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Transonic Shocks for the Full Compressible Euler System in a General Two-Dimensional De Laval Nozzle

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Abstract

In this paper, we study the transonic shock problem for the full compressible Euler system in a general two-dimensional de Laval nozzle as proposed in Courant and Friedrichs (Supersonic flow and shock waves, Interscience, New York, 1948): given the appropriately large exit pressure $p_e(x)$, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the diverging part of the nozzle, a shock front intervenes and the gas is compressed and slowed down to subsonic speed so that the position and the strength of the shock front are automatically adjusted such that the end pressure at the exit becomes $p_e(x)$. We solve this problem completely for a general class of de Laval nozzles whose divergent parts are small and arbitrary perturbations of divergent angular domains for the full steady compressible Euler system. The problem can be reduced to solve a nonlinear free boundary value problem for a mixed hyperbolic–elliptic system. One of the key ingredients in the analysis is to solve a nonlinear free boundary value problem in a weighted Hölder space with low regularities for a second order quasilinear elliptic equation with a free parameter (the position of the shock curve at one wall of the nozzle) and non-local terms involving the trace on the shock of the first order derivatives of the unknown function.

1. Introduction

A general de Laval nozzle plays a fundamental role in the operation of turbines, wind tunnels and rockets. A de Laval nozzle consists of a converging "entry" section and a diverging "exhaust" section (see Fig. 1). In Section 147 of [9], the following transonic shock phenomenon in a de Laval nozzle was described by Courant–Friedrichs: given the appropriately large exit pressure $p_e(x)$, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the diverging part of the nozzle, a shock front intervenes and the gas is compressed and slowed

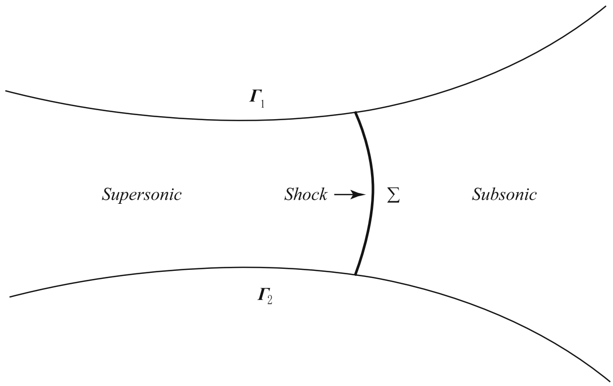

Fig. 1. De Laval Nozzle

down to subsonic speed so that the position and the strength of the shock front are automatically adjusted such that the end pressure at the exit becomes $p_e(x)$. In this paper, we will solve such a problem by establishing the existence and structural stability of a transonic shock solution in a general de Laval nozzle for the two-dimensional full steady compressible Euler system when $p_e(x)$ lies in a suitable scope.

The two-dimensional full steady compressible Euler system is

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) + \nabla P = 0, \\ \operatorname{div}(\rho(e + \frac{1}{2}|u|^2)u + uP) = 0, \\ P = P(\rho, S), \quad e = e(\rho, S), \end{cases} \tag{1.1}$$

where $u = (u_1, u_2)$, ρ, P, e and S stand for the velocity, density, pressure, internal energy and specific entropy, respectively. Moreover, $P = P(\rho, S)$ and $e = e(\rho, S)$ are smooth in their arguments. In particular, $\partial_\rho P(\rho, S) > 0$ and $\partial_S e(\rho, S) > 0$ for $\rho > 0$, and $c(\rho, S) = \sqrt{\partial_\rho P(\rho, S)}$ is called the local sound speed.

For the ideal polytropic gas, the equations of state are given by

$$P = D\rho^\gamma e^{\frac{S}{C_v}} \quad \text{and} \quad e = \frac{P}{(\gamma - 1)\rho},$$

with D, C_v and γ $(1 < \gamma < 3)$ being positive constants.

Since the divergent part of the de Laval nozzle to be considered in this paper is a general small perturbation of a straight angular domain, as in [30], it is more convenient to use the polar coordinates

$$(x_1, x_2) = (r\cos\theta, r\sin\theta), \tag{1.2}$$

$$(u_1, u_2) = (U_1\cos\theta - U_2\sin\theta, U_1\sin\theta + U_2\cos\theta) \tag{1.3}$$

to rewrite (1.1) as

$$\begin{cases} \partial_r(r\rho U_1) + \partial_\theta(\rho U_2) = 0, \\ \partial_r(\rho U_1^2 + P) + \frac{1}{r}\partial_\theta(\rho U_1 U_2) + \frac{1}{r}(\rho U_1^2 - \rho U_2^2) = 0, \\ \partial_r(\rho U_1 U_2) + \frac{1}{r}\partial_\theta(\rho U_2^2 + P) + \frac{2}{r}\rho U_1 U_2 = 0, \\ \partial_r\big(r(e + \frac{1}{2}|U|^2 + \frac{P}{\rho})\rho U_1\big) + \partial_\theta\big((e + \frac{1}{2}|U|^2 + \frac{P}{\rho})\rho U_2\big) = 0. \end{cases} \tag{1.4}$$

We now describe the class of two-dimensional de Laval nozzles to be investigated in this paper. Assume that the nozzle walls Γ_1 and Γ_2 are $C^{2,\alpha}$-regular with $\alpha \in (0, 1)$ for $X_0 - 1 < r = \sqrt{x_1^2 + x_2^2} < X_0 + 1$ $(X_0 > 1)$, where Γ_1^1 and Γ_2^1 include the walls for the converging part of the nozzle, while Γ_1^2 and Γ_2^2 consist of the divergent part of the nozzle (see Fig. 2). More precisely, let $\Gamma_i^2 (i = 1, 2)$ be represented by

$$\theta = (-1)^i\theta_0 + f_i(r), \quad r \geqq X_0 \tag{1.5}$$

with

$$f_i(r) \in C^{2,\alpha}[X_0, X_0 + 1] \quad \text{and} \quad \|f_i\|_{C^{2,\alpha}[X_0, X_0+1]} \leqq \varepsilon, \tag{1.6}$$

where $\varepsilon > 0$ is a suitably small constant, and $0 < \theta_0 < \frac{\pi}{2}$ is a fixed constant. In addition, for convenience, we will assume $f_i^{(j)}(X_0) = 0 (0 \leqq j \leqq 2)$ (this is to guarantee that the supersonic incoming flow is $C^{2,\alpha}$ smooth when

$$(U_1^-, U_2^-, P^-, S^-)(r, \theta)|_{r=X_0} \equiv (U_1^-, U_2^-, P^-, S^-)(X_0)$$

is given and the corresponding Bernoulli's constant is uniform).

Suppose that the transonic shock curve Σ and the flow behind Σ are represented by $r = \xi(\theta)$ and $(U_1^+(r, \theta), U_2^+(r, \theta), P^+(r, \theta), S^+(r, \theta))$ respectively. Then the Rankine–Hugoniot conditions on Σ are

$$\begin{cases} [\rho U_1] - \frac{\xi'(\theta)}{\xi(\theta)}[\rho U_2] = 0, \\ [\rho U_1^2 + P] - \frac{\xi'(\theta)}{\xi(\theta)}[\rho U_1 U_2] = 0, \\ [\rho U_1 U_2] - \frac{\xi'(\theta)}{\xi(\theta)}[\rho U_2^2 + P] = 0, \\ \Big[\rho U_1\Big(e + \frac{1}{2}|U|^2 + \frac{P}{\rho}\Big)\Big] - \frac{\xi'(\theta)}{\xi(\theta)}\Big[\rho U_2\Big(e + \frac{1}{2}|U|^2 + \frac{P}{\rho}\Big)\Big] = 0. \end{cases} \tag{1.7}$$

In addition, the following physical entropy condition should be satisfied (see [9]):

$$S^+(r, \theta) > S^-(r, \theta) \quad on \quad r = \xi(\theta). \tag{1.8}$$

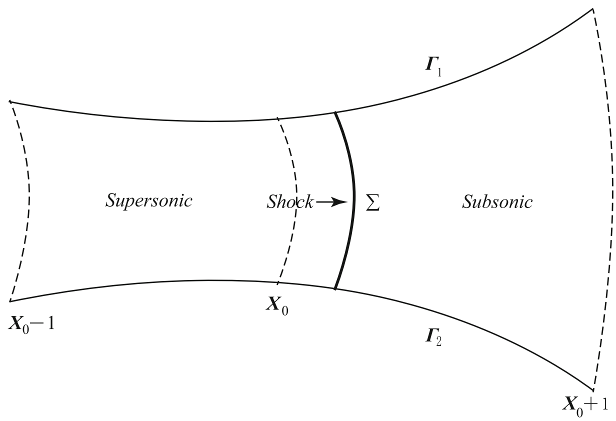

Fig. 2. Transonic shock in a nozzle

On the exit of the nozzle, the end pressure is prescribed by

$$P^+(X_0 + 1, \theta) = P_e + \varepsilon P_0(\theta) \tag{1.9}$$

with $P_0(\theta) \in C^{2,\alpha}[-\theta_0 + f_1(X_0 + 1), \theta_0 + f_2(X_0 + 1)]$ satisfying

$$\|P_0(\theta)\|_{C^{2,\alpha}} \leqq C, \tag{1.10}$$

and the constant P_e denoting the end pressure for which a symmetric transonic shock exists uniquely at the position $r = r_0$ with $r_0 \in (X_0, X_0 + 1)$ and the supersonic incoming flow is given by $(U_0^-(r), P_0^-(r), S_0^-)$ in the nozzle $\{(r, \theta) : X_0 \leqq r \leqq X_0 + 1, -\theta_0 \leqq \theta \leqq \theta_0\}$. For more details with respect to P_e and r_0, one can see Section 147 in [9] or Theorem 1.1 of [30].

The walls of the nozzle are assumed to be solid so that

$$U_2 = r f_i'(r) U_1 \quad \text{on} \quad \Gamma_i^2 (i = 1, 2). \tag{1.11}$$

By (1.6) and (1.11), one can show easily as in [15] that for suitably small ε, the supersonic incoming flow $(U_1^-, U_2^-, P^-, S^-)(r, \theta)$ can be extended into the domain $\Omega = \{(r, \theta) : X_0 < r < X_0 + 1, -\theta_0 + f_1(r) < \theta < \theta_0 + f_2(r)\}$ and satisfies

$$\|(U_1^-, U_2^-, P^-, S^-)(r, \theta) - (U_0^-(r), 0, P_0^-(r), S_0^-)\|_{C^{2,\alpha}(\bar{\Omega})} \leqq C\varepsilon, \tag{1.12}$$

here S_0^- is some fixed constant. Furthermore, without loss of generality and for the simplicity of computations, it will be assumed that

$$(U_1^-, U_2^-, P^-, S^-)(X_0, \theta) \equiv (U_0^-(X_0), 0, P_0^-(X_0), S_0^-)$$

is independent of the variable θ.

As has been stated in Section 147 of [9] or Theorem 1.1 of [30], for the given supersonic incoming flow $(U_0^-(r), P_0^-(r), S_0^-)$ in the nozzle

$$\{(r, \theta) : X_0 \leqq r \leqq X_0 + 1, -\theta_0 \leqq \theta \leqq \theta_0\},$$

there exits a unique radial symmetric transonic shock solution for the given constant end pressure P_e in a suitable scope. Furthermore, the position of the shock, $r = r_0$, depends monotonically on P_e. This solution will be called the background solution in this paper. Denote by $(U_0^+(r), P_0^+(r), S_0^+)$ (S_0^+ is a constant) the subsonic part of the background solution for $r_0 < r < X_0 + 1$, which can be extended into the domain $\{r : r_0 - \delta_0 \leqq r \leqq X_0 + 1\}$ ($\delta_0 > 0$ is some constant depending only on the supersonic incoming flow, see Theorem 1.1 of [30]) and the corresponding extension will be denoted by $(\hat{U}_0^+(r), \hat{P}_0^+(r), S_0^+)$.

With these preparations, the main results in this paper can be stated as follows:

Theorem 1.1. *Under the assumptions* (1.5)–(1.6)*, there exists an* $\varepsilon_0 > 0$ *such that for* $\varepsilon < \varepsilon_0$*, the problem* (1.4) *with* (1.7)–(1.12) *has a unique solution*

$$(U_1^+(r,\theta), U_2^+(r,\theta), P^+(r,\theta), S^+(r,\theta); \xi(\theta))$$

with the following properties:

(i). $\xi(\theta) \in C^{2,\alpha}(\theta^1, \theta^2) \cap C^{1,\alpha}[\theta^1, \theta^2]$. *Furthermore, it holds that*

$$\|\xi(\theta) - r_0\|_{C^{1,\alpha}[\theta^1,\theta^2]} \leqq C_0\varepsilon, \tag{1.13}$$

where $(r^i, \theta^i)(i = 1, 2)$ *stand for the intersection points of the shock curve with the nozzle walls and* C_0 *is a positive generic constant depending only on the supersonic incoming flow.*

(ii). $(U_1^+(r,\theta), U_2^+(r,\theta), P^+(r,\theta), S^+(r,\theta)) \in C^{1,\alpha}(\Omega_+) \cap C^\alpha(\bar{\Omega}_+)$*, and*

$$\|(U_1^+, U_2^+, P^+, S^+)(r,\theta) - (\hat{U}_0^+(r), 0, \hat{P}_0^+(r), S_0^+)\|_{C^\alpha(\bar{\Omega}_+)} \leqq C_0\varepsilon, \tag{1.14}$$

where Ω_+ *is the subsonic region given by*

$$\Omega_+ = \{(r,\theta) : \xi(\theta) < r < X_0 + 1, -\theta_0 + f_1(r) < \theta < \theta_0 + f_2(r)\}. \tag{1.15}$$

Remark 1.1. For general de Laval nozzles, the $C^\alpha(\bar{\Omega}_+)$ regularity of the subsonic flow in Theorem 1.1 is optimal even if $f_i(r)(i = 1, 2)$ is C^∞ smooth. This fact has been shown in Lemma 3.3 and Remark 3.2 of [27]. On the other hand, if the divergent part of the walls of the nozzle is straight, then the regularities of solution can be improved to $C^{2,\alpha}$ or even higher, see [5] and [30].

Remark 1.2. It should be emphasized that there is no restriction on the angle θ_0 of the divergent part of the nozzle in Theorem 1.1. Thus, we have removed the crucial assumption that the nozzle is nearly flat which was required in [5–8,17,27–30] and played an important role in their analysis since in this case, the principal linearized problem has constant coefficients.

Remark 1.3. Besides the assumption that the nozzle must be slowly varying, most of the previous known uniqueness and existence results (except [17]) require that the boundary condition at the exit has to be modified (other than the given exit pressure in [9]) and/or the shock curve must go through a fixed point on the wall of the nozzle in advance which makes the transonic shock problem ill-posed in general as shown in [27,29]. This paper has removed all these nonphysical assumptions and thus solved the transonic shock problem posed originally by Courant and Friedrichs [9] for a general de Laval nozzle.

Remark 1.4. Due to the low regularity of the transonic shock solution in general de Laval nozzles, the methods in [17,18] fail to work for the general case considered in this paper. In [17,18], the compatible conditions for the resulting second order elliptic equation can be verified in the iteration scheme, which is based on the geometric property of the straight nozzle. So the uniqueness and existence result can be obtained in the framework that the approximate solutions are bounded in $C^{2,\alpha}(\bar{\Omega}_+)-$norm and the related iteration scheme is contractible in $C^{1,\alpha}(\bar{\Omega}_+)-$norm.

Remark 1.5. It should be noted that the geometry of the nozzle with

$$f_i^{(j)}(X_0) = 0 (0 \leqq j \leqq 2)$$

and symmetry property of the supersonic incoming flow at X_0 are assumed just for simplicity in the presentation of the main ideas. In fact, these can be relaxed by modifying slightly the analysis here, see [26] for details.

Physically interesting problems involving transonic shocks have been investigated extensively in many important situations (see [3–11,14,16,21–24,26–35] and the references therein). Steady transonic shocks in multi-dimensional nozzles have been also studied recently by many authors for various boundary conditions ([5–8,16–18,21,26–30,34] and so on). Most of these known works deal with the uniqueness and existence of piecewise smooth transonic shock patterns in nozzles with slowly-varying cross sections either for various exit boundary conditions other than (1.9), or the given exit pressure condition (1.9), but with the additional condition that the shock curve is required to go through a fixed point a prior. Until recently, the first positive result confirming the transonic shock pattern proposed by COURANT and FRIEDRICHS [9] had been established by the authors in [17,18] for a class of two-dimensional finite nozzles with straight divergent parts without additional assumptions on the shock position. In this paper, we will study the Courant–Friedrich's transonic shock pattern in a class of general de Laval nozzles described in (1.5)–(1.6) and thus remove the stringent condition that the divergent part of the nozzle is straight in [17,18]. As is pointed out in Remark 1.1 and Remark 1.3, some new phenomena appear here.

We now make some comments on the key ingredients of the analysis in this paper. Since the supersonic flow with the given entrance conditions can be obtained in the whole nozzle, the transonic shock problem of Courant–Friedrich's is reduced to a free boundary value problem (with the transonic shock curve as the free boundary) of the full steady Euler system on the subsonic domain, where the Euler system is a mixed hyperbolic–elliptic system. Since the hyperbolic modes depend only on particle pathes, it turns out to be crucial to be able to introduce a globally defined Lagrangian coordinate to straighten stream lines. In the Lagrangian coordinate, the two-dimensional steady full Euler system on a subsonic domain can be decomposed formally as: a first order elliptic system for the pressure and the angular velocity, and hyperbolic transport equations for the entropy and the Bernoulli's function. However, the given exit pressure condition (1.9) becomes a nonlinear non-local boundary condition in the Lagrangian coordinate. To overcome this difficulty, one may introduce a potential like function ϕ which satisfies a second order nonlinear elliptic equation in the subsonic domain and a localized boundary condition at the exit of the nozzle. Reformulating the problem by first solving the hyperbolic equations for the entropy and the Bernoulli's function and fixing the free boundary, we can obtain a nonlinear boundary value problem on a fixed domain for a second nonlinear elliptic equation containing non-local terms (due to the hyperbolic modes) and an unknown constant (the intersection point of the shock curve with a wall of the nozzle), that is, (3.33), and an ordinary differential equation for the shock curve, (3.3). We then apply the contraction principle to study such a nonlinear problem. The key here is to design a careful iteration scheme which approximates the downstream subsonic solutions and the shock curve simultaneously. The main ingredients in our analysis are to investigate the solvability and a priori estimates for a linear boundary value problem of a second order linear elliptic equation with non-local terms and an unknown constant (see (4.1)) in a weighted Hölder space with lower regularities. Note that non-local terms in the problem (4.1) contain the trace of the first order derivative of the unknown function, it seems to be difficult to apply the Lax-Milgram theorem and the Fredholm alternative to obtain the solvability of the H^1-solution as in [19,21], where the facts that the non-local terms contain only the trace of unknown function itself and the solution has $C^{2,\alpha}$ regularity are crucial.

The main advantage of the Euler–Lagrange coordinate transformation defined by (2.1) in the next section is to straighten the stream lines. For nozzles with straight divergent walls in [17,18], since the compatible conditions are satisfied at the corner points in the subsonic domain, then the uniform $C^{2,\alpha}$ estimates for the subsonic flow are obtained. Due to the loss of regularities in the difference of two different stream lines, the contractible estimates can be obtained only in the $C^{1,\alpha}$ norm. So one can use the Banach contractible mapping theorem to obtain the existence and uniqueness of the transonic shock solution. For general de Laval nozzles in this paper, such compatible conditions at the corners are lost. At the corner points intersected by the free shock curve and the curved nozzle walls, the C^α regularity for the entropy is optimal. Then by the transport equation for the entropy and the fact that the walls of the nozzle are stream lines, the entropy has only C^α regularity near the walls of the nozzle. The low regularity of the entropy will affect the regularities of other physical quantities (the velocity and the pressure). And then it is difficult to obtain the contractible estimates in suitably weighted spaces. Fortunately, since the stream lines are all in the y_1-direction under the Euler–Lagrange coordinate, the loss of regularities in the difference of two different stream lines can be avoided. So the uniform estimates and contractible estimates can be done in the same weighted Hölder space and finally one can prove that the transonic shock solution with global C^α regularity exists uniquely.

