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Torus localization and wall crossing for cosection localized virtual cycles ☆

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A B S T R A C T

The theory of virtual fundamental class defines important invariants such as the Gromov–Witten and the Donaldson–Thomas invariants. It has been generalized to the cosection localized virtual cycle which has applications in Seiberg–Witten, Fan–Jarvis–Ruan–Witten and other invariants. In this paper, we prove the formulas of virtual pullback, torus localization and wall crossing for cosection localized virtual cycles.

1. Introduction

Since its introduction in 1995 by Li–Tian [30] and Behrend–Fantechi [1], the theory of virtual fundamental classes has played a key role in algebraic geometry, defining impor-

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tant invariants such as Gromov–Witten invariants and Donaldson–Thomas invariants. Quite a few methods for handling virtual fundamental classes were discovered such as torus localization [15], degeneration [28], virtual pullback [31] and cosection localization [21]. Often combining these methods turns out to be quite effective. The purpose of this paper is to prove

(1) virtual pullback formulas,
(2) a torus localization formula and
(3) a wall crossing formula

for cosection localized virtual cycles. Our results can be thought of as generalizations of the corresponding results for the ordinary virtual cycles because when the cosection is trivial, these formulas coincide with those for the ordinary virtual cycles. For (2), we remove a technical assumption in [15] on the existence of an equivariant global embedding into a smooth DM stack.

A DM stack X is equipped with the intrinsic normal cone $\mathfrak{C}_X$ which is étale locally $[C_{U/V}/T_V|_U]$ if $U \to X$ is étale and $U \hookrightarrow V$ is an embedding into a smooth variety, where $C_{U/V}$ is the normal cone of U in V. A perfect obstruction theory [1] gives us a vector bundle stack $\mathcal{E}$ together with an embedding $\mathfrak{C}_X \subset \mathcal{E}$. The virtual fundamental class [1,30] is then defined by applying the Gysin map [25] to the intrinsic normal cone

$$[X]^{\mathrm{vir}} = 0^!_{\mathcal{E}}[\mathfrak{C}_X].$$

When there is a torus action on X with respect to which the perfect obstruction theory is equivariant, then under suitable assumptions the virtual fundamental class is localized to the fixed locus $F = X^{\mathbb{C}^*}$ [15, (1)]:

$$[X]^{\mathrm{vir}} = \imath_* \frac{[F]^{\mathrm{vir}}}{e(N^{\mathrm{vir}})} \in A^{\mathbb{C}^*}_* X \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, t^{-1}]. \tag{1.1}$$

Here $\imath : F \to X$ is the inclusion and t is the generator of the equivariant ring of $\mathbb{C}^*$.

The construction of the virtual fundamental class can be relativized for morphisms $f : X \to Y$ to give the virtual pullback

$$f^! : A_*(Y) \to A_*(X)$$

when the intrinsic normal cone $\mathfrak{C}_{X/Y}$ is embedded into a vector bundle stack $\mathcal{E}$ on X. When the perfect obstruction theories of X and Y are compatible with $\mathcal{E}$, the virtual pullback gives us the formula [31, Corollary 4.9]

$$f^![Y]^{\mathrm{vir}} = [X]^{\mathrm{vir}}. \tag{1.2}$$

A wall crossing formula [22, Theorem 1.1] compares $[M_+]^{\mathrm{vir}}$ with $[M_-]^{\mathrm{vir}}$ when M_+ and M_- are suitable open DM substacks of the quotient $[X/\mathbb{C}^*]$ of a DM stack X.

Let $\mathcal{E}$ be the vector bundle stack associated to the perfect obstruction theory of X as before. The cosection localization says that when there is an open $U \subset X$ and a surjective $\sigma : \mathcal{E}|_U \to \mathcal{O}_U$, we can define a cosection localized virtual fundamental class

$$[X]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*(X(\sigma)) \quad \text{where } X(\sigma) = X - U$$

which satisfies the expected properties of virtual cycles such as deformation invariance and

$$\imath_*[X]^{\mathrm{vir}}_{\mathrm{loc}} = [X]^{\mathrm{vir}} \in A_*(X) \quad \text{where } \imath : X(\sigma) \hookrightarrow X.$$

The construction of $[X]^{\mathrm{vir}}_{\mathrm{loc}}$ in [21] is obtained in two steps:

- (cone reduction) the intrinsic normal cone $\mathfrak{C}_X$ has support contained in $\mathcal{E}(\sigma)$ where $\mathcal{E}(\sigma) = \mathcal{E}|_{X(\sigma)} \cup \ker(\mathcal{E}|_U \to \mathcal{O}_U)$[1];
- (localized Gysin map) there is a cosection localized Gysin map

$$0^!_{\mathcal{E},\mathrm{loc}} : A_*(\mathcal{E}(\sigma)) \longrightarrow A_*(X(\sigma))$$

  compatible with the usual Gysin map.

Then the cosection localized virtual fundamental class is defined as

$$[X]^{\mathrm{vir}}_{\mathrm{loc}} = 0^!_{\mathcal{E},\mathrm{loc}}[\mathfrak{C}_X].$$

The cosection localized virtual fundamental class turned out to be quite useful [2,3,5,6,10,13,14,16,18–20,24,29,32,34,35]. For further applications, it is important to have cosection localized analogues for torus localization formula, virtual pullback and wall crossing formulas. For instance, recently there arose a tremendous interest in the Landau–Ginzburg theory whose key invariants such as the Fan–Jarvis–Ruan–Witten invariants [11] are defined algebro-geometrically by cosection localized virtual cycles [6,9]. The MSP fields developed in [7,8] will provide a new effective theory to evaluate all genus Gromov–Witten invariants of the quintic Calabi–Yau threefolds, which on one hand is a field theory taking values in the master space bridging the two geometric invariant theoretic quotients of $[\mathbb{C}^6/\mathbb{C}^*]$, and on the other hand relies on the torus localization of cosection localized virtual cycles, which is proved in this paper.

In §2, we prove the cosection localized virtual pullback formulas (cf. Theorems 2.6 and 2.10). The proofs in [31] work with necessary modifications as long as the rational equivalences used in the proofs lie in the suitable substacks for localized Gysin maps.

[1] When $\sigma : \mathcal{O}b_X \dashrightarrow \mathcal{O}_X$ is meromorphic and U is not explicitly stated, we understand U to be the maximal open subset where σ is regular and surjective over U. In this paper, we adopt the convention $\ker[\sigma] = \mathcal{E}|_{X(\sigma)} \cup \ker(\mathcal{E}|_U \to \mathcal{O}_U)$.

In §3, we prove the torus localization formula for cosection localized virtual fundamental classes (cf. Theorem 3.5). In this new proof, we do not require (1) the existence of an equivariant global embedding of X into a smooth DM stack nor (2) the existence of a global resolution of the perfect obstruction theory as in [15]. The technical condition (1) is completely gone while (2) is significantly weakened to (2′) the existence of a global resolution of the virtual normal bundle N^{vir} only on the fixed locus F. Finally, in §4, we prove a wall crossing formula for cosection localized virtual fundamental classes. We remark that in [20], a degeneration formula for the cosection localized virtual fundamental classes was proved and it was effectively used to prove Maulik–Pandharipande's formulas for Gromov–Witten invariants of spin surfaces.

All schemes or DM stacks in this paper are defined over the complex number field $\mathbb{C}$. All Chow groups in this paper have coefficients in the rational number field $\mathbb{Q}$.

2. Virtual pullback for cosection localized virtual cycles

In this section, we show that Manolache's virtual pullback formula [31] holds for cosection localized virtual cycles (cf. Theorems 2.6 and 2.10).

2.1. Virtual pullback of cosection localized virtual cycle

Let $f : X \to Y$ be a morphism of DM stacks. Let $\phi_X : E_X \to \mathbb{L}_X$ and $\phi_Y : E_Y \to \mathbb{L}_Y$ be (relative) perfect obstruction theories that fit into a commutative diagram of distinguished triangles

$$\begin{tikzcd} f^*E_Y \arrow[r, "\varphi"] \arrow[d, "f^*\phi_Y"'] & E_X \arrow[r] \arrow[d, "\phi_X"'] & E_{X/Y} \arrow[r] \arrow[d, "\phi_{X/Y}"] & {} \\ f^*\mathbb{L}_Y \arrow[r] & \mathbb{L}_X \arrow[r] & \mathbb{L}_{X/Y} \arrow[r] & {} . \end{tikzcd} \tag{2.1}$$

Definition 2.1. We say $f : X \to Y$ is *virtually smooth* if $E_{X/Y}$ is of perfect amplitude contained in $[-1, 0]$.

By [31, §3.2], if f is virtually smooth, then $\phi_{X/Y}$ is a perfect obstruction theory. By [1], the perfect obstruction theory $\phi_{X/Y} : E_{X/Y} \to \mathbb{L}_{X/Y}$ gives us an embedding of the intrinsic normal sheaf into the vector bundle stack

$$h^1/h^0(\mathbb{L}_{X/Y}^\vee) \hookrightarrow h^1/h^0(E_{X/Y}^\vee) =: \mathcal{E}_{X/Y}.$$

Moreover the intrinsic normal cone of the morphism f is naturally embedded into the intrinsic normal sheaf

$$\mathfrak{C}_{X/Y} \hookrightarrow \mathcal{E}_{X/Y}.$$

Let $\mathcal{O}b_X = h^1(E_X^\vee)$ and $\mathcal{O}b_Y = h^1(E_Y^\vee)$ be the obstruction sheaves and let $\sigma_Y : \mathcal{O}b_Y \dashrightarrow \mathcal{O}_Y$ be a meromorphic cosection, i.e. a surjective homomorphism $\sigma_Y : \mathcal{O}b_U \twoheadrightarrow \mathcal{O}_U$ over an open $U \subset Y$. The morphism $f^*E_Y \to E_X$ induces a homomorphism $\mathcal{O}b_X = h^1(E_X^\vee) \to h^1(f^*E_Y^\vee) = f^*h^1(E_Y^\vee) = f^*\mathcal{O}b_Y$. Hence σ_Y induces a meromorphic cosection

$$\sigma_X : \mathcal{O}b_X \longrightarrow f^*\mathcal{O}b_Y \dashrightarrow f^*\mathcal{O}_Y = \mathcal{O}_X \tag{2.2}$$

of $\mathcal{O}b_X$. We call σ_X the cosection induced from σ_Y.

