.

Distributions of points on non-extensible closed curves in $\mathbb{R}^3$ realizing maximum total energies

参赛学生

王中子 Zhongzi Wang

北京中国人民大学附中高一 24 班

指导老师

郑绍远 Shiu-Yuen Cheng

北京清华大学数学系

带队老师

待定

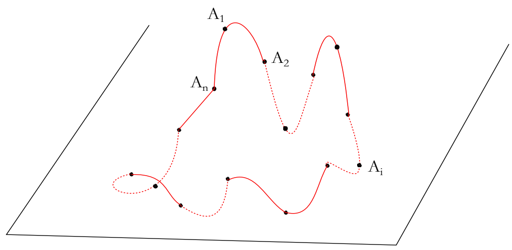
北京中国人民大学附中

Abstract: Distributions of points under certain conditions are widely concerned by people. **Our motivation:** Let G_n be a non-extensible, flexible closed curve of length n in $\mathbb{R}^3$ with n particles $A_1,\ldots\ ,A_n$ evenly fixed (according to arc length of G_n) on the curve. Let $f_\alpha(x) = x^\alpha$ for $\alpha > 0$, $f_0(x) = \ln x$, $f_\alpha(x) = -x^\alpha$ for $\alpha < 0$, where $x \geq 0$. Let d be the distance in $\mathbb{R}^3$. Define the total energy

$$E_n^\alpha(G_n) = \frac{1}{2}\sum_{p\neq q} f_\alpha(d(A_p, A_q)).$$

Problem 1.1. *What is the shape of G_n when the total energy reaches the maximum?*

Note $E_n^\alpha(G_n)$ relies only the positions of particles of G_n, but positions of those particles are constrained by the non-extensible curve.

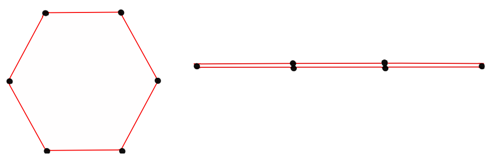


The famous Thomson type problem, which considers the distribution of n points on the unit sphere in $\mathbb{R}^3$ under essentially the same energy functions f_α, is an inspiration of the distribution problem we studied here.

We denote the maximum of the total energy E_n^α by **max**E_n^α. We will verify the existence of maxE_n^α (**Theorem 2.1**) and prove each G_n realizing $maxE_n^\alpha$ must be a Γ_n, a convex n-gon (may be degenerated) with edge length 1 (**Theorem 3.1**).

Problem 1.2. *What is the shape of Γ_n when the total energy reaches the maximum?*

There are two special shapes for Γ_n: the regular n-gon Γ_n^o, and the double straight arc Γ_n^- (only defined for even n).



For $n = 4$, the Problem is completely solved (**Example 4.3**).

We will prove for given n, $E_n^\alpha(\Gamma_n)$ is maximum if and only if $\Gamma_n = \Gamma_n^o$ for large enough negative α (**Theorem 5.6**); and for given even n, $E_n^\alpha(\Gamma_n)$ is maximum if and only if $\Gamma_n = \Gamma_n^-$ for large enough positive α. (**Theorem 6.1**)

Theorem 5.6 follows from **Theorem 5.1**: If Γ_n satisfies a bending condition, then $E_n^\alpha(\Gamma_n)$ is maximum if and only if $\Gamma_n = \Gamma_n^o$ for $\alpha \leq 1$. All central symmetry Γ_n satisfy this bending condition.

For each even n, Γ_n realizing maxE_n^α we found so far are only Γ_n^o and Γ_n^-. But there are infinitely many Γ_5 realizing maxE_5^α as α varies (**Proposition 6.7**).

We also add some information on the Thomson type problem (**Theorem 7.2**).

Distributions of points on non-extensible closed curves in $\mathbb{R}^3$ realizing maximum total energies

Contents

1 Introduction

Distributions of points under certain conditions are widely concerned by people. A motivation of our study is as below:

Let f be an energy function which is increasing about the distance d in $\mathbb{R}^3$. Let G_n be a non-extensible, flexible closed curve with n particles $A_1,\dots,A_n$ evenly fixed on the curve. Put G_n into $\mathbb{R}^3$. Define the total energy

$$E_n^f(G_n) = \frac{1}{2}\sum_{p\neq q} f(d(A_p, A_q)),$$

where d is the distance in $\mathbb{R}^3$.

Problem 1.1. *What is the shape of G_n when the total energy reaches the maximum?*

Note $E_n^f(G_n)$ relies only the positions of particles of G_n, but positions of those particles are constrained by the non-extensible curve.

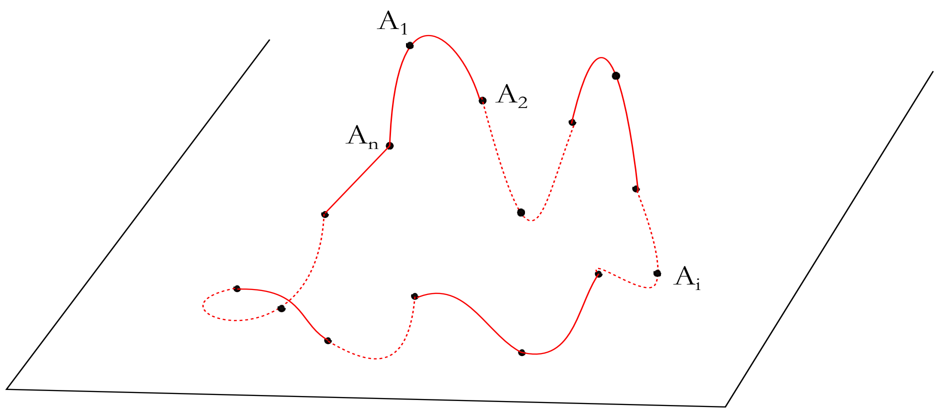


Figure 1

Mathematically, let G_n be a circle G of length n with n vertices $A_1, \dots, A_n$ attached consecutively so that the distance between A_i and A_{i+1} is 1 along G, $A_{i+n} = A_i$. Both G and $\mathbb{R}^3$ have their own standard metrics. Call a map $g : G_n \to \mathbb{R}^3$ is non-extensible, if g does not extend the length of any portion of G. More precisely, we assume $g : G_n \to \mathbb{R}^3$ is differentiable with finitely many exception points of G_n. Call $g : G_n \to \mathbb{R}^3$ is non-extensible, if the modulus of first derivative $|g'(x)| = 1$ for any differentiable point $x \in G$. We will consider E_n^f on $g(G_n)$ for any non-extensible map $g : G_n \to \mathbb{R}^3$, and call $g(G_n) \subset \mathbb{R}^3$ is allowable.

For simplicity, we will often use $G_n \subset \mathbb{R}^3$ to denote $g(G_n) \subset \mathbb{R}^3$, A_i to denote $g(A_i)$ and so on. In particular $d(A_i, A_{i+1}) \leq 1$. We will rewrite (1.1) as

$$E_n^f(G_n) = \frac{1}{2}\sum_{p\neq q} f(|A_p - A_q|) = \sum_{p<q} f(|A_p - A_q|) \qquad (1.2)$$

where each A_p is considered as a vector in $\mathbb{R}^3$ and $|A_p|$ is the length of A_p.

The following family of energy functions $f_\alpha, \alpha \in \mathbb{R}$ are often appeared in geometry and physics.

$$f_\alpha(x) = \begin{cases} x^\alpha, & \alpha > 0; \\ \ln x, & \alpha = 0; \\ -x^\alpha, & \alpha < 0. \end{cases} \qquad (1.3)$$

Remark 1.2. Cases $\alpha = -1$ and $\alpha = 2$ have physics means: Case $\alpha = -1$ was first consider by Thomson for points in the unit 2-sphere [To]. In our setting $-E_n^{-1}(G_n)$ is the total electric potential energy of G_n, where each vertex of G_n has unit charge, and there is no charge on the edges. For $\alpha = 2$, $E_n^2(G_n)$ is the moment of inertia of G_n about its mass center (Remark 8.9), where each vertex of G_n has unit mass, and there is no mass on the edges. According to [HLP], a most direct case $\alpha = 1$ was first considered by Toth for points with mutully distances ≤ 1 [To].

Below we will simply denote $E_n^{f_\alpha}$ by E_n^α. First we will verify the following two results.

Theorem 2.1. *For each α and n, the maximum of $E_n^\alpha(G_n)$ exists among all allowable $G_n \subset \mathbb{R}^3$.*

Theorem 3.1. *Suppose $E_n^f(G_n)$ reaches the maximum. Then G_n is a convex n-gon (may be degenerated) with edge length 1.*

The verification of Theorem 3.1 is longer and subtler than we first thought.

Below we use $\prod_n = \{\Gamma_n\}$ to denote the set of all convex n-gons with edge length 1 in the plane. With Theorem 3.1, Problem 1.1 is transformed to the following

Problem 1.3. *What is the shape of Γ_n when the total energy reaches the maximum?*

Below we always assume that the integer $n \geq 4$. (For $n = 2$ or $n = 3$ the answer is obvious). We often use $\max E_n^\alpha$ to denote the maximum of E_n^α.

There are two special shapes for Γ_n: **the regular n-gon Γ_n^o, and the double straight arc** Γ_n^- (only for even n, see Figure 2 right for $n = 6$, where two lines are coincided indeed. See Section 4 for the precise definition). We will see $E_4^\alpha(\Gamma_4)$ reaches the maximum at Γ_4^o for $\alpha < 2$ and at Γ_4^- for $\alpha > 2$, and $E_4^2(\Gamma_4)$ is a constant for all Γ_4 (Example 4.3).

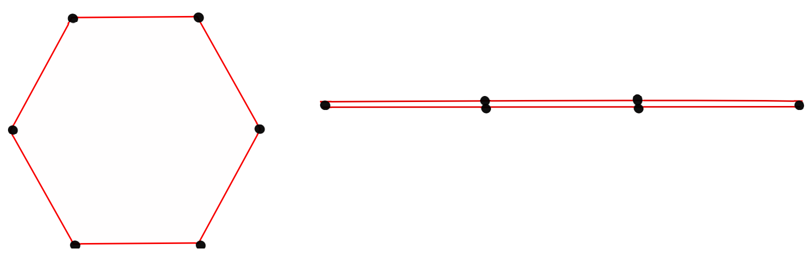


Figure 2

For general n we have

Theorem 5.6. *For given n, there is an $\alpha_* < 0$ (depends on n) such that $E_n^\alpha(\Gamma_n)$ reaches the maximum if and only if Γ_n is the regular n-gon for $\alpha < \alpha_*$.*

Theorem 6.1. *For given even $n > 0$, there is an $\alpha^* > 0$ (depends on n) such that $E_n^\alpha(\Gamma_n)$ reaches the maximum if and only if $\Gamma_n = \Gamma_n^-$ when $\alpha > \alpha^*$.*

The proofs of Theorem 5.6 and Theorem 6.1 are quite different: Theorem 6.1 follows from a rather complicated estimation (see Section 6), while Theorem 5.6 follows from Theorem 5.1 below whose proof needs a decomposition of $E_n^\alpha(\Gamma_n)$ (see Section 5). Let $[x]$ be the maximum integer not bigger than x.

Theorem 5.1: *Suppose the sum of any consecutive $[n/2] - 1$ exterior angles of Γ_n is no more than π. Then $E_n^\alpha(\Gamma_n)$ is the maximum if and only if $\Gamma_n = \Gamma_n^o$, the regular n-gon of edge length 1 for $\alpha \leq 1$.*

A direct consequences of Theorem 5.1 is that if Γ_n is central symmetry, then $E_n^\alpha(\Gamma_n)$ is maximum if and only if $\Gamma_n = \Gamma_n^o$ for $\alpha \leq 1$. (Corollary 5.8).

By Theorem 5.6 and Theorem 6.1, for each even number n larger than 2, the equation $E^{\alpha_n}(\Gamma_n^o) = E^{\alpha_n}(\Gamma_n^-)$ always has solutions. So far the Γ_n reaches $\max E_n^\alpha$ we proved are only Γ_n^o and Γ_n^-. We are interested to find some other Γ_n realizing $\max E_n^\alpha$.

For odd n, the situation is different. Let Γ_n^Δ denote the unique Γ_n which is an isoceles triangle with base length 1. We observed that E_5^α never reaches maximum at Γ_5^Δ for any α (Propositions 6.6), and based on this observation we have

Proposition 6.7. *There are infinitely many Γ_5 realizing $\max E_5^\alpha$ as α varies.*

We believe this is true for any odd n.

Beyond the several results listed above, Problem 1.3 is open in general. Some efforts are made to get some local results. An example is below. Note 6 is the next even number after 4, and central symmetry condition allow us to deal with calculus of only three variables, then some elementary tricks can apply.

Proposition 8.1. *If a central symmetry Γ_6 realizes $\max E_6^\alpha$ for $\alpha \geq 6$, then $\Gamma_6 = \Gamma_6^-$.*

The Thomson type problem, which considers the distribution of n points on the unit sphere in $\mathbb{R}^3$ under the energy functions f_α given by (1.3), is an inspiration of the distribution problem we studied in this note. The problem was first raised by Thomson for $\alpha = -1$ [Th], and later generalized to all $\alpha \in \mathbb{R}$. Smale put Thomson's problem in his problem list for 21st century [Sm]. There many studies on Thomson type problem, see [AP], [BH], [PB] and their references. For concrete n, the precise distributions of n point which realize the extremum is known only for few small n.

Mathematically, Thomson type problems can be raised for unit sphere S^m of $\mathbb{R}^{m+1}$ for any integer $m > 0$. Now we can also add some information to the Thomson type problem. One sample result is the following (where we use $E_n^\alpha(m)$ to denote the corresponding totoal energy).

Theorem 7.2. *Let $A_1, ..., A_n$ be n points on the unit sphere S^m. Then*

(1) For $\alpha = 2$, $A_1, ..., A_n$ realize the $\max E_n^2(m)$ if and only $\sum_{i=1}^n A_i = 0$, in particular there are infinitely many distributions to realize $\max E_n^2(m)$.

(2) For $\alpha > 2$ and n even, $A_1, ..., A_n$ realize the $\max E_n^\alpha(m)$ if and only if they stay evenly in the two ends of a diameter of S^m.

(3) For $\alpha < 2$ and $n \leq m+2$, $A_1, ..., A_n$ realize the $maxE_n^\alpha(m)$ if and only if they are the vertices regular $(n-1)$-simplex inscribed in $S^{n-2} = S^m \cap \mathbb{R}^{n-1}$, where $\mathbb{R}^{n-1}$ is a subspace of $\mathbb{R}^{m+1}$ passing the origin.

(3) is known at least for $\alpha = -1, m = 2$ and $\alpha = 1$, any integer $m > 0$, $n = m+2$, see [PB], [BH] for example.

The paper is developed as the table of content. All calculus used can be found in [St], or [LZ] in Chinese; some basic topology of Euclidean spaces can be found in [Ar], or [Yo] in Chinese. Several classical inequalities are well known, can be found in [HLP].

2 The existence of the maximum for E_n^α

Theorem 2.1. *For each α and n, the maximum of $E_n^\alpha(G_n)$ exists among all allowable $G_n \subset \mathbb{R}^3$.*

We use some basic topology of Euclidean space (see [Ar] or [Yo]) to prove Theorem 2.1.

First note every subset of $\mathbb{R}^n$ with the metric given by $\mathbb{R}^n$ become a metric space. For $x \in \mathbb{R}^n$ and $\epsilon > 0$, an open ϵ-neighborhood of x in $\mathbb{R}^n$ is defined as $U_\epsilon(x) = \{y | d(x, y) < \epsilon\}$. Call a subset $X \subset \mathbb{R}^n$ is open, if each $x \in X$ has an $U_\epsilon(x) \subset X$ for some $\epsilon > 0$. Let $\bar{X}$ denote the complement of X in $\mathbb{R}^n$. Call a subset $X \subset \mathbb{R}^n$ is closed if $\bar{X}$ is open. It is easy to verify that the union (intersection) of open (closed) sets is open (closed), and intersection (union) of finitely many open (closed) set is open (closed).

Call a subset $X \subset \mathbb{R}^n$ is compact, if every infinite sequence in X contains a convergent sub-sequence with limit in X. X compact implies that X is closed, and the intersection of a compact subset and a closed set is compact.