The rest of the paper is organized as follows. In Section 2, we will show first that the Lagrange-transformation can be globally defined for the transonic shock solution in Theorem 1.1 and then reformulate the two-dimensional problem (1.4) with (1.7)–(1.12) to obtain a second quasi-linear elliptic equation for a potential function ϕ, which is coupled by an equation for the shock curve $z_1 = \psi(z_2)$ together with some nonlinear boundary conditions. See (2.19)–(2.25) in Section 2.

In Section 3, we analyze and simplify the resulting problem in Section 2 further to obtain a second order elliptic equation containing some non-local terms involving the trace on the shock of the first order derivatives of the unknown function with an unknown constant and an ordinary differential equation for the shock position, see (3.33) in Section 3. In Section 4, we focus on the linearized elliptic equation of (3.33) which contains an unknown constant and some nonlocal terms. By establishing a priori estimates in a suitably weighted Hölder space, we can show that such a linear problem can be solved uniquely and the unknown constant can be also determined simultaneously. In Section 5, making use of the uniform estimates for the linearized problem in Section 4, we can define an iteration scheme to complete the proofs on Theorem 2.1 and Theorem 1.1. In addition, some elementary computations and properties, which are used in Sections 3 and 4, are given in the Appendix A and Appendix B, respectively.

2. The Lagrange-Transformation and Reformulation of the Transonic Shock Problem

In this section, we reformulate the nonlinear problem (1.4) with (1.7)–(1.12) to obtain a second order elliptic equation for a potential function ϕ with an unknown constant and some nonlocal terms, a first order partial differential equation for the special entropy S^+, and an algebraic equation for the density ρ^+. To this end, it is technically important to use the Lagrange coordinates $(y_1, y_2) = (r, y_2(r, \theta))$ as in [25]. Based on the structure of the desired transonic flow pattern, the function $y_2 = y_2(r, \theta)$ can be piecewisely defined as

$$\begin{cases} \dfrac{\partial y_2}{\partial r} = -\rho^- U_2^-, \quad \dfrac{\partial y_2}{\partial \theta} = r\rho^- U_1^-, \quad \text{if} \quad (r, \theta) \in \bar{\Omega}_-, \\ \dfrac{\partial y_2}{\partial r} = -\rho^+ U_2^+, \quad \dfrac{\partial y_2}{\partial \theta} = r\rho^+ U_1^+, \quad \text{if} \quad (r, \theta) \in \bar{\Omega}_+, \\ y_2(X_0, -\theta_0) = 0, \quad y_2(X_0 + 1, -\theta_0 + f_1(X_0 + 1)) = 0, \end{cases} \tag{2.1}$$

where

$$\Omega_- = \{(r, \theta) : X_0 - 1 < r < \xi(\theta), -\theta_0 + f_1(r) < \theta < \theta_0 + f_2(r)\},$$
$$\Omega_+ = \{(r, \theta) : \xi(\theta) < r < X_0 + 1, -\theta_0 + f_1(r) < \theta < \theta_0 + f_2(r)\}.$$

It follows from the expected regularity for the piecewise smooth solution

$$(U_1^\pm, U_2^\pm, P^\pm, S^\pm)$$

in Theorem 1.1 and the first equation in (1.4) that $y_2(r, \theta)$ is well-defined in the domains $\bar{\Omega}_\pm$, respectively. We now illustrate that $y_2(r, \theta)$ is actually well-defined in $\bar{\Omega} = \{(r, \theta) : X_0 \leqq r \leqq X_0 + 1, -\theta_0 + f_1(r) \leqq \theta \leqq \theta_0 + f_2(r)\}$ and belongs to $Lip(\bar{\Omega})$.

It is noted that along the nozzle walls $\Gamma_i^2 (i = 1, 2)$,

$$\frac{\mathrm{d}}{\mathrm{d}r} y_2(r, (-1)^i \theta_0 + f_i(r)) = \frac{\partial y_2}{\partial r} + f_i'(r) \frac{\partial y_2}{\partial \theta} = -\rho U_2 + f_i'(r) r\rho U_1.$$

Thus, the solid wall boundary condition (1.11) yields

$$\frac{\mathrm{d}}{\mathrm{d}r} y_2(r, (-1)^i\theta_0 + f_i(r)) = 0 \qquad \text{on} \quad \Gamma_i^2 (i = 1, 2).$$

This, together with boundary conditions in (2.1), shows that

$$y_2(r, -\theta_0 + f_1(r)) = 0$$

holds true on the whole Γ_1^2. Similarly, one has

$$\begin{cases} y_2(r, \theta_0 + f_2(r)) = M & \text{for} \quad r \in [X_0 - 1, r^2], \\ y_2(r, \theta_0 + f_2(r)) = M_1 & \text{for} \quad r \in [r^2, X_0 + 1], \end{cases}$$

where M and M_1 are two constants to be determined, and (r^2, θ^2) is the intersection point of $r = \xi(\theta)$ with Γ_2^2.

To show that $y_2(r, \theta)$ is well-defined in $\bar{\Omega}$ and belongs to $Lip(\bar{\Omega})$, one needs to verify

$$\frac{\partial y_2(\xi(\theta) + 0, \theta)}{\partial\theta} = \frac{\partial y_2(\xi(\theta) - 0, \theta)}{\partial\theta}, \tag{2.2}$$

where the notations $y_2(\xi(\theta) \pm 0, \theta)$ stand for the limiting values of $y_2(r, \theta)$ when (r, θ) tends to $(\xi(\theta), \theta)$ in $\bar{\Omega}_\pm$, respectively.

Indeed, it follows from a direct computation that on $r = \xi(\theta)$,

$$\begin{aligned} &\frac{\partial y_2(\xi(\theta) \pm 0, \theta)}{\partial\theta} \\ &= \frac{\partial y_2}{\partial r}(\xi(\theta) \pm 0, \theta)\xi'(\theta) + \frac{\partial y_2}{\partial\theta}(\xi(\theta) \pm 0, \theta) \\ &= -\xi'(\theta)\rho^\pm U_2^\pm + \xi(\theta)\rho^\pm U_1^\pm. \end{aligned}$$

Combining this with the jump condition (1.7) yields (2.2) directly. It follows from (2.1) and (2.2) that $M_1 = M$ with

$$M = X_0 \int_{-\theta_0 + f_1(X_0)}^{\theta_0 + f_2(X_0)} \rho^-(X_0, \theta) U_1^-(X_0, \theta)\mathrm{d}\theta = 2X_0\theta_0\rho_0^-(X_0)U_0^-(X_0)$$

and further $y_2(r, \theta) \in C(\bar{\Omega})$. This, together with Theorem 1.1 (ii), implies that $y_2(r, \theta) \in Lip(\bar{\Omega}) \cap C^{2,\alpha}(\Omega_-) \cap C^{2,\alpha}(\Omega_+)$. The Jacobian matrix of the transformation, $(r, \theta) \mapsto (r, y_2(r, \theta)) \equiv (y_1, y_2)$, is

$$\frac{\partial(y_1, y_2)}{\partial(r, \theta)} = \begin{pmatrix} 1 & 0 \\ -\rho U_2 & r\rho U_1 \end{pmatrix}$$

which is reversible.

This coordinate transformation is called the Lagrange transformation. Under such a transformation, the domain

$$\bar{\Omega} = \{(r, \theta) : X_0 \leqq r \leqq X_0 + 1, -\theta_0 + f_1(r) \leqq \theta \leqq \theta_0 + f_2(r)\}$$

is changed into $[X_0, X_0+1]\times[0,M]$, and the system (1.4) can be transformed into

$$\begin{cases} \partial_{y_1}\Big(\dfrac{1}{y_1\rho U_1}\Big)-\partial_{y_2}\Big(\dfrac{U_2}{y_1U_1}\Big)=0,\\ \partial_{y_1}\Big(U_1+\dfrac{P}{\rho U_1}\Big)-\partial_{y_2}\Big(\dfrac{PU_2}{U_1}\Big)-\dfrac{P}{y_1\rho U_1}-\dfrac{\rho(U_2)^2}{y_1\rho U_1}=0,\\ \partial_{y_1}\big(y_1U_2\big)+\partial_{y_2}\big(y_1P\big)=0,\\ \partial_{y_1}\Big(e+\dfrac12(U_1)^2+\dfrac12(U_2)^2+\dfrac{P}{\rho}\Big)=0. \end{cases} \tag{2.3}$$

Moreover, the nozzle walls Γ_1^2 and Γ_2^2 are changed into $y_2=0$ and $y_2=M$, respectively.

Suppose that the transonic shock curve Σ and the flow field behind Σ are denoted by $y_1=\psi(y_2)$ and $(U_1^+(y), U_2^+(y), P^+(y), S^+(y))$, respectively, in the (y_1,y_2) coordinate. Then the Rankine–Hugoniot conditions on Σ become

$$\begin{cases} \Big[\dfrac{1}{\rho U_1}\Big]+\psi'(y_2)\Big[\dfrac{U_2}{U_1}\Big]=0,\\ \Big[U_1+\dfrac{P}{\rho U_1}\Big]+\psi'(y_2)\Big[\dfrac{PU_2}{U_1}\Big]=0,\\ [U_2]-\psi'(y_2)[P]=0,\\ \Big[e+\dfrac12U_1^2+\dfrac12U_2^2+\dfrac{P}{\rho}\Big]=0. \end{cases} \tag{2.4}$$

In addition, the conditions (1.8)–(1.11) become

$$S^+(y)>S^-(y)\quad\text{on}\quad y_1=\psi(y_2), \tag{2.5}$$

$$P^+(X_0+1,y_2)=P_{\mathrm e}+\varepsilon P_0(\theta(X_0+1,y_2)) \tag{2.6}$$

and

$$U_2^+=y_1f_i'(y_1)U_1^+\quad\text{on}\quad \Gamma_i^2\quad(i=1,2), \tag{2.7}$$

where the function $\theta=\theta(X_0+1,y_2)$ is given by

$$\theta(X_0+1,y_2)=-\theta_0+f_1(X_0+1)+\int_0^{y_2}\frac{1}{(X_0+1)(\rho^+U_1^+)(X_0+1,s)}\mathrm{d}s, \tag{2.8}$$

following from $\partial_{y_2}\theta=\dfrac{1}{y_1\rho^+U_1^+}$ for $y\in\{y : y_1\in(\psi(y_2),X_0+1), y_2\in(0<M)\}$ and $\theta(X_0+1,0)=-\theta_0+f_1(X_0+1)$. Here, it should be emphasized that the exit pressure condition (2.6) becomes non-local and nonlinear in the Lagrange coordinate and will require special care, see Part 4 of Section 3.

To simplify the problem (2.3)–(2.7) further, one notes the first equation in (2.3) and may introduce a potential function to reduce the system (2.3) into a second order equation with a nonlinear boundary condition at the exit.

Indeed, the first equation in (2.3) implies that there exists a potential function $\phi(y)$ such that

$$\partial_{y_1}\phi = \frac{U_2}{y_1 U_1}, \quad \partial_{y_2}\phi = \frac{1}{y_1 \rho U_1}, \quad \phi(X_0+1,0)=0. \tag{2.9}$$

Due to the fourth equalities in (2.3) and (2.4), the following Bernoulli's law holds:

$$\frac{1}{2}U_1^2 + \frac{1}{2}U_2^2 + \frac{\gamma}{\gamma-1}\frac{P}{\rho} = B \tag{2.10}$$

with $B = \frac{1}{2}(U_0^-)^2(X_0) + \frac{\gamma}{\gamma-1}\frac{P_0^-(X_0)}{\rho_0^-(X_0)}$.

Since the entropy S is invariant along each stream line behind the shock, the equation of state implies

$$P = A(y_2)\rho^\gamma, \tag{2.11}$$

here $A(y_2)$ is a function depending only on y_2 in the subsonic region. It is noted that the function $A(y_2)$ is to be determined by the incoming flow

$$(U_1^-(y), U_2^-(y), P^-(y), S^-(y))$$

and the Rankine–Hugoniot conditions (2.4) on the shock Σ, given by $y_1 = \psi(y_2)$.

Thus,

$$\frac{1+(y_1\partial_{y_1}\phi)^2}{2(y_1\rho\partial_{y_2}\phi)^2} + \frac{\gamma}{\gamma-1}A(y_2)\rho^{\gamma-1} = B. \tag{2.12}$$

In addition, the third equation in (2.3) gives that

$$\partial_{y_1}\left(\frac{y_1\partial_{y_1}\phi}{\rho\partial_{y_2}\phi}\right) + \partial_{y_2}\big(y_1 A(y_2)\rho^\gamma\big) = 0. \tag{2.13}$$

The Rankine–Hugoniot conditions (2.4) on $y_1 = \psi(y_2)$ can be rewritten as

$$\begin{cases} \psi'(y_2) = \big(\dfrac{\partial_{y_1}\phi}{\rho\partial_{y_2}\phi} - U_2^-(\psi(y_2),y_2)\big)/\big(A\rho^\gamma - P^-(\psi(y_2),y_2)\big), \\ \partial_{y_2}\phi = \dfrac{1}{\psi(y_2)(\rho^-U_1^-)(\psi(y_2),y_2)} - \left(\partial_{y_1}\phi - \dfrac{U_2^-(\psi(y_2),y_2)}{\psi(y_2)U_1^-(\psi(y_2),y_2)}\right)\psi'(y_2), \\ \dfrac{1}{\psi(y_2)\rho\partial_{y_2}\phi} + \psi(y_2)\partial_{y_2}\phi A\rho^\gamma = U_1^-(\psi(y_2),y_2) + \dfrac{P^-(\psi(y_2),y_2)}{(\rho^-U_1^-)(\psi(y_2),y_2)} \\ \qquad - \left(\psi(y_2)A\rho^\gamma\partial_{y_1}\phi - \dfrac{(P^-U_2^-)(\psi(y_2),y_2)}{U_1^-(\psi(y_2),y_2)}\right)\psi'(y_2). \end{cases} \tag{2.14}$$

And the boundary conditions (2.7) become

$$\partial_{y_1}\phi = f_i'(y_1) \quad \text{on} \quad y_2 = (i-1)M, \quad i = 1,2. \tag{2.15}$$

Most importantly, due to (2.8), the integral-type boundary condition (2.6) on the exit of the nozzle becomes a nonlinear boundary condition

$$P = P_{\mathrm{e}} + \varepsilon P_0(-\theta_0 + f_1(X_0+1) + \phi(X_0+1, y_2)) \quad \text{on} \quad y_1 = X_0+1. \tag{2.16}$$

In order to treat the free boundary problem (2.12)–(2.16), one may first convert it into a fixed boundary problem. Thus, set

$$\begin{cases} z_1 = \dfrac{y_1 - \psi(y_2)}{X_0 + 1 - \psi(y_2)} N, \\ z_2 = y_2 \end{cases} \tag{2.17}$$

with $N = X_0 + 1 - r_0$.

Under the transformation (2.17), the subsonic domain

$$\{(y_1, y_2) : \psi(y_2) < y_1 < X_0 + 1, 0 < y_2 < M\}$$

is changed into

$$E_+ = \{(z_1, z_2) : 0 < z_1 < N, 0 < z_2 < M\}. \tag{2.18}$$

The equation (2.13) becomes such a divergence form

$$\partial_{z_1}\big(N_1(\nabla\phi, A, \psi)\big) + \partial_{z_2}\big(N_2(\nabla\phi, A, \psi)\big) + N_3(\nabla\phi, A, \psi) = 0, \tag{2.19}$$

$$\begin{cases} N_1 = \left(\dfrac{N}{X_0 + 1 - \psi(z_2)}\right)^2 \dfrac{\partial_{z_1}\phi}{\rho\big(\partial_{z_2}\phi + \frac{(z_1-N)\psi'(z_2)}{X_0+1-\psi(z_2)}\partial_{z_1}\phi\big)} \\ \qquad + \dfrac{(z_1 - N)\psi'(z_2)}{X_0 + 1 - \psi(z_2)} A(z_2)\rho^\gamma, \\ N_2 = A(z_2)\rho^\gamma - A_0^+(\rho_0^+(z_1))^\gamma, \\ N_3 = \dfrac{\dfrac{N\partial_{z_1}\phi}{X_0 + 1 - \psi(z_2)}}{\big(\psi(z_2) + \frac{z_1}{N}(X_0 + 1 - \psi(z_2))\big)\rho\big(\partial_{z_2}\phi + \frac{z_1-N}{X_0+1-\psi(z_2)}\psi'(z_2)\partial_{z_1}\phi\big)} \\ \qquad - \dfrac{\psi'(z_2)}{X_0 + 1 - \psi(z_2)} A(z_2)\rho^\gamma. \end{cases} \tag{2.20}$$

While the Rankine–Hugoniot conditions, (2.14), can be rewritten as for $z_1 = 0$ and $z_2 \in [0, M]$,

$$\begin{aligned} \psi'(z_2) = &\frac{1}{A(z_2)\rho^\gamma - P^-(\psi(z_2), z_2)} \\ &\times\left(\frac{\frac{N}{X_0+1-\psi(z_2)}\partial_{z_1}\phi}{\rho\big(\partial_{z_2}\phi - \frac{N}{X_0+1-\psi(z_2)}\psi'(z_2)\partial_{z_1}\phi\big)} - U_2^-(\psi(z_2), z_2)\right) \end{aligned} \tag{2.21}$$

and

$$\begin{cases} \partial_{z_2}\phi \\ \quad = \dfrac{1}{\psi(z_2)(\rho^- U_1^-)(\psi(z_2), z_2)} + \dfrac{U_2^-(\psi(z_2), z_2)}{\psi(z_2)U_1^-(\psi(z_2), z_2)}\psi'(z_2), \\ \dfrac{1}{\psi(z_2)\rho\big(\partial_{z_2}\phi - \frac{N}{X_0+1-\psi(z_2)}\psi'(z_2)\partial_{z_1}\phi\big)} \\ \quad + \psi(z_2)\left(\partial_{z_2}\phi - \dfrac{N}{X_0+1-\psi(z_2)}\psi'(z_2)\partial_{z_1}\phi\right)A(z_2)\rho^\gamma \\ \quad = U_1^-(\psi(z_2), z_2) + \dfrac{P^-(\psi(z_2), z_2)}{(\rho^- U_1^-)(\psi(z_2), z_2)} \\ \quad - \psi'(z_2)\left(\psi(z_2)A(z_2)\rho^\gamma \dfrac{N\partial_{z_1}\phi}{X_0+1-\psi(z_2)} - \dfrac{(P^- U_2^-)(\psi(z_2), z_2)}{U_1^-(\psi(z_2), z_2)}\right). \end{cases} \tag{2.22}$$

Meanwhile, the fixed wall conditions (2.15) and the given exit pressure condition (2.16) are changed, respectively, into

$$\partial_{z_1}\phi = f_i'\big(\psi((i-1)M) + \frac{X_0+1-\psi((i-1)M)}{N}\big) \tag{2.23}$$

on $z_2 = (i-1)M (i = 1, 2)$ and

$$P = P_e + \varepsilon P_0(-\theta_0 + f_1(X_0+1) + \phi(N, z_2)\big). \tag{2.24}$$

In addition, the Bernoulli's law (2.12) becomes

$$\frac{1 + (\psi(z_2) + \frac{z_1}{N}(X_0+1-\psi(z_2)))^2\big(\frac{N}{X_0+1-\psi(z_2)}\partial_{z_1}\phi\big)^2}{2\rho^2(\psi(z_2) + \frac{z_1}{N}(X_0+1-\psi(z_2)))^2(\partial_{z_2}\phi + \frac{z_1-N}{X_0+1-\psi(z_2)}\psi'(z_2)\partial_{z_1}\phi)^2} + \frac{\gamma}{\gamma-1}A(z_2)\rho^{\gamma-1} = B. \tag{2.25}$$

Thus, we have transformed the free boundary value problem (2.3)–(2.7) into a boundary value problem (2.19)–(2.25) on the fixed rectangle E_+. To prove Theorem 1.1, we will use some weighted Hölder spaces introduced in [12].