Lemma 2.2. *Let $X_U = X \times_Y U$. In case f is virtually smooth, then $\sigma_X : \mathcal{O}b_{X_U} \to \mathcal{O}_{X_U}$ is surjective.*

Proof. From the distinguished triangle $E_{X/Y}^\vee \to E_X^\vee \to f^*E_Y^\vee \xrightarrow{+1}$, we obtain an exact sequence

$$\cdots \longrightarrow \mathcal{O}b_X \longrightarrow f^*\mathcal{O}b_Y \longrightarrow h^2(E_{X/Y}^\vee) = 0$$

where the identity holds because $E_{X/Y}$ is perfect of amplitude $[-1, 0]$. Since $\mathcal{O}b_X \to f^*\mathcal{O}b_Y$ is surjective, (2.2) is surjective because $f^*\sigma_Y$ is also surjective over X_U by assumption. □

Definition 2.3. Let $Y(\sigma) = Y - U$, a closed substack of Y, and let $X(\sigma) = Y(\sigma) \times_Y X$. They are the loci where the cosections may not be surjective.

By the definition, we have a Cartesian square

$$\begin{tikzcd} X(\sigma) \arrow[r, "f_\sigma"] \arrow[d, "\imath'"'] & Y(\sigma) \arrow[d, "\imath"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{2.3}$$

where the vertical arrows are inclusion maps.

We recall Manolache's virtual pullback.

Definition 2.4. [31, Construction 3.6] Suppose we have an embedding of the intrinsic normal cone $\mathfrak{C}_{X/Y}$ of $f : X \to Y$ into a vector bundle stack $\mathcal{E}_{X/Y}$. Consider a fiber product diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "p"'] & Y' \arrow[d, "q"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Then the *virtual pullback* with respect to the inclusion $\mathfrak{C}_{X/Y} \subset \mathcal{E}_{X/Y}$ is defined as the composite

$$f^! : A_*(Y') \to A_*(\mathfrak{C}_{X'/Y'}) \to A_*(\mathfrak{C}_{X/Y} \times_X X') \to A_*(p^*\mathcal{E}_{X/Y}) \to A_{*+d}(X'),$$

where the four arrows are as follows: the first is $[B] \mapsto [\mathfrak{C}_{B\times_{Y'}X'/B}]$; the second is via the inclusion $\mathfrak{C}_{X'/Y'} \to \mathfrak{C}_{X/Y} \times_X X'$; the third is from the inclusion $\mathfrak{C}_{X/Y} \subset \mathcal{E}_{X/Y}$; the last is the Gysin map $0^!_{\mathcal{E}_{X/Y}} : A_*(p^*\mathcal{E}_{X/Y}) \to A_{*+d}(X')$ for $p^*\mathcal{E}_{X/Y}$. Here $d = -\operatorname{rank}\mathcal{E}_{X/Y}$, the rank of $E_{X/Y}$.

If X is not connected, we consider each connected component separately.

Letting $X' = X$ and $Y' = Y$, we get $f^! : A_*(Y) \to A_{*+d}(X)$. Letting $X' = X(\sigma)$ and $Y' = Y(\sigma)$, we obtain $f^!_\sigma : A_*(Y(\sigma)) \to A_*(X(\sigma))$. By [31, Theorem 2 (i)], these fit into a commutative diagram

$$\begin{tikzcd} A_*(Y(\sigma)) \arrow[r, "f^!_\sigma"] \arrow[d, "\imath_*"'] & A_{*+d}(X(\sigma)) \arrow[d, "\imath'_*"] \\ A_*(Y) \arrow[r, "f^!"'] & A_{*+d}(X) \end{tikzcd} \tag{2.4}$$

We need the following analogue of [31, Lemma 4.7].

Lemma 2.5. *Let $f : X \to Y$ be a morphism of DM stacks and $\mathcal{N}$ be a vector bundle stack on X equipped with an inclusion $\mathfrak{C}_{X/Y} \subset \mathcal{N}$ that gives us the virtual pullback $f^!_\sigma$. Let $\mathcal{E}$ be a vector bundle stack on Y with the zero section $0_\mathcal{E} : Y \to \mathcal{E}$. Let $U \subset Y$ be open and $\sigma : \mathcal{E}|_U \to \mathcal{O}_U$ be a surjective map of vector bundle stacks. Let $Y(\sigma)$ and $X(\sigma)$ be as before, and let $f_\sigma : X(\sigma) \to Y(\sigma)$ be the induced morphism. Let $\mathcal{E}(\sigma) = \mathcal{E}|_{Y(\sigma)} \cup \ker(\sigma)$. Then $\mathfrak{C}_{X/\mathcal{E}} \subset f^*\mathcal{E} \oplus \mathcal{N}$, where $\mathfrak{C}_{X/\mathcal{E}}$ denotes the normal cone of the morphism $0_\mathcal{E} \circ f : X \to Y \to \mathcal{E}$. Moreover for each cycle $B \subset \mathcal{E}(\sigma)$,*

$$f^!_\sigma 0^!_{\mathcal{E},\mathrm{loc}}[B] = 0^!_{f^*\mathcal{E}\oplus\mathcal{N},\mathrm{loc}}[\mathfrak{C}_{X\times_\mathcal{E} B/B}] \quad \text{in } A_*(X(\sigma)), \tag{2.5}$$

where $0^!_{\mathcal{E},\mathrm{loc}}$ and $0^!_{f^*\mathcal{E}\oplus\mathcal{N},\mathrm{loc}}$ denote the localized Gysin maps with respect to the cosections σ and $(f^*\sigma, 0) : f^*\mathcal{E} \oplus \mathcal{N}|_{f^{-1}(U)} \to \mathcal{O}_{f^{-1}(U)}$ respectively. Here $f^!_\sigma$ is the virtual pullback with respect to the inclusion $\mathfrak{C}_{X/Y} \subset \mathcal{N}$.

Proof. The inclusion $\mathfrak{C}_{X/\mathcal{E}} \subset f^*\mathcal{E} \oplus \mathcal{N} = f^*\mathcal{E} \times_X \mathcal{N}$ is constructed using the identity $\mathfrak{C}_{X/\mathcal{E}} = f^*\mathcal{E} \times_X \mathfrak{C}_{X/Y}$ proved in [31, Example 2.37]. We prove the last statement. Since both sides of (2.5) are additive, we may assume B is integral.

If $B \subset \mathcal{E}|_{Y(\sigma)}$, the localized Gysin maps are the ordinary Gysin maps and hence the lemma follows from [31, Lemma 4.7]. So we may suppose $B \not\subseteq \mathcal{E}|_{Y(\sigma)}$. Further, by shrinking Y if necessary, we can assume that Y is integral and $B \to Y$ is dominant.

By [21, §2], we can choose a projective variety Z, a generic finite and proper morphism $\rho : Z \to Y$, a Cartier divisor D on Z that fits into the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\rho"] & Y \\ D \arrow[u, hook] \arrow[r, "\rho_\sigma"] & Y(\sigma) \arrow[u, hook] \end{tikzcd}$$

and $\rho^*\sigma$ extends to a surjective map $\tilde{\sigma} : \rho^*\mathcal{E} \to \mathcal{O}_Z(D)$, where by abuse of notation we think of $\mathcal{O}_Z(D)$ as the total space of the line bundle $\mathcal{O}_Z(D)$.

Let $\tilde{\mathcal{E}} = \ker(\tilde{\sigma})$, which is a bundle stack over Z, and choose an integral $\tilde{B} \subset \tilde{\mathcal{E}}$ such that $\tilde{\rho}_*[\tilde{B}] = k[B]$ for some $k > 0$, where $\tilde{\rho} : \tilde{\mathcal{E}} \to \mathcal{E}(\sigma) \subset \mathcal{E}$ is the composition $\tilde{\mathcal{E}} \subset \rho^*\mathcal{E} \to \mathcal{E}$. Note that the properness of ρ implies that of $\rho^*\mathcal{E} \to \mathcal{E}$ and thus $\tilde{\rho}$ is proper. Then by the definition in [21, §2],

$$0^!_{\mathcal{E},\mathrm{loc}}[B] = \frac{1}{k}\rho_{\sigma*}(0^!_{\tilde{\mathcal{E}}}[\tilde{B}] \cdot D),$$

where $\cdot D$ denotes the refined intersection for the inclusion $D \subset Z$ defined in [12, Chapter 6].

We further simplify the situation as follows. Because $\tilde{\mathcal{E}}$ is a vector bundle stack over the DM stack Z, the flat pullback $A_*(Z) \to A_*(\tilde{\mathcal{E}})$ is an isomorphism by [25, Proposition 5.3.2]. Therefore there exist integral $Z_i \subset Z$ and rational r_i so that $[\tilde{B}] = \sum r_i[\tilde{\mathcal{E}}|_{Z_i}] \in A_*\tilde{\mathcal{E}}$. Because rational equivalence in $\tilde{\mathcal{E}}$ induces a rational equivalence in $\mathcal{E}(\sigma)$, to prove the theorem, we only need to consider the case where $\tilde{B} = \tilde{\mathcal{E}}$. Thus the above identity becomes $0^!_{\mathcal{E},\mathrm{loc}}[B] = \frac{1}{k}\rho_{\sigma*}[D]$.