Theorem 2.2. *(1) (Heine-Borel theorem) For a subset $X \subset \mathbb{R}^n$, X is compact if and only if X is closed and bounded.*

(2) A continuous real-valued function defined on a compact subset is bounded and reaches its bounds.

Proof of theorem 2.1. In this proof, we use $(x_1, x_2, ..., x_n)$, where x_i is a vector in $\mathbb{R}^3$, to denote of the image of the vertices under allowable maps $g : G_n \to \mathbb{R}^3$.

Now we define

$$\begin{aligned} B_1 &: |x_2 - x_1| \leq 1, \\ B_2 &: |x_3 - x_2| \leq 1, \\ &..., \\ B_{n-1} &: |x_n - x_{n-1}| \leq 1, \\ B_n &: |x_1 - x_n| \leq 1. \end{aligned}$$

Positions of $(x_1, x_2, ..., x_n)$ form a subset $U' \subset (\mathbb{R}^3)^n$ which defined by

$$B' = \cap_{i=1}^n B_i.$$

Since each B_i, defined by $\leq$, is closed, their intersection B' is closed.

Since $E_n^\alpha(G_n)$ is invariant under Euclidean transformations, so we may assume that $x_1 = 0$. Note $B'' \subset (\mathbb{R}^3)^n$ defined by $x_1 = 0$ is also a closed subset. Let

$$B = B' \cap B''.$$

B is also closed.

To consider the value of E_n^α, we need only restrict our attention on B. Since

$$|x_i - x_1| \leq i - 1 < n,$$

we have $|x_i| < n$, so

$$d((x_1, x_2, ..., x_n), 0)^2 = x_1^2 + x_2^2 + ... + x_n^2 \leq n^3,$$

hence B is bounded.

By Heine-Borel theorem, a closed bounded subset of Euclidean space is compact. So B is compact.

If $\alpha > 0$, then $E_n^\alpha(G_n) = \sum_{i<j} |x_i - x_j|^\alpha$ is continuous function defined on B. By Theorem 2.2 (2), B is compact implies that E_n^α has a maximum on B.

If $\alpha \leq 0$, then $E_n^\alpha(G_n)$ is defined only on $B_D = B \setminus D$ (those points in B but not in D), where

$$D = \cup_{i \neq j} D_{i,j},$$

$$D_{i,j} = \{(x_1, x_2, ..., x_n)|\ x_i = x_j,\ i \neq j\}.$$

Clearly $D_{i,j}$ is closed, so D, as a finite union of closed set is also closed. Hence $B \setminus D$ is not closed.

Now for $i \neq j$ and some $\epsilon > 0$, let

$$B_{i,j}^\epsilon = \{(x_1, x_2, ..., x_n)|\ |x_i - x_j| \geq \epsilon\},$$

then $B_{i,j}^\epsilon$ is a closed subset. Let

$$C^\epsilon = \cap_{i,j} B_{i,j}^\epsilon.$$

Then C^ϵ is a closed subset. Let

$$B^\epsilon = B \cap C^\epsilon.$$

A closed subset of a compact set is compact. So B^ϵ is compact. For any $(x_1, x_2, ..., x_n) \in U^\epsilon$, by definition $|x_i - x_j| \geq \epsilon$ for any $i \neq j$. So $E_n^\alpha(G_n)$ is defined on $B^\epsilon \subset B \setminus D$ for all $\alpha \leq 0$. By the same reason as before, E_n^α has a maximum on B^ϵ.

Once the (ordered) vertices of G_n belong to B^ϵ, we will simply to write $G_n \in B^\epsilon$. Below we assume that $\epsilon < 1$. Let Γ_n^o denote regular n-gon of edge length 1. Then $\Gamma_n^o \in B^\epsilon$. Suppose $E_n^\alpha(G_n^o) = l$.

When $\alpha < 0$, pick ϵ so that $-\epsilon^\alpha < l$. If $G_n \notin B^\epsilon$, then we have some $k \neq m$ such that $|x_k - x_m| < \epsilon$. Therefore

$$E_n^\alpha(G_n) = -\sum_{i<j} |x_i - x_j|^\alpha \leq -|x_k - x_m|^\alpha < -\epsilon^\alpha < l = E_n^\alpha(\Gamma_n^o).$$

When $\alpha = 0$, pick ϵ so that $\ln\epsilon + (\frac{n(n-1)}{2} - 1)\ln n < l$. If $G_n \notin B^\epsilon$, then we have some $k < m$ such that $|x_k - x_m| < \epsilon$. Clearly $|x_i - x_j| \leq n$. Therefore

$$E_n^0(G_n) = \sum_{i<j} \ln|x_i - x_j| =$$

$$\ln|x_k - x_m| + \sum_{i<j,(i,j)\neq(k,m)} \ln|x_i - x_j|$$

$$< \ln\epsilon + (\frac{n(n-1)}{2} - 1)\ln n < l = E_n^\alpha(\Gamma_n^o).$$

In either case, the value of $E_n^\alpha(G_n)$ on $B_D \setminus B^\epsilon$ is bounded by $E_n^\alpha(G_n^o)$. Since G_n^o is in B^ϵ, the maximum value of E_n^α on B^ϵ is the maximum value on $B \setminus D$.

So for each $\alpha \leq 0$ the maximum value of E_n^α exists.

Finally for each $\alpha \in \mathbb{R}$, the maximum value of E_n^α exists. $\square$

3 $E_n^\alpha(G_n)$ maximum implies G_n is Γ_n, a convex n-gon of edge length 1

A subset $X \subset \mathbb{R}^n$ is convex if it contains the line segments connecting each pair of its points. The convex hull of X is the (unique) minimal convex set containing X. Suppose S is a set of finitely many points. The convex hull of S forms a convex polygon if $S \subset \mathbb{R}^2$ and forms a convex polytope if $S \subset \mathbb{R}^3$.

The concept "convex polygon" are often used in two ways: either a 2-dimensional convex polygon, or its 1-dimensional boundary. People usually can understand the means from the context. Some time, we will indicate a convex polygon is 1 or 2 dimensional. We call polygon with n sides a n-gon.

For our purpose, we allow convex polygon to be degenerated. Precisely if the points of S are in a line in $\mathbb{R}^2$, the convex hull is the line segment joining the outermost two points P_1 and P_2. However we will consider it as a degenerated convex polygon, rather than a straight arc. Its boundary is still a closed curve which consists of two coincided straight arcs connecting P_1 and P_2, and the exterior angles at P_1 and P_2 are π.

Some explanations will be helpful: When S is in $\mathbb{R}^2$, we may imagine stretching a rubber band G so that it surrounds the set S and then releasing it, finally G will become a convex polygon which encloses the convex hull of S (see the right of Figure 3). When S is in a line, the rubber band becomes a two stretched segments between the leftmost and rightmost points (see the left of Figure 3).

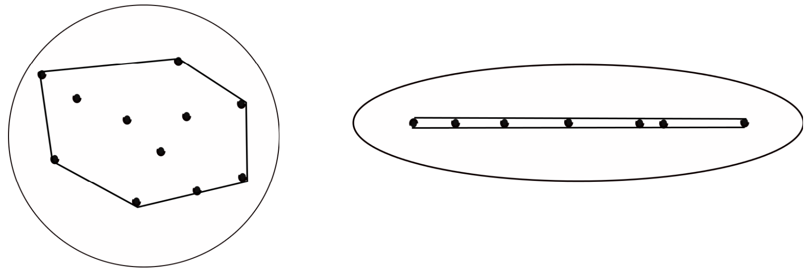


Figure 3

Theorem 3.1. *Suppose $E_n^f(G_n)$ reaches the maximum. Then G_n is a Γ_n, a convex n-gon with edge length 1.*

Recall the vertices $A_1, ..., A_n$ are cyclicly consecutive in G.

Theorem 3.1 follows from the following proposition whose statement gives the steps of the proof.

Proposition 3.2. *Suppose E_n^f reaches the maximum at G_n (the image of some allowable map $g : G_n \to \mathbb{R}^3$). Then*

(i) All vertices of G_n are in the same plane;

(ii) All vertices of G_n are vertices of a convex polygon C.

(iii) There is another Γ_n (the image of another allowable map $g' : G_n \to \mathbb{R}^3$) such that

(a) the vertices of Γ_n and the vertices of G_n are coincided, counting the multiplicity, in particular E_n^f also reaches the maximum at Γ_n;

(b) the vertices of $A_1, ..., A_n$ of Γ_n is cyclicly consecutive in the boundary of the convex polygon C in (ii).

(iv) Γ_n in (iii) is a convex n-gon of edge length 1.

(v) The original G_n is a convex n-gon of edge length 1.

Proof. (i) Let $\bar{C}$ be the convex hull of those vertices of G_n (the edges of G_n usually are not in $\bar{C}$). If those vertices are not contained in any plane, then $\bar{C}$ is a 3-dimensional polyhedron, and we pick a face of $\bar{C}$ and denote the plane containing this face by Π.

Denoted the vertices in Π by $P_1, P_2, \ldots, P_k$ according to their orders in the curve G. Note all remaining vertices are in one side of Π.

If P_1, P_2,..., P_k are not consecutive in G, we may assume that P_1, P_2 are not consecutive in G. Then P_1 and P_2 divide G_n into two parts G' and G'', each part contains some vertices not in Π, see Figure 4. Now reflect G' about Π we get a new distrubution of G_n. To compare with the old distribution, the distances $d(P', P'')$ increases for each vertex P' of G' and P'' of G'' who are not in Π; and the distance of any remaining two vertices are not changed. So for the new distribution $E_n^f(G_n)$ is larger.

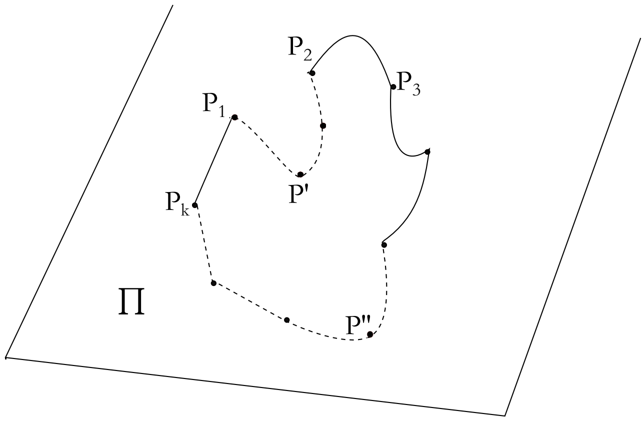


Figure 4

Suppose now P_1, P_2, ..., P_k are consecutive in G. Let C be the convex hull of P_1, P_2,..., P_k in Π. Then ∂C, the boundary of C, is a convex polygon in Π. There are two vertices, say P_i and P_j, consecutive in ∂C but not consecutive in G (otherwise all vertices of G_n are already in Π). Then we can rotate Π along the line L passes P_i and P_j a very small angle so that except P_i and P_j, all vertices of G_n are below Π (note G_n is invariant when we rotate Π), see Figure 5. P_i and P_j divide G_n into two parts G' and G'', each part contains some point not in Π. Now we can repeat the same argument in the last graph to show $E_n^f(G_n)$ can not be the maximum.

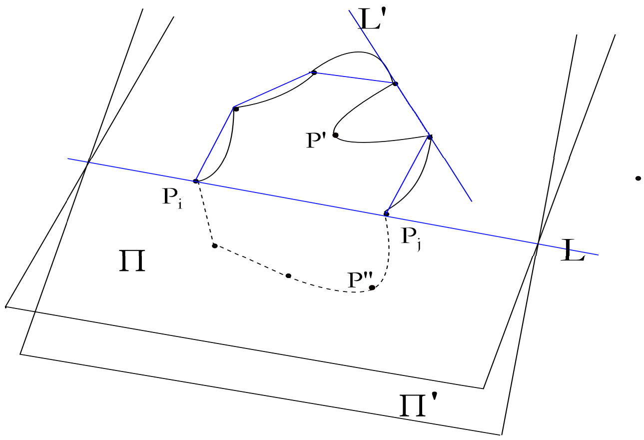


Figure 5

We have proved that all vertices of G_n are in the same plane when $E_n^f(G_n)$ reaches the maximum.

(ii) By (1), we assume now all vertices of G_n are in the plane Π. Suppose some vertex P' of G_n is in the interior of C (still refer Figure 5). Then again some line L' in Π (see Figure 5) contains an edge of C which divides G_n into two parts G' and G'', each part contains some point not in L'. Since the position of edges of G_n do not affects $E_n^f(G_n)$, for convenience, we may assume that G_n is in Π. Reflect G' about L', we can repeat the same argument as in (i) to show $E_n^f(G_n)$ can not be the maximum.

We have proved that all vertices of G_n are vertices of the convex polygon.

Below we will still use $A_1, ..., A_n$ to replace $P_1, ..., P_k$.

(iii) In the conclusion of (ii), the cyclic order of vertices in ∂C usually are not the same as the their cyclic order in G_n, see Figure 6. Also may be $A_i = A_j$ on ∂C, and C can be degenerated, see Figure 8. Recall $E_n^{\alpha}(G_n)$ relies only the positions of all vertices (counting the multiplicity) of G_n. To prove (iii), we first to prove the following

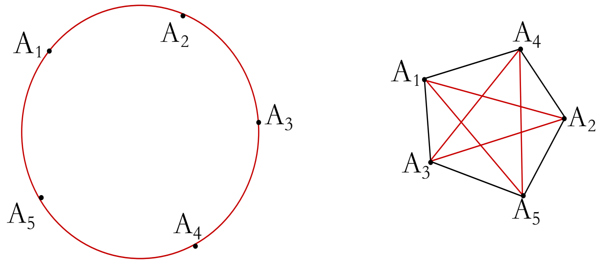


Figure 6

Lemma 3.3. *Suppose C is not degenerated. Then any two consecutive vertices in ∂C has distance no more than 1.*

Proof. In this case ∂C is convex polygon as in Figure 7. Suppose L is a maximum straight arc in ∂C. Then the two end points of L are vertices of ∂C. To prove the lemma, we need only to show that any two consecutive vertices in L has distance no more than 1.

Let S be all vertices of C in L. We first claim the points of S are consecutive in G_n. Precisely, for any two vertices A_i and A_{i+k} are in L, then either $A_{i+1}, ..., A_{i+k-1}$, or $A_{i+k+1}, ..., A_{i-1}$ must be in L. Otherwise we have some $A_j \in \{A_{i+1}, ..., A_{i+k-1}\}$ and $A_l \in \{A_{i+k+1}, ..., A_{i-1}\}$, both A_j and A_l are not in L. Then A_i and A_{i+k} divide G_n into G' and G'' with $A_j \in G'$ and $A_l \in G''$. Both of A_j and A_l must be in one side of L. Just reflect G' about the line containing L, we can argue as before to get $E_n^f(G_n)$ is not the maximum.

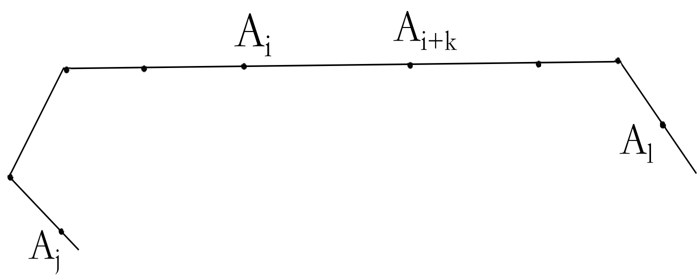


Figure 7

Now we prove that any two consecutive vertices in L has distance no more than 1. Suppose A_i and A_{i+k} are two consecutive vertices in L. We may suppose L is in horizontal position and A_i is on the left of A_{i+k}, see Figure 7. By the claim in the last paragraph, either $A_{i+1}, ..., A_{i+k-1}$, or $A_{i+k+1}, ..., A_{i-1}$ must be in L. We may assume that $A_{i+1}, ..., A_{i+k-1}$ are in L. Let j be the minimal integer such that A_{i+j} is not in the left side of A_{i+k}, $j = 1, ..., k$. Then A_{i+j-1} must be in left side of A_{i+k}. This implies that $A_{i+j-1}A_{i+j}$ covers A_iA_{i+k}. Then $d(A_{i+j-1}, A_{i+j}) \leq 1$ implies that $d(A_i, A_{i+k}) \leq 1$. This finishes the proof of the lemma. $\square$

Remark 3.4. In the last paragraph, we verified if the vertices of G_n on the maximum straight arc L in ∂C are consecutive in G_n, then any two consecutive vertices has distance no more than 1. This fact will also be used for degenerated case.

