem* 1.1 *such that, if (ρ, u) is a smooth solution of* (1.1),(1.3),(1.4) *on $\mathbb{R}^3 \times (0, T]$ satisfying* (3.4), *then*

$$\sup_{0\le t\le T} \|\rho(t)\|_{L^\infty} \le \frac{7\overline{\rho}}{4}$$

provided $C_0 \le \varepsilon$.

PROOF. Rewrite the equation of the mass conservation $(1.1)_1$ as

$$D_t\rho = g(\rho) + b'(t),$$

where

$$D_t\rho \triangleq \rho_t + u\cdot\nabla\rho, \quad g(\rho) \triangleq -\frac{a\rho}{2\mu+\lambda}(\rho^\gamma - \widetilde{\rho}^\gamma),$$

$$b(t) \triangleq -\frac{1}{2\mu+\lambda}\int_0^t \rho F\,dt.$$

For $t \in [0, \sigma(T)]$, one deduces from Lemma 2.2, (2.6), (3.36), (3.20), (3.21), and (2.3) that for δ_0 as in (3.3) and for all $0 \le t_1 < t_2 \le \sigma(T)$,

$$\begin{aligned}
&|b(t_2) - b(t_1)| \\
&\quad\le C\int_0^{\sigma(T)} \|(\rho F)(\cdot,t)\|_{L^\infty}\,dt \\
&\quad\le C(\overline{\rho})\int_0^{\sigma(T)} \|F(\cdot,t)\|_{L^6}^{1/2}\,\|\nabla F(\cdot,t)\|_{L^6}^{1/2}\,dt \\
&\quad\le C(\overline{\rho})\int_0^{\sigma(T)} \|\rho^{1/2}\dot{u}\|_{L^2}^{1/2}\,\|\nabla\dot{u}\|_{L^2}^{1/2}\,dt \\
&\quad\le C(\overline{\rho})\int_0^{\sigma(T)} t^{-(2-\beta)/4}\|\rho\dot{u}\|_{L^2}^{1/2}\big(t^{2-\beta}\|\nabla\dot{u}\|_{L^2}^2\big)^{1/4}\,dt \\
&\quad\le C(\overline{\rho}, M)\bigg(\int_0^{\sigma(T)} t^{-(2-\beta)/3}\|\rho\dot{u}\|_{L^2}^{2/3}\,dt\bigg)^{3/4} \\
&\quad= C(\overline{\rho}, M)\bigg(\int_0^{\sigma(T)} t^{-[(2-\beta)(-\delta_0+2/3)+\delta_0]} \\
&\qquad\qquad\cdot\big(t^{2-\beta}\|\rho^{1/2}\dot{u}\|_{L^2}^2\big)^{-\delta_0+1/3}\big(t\|\rho^{1/2}\dot{u}\|_{L^2}^2\big)^{\delta_0}\,dt\bigg)^{3/4} \le
\end{aligned}$$

$$\le C(\overline{\rho}, M)(A_1(\sigma(T)))^{3\delta_0/4}$$

$$\le C(\overline{\rho}, M)C_0^{3\delta_0/8},$$

provided $C_0 \le \varepsilon_2$. Therefore, for $t \in [0, \sigma(T)]$, one can choose N_0 and N_1 in (2.10) as follows:

$$N_1 = 0, \quad N_0 = C(\overline{\rho}, M)C_0^{3\delta_0/8},$$

and $\overline{\zeta} = \widetilde{\rho}$ in (2.11). Then

$$g(\zeta) = -\frac{a\zeta}{2\mu+\lambda}(\zeta^\gamma - \widetilde{\rho}^\gamma) \le -N_1 = 0 \quad \text{for all } \zeta \ge \overline{\zeta} = \widetilde{\rho}.$$

Lemma 2.4 thus yields that

$$\sup_{t\in[0,\sigma(T)]} \|\rho\|_{L^\infty} \le \max\{\overline{\rho}, \widetilde{\rho}\} + N_0 \le \overline{\rho} + C(\overline{\rho}, M)C_0^{3\delta_0/8} \le \frac{3\overline{\rho}}{2}, \tag{3.46}$$

provided

$$C_0 \le \min\{\varepsilon_2, \varepsilon_3\}$$
$$\text{for } \varepsilon_3 \triangleq \left(\frac{\overline{\rho}}{2C(\overline{\rho}, M)}\right)^{8/(3\delta_0)} = \left(\frac{\overline{\rho}}{2C(\overline{\rho}, M)}\right)^{32\beta/(3(2\beta-1))}.$$

On the other hand, for $t \in [\sigma(T), T]$, one deduces from Lemma 2.2, (3.36), (3.6), and (2.6) that for all $\sigma(T) \le t_1 \le t_2 \le T$,

$$\begin{aligned}
|b(t_2) - b(t_1)| &\le C(\overline{\rho}) \int_{t_1}^{t_2} \|F(\cdot, t)\|_{L^\infty}\, dt \\
&\le \frac{a}{2\mu+\lambda}(t_2 - t_1) + C(\overline{\rho}) \int_{\sigma(T)}^{T} \|F(\cdot, t)\|_{L^\infty}^{8/3}\, dt \\
&\le \frac{a}{2\mu+\lambda}(t_2 - t_1) + C(\overline{\rho}) \int_{\sigma(T)}^{T} \|F(\cdot, t)\|_{L^2}^{2/3} \|\nabla F(\cdot, t)\|_{L^6}^2\, dt \\
&\le \frac{a}{2\mu+\lambda}(t_2 - t_1) + C(\overline{\rho})C_0^{1/6} \int_{\sigma(T)}^{T} \|\nabla \dot{u}(\cdot, t)\|_{L^2}^2\, dt \\
&\le \frac{a}{2\mu+\lambda}(t_2 - t_1) + C(\overline{\rho})C_0^{2/3},
\end{aligned}$$

provided $C_0 \le \varepsilon_2$. Therefore, one can choose N_1 and N_0 in (2.10) as

$$N_1 = \frac{a}{2\mu+\lambda}, \quad N_0 = C(\overline{\rho})C_0^{2/3}.$$

Note that

$$g(\zeta) = -\frac{a\zeta}{2\mu+\lambda}(\zeta^\gamma - \widetilde{\rho}^\gamma) \le -N_1 = -\frac{a}{2\mu+\lambda} \quad \text{for all } \zeta \ge \widetilde{\rho} + 1.$$