Consider the Cartesian diagrams

$$\begin{tikzcd} D \arrow[r, "\rho_\sigma"] & Y(\sigma) \arrow[r, "\subset"] & Y \\ D' \arrow[u, "f'_\sigma"] \arrow[r, "\rho'_\sigma"'] & X(\sigma) \arrow[u, "f_\sigma"'] \arrow[r, "\subset"'] & X \arrow[u, "f"'] \end{tikzcd}$$

where $D' = D \times_{Y(\sigma)} X(\sigma)$. Since virtual pullbacks commute with pushforwards (cf. [31, Theorem 4.1 (i)]), we have

$$f^!_\sigma 0^!_{\mathcal{E},\mathrm{loc}}[B] = \frac{1}{k} f^!_\sigma \rho_{\sigma*}[D] = \frac{1}{k}\rho'_{\sigma*} {f'_\sigma}^![D] = \frac{1}{k}\rho'_{\sigma*} 0^!_{\mathcal{N}|_{D'}}[\mathfrak{C}_{D'/D}]. \tag{2.6}$$

Therefore the lemma will follow if we prove (cf. the right side of (2.5))

$$0^!_{f^*\mathcal{E}\oplus\mathcal{N},\mathrm{loc}}[\mathfrak{C}_{X\times_\mathcal{E} B/B}] = \frac{1}{k}\rho'_{\sigma*} 0^!_{\mathcal{N}|_{D'}}[\mathfrak{C}_{D'/D}]. \tag{2.7}$$

The rest of the proof is devoted to a proof of (2.7).

Consider the commutative diagram

$$\begin{tikzcd} Z' \arrow[r, "f'"] \arrow[d, "\rho'"] & Z \arrow[r, "0"] \arrow[d, "\rho"] & \tilde{\mathcal{E}} \arrow[r, equal] \arrow[d, "\tilde{\rho}"] & \tilde{B} \\ X \arrow[r, "f"] & Y \arrow[r, "0"] & \mathcal{E}(\sigma) & B \arrow[l] \end{tikzcd}$$

where $Z' := X \times_Y Z$. Let

$$\tilde{\rho}^f : f'^*\tilde{\mathcal{E}} \to f^*\mathcal{E}(\sigma)$$

denote the pullback of $\tilde{\rho}$. We claim that

$$\tilde{\rho}^f_*[\mathfrak{C}_{X\times_{\mathcal{E}}\tilde{\mathcal{E}}/\tilde{\mathcal{E}}}] = k[\mathfrak{C}_{X\times_{\mathcal{E}}B/B}]. \tag{2.8}$$

Indeed, by construction, we have $\tilde{\rho}_*[\tilde{\mathcal{E}}] = k[B]$, and $X \times_{\mathcal{E}} \tilde{\mathcal{E}} = Z'$, $X \times_{\mathcal{E}} B = Z$ and $Z' \cong Z \times_B \tilde{B}$. As B and $\tilde{B}$ are integral, if we let $M^0_{\cdot/\cdot}$ denote the deformation to the normal cone [12,25], the (functorially associated) morphism $M^0_{Z'/\tilde{B}} \to M^0_{Z/B}$ is a degree k map. Applying [12, Theorem 6.2(a)] to the fiber diagram

$$\begin{tikzcd} \mathfrak{C}_{Z'/\tilde{B}} \arrow[r, "\lambda"] \arrow[d] & \mathfrak{C}_{Z/B} \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ M^0_{Z'/\tilde{B}} \arrow[r] & M^0_{Z/B} \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

we obtain $\lambda_*[\mathfrak{C}_{Z'/\tilde{B}}] = k[\mathfrak{C}_{Z/B}]$. Thus (2.8) follows.

By the definition of the localized Gysin map [21, Definition 3.2 and (2.1)], we have

$$0^!_{f^*\mathcal{E}\oplus\mathcal{N},\mathrm{loc}}[\mathfrak{C}_{X\times_{\mathcal{E}}B/B}] = \frac{1}{k}\rho'_{\sigma*}\left(0^!_{f'^*\tilde{\mathcal{E}}\oplus\rho'^*\mathcal{N}}[\mathfrak{C}_{X\times_{\mathcal{E}}\tilde{\mathcal{E}}/\tilde{\mathcal{E}}}]\cdot D'\right).$$

Here by $\cdot D'$ we mean the intersection with $D \subset Z$ via the Cartesian square

$$\begin{tikzcd} D' \arrow[r] \arrow[d] & Z' \arrow[d, "f'"] \\ D \arrow[r] & Z. \end{tikzcd}$$

Since $X \times_{\mathcal{E}} \tilde{\mathcal{E}} = Z'$, by [31, Example 2.37]

$$\mathfrak{C}_{X\times_{\mathcal{E}}\tilde{\mathcal{E}}/\tilde{\mathcal{E}}} = \mathfrak{C}_{Z'/\tilde{\mathcal{E}}} = f'^*\tilde{\mathcal{E}} \times_{Z'} \mathfrak{C}_{Z'/Z}.$$

By the commutativity of Gysin maps, we have

$$0^!_{f^*\mathcal{E}\oplus\mathcal{N},\mathrm{loc}}[\mathfrak{C}_{X\times_{\mathcal{E}}B/B}] = \frac{1}{k}\rho'_{\sigma*}\big(0^!_{\mathcal{N}}[\mathfrak{C}_{Z'/Z}]\cdot D'\big) = \frac{1}{k}\rho'_{\sigma*}0^!_{\mathcal{N}|_{D'}}\left([\mathfrak{C}_{Z'/Z}]\cdot D'\right).$$

Let $L = \mathcal{O}_Z(D)$ and let $L' = f'^*L$. We now prove

$$[\mathfrak{C}_{D'/D}] = [\mathfrak{C}_{Z'/Z}]\cdot D'.$$

Indeed, by Vistoli's rational equivalence [36, Lemma 3.16] (also see [26, Proposition 4]) and $\mathfrak{C}_{D/Z} \cong L|_D$, we have

$$\begin{aligned}[\mathfrak{C}_{Z'/Z}]\cdot D' &= 0^!_{L'}[\mathfrak{C}_{\mathfrak{C}_{Z'/Z}\times_{Z'}D'/\mathfrak{C}_{Z'/Z}}] = 0^!_{L'}[\mathfrak{C}_{\mathfrak{C}_{D/Z}\times_D D'/\mathfrak{C}_{D/Z}}]\\ &= 0^!_{L'}[\mathfrak{C}_{D'/D}\times_{D'} L'|_{D'}] = [\mathfrak{C}_{D'/D}].\end{aligned}$$

Therefore, we have

$$0^!_{f^*\mathcal{E}\oplus\mathcal{N},\mathrm{loc}}[\mathfrak{C}_{X\times_{\mathcal{E}}B/B}] = \frac{1}{k}\rho'_{\sigma*}0^!_{\mathcal{N}|_{D'}}[\mathfrak{C}_{D'/D}].$$

This proves the desired equality $f^!_\sigma 0^!_{\mathcal{E},\mathrm{loc}}[B] = 0^!_{f^*\mathcal{E}\oplus\mathcal{N},\mathrm{loc}}[\mathfrak{C}_{X\times_{\mathcal{E}}B/B}]$. □

The following is a cosection localized analogue of [31, Corollary 4.9].

Theorem 2.6. *Let $f : X \to Y$ be a virtually smooth morphism of DM stacks, and let $\sigma : \mathcal{O}b_Y \to \mathcal{O}_Y$ be a cosection. Then*

$$f^!_\sigma[Y]^{\mathrm{vir}}_{\mathrm{loc}} = [X]^{\mathrm{vir}}_{\mathrm{loc}}.$$

Proof. The proofs of Theorem 4.8 (functoriality) and Corollary 4.9 in [31] work with necessary modifications. The reader is invited to go through the proofs in [31] with the proof of deformation invariance [21, Theorem 5.2] for cosection localized virtual cycles in mind. With Lemma 2.5 at hand, one will find that the theorem follows from Lemma 2.7 below. □

Lemma 2.7. *Let M^0_Y be the deformation space from the reduced point $\{pt\}$ to the intrinsic normal cone $\mathfrak{C}_Y$ and $c(u)$ be the cone of the morphism*

$$u = (x_0\cdot\mathrm{id}, x_1\cdot\varphi) : p^*f^*E_Y \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow p^*f^*E_Y \oplus p^*E_X \quad \text{in} \quad D(X\times\mathbb{P}^1)$$

where x_0, x_1 are the homogeneous coordinates of $\mathbb{P}^1$; p, q are projections from $X\times\mathbb{P}^1$ to X and $\mathbb{P}^1$ respectively; $\varphi : f^*E_Y \to E_X$ is the morphism in (2.1). Then we have the inclusion of the support

$$\operatorname{Supp}\mathfrak{C}_{X\times\mathbb{P}^1/M^0_Y} \subset \ker\left[h^1/h^0(c(u)^\vee) \longrightarrow q^*\mathcal{O}_{\mathbb{P}^1}(-1)\right]. \tag{2.9}$$

Proof. Let $\mathcal{E}_X = h^1/h^0(E_X^\vee)$ and $\mathcal{E}_Y = h^1/h^0(E_Y^\vee)$. The morphism

$$h^1/h^0(c(u)^\vee) \longrightarrow q^*\mathcal{O}_{\mathbb{P}^1}(-1)$$

comes from the commutative diagram

$$\begin{tikzcd} h^1/h^0(c(u)^\vee) \arrow[r] \arrow[d] & p^*f^*\mathcal{E}_Y \oplus p^*\mathcal{E}_X \arrow[r] \arrow[d] & p^*f^*\mathcal{E}_Y \otimes q^*\mathcal{O}_{\mathbb{P}^1}(1) \arrow[d] \\ q^*\mathcal{O}_{\mathbb{P}^1}(-1) \arrow[r] & q^*\mathcal{O}_{\mathbb{P}^1} \oplus q^*\mathcal{O}_{\mathbb{P}^1} \arrow[r] & q^*\mathcal{O}_{\mathbb{P}^1}(1) \end{tikzcd}$$

where the middle and right vertical arrows come from the cosection σ.