Suppose C is non-degenerated and the vertices of G_n appear in ∂C consecutively as $Q_1, Q_2, ..., Q_l$ with multiplicity $q_1, q_2, ..., q_l$, $\sum_{i=1}^{l} q_i = n$. By Lemma 3.3, $d(Q_i, Q_{i+1}) \leq 1$. Then there is a non-extensible map $g' : G_n \to \Pi$ which sends the first q_1 vertices A_1,... ,A_{q_1} to Q_1, the next q_2 vertices $A_{q_1+1}, ..., A_{q_1+q_2}$ to Q_2,...., and the last q_l vertices are sent to Q_l. Clearly Γ_n, the image of g', satisfies both (a) and (b) of (iii).

Suppose now C is degenerated. As we discussed in the begin of this section, ∂C consists of two coincided straight arcs C_1 and C_2, and ∂C first travel first along C_1 then along C_2. So it makes sense to about the cyclic order in ∂C in degenerated case.

We may assume that A_1 at one end and A_k at another end. The right-up of Figure 8 illustrates how G_7 maps to C when we view C as a straight arc.

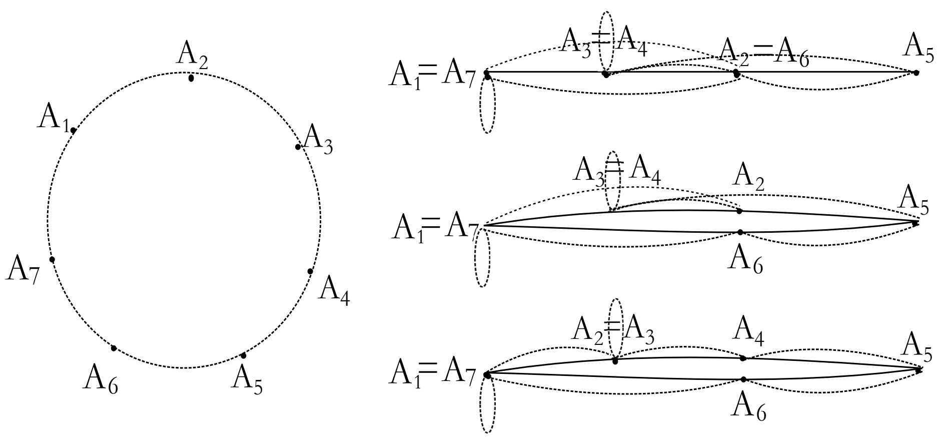


Figure 8

Since C_1 and C_2 are coincide, we assume that the image of all vertices from A_1 to A_k in G_n stay in C_1 and the image of all vertices from A_{k+1} to A_n in G_n stay in C_2. The the right-middle of Figure 8 illustrates how G_7 maps to $\partial C = C_1 \cup C_2$ (in the figure we slightly bend C_1 and C_2 so that their interiors are disjoint).

Suppose the vertices in C_1 from A_1 to A_k appears as $Q_1, Q_2, ..., Q_l$ with multiplicity $q_1, q_2, ..., q_l$, $\sum_{i=1}^{l} q_i = k$. Since all vertices in C_1 (resp. C_2) are consecutive in G_n, by Remark 3.4, we have $d(Q_i, Q_{i+1}) \leq 1$ for $i = 1, ..., l-1$. So there is a non-extensible map $g' : G_n \to \Pi$ which send the part of G_n from A_1 to A_k to C_1 so that the first q_1 vertices are sent to Q_1, the next q_2 vertices are sent to Q_2,...., and the last q_l vertices are sent to Q_l. Then g' maps the vertices in G_n from A_{k+1} to A_n to C_2 in similar way. Clearly Γ_n, the image of g', satisfies the conclusion of (iii).

The right-down of Figure 8 illustrates how G_7 maps to C which preserves the cyclic orders.

(iv) By (iii) we may assume that the vertices of in ∂C are in the cyclic order A_1, A_2,..., A_n.

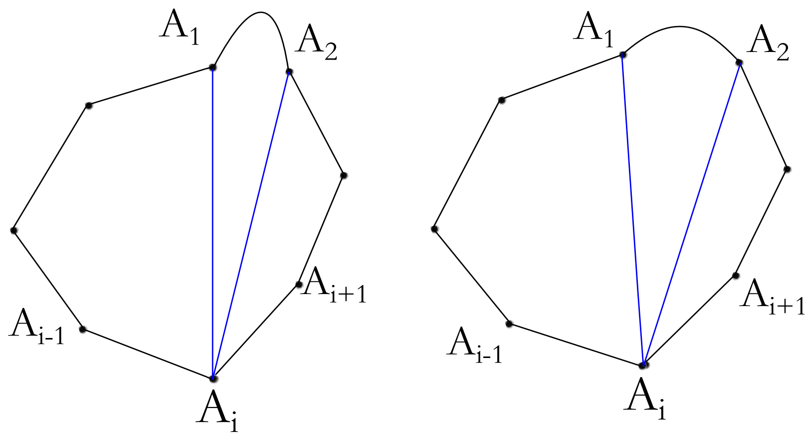


Figure 9

Suppose the distance of two consecutive vertices in Γ_n, say A_1 and A_2, is less than 1, that is the unique edge e of Γ_n connecting A_1 and A_2 is not straight. Let A_i be the vertex such that $d(A_i, A_1)$ is maximum. Then the angle $\angle A_{i-1}A_iA_{i+1}$ must be less than π (otherwise contradicts that $d(A_i, A_1)$ is maximum). Now e and A_i divide Γ_n into two parts G' and G'', G' contains A_1 and G'' contains A_2. Let G^1 be the union of G' and the segment A_1A_i and G^2 be the union of G'' and the segment A_2A_i. Now keep both G^1 and G^2 rigid. Then rotate slightly G^2 around A_i to increase the angle $\angle A_{i-1}A_iA_{i+1}$ slightly but still less then π. We can do this since the unique edge e connecting G^1 and G^2 is not straight. Since each G^1 and G^2 are rigid, and the angle $\angle A_{i-1}A_iA_{i+1}$ is increasing but still less then π, it is easy to see the distance for points in G^1 is not changed, the distance for points in G^2 is not changed, but for each A_k in G^1, A_l in G^2, $k, l \neq i$, the distance $d(A_k, A_l)$ is increasing by using cosine theorem. So $E_n^f(\Gamma_n)$ can not be the maximum.

We have proved (iv), that is Γ_n is a convex n-gon of edge length 1.

(v) The idea of the verification is easy:

1. Γ_n is a convex n-gon C with edge length 1, the length of ∂C is n.

2. ∂C is the unique shortest loop passing all vertices of Γ_n.

3. The vertices of G_n are coincided the vertices of Γ_n, and and G_n is a loop of length n.

So G_n must be coincided with ∂C, and indeed G_n must be coincided with Γ_n as a polygon.

We finish the verification of (v), therefore the proof of the proposition. $\square$

When G_n is a (1-dimensional) convex n-gon with edge length 1, we will denote G_n by Γ_n and G by Γ.

4 Basic facts, the classification for $n = 4$

From now on, we always assume that Γ_n is a convex n-gon with each edge of length 1 in $\mathbb{R}^2$. We often use $\max E_n^\alpha$ to denote the maximum value of E_n^α below.

Suppose Γ_n has vertices $A_1, ..., A_n$ and the exterior angle at A_i is θ_i. There are two extreme shapes for Γ_n: one is the regular n-gon, denoted as Γ_n^o, which can be defined by $\theta_1 = ... = \theta_n$; another the double straight arc, defined for only $n = 2m$, denoted as Γ_n^-, which can be defined by either $\theta_i = \theta_{i+m} = \pi$ for some i, or some diagonal has length m. Γ_{2m}^- is shown in Figure 10.

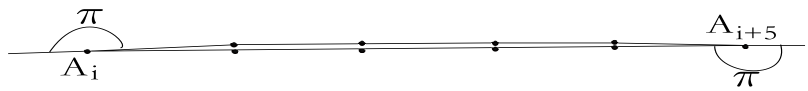


Figure 10

Proposition 4.1. *(1) (Jensen inequality) Suppose f is a convex function on $[a, b]$, $\theta_i \in [a, b]$. Then*

$$\frac{\sum_{i=1}^n f(\theta_i)}{n} \leq f(\frac{\sum_{i=1}^n \theta_i}{n}),$$

and the equality holds if and only if $\theta_1 = \theta_2 = ... = \theta_n$.

(2) (Karamata inequality) Suppose g is a concave function on $[a, b]$ and there are n variables $x_1, x_2, ..., x_n \in [a, b]$ with a fixed sum. Then the value $\sum_{i=1}^n g(x_i)$ reaches the maximum if and only if at least n − 1 variables are at endpoints.

Lemma 4.2. *$f_\alpha(x)$ is an increasing function; furthermore $f_a(x)$ is convex when $\alpha < 1$ and is concave when $a > 1$.*

Proof. First calculate the first derivative of f_α:

$$f'_\alpha(x) = \begin{cases} \alpha x^{\alpha-1}, & \alpha > 0; \\ 1/x, & \alpha = 0; \\ -\alpha x^{\alpha-1}, & \alpha < 0. \end{cases}$$

f'_α is always positive, hence f^α is an increasing function.

Then calculate the second derivative of f_α

$$f''_\alpha(x) = \begin{cases} \alpha(\alpha-1)x^{\alpha-2}, & \alpha > 0; \\ -1/x^2, & \alpha = 0; \\ -\alpha(\alpha-1)x^{\alpha-2}, & \alpha < 0. \end{cases}$$

f''_α is negative when $\alpha < 1$, hence f_α is convex when $\alpha < 1$. f''_α is positive when $\alpha > 1$, hence f_α is concave when $\alpha > 1$. □

The following decomposition of E_n^α plays significant roles in this note.

$$E_n^\alpha(\Gamma_n) = \sum_{k=1}^{[n/2]} \mu_{n,k} E_{n,k}^\alpha(\Gamma_n), \qquad (4.1)$$

where

$$E^{\alpha}_{n,k}(\Gamma_n) = \sum_{i=1}^{n} f_{\alpha}(|A_i - A_{i+k}|). \qquad (4.2)$$

where $\mu_{n,k} = 1/2$ if n is even and $k = n/2$ and $= 1$ for the remaining cases.

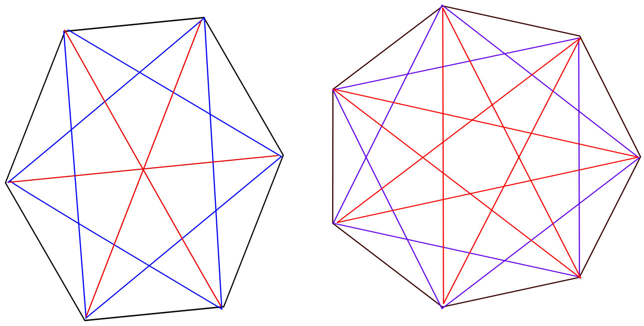


Figure 11

In Figure 11, the interactions of $E^{\alpha}_{n,k}$ along the black lines for $(n,k) = (6,1),(7,1)$, along the blue lines for $(n,k) = (6,2),(7,2)$, and along the red lines for $(n,k) = (6,3),(7,3)$.

Note $E^{\alpha}_{n,1}(\Gamma_n)$ is a constant. Precisely

$$E^{\alpha}_{n,1}(\Gamma_n) = \sum_{i=1}^{n} f^{\alpha}(|A_i - A_{i+1}|) = \begin{cases} n, & \alpha > 0; \\ 0 & \alpha = 0; \\ -n, & \alpha < 0. \end{cases} \qquad (4.3)$$

Some times it is more brief just to consider $\bar{E}^{\alpha}_{n}(\Gamma_n) = \sum_{k=2}^{[n/2]} E^{\alpha}_{n,k}(\Gamma_n)$.

Example 4.3. We will classify when Γ_4 realizing $\max E^{\alpha}_{4}$.

As we just discussed, $E^{\alpha}_{4}(\Gamma_4) = E^{\alpha}_{4,1}(\Gamma_n) + E^{\alpha}_{4,2}(\Gamma_n)$ and $E^{\alpha}_{4,1}(\Gamma_n)$ is a constant for given (n,α). So we need only to classify when Γ_4 realizing $\max E^{\alpha}_{4,2} = \bar{E}^{\alpha}_{4}$.

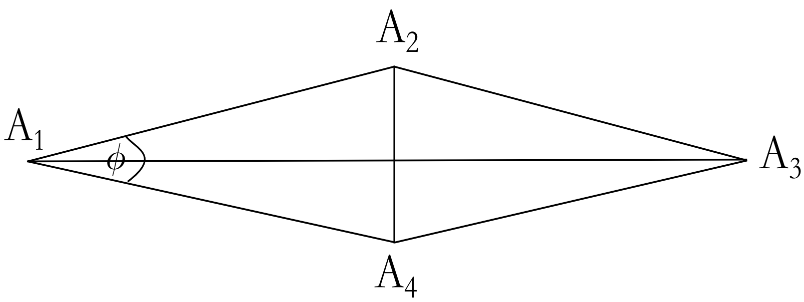


Figure 12

Let the inner angle at A_1 be ϕ. Then Γ_4 is determined by ϕ, and we have

$$\begin{aligned} E^{\alpha}_{4,2}(\Gamma_4) &= |A_1A_3|^{\alpha} + |A_2A_4|^{\alpha} \\ &= (2\cos\phi/2)^{\alpha} + (2\sin\phi/2)^{\alpha} \end{aligned}$$

$$= 2^\alpha (\cos^2 \phi/2)^{\alpha/2} + (\sin^2 \phi/2)^{\alpha/2})$$
$$= 2^\alpha (t^{\alpha/2} + (1-t)^{\alpha/2})$$

where $t = \cos^2 \phi/2$.

Note f_α is a convex function if $\alpha < 1$ and a concave function if $\alpha > 1$ by Lemma 4.2 .

If $\alpha < 2$, then $\alpha/2 < 1$, we can apply by Jenson inequality to get that $E^\alpha_{4,2}(\Gamma_4)$ reached the maximum if and only if $t = 1/2$, that is $\cos^2 \phi/2 = 1/2$, that is $\phi = \pi/2$ and therefore $E^\alpha_{4,2}(\Gamma_4)$ reaches the maximum if and only if $\Gamma_4 = \Gamma^o_n$. Moreover $E^\alpha_{4,2}(\Gamma_4) = 2^\alpha(\frac{1}{2}^{\alpha/2} + \frac{1}{2}^{\alpha/2}) = 2^{\alpha/2+1}$ if $\alpha > 0$, $E^\alpha_{4,2}(\Gamma_4) = -2^{\alpha/2+1}$ if $\alpha < 0$ and $E^0_4(\Gamma^o_n) = \ln 2$.

If $\alpha > 2$, then $\alpha/2 > 1$, we can apply Karamata inequality to get $E^\alpha_{4,2}(\Gamma_4)$ reached the maximum if and only if $t = 0$ or 1, that is $\cos^2 \phi/2 = 0$ or 1, that is $\phi = 0$ or π, and therefore $E^\alpha_{4,2}(\Gamma_4)$ reaches the maximum if and only if $\Gamma_4 = \Gamma^-_n$. Moreover $E^\alpha_{4,2}(\Gamma^-_4) = 2^\alpha$.

When $\alpha = 2$, then $\alpha/2 = 1$, and $E^\alpha_4(\Gamma_4)$ is constant 8.