So one can set $\overline{\zeta} = \widetilde{\rho} + 1$ in (2.11). Lemma 2.4 and (3.46) thus yield that

$$\sup_{t\in[\sigma(T),T]} \|\rho\|_{L^\infty} \le \max\left\{\frac{3\overline{\rho}}{2}, \widetilde{\rho}+1\right\} + N_0 \le \frac{3\overline{\rho}}{2} + C(\overline{\rho})C_0^{2/3} \le \frac{7\overline{\rho}}{4}, \tag{3.47}$$

provided

$$C_0 \le \varepsilon \triangleq \min\{\varepsilon_2, \varepsilon_3, \varepsilon_4\} \quad \text{for } \varepsilon_4 \triangleq \left(\frac{\overline{\rho}}{4C(\overline{\rho})}\right)^{3/2}. \tag{3.48}$$

The combination of (3.46) with (3.47) completes the proof of Lemma 3.6. □

From now on, we will always assume that the initial energy C_0 satisfies (3.48) and the positive constant C may depend on

$$T,\ \|\rho_0^{1/2} g\|_{L^2},\ \|\nabla g\|_{L^2},\ \|\nabla u_0\|_{H^2},\ \|\rho_0 - \widetilde{\rho}\|_{H^3},\ \|P(\rho_0) - P(\widetilde{\rho})\|_{H^3},$$

besides μ, λ, $\widetilde{\rho}$, a, γ, $\overline{\rho}$, β, and M, where g is as in (1.9).

Next, we will derive important estimates on the spatial gradient of the smooth solution (ρ, u).

LEMMA 3.7. *The following estimates hold*:

$$\sup_{0\le t\le T} \int \rho|\dot{u}|^2\,dx + \int_0^T\int |\nabla\dot{u}|^2\,dx\,dt \le C, \tag{3.49}$$

$$\sup_{0\le t\le T} (\|\nabla\rho\|_{L^2\cap L^6} + \|\nabla u\|_{H^1}) + \int_0^T \|\nabla u\|_{L^\infty}\,dt \le C. \tag{3.50}$$

PROOF. Taking $\theta = 1$ in (3.32) together with (3.36) gives

$$\sup_{t\in[0,T]} \|\nabla u\|_{L^2}^2 + \int_0^T\int \rho|\dot{u}|^2\,dx\,dt \le C. \tag{3.51}$$

Taking $m = 0$ in (3.19), one can deduce from Gagliardo-Nirenberg's inequality (2.2), (2.6), (3.51), and (3.19) that

$$\begin{aligned}
&\left(\int \rho|\dot{u}|^2\,dx\right)_t + \mu\int|\nabla\dot{u}|^2\,dx + (\mu+\lambda)\int(\operatorname{div}\dot{u})^2\,dx \\
&\quad\le C\|\nabla u\|_{L^4}^4 + C(\overline{\rho})\|\nabla u\|_{L^2}^2 \\
&\quad\le C\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}^3 + C \\
&\quad\le C\left(\|F\|_{L^6}^3 + \|\omega\|_{L^6}^3 + \|P - P(\widetilde{\rho})\|_{L^6}^3\right) + C \\
&\quad\le C\left(\|\nabla F\|_{L^2}^3 + \|\nabla\omega\|_{L^2}^3\right) + C \\
&\quad\le C\|\rho\dot{u}\|_{L^2}^3 + C \\
&\quad\le C\|\rho^{1/2}\dot{u}\|_{L^2}^4 + C.
\end{aligned} \tag{3.52}$$

Taking into account the compatibility condition (1.9), we can define

$$\sqrt{\rho}\dot{u}(x, t = 0) = \sqrt{\rho_0}\, g. \tag{3.53}$$

Then (3.49) follows from (3.51)–(3.53) and Gronwall's inequality.

Next, we prove (3.50) by using Lemma 2.5 as in [18]. For $2 \le p \le 6$, $|\nabla\rho|^p$ satisfies

$$\begin{aligned}&(|\nabla\rho|^p)_t + \operatorname{div}(|\nabla\rho|^p u) + (p-1)|\nabla\rho|^p \operatorname{div} u\\ &\quad + p|\nabla\rho|^{p-2}(\nabla\rho)^t \nabla u(\nabla\rho) + p\rho|\nabla\rho|^{p-2}\nabla\rho\cdot\nabla \operatorname{div} u = 0.\end{aligned}$$

Thus,

$$\begin{aligned}\partial_t\|\nabla\rho\|_{L^p} &\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^p} + C\|\nabla^2 u\|_{L^p}\\ &\le C(1+\|\nabla u\|_{L^\infty})\|\nabla\rho\|_{L^p} + C\|\rho\dot{u}\|_{L^p}\end{aligned} \tag{3.54}$$

due to

$$\|\nabla^2 u\|_{L^p} \le C(\|\rho\dot{u}\|_{L^p} + \|\nabla P\|_{L^p}), \tag{3.55}$$

which follows from the standard L^p-estimate for the following elliptic system:

$$\mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u = \rho\dot{u} + \nabla P, \quad u \to 0 \text{ as } |x| \to \infty. \tag{3.56}$$

It follows from Lemma 2.5 and (3.55) that

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\le C(\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty})\log(e + \|\nabla^2 u\|_{L^6})\\ &\quad + C\|\nabla u\|_{L^2} + C\\ &\le C(\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty})\log(e + \|\dot{u}\|_{L^6} + \|\nabla P\|_{L^6}) + C\\ &\le C(\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty})\log(e + \|\nabla\dot{u}\|_{L^2})\\ &\quad + C(\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty})\log(e + \|\nabla\rho\|_{L^6}) + C.\end{aligned} \tag{3.57}$$

Set

$$\begin{aligned}f(t) &\triangleq e + \|\nabla\rho\|_{L^6},\\ g(t) &\triangleq 1 + (\|\operatorname{div} u\|_{L^\infty} + \|\omega\|_{L^\infty})\log(e + \|\nabla\dot{u}\|_{L^2}) + \|\nabla\dot{u}\|_{L^2}.\end{aligned}$$