By the double deformation space construction (cf. [23, (7)]),

$$\mathfrak{C}_{X\times\mathbb{P}^1/M_Y^0} \times_{\mathbb{P}^1} (\mathbb{P}^1 - \{(0:1)\}) = \mathfrak{C}_X \times (\mathbb{P}^1 - \{(0:1)\}).$$

By the cone reduction (cf. [21, Proposition 4.3 and Corollary 4.5]), we have the inclusion of the support

$$\operatorname{Supp}\mathfrak{C}_X \subset \ker\left[h^1/h^0(E_X^\vee) \to \mathcal{O}_X\right].$$

Hence (2.9) holds over $\mathbb{P}^1 - \{(0:1)\}$.

Let $D = \mathfrak{C}_{X\times\mathbb{P}^1/M_Y^0} \times_{\mathbb{P}^1} (\mathbb{P}^1 - \{(1:0)\})$ be open in $\mathfrak{C}_{X\times\mathbb{P}^1/M_Y^0}$ containing the fiber over $(0:1)$. By a diagram chase,

$$h^1/h^0(c(u)^\vee)|_{\{(0:1)\}} \cong f^*h^1/h^0(E_Y^\vee) \times_X h^1/h^0(E_{X/Y}^\vee)$$

and the homomorphism in (2.9) over the point $(0:1)$ is

$$(f^*\sigma, 0) : f^*h^1/h^0(E_Y^\vee) \times_X h^1/h^0(E_{X/Y}^\vee) \longrightarrow \mathcal{O}_X.$$

Therefore the lemma follows if we show that irreducible components A of D lying over $X \times \{(0:1)\}$ have support contained in

$$\ker\left(f^*h^1/h^0(E_Y^\vee) \xrightarrow{f^*\sigma} \mathcal{O}_X\right) \times_X h^1/h^0(E_{X/Y}^\vee). \tag{2.10}$$

Let A be an irreducible component of D lying over $X \times \{(0:1)\}$ and let a be a general closed point in A. We claim that a is contained in (2.10). Since the problem is local, using the argument as that in [21, (5.10)–(5.11)] and [21, p. 19], we may assume X, Y are affine, equipped with embeddings $X \subset V$, $Y \subset W$ into smooth affine varieties that fit into a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"'] & V \arrow[r, "(\gamma\circ g,\eta)"] \arrow[d, "g"] & \mathbb{A}^m \times \mathbb{A}^n \arrow[d, "pr_1"] \\ Y \arrow[r, hook] & W \arrow[r, "\gamma"] & \mathbb{A}^m \end{tikzcd}$$

such that the morphism $g : V \to W$ is smooth, $X = \mathrm{zero}(\gamma \circ g, \eta)$, $Y = \mathrm{zero}(\gamma)$ and

$$E_Y = [\mathcal{O}_Y^{\oplus m} \xrightarrow{d\gamma} \Omega_W|_Y], \quad E_{X/Y} = [\mathcal{O}_X^{\oplus n} \xrightarrow{d\eta} \Omega_{V/W}|_X].$$

(Cf. [23, (11)].) Since we have nothing to prove when $\sigma = 0$ at general points of the irreducible component A, we may assume that σ is surjective.

To prove the claim, we recall the double deformation space construction for D (cf. [23]): Let Γ be the graph of the morphism

$$V \times (\mathbb{A}^1_t - \{0\}) \times (\mathbb{A}^1_s - \{0\}) \longrightarrow \mathbb{A}^m \times \mathbb{A}^n, \quad (v, t, s) \mapsto ((ts)^{-1}\gamma \circ g(v), t^{-1}\eta(v))$$

and let $\bar{\Gamma}$ be the closure of Γ in $V \times \mathbb{A}^1_t \times \mathbb{A}^1_s \times \mathbb{A}^m \times \mathbb{A}^n$. Here $\mathbb{A}^1_t$ and $\mathbb{A}^1_s$ denote the affine line with local coordinates t and s respectively. Then we have

$$\bar{\Gamma} \times_{\mathbb{A}^1_t} \{0\}/(pr_V^* T_V|_{\bar{\Gamma} \times_{\mathbb{A}^1_t} \{0\}}) = D.$$

Now we can prove the claim. Since $D \subset (\bar{\Gamma} - \Gamma)/(pr_V^* T_V|_{\bar{\Gamma}-\Gamma})$, we may choose a smooth pointed curve $(\Delta, 0)$ with local coordinate δ and a morphism $\rho : \Delta \to \bar{\Gamma}$ such that $\rho(\Delta - \{0\}) \subset \Gamma$ and $\rho(0)$ represents $a \in A$. Let $t_\Delta : \Delta \xrightarrow{\rho} \bar{\Gamma} \xrightarrow{pr_t} \mathbb{A}^1_t$ and $s_\Delta : \Delta \xrightarrow{\rho} \bar{\Gamma} \xrightarrow{pr_s} \mathbb{A}^1_s$ denote the compositions of ρ and the projections to $\mathbb{A}^1_t$ and $\mathbb{A}^1_s$ respectively. Let $\rho_V : \Delta \to V$ denote the composition of ρ with the projection $pr_V : \bar{\Gamma} \to V$ and let $v_0 = \rho_V(0) \in X$. Then $\rho(0) = (v_0, 0, 0, v_1, v_2)$,

$$v_1 = \lim_{\delta \to 0} (t_\Delta s_\Delta)^{-1} \cdot \gamma \circ g \circ \rho_V \in \mathbb{A}^m, \quad v_2 = \lim_{\delta \to 0} t_\Delta^{-1} \cdot \eta \circ \rho_V \in \mathbb{A}^n. \tag{2.11}$$

Since $\mathcal{O}_Y^{\oplus m} \twoheadrightarrow \mathcal{O}b_Y \xrightarrow{\sigma} \mathcal{O}_Y$ is surjective, by copying the proofs of Lemma 4.4 and Corollary 4.5 in [21], we find that v_1 represents a point in

$$\ker\left(f^* h^1/h^0(E_Y^\vee) \xrightarrow{f^*\sigma} \mathcal{O}_X\right)$$

and v_2 a point in $h^1/h^0(E_{X/Y}^\vee)$. Therefore a lies in (2.10). This proves the lemma. □

2.2. Cosection localized virtual pullback

In §2.1, we considered the virtual pullback of a cosection localized virtual fundamental class when there is a cosection $\sigma : \mathcal{O}b_Y \dashrightarrow \mathcal{O}_Y$ on Y that induces a cosection $\mathcal{O}b_X \to f^*\mathcal{O}b_Y \xrightarrow{\sigma} \mathcal{O}_X$ on X. Actually there is another way to combine virtual pullback with cosection localization. Consider the case when there is a cosection $\sigma : \mathcal{O}b_X \dashrightarrow \mathcal{O}_X$ that induces a cosection $\tilde\sigma : \mathcal{O}b_{X/Y} \to \mathcal{O}b_X \dashrightarrow \mathcal{O}_X$ of the relative obstruction sheaf. In this subsection, we define the cosection localized virtual pullback (cf. Definition 2.9)

$$f^!_\sigma : A_*(Y) \longrightarrow A_*(X(\tilde\sigma))$$

for a virtually smooth morphism $f : X \to Y$ where $X(\tilde\sigma)$ is the locus where $\tilde\sigma$ is not surjective, and prove the cosection localized virtual pullback formula (cf. Theorem 2.10).

We let $f : X \to Y$ be a virtually smooth morphism between DM stacks as before; we let $\sigma = \sigma_X : \mathcal{O}b_X \dashrightarrow \mathcal{O}_X$ be a cosection, and form the (composite)

$$\tilde\sigma = \sigma_{X/Y} : \mathcal{O}b_{X/Y} = h^1(E^\vee_{X/Y}) \longrightarrow h^1(E^\vee_X) = \mathcal{O}b_X \dashrightarrow \mathcal{O}_X.$$

Let $X(\sigma)$ and $X(\tilde\sigma)$ denote the loci where the cosections are not surjective, i.e. either zero or undefined. Then by definition, we have an inclusion

$$\jmath : X(\sigma) \hookrightarrow X(\tilde\sigma).$$

We let $\mathcal{K} = h^1/h^0(E^\vee_{X/Y})$, and let $\mathcal{E}_X = h^1/h^0(E^\vee_X)$, etc. Let $\tilde U = X - X(\tilde\sigma)$ and $U = X - X(\sigma)$ denote the maximal open subsets where $\tilde\sigma$ and σ respectively are surjective. Then $\tilde\sigma$ (resp. σ) induces a surjective morphism $\tilde\sigma_{\tilde U} : \mathcal{K}|_{\tilde U} \to \mathcal{O}_{\tilde U}$ (resp. $\sigma_U : \mathcal{E}|_U \to \mathcal{O}_U$). As before, we denote $\mathcal{E}_X(\sigma) = \ker[\sigma]$, and

$$\mathcal{K}(\tilde\sigma) := \mathcal{K}|_{X(\tilde\sigma)} \cup \ker[\tilde\sigma_{\tilde U} : \mathcal{K}|_{\tilde U} \to \mathcal{O}_{\tilde U}].$$