By the discussion above and (4.3) we have the following classification for $n = 4$:

$$max E^\alpha_4(\Gamma_4) = \begin{cases} 4 + 2^\alpha, & \Gamma_4 = \Gamma^-_4 & \alpha > 2; \\ 8 & \text{any } \Gamma_4 & \alpha = 2 \\ 4 + 2^{\frac{\alpha}{2}+1} & \Gamma_4 = \Gamma^o_4 & 0 < \alpha < 2; \\ \ln 2, & \Gamma_4 = \Gamma^o_4 & \alpha = 0. \\ -4 - 2^{\frac{\alpha}{2}+1} & \Gamma_4 = \Gamma^o_4 & \alpha < 0. \end{cases} \tag{4.4}$$

Example 4.4. This example provides some solutions of $E^{\alpha_n}_n(\Gamma^o_n) = E^{\alpha_n}_n(\Gamma^-_n)$ for $n = 6, 8$.

$$E^2_6(\Gamma^o_6) = 36 > 33 = E^2_6(\Gamma^-_6)$$
$$E^3_6(\Gamma^o_6) < 63 < 67 = E^3_6(\Gamma^-_6).$$

$E^{\alpha_6}_6(\Gamma^o_6) = E^{\alpha_6}_6(\Gamma^-_6)$ for some $\alpha_6 \in (2.5525, 2.5529)$.

$$E^2_8(\Gamma^o_8) > 109 > 97 = E^2_8(\Gamma^-_8)$$
$$E^3_8(\Gamma^o_8) < 243 < 248 = E^3_6(\Gamma^-_6).$$

$E^{\alpha_8}_8(\Gamma^o_8) = E^{\alpha_8}_8(\Gamma^-_8)$ for some $\alpha_8 \in (2.878, 2.879)$.

5 When the regular n-gon Γ^o_n realizing maxE^α_n

5.1 Γ^o_n realizing maxE^α_n if Γ_n not bending fast for $\alpha \leq 1$

Let $[x]$ be the maximum integer not bigger than x.

Theorem 5.1. *Suppose the sum of any consecutive $[n/2] - 1$ exterior angles of Γ_n is no more than π. Then $E^\alpha_n(\Gamma_n)$ is the maximum if and only if $\Gamma_n = \Gamma^o_n$, the regular n-gon of edge length 1 for $\alpha \leq 1$.*

Definition 5.2. Suppose Γ_n is a convex n-gon with each edge of length 1. Say Γ_n satisfying the condition $k*$, if the sum of any consecutive $k-1$ exterior angles is no more than π.

The proof of Theorem 5.1 follows from the following results whose statement gives the steps of the proof.

Proposition 5.3. *(1) Suppose Γ_n satisfies the condition $k*$, $1 < k \le [n/2]$. Then $E^1_{n,k}(\Gamma_n)$ is maximum if and only if $\Gamma_n = \Gamma^o_n$.*

(2) Suppose Γ_n satisfies the condition $k*$, $1 < k \le [n/2]$. Then $E^\alpha_{n,k}(\Gamma_n)$ is maximum if and only if $\Gamma_n = \Gamma^o_n$ for all $\alpha < 1$.

(3) Suppose Γ_n satisfies the condition $[n/2]*$. Then $E^\alpha_n(\Gamma_n)$ is maximum if and only if $\Gamma_n = \Gamma^o_n$ for all $\alpha \le 1$.

Proof. (1) We often use $E_{n,k}$ to denote $E^1_{n,k}$ in the proof.

Suppose Γ_n have vertices $A_1, ..., A_n$ in the clockwise order, and the exterior angle at A_i is θ_i. Then $\sum_{i=1}^n \theta_i = 2\pi$.

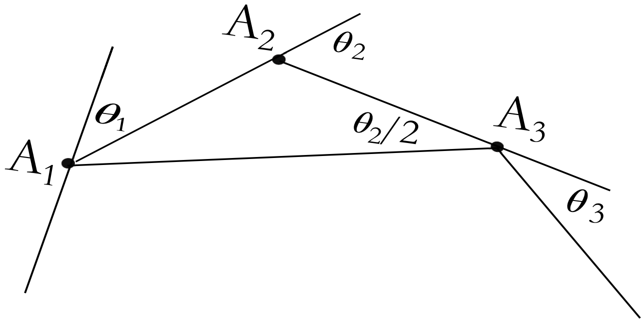


Figure 13

Form now on, for two vectors A and B, we will often use AB to denote $B - A$, the vector from A to B.

We first prove (1) for $k = 2$: Since $|A_1A_2| = |A_2A_3| = 1$, it is easy to see the angle $\angle A_1A_3A_2$ is $\theta_2/2$, see Figure 5, and therefore $|A_1A_3| = 2\cos\frac{\theta_2}{2}$. Similarly we have

$$|A_iA_{i+2}| = 2\cos\frac{\theta_{i+1}}{2}. \qquad (5.1)$$

Then

$$E_{n,2}(\Gamma_n) = \sum_{i=1}^{n} |A_iA_{i+2}| = \sum_{i=1}^{n} 2\cos\frac{\theta_{i+1}}{2}. \qquad (5.2)$$

Since Γ_n is convex, each $\theta_i \le \pi$, so $\theta_i/2 \le \pi/2$, so Γ_n satisfies the condition $2*$. Since the function $\cos x$ is convex on the interval $[0, \pi/2]$, by Proposition 4.1 (1), $E_{n,2}$ is maximum if and only if that $\theta_1 = ... = \theta_n$, that is Γ_n is the regular n-gon Γ^o_n.

To make the proof more clear, now we prove (1) for $k = 3$.

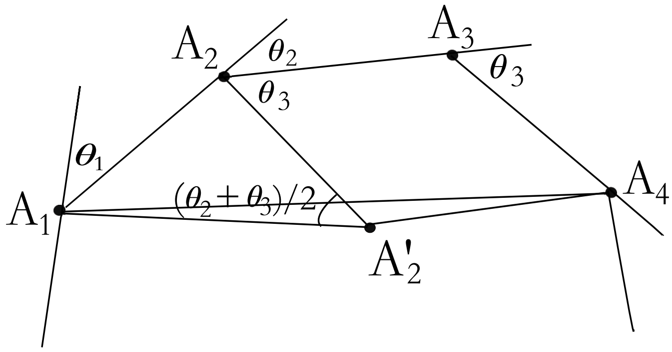


Figure 14

Parallel shift A_2A_3 to $A_2'A_4$, see Figure 5. Then $|A_2'A_4| = 1$ and $|A_1A_2'| = 2\cos(\theta_2+\theta_3)/2$ as we see from the proof of $k = 2$.

$$|A_1A_4| = |A_1A_2' + A_2'A_4| \leq |A_1A_2'| + |A_2'A_4| = 2\cos\frac{\theta_2+\theta_3}{2} + 1 \qquad (5.3)$$

Similarly we have

$$|A_iA_{i+3}| \leq 2\cos\frac{\theta_{i+1}+\theta_{i+2}}{2} + 1 \qquad (5.4)$$

$$E_{n,3}(\Gamma_n) = \sum_{i=1}^{n} |A_iA_{i+3}| \leq \sum_{i=1}^{n}(2\cos\frac{\theta_{i+1}+\theta_{i+2}}{2} + 1) \qquad (5.5)$$

Since Γ_n satisfies the condition 3∗, each $\theta_{i+1}+\theta_{i+2} \leq \pi$, so $(\theta_{i+1}+\theta_{i+2})/2 \leq \pi/2$. Note $\sum_{i=1}^{n}(\theta_{i+1}+\theta_{i+2})/2 = 2\pi$. Since the function $\cos x$ is convex on the interval $[0, \pi/2]$, by Proposition 4.1 (1), $E_{n,3}$ is maximum if and only if that $(\theta_i+\theta_{i+1})/2$ are the same for all i. This is true when $\Gamma_n = \Gamma_n^o$. Moreover when $\Gamma_n = \Gamma_n^o$, the $\leq$ in (5.5) becomes $=$. So $E_{n,3}(\Gamma_n)$ reaches the maximum when $\Gamma_n = \Gamma_n^o$. On the other hand, if $E_{n,3}(\Gamma_n)$ reaches the maximum, then the $\leq$ in (5.4) must be $=$, for example $i = 1$, the $\leq$ in (5.3) must be $=$, which implies that A_1, A_2', A_4 are in the same line, which implies A_1A_2' is parallel to A_2A_3, which implies $\frac{\theta_2+\theta_3}{2} = \theta_3$, that is $\theta_2 = \theta_3$. Similarly, for each i we have $\theta_i = \theta_{i+1}$, that is to say $\theta_1 = ... = \theta_n$, that is $\Gamma_n = \Gamma_n^o$. We have proved that $E_{n,3}(\Gamma_n)$ reaches the maximum if and only if $\Gamma_n = \Gamma_n^o$.

Now we assume (1) is proved for positive integers $\leq k-1$, and we will prove it for k. Parallel shift the path $A_2A_3......A_{k-1}A_k$ to the path $A_2'A_3'......A_{k-1}'A_{k+1}$ as in Figure 15. Then

$$\begin{aligned}|A_1A_{k+1}| &= |A_1A_2' + A_2'A_{k+1}| \\ &\leq |A_1A_2'| + |A_2'A_{k+1}| = |A_1A_2'| + |A_2A_k| \qquad (5.6)\end{aligned}$$

As we observed for the cases for $k = 2, 3$ we have

$$|A_1A_2'| = 2\cos\frac{\sum_{i=2}^{k}\theta_i}{2}$$

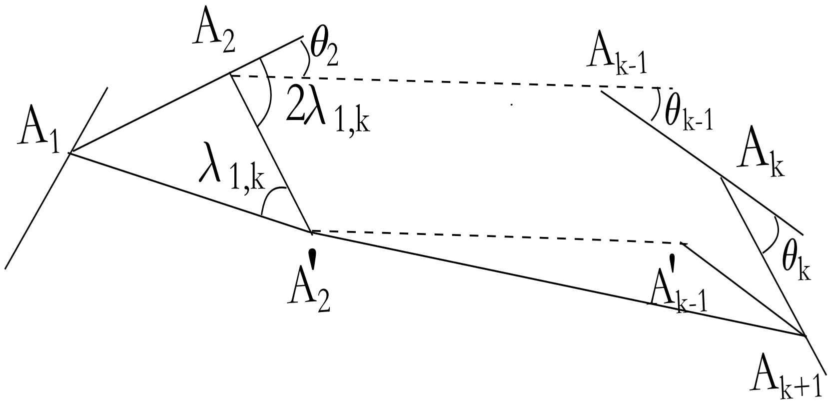


Figure 15

So we have

$$|A_1A_{k+1}| \leq 2\cos\frac{\sum_{j=2}^{k}\theta_j}{2} + |A_2A_k|$$

Similarly we have

$$|A_iA_{k+i}| \leq 2\cos\frac{\sum_{j=i+1}^{k+i-1}\theta_j}{2} + |A_{i+1}A_{i+k-1}| \qquad (5.7)$$

Let

$$\lambda_{i,k} = \frac{1}{2}\sum_{j=i+1}^{k+i-1}\theta_j. \qquad (5.8)$$

Then

$$E_{n,k}(\Gamma_n) = \sum_{i=1}^{n}|A_iA_{i+k}| \leq 2\sum_{i=1}^{n}\cos\lambda_{i,k} + \sum_{i=1}^{n}|A_{i+1}A_{i+k-1}|.$$

$$= 2\sum_{i=1}^{n}\cos\lambda_{i,k} + E_{n,k-2}(\Gamma_n). \qquad (5.9)$$

Since Γ_n satisfies the condition $k*$, we have $\sum_{j=i+1}^{k+i-1}\theta_j \in [0,\pi]$, and hence

$$\lambda_{i,k} \in [0,\pi/2].$$

Since $\sum_{i=1}^{n}\theta_i = 2\pi$, we have

$$\sum_{i=1}^{n}\lambda_{i,k} = (k-1)\pi. \qquad (5.10)$$

Since the function $\cos x$ is convex on $[0,\pi/2]$, by Propostion 4.1 (1), $2\sum_{i=1}^{n}\cos\lambda_{i,k}$ obtains the maximum if and only if all $\lambda_{i,k}$ are equal, which is true when $\Gamma_n = \Gamma_n^o$. Moreover the $\leq$ in (5.6) and in (5.9) become $=$ when $\Gamma_n = \Gamma_n^o$. So $E_{n,k}(\Gamma_n)$ reaches the maximum when $\Gamma_n = \Gamma_n^o$.

Moreover Γ_n satisfying the condition $k*$ implies that Γ_n satisfying the condition $(k-2)*$. By induction hypothesis, $E_{n,k-2}(\Gamma_n)$ is maximum if and only if Γ_n is the regular n-gon.

By (5.9), $E_{n,k}(\Gamma_n)$ obtained the maximum if and only if Γ_n is the regular n-gon. We have proved (1).

(2) The proof relies Proposition 5.3 (1) and Lemma 4.2 and next two Lemmas, which will be also used in other places.

Lemma 5.4. *Suppose Q is a subset of $\mathcal{M}_n$ and $\Gamma_n^o \in Q$. Suppose (i) $v_1, ..., v_n$ are n values come from $\Gamma_n \subset \mathbb{R}^3$, and $v_1 = ... = v_n$ if $\Gamma_n = \Gamma_n^o$, (ii) for some function g, $g(\Gamma_n) = \sum_{i=1}^n g(v_i)$ reaches the maximum on Q if and only if $\Gamma_n = \Gamma_n^o$. Then $hg(\Gamma_n) = \sum_{i=1}^n h \circ g(v_i)$ reaches the maximum on Q if and only if $\Gamma_n = \Gamma_n^o$, where h is a convex increasing function.*

Proof. Since h is convex, by Lemma 4.2 (1) we have

$$\frac{\sum_{i=1}^n h(g(v_i))}{n} \leq h(\frac{\sum_{i=1}^n g(v_i)}{n}) \qquad (5.11)$$

Suppose the maximum of $g(\Gamma_n)$ on Q is M_o. Since h is increasing, we have

$$h(\frac{\sum_{i=1}^n g(v_i)}{n}) \leq h(\frac{M_o}{n}) \qquad (5.12)$$

Then

$$\sum_{i=1}^n h(g(v_i)) \leq nh(\frac{M_o}{n}) \qquad (5.13)$$

If $\Gamma_n = \Gamma_n^o$, then $v_1 = ... = v_n$, hence $g(v_1) = ... = g(v_n)$. Then (5.11) becomes an equality by Lemma 4.2 (1) and the convexity of h, and (5.12) becomes an equality by the assumption on g. So (5.13) becomes an equality. That is to say on Q, hg reaches the maximum $nh(\frac{M_o}{n})$ at Γ_n^o.

If (5.13) becomes an equality for some $\Gamma_n \in Q$, (5.12) must also become an equality. Since h is increasing, we have $g(\Gamma_n) = \sum_{i=1}^n g(v_i)$ reaches the maximum M_o. Since $\Gamma_n \in Q$, $\Gamma_n = \Gamma_n^o$ by the assumption. □

Lemma 5.5. *Suppose Q is a subset of $\mathcal{M}_n$ and $\Gamma_n^o \in Q$. If $E_{n,k}^{\alpha}(\Gamma_n)$ reaches the maximum on Q if and only $\Gamma_n = \Gamma_n^o$, then $E_{n,k}^{\beta}(\Gamma_n)$ reaches the maximum on Q if and only $\Gamma_n = \Gamma_n^o$ for $\beta < \alpha$.*

Proof. By Lemma 4.2 $f_\alpha(x)$ is an increasing function; furthermore $f_\alpha(x)$ is convex when $\alpha < 1$. Recall

$$E_{n,k}^{\alpha}(\Gamma_n) = \sum_{i=1}^n f_\alpha(|A_i - A_{i+k}|).$$

We may assume that $\alpha > 0$ (we only use the case $\alpha > 0$, and the other cases are simpler). Then we define $g(x) = f_\alpha(x)$ and

$$h(x) = \begin{cases} x^{\beta/\alpha} = f_{\beta/\alpha}(x), & \beta > 0; \\ ln(x^{1/\alpha}) = 1/\alpha f_{\beta/\alpha}(x), & \beta = 0; \\ -x^{\beta/\alpha} = f_{\beta/\alpha}(x),, & \beta < 0. \end{cases} \tag{5.14}$$

Then one can verify

$$E^{\beta}_{n,k}(\Gamma_n) = \sum_{i=1}^{n} f_\beta(|A_i - A_{i+k}|) = \sum_{i=1}^{n} h \circ f_\alpha(|A_i - A_{i+k}|).$$

Since $\beta < \alpha$, in each case we have $\beta/\alpha \leq 1$ and therefore h is convex and increasing. Clearly $v_i = |A_i - A_{i+k}|$ meets the condition (i) and g meet the condition (ii) in Lemma 5.4 by our assumption. Then by Lemma 5.4, $E^{\beta}_{n,k}(\Gamma_n)$ reaches the maximum if and only $\Gamma_n = \Gamma^o_n$ for $\beta < \alpha$. □

Now we are going to prove Proposition 5.3 (2).