Combining (3.57) with (3.54) and setting $p = 6$ in (3.54), one gets

$$f'(t) \le Cg(t)f(t) + Cg(t)f(t)\log f(t) + Cg(t),$$

which yields

$$(\log f(t))' \le Cg(t) + Cg(t)\log f(t) \tag{3.58}$$

due to $f(t) > 1$. Note that (2.5), Lemma 2.2, (2.6), (3.49), and Lemma 3.6 imply

$$
\begin{aligned}
\int_0^T g(t)dt &\le C\int_0^T (\|\operatorname{div} u\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2)dt + C \\
&\le C\int_0^T \big(\|F\|_{L^\infty}^2 + \|P - P(\widetilde{\rho})\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2\big)dt + C \\
&\le C\int_0^T \big(\|F\|_{L^2}^2 + \|\nabla F\|_{L^6}^2 + \|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^6}^2\big)dt + C \\
&\le C\int_0^T \|\nabla\dot{u}\|_{L^2}^2\, dt + C \\
&\le C,
\end{aligned} \tag{3.59}
$$

which, together with (3.58) and Gronwall's inequality, shows that

$$\sup_{0\le t\le T} f(t) \le C.$$

Consequently,

$$\sup_{0\le t\le T} \|\nabla\rho\|_{L^6} \le C. \tag{3.60}$$

As a consequence of (3.57), (3.59), and (3.60), one obtains

$$\int_0^T \|\nabla u\|_{L^\infty}\, dt \le C. \tag{3.61}$$

Next, taking $p = 2$ in (3.54), one gets by using (3.61), (3.51) and Gronwall's inequality that

$$\sup_{0\le t\le T} \|\nabla\rho\|_{L^2} \le C,$$

which, together with (3.55), (3.49), (3.51), (3.60), and (3.61) gives (3.50). The proof of Lemma 3.7 is completed. □

Lemmas 3.8 through 3.11 below deal with the higher-order estimates of the solutions that are needed to guarantee the extension of a local classical solution to a global one. The proofs are similar to the ones in [21], and we sketch them here for completeness.

LEMMA 3.8. *The following estimates hold*:

$$\sup_{0\le t\le T} \int \rho|u_t|^2\, dx + \int_0^T\int |\nabla u_t|^2\, dx\, dt \le C, \tag{3.62}$$

$$\sup_{t\in[0,T]} (\|\rho - \widetilde{\rho}\|_{H^2} + \|P(\rho) - P(\widetilde{\rho})\|_{H^2}) \le C. \tag{3.63}$$

PROOF. Estimate (3.62) follows directly from the following simple facts:

$$\begin{aligned}\int \rho|u_t|^2\,dx &\le \int \rho|\dot{u}|^2\,dx + \int \rho|u\cdot\nabla u|^2\,dx\\ &\le C + C\|\rho^{1/2}u\|_{L^2}\,\|u\|_{L^6}\,\|\nabla u\|_{L^6}^2\\ &\le C,\end{aligned}$$

and

$$\begin{aligned}\|\nabla u_t\|_{L^2}^2 &\le \|\nabla\dot{u}\|_{L^2}^2 + \|\nabla(u\cdot\nabla u)\|_{L^2}^2\\ &\le \|\nabla\dot{u}\|_{L^2}^2 + C\|u\|_{L^\infty}^2\,\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4\\ &\le \|\nabla\dot{u}\|_{L^2}^2 + C\end{aligned}$$

due to Lemma 3.7.

Next, we prove (3.63). Note that P satisfies

$$P_t + u\cdot\nabla P + \gamma P\,\mathrm{div}\,u = 0, \tag{3.64}$$

which, together with $(1.1)_1$ and a simple computation, yields that

$$\begin{aligned}&\frac{d}{dt}\big(\|\nabla^2 P\|_{L^2}^2 + \|\nabla^2\rho\|_{L^2}^2\big)\\ &\quad\le C(1+\|\nabla u\|_{L^\infty})\big(\|\nabla^2 P\|_{L^2}^2 + \|\nabla^2\rho\|_{L^2}^2\big)\\ &\qquad + C\|F\|_{H^2}^2 + C\|\omega\|_{H^2}^2 + C,\end{aligned} \tag{3.65}$$

where we have used the following simple fact:

$$\begin{aligned}\|\nabla u\|_{H^m} &\le C(\|\mathrm{div}\,u\|_{H^m} + \|\omega\|_{H^m})\\ &\le C(\|F\|_{H^m} + \|\omega\|_{H^m} + \|P-P(\tilde{\rho})\|_{H^m}) \quad \text{for } m=1,2.\end{aligned}$$

Noticing that F and ω satisfy (2.4), we get by the standard L^2-estimate for an elliptic system, (3.49), and (3.50) that

$$\begin{aligned}\|F\|_{H^2} + \|\omega\|_{H^2} &\le C(\|F\|_{L^2} + \|\omega\|_{L^2} + \|\rho\dot{u}\|_{L^2} + \|\nabla(\rho\dot{u})\|_{L^2})\\ &\le C(1 + \|\nabla\rho\|_{L^3}\|\dot{u}\|_{L^6} + \|\nabla\dot{u}\|_{L^2})\\ &\le C(1+\|\nabla\dot{u}\|_{L^2}),\end{aligned}$$

which, together with (3.65), Lemma 3.7, and Gronwall's inequality gives directly

$$\sup_{t\in[0,T]}\big(\|\nabla^2 P\|_{L^2} + \|\nabla^2\rho\|_{L^2}\big) \le C.$$

Thus the proof of Lemma 3.8 is completed. □

LEMMA 3.9. *The following estimates hold*:

$$\sup_{0\le t\le T}(\|\rho_t\|_{H^1} + \|P_t\|_{H^1}) + \int_0^T \big(\|\rho_{tt}\|_{L^2}^2 + \|P_{tt}\|_{L^2}^2\big)dt \le C, \tag{3.66}$$

$$\sup_{0\le t\le T}\int |\nabla u_t|^2\,dx + \int_0^T\int \rho u_{tt}^2\,dx\,dt \le C. \tag{3.67}$$