Lemma 2.8. *We have*

$$\operatorname{Supp} \mathfrak{C}_{X/Y} \subset \mathcal{K}(\tilde\sigma).$$

Proof. We apply the functoriality of the h^1/h^0 construction to (2.1) to obtain the commutative diagram

$$\begin{tikzcd} \mathfrak{C}_{\tilde U/Y} \arrow[r, "\subset"] \arrow[d] & h^1/h^0(\mathbb{L}^\vee_{X/Y})|_{\tilde U} \arrow[r, "\subset"] \arrow[d] & h^1/h^0(E^\vee_{X/Y})|_{\tilde U} = \mathcal{K}|_{\tilde U} \arrow[r, "\tilde\sigma"] \arrow[d] & \mathcal{O}_{\tilde U} \arrow[d, "="'] \\ \mathfrak{C}_{\tilde U} \arrow[r, "\subset"] & h^1/h^0(\mathbb{L}^\vee_X)|_{\tilde U} \arrow[r, "\subset"] & h^1/h^0(E^\vee_X)|_{\tilde U} = \mathcal{E}_X|_{\tilde U} \arrow[r, "\sigma"] & \mathcal{O}_{\tilde U} \end{tikzcd} \tag{2.12}$$

Like before, we have $\operatorname{Supp} \mathfrak{C}_X \subset \mathcal{E}_X(\sigma)$. Since $\tilde\sigma$ is induced from $\sigma = \sigma_X$, the lemma follows. □

We now define the cosection localized virtual pullback. We let $Y' \to Y$ be a morphism of stacks where Y' has stratification by global quotients. We form the Cartesian product

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "q"] & Y, \end{tikzcd}$$

and let $\hat{\sigma} : p^*\mathcal{K} \dashrightarrow \mathcal{O}_{X'}$ be the pullback of $\tilde{\sigma} : \mathcal{K} \dashrightarrow \mathcal{O}_X$. Then $\hat{\sigma}$ is surjective away from

$$X'(\tilde{\sigma}) := X(\tilde{\sigma}) \times_X X'$$

and

$$p^*\mathcal{K}(\hat{\sigma}) = \ker[\hat{\sigma} : p^*\mathcal{K} \dashrightarrow \mathcal{O}_{X'}] = \mathcal{K}(\tilde{\sigma}) \times_X X'.$$

Consider the composite

$$\iota : (\mathfrak{C}_{X'/Y'})_{\mathrm{red}} \subset (\mathfrak{C}_{X/Y} \times_X X')_{\mathrm{red}} \subset \mathcal{K}(\tilde{\sigma}) \times_X X' = p^*\mathcal{K}(\hat{\sigma}),$$

where the first inclusion follows from the definition of X', and the second inclusion follows from Lemma 2.8.

Definition 2.9. We define the *cosection localized virtual pullback* to be

$$f^!_\sigma : A_*Y' \xrightarrow{\epsilon} A_*\mathfrak{C}_{X'/Y'} \xrightarrow{\iota_*} A_*(p^*\mathcal{K}(\hat{\sigma})) \xrightarrow{0^!_{p^*\mathcal{K},\mathrm{loc}}} A_*(X'(\tilde{\sigma})),$$

where ϵ is defined on the level of cycles by $\epsilon(\sum n_i[V_i]) = \sum n_i[\mathfrak{C}_{V_i \times_{Y'} X'/V_i}]$.

Note that the way that [31, Theorem 2.31] was applied to [31, Construction 3.6] can also be applied here to conclude that ϵ descends to maps between Chow groups.

We have the following virtual pullback formula.

Theorem 2.10. *Let $f : X \to Y$ be a virtually smooth morphism, $\sigma : \mathcal{O}b_X \to \mathcal{O}_X$ be a cosection and $\tilde{\sigma} : \mathcal{O}b_{X/Y} \to \mathcal{O}b_X \xrightarrow{\sigma} \mathcal{O}_X$ be the induced cosection. Let $\jmath : X(\sigma) \to X(\tilde{\sigma})$ be the inclusion of zero loci of σ and $\tilde{\sigma}$. Then we have*

$$f^!_\sigma[Y]^{\mathrm{vir}} = \jmath_*[X]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*(X(\tilde{\sigma})).$$

The proof is completely parallel to that of Theorem 2.6, so we only provide a sketch. We need the following analogue of Lemma 2.5.

Lemma 2.11. *Let $f : X \to Y$ be a morphism of DM stacks and $\mathcal{K}$ be a vector bundle stack on X such that $\mathfrak{C}_{X/Y} \subset \mathcal{K}$. Let $\mathcal{F}$ be a vector bundle stack on Y with the zero section $0_\mathcal{F} : Y \to \mathcal{F}$. Let $U \subset X$ be open and $\tilde{\sigma} : \mathcal{K}|_U \to \mathcal{O}_U$ be a surjective map of vector bundle stacks. Let $X(\tilde{\sigma}) = X - U$. Then for each irreducible $B \subset \mathcal{F}$,*

$$f^!_\sigma 0^!_{\mathcal{F}}[B] = 0^!_{f^*\mathcal{F}\oplus\mathcal{K},\mathrm{loc}}[\mathfrak{C}_{X\times_{\mathcal{F}}B/B}] \quad \textit{in } A_*(X(\tilde\sigma))$$

where $0^!_{f^*\mathcal{F}\oplus\mathcal{K},\mathrm{loc}}$ *denotes the localized Gysin map with respect to the cosection* $(0,\tilde\sigma) : f^*\mathcal{F}\oplus\mathcal{K}|_{f^{-1}(U)} \to \mathcal{O}_{f^{-1}(U)}$.

Proof. By the pullback isomorphism $A_*(Y)\to A_*(\mathcal{F})$, we may assume that there is an irreducible $\bar B\subset Y$ such that $B=\mathcal{F}|_{\bar B}=\mathcal{F}\times_Y \bar B$ and $0^!_{\mathcal{F}}[B]=\bar B$. The left side is

$$f^!_\sigma 0^!_{\mathcal{F}}[B] = f^!_\sigma[\bar B] = 0^!_{\mathcal{K},\mathrm{loc}}[\mathfrak{C}_{\bar B\times_Y X/\bar B}] = 0^!_{f^*\mathcal{F}\oplus\mathcal{K},\mathrm{loc}}[f^*\mathcal{F}\times_X \mathfrak{C}_{\bar B\times_Y X/\bar B}].$$

Since $B=\mathcal{F}\times_Y\bar B$, $f^*\mathcal{F}\times_X\mathfrak{C}_{\bar B\times_Y X/\bar B}=\mathfrak{C}_{B\times_{\mathcal{F}}X/B}$. Hence the lemma follows. □

Theorem 2.10 follows from the following.

Proposition 2.12. *Suppose* $Y' = h^1/h^0(E_Y^\vee)=\mathcal{F}$ *so that* $p|_{X'(\tilde\sigma)} : X'(\tilde\sigma)\to X(\tilde\sigma)$ *is a bundle stack and that we have the Gysin map* $0^!_{\mathcal{F}} : A_*(X'(\tilde\sigma)) \longrightarrow A_*(X(\tilde\sigma))$*. Then we have*

$$0^!_{\mathcal{F}} f^!_\sigma[\mathfrak{C}_Y] = \jmath_*[X]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*(X(\tilde\sigma)).$$

Proof of Theorem 2.10. By Lemma 2.11 and Proposition 2.12,

$$f^!_\sigma[Y]^{\mathrm{vir}} = f^!_\sigma 0^!_{\mathcal{F}}[\mathfrak{C}_Y] = 0^!_{f^*\mathcal{F}\oplus\mathcal{K},\mathrm{loc}}[\mathfrak{C}_{X\times_{\mathcal{F}}\mathfrak{C}_Y/\mathfrak{C}_Y}] = 0^!_{\mathcal{F}} f^!_\sigma[\mathfrak{C}_Y] = \jmath_*[X]^{\mathrm{vir}}_{\mathrm{loc}}.$$

This proves the theorem. □

Proof of Proposition 2.12. The proof is almost identical to that of Theorem 2.6, so we only point out the difference. The construction of the double deformation space and the cone $c(u)$ is identical and we have a commutative diagram

$$\begin{tikzcd} h^1/h^0(c(u)^\vee) \arrow[r] \arrow[d] & p^*f^*\mathcal{E}_Y\oplus p^*\mathcal{E}_X \arrow[r] \arrow[d, "(0,\sigma)"] & p^*f^*\mathcal{E}_Y\otimes q^*\mathcal{O}_{\mathbb{P}^1}(1) \\ q^*\mathcal{O}_{\mathbb{P}^1} \arrow[r, "="] & q^*\mathcal{O}_{\mathbb{P}^1} & \end{tikzcd}$$

where the vertical arrows are defined by σ. Again it suffices to show

$$\operatorname{Supp}\mathfrak{C}_{X\times\mathbb{P}^1/M_Y^\circ} \subset \ker[h^1/h^0(c(u)^\vee) \dashrightarrow q^*\mathcal{O}_{\mathbb{P}^1}]. \tag{2.13}$$

Now the proof continues exactly the same as the proof of Theorem 2.6. By the cone reduction in [21, §2], (2.13) holds over the open $\mathbb{P}^1-\{(0:1)\}$. To prove (2.13) over the point $(0:1)$, we consider a general point a in any irreducible component A of D lying over $(0:1)$ and use the local construction of the double deformation space. After choosing a morphism ρ from a smooth pointed curve $(\Delta,0)$ with $\rho(0)$ representing a, one finds that we only have to check that v_2 represents a point in

$$\ker\left(\tilde{\sigma} : h^1/h^0(E_{X/Y}^\vee) \longrightarrow \mathcal{E}_X \overset{\sigma}{\dashrightarrow} \mathcal{O}_X\right).$$