Let Q be a subset of $\mathcal{M}_n$ defines by that Γ_n satisfies the condition $k*$. By Proposition 5.3 (1), $E^1_{n,k}(\Gamma_n) = \sum_{i=1}^{n} g(|A_i - A_{i+k}|)$ reaches the maximum if and only if $\Gamma_n = \Gamma^o_n$, then by Lemma 5.5, $E^{\alpha}_{n,k}(\Gamma_n)$ reaches the maximum if and only $\Gamma_n = \Gamma^o_n$ for $\alpha \leq 1$.

(3) Suppose Γ_n satisfies the condition $[n/2]*$. Then Γ_n satisfies the condition $k*$, $1 < k \leq [n/2]$.

If $\Gamma_n \neq \Gamma^o_n$, then $E^{\alpha}_{n,k}(\Gamma_n) < E^{\alpha}_{n,k}(\Gamma^o_n)$ for $\alpha \leq 1$ by Proposition 5.3 (1) and (2). So $E^{\alpha}_n(\Gamma_n) = \sum_{k=1}^{[n/2]} \mu_{n,k} E^{\alpha}_{n,k}(\Gamma_n)$ reaches the maximum if and only Γ_n is the regular n-gon. for $\alpha \leq 1$. □

5.2 Γ^o_n realizing maxE^α_n for large negative α, more corollaries

Theorem 5.6. *For given n, there is an $\alpha_* < 0$ (depends on n) such that $E^{\alpha}_n(\Gamma_n)$ reaches the maximum if and only if Γ_n is the regular n-gon for $\alpha < \alpha_*$.*

Proof. For each Γ_n in the neighborhood U_n of Γ^o_n defined below

$$U_n = \{\Gamma_n | \theta_i \leq \frac{\pi}{[n/2]-1} \text{for all exterior angle } \theta_i \text{ of } \Gamma_n\},$$

we have proved that $E^{\alpha}(\Gamma_n)$ is no more than $E^{\alpha}(\Gamma^o_n)$ for any $\alpha < 0$, and $E^{\alpha}(\Gamma_n) = E^{\alpha}(\Gamma^o_n)$ if and only if $G_n) = \Gamma^o_n$.

Let $\alpha < 0$. Suppose $\Gamma_n \notin U_n$. We may assume that $\theta_1 > \frac{\pi}{[n/2]-1}$. Then (see the proof of Proposition 5.3 (1))

$$|A_n A_2| = 2\cos\theta_1/2 < 2\cos\frac{\pi}{2([n/2]-1)}$$

Hence $\bar{E}^{\alpha}_n(\Gamma_n) \leq -|A_n A_2|^{\alpha} < -(2\cos\frac{\pi}{2([n/2]-1)})^{\alpha}$

On the other hand, for the regular n-gon Γ^o_n, let $A_i A_j$ be one of $\frac{n(n-3)}{2}$ diagonals, then $|A_i A_j| \geq |A_n A_2|$, hence $|A_i A_j|^{\alpha} \leq |A_n A_2|^{\alpha}$, since $\alpha < 0$. Then

$$\bar{E}_n^\alpha(\Gamma_n^o) \geq -\frac{n(n-3)}{2}|A_nA_2|^\alpha = -\frac{n(n-3)}{2}(2\cos\frac{\pi}{n})^\alpha.$$

To show $\bar{E}_n^\alpha(\Gamma_n) < \bar{E}_n^\alpha(\Gamma_n^o)$ for $\alpha < 0$ small enough, we need only to find $\alpha < 0$ such that

$$-(2\cos\frac{\pi}{2([n/2]-1)})^\alpha < -\frac{n(n-3)}{2}(2\cos\frac{\pi}{n})^\alpha,$$

which is equivalent to

$$(\frac{\cos\frac{\pi}{2([n/2]-1)}}{\cos\frac{\pi}{n}})^\alpha > \frac{n(n-3)}{2},$$

which is equivalent to

$$(\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2([n/2]-1)}})^\alpha < \frac{2}{n(n-3)},$$

which is equivalent to

$$\alpha < \log_{\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2([n/2]-1)}}}\frac{2}{n(n-3)} = -\log_{\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2([n/2]-1)}}}\frac{n(n-3)}{2}$$

Let $\alpha_* = -\log_{\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2([n/2]-1)}}}\frac{n(n-3)}{2}$. Then any $\alpha < \alpha_*$, the conclusion of Theorem 5.6 holds. □

Remark 5.7. Using Taylor expansion, we can get more concrete estimation of α_* for larger n. We first rewrite α_* as

$$-\alpha_* = \frac{\ln\frac{n(n-3)}{2}}{\ln\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{(n-m)}}} = \frac{\ln\frac{n(n-3)}{2}}{\ln\cos(\pi/n) - \ln\cos(\pi/(n-m))},$$

where m is 2 if n even and is 3 if n odd.

Recall Taylor expansion:

$$\cos x = 1 - \frac{x^2}{2!} + ... + (-1)^n\frac{x^{2n}}{(2n)!} + (-1)^{n+1}\frac{\cos\eta}{(2n)!}x^{2n}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - .... + (-1)^n\frac{x^{n-1}}{n!} + (-1)^n\frac{x^{n+1}}{(n+1)(1+\eta)^{n+1}}$$

where $\eta \in [0, x]$.

First we have $\ln\frac{n(n-3)}{2} = \ln n + \ln(n-3) - \ln 2$ which is appoximate $2\ln n$ for large n. Using Taylor expansions above, we have

$$\ln\cos(\pi/n) - \ln\cos(\pi/(n-m)$$

$$= \ln(1 - \frac{1}{2}(\pi/n)^2 + O(\frac{1}{n^4})) - \ln(1 - \frac{1}{2}(\pi/(n-m))^2 + O(\frac{1}{n^4}))$$
$$= \frac{1}{2}(\pi/(n-m))^2 - \frac{1}{2}(\pi/n)^2) + O(\frac{1}{n^4})$$
$$= \frac{1}{2}\pi^2(1/(n-m))^2 - (1/n)^2) + O(\frac{1}{n^4}) = \pi^2 n^{-3} m + O(\frac{1}{n^4})$$

Therefore α_* is approximately

$$-\frac{2\ln n}{\pi^2 n^{-3} m} = -\frac{2n^3 \ln n}{\pi^2 m},$$

for large n, and more precisely which is approximately

$$-\frac{1}{\pi^2} n^3 \ln n \text{ when } n \text{ even and } -\frac{2}{3\pi^2} n^3 \ln n \text{ when } n \text{ odd.}$$

.

We give another two corollaries of Theorem 5.1

If Γ_n is central symmetry, clearly Γ_n satisfies the condition $[n/2]*$.

Corollary 5.8. *Suppose Γ_n is central symmetry. $E_n^\alpha(\Gamma_n)$ reaches the maximum if and only if Γ_n is the regular n-gon for $\alpha \leq 1$.*

Corollary 5.9. *Γ_n^o realizing $maxE_n^\alpha$ for $n = 5, 6$ and $\alpha \leq 1$*

Proof. For $n = 5$, $[5/2]* = 2*$, and each convex polygon satisfies condition 2*.

For $n = 6$, $[6/2]* = 3*$ and the condition 3* implies the sum of any two consecutive exterior angle is no more than π. This fact is included in the next lemma.

Lemma 5.10. *The sum of any two consecutive exterior angles of Γ_n is no more than π when n even.*

Proof. To prove the lemma, we need the following fact in plane geometry:

(*) Suppose $ABCD$ is a 4-gon shown as in Figure 16, where $AD = CB$ and $\theta_1, \theta_2 \in (0, \pi)$, $\theta_1 + \theta_2 \in (\pi, 2\pi)$, then $|AB| > |DC|$.

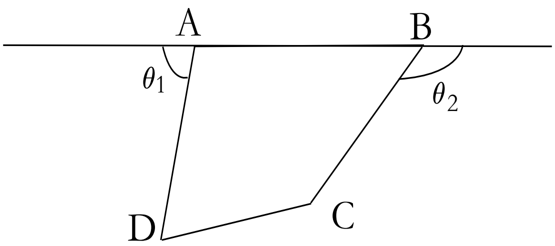
There are should be many proofs for (*). One can calculate that

$$|CD|^2 = |AB|^2 + 4|AD|^2 \cos\frac{\theta_1 + \theta_2}{2} \cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2}$$

Since $\theta_1/2, \theta_2/2 \in (0, \pi/2)$, so $\cos\theta_1 > 0$, $\cos\theta_1 > 0$, and $\frac{\theta_1+\theta_2}{2} \in (\pi/2, \pi)$, $\cos\frac{\theta_1+\theta_2}{2} < 0$, that is $|AB| > |DC|$.

Suppose the lemma is not true, we may write $n = 2m$ and assume that the sum of two exterior angles at A_1 and A_{2m}, is more than π. See Figure 16. So we have $\angle A_2 A_1 A_{2m} + \angle A_1 A_{2m} A_{2m-1} < \pi$. From the convexity of Γ_n, one can verify that in general

$$\angle A_{i+1} A_i A_{2m-i+1} + \angle A_i A_{2m-i+1} A_{2m-i} < \pi, \quad i = 1, ..., m-1$$



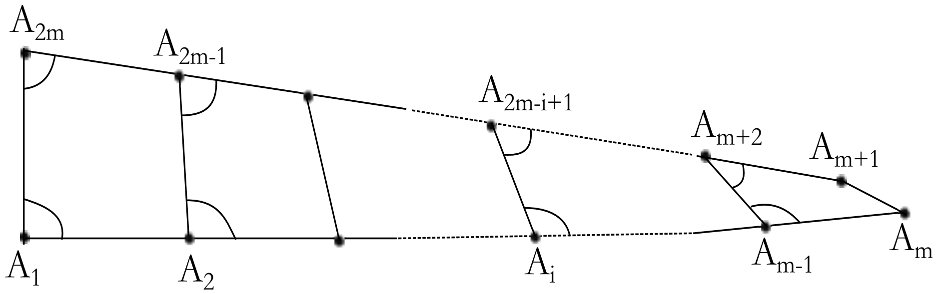


Figure 16

Clear we also have

$$|A_iA_{i+1}| = |A_{2m-i+1}A_{2m-i+2}| = 1, \;\; i = 1, ..., m-1$$

Now apply fact (*) to each 4-side gon $A_{i+1}A_iA_{2m-i+1}A_{2m-i+2}$ inductively, $i = 1, ..., m-1$, we get

$$|A_1A_{2m}| > |A_2A_{2m-1}| > .... > |A_{m-1}A_{m+2}| > |A_mA_{m+1}|.$$

This contradicts that $A_1A_{2m} = 1 = A_mA_{m+1}$. $\square$

$\square$

6 Which Γ_n realizing maxE_n^α for large positive α

6.1 For even n, the double straight arc Γ_n^- realizing maxE_n^α for large $\alpha > 0$

Suppose n even is given. For a given $\Gamma_n \neq \Gamma_n^-$, since Γ_n^- has a diagonal of length $n/2$ and all diagonals of Γ_n has length smaller than $n/2$, it is not hard to see there exist large positive α such that $E_n^\alpha(\Gamma_n^-) > E_n^\alpha(\Gamma_n)$. However the following result claim there is α such that $E_n^\alpha(\Gamma_n^-) > E_n^\alpha(\Gamma_n)$ for any $\Gamma_n \in \prod_n$, $\Gamma_n \neq \Gamma_n^-$.

Theorem 6.1. *For given even $n > 0$, there is an α^* such that $E_n^\alpha(\Gamma_n)$ reaches the maximum implies that $\Gamma_n = \Gamma_n^-$ when $\alpha > \alpha^*$.*

Theorem 6.1 follows from Propositon 6.2 and Propositon 6.3 below.

Simply speaking, Proposition 6.2 claims that for any "δ-neighborhood" of Γ_n^-, there is $\alpha_1(\delta) > 0$ such that Γ_n which realizes maxE_n^α must be in this neighborhood when $\alpha > \alpha_1$; Proposition 6.3 claims that there is a "δ-neighborhood" of Γ_n^- and $\alpha_2(\delta) > 0$, such that Γ_n which realizes maxE_n^α must be Γ_n^- when Γ_n in this neighborhood and $\alpha > \alpha_2$.

Proposition 6.2. *For a given even n and $0 < \delta < 1$, there exists $\alpha_1 = \alpha_1(n, \delta) > 0$ such that when $\alpha > \alpha_1$, if Γ_n realizing the maximum of E_n^α, then*

(1) the longest diagonal must be $A_iA_{i+\frac{n}{2}}$ for some i;

(2) $\eta_i, \eta_{i+\frac{n}{2}} \leq \delta$, where η_j is the inner angle at A_j.

Proposition 6.3. *There exist a constant c_n and a function $\alpha_2(\delta)$ satisfying the follwoing condition: For any $0 < \delta_n < c_n$ and any Γ_n, if*

(1) the longest diagonal of Γ_n is $A_iA_{i+\frac{n}{2}}$ for some i,

(2) $\eta \in [0, \delta_n)$, where $\eta = max\ \{\eta_i, \eta_{i+\frac{n}{2}}\}$, η_i is the inner angle at A_i.

Then Γ_n reaches max$E_n^{2\alpha}$ implies that $\Gamma_n = \Gamma_n^-$ when $\alpha > \alpha_2(\delta_n)$.

Proof of Theorem 6.1 from Propositon 6.2 and Propositon 6.3. First choose δ_n provided by Proposition 6.3. Let $\alpha^* = 2\max\{\alpha_1(\delta_n), \alpha_2(\delta_n)\}$, where $\alpha_1(\delta_n), \alpha_2(\delta_n)$ are provided by Proposition 6.2 and Proposition 6.3 respectively.

Suppose for $\alpha > \alpha^*$, Γ_n realizes the maximum of E_n^α. Since $\alpha > \alpha_1(\delta_n)$, by Proposition 6.2, the largest diagnol of Γ_n is $A_iA_{i+\frac{n}{2}}$ and $\eta = \max\{\eta_i, \eta_{i+\frac{n}{2}}\} \in [0, \delta_n)$. Now we apply Proposition 6.3 to Γ_n. Since $\eta \in [0, \delta_n)$ and $\frac{\alpha}{2} > \alpha_2(\delta_n)$, we have $\eta = 0$, that is $\Gamma_n = \Gamma_n^-$. □

Proof of Propositon 6.2. Suppose A_iA_{i+k} is a longest diagonal and $|A_iA_{i+k}| = L$. Suppose Γ_n realizes the maximum of E_n^α. Then $E_n^\alpha(\Gamma_n) \geq E_n^\alpha(\Gamma_n^-)$ implies that

$$\frac{n(n-3)}{2}L^\alpha \geq \bar{E}_n^\alpha(\Gamma_n) \geq \bar{E}_n^\alpha(\Gamma_n^-) \geq (n/2)^\alpha$$

, which further implies that

$$L > \frac{n}{2}(\frac{2}{n(n-3)})^{\frac{1}{\alpha}}.$$

If $k \leq n/2 - 1$, then $|A_iA_{i+k}| = L \leq n/2 - 1$, that is

$$(\frac{2}{n(n-3)})^{\frac{1}{\alpha}} < \frac{2}{n}L \leq \frac{2}{n}(\frac{n}{2} - 1) = 1 - \frac{2}{n},$$

Denote by $\lambda_n = \frac{2}{n(n-3)}$. Since $\lambda_n \in (0, 1)$, we have

$$\frac{1}{\alpha} > \log_{\lambda_n}(1 - \frac{2}{n})$$

which implies

$$\alpha < (\log_{\lambda_n}(1 - \frac{2}{n}))^{-1} = \alpha'$$

Figure 17

If $\alpha > \alpha'$, then k must be $n/2$.