PROOF. We first prove (3.66). One deduces from (3.64) and (3.50) that

$$\|P_t\|_{L^2} \le C\|u\|_{L^\infty}\|\nabla P\|_{L^2} + C\|\nabla u\|_{L^2} \le C. \tag{3.68}$$

Differentiating (3.64) yields

$$\nabla P_t + u \cdot \nabla\nabla P + \nabla u \cdot \nabla P + \gamma \nabla P \operatorname{div} u + \gamma P \nabla \operatorname{div} u = 0.$$

Hence, by (3.50) and (3.63), one gets

$$\begin{aligned}\|\nabla P_t\|_{L^2} &\le C\|u\|_{L^\infty}\|\nabla^2 P\|_{L^2} + C\|\nabla u\|_{L^3}\|\nabla P\|_{L^6} + C\|\nabla^2 u\|_{L^2}\\ &\le C.\end{aligned} \tag{3.69}$$

The combination of (3.68) with (3.69) implies

$$\sup_{0\le t\le T}\|P_t\|_{H^1} \le C. \tag{3.70}$$

Note that P_{tt} satisfies

$$P_{tt} + \gamma P_t \operatorname{div} u + \gamma P \operatorname{div} u_t + u_t \cdot \nabla P + u \cdot \nabla P_t = 0. \tag{3.71}$$

Thus, one gets from (3.71), (3.70), (3.50), and (3.62) that

$$\begin{aligned}&\int_0^T \|P_{tt}\|_{L^2}^2\,dt\\ &\quad\le C\int_0^T (\|P_t\|_{L^6}\|\nabla u\|_{L^3} + \|\nabla u_t\|_{L^2} + \|u_t\|_{L^6}\|\nabla P\|_{L^3} + \|\nabla P_t\|_{L^2})^2\,dt\\ &\quad\le C.\end{aligned}$$

One can handle ρ_t and ρ_{tt} similarly. Thus (3.66) is proved.

Next, we prove (3.67). Differentiating $(1.1)_2$ with respect to t, then multiplying the resulting equation by u_{tt}, one gets after integration by parts that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int(\mu|\nabla u_t|^2 + (\lambda+\mu)(\operatorname{div} u_t)^2)dx + \int \rho u_{tt}^2\,dx\\ &= \frac{d}{dt}\left(-\frac{1}{2}\int \rho_t|u_t|^2\,dx - \int \rho_t u \cdot \nabla u \cdot u_t\,dx + \int P_t \operatorname{div} u_t\,dx\right)\\ &\quad + \frac{1}{2}\int \rho_{tt}|u_t|^2\,dx + \int(\rho_t u\cdot\nabla u)_t \cdot u_t\,dx - \int \rho u_t \cdot \nabla u \cdot u_{tt}\,dx\\ &\quad - \int \rho u \cdot \nabla u_t \cdot u_{tt}\,dx - \int P_{tt} \operatorname{div} u_t\,dx\\ &\triangleq \frac{d}{dt} I_0 + \sum_{i=1}^5 I_i.\end{aligned} \tag{3.72}$$

It follows from $(1.1)_1$, (3.50), (3.66), and (3.62) that

$$
\begin{aligned}
|I_0| &= \left| -\frac{1}{2} \int \rho_t |u_t|^2 \, dx - \int \rho_t u \cdot \nabla u \cdot u_t \, dx + \int P_t \operatorname{div} u_t \, dx \right| \\
&\le \left| \int \operatorname{div}(\rho u) |u_t|^2 \, dx \right| + C \|\rho_t\|_{L^3} \|u \cdot \nabla u\|_{L^2} \|u_t\|_{L^6} \\
&\quad + C \|P_t\|_{L^2} \|\nabla u_t\|_{L^2} \\
&\le C \int \rho |u| \, |u_t| \, |\nabla u_t| dx + C \|\nabla u_t\|_{L^2} \\
&\le C \|u\|_{L^6} \|\rho^{1/2} u_t\|_{L^2}^{1/2} \|u_t\|_{L^6}^{1/2} \|\nabla u_t\|_{L^2} + C \|\nabla u_t\|_{L^2} \\
&\le \delta \|\nabla u_t\|_{L^2}^2 + C_\delta,
\end{aligned} \tag{3.73}
$$

$$
\begin{aligned}
2|I_1| &= \left| \int \rho_{tt} |u_t|^2 \, dx \right| \\
&= \left| \int (\rho_t u + \rho u_t) \cdot \nabla(|u_t|^2) dx \right| \\
&\le C \big(\|\rho_t\|_{L^3} \|u\|_{L^\infty} + \|\rho^{1/2} u_t\|_{L^2}^{1/2} \|u_t\|_{L^6}^{1/2} \big) \|u_t\|_{L^6} \|\nabla u_t\|_{L^2} \\
&\le C \|\nabla u_t\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^{5/2} \\
&\le C \|\nabla u_t\|_{L^2}^4 + C,
\end{aligned} \tag{3.74}
$$

and

$$
\begin{aligned}
|I_2| &= \left| \int (\rho_t u \cdot \nabla u)_t \cdot u_t dx \right| \\
&= \left| \int (\rho_{tt} u \cdot \nabla u \cdot u_t + \rho_t u_t \cdot \nabla u \cdot u_t + \rho_t u \cdot \nabla u_t \cdot u_t) dx \right| \\
&\le \|\rho_{tt}\|_{L^2} \|u \cdot \nabla u\|_{L^3} \|u_t\|_{L^6} + \|\rho_t\|_{L^2} \||u_t|^2\|_{L^3} \|\nabla u\|_{L^6} \\
&\quad + \|\rho_t\|_{L^3} \|u\|_{L^\infty} \|\nabla u_t\|_{L^2} \|u_t\|_{L^6} \\
&\le C \|\rho_{tt}\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2.
\end{aligned} \tag{3.75}
$$