For $x = (x_1, x_2)$ and $y = (y_1, y_2) \in E_+$, set

$$d_x = \min\{x_2, M - x_2\}, d_{x,y} = \min\{d_x, d_y\}.$$

For $m \in \mathbb{N} \cup \{0\}$, $0 < \alpha < 1$, $\sigma \in \mathbb{R}$ and $u \in C^{m,\alpha}(E_+)$, one defines

$$[u]^{(\sigma)}_{k,0;E_+} = \sup_{|\beta|=k} d_x^{\max(k+\sigma,0)}|D^\beta u|, \quad k = 0, \dots, m;$$

$$[u]^{(\sigma)}_{m,\alpha;E_+} = \sup_{|\beta|=m} d_{x,y}^{\max(m+\alpha+\sigma,0)}\frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha};$$

$$\|u\|^{(\sigma)}_{m,\alpha;E_+} = \sum_{i=0}^m [u]^{(\sigma)}_{i,0;E_+} + [u]^{(\sigma)}_{m,\alpha;E_+}$$

with the corresponding function space defined as

$$H^{(\sigma)}_{m,\alpha}(E_+) = \{u \in C^{m,\alpha}(E_+) : \|u\|^{(\sigma)}_{m,\alpha;E_+} < +\infty\}.$$

Similarly, for $\Gamma = (0, M)$ and $v(z_2) \in C^{m,\alpha}(\Gamma)$, one can define $[v]^{(\sigma)}_{m,0;\Gamma}$, $[v]^{(\sigma)}_{m,\alpha;\Gamma}$, $\|v\|^{(\sigma)}_{m,\alpha;\Gamma}$ and $H^{(\sigma)}_{m,\alpha}(\Gamma)$, respectively.

In addition, in the case of no confusion, we often neglect the subscripts E_+ or Γ in the norms $\|u\|^{(\sigma)}_{m,\alpha;E_+}$ or $\|v\|^{(\sigma)}_{m,\alpha;\Gamma}$, respectively.

Remark 2.1. As shown in [12], for any nonnegative integers m and l with $m \geqq l$ and $\alpha, \beta \in (0, 1)$ with $\alpha \geqq \beta$, the space $H^{(-l-\beta)}_{m,\alpha}(E_+)$ can be embedded into the Hölder space $C^{l,\beta}(\bar{E}_+)$. Moreover, the following estimate holds

$$\|U\|_{C^{l,\beta}(\bar{E}_+)} \leqq C\|U\|^{(-l-\beta)}_{m,\alpha} \quad \forall\, U \in H^{(-l-\beta)}_{m,\alpha}(E_+).$$

It is noted that the Lagrange coordinate (y_1, y_2) and the Euler coordinate are equivalent when the solution belongs to $Lip(\bar{\Omega}) \cap C^{2,\alpha}(\Omega_-) \cap C^{2,\alpha}(\Omega_+)$, which has been shown in the discussions of (2.1)–(2.2). Then by a direct verification, Theorem 1.1 follows from the following result:

Theorem 2.1. *Under the assumptions* (1.5)–(1.6)*, there exists an* $\varepsilon_0 > 0$ *such that for* $\varepsilon < \varepsilon_0$*, the boundary value problem* (2.19)–(2.25) *has a unique solution* $(\phi(z); A(z_2), \psi(z_2)) \in H^{(-1-\alpha)}_{2,\alpha}(E_+) \times H^{(-\alpha)}_{1,\alpha}(0, M) \times H^{(-1-\alpha)}_{2,\alpha}(0, M)$ *with the following estimates*

$$\|\phi(z) - \phi^+_0(z_2)\|^{(-1-\alpha)}_{2,\alpha} + \|A(z_2) - A^+_0\|^{(-\alpha)}_{1,\alpha} + \|\psi(z_2) - r_0\|^{(-1-\alpha)}_{2,\alpha} \leqq C\varepsilon, \tag{2.26}$$

where $\phi^+_0(z_2) = \dfrac{z_2}{r_0\rho^+_0(r_0)U^+_0(r_0)}$, $A^+_0 = De^{\frac{S^+_0}{c_v}}$, *and the positive constant* C *depends only on* α, θ_0 *and the supersonic incoming flow.*

Remark 2.2. In fact, it can be verified directly that $\phi^+_0(z_2)$ is the potential function defined by (2.9) corresponding to the background transonic shock solution $(U^\pm_0, 0, P^\pm_0, S^\pm_0; r_0)$.

Remark 2.3. It follows from the estimate (2.26) that the solution

$$(U^+_1, U^+_2, P^+, S^+; \psi(y_2))$$

of the problem (2.3) with (2.4)–(2.7) satisfies

$$\|(U^+_1, U^+_2, P^+, S^+) - (U^+_0, 0, P^+_0, S^+_0)\|_{C^\alpha} + \|\psi(y_2) - r_0\|_{C^{1,\alpha}} \leqq C\varepsilon.$$

Thus, $(r, \theta) = (y_1, \theta(y))$ defined by the inverse transform of (2.9) and (2.1), is in $C^{1,\alpha}$, and the solution of the problem (1.4) with (1.7)–(1.12) is obtained, which satisfies the estimates in Theorem 1.1.

3. Reductions of the Problem

In this section, we reformulate the nonlinear boundary problem (2.19)–(2.25) so that an elaborate iteration scheme can be constructed in Section 5.

For $z \in E_+$, set $W = (W_1, W_2, W_3)$ with

$$\begin{cases} W_1(z) = \phi(\psi(z_2) + \dfrac{X_0 + 1 - \psi(z_2)}{N} z_1, z_2) - \phi_0^+(z_2), \\ W_2(z_2) = A(z_2) - A_0^+, \\ W_3(z_2) = \psi(z_2) - r_0. \end{cases} \tag{3.1}$$

Define

$$\Xi_\delta = \{W : \|(W_1, W_3)\|_{2,\alpha}^{(-1-\alpha)} + \|W_2\|_{1,\alpha}^{(-\alpha)} \leqq \delta, W_1(N, 0) = 0\}, \tag{3.2}$$

with δ to be determined. Note that $W_1(N, 0) = 0$ is due to the normalization condition in (2.9).

It is clear that for any $W = (W_1, W_2, W_3)$ and $W^i = (W_1^i, W_2^i, W_3^i) \in \Xi_\delta (i = 1, 2)$, one can obtain the corresponding $(\phi, A(z_2), \psi(z_2), \rho)$ and $(\phi_i, A_i(z_2), \psi_i(z_2), \rho_i(z))$, respectively, by (3.1) and (2.25).

Our main strategy to solve the boundary value problem (2.19)–(2.25) is by an elaborate iteration scheme to decouple effectively the hyperbolic mode from the subsonic Euler system and take into account the location of the transonic shock. To this end, we will transform the problem (2.19)–(2.25) further to obtain the principle parts. This will be done by the following five parts.

For convenience, in what follows, we will use the conventions that:

$O(\kappa)$ means that there exists a generic constant $C > 0$ depending only on the supersonic incoming flow and r_0 such that $\|O(\kappa)\|_{1,\alpha}^{(-\alpha)} \leqq C\kappa$.

Part 1. Reduction for the Shock Location

It follows from (2.21) that

$$W_3'(z_2) = a_1 \partial_{z_1} W_1 + F_1(\nabla W_1, W_2, W_3) \quad \text{on} \quad z_1 = 0, \tag{3.3}$$

where $a_1 = \dfrac{1}{(A_0^+(\rho_0^+)^\gamma(r_0) - P_0^-(r_0))\rho_0^+(r_0)\partial_{z_2}\phi_0^+} > 0$, and

$$\begin{aligned} &F_1(\nabla W_1, W_2, W_3) \\ &= \frac{1}{A(z_2)\rho^\gamma - P^-(\psi(z_2), z_2)} \times \frac{\frac{N}{X_0+1-\psi(z_2)}\partial_{z_1} W_1}{\rho\big(\partial_{z_2}\phi - \dfrac{N}{X_0 + 1 - \psi(z_2)}\psi'(z_2)\partial_{z_1}\phi\big)} \\ &\quad - \frac{1}{P_0^+(r_0) - P_0^-(r_0)} \frac{\partial_{z_1} W_1}{\rho_0^+ \partial_{z_2}\phi_0^+} - \frac{U_2^-(\psi(z_2), z_2)}{A(z_2)\rho^\gamma - P^-(\psi(z_2), z_2)}. \end{aligned}$$

We claim that $F_1(\nabla W_1, W_2, W_3)$ is a higher order error term.

Indeed, set

$$\begin{cases} M_1(\nabla W_1, W_2, W_3) = \rho_0^+\partial_{z_2}\phi_0^+ - \rho\partial_{z_2}\phi + \dfrac{N}{X_0+1-\psi(z_2)}\rho W_3'(z_2)\partial_{z_1}W_1, \\ M_2(\nabla W_1, W_2, W_3) = \rho\big(\partial_{z_2}\phi - \dfrac{N}{X_0+1-\psi(z_2)}W_3'(z_2)\partial_{z_1}W_1\big), \\ M_3(\nabla W_1, W_2, W_3) = P_0^+(r_0) - A(z_2)\rho^\gamma + P^-(\psi(z_2), z_2) - P_0^-(r_0), \\ M_4(\nabla W_1, W_2, W_3) = A(z_2)\rho^\gamma - P^-(\psi(z_2), z_2). \end{cases} \tag{3.4}$$

Then $F_1(\nabla W_1, W_2, W_3)$ can be expressed as

$$\begin{aligned} &F_1(\nabla W_1, W_2, W_3) \\ &= \frac{1}{P_0^+(r_0) - P_0^-(r_0)} \times \Bigg\{ \frac{\partial_{z_1}W_1 M_1(\nabla W_1, W_2, W_3)}{M_2(\nabla W_1, W_2, W_3)} \\ &\quad + \Bigg(\frac{\frac{N}{X_0+1-\psi(z_2)}\partial_{z_1}W_1}{M_2(\nabla W_1, W_2, W_3)} - U_2^-(\psi(z_2), z_2)\Bigg) \frac{M_3(\nabla W_1, W_2, W_3)}{M_4(\nabla W_1, W_2, W_3)} \\ &\quad - U_2^-(\psi(z_2), z_2) + \frac{\partial_{z_1}W_1}{M_2(\nabla W_1, W_2, W_3)} \times \frac{W_3}{X_0+1-\psi(z_2)}\Bigg\}. \end{aligned} \tag{3.5}$$

A direct computation shows that

$$\begin{cases} M_1(\nabla W_1^i, W_2^i, W_3^i) \\ \quad = -\partial_{z_2}\phi_0^+(\rho_i - \rho_0^+) - \rho_i\partial_{z_2}W_1^i + \dfrac{N}{X_0+1-\psi}\rho_i(W_3^i)'\partial_{z_1}W_1^i, \quad i = 1, 2, \\ M_1(\nabla W_1^1, W_2^1, W_3^1) - M_1(\nabla W_1^2, W_2^2, W_3^2) \\ \quad = \partial_{z_2}\phi_1(\rho_1 - \rho_2) + \rho_2\partial_{z_2}(W_1^1 - W_1^2) \\ \qquad + \dfrac{N}{(X_0+1-\psi_1)(X_0+1-\psi_2)}\rho_1(W_3^1)'\partial_{z_1}W_1^1(W_3^1 - W_3^2) \\ \qquad + \dfrac{N}{X_0+1-\psi_2}\Big((W_3^1)'\partial_{z_1}W_1^1(\rho_1-\rho_2) + \rho_2\partial_{z_1}W_1^1(W_3^1 - W_3^2)' \\ \qquad + \rho_2(W_3^2)'\partial_{z_1}(W_1^1 - W_1^2)\Big). \end{cases}$$

Combining this with the definition of the space Ξ_δ and the estimate (A.3)–(A.4) in Lemma A.2 yields

$$\begin{cases} M_1(\nabla W_1^i, W_2^i, W_3^i) \\ \quad = O(1)(\rho_i - \rho_0^+) + O(1)\nabla W_1^i + O(\delta^2)W_3^i(z_2) + O(\delta)(W_3^i)'(z_2) \\ \quad = O(\delta), \\ M_1(\nabla W_1^1, W_2^1, W_3^1) - M_1(\nabla W_1^2, W_2^2, W_3^2) \\ \quad = O(1)(\rho_1 - \rho_2) + O(1)\nabla(W_1^1 - W_2^1) + O(\delta^2)(W_3^1 - W_3^2) \\ \qquad + O(\delta)(W_3^1 - W_3^2)'(z_2). \end{cases}$$

Similarly, one has

$$\begin{cases} M_1(\nabla W_1^i, W_2^i, W_3^i) = O(\delta), \\ M_2(\nabla W_1^i, W_2^i, W_3^i) = O(1)(O(1) + O(\delta^2)) = O(1), \\ M_3(\nabla W_1^i, W_2^i, W_3^i) = O(1)(\rho_i - \rho_0^+) + O(1)W_2^i \\ \qquad\qquad + \int_0^1 (P_0^-)'(s\psi_i(z_2) + (1-s)r_0)\mathrm{d}s \cdot W_3^i \\ \qquad\qquad + P^-(\psi_i(z_2), z_2) - P_0^-(\psi_i(z_2)) \\ \qquad = O(\delta) + O(\varepsilon), \\ M_4(\nabla W_1^i, W_2^i, W_3^i) = O(1), \end{cases} \tag{3.6}$$

and

$$\begin{cases} M_j(\nabla W_1^1, W_2^1, W_3^1) - M_j(\nabla W_1^2, W_2^2, W_3^2) \\ = O(1)(\rho_1 - \rho_2) + O(1)\nabla(W_1^1 - W_2^1) \\ \quad + O(\delta^2)(W_3^1 - W_3^2) + O(\delta)(W_3^1 - W_3^2)'(z_2), \quad j = 1, 2 \\ M_k(\nabla W_1^1, W_2^1, W_3^1) - M_k(\nabla W_1^2, W_2^2, W_3^2) \\ = O(1)(\rho_1 - \rho_2) + O(1)(W_2^1 - W_2^2) \\ \quad + \int_0^1 (\partial_1 P^-)(s\psi_1(z_2) + (1-s)\psi_2(z_2), z_2)\mathrm{d}s \cdot (W_3^1 - W_3^2) \\ = O(1)(\rho_1 - \rho_2) + O(1)(W_2^1 - W_2^2) + O(1)(W_3^1 - W_3^2), \quad k = 3, 4. \end{cases} \tag{3.7}$$

Substituting (3.6)–(3.7) into (3.5) and applying Lemma A.1 yield

$$\begin{cases} F_1(\nabla W_1^i, W_2^i, W_3^i) \\ \quad = O(\delta^2) + O(\delta(\delta + \varepsilon)) + O(\varepsilon) \\ \quad = O(\delta^2 + \varepsilon), \\ F_1(\nabla W_1^1, W_2^1, W_3^1) - F_1(\nabla W_1^2, W_2^2, W_3^2) \\ \quad = O(\delta)\partial_{z_1}(W_1^1 - W_1^2) \\ \qquad + O(\delta + \varepsilon)\sum_{i=1}^{4}\big(M_i(\nabla W_1^1, W_2^1, W_3^1) - M_i(\nabla W_1^2, W_2^2, W_3^2)\big) \\ \qquad + O(\delta + \varepsilon)(W_3^1 - W_3^2) + O(1)(U_2^-(\psi_1(z_2), z_2) - U_2^-(\psi_2(z_2), z_2)). \end{cases}$$

Thus it follows from (1.12), (3.7), Lemma A.1 and Lemma A.2 that

$$\|F_1(\nabla W_1^i, W_2^i, W_3^i)\|_{1,\alpha}^{(-\alpha)} \leqq C_0(\delta^2 + \varepsilon), \quad i = 1, 2, \tag{3.8}$$

and

$$\begin{aligned} &\|F_1(\nabla W_1^1, W_2^1, W_3^1) - F_1(\nabla W_1^2, W_2^2, W_3^2)\|_{1,\alpha}^{(-\alpha)} \\ &\quad \leqq C_0(\delta + \varepsilon)\left(\|(\nabla(W_1^1 - W_1^2), W_2^1 - W_2^2)\|_{1,\alpha}^{(-\alpha)} + \|W_3^1 - W_3^2\|_{2,\alpha}^{(-1-\alpha)}\right). \end{aligned} \tag{3.9}$$

This verifies that $F_1(\nabla W_1, W_2, W_3)$ is a higher order error term, which can be controlled as seen later.

Part 2. Reduction of the Boundary Condition for the Momentum on the Shock Curve

It follows from the first equation in (2.22) that

$$\partial_{z_2} W_1(0, z_2) = F_2(\nabla W_1, W_2, W_3), \tag{3.10}$$

where

$$F_2(\nabla W_1, W_2, W_3) = \frac{1}{\psi(z_2)(\rho^- U_1^-)(\psi(z_2), z_2)} - \frac{1}{r_0(\rho_0^- U_0^-)(r_0)} + \frac{\psi'(z_2) U_2^-(\psi(z_2), z_2)}{\psi(z_2)(\rho^- U_1^-)(\psi(z_2), z_2)}.$$

Since

$$\frac{1}{\psi(z_2)(\rho_0^- U_0^-)(\psi(z_2), z_2)} = \frac{1}{r_0(\rho_0^- U_0^-(r_0))},$$

then F_2 can be estimated as for $F_1(\nabla W_1, W_2, W_3)$ to obtain

$$\|F_2(\nabla W_1^i, W_2^i, W_3^i)\|_{1,\alpha}^{(-\alpha)} \leqq C_0(\delta^2 + \varepsilon), \qquad i = 1, 2, \tag{3.11}$$

and

$$\|F_2(\nabla W_1^1, W_2^1, W_3^1) - F_2(\nabla W_1^2, W_2^2, W_3^2)\|_{1,\alpha}^{(-\alpha)} \leqq C_0(\delta + \varepsilon)\Big(\| \big(\nabla(W_1^1 - W_1^2), W_2^1 - W_2^2\big) \|_{1,\alpha}^{(-\alpha)} + \|W_3^1 - W_3^2\|_{2,\alpha}^{(-1-\alpha)}\Big). \tag{3.12}$$

Part 3. Reduction of the Boundary Condition for the Entropy on the Shock Curve

It follows from the second equation in (2.22) that

$$\begin{aligned} &\frac{1}{\psi(z_2) M_2(\nabla W_1, W_2, W_3)} + \psi(z_2) A(z_2) \rho^{\gamma-1} M_2(\nabla W_1, W_2, W_3) \\ &= U_1^-(\psi(z_2), z_2) + \frac{P^-(\psi(z_2), z_2)}{(\rho^- U_1^-)(\psi(z_2), z_2)} \\ &\quad -\psi'(z_2)\psi(z_2)\left(\frac{N}{X_0 + 1 - \psi(z_2)} A(z_2)\rho^\gamma \partial_{z_1}\phi - \frac{(P^- U_2^-)(\psi(z_2), z_2)}{\psi(z_2) U_1^-(\psi(z_2), z_2)}\right) \end{aligned} \tag{3.13}$$

with $M_2(\nabla W_1, W_2, W_3)$ defined in (3.4).