Since $|A_{i+1}A_{i+k-1}| \le n/2 - 2$, we have (see Figure 17)

$$|A_iA_{i+k}| \le n/2 - 2 + \cos\beta_1 + \cos\beta_2 = n/2 - (1-\cos\beta_1) - (1-\cos\beta_2)$$

$$|A_iA_{i+k}| \le n/2 - 2 + \cos\beta_1' + \cos\beta_2' = n/2 - (1-\cos\beta_1') - (1-\cos\beta_2')$$

So $n/2 - L \ge (1-\cos\beta_1) + (1-\cos\beta_2) \ge 1 - \cos\beta_1 = 2\sin^2\frac{\beta_1}{2}$, and then

$$\left(\frac{n}{4}(1-\lambda_n^{\frac{1}{\alpha}})\right)^{\frac{1}{2}} > \left(\frac{n}{4} - \frac{L}{2}\right)^{\frac{1}{2}} > \sin\frac{\beta_1}{2}$$

Let α'' be the solution of the following equation of x.

$$\frac{1}{4}\delta = arc\sin\left(\frac{n}{4}(1-\lambda_n^{\frac{1}{x}})\right)^{\frac{1}{2}}$$

Let α_1 be the maximum of α' and α''. Then, when $\alpha > \alpha_1$, since $arc\sin(\frac{n}{4}(1-\lambda_n^{\frac{1}{x}}))^{\frac{1}{2}}$ is a decreasing function of x ($z = arc\sin y$ is increasing and $y = (\frac{n}{4}(1-\lambda_n^{\frac{1}{x}}))^{\frac{1}{2}}$ is decreasing), we have

$$\frac{\beta_1}{2} < arc\sin\left(\frac{n}{4}(1-\lambda_n^{\frac{1}{\alpha}})\right)^{\frac{1}{2}} < arc\sin\left(\frac{n}{4}(1-\lambda_n^{\frac{1}{\alpha''}})\right)^{\frac{1}{2}} = \frac{\delta}{4}.$$

So $\beta_1 < \frac{\delta}{2}$. Similarly $\beta_2, \beta_1', \beta_2' < \frac{\delta}{2}$

So

$$\eta_i = \beta_1 + \beta_1' \le \delta, \;\; \eta_{i+n/2} = \beta_2 + \beta_2' \le \delta$$

□

Proof Propositon 6.3. Propositon 6.3 follows from two lemmas below.

Lemma 6.4. *Under the assumption about Γ_n in Proposition 6.3, there is a function F_α such that*

$$E_n^{2\alpha}(\Gamma_n) \le F_\alpha(\Gamma_n) = F_\alpha(u),$$

where $u = \sin^2\frac{\eta}{4}$, and

$$F_\alpha(0) = F_\alpha(\Gamma_n^-) = E_n^{2\alpha}(\Gamma_n^-)$$

Proof. Let $n = 2k$. We may assume that A_0A_k is the longest diagonal of Γ_n. We assume Γ_n is on the $x-y$ plane, A_0 is the origin and A_0A_k is along the direction of x-axis. Let $A_j = (x_j, y_j)$. Let

$$h = max\{|y_j||1 \le j \le n\}.$$

Then $h \le k \sin \eta$. Denote $\angle A_kA_0A_1$ by β. May assume $\angle A_1A_0A_{n-1} = \eta$ and $\beta \ge \frac{\eta}{2}$. Then

$$|A_0A_k| \le |x_1 - x_0| + |x_2 - x_1| + ... + |x_k - x_{k-1}|$$
$$\le \cos\beta + k - 1 \le \cos\frac{\eta}{2} - 1 + k = k - 2\sin^2\frac{\eta}{4} \qquad (6.1)$$

Also we have

$$|A_iA_j| \le ((x_i - x_j)^2 + (y_i - y_j)^2)^{1/2} \le ((x_i - x_j)^2 + (2h)^2)^{1/2}$$
$$\le ((x_i - x_j)^2 + (2k\sin\eta)^2)^{1/2} \qquad (6.2)$$

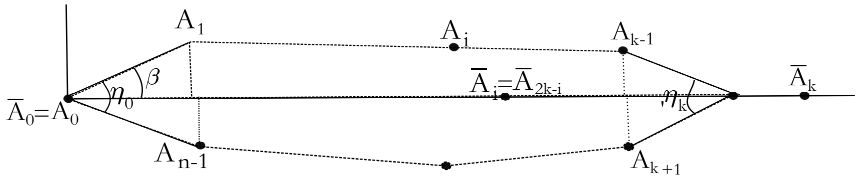


Figure 18

Let $S = \{(i, j)|i < j, (i, j) \ne (0, k)\}$. Then

$$E_n^{2\alpha}(\Gamma_n) = \sum_{i<j} |A_iA_j|^{2\alpha} = |A_0A_k|^{2\alpha} + \sum_S |A_iA_j|^{2\alpha} \qquad (6.3)$$

By (6.1) and (6.2), the right side of (6.3) can be enlarged to obtain

$$E_n^{2\alpha}(\Gamma_n) \le |k - 2\sin^2\frac{\eta}{4}|^{2\alpha} + \sum_S |(x_i - x_j)^2 + (2k\sin\eta)^2|^{\alpha} \qquad (6.4)$$

Let $t = \sin\frac{\eta}{4}$. For $\eta \in [0, \pi/2]$, we have

$$\sin\eta = 2\cos\eta/2\sin\eta/2 \le 2\sin\eta/2 = 4\cos\eta/4\sin\eta/4 \le 4\sin\eta/4 \le 4t \qquad (6.5)$$

Substitute $t = \sin\frac{\eta}{4}$ and apply the enlargement (6.5), we have

$$E_n^{2\alpha}(\Gamma_n) \le |k - 2t^2|^{2\alpha} + \sum_S |(x_i - x_j)^2 + 64k^2t^2|^{\alpha} \qquad (6.6)$$

Denote the vertices of Γ_n^- by $\bar{A}_i$. Suppose $\bar{A}_0$ and $\bar{A}_k$ are the ends of Γ_n^-. Put Γ_n^- along x-axis with$\bar{A}_0$ coincides with the origin. Let $(\bar{x}_i, 0)$ be the coordinates of the vertices $\bar{A}_i$. Note $|x_i - x_{i-1}| = \cos\beta_i$ is the length of the projection of $A_{i-1}A_i$ on the x-axis, where β_i is the acute angle between $A_{i-1}A_i$ and the x-axis. From the convexity of Γ_n and $\eta = \max\{\eta_0, \eta_k\}$, we have $\cos\beta_i \geq \cos\eta$. Then when $0 \leq i \leq k$, we have

$$|x_i - \bar{x}_i| = |(x_i - x_{i-1}) + (x_{i-1} - x_{i-2}) + ... + (x_2 - x_1) + x_1 - \bar{x}_i| =$$

$$|\cos\beta_i + \cos\beta_{i-1} + ... + \cos\beta_2 + \cos\beta_1 - i| \leq i(1 - \cos\eta) \leq k(1 - \cos\eta) \qquad (6.7)$$

By symmetry argument, (6.7) also holds for $k + 1 \leq i \leq 2k - 1 = n - 1$.

By the triangular inequality and (6.7) we have

$$||x_i - x_j| - |\bar{x}_i - \bar{x}_j|| \leq |x_i - \bar{x}_i| + |x_j - \bar{x}_j| \leq 2k(1 - \cos\eta)$$

$$= 4k\sin^2\frac{\eta}{2} = 16k\sin^2\frac{\eta}{4}\cos^2\frac{\eta}{4} \leq 16k\sin^2\frac{\eta}{4} = 16kt^2$$

So we have

$$|x_i - x_j| \leq |\bar{x}_i - \bar{x}_j| + 16kt^2 \qquad (6.8)$$

Put (6.8) into (6.6) we have

$$E_n^{2\alpha}(\Gamma_n) \leq |k - 2t^2|^{2\alpha} + \sum_S (|\bar{x}_i - \bar{x}_j| + 16kt^2)^2 + 64k^2t^2|)^\alpha \qquad (6.9)$$

Fix a $1 > \delta_0 > 0$ for moment. If $\eta \in [0, 4\delta_0)$, then $t = \sin\frac{\eta}{4} \in [0, \delta_0)$, and $t^2 \in [0, \delta_0^2)$. Then we have

$$|k - 2t^2|^{2\alpha} = (k^2 - 4kt^2 + 4t^4)^\alpha \leq (k^2 - 4kt^2 + 4t^2\delta_0^2)^\alpha = (k^2 - C_1t^2)^\alpha \qquad (6.10)$$

where $C_1 = 4k - 4\delta_0^2$, and

$$\sum_S (|\bar{x}_i - \bar{x}_j| + 16kt^2)^2 + 64k^2t^2|)^\alpha = \sum_S (|\bar{x}_i - \bar{x}_j|^2 + 32|\bar{x}_i - \bar{x}_j|kt^2 + 256k^2t^4 + 64k^2t^2)^\alpha$$

$$\leq \sum_S (|\bar{x}_i - \bar{x}_j|^2 + 256k^2t^4 + 96k^2t^2)^\alpha \leq \sum_S (|\bar{x}_i - \bar{x}_j|^2 + 256k^2t^2\delta_0^2 + 96k^2t^2)^\alpha$$

$$= \sum_S (|\bar{x}_i - \bar{x}_j|^2 + (256k^2\delta_0^2 + 96k^2)t^2)^\alpha = \sum_S (|\bar{x}_i - \bar{x}_j|^2 + C_2t^2)^\alpha \qquad (6.11)$$

where $C_2 = 256k^2\delta_0^2 + 96k^2$.

Substitute (6.10) and (6.11) into (6.9) we get

$$E_n^{2\alpha}(\Gamma_n) \leq (k^2 - C_1u)^\alpha + \sum_S (|\bar{x}_i - \bar{x}_j|^2 + C_2u)^\alpha \qquad (6.12)$$

where C_1 and C_2 are constant for given n and δ_0. Let $u = t^2 = \sin^2\frac{\eta}{4}$ and let

$$F_\alpha(\Gamma_n) = F_\alpha(u) = (k^2 - C_1u)^\alpha + \sum_S (|\bar{x}_i - \bar{x}_j|^2 + C_2u)^\alpha \qquad (6.13)$$

Clearly $E_n^{2\alpha}(\Gamma_n) \leq F_\alpha(\Gamma_n) = F_\alpha(u)$, and $F_\alpha(\Gamma_n^-) = F_\alpha(0)$.

□

Lemma 6.5. *For F_α given in Lemma 6.4, there are $\delta_n > 0$ and $\alpha_2(\delta_n) > 0$ such that u reaches $maxF_\alpha$ implies that $u = 0$ when $u \in [0, \frac{\delta_n^2}{16})$ and $\alpha > \alpha_2$.*

Proof. Now we consider $F_\alpha(u)$ on $[0, \delta^2)$ with $\delta \leq \delta_0$

$$F'_\alpha(u) = \alpha(k^2 - C_1u)^{\alpha-1}(-C_1) + \sum_S \alpha(|\bar{x}_i - \bar{x}_j|^2 + C_2u)^{\alpha-1}C_2$$

$$\leq \alpha(-(k^2 - C_1\delta^2)^{\alpha-1}C_1 + \sum_S (|\bar{x}_i - \bar{x}_j|^2 + C_2\delta^2)^{\alpha-1}C_2)$$

$$\leq \alpha(-(k^2 - C_1\delta^2)^{\alpha-1}C_1 + (\frac{n(n-1)}{2} - 1)((k-1)^2 + C_2\delta^2)^{\alpha-1}C_2)$$

$$= \alpha(-(k^2 - C_1\delta^2)^{\alpha-1}C_1 + C_3((k-1)^2 + C_2\delta^2)^{\alpha-1}) \qquad (6.14)$$

where $C_3 = (\frac{n(n-1)}{2} - 1)C_2$.

To get $F'_\alpha(u) < 0$, we need only to choose δ and α such that

$$(k^2 - C_1\delta^2)^{\alpha-1}C_1 > C_3((k-1)^2 + C_2\delta^2)^{\alpha-1}$$

That is

$$(\frac{(k^2 - C_1\delta^2)}{((k-1)^2 + C_2\delta^2)})^{\alpha-1} = \frac{(k^2 - C_1\delta^2)^{\alpha-1}}{((k-1)^2 + C_2\delta^2)^{\alpha-1}} > \frac{C_3}{C_1} \qquad (6.15)$$

To make (6.15) hold for large enough α, we need only

$$\frac{(k^2 - C_1\delta^2)}{((k-1)^2 + C_2\delta^2)} > 1 \qquad (6.16)$$

To solve the inequality (6.16) we need

$$\delta^2 < \frac{k^2 - (k-1)^2}{C_1 + C_2} \qquad (6.17)$$

So once (6.17) is hold, (6.15) is hold if

$$\alpha > \log_{\frac{(k^2 - C_1\delta^2)}{((k-1)^2 + C_2\delta^2)}} \frac{C_3}{C_1} + 1 \qquad (6.18)$$

Let c_n in Proposition 6.3 to be

$$c_n = min\{4(\frac{k^2 - (k-1)^2}{C_1 + C_2})^{1/2}, 4\delta_0\}). \qquad (6.19)$$

Then for any $\delta_n \leq c_n$, we have

$$\frac{\delta_n^2}{16} < min\{\frac{k^2 - (k-1)^2}{C_1 + C_2}, \delta_0^2\} \qquad (6.20)$$

Let

$$\alpha_2 = \alpha_2(\delta_n) = \log_{\frac{(k^2 - C_1\frac{\delta_n^2}{16})}{((k-1)^2 + C_2\frac{\delta_n^2}{16}}} \frac{C_3}{C_1} + 1 \qquad (6.21)$$

So $F'_\alpha(u) < 0$ when $u \in [0, \frac{\delta_n^2}{16})$ (that is when $\eta \in [0, \delta_n)$) and $\alpha > \alpha_2$. That is F_α is decreasing for $u \in [0, \delta_n^2)$. It follows that $F_\alpha(u)$ reaches the maximum if and only if $u = 0$, (that is $\eta = 0$).

Note that $E_n^{2\alpha}(\Gamma_n) \le F_\alpha(\Gamma_n) = F_\alpha(u)$, and $E_n^{2\alpha}(\Gamma_n^-) \le F_\alpha(\Gamma_n^-) = F_\alpha(0)$. So $E_n^{2\alpha}(\Gamma_n)$ realizes the maximum if and only if $\Gamma_n = \Gamma_n^-$. $\square$

$\square$

6.2 A sample of odd n: Infinitely many Γ_5 realizing maxE_5^α for large $\alpha > 0$

Proposition 6.6. *E_5^α does not reach the maximum at Γ_5^Δ for any α.*

Proof. We may assume that $\Gamma_5^\Delta = A_1A_2A_3A_4A_5$ and put it in symmetry position about $x-$axie shown as Figure 19. We fix the base A_3A_4 in the whole proof. Denote the angle $\angle A_4A_3A_2 = \angle A_5A_4A_3 = \eta_0$ (clearly $\cos\eta_0 = \frac{1}{4}$). Now we rotate A_3A_2 to A_3A_2' (respectively A_4A_5 to A_4A_5') to increase an small angle θ, we get a new $\Gamma_5(\theta) = A_1'A_2'A_3'A_4'A_5'$ in red, with $A_3' = A_3$ and $A_4' = A_4$, see Figure 19.

We will prove
$$E_5^\alpha(\Gamma_5(\theta)) > E_5^\alpha(\Gamma_5^\Delta)$$
for small θ. The reason is simple: One can see directly from Figure 19 when we change Γ_5^Δ to $\Gamma_5(\theta)$, $|A_2A_4|$, $|A_3A_5|$, $|A_2A_5|$ are increasing, $|A_1A_3|$, $|A_1A_4|$ are decreasing. But for small θ, those increasing are in the order of θ, and those decreasing are in order of θ^2. The detailed verification of the last sentence is as below.

Denote the $\angle A_1'A_3'A_2' = \theta'$, then the $\angle A_1'A_3'A_1 = \theta' - \theta$. Note first
$$|A_1'A_3'| = 2\cos\theta' = 2 - O(\theta'^2) = |A_1A_3| - O(\theta'^2)$$
the second equality use Taylor expansion. Note then if we consider triangle $A_1A_3A_1'$, then the angle opposite the edge A_1A_1' is $\theta' - \theta$ and we have
$$|A_1A_3| - |A_1'A_3'| = O(\theta' - \theta)$$
.