Cauchy's inequality gives

$$
\begin{aligned}
|I_3|+|I_4| &= \left|\int \rho u_t\cdot\nabla u\cdot u_{tt}\,dx\right| + \left|\int \rho u\cdot\nabla u_t\cdot u_{tt}\,dx\right| \\
&\le C\|\rho^{1/2}u_{tt}\|_{L^2}(\|u_t\|_{L^6}\|\nabla u\|_{L^3}+\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}) \\
&\le \delta\|\rho^{1/2}u_{tt}\|_{L^2}^2 + C_\delta\|\nabla u_t\|_{L^2}^2
\end{aligned}
\tag{3.76}
$$

and

$$
\begin{aligned}
|I_5| = \left|\int P_{tt}\operatorname{div} u_t\,dx\right| &\le \|P_{tt}\|_{L^2}\|\operatorname{div} u_t\|_{L^2} \\
&\le C\|P_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2.
\end{aligned}
\tag{3.77}
$$

Due to the regularity of the local solution (2.1), $t\nabla u_t\in C([0,T_*];L^2)$. Thus

$$
\begin{aligned}
\|\nabla u_t(\cdot,T_*/2)\|_{L^2} &\le \frac{2}{T_*}\|t\nabla u_t\|_{L^\infty(0,T_*;L^2)} \\
&\le C,
\end{aligned}
\tag{3.78}
$$

where C may also depend on $\|\nabla g\|_{L^2}$.

Collecting the estimates (3.73)–(3.78), one deduces from (3.72), (3.66), (3.62), and Gronwall's inequality that

$$
\sup_{T_*/2\le t\le T}\|\nabla u_t\|_{L^2} + \int_{T_*/2}^{T}\int \rho|u_{tt}|^2\,dx\,dt \le C. \tag{3.79}
$$

On the other hand, (2.1) gives

$$
\sup_{0\le t\le T_*/2}\|\nabla u_t\|_{L^2} + \int_0^{T_*/2}\int \rho|u_{tt}|^2\,dx\,dt \le C. \tag{3.80}
$$

The combination of (3.79) with (3.80) yields (3.67) immediately. This completes the proof of Lemma 3.9. □

LEMMA 3.10. *It holds that*

$$
\sup_{t\in[0,T]}(\|\rho-\tilde\rho\|_{H^3}+\|P-P(\tilde\rho)\|_{H^3})\le C, \tag{3.81}
$$

$$
\sup_{t\in[0,T]}(\|\nabla u_t\|_{L^2}+\|\nabla u\|_{H^2}) + \int_0^T(\|\nabla u\|_{H^3}^2+\|\nabla u_t\|_{H^1}^2)dt \le C. \tag{3.82}
$$

PROOF. It follows from (3.67) and (3.50) that

$$
\begin{aligned}
\|\nabla(\rho\dot u)\|_{L^2} &\le \||\nabla\rho|\,|u_t|\|_{L^2} + \|\rho\nabla u_t\|_{L^2} + \||\nabla\rho|\,|u|\,|\nabla u|\|_{L^2} \\
&\quad + \|\rho|\nabla u|^2\|_{L^2} + \|\rho|u|\,|\nabla^2 u|\|_{L^2} \\
&\le \|\nabla\rho\|_{L^3}\,\|u_t\|_{L^6} + C\|\nabla u_t\|_{L^2} + C\|\nabla\rho\|_{L^3}\,\|u\|_{L^\infty}\,\|\nabla u\|_{L^6} \\
&\quad + C\|\nabla u\|_{L^3}\,\|\nabla u\|_{L^6} + C\|u\|_{L^\infty}\,\|\nabla^2 u\|_{L^2} \\
&\le C,
\end{aligned}
$$

which together with (3.49) gives

$$\sup_{0\le t\le T} \|\rho\dot u\|_{H^1} \le C. \tag{3.83}$$

The standard H^1-estimate for elliptic system (3.56) gives

$$\begin{aligned}\|\nabla^2 u\|_{H^1} &\le C\|\mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u\|_{H^1}\\ &= C\|\rho\dot u + \nabla P\|_{H^1}\\ &\le C(\|\rho\dot u\|_{H^1} + \|\nabla P\|_{H^1})\\ &\le C\end{aligned} \tag{3.84}$$

due to $(1.1)_2$, (3.83), and (3.63). As a consequence of (3.50) and (3.84), one has

$$\sup_{0\le t\le T} \|\nabla u\|_{H^2} \le C. \tag{3.85}$$

Therefore, the standard L^2-estimate for elliptic system (3.50) and Lemma 3.9 yield that

$$\begin{aligned}\|\nabla^2 u_t\|_{L^2} &\le C\|\mu\Delta u_t + (\mu+\lambda)\nabla \operatorname{div} u_t\|_{L^2}\\ &= \big\|\rho u_{tt} + \rho_t u_t + \rho_t u\cdot\nabla u + \rho u_t\cdot\nabla u\\ &\qquad + \rho u\cdot\nabla u_t + \nabla P_t\big\|_{L^2}\\ &\le C(\|\rho u_{tt}\|_{L^2} + \|\rho_t\|_{L^3}\|u_t\|_{L^6} + \|\rho_t\|_{L^3}\|u\|_{L^\infty}\|\nabla u\|_{L^6})\\ &\quad + C(\|u_t\|_{L^6}\|\nabla u\|_{L^3} + \|u\|_{L^\infty}\|\nabla u_t\|_{L^2} + \|\nabla P_t\|_{L^2})\\ &\le C\|\rho u_{tt}\|_{L^2} + C,\end{aligned} \tag{3.86}$$

which, together with (3.67), implies

$$\int_0^T \|\nabla u_t\|_{H^1}^2\,dt \le C. \tag{3.87}$$

Applying the standard H^2-estimate for elliptic system (3.56) again leads to

$$\begin{aligned}\|\nabla^2 u\|_{H^2} &\le C\|\mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u\|_{H^2}\\ &\le C\|\rho\dot u\|_{H^2} + C\|\nabla P\|_{H^2}\\ &\le C + C\|\nabla u_t\|_{H^1} + C\|\nabla^3 P\|_{L^2},\end{aligned} \tag{3.88}$$

where one has used (3.83) and the following simple facts:

$$\begin{aligned}\|\nabla^2(\rho u_t)\|_{L^2} &\le C(\||\nabla^2\rho|\,|u_t|\|_{L^2} + \||\nabla\rho|\,|\nabla u_t|\|_{L^2} + \|\nabla^2 u_t\|_{L^2})\\ &\le C(\|\nabla^2\rho\|_{L^2}\|\nabla u_t\|_{H^1} + \|\nabla\rho\|_{L^3}\|\nabla u_t\|_{L^6} + \|\nabla^2 u_t\|_{L^2})\\ &\le C + C\|\nabla u_t\|_{H^1},\end{aligned}$$

and

$$\begin{aligned}\|\nabla^2(\rho u\cdot\nabla u)\|_{L^2} &\le C(\||\nabla^2(\rho u)|\,|\nabla u|\|_{L^2}+\||\nabla(\rho u)|\,|\nabla^2 u|\|_{L^2}+\|\nabla^3 u\|_{L^2})\\ &\le C(1+\|\nabla^2(\rho u)\|_{L^2}\|\nabla u\|_{H^2}+\|\nabla(\rho u)\|_{L^3}\|\nabla^2 u\|_{L^6})\\ &\le C(1+\|\nabla^2\rho\|_{L^2}\|u\|_{L^\infty}+\|\nabla\rho\|_{L^6}\|\nabla u\|_{L^3}+\|\nabla^2 u\|_{L^2})\\ &\le C\end{aligned}$$

due to (3.63) and (3.85). By using (3.85), (3.88), and (3.63), one may get that

$$\begin{aligned}\big(\|\nabla^3 P\|_{L^2}^2\big)_t &\le C\big(\||\nabla^3 u|\,|\nabla P|\|_{L^2}+\||\nabla^2 u|\,|\nabla^2 P|\|_{L^2}\\ &\qquad+\||\nabla u|\,|\nabla^3 P|\|_{L^2}+\|\nabla^4 u\|_{L^2}\big)\|\nabla^3 P\|_{L^2}\\ &\le C\big(\|\nabla^3 u\|_{L^2}\|\nabla P\|_{H^2}+\|\nabla^2 u\|_{L^3}\|\nabla^2 P\|_{L^6}\\ &\qquad+\|\nabla u\|_{L^\infty}\|\nabla^3 P\|_{L^2}\big)\|\nabla^3 P\|_{L^2}\\ &\quad+C(1+\|\nabla^2 u_t\|_{L^2}+\|\nabla^3 P\|_{L^2})\|\nabla^3 P\|_{L^2}\\ &\le C+C\|\nabla u_t\|_{H^1}^2+C\|\nabla^3 P\|_{L^2}^2,\end{aligned}$$

which, together with Gronwall's inequality and (3.87), yields that

$$\sup_{0\le t\le T}\|\nabla^3 P\|_{L^2}\le C. \tag{3.89}$$

Collecting estimates (3.87)–(3.89) and (3.63) shows that

$$\sup_{0\le t\le T}\|P-P(\tilde\rho)\|_{H^3}+\int_0^T\|\nabla u\|_{H^3}^2\,dt\le C. \tag{3.90}$$

It is easy to check that similar arguments work for $\rho-\tilde\rho$ by using (3.90). Hence,

$$\sup_{0\le t\le T}\|\rho-\tilde\rho\|_{H^3}\le C. \tag{3.91}$$

Combining (3.90) with (3.91) shows (3.81). Estimate (3.82) thus follows from (3.67), (3.85), (3.87), and (3.90). Hence the proof of Lemma 3.10 is finished. □

LEMMA 3.11. *For any $\tau\in(0,T)$, there exists some positive constant $C(\tau)$ such that*

$$\sup_{\tau\le t\le T}(\|\nabla u_t\|_{H^1}+\|\nabla^4 u\|_{L^2})+\int_\tau^T\int|\nabla u_{tt}|^2\,dx\,dt\le C(\tau). \tag{3.92}$$

PROOF. Differentiate $(1.1)_2$ with respect to t twice to get

$$\begin{aligned}&\rho u_{ttt}+\rho u\cdot\nabla u_{tt}-\mu\Delta u_{tt}-(\mu+\lambda)\nabla\operatorname{div}u_{tt}\\ &\quad=2\operatorname{div}(\rho u)u_{tt}+\operatorname{div}(\rho u)_t u_t-2(\rho u)_t\cdot\nabla u_t\\ &\qquad-(\rho_{tt}u+2\rho_t u_t)\cdot\nabla u-\rho u_{tt}\cdot\nabla u-\nabla P_{tt}.\end{aligned} \tag{3.93}$$

Multiplying (3.93) by u_{tt} and then integrating the resulting equation over $\mathbb{R}^3$, one gets after integration by parts that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|u_{tt}|^2\,dx + \int(\mu|\nabla u_{tt}|^2 + (\mu+\lambda)(\operatorname{div} u_{tt})^2)dx \\
&\quad = -4\int u_{tt}^i \rho u\cdot\nabla u_{tt}^i dx - \int(\rho u)_t\cdot[\nabla(u_t\cdot u_{tt}) + 2\nabla u_t\cdot u_{tt}]dx \\
&\qquad - \int(\rho_{tt}u + 2\rho_t u_t)\cdot\nabla u\cdot u_{tt}\,dx - \int \rho u_{tt}\cdot\nabla u\cdot u_{tt}\,dx \\
&\qquad + \int P_{tt}\operatorname{div} u_{tt}\,dx \\
&\quad \triangleq \sum_{i=1}^{5} J_5.
\end{aligned}
\tag{3.94}
$$

We estimate each J_i, $i = 1,\dots,5$, as follows: Hölder's inequality gives

$$
\begin{aligned}
|J_1| &\le C\|\rho^{1/2}u_{tt}\|_{L^2}\,\|\nabla u_{tt}\|_{L^2}\,\|u\|_{L^\infty} \\
&\le \delta\|\nabla u_{tt}\|_{L^2}^2 + C_\delta\|\rho^{1/2}u_{tt}\|_{L^2}^2.
\end{aligned}
\tag{3.95}
$$