Note that by direct computation, one may get

$$
\begin{aligned}
&\frac{1}{\psi(z_2)M_2(\nabla W_1, W_2, W_3)} - \frac{1}{r_0 M_2(0)} \\
&\quad = -\frac{1}{r_0^2 M_2^2(0)}(\psi(z_2)M_2(\nabla W_1, W_2, W_3) - r_0 M_2(0)) \\
&\qquad +2\int_0^1\int_0^1 \frac{\big(\psi(z_2)M_2(\nabla W_1, W_2, W_3) - r_0 M_2(0)\big)^2 s}{\big(ts\psi(z_2)M_2(\nabla W_1, W_2, W_3) + (1-st)r_0 M_2(0)\big)^3}\mathrm{d}t\mathrm{d}s \\
&\quad = -\frac{1}{r_0^2\rho_0^+(r_0)\partial_{z_2}\phi_0^+}W_3 - \frac{1}{r_0(\rho_0^+(r_0))^2\partial_{z_2}\phi_0^+}(\rho - \rho_0^+) \\
&\qquad -\frac{1}{r_0\rho_0^+(r_0)(\partial_{z_2}\phi_0^+)^2}\partial_{z_2}W_1 + O(\delta^2+\varepsilon)
\end{aligned}
\tag{3.14}
$$

and

$$
\begin{cases}
\psi(z_2)A(z_2)\rho^{\gamma-1}M_2(\nabla W_1, W_2, W_3) - r_0 A_0^+(\rho_0^+(r_0))^{\gamma-1}M_2(0,0,0) \\
\quad = r_0(\rho_0^+(r_0))^\gamma\partial_{z_2}\phi_0^+ W_2(z_2) + A_0^+(\rho_0^+(r_0))^\gamma\partial_{z_2}\phi_0^+ W_3(z_2) \\
\qquad + \gamma r_0 A_0^+(\rho_0^+(r_0))^{\gamma-1}\partial_{z_2}\phi_0^+(\rho - \rho_0^+(r_0)) + r_0 A_0^+(\rho_0^+(r_0))^\gamma\partial_{z_2}W_1 \\
\qquad + O(\delta^2+\varepsilon), \\
U_1^-(\psi(z_2), z_2) + \dfrac{P^-(\psi(z_2), z_2)}{(\rho^- U_1^-)(\psi(z_2), z_2)} - U_0^-(r_0) - \dfrac{P_0^-(r_0)}{(\rho_0^- U_0^-)(r_0)} \\
\quad = \dfrac{P_0^-(r_0)}{r_0(\rho_0^- U_0^-)(r_0)}W_3(z_2) + O(\delta^2+\varepsilon),
\end{cases}
\tag{3.15}
$$

where one has used (1.12) and (2.3) for the background solution.

On the other hand, it follows from (A.13) and $\partial_{z_2}\phi_0^+ = \dfrac{1}{(r_0+z_1)\rho_0^+U_0^+}$ that

$$
\begin{aligned}
&\rho - \rho_0^+(r_0) \\
&\quad = \frac{\rho_0^+(r_0)}{c^2(\rho_0^+(r_0)) - (U_0^+(r_0))^2}\bigg(\frac{(U_0^+(r_0))^2}{r_0}W_3(z_2) \\
&\qquad + r_0\rho_0^+(r_0)(U_0^+(r_0))^3\partial_{z_2}W_1 - \frac{\gamma}{\gamma-1}(\rho_0^+(r_0))^{\gamma-1}W_2(0,z_2)\bigg) \\
&\qquad + O(\delta^2+\varepsilon).
\end{aligned}
\tag{3.16}
$$

Substituting (3.10) and (3.14)–(3.16) into (3.13) yields

$$
W_2(z_2) = a_2 W_3(z_2) + F_3(\nabla W_1, W_2, W_3), \tag{3.17}
$$

where $a_2 = \dfrac{(\gamma-1)(P_0^+(r_0) - P_0^-(r_0))}{r_0(\rho_0^+(r_0))^\gamma} > 0$, and $F_3(\nabla W_1, W_2, W_3)$ satisfies

$$
\|F_3(\nabla W_1^i, W_2^i, W_3^i)\|_{1,\alpha}^{(-\alpha)} \leqq C_0(\delta^2+\varepsilon), \quad i = 1, 2, \tag{3.18}
$$

and

$$\begin{aligned}&\|F_3(\nabla W_1^1, W_2^1, W_3^1) - F_3(\nabla W_1^2, W_2^2, W_3^2)\|_{1,\alpha}^{(-\alpha)}\\&\leqq C_0(\delta+\varepsilon)\Big(\|(\nabla(W_1^1-W_1^2), W_2^1-W_2^2)\|_{1,\alpha}^{(-\alpha)}+\|W_3^1-W_3^2\|_{2,\alpha}^{(-1-\alpha)}\Big).\end{aligned}\tag{3.19}$$

Part 4. Reduction of the Given Exit Pressure Condition

Note that (2.24) implies that on $z_1 = N$,

$$\begin{aligned}&\rho-\rho_0^+\\&=\frac{1}{\gamma}(\frac{P_{\mathrm{e}}}{A_0^+})^{\frac{1-\gamma}{\gamma}}\Big(-\frac{P_{\mathrm{e}}}{(A_0^+)^2}W_2+\frac{\varepsilon P_0(\theta(X_0+1,z_2))}{A_0^+}\Big)+O(\delta^2+\varepsilon),\end{aligned}\tag{3.20}$$

where the term $O(\delta^2+\varepsilon)$ follows from the estimates in Lemma A.2-Lemma A.3 and the expression

$$\theta(X_0+1,z_2) = -\theta_0 + f_1(X_0+1) + \int_0^{z_2}\partial_{z_2}\phi(N,s)\mathrm{d}s.\tag{3.21}$$

Due to (1.10), one can derive that

$$\begin{cases}\|P_0(\theta(N,z_2))\|_{1,\alpha}^{(-\alpha)}\leqq C_0,\\ \|P_0(\theta_1(N,z_2))-P_0(\theta_2(N,z_2))\|_{1,\alpha}^{(-\alpha)}\leqq C_0\|\partial_{z_2}(W_1^1-W_1^2)\|_{1,\alpha}^{(-\alpha)},\end{cases}\tag{3.22}$$

where $\theta(N,z_2)$ and $\theta_i(N,z_2)(i=1,2)$ are defined by (3.21) with ϕ and ϕ_i, respectively.

By an analogous treatment as in (3.16), we can obtain from (A.3) in Appendix A and (3.20) that on $z_1 = N$

$$\begin{aligned}&\frac{1}{(X_0+1)^2(\rho_0^+)^2(\partial_{z_2}\phi_0^+)^3}\partial_{z_2}W_1\\&=\frac{1}{\gamma}\Big(\frac{1}{(X_0+1)^2(\rho_0^+)^3(\partial_{z_2}\phi_0^+)^2}+\frac{\gamma}{\gamma-1}A_0^+(\rho_0^+)^{\gamma-2}\Big)\\&\quad\times\Big(\frac{P_{\mathrm{e}}}{A_0^+}\Big)^{\frac{1-\gamma}{\gamma}}\frac{P_{\mathrm{e}}W_2-\varepsilon A_0^+P_0(\theta(N,z_2))}{(A_0^+)^2}+O(\delta^2+\varepsilon).\end{aligned}\tag{3.23}$$

It follows from (3.22) and (3.23) that

$$\partial_{z_2}W_1(N,z_2) = a_3W_2(z_2) + F_4(\nabla W_1, W_2, W_3)\quad\text{on}\quad z_1 = N,\tag{3.24}$$

where

$$a_3=\frac{1}{\gamma}\frac{P_{\mathrm{e}}^{1/\gamma}}{(A_0^+)^{(\gamma+1)/\gamma}}\Big(\frac{\partial_{z_2}\phi_0^+}{\rho_0^+}+\frac{\gamma}{\gamma-1}(X_0+1)^2A_0^+(\rho_0^+)^\gamma(\partial_{z_2}\phi_0^+)^3\Big)>0,$$

moreover, $F_4(\nabla W_1, W_2, W_3)$ satisfies

$$\|F_4(\nabla W_1^i, W_2^i, W_3^i)\|_{1,\alpha}^{(-\alpha)}\leqq C_0(\delta^2+\varepsilon),\quad i=1,2,\tag{3.25}$$

and

$$\|F_4(\nabla W_1^1, W_2^1, W_3^1) - F_4(\nabla W_1^2, W_2^2, W_3^2)\|_{1,\alpha}^{(-\alpha)}$$
$$\leqq C_0(\delta+\varepsilon)\Big(\|(\nabla(W_1^1-W_1^2), W_2^1 - W_2^2)\|_{1,\alpha}^{(-\alpha)}+\|W_3^1-W_3^2\|_{2,\alpha}^{(-1-\alpha)}\Big). \tag{3.26}$$

Part 5. Reduction of the Elliptic Problem

Rewrite (2.19) as

$$\begin{aligned}
&\sum_{j=1}^{2}\partial_{z_j}\big(a_{j+3}(z_1)\partial_{z_j}W_1\big)+a_6(z_1)\partial_{z_1}W_1+a_7(z_1)W_2'(z_2)\\
&\quad+\partial_{z_1}\Big(\frac{z_1-N}{N}A_0^+(\rho_0^+)^\gamma W_3'(z_2)\Big)+\partial_{z_2}\Big(\frac{(1-\frac{z_1}{N})a_5(z_1)\partial_{z_2}\phi_0^+}{r_0+z_1}W_3(z_2)\Big)\\
&\quad-\frac{1}{N}A_0^+(\rho_0^+(z_1))^\gamma W_3'(z_2)\\
&\;=\sum_{k=1}^{2}\partial_{z_k}F_{k+4}(\nabla W_1, W_2, W_3)+F_7(\nabla W_1, W_2, W_3)\quad\text{in}\quad E_+,
\end{aligned} \tag{3.27}$$

where

$$\begin{cases}
a_4(z_1)=\dfrac{1}{\rho_0^+(z_1)\partial_{z_2}\phi_0^+}>0,\\
a_5(z_1)=\dfrac{c^2(A_0^+,\rho_0^+(z_1))}{\big(c^2(A_0^+,\rho_0^+(z_1))-(U_0^+(z_1))^2\big)\rho_0^+(z_1)(r_0+z_1)^2(\partial_{z_2}\phi_0^+)^3}>0,\\
a_6(z_1)=\dfrac{1}{(r_0+z_1)\rho_0^+(z_1)\partial_{z_2}\phi_0^+},\\
a_7(z_1)=-\dfrac{(\rho_0^+(z_1))^\gamma}{c^2(A_0^+,\rho_0^+(z_1))-(U_0^+(z_1))^2}\Big[\dfrac{c^2(A_0^+,\rho_0^+)}{\gamma-1}+(U_0^+)^2\Big](z_1)<0,
\end{cases} \tag{3.28}$$

and

$$\begin{cases}
F_5(\nabla W_1, W_2, W_3)=-N_1(\nabla\phi, A, \psi)+a_4(z_1)\partial_{z_1}W_1+\dfrac{z_1-N}{N}A_0^+(\rho_0^+)^\gamma W_3'(z_2),\\
F_6(\nabla W_1, W_2, W_3)=-N_2(\nabla\phi, A, \psi)+a_5(z_1)\partial_{z_2}W_1\\
\qquad\qquad+\dfrac{(1-\frac{z_1}{N})a_5(z_1)\partial_{z_2}\phi_0^+}{r_0+z_1}W_3(z_2)+a_7(z_1)W_2(z_2),\\
F_7(\nabla W_1, W_2, W_3)=-N_3(\nabla\phi, A, \psi)+a_6(z_1)\partial_{z_1}W_1-\dfrac{1}{N}A_0^+(\rho_0^+(z_1))^\gamma W_3'(z_2).
\end{cases} \tag{3.29}$$

For the last three terms in the left hand side of (3.27), due to Bernoulli's law, (2.25) for the background solution, a direct calculation yields

$$\begin{aligned}
&\partial_{z_1}\Big(\frac{z_1-N}{N}A_0^+(\rho_0^+)^\gamma W_3'(z_2)\Big)+\partial_{z_2}\Big(\frac{(1-\frac{z_1}{N})a_5(z_1)\partial_{z_2}\phi_0^+}{r_0+z_1}W_3(z_2)\Big)\\
&\quad-\frac{1}{N}A_0^+(\rho_0^+(z_1))^\gamma W_3'(z_2)\\
&=\frac{(1-\frac{z_1}{N})a_5(z_1)\partial_{z_2}\phi_0^+}{r_0+z_1}W_3'(z_2)-\gamma(1-\frac{z_1}{N})A_0^+(\rho_0^+)^{\gamma-1}\partial_{z_1}\rho_0^+W_3'(z_2)\\
&=0.
\end{aligned}$$

This is one of the main observations here, which makes it possible to define $F_i(i=5,6,7)$ in (3.29) so that each F_i is a high order as shown below.

Thus, (3.27) takes the form

$$\begin{aligned}
&\sum_{j=1}^{2}\partial_{z_j}\big(a_{j+3}(z_1)\partial_{z_j}W_1\big)+a_6(z_1)\partial_{z_1}W_1+a_7(z_1)W_2'(z_2)\\
&\quad=\sum_{k=1}^{2}\partial_{z_k}F_{k+4}(\nabla W_1,W_2,W_3)+F_7(\nabla W_1,W_2,W_3)\quad\text{in}\quad E_+.
\end{aligned}$$

Next, we analyze the properties of $F_j(\nabla W_1,W_2,W_3)$ $(j=5,6,7)$ defined in (3.29).

Since it follows from the definition Ξ_δ and the estimate (A.14) imply that

$$\begin{aligned}
&N_1(\nabla\phi_i,A_i,\psi_i)\\
&\quad=\big(\frac{N}{N+O(\delta)}\big)^2\frac{\partial_{z_1}W_1^i}{(\rho_0^++O(\delta))(\partial_{z_2}\phi_0^++O(\delta))}\\
&\qquad+\frac{z_1-N}{N+O(\delta)}(A_0^++O(\delta))(\rho_0^++O(\delta))^\gamma(W_3^i)'(z_2)\\
&\quad=\frac{1}{\rho_0^+\partial_{z_2}\phi_0^+}\partial_{z_1}W_1^i+(\frac{z_1}{N}-1)A_0^+(\rho_0^+)^\gamma(W_3^i)'(z_2)+O(\delta^2),
\end{aligned}$$

so substituting this into the expression of $F_5(\nabla W_1,W_2,W_3)$ yields

$$\|F_5(\nabla W_1^i,W_2^i,W_3^i)\|_{1,\alpha}^{(-\alpha)}\leqq C\delta^2\leqq C(\delta^2+\varepsilon).$$

Similarly, we can obtain for $j=5,6,7$,

$$\|F_j(\nabla W_1^i,W_2^i,W_3^i)\|_{1,\alpha}^{(-\alpha)}\leqq C_0\big(\delta^2+\varepsilon\big),\qquad i=1,2 \tag{3.30}$$

and for $j=5,6,7$,

$$\begin{aligned}
&\|F_j(\nabla W_1^1,W_2^1,W_3^1)-F_j(\nabla W_1^2,W_2^2,W_3^2)\|_{1,\alpha}^{(-\alpha)}\\
&\quad\leqq C_0(\delta+\varepsilon)\big(\|\Big(\nabla(W_1^1-W_1^2),W_2^1-W_2^2\Big)\|_{1,\alpha}^{(-\alpha)}+\|W_3^1-W_3^2\|_{2,\alpha}^{(-1-\alpha)}\big).
\end{aligned}\tag{3.31}$$

On the other hand, the boundary condition (2.23) can be rewritten as

$$\begin{cases} \partial_{z_1} W_1(z_1, 0) = f_1'\left(\psi(0) + \dfrac{1}{N}(X_0 + 1 - \psi(0))z_1\right), \\ \partial_{z_1} W_1(z_1, M) = f_2'\left(\psi(M) + \dfrac{1}{N}(X_0 + 1 - \psi(M))z_1\right). \end{cases} \tag{3.32}$$

Based on these reductions in Part 1–Part 5, the nonlinear problem, (2.19)–(2.25), is reduced to the following problem:

$$\begin{cases} \displaystyle\sum_{j=1}^{2} \partial_{z_j}\big(a_{j+3}(z_1)\partial_{z_j} W_1\big) + a_6(z_1)\partial_{z_1} W_1 + a_1 a_2 a_7(z_1)\partial_{z_1} W_1(0, z_2) \\ \displaystyle\quad = \sum_{k=1}^{2} \partial_{z_k} F_{k+7}(\nabla W_1, W_2, W_3) + F_{10}(\nabla W_1, W_2, W_3) \quad \text{in} \quad E_+, \\ \partial_{z_2} W_1(0, z_2) = F_2(\nabla W_1, W_2, W_3)(0, z_2), \\ \displaystyle\partial_{z_2} W_1(N, z_2) = a_2 a_3\left(W_3(0) + a_1 \int_0^{z_2} \partial_{z_1} W_1(0, s)\mathrm{d}s\right) \\ \qquad\qquad + F_{11}(\nabla W_1, W_2, W_3)(0, z_2), \\ \partial_{z_1} W_1(z_1, 0) = f_1'(\psi(0) + \dfrac{1}{N}(X_0 + 1 - \psi(0))z_1), \\ \partial_{z_1} W_1(z_1, M) = f_2'(\psi(M) + \dfrac{1}{N}(X_0 + 1 - \psi(M))z_1), \\ W_1(N, 0) = 0, \end{cases} \tag{3.33}$$

coupled with (3.3) and (3.17), where

$$\begin{cases} F_8(\nabla W_1, W_2, W_3) = F_5(\nabla W_1, W_2, W_3), \\ F_9(\nabla W_1, W_2, W_3) = F_6(\nabla W_1, W_2, W_3) - a_7(z_1)\partial_{z_2} F_3(\nabla W_1, W_2, W_3)(0, z_2), \\ F_{10}(\nabla W_1, W_2, W_3) = F_7(\nabla W_1, W_2, W_3) - a_2 a_7(z_1) F_1(\nabla W_1, W_2, W_3)(0, z_2), \\ \displaystyle F_{11}(\nabla W_1, W_2, W_3)(0, z_2) = a_1 a_2 a_3 \int_0^{z_2} F_1(\nabla W_1, W_2, W_3)(0, s)\mathrm{d}s \\ \qquad\qquad + a_3 F_3(\nabla W_1, W_2, W_3)(0, z_2) \\ \qquad\qquad + F_4(\nabla W_1, W_2, W_3)(N, z_2). \end{cases}$$

Furthermore, it follows from the estimates (3.8)–(3.9), (3.18)–(3.19), (3.25)–(3.26) and (3.30)–(3.31) that for each $W^i \in \Xi_\delta (i = 1, 2)$,

$$\sum_{j=8}^{11} \|F_j(\nabla W_1^i, W_2^i, W_3^i)\|_{1,\alpha}^{(-\alpha)} \leqq C_0\big(\delta^2 + \varepsilon\big), \quad i = 1, 2 \tag{3.34}$$

and

$$\sum_{j=8}^{11} \|F_j(\nabla W_1^1, W_2^1, W_3^1) - F_j(\nabla W_1^2, W_2^2, W_3^2)\|_{1,\alpha}^{(-\alpha)}$$
$$\leqq C_0(\delta+\varepsilon)\Big(\|(\nabla(W_1^1 - W_1^2), W_2^1 - W_2^2)\|_{1,\alpha}^{(-\alpha)} + \|W_3^1 - W_3^2\|_{2,\alpha}^{(-1-\alpha)}\Big). \tag{3.35}$$

In order to solve the nonlinear problem (3.33) coupled with (3.3) and (3.17), we first consider a related linear problem corresponding to the nonlinear problem (3.33) in the next section. In addition,it should be emphasized that the boundary conditions on $W_1(z)$ in (3.33) are the "tangent differential" conditions other than the usual Neumann boundary conditions, moreover, the positivity of $a_1, a_2, a_3, a_4(z_1), a_5(z_1)$ and the negativity of $a_7(z_1)$ in (3.33) will play a crucial role in showing its solvability and deriving the related estimates (see the problem (4.1) with the assumptions (4.2) and the Proposition 4.4 in Section 4 below).