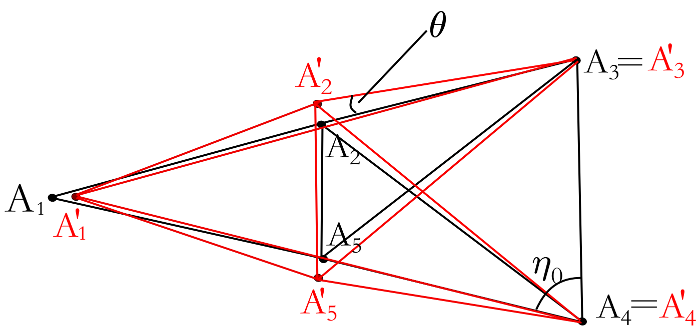


Figure 19

Compare the last two formula we have $\theta - \theta' = O(\theta'^2)$, that is θ and θ' are infinitesimal quantities of the same order and we have

$$|A_1A_3| - |A_1'A_3'| = O(\theta^2)$$

Similarly

$$|A_4A_1| - |A_4'A_1'| = O(\theta^2)$$

Since $|A_2A_3| = |A_2'A_3'| = |A_3A_4|$, we have $A_2A_4 = 2\sin\frac{\eta_0}{2}$ and

$$\begin{aligned} A_2'A_4' &= 2\sin\frac{\eta_0+\theta}{2} = 2(\cos\frac{\theta}{2}\sin\frac{\eta_0}{2} + \cos\frac{\eta_0}{2}\sin\frac{\theta}{2}) \\ &= 2\sin\frac{\eta_0}{2} + C\theta + O(\theta^2) = 2\sin\frac{\eta_0}{2} + C\theta + O(\theta^2) \end{aligned}$$

So $|A_2'A_4'| - |A_2A_4| = C\theta + O(\theta^2)$, and similarly $|A_3'A_5'| - |A_3A_5| = C\theta + O(\theta^2)$, where $C = \cos\frac{\eta_0}{2} > 0$. By the same reason, we have $|A_2'A_5'| - |A_2A_5| = C'\theta + O(\theta^2)$, where $C' = 4\cos\frac{\eta}{2} > 0$.

Note $\Gamma_5 = \Gamma_5(\theta)$ and $\Gamma_5^\Delta = \Gamma_5(0)$. Make a Taylor expansion of $E_5^\alpha(\Gamma_5)$ at $E_5^\alpha(\Gamma_5^\Delta)$ as a function of θ, we have

$$\begin{aligned} E_5^\alpha(\Gamma_5) - 5 &= 2|A_1'A_3'|^\alpha + 2|A_2'A_4'|^\alpha + |A_2'A_5'|^\alpha \\ &= 2|A_1A_3|^\alpha + 2f_\alpha'(|A_1A_3|)O(\theta^2) \\ &+2|A_2A_4|^\alpha + 2f_\alpha'(|A_2A_4|)(C\theta + O(\theta^2)) \\ &+|A_2A_5|^\alpha + f_\alpha'(|A_2A_5|)(C'\theta + O(\theta^2)) \\ &= E_5^\alpha(\Gamma_5^\Delta) - 5 + C''\theta + O(\theta^2) \end{aligned}$$

where $C'' = 2f_\alpha'(|A_2A_4|)C + f_\alpha'(|A_2A_5|)C'$. By Lemma 4.2, $f_\alpha' > 0$ for any α, so there is small $\delta > 0$ such that when $\theta \in (0, \delta)$, $E_5^\alpha(\Gamma_5(\theta))$ is an increasing function about θ. So E_5^α does not reach the maximum at $\Gamma_5^\Delta = \Gamma_5(0)$. $\square$

Proposition 6.7. *There is an increasing sequence $\{\alpha_i\}$ of real numbers such that*

(1) For each α_i, there is $\Gamma_5^{a_i}$ realizing max$E_5^{\alpha_i}$,

(2) $\Gamma_5^{a_i} \neq \Gamma_5^{a_j}$ for $i < j$.

Proof. Note the longest diameter of any Γ_5 is no more than 2, and $\Gamma_5 = \Gamma_5^\Delta$ if and only if Γ_5 has two diagnals of length 2. Then it is not hard to see the following

Claim: If $\Gamma_5 \neq \Gamma_5^\Delta$, then there exist $\alpha^* > 0$ such that $E_5^\alpha(\Gamma_5^\Delta) > E_5^\alpha(\Gamma_5)$ when $\alpha > \alpha^*$.

Pick any increasing sequence $\{\alpha_i\}$ of real with α_i tends infinite as n tends infinite. For each α_i, there is Γ_5, denoted by $\Gamma_5^{\alpha_i}$, realizing max$E_5^{\alpha_i}$. By Proposition 6.6 and its proof, we have

$$E_5^{\alpha_i}(\Gamma_5^{\alpha_i}) > E_5^{\alpha_i}(\Gamma_5^\Delta).$$

The infinite sequence $\{\Gamma_5^{\alpha_i}\}$ must contains infinitely many different Γ_5. Otherwise passing to a subsequence, we may assume that $\Gamma_5^{\alpha_i} = \Gamma_5^*$ for all i. Hence

$$E_5^{\alpha_i}(\Gamma_5^*) > E_5^{\alpha_i}(\Gamma_5^\Delta). \qquad (6.22)$$

Since $\Gamma_5^* \neq \Gamma_5^\Delta$, and in (6.22) α_i can be arbitrary large, it contradicts the claim. $\square$

Example 6.8. We will compare E_5^α at Γ_5^o, Γ_5^Δ and $\Gamma_5(\theta)$ (defined in the proof of Proposition 6.6) for some θ at some α to get more concrete feeling of the last two propositions.

Choose θ so that $|A_2'A_5'| = 0.52$ in the proof of Proposition 6.6. Note $|A_2A_5| = 0.5$ in Γ_5^Δ. Hence $\theta = arc\cos 0.24 - arc\cos 0.25 \in (0.59907^o, 0.59908^o)$.

Then for $\alpha = 3.54$, we have $E_5^\alpha(\Gamma_5(\theta)) > E_5^\alpha(\Gamma_5^o) > E_5^\alpha(\Gamma_5^\Delta)$, since

$$E_5^\alpha(\Gamma_5^o) \in (27.46, 27.47),\ E_5^\alpha(\Gamma_5^\Delta) \in (27.44, 27.45),\ E_5^\alpha(\Gamma_5(\theta)) \in (27.51, 27.52);$$

and for $\alpha = 3.55$, $E_5^\alpha(\Gamma_5(\theta)) > E_5^\alpha(\Gamma_5^\Delta) > E_5^\alpha(\Gamma_5^o)$, since

$$E_5^\alpha(\Gamma_5^o) \in (27.59, 27.60),\ E_5^\alpha(\Gamma_5^\Delta) \in (27.61, 27.62),\ E^\alpha(\Gamma_5(\theta)) \in (27.68, 27.69);$$

and for $\alpha = 4$, $E_5^\alpha(\Gamma_5^\Delta) > E_5^\alpha(\Gamma_5(\theta)) > E_5^\alpha(\Gamma_5^o)$, since

$$E_5^\alpha(\Gamma_5^o) \in (89.72, 89.73),\ E_5^\alpha(\Gamma_5^\Delta) \in (134.76, 134.77),\ E_5^\alpha(\Gamma_5(\theta)) \in (134.65, 134.66).$$

7 Back to Thomson type problems

Thomson type problem considers the distribution of n points on the unit sphere in $\mathbb{R}^3$ under the energy functions f_α given by (1.1). The problem was first raised by Thomson for $\alpha = -1$ [Th], and later generalized to all $\alpha \in \mathbb{R}$. There many studies on Thomson type problem, see [AP], [BH], [PB], [Sm] and their references. Mathematically, Thomson type problems can be raised for unit sphere S^m of $\mathbb{R}^{m+1}$ for any integer $m > 0$.

Problem 7.1. *Let $A_1, ..., A_n$ be n points on the unit sphere S^m. What is the distribution of those n points on S^m when the total energy*

$$E_n^\alpha(m) = \sum_{p \neq q} f_\alpha(|A_p - A_q|) \qquad (7.1)$$

reaches the maximum?

Indeed Thomson type problem is an inspiration of the distribution problem we studied in this note. Inspired by our study we can also add some information to the Thomson type problem.

Theorem 7.2. *Let $A_1, ..., A_n$ be n points on the unit sphere S^m. Then*

(1) For $\alpha = 2$, $A_1, ..., A_n$ realize the $maxE_n^2(m)$ if and only $\sum_{i=1}^n A_i = 0$, in particular there are infinitely many distributions to realize $maxE_n^2(m)$.

(2) For $\alpha > 2$ and n even, $A_1, ..., A_n$ realize the $maxE_n^\alpha(m)$ if and only if they stay evenly in the two ends of a diameter of S^m.

(3) For $\alpha < 2$ and $n \leq m + 2$, $A_1, ..., A_n$ realize the $maxE_n^\alpha(m)$ if and only if they are the vertices regular $(n-1)$-simplex inscribed in $S^{n-2} = S^m \cap \mathbb{R}^{n-1}$, where $\mathbb{R}^{n-1}$ is a subspace of $\mathbb{R}^{m+1}$ passing the origin.

Note a regular m-simplex is the convex hull of $m+1$ points $A_1, ..., A_{m+1}$ with $d(A_i, A_j)$ a constant C for $i \neq j$. A way to see a regular m-simplex is: Pick an orthorgonal coordinate O-$x_1....x_{m+1}$ of $\mathbb{R}^{m+1}$. Then the n-dimensional space defined by $x_1 + x_2 + ... + x_{n+1} = 1$ intersections each x_i-axies in the unit A_i. Then $d(A_i, A_j) = 2^{\frac{1}{2}}$ for $i \neq j$, and those $A_1, ..., A_{m+1}$ form a regular m-simplex. Clearly those A_i has the same distance to their barycenter $(\frac{1}{m+1}, ..., \frac{1}{m+1})$, so they inscribe an $(m-1)$-sphere. Moreover when they inscribe an $(m-1)$-sphere, their barycenter is the center of the $(m-1)$-sphere.

Proof. (1) For $\alpha = 2$, we have

$$E_n^2 = \sum_{i<j} |A_i - A_j|^2 = \sum_{i<j} (A_i - A_j)^2$$

$$= \sum_{i<j} (2 - 2A_iA_j) = 2C_n^2 - \sum_{i<j} 2A_iA_j$$

We also have

$$(\sum_{i=1}^{n} A_i)^2 = n + \sum_{i<j} 2A_iA_j$$

So

$$E_n^2 = 2C_n^2 + n - (\sum_{i=1}^{n} A_i)^2 = n^2 - (\sum_{i=1}^{n} A_i)^2 \qquad (7.2)$$

Hence $E_n^2 \leq 2C_n^2 + n$ and the equality hold if and only if $\delta = \sum_{i=1}^{n} A_i = 0$. That is their mess center in the origin O if consider each point has a unit mass.

Now we are going to prove (2) and (3) from the conclusion of (1).

For simplicity, we call the distributions in (2) and (3) even pole distributions and regular n-simplex distributions respectively. Note both those two distributions satisfy $\sum_{i=1}^{n} A_i = 0$.

Now let $f_\alpha(x) = g_\alpha(x^2)$, where

$$g_\alpha(x) = \left\{ \begin{array}{cc} x^{\alpha/2}, & \alpha > 0; \\ \ln x^{1/2}, & \alpha = 0; \\ -x^{\alpha/2}, & \alpha < 0. \end{array} \right\}$$

By Lemma 4.2, g_α is increasing; moreover is convex if $\alpha < 2$, and concave if $\alpha > 2$. Now we have

$$E_n^\alpha = \sum_{i<j} f_\alpha(|A_i - A_j|) = \sum_{i<j} g_\alpha(|A_i - A_j|^2) \qquad (7.3)$$

(2) Now $\alpha > 2$ and $n = 2k$, and g is concave.

We need a fact derived from Karamate Lemma (Proposition 4.1 (2)).

Lemma 7.3. *Suppose g is a concave function on $[0, 4]$ and there are n variables $x_1, x_2, ..., x_n \in [0, 4]$ with a fixed sum $4q$ for some positive integer q. Then the value $\sum_{i=1}^{n} g(x_i)$ reaches the maximum if and only if all x_i is either 0 or 4.*

Proof. Under the same condition, Karamata Lemma claim that the value $\sum_{i=1}^{n} g(x_i)$ reaches the maximum if and only if at most one x_i neither 0 nor 4. If one x_i is neither 0 nor 4, then their sum can not be $4q$. So each x_i is either 0 or 4. $\square$

Let $l = \sum_{i<j} |A_i - A_j|^2$. By the conclusion of (1), $l + \delta = 2C_{2k}^2 + 2k = 4k^2$.

Choose $\delta_{i,j} \geq 0$ such that $\sum_{i<j} \delta_{i,j} = \delta$ and $x_{i,j} = |A_i - A_j|^2 + \delta_{i,j} \leq 4$.

Since g is increasing, we have

$$\sum_{i<j} g_\alpha(|A_i - A_j|^2) \leq \sum_{i<j} g_\alpha(|A_i - A_j|^2 + \delta_{i,j}) = \sum_{i<j} g_\alpha(x_{i,j}) \qquad (7.4)$$

and the equality hold if and only if $\delta = \sum_{i=1}^{n} A_i = 0$.

Since $x_{i,j} \in [0,4]$, $\sum_{i<j} x_{i,j} = 4k^2$, g is concave, we can apply Lemma 7.3. Note first the conclusion of Lemma 7.3 and $\sum_{i<j} x_{i,j} = 4k^2$ implies the when $\sum_{i<j} g(x_{i,j})$ reaches the maximum, the number of $x_{i,j}$ which equals 4 is k^2. Now apply Lemma 7.3 we have

$$\sum_{i<j} g_\alpha(x_{i,j}) \leq (C_n^2 - k^2) g_\alpha(0) + k^2 g_\alpha(4) \qquad (7.5)$$

Suppose those A_i's are in even poles distribution, then clearly $\delta = \sum_{i=1}^{n} A_i = 0$, therefore $x_{i,j} = |A_i - A_j|^2$; furthermore there are k^2 $x_{i,j}$ which equals 4, and remaining $x_{i,j}$ equal to 0. So both (7.4) and (7.5) become equalities, which implies the even poles distribution realizing the $\max E_n^a$.

Suppose $\sum_{i<j} g_\alpha(|A_i - A_j|^2)$ reaches the maximum for a distribution of those A_i's. Then (7.4) becomes equality, that is $\delta = 0$ and $x_{i,j} = |A_i - A_j|^2$. Next (7.5) becomes equality, that is there are k^2 $|A_i - A_j|^2$ which equal to 4, and other $|A_i - A_j|^2$ are zero, which implies that all A_i evenly stay in the two ends of a diameter.

(3) Now $\alpha < 2$: We have

$$\frac{1}{C_n^2} \sum_{i<j} g_\alpha(|A_i - A_j|^2) \leq g_\alpha\left(\frac{\sum_{i<j} |A_i - A_j|^2}{C_n^2}\right) \leq g_\alpha\left(\frac{2C_n^2 + n}{C_n^2}\right)$$

The first inequality follows from that g is convex and Lemma 4.1 (1), and the second inequality follows from that g is increasing and the conclusion of (1).

Suppose now and $n \leq m + 2$.

When those points are in regular n-simplex distribution, $|A_i - A_j|$ are equal for all $i \neq j$, therefore the first equality holds, and moreover $\delta = \sum_{i=1}^{n} A_i = 0$ therefore the second equality hold. So the regular n-simplex distribution reaches the $\max E_n^\alpha$.

On the other hand if first equality holds, we must have $|A_i - A_j|$ are equal for all $i \neq j$ by Lemma 4.1 (1). If the second equality hold, then we have $\delta = \sum_{i=1}^{n} A_i = 0$, and all those A_i stay in some $\mathbb{R}^{n-1} \subset \mathbb{R}^{m+1}$. So those A_i stay in $S^{n-2} = S^m \cap \mathbb{R}^{n-1}$ which is in a regular n-simplex distribution. $\square$

8 Some miscellaneous results

8.1 E_6^α reaches max at Γ_6^- for central symmetry Γ_6 when $\alpha \geq 6$

Proposition 8.1. *Suppose* Γ_6 *is central symmetry. Then for* $\alpha \geq 6$, $E_6^\alpha(\Gamma_6)$ *maximum implies that* $\Gamma_6 = \Gamma_6^-$.

Since $E_6^\alpha(\Gamma_6) = E_{6,1}^\alpha(\Gamma_6) + E_{6,2}^\alpha(\Gamma_6) + E_{6,3}^\alpha(\Gamma_6)$ and $E_{6,1}^\alpha(\Gamma_6) = 6$ for $\alpha > 0$, Proposition 8.1 follow from Proposition 8.2 and Proposition 8.4 below.

Proposition 8.2. *Suppose* Γ_6 *is central symmetry. Then for* $\alpha \geq 6$, $E_{6,2}^\alpha(\Gamma_6)$ *maximum implies that* $\Gamma_6 = \Gamma_6^-$.