It follows from (3.62), (3.66), (3.67), and (3.50) that

$$
\begin{aligned}
|J_2| &\le C(\|\rho u_t\|_{L^3} + \|\rho_t u\|_{L^3})(\|u_{tt}\|_{L^6}\,\|\nabla u_t\|_{L^2} + \|\nabla u_{tt}\|_{L^2}\,\|u_t\|_{L^6}) \\
&\le C\big(\|\rho^{1/2}u_t\|_{L^2}^{1/2}\,\|u_t\|_{L^6}^{1/2} + \|\rho_t\|_{L^6}\,\|u\|_{L^6}\big)\|\nabla u_{tt}\|_{L^2} \\
&\le \delta\|\nabla u_{tt}\|_{L^2}^2 + C_\delta,
\end{aligned}
\tag{3.96}
$$

$$
\begin{aligned}
|J_3| &\le C(\|\rho_{tt}\|_{L^2}\,\|u\|_{L^\infty}\,\|\nabla u\|_{L^3} + \|\rho_t\|_{L^6}\,\|u_t\|_{L^6}\,\|\nabla u\|_{L^2})\|u_{tt}\|_{L^6} \\
&\le \delta\|\nabla u_{tt}\|_{L^2}^2 + C_\delta\|\rho_{tt}\|_{L^2}^2,
\end{aligned}
\tag{3.97}
$$

and

$$
\begin{aligned}
|J_4| + |J_5| &\le C\|\rho u_{tt}\|_{L^2}\,\|\nabla u\|_{L^3}\,\|u_{tt}\|_{L^6} + C\|P_{tt}\|_{L^2}\,\|\nabla u_{tt}\|_{L^2} \\
&\le \delta\|\nabla u_{tt}\|_{L^2}^2 + C_\delta\|\rho^{1/2}u_{tt}\|_{L^2}^2 + C_\delta\|P_{tt}\|_{L^2}^2.
\end{aligned}
\tag{3.98}
$$

For any $\tau \in (0, T_*)$, since $t^{1/2}\sqrt{\rho}\,u_{tt} \in L^\infty(0, T_*; L^2)$ by (2.1), there exists some $t_0 \in (\tau/2, \tau)$ such that

$$
\begin{aligned}
\int \rho|u_{tt}|^2\,dx(t_0) &\le \frac{1}{t_0}\|t^{1/2}\sqrt{\rho}\,u_{tt}\|_{L^\infty(0,T_*;L^2)}^2 \\
&\le C(\tau).
\end{aligned}
\tag{3.99}
$$

Substituting (3.95)–(3.98) into (3.94) and choosing δ suitably small, one obtains by using (3.66) and (3.99) and Gronwall's inequality that

$$
\sup_{t_0\le t\le T}\int \rho|u_{tt}|^2\,dx + \int_{t_0}^{T}\int|\nabla u_{tt}|^2\,dx\,dt \le C(\tau),
$$

which, together with (3.86) and (3.67), yields that

$$\sup_{\tau\le t\le T}\|\nabla u_t\|_{H^1}+\int_\tau^T\int|\nabla u_{tt}|^2\,dx\,dt\le C(\tau) \tag{3.100}$$

due to $t_0<\tau$. Now, (3.92) follows from (3.88), (3.100), and (3.81). We have finished the proof of Lemma 3.11. □

4 Proofs of Theorems 1.1 and 1.2

With all the a priori estimates in Section 3 at hand, we are ready to prove the main results of this paper.

PROOF OF THEOREM 1.1. By Lemma 2.1, there exists a $T_*>0$ such that the Cauchy problem (1.1),(1.3),(1.4) has a unique classical solution (ρ,u) on $\mathbb{R}^3\times(0,T_*]$. We will use the a priori estimates, Proposition 3.1, and Lemmas 3.10 and 3.11 to extend the local classical solution (ρ,u) to all time.

First, it follows from (3.1), (3.2), (3.35), and (1.8) that

$$A_1(0)+A_2(0)=0,\quad A_3(0)\le C_0^{\delta_0},\quad \rho_0\le\bar\rho,$$

due to $C_0\le\varepsilon$. Therefore, there exists a $T_1\in(0,T_*]$ such that (3.4) holds for $T=T_1$.

Next, we set

$$T^*=\sup\{T\mid (3.4)\text{ holds}\}. \tag{4.1}$$

Then $T^*\ge T_1>0$. Hence, for any $0<\tau<T\le T^*$ with T finite, it follows from Lemmas 3.10 and 3.11 that

$$\nabla u_t,\nabla^3u\in C([\tau,T];L^2\cap L^4),\quad \nabla u,\nabla^2u\in C([\tau,T];L^2\cap C(\overline{\mathbb{R}^3})), \tag{4.2}$$

where we have used the standard embedding

$$L^\infty(\tau,T;H^1)\cap H^1(\tau,T;H^{-1})\hookrightarrow C([\tau,T];L^q)\quad\text{for any } q\in[2,6).$$

Due to (3.62), (3.67), and (3.92), one can get

$$\begin{aligned}
&\int_\tau^T\|(\rho|u_t|^2)_t\|_{L^1}\,dt\\
&\le\int_\tau^T(\|\rho_t|u_t|^2\|_{L^1}+2\|\rho u_t\cdot u_{tt}\|_{L^1})dt\\
&\le C\int_\tau^T\big(\|\rho|\mathrm{div}\,u|\,|u_t|^2\|_{L^1}\\
&\qquad+\||u|\,|\nabla\rho|\,|u_t|^2\|_{L^1}+\|\rho^{1/2}u_t\|_{L^2}\,\|\rho^{1/2}u_{tt}\|_{L^2}\big)dt\\
&\le C\int_\tau^T\big(\|\rho|u_t|^2\|_{L^1}\,\|\nabla u\|_{L^\infty}+\|u\|_{L^6}\,\|\nabla\rho\|_{L^2}\,\|u_t\|_{L^6}^2+\|\rho^{1/2}u_{tt}\|_{L^2}\big)dt\\
&\le C,
\end{aligned}$$

which yields

$$\rho^{1/2} u_t \in C([\tau, T]; L^2).$$

This, together with (4.2), gives

$$\rho^{1/2}\dot{u}, \quad \nabla\dot{u} \in C([\tau, T]; L^2). \tag{4.3}$$

Next we claim that

$$T^* = \infty. \tag{4.4}$$

Otherwise, $T^* < \infty$. Then by Proposition 3.1, (3.5) holds for $T = T^*$. It follows from Lemmas 3.10 and 3.11 and (4.3) that $(\rho(x, T^*), u(x, T^*))$ satisfies (1.7)–(1.9) except $u(\cdot, T^*) \in \dot{H}^\beta$, where $g(x) \triangleq \dot{u}(x, T^*)$, $x \in \mathbb{R}^3$. Thus, Lemma 2.1 implies that there exists some $T^{**} > T^*$ such that (3.4) holds for $T = T^{**}$, which contradicts (4.1). Hence (4.4) holds. Lemmas 2.1, 3.10, and 3.11 and (4.2) thus show that (ρ, u) is in fact the unique classical solution defined on $\mathbb{R}^3 \times (0, T]$ for any $0 < T < T^* = \infty$.