4. Solvability and A Priori Estimates on Some Second Order Linear Elliptic Equation with Nonlocal Terms and an Unknown Constant

The key step to solve the nonlinear boundary value problem (3.33) is to consider the following second order linear elliptic boundary value problem with an unknown parameter and nonlocal terms

$$\begin{cases} \sum_{i=1}^{2} \partial_{z_i}\big(b_i(z_1)\partial_{z_i} u\big) + b_3(z_1)\partial_{z_1} u + b_4(z_1)\partial_{z_1} u(0, z_2) \\ \quad = \sum_{j=1}^{2} \partial_{z_j} h_j(z) + h_3(z) \quad \text{in} \quad E_+, \\ \partial_{z_2} u(0, z_2) = g_1(z_2), \\ \partial_{z_2} u(N, z_2) = b_5\Big(\kappa + \int_0^{z_2} \partial_{z_1} u(0, s)\mathrm{d}s\Big) + g_2(z_2), \\ \partial_{z_1} u(z_1, 0) = g_3(z_1), \\ \partial_{z_1} u(z_1, M) = g_4(z_1), \\ u(N, 0) = 0, \end{cases} \tag{4.1}$$

where $b_i(z_1) \in C^\infty[0, N](1 \leqq i \leqq 4)$, $g_j \in H_{1,\alpha}^{(-\alpha)}(0, M)\ (j = 1, 2)$, $g_k \in C^{1,\alpha}[0, N](k = 3, 4)$ and $h_l \in H_{1,\alpha}^{(-\alpha)}(E_+)(l = 1, 2, 3)$. Assume further that there exist two positive constants, b_0 and B_0, with $b_0 < 1 < B_0$ such that

$$\sum_{j=1}^{4} \|b_j\|_{C^3} + |b_5| \leqq B_0, \quad b_i(z_1) \geqq b_0(i = 1, 2, 5), \quad b_4 \leqq -b_0. \tag{4.2}$$

For convenience, we set

$$u_1 = u - \left(\frac{z_2}{M}\int_N^{z_1} g_4(s)\mathrm{d}s + (1-\frac{z_2}{M})\int_N^{z_1} g_3(s)\mathrm{d}s\right). \tag{4.3}$$

Then (4.1) is reduced to

$$\begin{cases} \sum_{i=1}^{2}\partial_{z_i}\big(b_i(z_1)\partial_{z_i}u_1\big) + b_3(z_1)\partial_{z_1}u_1 + b_4(z_1)\partial_{z_1}u_1(0,z_2) \\ \qquad = \sum_{j=1}^{2}\partial_{z_j}h_{3+j}(z) + h_6(z) \quad \text{in} \quad E_+, \\ \partial_{z_2}u_1(0,z_2) = g_5(z_2), \\ \partial_{z_2}u_1(N,z_2) = b_5\big(\kappa + \int_0^{z_2}\partial_{z_1}u_1(0,s)\mathrm{d}s\big) + g_6(z_2), \\ \partial_{z_1}u_1(z_1,0) = 0, \\ \partial_{z_1}u_1(z_1,M) = 0, \\ u_1(N,0) = 0, \end{cases} \tag{4.4}$$

where

$$\begin{cases} g_5(z_2) = g_1(z_2) + \frac{1}{M}\int_0^N (g_4(s) - g_3(s))\mathrm{d}s, \\ g_6(z_2) = g_2(z_2) + b_5\left(\frac{z_2^2}{2M}g_4(0) + (z_2 - \frac{z_2^2}{2M})g_3(0)\right), \\ h_4(z) = h_1(z) - b_1(z_1)\left(\frac{z_2}{M}g_4(z_1) + (1-\frac{z_2}{M})g_3(z_1)\right), \\ h_5(z) = h_2(z), \\ h_6(z) = h_3(z) - b_3(z_1)\left(\frac{z_2}{M}g_4(z_1) + (1-\frac{z_2}{M})g_3(z_1)\right) \\ \qquad\quad - b_4(z_1)\left(\frac{z_2}{M}g_4(0) + (1-\frac{z_2}{M})g_3(0)\right). \end{cases} \tag{4.5}$$

Note that the problem (4.4) is not a standard one for second order linear elliptic equations since it involves a free constant κ and non-local terms containing $\partial_{z_1}u(0,z_2)$ and $\int_0^{z_2}\partial_{z_1}u(0,s)\mathrm{d}s$. Thus, it seems to be difficult to use the Lax-Milgram and Fredholm Alternative Theorem to obtain the solvability in $H^1(E_+)$ as in [19]. To overcome this difficulty, we intend to apply a continuity method in Chapter 5 of [13] to obtain the existence of (4.4). The key here is to obtain some a priori estimates of the solution $u_1(z)$ in the weighted Hölder spaces introduced in Section 2.

To this end, we first need a lemma.

Lemma 4.1. *If* $g_6(z_2) \in H^{(-\alpha)}_{1,\alpha}(0, M)$ *and* $h_j(z) \in H^{(-\alpha)}_{1,\alpha}(E_+)(j = 4, 5, 6)$*, then the following boundary value problem*

$$\begin{cases} L(u_2) = \sum_{i=1}^{2} \partial_{z_i}\big(b_i(z_1)\partial_{z_i}u_2\big) + b_3(z_1)\partial_{z_1}u_2 \\ \qquad = \sum_{j=1}^{2} \partial_{z_j}h_{j+3}(z) + h_6(z), \quad in \quad E_+, \\ \partial_{z_1}u_2(0, z_2) = 0, \\ u_2(N, z_2) = \int_0^{z_2} g_6(s)\mathrm{d}s, \\ u_2(z_1, 0) = 0, \\ u_2(z_1, M) = \int_0^{M} g_6(s)\mathrm{d}s \end{cases} \tag{4.6}$$

has a unique solution $u_2(z) \in H^{(-1-\alpha)}_{2,\alpha}(E_+)$ *such that*

$$\|u_2\|^{(-1-\alpha)}_{2,\alpha} \leqq C\left(\sum_{k=4}^{6} \|h_k\|^{(-\alpha)}_{1,\alpha} + \|g_6\|^{(-\alpha)}_{1,\alpha}\right). \tag{4.7}$$

Proof. As shown in Chapter 8 of [13], (4.6) has a unique solution

$$u_2 \in C^{2,\alpha}(E_+) \cap C^0(\bar{E}_+).$$

It remains to show that u_2 has a higher regularity in $\bar{E}_+$ and admits the estimate (4.7). The idea of this proof is somewhat similar to that in [12] and [20]. However, for the reader's convenience, we still give the details of the proof.

Set

$$\hat{u}_2(z) = u_2(z) - \int_0^{z_2} g_6(s)\mathrm{d}s, \tag{4.8}$$

then the problem (4.6) is equivalent to

$$\begin{cases} L(\hat{u}_2)=\sum_{i=1}^{2} \partial_{z_i}\big(b_i(z_1)\partial_{z_i}\hat{u}_2\big)+b_3(z_1)\partial_{z_1}\hat{u}_2=\sum_{j=1}^{2} \partial_{z_j}\hat{h}_{j+3}(z)+\hat{h}_6(z) \quad \text{in} \quad E_+, \\ \partial_{z_1}\hat{u}_2(0, z_2) = \hat{u}_2(N, z_2) = 0, \\ \hat{u}_2(z_1, 0) = \hat{u}_2(z_1, M) = 0 \end{cases} \tag{4.9}$$

with

$$\hat{h}_k(z) = h_k(z) \quad \text{for } k = 4, 6, \quad \hat{h}_5(z) = h_5(z) - b_2(z_1)g_6(z_2). \tag{4.10}$$

In order to derive the estimates on the solution $\hat{u}_2$ of (4.9), we will carry out the following four steps.

Step 1. The L^∞ estimate of $\hat{u}_2$

Set

$$v_1 = G_1\Big(1 + (z_1 - \frac{N}{2})^2 + G_2\big(z_2^\alpha + (M - z_2)^\alpha\big)\Big),$$

where $G_1 = \sum_{j=4}^{6} \|\hat{h}_j\|_{1,\alpha}^{(-\alpha)}$ and $G_2 = \dfrac{4}{\alpha(1-\alpha)} M^{2-\alpha}(1+N)\dfrac{B_0}{b_0}$.

Then a direct computation yields

$$\begin{aligned} L(v_1) &= -b_2(z_1)G_1G_2\alpha(1-\alpha)\big(z_2^{\alpha-2} + (M-z_2)^{\alpha-2}\big) \\ &\quad + \Big(\big(b_1'(z_1) + b_3(z_1)\big)(2z_1 - N) + 2b_1(z_1)\Big)G_1 \\ &\leqq -b_0G_1G_2\alpha(1-\alpha)\big(z_2^{\alpha-2} + (M-z_2)^{\alpha-2}\big) + 2B_0(1+N)G_1 \\ &\leqq -\frac{1}{2}b_0G_1G_2\alpha(1-\alpha)\big(z_2^{\alpha-2} + (M-z_2)^{\alpha-2}\big). \end{aligned}$$

Thus, for a suitably large positive constant C independent of $\|\hat{h}_j\|_{1,\alpha}^{(-\alpha)}$ $(j = 4, 5, 6)$, we have

$$\begin{cases} \hat{L}(Cv_1 \pm \hat{u}_2) \leqq 0 \quad \text{in} \quad E_+, \\ \partial_{z_1}(Cv_1 \pm \hat{u}_2)(0, z_2) < 0, \\ (Cv_1 \pm \hat{u}_2)(N, z_2) > 0, \\ (Cv_1 \pm \hat{u}_2)(z_1, 0) = (Cv_1 \pm \hat{u}_2)(z_1, M) > 0. \end{cases}$$

This, together with the comparison principle, yields

$$\|\hat{u}_2\|_{L^\infty} \leqq C\sum_{j=4}^{6} \|\hat{h}_j\|_{1,\alpha}^{(-\alpha)}. \tag{4.11}$$

Step 2. The weighted L^∞ estimates of $\hat{u}_2$ near the corner points $(N, 0)$ and (N, M) of E_+

In this step, we focus only on the analysis of $\hat{u}_2$ near the corner point $(N, 0)$ since the corner point (N, M) can be treated similarly.

Following the arguments in [1,2], one can define a comparison function in polar coordinates as follows

$$v_2(\tilde{r}, \tilde{\theta}) = \frac{2}{\sin(\frac{1-\alpha}{2})}\tilde{r}^{1+\alpha}\sin\Big(m(\tilde{\theta})\Big) - \tilde{r}^{1+\alpha}\sin^{1+\alpha}\tilde{\theta}, \quad \frac{\pi}{2} \leqq \tilde{\theta} \leqq \pi, \tag{4.12}$$

where $\tilde{r} = \sqrt{(z_1 - N)^2 + z_2^2}$, $\tilde{\theta} = \arctan\dfrac{z_2}{z_1 - N} + \pi$, and

$$m(\tilde{\theta}) \equiv \frac{1-\alpha}{2} + \frac{3+\alpha}{2}(\tilde{\theta} - \frac{\pi}{2}) \quad \text{with} \quad \tilde{\theta} \in [\frac{\pi}{2}, \pi].$$

Due to $m(\frac{\pi}{2})+m(\pi)>\pi$ and $\frac{1}{2}m(\frac{\pi}{2})+m(\pi)<\pi$, then

$$v_2(\tilde r,\tilde\theta)\geqq \tilde r^{1+\alpha}\left(\frac{2\sin(m(\pi))}{\sin(m(\frac{\pi}{2}))}-1\right)\geqq \tilde r^{1+\alpha}\big(\frac{1}{\cos(\frac{1-\alpha}{4})}-1\big). \tag{4.13}$$

Without loss of generality, we assume that

$$b_1(N)=b_2(N)=1. \tag{4.14}$$

A simple computation shows that

$$\begin{aligned}\Delta v_2=&\frac{(\alpha-1)(3\alpha+5)}{2\sin\big(\frac{1-\alpha}{2}\big)}\tilde r^{\alpha-1}\sin\big(m(\tilde\theta)\big)-\alpha(1+\alpha)\tilde r^{\alpha-1}\sin^{\alpha-1}(\tilde\theta)\\ \leqq&-\alpha(1+\alpha)\tilde r^{\alpha-1}\sin^{\alpha-1}(\tilde\theta).\end{aligned} \tag{4.15}$$

Let $r_0>0$ be small and fixed and set $B^+_{r_0}=B_{r_0}((N,0))\cap E_+$. Then it follows from (4.11), (4.13) and (4.15) that there exits a suitably large positive constant C independent of $\|\hat h_j\|^{(-\alpha)}_{1,\alpha}$ $(j=4,5,6)$ such that

$$\begin{cases}\hat L(\tilde C v_2\pm\hat u_2)\leqq 0 & \text{in}\quad B^+_{r_0},\\ \tilde C v_2\pm\hat u_2>0 & \text{on}\quad \partial B^+_{r_0}\cap\partial E_+,\\ \tilde C v_2\pm\hat u_2>0 & \text{on}\quad \partial B^+_{r_0}\cap\{\tilde r=r_0\},\end{cases}$$

where $\tilde C=Cr_0^{-1-\alpha}\sum_{j=4}^6\|\hat h_j\|^{(-\alpha)}_{1,\alpha}$. Then it follows from the comparison principle that

$$|\hat u_2(z)|\leqq C\sum_{j=4}^{6}\|\hat h_j\|^{(-\alpha)}_{1,\alpha}\tilde r^{1+\alpha}\quad\text{in}\quad B^+_{r_0}. \tag{4.16}$$

Step 3. Some estimates for a related auxiliary problem

To estimate $\hat u_2$ near the left corner points, $(0,0)$ and $(0,M)$, in E_+, we need to consider the following boundary value problem with the homogeneous Neumann boundary condition at $z_1=N$,

$$\begin{cases}L(v)=\sum_{i=1}^{2}\partial_{z_i}\big(b_i(z_1)\partial_{z_i}v\big)+b_3(z_1)\partial_{z_1}v=\sum_{j=1}^{2}\partial_{z_j}\hat h_{j+3}(z)+\hat h_6(z)\quad\text{in}\quad E_+,\\ \partial_{z_1}v(0,z_2)=\partial_{z_1}v(N,z_2)=0,\\ v(z_1,0)=v(z_1,M)=0.\end{cases} \tag{4.17}$$

As shown in Chapter 8 of [13], (4.17) has a unique solution

$$v\in C^{2,\alpha}(E_+)\cap C^{1,\alpha}(\bar E_+\backslash\{(0,0),(0,M),(N,0),(N,M)\})\cap C(\bar E_+).$$

In a similar way as in Step 1, one can prove

$$\|v\|_{L^\infty}\leqq C\sum_{i=4}^{6}\|\hat h_i\|^{(-\alpha)}_{1,\alpha}. \tag{4.18}$$

Set

$$\begin{cases} \tilde{v}(z)=v(z),\ \tilde{b}_i(z_1)=b_i(z_1)(i=1,2),\ \tilde{h}_4(z)=\hat{h}_4(z)-\hat{h}_4(0,z_2),\ \tilde{h}_5(z)=\hat{h}_5(z), \\ \tilde{h}_6(z)=\hat{h}_6(z)-(b_1'(z_1)+b_3(z_1))\partial_{z_1}v(z), \quad \text{for } 0\leqq z_1\leqq N; \\ \tilde{v}(z)=\tilde{v}(-z_1,z_2),\ \tilde{b}_i(z_1)=\tilde{b}_i(-z_1)(i=1,2),\ \tilde{h}_4(z)=-\tilde{h}_4(-z_1,z_2), \\ \tilde{h}_5(z)=\tilde{h}_5(-z_1,z_2), \tilde{h}_6(z)=\tilde{h}_6(-z_1,z_2), \quad \text{for } -N\leqq z_1\leqq 0; \\ \tilde{v}(z)=\tilde{v}(2N-z_1,z_2),\ \tilde{b}_i(z_1)=\tilde{b}_i(2N-z_1)(i=1,2),\ \tilde{h}_4(z)=-\tilde{h}_4(2N-z_1,z_2), \\ \tilde{h}_5(z)=h_5(2N-z_1,z_2),\ \tilde{h}_6(z)=\tilde{h}_6(2N-z_1,z_2), \quad \text{for } N\leqq z_1\leqq 2N. \end{cases} \tag{4.19}$$

Then $\tilde{v}$ solves the following problem

$$\begin{cases} \tilde{L}(\tilde{v}) \triangleq \sum_{i=1}^{2} \tilde{b}_i(z_1)\partial_{z_i}^2\tilde{v} = \sum_{j=1}^{2} \partial_{z_j}\tilde{h}_{j+3}(z)+\tilde{h}_6(z) \\ \qquad\qquad\qquad \text{in} \quad \tilde{E}_+=(-N,2N)\times(0,M), \\ \partial_{z_1}\tilde{v}(-N,z_2)=\partial_{z_1}\tilde{v}(2N,z_2)=0, \\ \tilde{v}(z_1,0)=\tilde{v}(z_1,M)=0. \end{cases}$$

It follows from the Schauder interior and boundary estimates in Theorem 8.32 and Corollary 8.36 of Chapter 8 of [13] and the definitions in (4.19) that

$$\begin{aligned} \|v\|_{C^{1,\alpha}(\bar{E}_+)} = \|\tilde{v}\|_{C^{1,\alpha}(\bar{E}_+)} &\leqq C\Big(\|\tilde{v}\|_{L^\infty} + \sum_{i=4}^{5}\|\tilde{h}_i\|_{C^\alpha(\overline{\tilde{E}_+})} + \|\tilde{h}_6\|_{L^\infty}\Big) \\ &\leqq C\Big(\|v\|_{L^\infty} + \sum_{j=4}^{6}\|\hat{h}_j\|_{1,\alpha}^{(-\alpha)} + \|\partial_{z_1}v\|_{L^\infty}\Big). \end{aligned}$$

Combining this with the estimate (4.18) and the interpolation formula yields

$$\|v\|_{C^{1,\alpha}(\bar{E}_+)} \leqq C\sum_{j=4}^{6}\|\hat{h}_j\|_{1,\alpha}^{(-\alpha)}. \tag{4.20}$$

Next, we derive the $H_{2,\alpha}^{(-1-\alpha)}(E_+)$ estimate on v.
By (4.17), $\partial_{z_1}v$ solves

$$\begin{cases} \dfrac{b_1(z_1)}{b_2(z_1)}\partial_{z_1}^2(\partial_{z_1}v) + \partial_{z_2}^2(\partial_{z_1}v) + \left((\dfrac{b_1(z_1)}{b_2(z_1)})' + \dfrac{b_1'(z_1)+b_3(z_1)}{b_2(z_1)}\right)\partial_{z_1}(\partial_{z_1}v) \\ \quad = \partial_{z_1}\left(\dfrac{\sum_{i=1}^{2}\partial_{z_i}\hat{h}_{i+3}(z)+\hat{h}_6(z)}{b_2(z_1)}\right) - \left(\dfrac{b_1'(z_1)+b_3(z_1)}{b_2(z_1)}\right)'\partial_{z_1}v \quad \text{in} \quad E_+, \\ (\partial_{z_1}v)(0,z_2)=(\partial_{z_1}v)(N,z_2)=0, \\ (\partial_{z_1}v)(z_1,0)=(\partial_{z_1}v)(z_1,M)=0. \end{cases}$$

Let $z^0 = (z_1^0, z_2^0)$ be any fixed point in E_+. Without loss of generality, we assume $z_2^0 \leqq M - z_2^0$ and set $E_+^0 \equiv E_+ \cap \{z : \frac{z_2^0}{2} < z_2 < \frac{3z_2^0}{2}\}$. By the Schauder estimates in Chapter 8 of [13] and the standard scaling argument, one has

$$\sum_{|\beta|=1}\Big((z_2^0)^{|\beta|}\|D^\beta \partial_{z_1} v\|_{L^\infty(E_+^0)} + (z_2^0)^{|\beta|+\alpha}\big[D^\beta \partial_{z_1} v\big]_{\alpha;E_+^0}\Big)$$
$$\leqq C\Big(\|\partial_{z_1} v\|_{L^\infty(E_+^0)} + (z_2^0)^\alpha \sum_{i=4}^{6}\|\hat{h}_i\|_{1,\alpha}^{(-\alpha)} + (z_2^0)^\alpha\|\partial_{z_1} v\|_{C^\alpha(\bar{E}_+)}\Big).$$

Combining this with the boundary conditions on $z_2 = 0$, $z_2 = M$ in (4.17) and the estimate (4.20) yields

$$\|v\|_{C^{1,\alpha}(\bar{E}_+)} + \sum_{|\beta|=1}\Big((z_2^0)^{|\beta|-\alpha}\|D^\beta \partial_{z_1} v\|_{L^\infty(E_+^0)} + (z_2^0)^{|\beta|}\big[D^\beta \partial_{z_1} v\big]_{\alpha;E_+^0}\Big)$$
$$\leqq C\sum_{i=4}^{6}\|\hat{h}_i\|_{1,\alpha}^{(-\alpha)}.$$

Namely,

$$\|v\|_{C^{1,\alpha}(\bar{E}_+)} + \|\partial_{z_1} v\|_{1,\alpha}^{(-\alpha)} \leqq C\sum_{i=4}^{6}\|\hat{h}_i\|_{1,\alpha}^{(-\alpha)}.$$

This, together with the equation (4.17), yields

$$\|v\|_{2,\alpha}^{(-1-\alpha)} \leqq C\sum_{i=4}^{6}\|\hat{h}_i\|_{1,\alpha}^{(-\alpha)}. \tag{4.21}$$

Step 4. The $C^{1,\alpha}(\bar{E}_+)$ estimate of $\hat{u}_2$ and the proof of (4.7).