Again this follows from the arguments in the proofs of Lemma 4.4 and Corollary 4.5 in [21]: When $v_2 \neq 0$, because $\lim_{\delta\to 0} s_\Delta = 0$, the image v_2' of v_2 in $h^1/h^0(E_X^\vee)$ under the tautological $h^1/h^0(E_{X/Y}^\vee) \longrightarrow \mathcal{E}_X$ lies in $\mathfrak{C}_X$. Using the cosection σ, we see that $v_2' \in \ker[h^1/h^0(E_X^\vee) \dashrightarrow \mathcal{O}_X]$. Because $\tilde{\sigma}$ is induced by σ, we obtain (2.13). This proves the proposition. $\square$

Remark 2.13. When $f : X \to Y$ is a morphism over a smooth Artin stack $\mathcal{S}$, sometimes it is more convenient to work with relative obstruction theories, say with $\mathbb{L}_X$ replaced by $\mathbb{L}_{X/\mathcal{S}}$, etc. If a relative cosection $\sigma_{Y/\mathcal{S}} : \mathcal{O}b_{Y/\mathcal{S}} \dashrightarrow \mathcal{O}_Y$ (resp. $\sigma_{X/\mathcal{S}} : \mathcal{O}b_{X/\mathcal{S}} \dashrightarrow \mathcal{O}_X$) descends to an absolute cosection $\sigma_Y : \mathcal{O}b_Y \dashrightarrow \mathcal{O}_Y$ (resp. $\sigma_X : \mathcal{O}b_X \dashrightarrow \mathcal{O}_X$), then all the statements and proofs in this section apply. (For descents of $\sigma_{Y/\mathcal{S}}$ and $\sigma_{X/\mathcal{S}}$, see [21, (4.3) and (4.5)], [4, Prop. 3.5], and [6, Prop. 3.4].)

3. Torus localization for cosection localized virtual cycles

In this section, we prove the torus localization formula for cosection localized virtual cycles (Theorem 3.5). We do not assume the existence of an equivariant global embedding or a global resolution of the perfect obstruction theory. When the cosection is trivial, our argument gives a new proof of the torus localization theorem in [15] without these assumptions.

Let X be a DM stack acted on by $T = \mathbb{C}^*$. Let F be the T-fixed locus, i.e. locally if $X = \mathrm{Spec}(A)$, then $F = \mathrm{Spec}(A/\langle A^{\mathrm{mv}}\rangle)$ where A^{mv} denotes the ideal generated by T-eigenfunctions with nontrivial characters. Let

$$\imath : F \longrightarrow X$$

be the inclusion map.

Let $D([X/T])$ denote the derived category of sheaves of $\mathcal{O}_{[X/T]}$-modules on the lisse-étale site of the algebraic stack $[X/T]$ (cf. [27, (13.2)], [27, Lemma 13.2.5], and [27, p. 126 line 1–7]). Since the construction of cotangent complexes in [17, Section 1.2.1] uses the canonical simplicial algebraic resolution of $\mathcal{O}_X$, it produces the cotangent complex $\mathbb{L}_X$ as a T-equivariant complex of T-equivariant locally free sheaves of $\mathcal{O}_X$-modules. This complex $\mathbb{L}_X$ induces an object in $D([X/T])$, which is identical to $\mathbb{L}_{[X/T]/[pt/T]}$, defined in [27, (17.3)] (cf. [33, Section 8]) because

$$\begin{array}{ccc} X & \longrightarrow & pt \\ \downarrow & & \downarrow \\ [X/T] & \longrightarrow & [pt/T] \end{array}$$

forms a fiber diagram.

Definition 3.1. A *T-equivariant perfect obstruction theory* consists of an object $E \in D([X/T])$ and a morphism $\phi : E \to \mathbb{L}_X$ in $D([X/T])$ that is a perfect obstruction theory on X.

Remark 3.2. In (2.1), if $f : X \to Y$ is a T-equivariant morphism between T-equivariant DM stacks equipped with T-equivariant perfect obstruction theories ϕ_X, ϕ_Y, and if all objects and arrows in (2.1) are T-equivariant (i.e. lie in $D([X/T])$) with cosection σ_Y also T-equivariant, then all the cone stacks and cycles in §2.1 are T-equivariant and all the operations are constructed T-equivariantly. For example, f_σ in (2.3) is T-equivariant and in Theorem 2.6 the identity $f_\sigma^![Y]^{\mathrm{vir}}_{\mathrm{loc}} = [X]^{\mathrm{vir}}_{\mathrm{loc}}$ holds for $[Y]^{\mathrm{vir}}_{\mathrm{loc}}$ in $A^T_*(Y(\sigma))$ and $[X]^{\mathrm{vir}}_{\mathrm{loc}}$ in $A^T_{*+d}(X(\sigma))$. Similar properties hold for §2.2 as well.

If A is a T-equivariant sheaf of $\mathcal{O}_F$-modules on F, we let A^{fix} denote the sheaf of T-fixed submodules and A^{mv} denote the subsheaf generated by T-eigensections with nontrivial characters. Given $E \in D([X/T])$, $\bar{E} := E|_F$ is a complex of T-equivariant sheaves on F, thus we can decompose $\bar{E} = \bar{E}^{\mathrm{fix}} \oplus \bar{E}^{\mathrm{mv}}$ into the fixed and moving parts. A T-equivariant chain map $\psi : \bar{E} \to \tilde{E}$ to an $\tilde{E} \in D([F/T])$ preserves such decompositions to give us $\psi^{\mathrm{fix}} : \bar{E}^{\mathrm{fix}} \to \tilde{E}^{\mathrm{fix}}$ and $\psi^{\mathrm{mv}} : \bar{E}^{\mathrm{mv}} \to \tilde{E}^{\mathrm{mv}}$. If ψ is a quasi-isomorphism, so are ψ^{fix} and ψ^{mv}. Therefore a T-equivariant perfect obstruction theory $\phi : E \to \mathbb{L}_X$ induces morphisms in $D([F/T])$

$$\phi^{\mathrm{fix}} : E|_F^{\mathrm{fix}} \longrightarrow \mathbb{L}_X|_F^{\mathrm{fix}} \quad \text{and} \quad \phi^{\mathrm{mv}} : E|_F^{\mathrm{mv}} \longrightarrow \mathbb{L}_X|_F^{\mathrm{mv}}.$$

Lemma 3.3. *Let the notation be as above. The composition $\phi_F : E|_F^{\mathrm{fix}} \to \mathbb{L}_X|_F^{\mathrm{fix}} \to \mathbb{L}_F$ of ϕ^{fix} and the natural morphism $\mathbb{L}_X|_F^{\mathrm{fix}} \to \mathbb{L}_F$ is a perfect obstruction theory of F.*

Proof. Since the issue is local, we may assume that F and X are affine. For any square zero extension $\Delta \to \bar{\Delta}$ of affine schemes with ideal sheaf J and a morphism $g : \Delta \to F$, let $\omega(g) \in \mathcal{E}xt^1(g^*\mathbb{L}_F, J)$ denote the composition $g^*\mathbb{L}_F \to \mathbb{L}_\Delta \to J[1]$ of the natural morphisms $g^*\mathbb{L}_F \to \mathbb{L}_\Delta$ from g and $\mathbb{L}_\Delta \to \mathbb{L}_{\Delta/\bar{\Delta}} \to \mathbb{L}^{\geq -1}_{\Delta/\bar{\Delta}} = J[1]$ from $\Delta \to \bar{\Delta}$. Let

$$\phi_F^*\omega(g) \in \mathrm{Ext}^1(g^*E|_F^{\mathrm{fix}}, J)$$

be the image of $\omega(g)$ by the map $\mathrm{Ext}^1(g^*\mathbb{L}_F, J) \to \mathrm{Ext}^1(g^*E|_F^{\mathrm{fix}}, J)$ induced from $\phi_F : E|_F^{\mathrm{fix}} \to \mathbb{L}_F$. Note that $\mathrm{Ext}^1(g^*E|_F^{\mathrm{fix}}, J)$ is a T-module and $\phi_F^*\omega(g)$ is T-invariant, where T acts on Δ, $\bar{\Delta}$ and J trivially.

By [1, Theorem 4.5], it suffices to show the following claim: the obstruction assignment $\phi_F^*(\omega(g))$ vanishes if and only if an extension $\bar{g} : \bar{\Delta} \to F$ of g exists; and if $\phi_F^*(\omega(g)) = 0$, then the extensions form a torsor under $\mathrm{Ext}^0(g^*E|_F^{\mathrm{fix}}, J)$.

Let $h : \Delta \to X$ be the composite of g with the inclusion $F \subset X$. Since $\phi : E \to \mathbb{L}_X$ is a perfect obstruction theory, h extends to $\bar{h} : \bar{\Delta} \to X$ if and only if $0 = \phi_X^*\omega(h) \in \mathrm{Ext}^1(h^*E, J)$. Because h factors through $F \subset X$ and J is an $\mathcal{O}_\Delta$-module,

$$\mathrm{Ext}^1(h^*E, J) = \mathrm{Ext}^1(g^*E|_F^{\mathrm{fix}}, J) \oplus \mathrm{Ext}^1(g^*E|_F^{\mathrm{mv}}, J),$$

as T-module, and further $\phi_X^*\omega(h)$ is T-invariant. Since $\phi_X^*\omega(h)^T = \phi_F^*\omega(g)$, we see that $\phi_F^*\omega(g) = 0$ if and only if h extends to $\bar{h} : \bar{\Delta} \to X$. Because T is reductive and both $\bar{\Delta}$ and X are affine, by applying the Reynolds operator, we can find a T-invariant extension $\bar{h}$, which necessarily factors through $F \subset X$. This proves that $\phi_F^*\omega(g)$ is an obstruction class to extending g to $\bar{g} : \bar{\Delta} \to F$.