Finally, to finish the proof of Theorem 1.1, it remains to prove (1.13).

Multiplying (3.40) by $4(P - P(\widetilde{\rho}))^3$ and integrating the resulting equality over $\mathbb{R}^3$, one has

$$\begin{aligned}(\|P - P(\widetilde{\rho})\|_{L^4}^4)'(t) = &-(4\gamma - 1) \int (P - P(\widetilde{\rho}))^4 \operatorname{div} u \, dx \\ &- \gamma \int P(\widetilde{\rho})(P - P(\widetilde{\rho}))^3 \operatorname{div} u \, dx,\end{aligned}$$

which yields that

$$\begin{aligned}\int_1^\infty \left|(\|P - P(\widetilde{\rho})\|_{L^4}^4)'(t)\right| dt &\le C \int_1^\infty \left(\|P - P(\widetilde{\rho})\|_{L^4}^4 + \|\nabla u\|_{L^4}^4\right) dt \\ &\le C\end{aligned} \tag{4.5}$$

due to (3.43). Combining (3.43) with (4.5) leads to

$$\lim_{t \to \infty} \|P - P(\widetilde{\rho})\|_{L^4} = 0,$$

which together with (3.6) implies

$$\lim_{t \to \infty} \int |\rho - \widetilde{\rho}|^q \, dx = 0$$

for all q satisfying (1.14). Note that (3.6) and (2.2) imply

$$\int \rho^{1/2} |u|^4 \, dx \le \left(\int \rho |u|^2 \, dx \right)^{1/2} \|u\|_{L^6}^3 \le C \|\nabla u\|_{L^2}^3.$$

Thus (1.13) follows provided that

$$\lim_{t \to \infty} \|\nabla u\|_{L^2} = 0. \tag{4.6}$$

Setting

$$I(t) \triangleq \frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{\lambda + \mu}{2} \|\operatorname{div} u\|_{L^2}^2,$$

choosing $m = 0$ in (3.9), and using (3.11) and (3.12), one has

$$|I'(t)| \le C \int \rho |\dot{u}|^2 \, dx + C \|\nabla u\|_{L^3}^3 + C C_0^{1/2} \|\nabla \dot{u}\|_{L^2}, \tag{4.7}$$

where one has used the following simple estimate:

$$\begin{aligned} |M_1| = \left| \int \dot{u} \cdot \nabla P \, dx \right| &= \left| \int (P - P(\tilde{\rho})) \operatorname{div} \dot{u} \, dx \right| \\ &\le C C_0^{1/2} \|\nabla \dot{u}\|_{L^2}. \end{aligned}$$

We thus deduce from (4.7), (3.36), and (3.43) that

$$\begin{aligned} \int_1^\infty |I'(t)|^2 \, dt &\le C \int_1^\infty \left(\|\rho^{1/2} \dot{u}\|_{L^2}^4 + \|\nabla u\|_{L^2}^2 \, \|\nabla u\|_{L^4}^4 + \|\nabla \dot{u}\|_{L^2}^2 \right) dt \\ &\le C \int_1^\infty \left(\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^4}^4 + \|\nabla \dot{u}\|_{L^2}^2 \right) dt \\ &\le C, \end{aligned}$$

which, together with

$$\int_1^\infty |I(t)|^2 \, dt \le C \int_1^\infty \|\nabla u\|_{L^2}^2 \, dt \le C,$$

implies (4.6). The proof of Theorem 1.1 is finished. □

PROOF OF THEOREM 1.2. The proof is similar to that of theorem 1.2 in [26]. We just sketch it here.

If the conclusion in Theorem 1.2 is false, then there exist some constant $C_1 > 0$ and a subsequence $\{t_{n_j}\}_{j=1}^\infty$, $t_{n_j} \to \infty$, such that $\|\nabla \rho(\cdot, t_{n_j})\|_{L^r} \le C_1$. Hence, the Gagliardo-Nirenberg inequality (2.3) yields that there exists some positive constant C independent of t_{n_j} such that for $a = r/(2r-3) \in (0,1)$,

$$\begin{aligned} \|\rho(x, t_{n_j}) - \tilde{\rho}\|_{C(\overline{\mathbb{R}^3})} &\le C \|\nabla \rho(x, t_{n_j})\|_{L^r}^a \, \|\rho(x, t_{n_j}) - \tilde{\rho}\|_{L^3}^{1-a} \\ &\le C C_1^a \|\rho(x, t_{n_j}) - \tilde{\rho}\|_{L^3}^{1-a}. \end{aligned} \tag{4.8}$$

Due to (1.13), the right-hand side of (4.8) goes to 0 as $t_{n_j} \to \infty$. Hence

$$\|\rho(x, t_{n_j}) - \tilde{\rho}\|_{C(\overline{\mathbb{R}^3})} \to 0 \quad \text{as } t_{n_j} \to \infty. \tag{4.9}$$

On the other hand, since (ρ, u) is a classical solution satisfying (1.12), there exists a unique particle path $x_0(t)$ with $x_0(0) = x_0$ such that

$$\rho(x_0(t), t) \equiv 0 \quad \text{for all } t \ge 0.$$

So, we conclude from this identity that

$$\|\rho(x, t_{n_j}) - \tilde{\rho}\|_{C(\overline{\mathbb{R}^3})} \ge |\rho(x_0(t_{n_j}), t_{n_j}) - \tilde{\rho}| \equiv \tilde{\rho} > 0,$$

which contradicts (4.9). This completes the proof of Theorem 1.2. □

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