It follows from (4.9) and (4.17) that $\hat{u}_2 - v$ satisfies

$$\begin{cases} L(\hat{u}_2 - v) = 0 \quad \text{in} \quad E_+, \\ \partial_{z_1}(\hat{u}_2 - v)(0, z_2) = 0, \\ (\hat{u}_2 - v)(N, z_2) = -v(N, z_2), \\ (\hat{u}_2 - v)(z_1, 0) = (u_2 - v)(z_1, M) = 0, \end{cases} \tag{4.22}$$

with $\|v(N, z_2)\|_{2,\alpha}^{(-1-\alpha)} \leqq C\sum_{i=4}^{6}\|\hat{h}_i\|_{1,\alpha}^{(-\alpha)}$ as shown in (4.21).

Set $W(z) = \hat{u}_2 - v + \partial_{z_2} v(N, 0)z_2$. Then $|W(N, z_2)| \leqq C\|v\|_{C^{1,\alpha}}|z_2|^{1+\alpha}$ and $W(z)$ solves

$$\begin{cases} L(W) = 0 \quad \text{in} \quad B_{\tilde{r}_0}^+, \\ W(z_1, 0) = 0, \\ W(N, z_2) = -v(N, z_2) + \partial_{z_2} v(N, 0)z_2. \end{cases}$$

As in Step 2, one can obtain

$$|W(z)| \leqq C\|v(N, z_2)\|_{C^{1,\alpha}}\tilde{r}^{1+\alpha} \quad \text{in} \quad B^+_{\tilde{r}_0} = B_{\tilde{r}_0}((N,0)) \cap E_+, \tag{4.23}$$

for some $\tilde{r}_0 > 0$.

For any fixed point $z^0 = (z_1^0, z_2^0) \in B^+_{\tilde{r}_0/2}$, let $d_{z^0} = \frac{1}{2}\sqrt{(z_1^0 - N)^2 + (z_2^0)^2}$. Then by Corollary 6.3 and Corollary 6.7 in [13], together with the scaling technique in the proof of Theorem 6.2 in [13], one has

$$\begin{aligned}
&\sum_{|\beta|=0}^{2} d_{z^0}^{|\beta|}\|D^\beta W\|_{L^\infty(B^+_{d_{z^0}/2}(z_0))} + \sum_{|\beta|=2} d_{z^0}^{2+\alpha}\big[D^\beta W\big]_{\alpha; B^+_{d_{z^0}/2}(z_0)} \\
&\leqq C\Bigg(\|W\|_{L^\infty(B^+_{d_{z^0}}(z_0))} + \sum_{|\mu|=0}^{2} d_{z^0}^{|\mu|}\|\partial^\mu_{z_2} W(N, z_2)\|_{L^\infty(d_{z^0}/2, 3d_{z^0})} \\
&\quad + \sum_{|\mu|=2} d_{z^0}^{2+\alpha}\big[\partial^\mu_{z_2} W(N, z_2)\big]_{\alpha; (d_{z^0}/2, 3d_{z^0})}\Bigg) \\
&\leqq C\big(\|W\|_{L^\infty(B_{d^+_{z^0}}(z_0))} + d_{z^0}^{1+\alpha}\|v(N, z_2)\|^{(-1-\alpha)}_{2,\alpha}\big).
\end{aligned} \tag{4.24}$$

Since $d_{z^0} \geqq \frac{|z_2^0|}{2}$, then substituting (4.21) and (4.23) into (4.24) yields

$$\begin{aligned}
&\sum_{0\leqq|\beta|\leqq 2} (z_2^0)^{(\beta-\alpha-1)_+}\|D^\beta W\|_{L^\infty(B^+_{d_{z^0}/2}(z^0))} + \sum_{|\beta|=2} z_2^0\big[D^\beta W\big]_{\alpha; B^+_{d_{z^0}/2}(z^0)} \\
&\leqq C\|v\|^{(-1-\alpha)}_{2,\alpha}.
\end{aligned} \tag{4.25}$$

Away from the corner points, by the interior and boundary estimates in Chapter 6 of [13] (or Theorem 5.1 in [12]), we can also obtain an analogous estimate as in (4.25). This, together with (4.8), (4.10) and (4.21), yields

$$\|u_2\|^{(-1-\alpha)}_{2,\alpha} \leqq C\left(\sum_{i=1}^{3}\|h_i\|^{(-\alpha)}_{1,\alpha} + \|g_6\|^{(-\alpha)}_{1,\alpha}\right).$$

Therefore, (4.7) is shown and the proof of Lemma 4.1 is completed. □

Based on Lemma 4.1, the second order elliptic equation in (4.4) can be changed into a homogeneous one.

Indeed, set

$$u_3 = u_1 - u_2.$$

Then

$$\begin{cases} \sum_{i=1}^{2} \partial_{z_i}\big(b_i(z_1)\partial_{z_i}u_3\big) + b_3(z_1)\partial_{z_1}u_3 + b_4(z_1)\partial_{z_1}u_3(0,z_2) = 0 \quad \text{in} \quad E_+, \\ \partial_{z_2}u_3(0,z_2) = g_7(z_2), \\ \partial_{z_2}u_3(N,z_2) = b_5\Big(\kappa + \int_0^{z_2} \partial_{z_1}u_3(0,s)\mathrm{d}s\Big), \\ u_3(z_1,0) = 0, \\ \partial_{z_1}u_3(z_1,M) = 0, \end{cases} \tag{4.26}$$

where

$$g_7(z_2) = g_5(z_2) - \partial_{z_2}u_2(0,z_2). \tag{4.27}$$

Note that (4.26) is a mixed Dirichlet-tangential derivative problem. The solvability conditions for this is

$$\int_0^M g_7(s)\mathrm{d}s = b_5\Big(M\kappa + \int_0^M \int_0^s \partial_{z_1}u_3(0,t)\mathrm{d}t\mathrm{d}s\Big), \tag{4.28}$$

which will be used to determine the unknown constant κ.

(4.26) is reduced to a Dirichlet problem by setting

$$u_4 = u_3 - \frac{z_2}{M}\int_0^M g_7(s)\mathrm{d}s, \quad g_8(z_2) = \int_0^{z_2} g_7(s)\mathrm{d}s - \frac{z_2}{M}\int_0^M g_7(s)\mathrm{d}s. \tag{4.29}$$

In this case, (4.26) becomes

$$\begin{cases} \sum_{i=1}^{2} \partial_{z_i}\big(b_i(z_1)\partial_{z_i}u_4\big) + b_3(z_1)\partial_{z_1}u_4 + b_4(z_1)\partial_{z_1}u_4(0,z_2) = 0 \quad \text{in} \quad E_+, \\ u_4(0,z_2) = g_8(z_2), \\ u_4(N,z_2) = b_5\Big(\int_0^{z_2}\int_0^s \partial_{z_1}u_4(0,t)\mathrm{d}t\mathrm{d}s - \frac{z_2}{M}\int_0^M\int_0^s \partial_{z_1}u_4(0,t)\mathrm{d}t\mathrm{d}s\Big), \\ u_4(z_1,0) = 0, \\ u_4(z_1,M) = 0, \end{cases} \tag{4.30}$$

where

$$\|g_8\|_{2,\alpha}^{(-1-\alpha)} \leqq C\|g_7\|_{1,\alpha}^{(-\alpha)}, \quad g_8(0) = g_8(M) = 0. \tag{4.31}$$

We now focus on the linear problem (4.30) in order to solve the problem (4.1). This will be done by a separation variable method together with the continuity method in a weighted Hölder space.

Lemma 4.2. (A priori estimate of $\|u_4\|_{2,\alpha}^{(-1-\alpha)}$) *Let* $u_4 \in H_{2,\alpha}^{(-1-\alpha)}(E_+)$ *be a solution to* (4.30). *Then*

$$\|u_4\|_{2,\alpha}^{(-1-\alpha)} \leqq C\|g_8\|_{2,\alpha}^{(-1-\alpha)}. \tag{4.32}$$

Proof. Note that $\{\sin(\frac{n}{M}\pi z_2)\}_{n=1}^{\infty}$ forms a complete orthogonal basis in $H_0^1[0, M]$ according to Theorem 8.37 in [13], u_4 is a solution of (4.30) in $H_{2,\alpha}^{(-1-\alpha)}$, and g_8 satisfies (4.31). It then follows from Remark B.1 in Appendix B that for any $\alpha' \in (0, \alpha)$

$$\begin{cases} u_4(z) = \sum_{n=1}^{\infty} X_n(z_1)\sin(\frac{n}{M}\pi z_2) \quad \text{in} \quad C^{1,\alpha'}([0, M]) \quad \text{for} \quad z_1 \in [0, N], \\ g_8(z_2) = \sum_{n=1}^{\infty} g_{8n}\sin(\frac{n}{M}\pi z_2) \quad \text{in} \quad C^{1,\alpha'}([0, M]) \end{cases} \tag{4.33}$$

and

$$\sum_{n=1}^{\infty} g_{8n}^2 + \sum_{n=1}^{\infty} n^2 g_{8n}^2 \leqq C(\|g_8\|_{2,\alpha}^{(-1-\alpha)})^2. \tag{4.34}$$

Moreover, due to $u_4(z) \in H_{2,\alpha}^{(-1-\alpha)}$, then for each positive integer n,

$$X_n(z_1) = \frac{2}{M}\int_0^M u_4(z_1, z_2)\sin(\frac{n}{M}\pi z_2)\mathrm{d}z_2 \in C^{1,\alpha}[0, N] \cap C^{2,\alpha}(0, N),$$

which solves the following problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}z_1}\Big(b_1(z_1)X_n'(z_1)\Big) + b_3(z_1)X_n'(z_1) - b_2(z_1)(\frac{n\pi}{M})^2 X_n(z_1) \\ \quad + b_4(z_1)X_n'(0) = 0 \quad \text{in} \quad (0, N), \\ X_n(0) = g_{8n}, \\ X_n(N) = -b_5\frac{M^2}{(n\pi)^2}X_n'(0). \end{cases} \tag{4.35}$$

Next, we establish the L^{∞}-norm estimate of $X_n(z_1)$. Without loss of generality, we assume $g_{8n} > 0$ since the analogous analysis can be given in the case of $g_{8n} \leqq 0$.

First, we claim that $X_n'(0) \leqq 0$.

If not, namely, $X_n'(0) > 0$, then it follows from the maximum principle that $X_n(z_1)$ achieves its positive maximum at $z_1 = 0$ due to $X_n(N) < 0$ and $b_4 \leqq -b_0$ by the assumption in (4.2). However, this is contrary to $X_n'(0) > 0$. Thus, $X_n'(0) \leqq 0$ holds true.

Second, we show that if $-\frac{g_{8n}}{b_0}(\frac{n\pi}{M})^2 \leqq X_n'(0) \leqq 0$, then

$$|X_n(z_1)| \leqq \left(1 + \frac{B_0}{b_0} + \frac{B_0}{b_0^2}\right)g_{8n}.$$

In fact, if $X_n(z_1)$ attains its positive maximum at some interior point $z_1^0 \in (0, N)$, then $X_n'(z_1^0) = 0$ and $X_n''(z_1^0) \leqq 0$. It follows from the equation in (4.35) that

$$b_1(z_1^0)X_n''(z_1^0) - b_2(z_1^0)(\frac{n\pi}{M})^2 X_n(z_1^0) + b_4(z_1^0)X_n'(0) = 0.$$

This yields

$$b_2(z_1^0)(\frac{n\pi}{M})^2 X_n(z_1^0) \leqq b_4(z_1^0)X_n'(0).$$

It follows from the assumptions on $b_2(z_1)$ and $b_4(z_1)$ in (4.2) that $X_n(z_1^0) \leqq \dfrac{B_0}{b_0^2} g_{8n}$. If $X_n(z_1)$ attains its negative minimum at some interior point $\bar{z}_1 \in (0, N)$, then $X_n'(\bar{z}_1) = 0$ and $X_n''(\bar{z}_1) \geqq 0$. Thus the equation in (4.35) shows that

$$b_2(\bar{z}_1)(\frac{n\pi}{M})^2 |X_n(\bar{z}_1)| \leqq -b_4(\bar{z}_1)X_n'(0) \leqq 0,$$

which yields a contradiction. Therefore,

$$|X_n(z_1)| \leqq \max\{X_n(z_1^0), X_n(0), X_n(N)\} \leqq \Big(1 + \frac{B_0}{b_0} + \frac{B_0}{b_0^2}\Big) g_{8n}.$$

Finally, we show that for $X_n'(0) < -\dfrac{g_{8n}}{b_0}(\dfrac{n\pi}{M})^2$, $|X_n(z_1)| \leqq (1+\dfrac{B_0^2}{b_0}+\dfrac{B_0^2}{b_0^2})g_{8n}$ holds.

Indeed, for $X_n'(0) < -\dfrac{g_{8n}}{b_0}(\dfrac{n\pi}{M})^2 < 0$, then $X_n(N) > g_{8n}$ holds. This means that $X_n(z_1)$ only attains its minimum at some interior point in $(0, N)$. It follows from the minimum principle that

$$-\frac{b_0}{B_0}(\frac{M}{n\pi})^2 X_n'(0) \leqq \min_{z_1 \in [0,N]} X_n(z_1).$$

On the other hand, $\min_{z_1 \in [0,N]} X_n(z_1) \leqq X_n(0) = g_{8n}$. Thus,

$$-\frac{B_0 g_{8n}}{b_0}(\frac{n\pi}{M})^2 \leqq X_n'(0) < 0.$$

It follows from this and the arguments in the second step that

$$|X_n(z_1)| \leqq \Big(1 + \frac{B_0^2}{b_0} + \frac{B_0^2}{b_0^2}\Big) g_{8n}.$$

Collecting all these cases yields

$$\|X_n\|_{L^\infty} \leqq C|g_{8n}|, \tag{4.36}$$

where the generic positive constant C is independent of n.

This, together with (4.33)–(4.34), shows that

$$\|u_4\|_{L^\infty} \leqq C\|g_8\|_{2,\alpha}^{(-1-\alpha)}. \tag{4.37}$$

In addition, one has

$$\begin{aligned}\|u_4\|_{C^{1,\alpha}} &\leqq C\big(\|u_4\|_{L^\infty} + \|g_8\|_{C^{1,\alpha}} + \|u_4(N, z_2)\|_{C^{1,\alpha}}\big)\\ &\leqq C\big(\|u_4\|_{L^\infty} + \|g_8\|_{C^{1,\alpha}} + \|\partial_{z_1} u_4(0, z_2)\|_{L^\infty}\big).\end{aligned}$$

Combining this with (4.37) and the interpolation inequality yields

$$\|u_4\|_{C^{1,\alpha}} \leqq C\|g_8\|_{2,\alpha}^{(-1-\alpha)}.$$

Finally, we have

$$\begin{aligned}
&\|u_4\|_{2,\alpha}^{(-1-\alpha)} \\
&\leqq C\bigg(\|u_4\|_{L^\infty} + \|b_4(z_1)\partial_{z_1}u_4(0,z_2)\|_{C^\alpha} + \|g_8\|_{2,\alpha}^{(-1-\alpha)} \\
&\quad + \Big\|\int_0^{z_2}\int_0^s \partial_{z_1}u_4(0,t)\mathrm{d}t\mathrm{d}s - \frac{z_2}{M}\int_0^M\int_0^s \partial_{z_1}u_4(0,t)\mathrm{d}t\mathrm{d}s\Big\|_{2,\alpha}^{(-1-\alpha)}\bigg) \\
&\leqq C\big(\|u_4\|_{L^\infty} + \|u_4\|_{C^{1,\alpha}} + \|g_8\|_{2,\alpha}^{(-1-\alpha)}\big) \\
&\leqq C\|g_8\|_{2,\alpha}^{(-1-\alpha)}.
\end{aligned}$$

Consequently, the proof of Lemma 4.2 is completed.

We now prove the existence by a continuity method, thus we start with the special case that the principal part of the elliptic operator is the Laplacian operator.

Lemma 4.3. *If $G(z_2) \in H_{2,\alpha}^{(-1-\alpha)}(0,M)$, $G(0) = G(M) = 0$, then the following problem*

$$\begin{cases}
\Delta U - \partial_{z_1}U(0,z_2) = 0 \quad in \quad E_+, \\
U(0,z_2) = G(z_2), \\
U(N,z_2) = b_5\Big(\int_0^{z_2}\int_0^s \partial_{z_1}U(0,s)\mathrm{d}s - \frac{z_2}{M}\int_0^M\int_0^s \partial_{z_1}U(0,s)\mathrm{d}s\Big), \\
U(z_1,0) = 0, \\
U(z_1,M) = 0
\end{cases} \tag{4.38}$$

has a unique solution $U \in H_{2,\alpha}^{(-1-\alpha)}(E_+)$ which satisfies

$$\|U\|_{2,\alpha}^{(-1-\alpha)} \leqq C\|G\|_{2,\alpha}^{(-1-\alpha)}. \tag{4.39}$$

Proof. Due to $G(z_2) \in H_{2,\alpha}^{(-1-\alpha)}(0,M)$ and $G(0) = G(M) = 0$, then by Remark B.1 in Appendix B, we have

$$G(z_2) = \sum_{k=1}^{\infty} G_k \sin(\frac{k}{M}\pi z_2) \quad in \quad C^{1,\alpha'}[0,M], \quad 0 < \alpha' < \alpha, \tag{4.40}$$

where $G_k = \frac{2}{M}\int_0^M G(z_2)\sin(\frac{k}{M}\pi z_2)\mathrm{d}z_2$ $(k = 1, 2, \ldots)$.

For each $k (k = 1, 2, \ldots)$, consider the following problem

$$\begin{cases}
\frac{\mathrm{d}^2}{\mathrm{d}z_1^2}Y_k(z_1) - (\frac{k}{M}\pi)^2 Y_k(z_1) - Y_k'(0) = 0, \\
Y_k(0) = G_k, \\
Y_k(N) = -b_5\frac{M^2}{(k\pi)^2}Y_k'(0).
\end{cases} \tag{4.41}$$

A solution to this problem is given by

$$Y_k(z_1) = C_k^1 \exp\Big(\frac{k}{M}\pi z_1\Big) + C_k^2 \exp\Big(-\frac{k}{M}\pi z_1\Big) - \frac{M}{k\pi}(C_k^1 - C_k^2)$$

with $C_k^i (i = 1, 2)$ satisfying

$$\begin{cases} \Big(1 - \dfrac{M}{k\pi}\Big)C_k^1 + \Big(1 + \dfrac{M}{k\pi}\Big)C_k^2 = G_k, \\ \Big(\exp(\dfrac{k}{M}\pi N) - (1 - b_5)\dfrac{M}{k\pi}\Big)C_k^1 \\ \quad + \Big(\exp(-\dfrac{k}{M}\pi N) + (1 - b_5)\dfrac{M}{k\pi}\Big)C_k^2 = 0. \end{cases} \tag{4.42}$$

Note that the determinant of the coefficient matrix in (4.42) is a negative number, which is less than $-2b_0\dfrac{M}{k\pi}$, then the algebraic system (4.42) has a unique solution (C_k^1, C_k^2) for each $k \in \mathbb{N}$ since $Y_k(z)$ is the unique solution to (4.11).