The part on the space of extensions $\bar{g}$ follows by the same argument. □

We let $E_F := E|_F^{\mathrm{fix}}$ and $N^{\mathrm{vir}} := (E|_F^{\mathrm{mv}})^\vee$. Since E is perfect, both the fixed part E_F and the moving part $E|_F^{\mathrm{mv}} = (N^{\mathrm{vir}})^\vee$ of $E|_F$ are perfect. They fit into the following diagram of distinguished triangles:

$$\begin{tikzcd} E|_F \arrow[r] \arrow[d] & E_F \arrow[r, "\tau"] \arrow[d] & (N^{\mathrm{vir}})^\vee[1] \arrow[r, "+1"] \arrow[d, "\mathfrak{f}"] & {} \\ \mathbb{L}_X|_F \arrow[r] & \mathbb{L}_F \arrow[r] & \mathbb{L}_{F/X} \arrow[r, "+1"] & {} \end{tikzcd} \tag{3.1}$$

The morphism $E|_F \to E_F$ induces a homomorphism

$$\mathcal{O}b_F = H^1(E_F^\vee) \longrightarrow H^1(E|_F^\vee) \cong H^1(E^\vee)|_F = \mathcal{O}b_X|_F.$$

Let $\sigma : \mathcal{O}b_X = H^1(E^\vee) \to \mathcal{O}_X$ be a T-equivariant cosection. Then σ induces a T-invariant cosection

$$\sigma_F : \mathcal{O}b_F \longrightarrow \mathcal{O}b_X|_F \longrightarrow \mathcal{O}_X|_F = \mathcal{O}_F$$

and we have a cosection localized virtual cycle $[F]^{\mathrm{vir}}_{\mathrm{loc}}$.

Definition 3.4. Suppose the virtual normal bundle N^{vir} admits a global resolution $[N_0 \to N_1]$ by locally free sheaves N_0 and N_1 over F. We define the Euler class $e(N^{\mathrm{vir}})$ of N^{vir} to be

$$e(N^{\mathrm{vir}}) = e(N_0)/e(N_1) \in A^*(F) \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}].$$

The goal of this section is to prove the following.

Theorem 3.5. *Let X be a DM stack acted on by T and let $E \to \mathbb{L}_X$ be an equivariant perfect obstruction theory on X. Let F be the T-fixed locus in X. Let $\sigma : \mathcal{O}b_X \to \mathcal{O}_X$ be a T-equivariant cosection on X. Suppose there is a global resolution $N^{\mathrm{vir}} \cong [N_0 \to N_1]$, where N_0 and N_1 are locally free sheaves on F (whose ranks may vary from component to component). Then we have*

$$[X]^{\mathrm{vir}}_{\mathrm{loc}} = \imath_* \frac{[F]^{\mathrm{vir}}_{\mathrm{loc}}}{e(N^{\mathrm{vir}})} \in A^T_* X(\sigma) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, t^{-1}].$$

Here the class $[F]^{\mathrm{vir}}_{\mathrm{loc}}$ is defined with respect to the induced perfect obstruction theory E_F and cosection σ_F.

Remark 3.6. In [15], the localization formula in Theorem 3.5 was proved for the ordinary virtual fundamental class under the following assumptions:

(1) X admits a global equivariant embedding into a smooth Y;
(2) the perfect obstruction theory E admits an equivariant global locally free resolution.

Both conditions are nontrivial unless X is a projective scheme. Recent development in moduli theory and enumerative geometry utilizes a plethora of moduli stacks for which (1) is often tedious to verify and hence it is desirable to give a proof without the assumption (1). Here we remove the first assumption entirely and weaken the second assumption to

(2′) the virtual normal bundle N^{vir} admits a global locally free resolution $[N_0 \to N_1]$ on the fixed locus F,

which is often easier to check. When $\sigma = 0$, Theorem 3.5 says that the torus localization in [15] works without the assumption (1) and with a much weaker (2′).

By our assumption that there is a resolution $[N_0 \to N_1]$ of N^{vir}, we find that the normal sheaf $N_{F/X}$ is contained in $h^1/h^0(N^{\mathrm{vir}}[-1]) = \ker\{N_0 \to N_1\}$, thus contained in N_0. Hence the normal cone $\mathfrak{C}_{F/X}$ is contained in N_0 as well. As in Definition 2.4, we define the virtual pullback

$$\imath^! : A_*(X(\sigma)) \to A_*(F(\sigma))$$

for the inclusion $\imath : F \to X$, by

$$[B] \longmapsto [\mathfrak{C}_{B\times_X F/B}] \longmapsto 0^!_{N_0}[\mathfrak{C}_{B\times_X F/B}].$$

The proof of Theorem 3.5 is attained through the following two lemmas.

Lemma 3.7. *Let $X(\sigma)$ and $F(\sigma)$ denote the vanishing loci of σ and σ_F respectively. Then $F(\sigma) = X(\sigma) \cap F$ and $\imath^![X]^{\mathrm{vir}}_{\mathrm{loc}} = [F]^{\mathrm{vir}}_{\mathrm{loc}} \cap e(N_1)$.*

Proof. The first identity follows from Lemma 2.2. We prove the second identity. By definitions, $[X]^{\mathrm{vir}}_{\mathrm{loc}} = 0^!_{\mathcal{E},\mathrm{loc}}[\mathfrak{C}_X]$ and $[F]^{\mathrm{vir}}_{\mathrm{loc}} = 0^!_{\mathcal{E}_F,\mathrm{loc}}[\mathfrak{C}_F]$, where $\mathcal{E} = h^1/h^0(E^\vee)$ and $\mathcal{E}_F = h^1/h^0(E_F^\vee)$.

We now modify the perfect obstruction theory of F to make $F \to X$ virtually smooth. As $(N^{\mathrm{vir}})^\vee[1] \cong [N_1^\vee \to N_0^\vee]$ is perfect of amplitude $[-2,-1]$, it fits in the natural distinguished triangle

$$N_0^\vee[1] \xrightarrow{\ \mathfrak{g}\ } (N^{\mathrm{vir}})^\vee[1] \xrightarrow{\ \mathfrak{h}\ } N_1^\vee[2] \xrightarrow{+1} .$$

Define $\tilde{E}_F$ by the distinguished triangle

$$\tilde{E}_F \xrightarrow{\ \alpha\ } E_F \xrightarrow{\ \mathfrak{h}\circ\tau\ } N_1^\vee[2] \xrightarrow{+1} , \tag{3.2}$$

where $\tau : E_F \to (N^{\mathrm{vir}})^\vee[1]$ is from (3.1). Taking cohomology via arbitrary base change of (3.2) shows $\tilde{E}_F$ (over F) is perfect of amplitude $[-1,0]$, $h^0(\alpha)$ is an isomorphism, and $h^{-1}(\alpha)$ is surjective. Applying the octahedral axiom one also obtains a diagram of distinguished triangles

$$\begin{tikzcd} E|_F \arrow[r, "\beta"] \arrow[d, "="] & \tilde{E}_F \arrow[r, "\beta'"] \arrow[d, "\alpha"] & N_0^\vee[1] \arrow[r, "+1"] \arrow[d, "\mathfrak{g}"] & {} \\ E|_F \arrow[r] & E_F \arrow[r, "\tau"] & (N^{\mathrm{vir}})^\vee[1] \arrow[r, "+1"] & {} . \end{tikzcd} \tag{3.3}$$

As the second row of (3.3) equals the first row of (3.1), combined they give

$$\begin{tikzcd} E|_F \arrow[r, "\beta"] \arrow[d, "\phi_X|_F"] & \tilde{E}_F \arrow[r] \arrow[d, "\phi_F"] & N_0^\vee[1] \arrow[r, "+1"] \arrow[d, "\phi_{F/X}"] & {} \\ \mathbb{L}_X|_F \arrow[r] & \mathbb{L}_F \arrow[r] & \mathbb{L}_{F/X} \arrow[r, "+1"] & {} . \end{tikzcd} \tag{3.4}$$

The just-stated properties of $h^0(\alpha)$ and $h^{-1}(\alpha)$ imply that ϕ_F is a perfect obstruction theory for F. Since $N_0^\vee[1]$ is of amplitude contained in $[-1,0]$, the inclusion $\imath : F \to X$ is virtually smooth via (3.4). Theorem 2.6 (cf. Remark 3.2) then implies

$$\imath^![X]^{\mathrm{vir}}_{\mathrm{loc}} = [F]^{\mathrm{vir}}_{\tilde{E}_F,\mathrm{loc}}, \tag{3.5}$$

where $[F]^{\mathrm{vir}}_{\tilde{E}_F,\mathrm{loc}}$ is the cosection localized virtual fundamental class of ϕ_F, via the cosection

$$\sigma' := \sigma|_F \circ h^1(\beta^\vee) : H^1(\tilde{E}_F^\vee) \longrightarrow \mathcal{O}_F,$$

here all objects and arrows above are in $D([X/T])$ or $D([F/T])$.