By (4.40) and (4.41), we can easily verify that

$$U_n(z) = \sum_{k=1}^{n} Y_k(z_1)\sin(\frac{k}{M}\pi z_2)$$

is a unique smooth solution of the following problem

$$\begin{cases} \Delta U_n - \partial_{z_1} U_n(0, z_2) = 0 \quad \text{in} \quad E_+, \\ U_n(0, z_2) = G^n(z_2), \\ U_n(N, z_2) = b_5\Big(\displaystyle\int_0^{z_2}\int_0^s \partial_{z_1} U_n(0, t)\mathrm{d}t\mathrm{d}s - \frac{z_2}{M}\int_0^M\int_0^s \partial_{z_1} U_n(0, t)\mathrm{d}t\mathrm{d}s\Big), \\ U_n(z_1, 0) = 0, \\ U_n(z_1, M) = 0, \end{cases}$$

with

$$G^n(z_2) = \sum_{k=1}^{n} G_k \sin(\frac{k}{M}\pi z_2) \quad \text{in} \quad C^{1,\alpha'}[0, M].$$

By the argument as in Lemma 4.2, one arrives at

$$\|U_n\|_{1,\alpha'} \leqq C\|G^n\|_{1,\alpha'}, \quad \|U_n - U_m\|_{1,\alpha'} \leqq C\|G^n - G^m\|_{1,\alpha'}. \tag{4.43}$$

Since $\{G^n(z_2)\}$ converges uniformly to $G(z_2)$ in $C^{1,\alpha'}[0, M]$, then there exists a unique $U(z) \in C^{1,\alpha'}(\bar{E}_+)$ which solves the problem (4.38). By the Schauder interior and boundary estimates in Chapter 8 of [13], it holds that

$$U(z) \in C^{2,\alpha}(E_+) \cap C^{1,\alpha}(\bar{E}_+).$$

In addition, by Lemma 4.2, $U(z)$ admits

$$\|U(z)\|_{2,\alpha}^{(-1-\alpha)} \leqq C\|G(z_2)\|_{2,\alpha}^{(-1-\alpha)}.$$

Thus, the proof of Lemma 4.3 is completed. □

Based on Lemma 4.1 to Lemma 4.3, we can show the solvability of (4.1) and give some related estimates.

Proposition 4.4. (Solvability and Estimates) *The problem* (4.1) *has a unique solution* $(u,\kappa)\in H^{(-1-\alpha)}_{2,\alpha}(E_+)\times\mathbb{R}$ *such that*

$$\|u\|^{(-1-\alpha)}_{2,\alpha}+|\kappa|\leqq C\Big(\sum_{i=1}^{2}\|g_i\|^{(-\alpha)}_{1,\alpha}+\sum_{j=3}^{4}\|g_j\|_{C^{1,\alpha}}+\sum_{k=1}^{3}\|h_k\|^{(-\alpha)}_{1,\alpha}\Big). \tag{4.44}$$

Proof. Due to (4.3), (4.5)–(4.6), and (4.26)–(4.29), it suffices to prove that the boundary value problem (4.30) has a unique solution $u_4\in H^{(-1-\alpha)}_{2,\alpha}(E_+)$ which satisfies

$$\|u_4\|^{(-1-\alpha)}_{2,\alpha}\leqq C\|g_8\|^{(-1-\alpha)}_{2,\alpha}. \tag{4.45}$$

To this end, we will employ the method of continuity (see Theorem 5.2 in [13]). First, we set

$$u_5=u_4-g_8(z_2)\Big(1-\big(\frac{z_1}{N}\big)^2\Big). \tag{4.46}$$

Then (4.30) is equivalent to

$$\begin{cases}\displaystyle\sum_{i=1}^{2}\partial_{z_i}\big(b_i(z_1)\partial_{z_i}u_5\big)+b_3(z_1)\partial_{z_1}u_5+b_4(z_1)\partial_{z_1}u_5(0,z_2)\\ \displaystyle\quad=\partial_{z_1}\Big(\frac{2z_1}{N}b_1(z_1)g_8(z_2)\Big)-\partial_{z_2}\Big(b_2(z_1)\big(1-(\frac{z_1}{N})^2\big)g_8'(z_2)\Big)\\ \displaystyle\qquad+2b_3(z_1)g_8(z_2)\frac{z_1}{N}\quad\text{in}\quad E_+,\\ u_5(0,z_2)=0,\\ \displaystyle u_5(N,z_2)=b_5\Big(\int_0^{z_2}\int_0^s\partial_{z_1}u_5(0,t)\mathrm{d}t\mathrm{d}s-\frac{z_2}{M}\int_0^M\int_0^s\partial_{z_1}u_5(0,t)\mathrm{d}t\mathrm{d}s\Big),\\ u_5(z_1,0)=0,\\ u_5(z_1,M)=0.\end{cases} \tag{4.47}$$

Next, we consider a family of operators $L_t(t\in[0,1])$ defined as

$$\begin{aligned}L_tv=&(1-t)\Big(\Delta v-\partial_{z_1}v(0,z_2)\Big)\\&+t\Big(\sum_{i=1}^{2}\partial_{z_i}\big(b_i(z_1)\partial_{z_i}v\big)+b_3(z_1)\partial_{z_1}v+b_4(z_1)\partial_{z_1}v(0,z_2)\Big),\end{aligned} \tag{4.48}$$

which maps the Banach space

$$\begin{aligned}B_1=\Big\{&v\in H^{(-1-\alpha)}_{2,\alpha}(E_+):\|v\|^{(-1-\alpha)}_{2,\alpha}<+\infty,\ v(0,z_2)=v(z_1,0)=v(z_1,M)=0,\\ &v(N,z_2)=b_5\Big(\int_0^{z_2}\int_0^s\partial_{z_1}v(0,t)\mathrm{d}t\mathrm{d}s-\frac{z_2}{M}\int_0^M\int_0^s\partial_{z_1}v(0,t)\mathrm{d}t\mathrm{d}s\Big)\Big\}\end{aligned} \tag{4.49}$$

into the normed linear space

$$B_2 = \left\{ \varphi = \sum_{i=1}^{2} \partial_{z_i} \varphi_i + \varphi_3 : \sum_{j=1}^{3} \|\varphi_j\|_{1,\alpha}^{(-\alpha)} < +\infty \right\} \tag{4.50}$$

with a norm defined as

$$\|\varphi\|_{B_2} = \inf_{\varphi = \sum_{i=1}^{2} \partial_{z_i} \varphi_i + \varphi_3} \sum_{j=1}^{3} \|\varphi_j\|_{1,\alpha}^{(-\alpha)}.$$

Then it follows from the arguments in Lemma 4.1 to Lemma 4.2 that for any $t \in [0, 1]$ and any $v \in B_1$,

$$\|v\|_{2,\alpha}^{(-1-\alpha)} \leqq C \|L_t v\|_{B_2}. \tag{4.51}$$

In addition, for any $g \in B_2$, the arguments for Lemma 4.1 and Lemma 4.3 show that the following problem

$$\begin{cases} L_0 v = g \quad \text{in} \quad E_+, \\ v \in B_1 \end{cases} \tag{4.52}$$

has a unique solution u satisfying the following estimate

$$\|v\|_{2,\alpha}^{(-1-\alpha)} \leqq C \|g\|_{B_2}. \tag{4.53}$$

Based on (4.51)–(4.53), it follows from Theorem 5.2 in [13] that (4.30) has a unique solution $u_4 \in H_{2,\alpha}^{(-1-\alpha)}(E_+)$ satisfying (4.45). On the other hand, it is noted that κ can be solved from (4.28) as $\kappa = \frac{1}{M}\big(\frac{1}{b_5}\int_0^M g_7(s)\mathrm{d}s - \int_0^M \int_0^s \partial_{z_1} u_3(0, t)\mathrm{d}t\mathrm{d}s\big)$. Therefore, the proof of Proposition 4.4 is completed. □

5. Proofs of Theorem 2.1 and Theorem 1.1

According to the reformulation in Section 3, we need to solve the nonlinear coupled system (3.3), (3.17) and (3.33). This will be done by an iteration scheme. Note that due to (3.1), for each $\hat{W} \in \Xi_\delta$, there corresponds to a triplet

$$(\hat{\phi}(z), \hat{A}(z_2), \hat{\psi}(z_2)). \tag{5.1}$$

The iteration procedure will be divided into the following three parts:

Part 1. The Determinations of the Approximate Stream Function and the Shock Location at one Nozzle Wall

For given $\hat{W} \in \Xi_\delta$, the new potential function $\bar{W}_1$ and the new shock location $\bar{W}_3(0)$ at the lower nozzle wall are obtained by solving the linearized problem associated with (3.33):

$$
\begin{cases}
\sum_{j=1}^{2} \partial_{z_j}\left(a_{j+3}(z_1)\partial_{z_j}\bar{W}_1\right) + a_6(z_1)\partial_{z_1}\bar{W}_1 + a_1a_2a_7(z_1)\partial_{z_1}\bar{W}_1(0, z_2) \\
\quad = \sum_{k=1}^{2} \partial_{z_k} F_{k+7}(\nabla\hat{W}_1, \hat{W}_2, \hat{W}_3) + F_{10}(\nabla\hat{W}_1, \hat{W}_2, \hat{W}_3) \quad \text{in} \quad E_+, \\
\partial_{z_2}\bar{W}_1(0, z_2) = F_2(\nabla\hat{W}_1, \hat{W}_2, \hat{W}_3)(0, z_2), \\
\partial_{z_2}\bar{W}_1(N, z_2) = a_2a_3\left(\bar{W}_3(0) + a_1\int_0^{z_2}\partial_{z_1}\bar{W}_1(0, s)\mathrm{d}s\right) \\
\qquad\qquad + F_{11}(\nabla\hat{W}_1, \hat{W}_2, \hat{W}_3)(0, z_2), \\
\partial_{z_1}\bar{W}_1(z_1, 0) = f_1'(\hat{\psi}(0) + \frac{1}{N}(X_0 + 1 - \hat{\psi}(0))z_1), \\
\partial_{z_1}\bar{W}_1(z_1, M) = f_2'(\hat{\psi}(M) + \frac{1}{N}(X_0 + 1 - \hat{\psi}(M))z_1), \\
\bar{W}_1(N, 0) = 0,
\end{cases}
\tag{5.2}
$$

where $\hat{\psi}$ is given in (5.1).

It follows from Proposition 4.4 that (5.2) has a unique solution

$$
(\bar{W}_1, \bar{W}_3(0)) \in H_{2,\alpha}^{(-1-\alpha)}(E_+) \times \mathbb{R}
$$

such that

$$
\begin{aligned}
&\|\bar{W}_1\|_{2,\alpha}^{(-1-\alpha)} + |\bar{W}_3(0)| \\
&\leqq C_0\left(\|F_2(\nabla\hat{W}_1, \hat{W}_2, \hat{W}_3)\|_{1,\alpha}^{(-\alpha)} + \sum_{i=8}^{11}\|F_i(\nabla\hat{W}_1, \hat{W}_2, \hat{W}_3)\|_{1,\alpha}^{(-\alpha)} + \sum_{j=1}^{2}\|f_i\|_{C^{2,\alpha}}\right) \\
&\leqq C_0\left(\delta^2 + \varepsilon\right),
\end{aligned}
\tag{5.3}
$$

where the last inequality follows from (3.11), (3.34) and (1.6).

Part 2. The Determination of the Approximate Shock Position

The new shock position, $\bar{W}_3$, can be obtained by solving the linearized equation of (3.3) as

$$
\bar{W}_3'(z_2) = a_1\partial_{z_1}\bar{W}_1 + F_1(\nabla\hat{W}_1, \hat{W}_2, \hat{W}_3),
\tag{5.4}
$$

with the initial data $\bar{W}_3(0)$ given in (5.3).

Thus, the solution $\bar{W}_3$ of (5.4) exists uniquely and satisfies

$$
\begin{aligned}
\|\bar{W}_3\|_{2,\alpha}^{(-1-\alpha)} &\leqq C_0\left(\|\bar{W}_1\|_{2,\alpha}^{(-1-\alpha)} + |\bar{W}_3(0)| + \|F_1(\nabla\hat{W}_1, \hat{W}_2, \hat{W}_3)\|_{1,\alpha}^{(-\alpha)}\right) \\
&\leqq C_0\left(\delta^2 + \varepsilon\right),
\end{aligned}
\tag{5.5}
$$

where the last inequality follows from (3.8) and (5.3).

Part 3. The Determination of the Approximate Entropy Function

Due to (3.17), one can define the new entropy $\bar{W}_2(z_2)$ by

$$\bar{W}_2(z_2) = a_2\bar{W}_3(z_2) + F_3(\nabla\hat{W}_1, \hat{W}_2, \hat{W}_3). \tag{5.6}$$

It follows from (3.18) and (5.5) that

$$\begin{aligned}\|\bar{W}_2\|_{1,\alpha}^{(-\alpha)} &\leqq C_0\Big(\|\bar{W}_3\|_{1,\alpha}^{(-\alpha)} + \|F_3(\nabla\hat{W}_1, \hat{W}_2, \hat{W}_3)\|_{1,\alpha}^{(-\alpha)}\Big)\\ &\leqq C_0\big(\delta^2 + \varepsilon\big).\end{aligned} \tag{5.7}$$

Based on those estimates in Part 1–Part 3, we can now show Theorem 2.1.

Proof of Theorem 2.1. Theorem 2.1 will be proved by the contractible mapping theorem based on the iteration scheme in (5.2), (5.4) and (5.6).

Indeed, due to (5.2)–(5.7), we can define a mapping T as follows

$$T(\hat{W}) = \bar{W}, \tag{5.8}$$

where $\hat{W} = (\hat{W}_1, \hat{W}_2, \hat{W}_3) \in \Xi_\delta$ and $\bar{W} = (\bar{W}_1, \bar{W}_2, \bar{W}_3)$. It follows from the estimates (5.3), (5.5) and (5.7) that T is a continuous mapping from Ξ_δ into itself for properly chosen $\delta = O(1)(\varepsilon) > 0$.

It suffices to show that T is contractible.

For any given two states $\hat{W}^1 = (\hat{W}_1^1, \hat{W}_2^1, \hat{W}_3^1)$ and $\hat{W}^2 = (\hat{W}_1^2, \hat{W}_2^2, \hat{W}_3^2)$ in Ξ_δ with the corresponding functions $(\hat{\phi}_1, \hat{A}_1, \hat{\psi}_1)$ and $(\hat{\phi}_2, \hat{A}_2, \hat{\psi}_2)$ respectively, we set

$$T(\hat{W}^i) = \bar{W}^i, \qquad i = 1, 2$$

with $\bar{W}^1 = (\bar{W}_1^1, \bar{W}_2^1, \bar{W}_3^1)$ and $\bar{W}^2 = (\bar{W}_1^2, \bar{W}_2^2, \bar{W}_3^2)$.

Due to (5.2) and the estimate (5.3), one has

$$\begin{aligned}&\|\bar{W}_1^1 - \bar{W}_1^2\|_{2,\alpha}^{(-1-\alpha)} + |\bar{W}_3^1(0) - \bar{W}_3^2(0)|\\ &\quad\leqq C_0\Big(\sum_{i=2,8}^{11}\|F_i(\nabla\hat{W}_1^1, \hat{W}_2^1, \hat{W}_3^1) - F_i(\nabla\hat{W}_1^2, \hat{W}_2^2, \hat{W}_3^2)\|_{1,\alpha}^{(-\alpha)}\\ &\qquad + \sum_{j=1}^{2}\|f_j\|_{C^{2,\alpha}}\|\hat{W}_3^1 - \hat{W}_3^2\|_{2,\alpha}^{(-1-\alpha)}\Big)\\ &\quad\leqq C_0\varepsilon\Big(\|\hat{W}_1^1 - \hat{W}_1^2\|_{2,\alpha}^{(-1-\alpha)} + \|\hat{W}_2^1 - \hat{W}_2^2\|_{1,\alpha}^{(-\alpha)} + \|\hat{W}_3^1 - \hat{W}_3^2\|_{2,\alpha}^{(-1-\alpha)}\Big),\end{aligned} \tag{5.9}$$

where in the last inequality, one has used (3.12), (3.35) and (1.6).

In addition, it follows from (5.4) and (5.5) that

$$\begin{aligned}&\|\bar{W}_3^1 - \bar{W}_3^2\|_{2,\alpha}^{(-1-\alpha)}\\ &\quad\leqq C_0\Big(\|\bar{W}_1^1 - \bar{W}_1^2\|_{2,\alpha}^{(-1-\alpha)} + |\bar{W}_3^1(0) - \bar{W}_3^2(0)|\\ &\qquad + \|F_1(\nabla\hat{W}_1^1, \hat{W}_2^1, \hat{W}_3^2) - F_1(\nabla\hat{W}_1^2, \hat{W}_2^2, \hat{W}_3^2)\|_{1,\alpha}^{(-\alpha)}\Big).\end{aligned}$$

Combining this with (3.9) and (5.9) yields

$$\|\bar{W}_3^1 - \bar{W}_3^2\|_{2,\alpha}^{(-1-\alpha)} \leqq C_0\varepsilon\bigg(\sum_{i=1,3}\|\hat{W}_i^1 - \hat{W}_i^2\|_{2,\alpha}^{(-1-\alpha)} + \|\hat{W}_2^1 - \hat{W}_2^2\|_{1,\alpha}^{(-\alpha)}\bigg). \tag{5.10}$$

Similarly, by (5.6)–(5.7) and the estimates (3.19) and (5.10), one can arrive at

$$\|\bar{W}_2^1 - \bar{W}_2^2\|_{1,\alpha}^{(-\alpha)} \leqq C_0\varepsilon\bigg(\sum_{i=1,3}\|\hat{W}_i^1 - \hat{W}_i^2\|_{2,\alpha}^{(-1-\alpha)} + \|\hat{W}_2^1 - \hat{W}_2^2\|_{1,\alpha}^{(-\alpha)}\bigg). \tag{5.11}$$

Therefore, the estimates (5.9)–(5.11) show that

$$\begin{aligned}&\bigg(\sum_{i=1,3}\|\bar{W}_i^1 - \bar{W}_i^2\|_{2,\alpha}^{(-1-\alpha)} + \|\bar{W}_2^1 - \bar{W}_2^2\|_{1,\alpha}^{(-\alpha)}\bigg)\\ &\quad\leqq C_0\varepsilon\bigg(\sum_{i=1,3}\|\bar{W}_i^1 - \bar{W}_i^2\|_{2,\alpha}^{(-1-\alpha)} + \|\bar{W}_2^1 - \bar{W}_2^2\|_{1,\alpha}^{(-\alpha)}\bigg),\end{aligned} \tag{5.12}$$

here the constant $C_0 > 0$ depends only on α, θ_0 and the supersonic incoming flow. Thus, for suitably small ε, (5.12) implies that the mapping T is contractible in Ξ_δ. This means that there exists a unique solution $W = (W_1, W_2, W_3) \in \Xi_\delta$ which solves the nonlinear problem (2.19)–(2.25), in particular, $\delta = O(\varepsilon)$ can be chosen. Consequently, the proof of Theorem 2.1 is completed. □

Proof of Theorem 1.1. This follows from Theorem 2.1, the Lagrange coordinate transformation determined by (2.1), the transformation (2.17), and the estimate (2.26). Thus, the proof of Theorem 1.1 is completed. □

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Appendix A. Some Basic Estimates Used in Section 3

In this Appendix, we give some basic estimates and computations which have been used in Section 3.