To evaluate the right hand side of (3.5), we take $h^1/h^0(\cdot^\vee)$ of the triangle (3.2); we apply [1, Proposition 2.7] to obtain a short exact sequence of abelian cone stacks over F:

$$h^1/h^0(E_F^\vee) \longrightarrow h^1/h^0(\tilde{E}_F^\vee) \longrightarrow N_1. \tag{3.6}$$

Since T acts on F trivially, for each object $D([F/T])$ that is perfect on F one can use its weight decomposition to represent it locally by a two term complex of locally free T-equivariant sheaves. Thus each term in (3.6) is a bundle stack over $[F/T]$, and so are the arrows in (3.6).

As the intrinsic normal cone of F is contained in $h^1/h^0(E_F^\vee)$, and the cosection σ', σ_F are compatible via α in (3.2), by using exactness of (3.6) we conclude that $[F]^{\mathrm{vir}}_{\tilde{E}_F,\mathrm{loc}} = [F]^{\mathrm{vir}}_{\mathrm{loc}} \cap e(N_1)$. Together with (3.5), this proves the second identity in the statement of the lemma. $\square$

Lemma 3.8. $\imath^! \imath_* \alpha = \alpha \cap e(N_0)$ *for* $\alpha \in A_*(F(\sigma)) \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}]$.

Proof. If B is a cycle in $F(\sigma)$, the normal cone of $B \cap F(\sigma)$ in B is B and $\imath^! \imath_* B = 0^!_{N_0} B = B \cap e(N_0)$ by the definition of virtual pullback $\imath^!$. $\square$

Now we can prove Theorem 3.5.

Proof of Theorem 3.5. By [25, Theorem 6.3.5],

$$\imath_* : A^T_*(F(\sigma)) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, t^{-1}] \longrightarrow A^T_*(X(\sigma)) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, t^{-1}]$$

is an isomorphism. Thus $\imath_* \alpha = [X]^{\mathrm{vir}}_{\mathrm{loc}}$ for some

$$\alpha \in A^T_*(F(\sigma)) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, t^{-1}] = A_*(F(\sigma)) \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}].$$

By Lemmas 3.7 and 3.8,

$$[F]^{\mathrm{vir}}_{\mathrm{loc}} \cap e(N_1) = \imath^! [X]^{\mathrm{vir}}_{\mathrm{loc}} = \imath^! \imath_* \alpha = \alpha \cap e(N_0).$$

Hence

$$\alpha = \frac{[F]^{\mathrm{vir}}_{\mathrm{loc}}}{e(N^{\mathrm{vir}})} \quad \text{and} \quad [X]^{\mathrm{vir}}_{\mathrm{loc}} = \imath_* \alpha = \imath_* \frac{[F]^{\mathrm{vir}}_{\mathrm{loc}}}{e(N^{\mathrm{vir}})}$$

as desired. $\square$

Example 3.9. Let $V = \mathbb{C}^d$ be a vector space with $(z_1, \cdots, z_d)$ being its standard coordinates. We let $T = \mathbb{C}^*$ act on V via $(z_i)^\alpha = (\alpha z_i)$. The global differentials dz_i give a trivialization of Ω_V, in the form $\Omega_V \cong V \times V^*$. Let $E = [T_V \xrightarrow{0} \Omega_V]$ and the cosection $\sigma : \Omega_V = V \times V^* \to \mathcal{O}_V$ be the tautological pairing. Then the vanishing loci of σ is $\{O\} \subset V$, the (reduced) origin $O \in V$; and under the induced T-action on Ω_V, σ is T-invariant. We observe

$$F = V^{\mathbb{C}^*} = \{O\}, \quad E|_F^\vee = [V \xrightarrow{0} V^*] = N^{\mathrm{vir}}, \quad \text{and} \quad E_F = [0 \to 0].$$

Hence $e(N^{\mathrm{vir}}) = (-1)^d$ and $[F]^{\mathrm{vir}}_{\mathrm{loc}}$ is the zero cycle $[O]$ consisting of one simple point O. Then by Theorem 3.5, we have

$$[V]^{\mathrm{vir}}_{\mathrm{loc}} = \frac{[F]^{\mathrm{vir}}_{\mathrm{loc}}}{e(N^{\mathrm{vir}})} = (-1)^d[O]$$

as expected from [21, Example 2.4].

In this example, if instead we consider a cosection $\sigma' : \Omega_V \to \mathcal{O}_V$ via $dz_1 \mapsto 1$ and $dz_{i>1} \mapsto 0$. Since σ' is surjective, we obtain $[V]^{\mathrm{vir}}_{\mathrm{loc}} = 0$. However, $\frac{[F]^{\mathrm{vir}}_{\mathrm{loc}}}{e(N^{\mathrm{vir}})} = (-1)^d[O]$ as before. Note that since the cosection σ' is not $\mathbb{C}^*$ invariant, Theorem 3.5 does not apply in this case.

4. Wall crossing formulas for cosection localized virtual cycles

In this section we provide a wall crossing formula for simple $\mathbb{C}^*$-wall crossings. The construction and proof are rather standard (cf. [22]).

Let X be a DM stack acted on by $T = \mathbb{C}^*$. Let $\phi : E \to \mathbb{L}_X$ be a T-equivariant perfect obstruction theory, together with an equivariant cosection $\sigma : \mathcal{O}b_X = h^1(E^\vee) \to \mathcal{O}_X$. Let

(1) F be the T-fixed locus in X;
(2) X^s be the open substack of $x \in X$ so that the orbit $T \cdot x$ is 1-dimensional and closed in X;
(3) $\Sigma^0_\pm = \{x \in X - (X^s \cup F) \,|\, \lim_{t\to 0} t^{\pm 1} \cdot x \in F\}$;
(4) $\Sigma_\pm = \Sigma^0_\pm \cup F$;
(5) $X_\pm = X - \Sigma_\mp \subset X$;
(6) $M_\pm = [X_\pm/T] \subset M = [X/T]$ are separated DM stacks.

Typically such pairs $M_\pm \subset M$ arise when we change the stability condition for a moduli problem. In this context, $\Sigma_\pm$ should be thought of as $\mp$-unstable loci while $X_\pm$ are $\pm$-stable parts. When the conditions (1)–(6) are satisfied, we say $M_\pm \subset M$ form a simple $\mathbb{C}^*$-wall crossing (cf. [22]).

Recall from §3 that we have the induced cosections $\sigma_F : \mathcal{O}b_F \to \mathcal{O}_F$. We then define the master space of the wall crossing $M_\pm$ to be

$$\mathfrak{M} = \left[X \times \mathbb{P}^1 - \Sigma_- \times \{0\} - \Sigma_+ \times \{\infty\}/\mathbb{C}^*\right]$$

where $\mathbb{C}^*$ acts on $X \times \mathbb{P}^1$ diagonally. Here the $\mathbb{C}^*$-action on $\mathbb{P}^1$ is $\lambda \cdot (a : b) = (a : \lambda b)$. The action of T on X induces an action of T on $\mathfrak{M}$ whose fixed locus is

$$M_+ \sqcup F \sqcup M_-$$

as is easy to check. Since $\mathbb{C}^*$ acts only on the component $\mathbb{P}^1$, the pullback of any sheaf on X by the projection $X \times \mathbb{P}^1 \to X$ is $\mathbb{C}^*$-equivariant and hence descends to the free quotient $\mathfrak{M}$. By pulling back the perfect obstruction theory $\phi : E \to \mathbb{L}_X$ and descending, we obtain a morphism $\bar{\phi} : \bar{E} \to \mathbb{L}_{\mathfrak{M}}$.

Lemma 4.1. *The morphism $\bar{\phi} : \bar{E} \to \mathbb{L}_{\mathfrak{M}}$ is a T-equivariant perfect obstruction theory of $\mathfrak{M}$. Moreover the pullback of σ descends to a T-equivariant cosection $\bar{\sigma} : \mathcal{O}b_{\mathfrak{M}} \to \mathcal{O}_{\mathfrak{M}}$.*

Proof. This is straightforward and we omit the proof. □

The cosection $\bar{\sigma}$ induces cosections on the fixed locus $M_\pm$ and F in $\mathfrak{M}$. We are ready to state the main result of this section.

Theorem 4.2. *Let the notation be as above. Suppose the virtual normal bundle N^{vir} admits a resolution $[N_0 \to N_1]$ by vector bundles on F. Then we have*

$$[M_+]^{\mathrm{vir}}_{\mathrm{loc}} - [M_-]^{\mathrm{vir}}_{\mathrm{loc}} = \mathrm{res}_{t=0} \frac{[F]^{\mathrm{vir}}_{\mathrm{loc}}}{e(N^{\mathrm{vir}})} \quad \text{in } A^T_*(\mathfrak{M}(\sigma)) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, t^{-1}].$$

Proof. Applying Theorem 3.5 to the master space $\mathfrak{M}$, we find that

$$[\mathfrak{M}]^{\mathrm{vir}}_{\mathrm{loc}} = \frac{[M_+]^{\mathrm{vir}}_{\mathrm{loc}}}{-t} + \frac{[M_-]^{\mathrm{vir}}_{\mathrm{loc}}}{t} + \frac{[F]^{\mathrm{vir}}_{\mathrm{loc}}}{e(N^{\mathrm{vir}})}$$

since the normal bundle of M_+ is trivial with weight 1 while that of M_- is trivial with weight -1 by construction. If we take $\mathrm{res}_{t=0}$, the left side vanishes because $[\mathfrak{M}]^{\mathrm{vir}}_{\mathrm{loc}} \in A^T_*(\mathfrak{M})$ has trivial principal part. Therefore the residue of the right side

$$-[M_+]^{\mathrm{vir}}_{\mathrm{loc}} + [M_-]^{\mathrm{vir}}_{\mathrm{loc}} + \mathrm{res}_{t=0} \frac{[F]^{\mathrm{vir}}_{\mathrm{loc}}}{e(N^{\mathrm{vir}})}$$

vanishes. This proves the theorem. □

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