Lemma A.1. *If* $Q_i(z) \in H_{1,\alpha}^{(-\alpha)}(E_+)(i = 1, 2)$*, then there exists a generic constant* $C > 0$ *such that*

$$\|Q_1(z)Q_2(z)\|_{1,\alpha}^{(-\alpha)} \leqq C\|Q_1\|_{1,\alpha}^{(-\alpha)}\|Q_2\|_{1,\alpha}^{(-\alpha)}. \tag{A.1}$$

In addition, if $|\frac{1}{Q_2(z)}| \leqq C$ *holds, then*

$$\|\frac{Q_1}{Q_2}\|_{1,\alpha}^{(-\alpha)} \leqq C\|Q_1\|_{1,\alpha}^{(-\alpha)}\|Q_2\|_{1,\alpha}^{(-\alpha)}. \tag{A.2}$$

Analogous conclusions hold for space $H_{1,\alpha}^{(-\alpha)}(\Gamma)$ *with* $\Gamma = (0, M)$.

Proof. This can be verified by direct computations, we omit the proof here. □

Let Ξ_δ be defined by (3.2). For $W^i = (W_1^i, W_2^i, W_3^i) \in \Xi_\delta (i = 1, 2)$, denote by $(\phi_i(z), A_i(z_2), \psi_i(z_2), \rho_i(z))$ $(i = 1, 2)$ the corresponding functions obtained from (3.1) and (2.25), respectively. Then one has:

Lemma A.2. *It holds that*

$$\|\rho_i\|_{1,\alpha}^{(-\alpha)} \leqq C \tag{A.3}$$

and

$$\|\rho_1-\rho_2\|_{1,\alpha}^{(-\alpha)} \leqq C\Big(\|\Big(\nabla(W_1^1-W_1^2), W_2^1-W_2^2\Big)\|_{1,\alpha}^{(-\alpha)}+\|W_3^1-W_3^2\|_{2,\alpha}^{(-1-\alpha)}\Big). \tag{A.4}$$

Proof. It follows from (2.25) and a direct computation that

$$\frac{M_5(V^i)}{M_6(V^i)} + \frac{\gamma}{\gamma-1}A_i(z_2)\rho_i^{\gamma+1} = B\rho_i^2, \quad i = 1, 2, \tag{A.5}$$

where $V^i = (\nabla W_1^i, W_2^i, W_3^i)$ and

$$\begin{cases} M_5(V^i) \\ \quad = 1 + \Big(\psi_i(z_2) + \frac{z_1}{N}(X_0 + 1 - \psi(z_2))\Big)^2\Big(\frac{N}{X_0 + 1 - \psi_i(z_2)}\Big)^2(\partial_{z_1}W_1^i)^2, \\ M_6(V^i) \\ \quad = 2\Big(\psi_i(z_2) + \frac{z_1}{N}(X_0 + 1 - \psi(z_2))\Big)^2 \\ \qquad \times\Big(\partial_{z_2}\phi_i + \frac{z_1 - N}{X_0 + 1 - \psi_i(z_2)}\psi_i'(z_2)\partial_{z_1}W_1^i\Big)^2. \end{cases} \tag{A.6}$$

Since $2B - \frac{\gamma(\gamma+1)}{\gamma-1}A_i(z_2)\rho_i^{\gamma-1} < 0$ for subsonic flows, then by the implicit function theorem and (A.5), there exists a smooth function $F(\cdot, \cdot)$ such that

$$\rho_i = F\Big(\frac{M_5(V^i)}{M_6(V^i)}, W_2^i\Big), \quad i = 1, 2. \tag{A.7}$$

We now prove (A.3).

First, it is easy to know

$$\|\rho_i\|_{C^\alpha} \leqq C. \tag{A.8}$$

In addition, due to $W^i \in \Xi_\delta$, then a direct computation yields

$$|M_6(V^i) - 2((r_0 + z_1)\partial_{z_2}\phi_0^+)^2| \leqq C\delta, \ \|M_j(V^i)\|_{C^\alpha} \leqq C(j = 5, 6).$$

This, together with Lemma A.1, yields

$$\|\nabla\rho_i\|_\alpha^{(1-\alpha)} = \left\|\nabla F\left(\frac{M_5(V^i)}{M_6(V^i)}, W_2^i\right) \cdot \left(\nabla\frac{M_5(V^i)}{M_6(V^i)}, \nabla W_2^i\right)\right\|_\alpha^{(1-\alpha)} \leqq C. \tag{A.9}$$

Consequently, (A.3) follows from (A.8)–(A.9) directly.
Next, we show (A.4).
Since

$$\begin{aligned}\rho_1 - \rho_2 &= \int_0^1 \nabla F\left(s\frac{M_5(V^1)}{M_6(V^1)} + (1-s)\frac{M_5(V^2)}{M_6(V^2)}, sW_2^1 + (1-s)W_2^2\right)\mathrm{d}s \\ &\quad\times\left(\frac{M_5(V^1)}{M_6(V^1)} - \frac{M_5(V^2)}{M_6(V^2)}, W_2^1 - W_2^2\right),\end{aligned}$$

and as in the proof of (A.3), one has

$$\left\|\int_0^1 \nabla F\left(s\frac{M_5(V^1)}{M_6(V^1)} + (1-s)\frac{M_5(V^2)}{M_6(V^2)}, sW_2^1 + (1-s)W_2^2\right)\mathrm{d}s\right\|_{1,\alpha}^{(-\alpha)} \leqq C.$$

Then

$$\|\rho_1 - \rho_2\|_{1,\alpha}^{(-\alpha)} \leqq C\left(\sum_{i=5,6}\|M_i(V^1) - M_i(V^2)\|_{1,\alpha}^{(-\alpha)} + \|W_2^1 - W_2^2\|_{1,\alpha}^{(-\alpha)}\right). \tag{A.10}$$

On the other hand,

$$M_5(V^1) - M_5(V^2) = O(\delta)\partial_{z_1}(W_1^1 - W_1^2) + O(\delta^2)(W_3^1 - W_3^2) \tag{A.11}$$

and

$$\begin{aligned}&M_6(V^1) - M_6(V^2) \\ &\quad= O(\delta)\partial_{z_1}(W_1^1 - W_1^2) + O(1)\partial_{z_2}(W_1^1 - W_1^2) \\ &\quad\quad+ O(1)(W_3^1 - W_3^2) + O(\delta)(W_3^1 - W_3^2)'(z_2),\end{aligned} \tag{A.12}$$

here the notation $O(\kappa)$ means that there exists a generic constant $C > 0$ such that $\|O(\kappa)\|_{1,\alpha}^{(-\alpha)} \leqq C\kappa$.

Thus, substituting (A.11) and (A.12) into (A.10) yields (A.4), and the proof of Lemma A.2 is completed. □

Lemma A.3. *It also holds that*

$$\begin{aligned}&\Big\|\rho_i-\rho_0^++\frac{\gamma}{\gamma-1}\frac{(\rho_0^+)^\gamma}{c^2(A_0^+,\rho_0^+)-(U_0^+)^2}W_2^i\\&\quad-\frac{1}{\rho_0^+\big(c^2(A_0^+,\rho_0^+)-(U_0^+)^2\big)}\\&\quad\times\bigg(\frac{\partial_{z_2}W_1^i}{(r_0+z_1)^2(\partial_{z_2}\phi_0^+)^3}+\frac{(1-\frac{z_1}{N})W_3^i}{(r_0+z_1)^3(\partial_{z_2}\phi_0^+)^2}\bigg)\Big\|_{1,\alpha}^{(-\alpha)}\\&\leqq C\delta^2.\end{aligned}\tag{A.13}$$

Proof. It follows from the estimate (A.4) that

$$\begin{aligned}&\|\rho_i-\rho_0^+\|_{1,\alpha}^{(-\alpha)}\\&\leqq C\bigg(\|\nabla W_1^i\|_{1,\alpha}^{(-\alpha)}+\|W_2^i\|_{1,\alpha}^{(-\alpha)}+\|W_3^i\|_{2,\alpha}^{(-1-\alpha)}\bigg)\leqq C\delta.\end{aligned}\tag{A.14}$$

By (2.25), one has

$$\begin{aligned}&B\rho_i^2-\frac{\gamma}{\gamma-1}A_i(z_2)(\rho_i)^{\gamma+1}\\&=\frac{1}{2(r_0+z_1)^2(\partial_{z_2}\phi_0^+)^2}-\frac{\partial_{z_2}W_1^i}{(r_0+z_1)^2(\partial_{z_2}\phi_0^+)^3}-\frac{(1-\frac{z_1}{N})W_3^i}{(r_0+z_1)^3(\partial_{z_2}\phi_0^+)^2}\\&\quad+\big(O(1)\nabla W_1^i+O(1)W_3^i+O(1)(W_3^i)'\big)^2.\end{aligned}\tag{A.15}$$

Similarly, the estimates (A.3) and (A.14) imply

$$\begin{aligned}&\big(B\rho_i^2-\frac{\gamma}{\gamma-1}A_i(z_2)\rho_i^{\gamma+1}\big)-\big(B(\rho_0^+)^2-\frac{\gamma}{\gamma-1}A_0^+(\rho_0^+)^{\gamma+1}\big)\\&=\bigg(\rho_0^+\big((U_0^+)^2-c^2(A_0^+,\rho_0^+)\big)+O(\delta)\bigg)(\rho_i-\rho_0^+)-\frac{\gamma}{\gamma-1}(\rho_0^+)^{\gamma+1}W_2^i\\&\quad+\big(O(1)\nabla W_1^i+O(1)W_2^i+O(1)W_3^i+O(1)(W_3^i)'\big)^2.\end{aligned}$$

Combining this with (A.15) yields (A.13) and thus the proof of Lemma A.3 is completed. □

Appendix B. A Property of Fourier Series

In this Appendix, we will give an elementary property for the Fourier series of periodic functions with $C^{1,\alpha}$ regularity.

It is well known that if $f(x)\in L^2[-\pi,\pi]$, then

$$f(x)=a_0+\sum_{k=1}^{\infty}\big(a_k\cos(kx)+b_k\sin(kx)\big)\quad\text{in}\quad L^2[-\pi,\pi]\tag{B.1}$$

with

$$\begin{cases} a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(s)\mathrm{d}s, \\ a_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(s)\cos(ks)\mathrm{d}s, \quad k = 1, 2, \dots \\ b_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(s)\sin(ks)\mathrm{d}s, \quad k = 1, 2, \dots. \end{cases} \tag{B.2}$$

Let $\{S_n(x; f)\}$ be the nth partial sum of the Fourier series of $f(x)$ are defined as

$$\begin{aligned} S_n(x; f) &= a_0 + \sum_{k=1}^{n} \Big(a_k \cos(kx) + b_k \sin(kx) \Big) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\big((n+1/2)t\big)}{\sin(t/2)} \hat{f}(x-t)\mathrm{d}t, \quad n = 0, 1, 2, \dots, \end{aligned} \tag{B.3}$$

where $\hat{f}(x)$ is the 2π periodic extension of $f(x)$.

For $0 < \alpha < 1$, and $g(x) \in C^{\alpha}[-\pi, \pi]$ with $g(-\pi) = g(\pi)$, $\{S_n(x; g)\}$ has the following property:

Proposition B.1. *For any $\alpha' \in (0, \alpha)$,*

$$\lim_{n\to\infty} \|S_n(x; g) - g(x)\|_{C^{\alpha'}[-\pi,\pi]} = 0. \tag{B.4}$$

Proof. Since $S_n(x; 1) \equiv 1$, so (B.3) yields

$$S_n(x; g) - g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\big((n+1/2)t\big)}{\sin(t/2)} (\hat{g}(x-t) - \hat{g}(x))\mathrm{d}t, \tag{B.5}$$

where $\hat{g}(x)$ is the 2π periodic extension of $g(x)$.

The proof on (B.4) is divided into two parts as follows.

Part 1. The estimate on $\|S_n(x; g) - g(x)\|_{L^\infty}$

For any $\lambda \in (0, 1)$,

$$\begin{aligned} |S_n(x; g) - g(x)| &\leqq \frac{1}{2\pi}\int_{|t|\leqq\lambda} \frac{1}{|\sin(t/2)|} |\hat{g}(x-t) - \hat{g}(x)|\mathrm{d}t \\ &\quad + \frac{1}{2\pi}\left| \int_{\lambda\leqq|t|\leqq\pi} \frac{\sin\big((n+1/2)t\big)}{\sin(t/2)} (\hat{g}(x-t) - \hat{g}(x))\mathrm{d}t \right| \\ &\equiv I_1 + I_2. \end{aligned} \tag{B.6}$$

Due to $\dfrac{|x|}{\pi} \leqq |\sin x|$ for $|x| \leqq 1$, then

$$I_1 \leqq \|g\|_{C^\alpha} \int_{|t|\leqq\lambda} |t|^{\alpha-1}\mathrm{d}t \leqq \frac{2}{\alpha}\|g\|_{C^\alpha}\lambda^{\alpha}. \tag{B.7}$$

Next we deal with I_2.
Since $\hat{g}(x) \in C^{\alpha}(\mathbb{R})$, there exists a function $\tilde{g}(x) \in C^{1,\alpha}(\mathbb{R})$ such that

$$\|\tilde{g}(x) - \hat{g}(x)\|_{C^\alpha} \leqq \lambda^2.$$

Thus

$$\begin{aligned} I_2 \leqq & \frac{1}{2\pi}\left|\int_{\lambda\leqq|t|\leqq\pi} \frac{\sin\big((n+1/2)t\big)}{\sin(t/2)}(\tilde{g}(x-t)-\tilde{g}(x))\mathrm{d}t\right| + \frac{2\lambda^2}{\sin(\lambda/2)} \\ = & I_{21} + \frac{2\lambda^2}{\sin(\lambda/2)}. \end{aligned} \tag{B.8}$$

Integration by parts in I_{21} gives

$$\begin{aligned} I_{21} \leqq & \frac{1}{\lambda(n+1/2)}\big|\tilde{g}(x-\lambda)-\tilde{g}(x)\big| + \frac{1}{\lambda(n+1/2)}\big|\tilde{g}(x+\lambda)-\tilde{g}(x)\big| \\ & + \frac{1}{(n+1/2)}\big|\tilde{g}(x-\pi)-\tilde{g}(x)\big| + \frac{1}{(n+1/2)}\big|\tilde{g}(x+\pi)-\tilde{g}(x)\big| \\ & + \frac{1}{n+1/2}\int_{\lambda\leqq|t|\leqq\pi} \frac{1}{|\sin(t/2)|}\big|\tilde{g}'(x-t)\big|\mathrm{d}t \\ & + \frac{1}{n+1/2}\int_{\lambda\leqq|t|\leqq\pi} \frac{1}{|\sin(t/2)|^2}\big|\tilde{g}(x-t)-\tilde{g}(x)\big|\mathrm{d}t \\ \leqq & \frac{1}{n+1/2}\left(\frac{8}{\lambda}+\frac{16\pi}{\lambda^2}\right)\|\tilde{g}\|_{C^1}. \end{aligned} \tag{B.9}$$

So for fixed $\lambda \in (0, 1)$, one has

$$I_{21} \to 0 \quad \text{as} \quad n \to \infty. \tag{B.10}$$

Combining (B.6)–(B.10) yields

$$\|S_n(x; g) - g(x)\|_{L^\infty} \to 0 \quad \text{as} \quad n \to \infty. \tag{B.11}$$

Part 2. The estimate on $[S_n(x; g) - g(x)]_{\alpha'}$

It follows from (B.5) that

$$\begin{aligned} & \big(S_n(x; g) - g(x)\big) - \big(S_n(y; g) - g(y)\big) \\ & = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\big((n+1/2)t\big)}{\sin(t/2)}\Big(\big(\hat{g}(x-t)-\hat{g}(x)\big) - \big(\hat{g}(y-t)-\hat{g}(y)\big)\Big)\mathrm{d}t. \end{aligned} \tag{B.12}$$

It can be verified directly that

$$\begin{aligned} & \big|\big(\hat{g}(x-t)-\hat{g}(x)\big) - \big(\hat{g}(y-t)-\hat{g}(y)\big)\big| \\ & \leqq 2\|g\|_{C^\alpha}|x-y|^{\alpha'}|t|^{\alpha-\alpha'}. \end{aligned} \tag{B.13}$$

For any $\lambda \in (0, 1)$, one has

$$
\begin{aligned}
&\big|\big(S_n(x;g)-g(x)\big)-\big(S_n(y;g)-g(y)\big)\big| \\
&\leqq \frac{1}{2\pi}\int_{|t|\leqq\lambda}\frac{1}{|\sin(t/2)|}\big|\big(\hat{g}(x-t)-\hat{g}(x)\big)-\big(\hat{g}(y-t)-\hat{g}(y)\big)\big|\mathrm{d}t \\
&\quad+\frac{1}{2\pi}\left|\int_{\lambda\leqq|t|\leqq\pi}\frac{\sin\big((n+1/2)t\big)}{\sin(t/2)}\big(\tilde{g}(x-t)-\tilde{g}(x)\big)-\big(\tilde{g}(y-t)-\tilde{g}(y)\big)\mathrm{d}t\right| \\
&\quad+\frac{2\lambda^2}{\sin(\lambda/2)}|x-y|^{\alpha} \\
&\equiv I_3+I_4+\frac{2\lambda^2}{\sin(\lambda/2)}|x-y|^{\alpha}.
\end{aligned}
\tag{B.14}
$$

Similar to the proof in (B.7), (B.13) yields

$$
I_3 \leqq \frac{4}{\alpha-\alpha'}\|g\|_{C^{\alpha}}\lambda^{\alpha-\alpha'}|x-y|^{\alpha'}. \tag{B.15}
$$

Following the arguments in (B.9) shows that

$$
I_4 \leqq \frac{1}{n+1/2}\left(\frac{8}{\lambda}+\frac{16\pi}{\lambda^2}\right)\|\tilde{g}\|_{C^{1,\alpha}}|x-y|^{\alpha}. \tag{B.16}
$$

It follows from (B.14)–(B.16) that

$$
[S_n(x;g)-g(x)]_{\alpha'} \to 0 \quad \text{as} \quad n\to\infty. \tag{B.17}
$$

Thus, (B.4) follows from (B.11) and (B.17), and so the proof of Proposition B.1 is completed. □

Remark B.1. If $h \in C^{1,\alpha}[0,\pi]$ with $h(0) = h(\pi) = 0$, let

$$
\bar{h}(x) = \begin{cases} h(x), & 0 \leqq x \leqq \pi, \\ -h(-x), & -\pi \leqq x \leqq 0, \end{cases} \tag{B.18}
$$

then the function $\bar{h}(x) \in C^{1,\alpha}[-\pi,\pi]$ and its $2\pi-$periodic extension $\bar{\bar{h}}(x)$ belongs to $C^{1,\alpha}(\mathbb{R})$. By Proposition $B.1$, for any $\alpha' \in (0,\alpha)$, it holds that

$$
\|S_n(x;\bar{h})-\bar{h}(x)\|_{C^{1,\alpha'}[-\pi,\pi]}+\|S_n(x;\bar{h})-h(x)\|_{C^{1,\alpha'}[0,\pi]} \to 0 \quad \text{as} \quad n\to\infty.
$$

Moreover, it also follows from (B.1)–(B.2) and the odd symmetric property of $\bar{h}(x)$ in (B.18) that for $x \in [-\pi,\pi]$,

$$
S_n(x;\bar{h}) = \sum_{k=1}^{n}\bar{h}_n\sin(nx) \qquad \text{with} \quad \bar{h}_n = \frac{2}{\pi}\int_0^{\pi}h(s)\sin(ks)\mathrm{d}s.
$$

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