Proof. Suppose Γ_6 has vertices $A_1, ..., A_6$ in the cyclic order Γ, and the exterior angle at A_i is θ_i. Γ_6 is determined by θ_i, $i = 1, 2, 3$, $\theta_1 + \theta_2 + \theta_3 = \pi$. We may always assume that $\theta_1 \leq \theta_2 \leq \theta_3$, (see Figure 20). As we see before that $|A_iA_{i+2}| = 2\cos\phi_i$, where $\phi_i = \frac{1}{2}\theta_i$ (see Figure 13). Now we have $\phi_1 + \phi_2 + \phi_3 = \pi/2$, $\phi_1 \leq \phi_2 \leq \phi_3$ and

$$\frac{1}{2}E_{n,2}^\alpha(\Gamma_6) = 2^\alpha((\cos\phi_1)^\alpha + (\cos\phi_2)^\alpha + (\cos\phi_3)^\alpha) \qquad (*)$$

We may also assume that $\phi_3 \leq \pi/2$, otherwise we already have $\Gamma_6 = \Gamma_6^-$.

Let $h(x) = (\cos x)^\alpha$, $x \in [0, \pi/2]$, we can calculate

$$h'(x) = \alpha \cos^{\alpha-1} x(-\sin x),$$

and

$$\begin{aligned} h''(x) &= \alpha(\alpha-1)\cos^{\alpha-2} x \sin^2 x + \alpha \cos^{\alpha-1} x(-\cos x) \\ &= \alpha(\alpha-1)\cos^{\alpha-2} x(1-\cos^2 x) + \alpha\cos^{\alpha-1} x(-\cos x) \\ &= \alpha(\alpha-1)(\cos^{\alpha-2} x - \cos^\alpha x) - \alpha\cos^\alpha x \\ &= \alpha(\cos x)^{\alpha-2}((\alpha-1) - \alpha\cos^2 x)). \end{aligned}$$

So for $\cos x \in (0, (\frac{\alpha-1}{a})^{\frac{1}{2}})$, $h''(x) > 0$, that is for $x \in (\beta, \pi/2)$, $h''(x) > 0$, where $\beta = arc\cos(\frac{\alpha-1}{a})^{\frac{1}{2}}$. Similarly for $x \in (0, \beta)$, $h''(x) < 0$. Now we rewrite (*) as

$$\frac{1}{2^{\alpha+1}}E_{6,2}^\alpha(\Gamma_6) = h(\phi_1) + h(\phi_2) + h(\phi_3)$$

Lemma 8.3. *We must have* $\phi_1 = \phi_2 \in [0, \beta]$ *and* $\phi_3 \in (\beta, \pi/2]$.

Proof. There are several cases to discuss:

(i) All $\phi_1, \phi_2, \phi_3 \in [\beta, \pi/2]$, then we must have $\phi_2 = \beta$. Otherwise push ϕ_2 to ϕ_2' closer to β (ϕ_1 may also be pushed to ϕ_1' to keep $\phi_1' \leq \phi_2'$) and push ϕ_3 to ϕ_3' closer to $\pi/2$. Since $h'' > 0$ on $(\beta, \pi/2)$, by Karamata inequality, $h(\phi_1') + h(\phi_2') + h(\phi_3') > h(\phi_1) + h(\phi_2) + h(\phi_3)$, so Γ_6 can not realizing the maximum.

(ii) $\phi_1 \in [0,\beta)$ and $\phi_3 \in (\beta, \pi/2]$, If $\phi_2 \in [\beta, \pi/2]$then we must have $\phi_2 = \beta$ by reason in the last paragrah. Then $\phi_1, \phi_2 \in [0,\beta]$. Since $h'' < 0$ on $(0,\beta)$, by Jensen inequality, to reach the maximum, we must have $\phi_1 = \phi_2$.

(iii) All $\phi_1, \phi_2, \phi_3 \in [0,\beta]$, then we have $\phi_1 + \phi_2 + \phi_3 \leq 3\beta$.

Recall $\cos\beta = (\frac{\alpha-1}{\alpha})^{\frac{1}{2}}$, since $\beta = arc\cos(\frac{\alpha-1}{\alpha})^{\frac{1}{2}}$. By $(\sin\beta)^2 + (\cos\beta)^2 = 1$ we have

$$\sin\beta = (\frac{1}{\alpha})^{\frac{1}{2}}.$$

When $\alpha \geq 6$, $(\frac{1}{\alpha})^{\frac{1}{2}} \leq (\frac{1}{6})^{\frac{1}{2}} < \frac{1}{2}$. Then by $\sin\beta < \frac{1}{2}$ and $\beta \in (0, \frac{\pi}{2})$, we have $\beta < \frac{\pi}{6}$, hence $3\beta < \frac{\pi}{2}$, which is impossible. $\square$

Now let $\phi = \phi_1 = \phi_2$, $\phi_3 = \pi/2 - 2\phi$. Denote

$$E_\alpha(\phi) = \frac{1}{2^{\alpha+1}} E^\alpha_{6,2}(\Gamma_6) = 2(\cos\phi)^\alpha + (\cos(\pi/2 - 2\phi))^\alpha$$

We have

$$E'_\alpha(\phi) = 2\alpha(-(\cos\phi)^{\alpha-1}\sin\phi + (\sin 2\phi)^{\alpha-1}\cos 2\phi).$$

Then Proposition 8.2 follows from the following

Claim: *When $\phi \in (0,\beta)$ and $\alpha > 6$, we have $E'(\phi) < 0$.*

To prove the claim, we need only to show

$$(\sin 2\phi)^{\alpha-1}\cos 2\phi - (\cos\phi)^{\alpha-1}\sin\phi < 0 \qquad (8.1)$$

Once $\alpha \geq 6$, as we see in the proof of Lemma 8.3, $\beta < \pi/6$, so $\cos 2\phi > 0$. Hence (8.1) is equivalent to

$$(\frac{\sin 2\phi}{\cos\phi})^{\alpha-1} < \frac{\sin\phi}{\cos 2\phi} \qquad (8.2)$$

By the formula $sin2\phi = 2\cos\phi\sin\phi$, (1) is equivalent to

$$(2\sin\phi)^{\alpha-1} < \frac{\sin\phi}{\cos 2\phi} \qquad (8.3)$$

We will show

$$(\sin\beta)^{\alpha-2} < 2^{-(\alpha-1)} \qquad (8.4)$$

if $\alpha \geq 6$. Then $(\sin\phi)^{\alpha-2} < (\sin\beta)^{\alpha-2} < 2^{-(\alpha-1)}$, so

$$(2\sin\phi)^{\alpha-1} = 2^{\alpha-1}\sin^{\alpha-2}\phi\sin\phi < \sin\phi.$$

Since $\cos 2\phi \leq 1$, so (8.3) follows.

Now we show (8.4) holds if $\alpha \geq 6$. Once $\alpha \geq 6$, we have $\frac{\alpha}{4} > 1$. So $\frac{1}{4}(\frac{\alpha}{4})^{\alpha-2} > \frac{1}{4}(\frac{\alpha}{4})^{6-2} = \frac{1}{4}(\frac{\alpha}{4})^4$, and $\frac{1}{4}(\frac{\alpha}{4})^4 \geq \frac{1}{4}(\frac{6}{4})^4 = \frac{81}{64} > 1$. So

$$\frac{1}{4}(\frac{\alpha}{4})^{\alpha-2} > 1;$$

by taking square roots on both sides, we have $\frac{1}{2}(\frac{\alpha^{\frac{1}{2}}}{2})^{\alpha-2} > 1$, which implies

$$((\frac{1}{\alpha})^{\frac{1}{2}})^{\alpha-2} < 2^{-(\alpha-1)} \qquad (8.5)$$

Put $\sin\beta = (\frac{1}{\alpha})^{\frac{1}{2}}$ into (8.5), we get (8.4). So the claim is proved.

So if $\alpha \geq 6$ then $E(\phi)$ is decreasing on $[0, \beta]$. We get $E_a(\phi)$ reaches the maximum on $\phi = 0$, that is $\phi_1 = \phi_2 = 0$, $\phi_3 = \pi/2$, which implies that $\Gamma_6 = \Gamma_6^-$. $\square$

Proposition 8.4. *Suppose Γ_6 is central symmetry. Then for $\alpha \geq 6$, $E_{6,3}^{\alpha}(\Gamma_6)$ maximum implies that $\Gamma_6 = \Gamma_6^-$.*

Proof. Suppose Γ_6 has vertices $A_1, ..., A_6$ in the cyclic order Γ, and the exterior angle at A_i is $\Theta = (\theta_1, \theta_2, \theta_3)$. Γ_6 is determined by θ_i, $i = 1, 2, 3$, $\theta_1 + \theta_2 + \theta_3 = \pi$. We define the bound points to be

$$B = \{\Gamma_6 | \theta_1\theta_2\theta_3 = 0\}$$

We may always assume that $\theta_1 \leq \theta_2 \leq \theta_3$, then for Γ_6^- we have $\theta_1 = \theta_2 = 0$ and $\theta_3 = \pi$. We have $A_1A_4 = A_1A_2 + A_2A_3 + A_3A_4$, so

$$|A_1A_4|^2 = |A_1A_2 + A_2A_3 + A_3A_4|^2$$

$$= |A_1A_2|^2 + |A_2A|_3^2 + |A_3A_4|^2 + 2\langle A_1A_2, A_2A_3\rangle + 2\langle A_2A_3, A_3A_4\rangle + 2\langle A_1A_2, A_3A_4\rangle .$$

Since $A_3A_4 = A_1A_6$ (as vectors), and $|A_iA_{i+1}| = 1$, we have

$$|A_1A_4|^2 = 3 + 2\cos\theta_2 + 2\cos\theta_3 + 2\cos(\pi - \theta_1) = 3 + 2\cos\theta_2 + 2\cos\theta_3 - 2\cos\theta_1$$

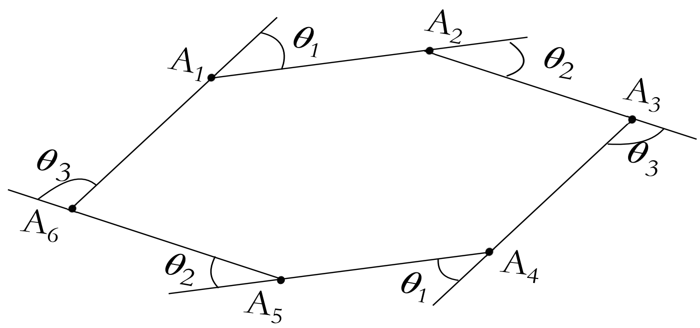


Figure 20

Similarly one can check in general

$$|A_iA_{i+3}|^2 = 3 + 2\cos\theta_{i+1} + 2\cos\theta_{i+2} - 2\cos\theta_i = \sigma_i \qquad (8.6)$$

For brief, we often use $E_\alpha(\Theta)$ to denote $E_{6,3}^{2\alpha}(\Gamma_6)$. Therefore

$$E_\alpha(\Theta) = E_{6,3}^{2\alpha}(\Gamma_6) = \sum_{i=1}^{3} |A_iA_{i+3}|^2 = \sum_{i=1}^{3} \sigma_i^\alpha \qquad (8.7)$$

For simple we denote

$$\Pi_i = \frac{\partial E_\alpha(\Theta)}{\partial\theta_i}, \ \Pi_{ij} = \frac{\partial\Pi_i}{\partial\theta_j} = \frac{\partial^2 E_\alpha(\Theta)}{\partial\theta_j\partial\theta_i}$$

From (8.6), one can calculate that

$$\frac{\partial \sigma_i^\alpha}{\partial \theta_i} = 2\alpha\sigma_i^{\alpha-1}\sin\theta_i,\ \frac{\partial \sigma_{i+j}^\alpha}{\partial \theta_i} = -2\alpha\sigma_{i+j}^{\alpha-1}\sin\theta_i,\ j = 1, 2.$$

So we have

$$\Pi_i = 2\alpha\sin\theta_i(\sigma_i^{\alpha-1} - \sigma_{i+1}^{\alpha-1} - \sigma_{i+2}^{\alpha-1}),\ i = 1, 2, 3 \qquad (8.8)$$

$$\Pi_{ii} = 2\alpha\cos\theta_i(\sigma_i^{\alpha-1} - \sigma_{i+1}^{\alpha-1} - \sigma_{i+2}^{\alpha-1}) + 2\alpha\sin\theta_i\frac{\partial}{\partial\theta_i}(\sigma_i^{\alpha-1} - \sigma_{i+1}^{\alpha-1} - \sigma_{i+2}^{\alpha-1}) \qquad (8.9)$$

$$\Pi_{ij} = 2\alpha\sin\theta_i\frac{\partial}{\partial\theta_j}(\sigma_i^{\alpha-1} - \sigma_{i+1}^{\alpha-1} - \sigma_{i+2}^{\alpha-1}), i \neq j \qquad (8.10)$$

Lemma 8.5. Γ_n^- *is a local maximum for* $E_\alpha(\Theta)$ *when* $\alpha \geq 3/2$.

Proof. We have

$$\Delta E_\alpha(\Theta)(\Delta\theta_1, \Delta\theta_2, \Delta\theta_3) = \sum_{i=1}^{3}\Pi_i\Delta\theta_i + \frac{1}{2}\sum_{i,j}\Pi_{ij}\Delta\theta_i\Delta\theta_j + o(\sum|\Delta\theta_i|^2).$$

Since $\theta_1 = \theta_2 = 0$ and $\theta_3 = \Pi$ for Γ_6^-, we have $\Pi_i = 0$, $\Pi_{ij} = 0, i \neq j$ at Γ_6^- by (8.8) and (8.10). Moreover for Γ_6^-, we have $\sigma_1 = \sigma_1 = 1$ and $\sigma_3 = 9$ (this can be seen directly from the picture of Γ_6^- or from (8.6)). Then we have $\Pi_{11} = \Pi_{22} = -2\alpha 9^{\alpha-1}$, and $\Pi_{33} = -2\alpha(9^{\alpha-1} - 2)$ by (8.9). Once $\alpha \geq 3/2$, we have $\Pi_{33} = -2\alpha(9^{\alpha-1} - 2) < 0$ by (8.9), therefore at Γ_n^- we have

$$\Delta E_\alpha(\Theta)(\Delta\theta_1, \Delta\theta_2, \Delta\theta_3) = \sum_{ii}\Pi_{ii}\Delta\theta_i\Delta\theta_i + o(\sum|\Delta\theta_i|^2) < 0.$$

So Γ_n^- is a local maximum of $E_\alpha(\Theta)$ when $\alpha > 3/2$. $\square$

Lemma 8.6. *If* Γ_6 *is not in* B*, and* $\theta_3 \geq \frac{2}{3}\Pi$*, then* Γ_n *is not a critical point for* $\alpha \geq 2$.

Proof. If Γ_6 is not in B and Γ_6 is a critical point, then from $dE_\alpha(\Theta) = \Pi_1 d\theta_1 + \Pi_2 d\theta_2 + \Pi_3 d\theta_3$ and $d\theta_1 + d\theta_2 + d\theta_3 = 0$, we have

$$0 = \Pi_1 d\theta_1 + \Pi_2 d\theta_2 - \Pi_3(d\theta_1 + d\theta_2) = (\Pi_1 - \Pi_2)d\theta_1 + (\Pi_2 - \Pi_3)d\theta_2$$

which implies $\Pi_1 = \Pi_2 = \Pi_3$. Let $K = \frac{\Pi_i}{2\alpha}$. From (8) we have

$$\sigma_i^{\alpha-1} - \sigma_{i+1}^{\alpha-1} - \sigma_{i+2}^{\alpha-1} = \frac{K}{\sin\theta_i},\ i = 1, 2, 3 \qquad (8.11)$$

Then we obtained

$$\sigma_i^{\alpha-1} = -\frac{K}{2}(\frac{1}{\sin\theta_{i+1}} + \frac{1}{\sin\theta_{i+2}}) \qquad (8.12)$$

Since $\theta_1 \le \theta_2 \le \theta_3$, we have $\sin\theta_1 \le \sin\theta_2 \le \sin\theta_3$ (this is clear if $\theta_3 \le \pi/2$, and if $\theta_3 > \pi/2$, then $\theta_2 \le \pi - \theta_3 < \pi/2$, hence $\sin\theta_2 \le \sin(\pi - \theta_3) = \sin(\theta_3)$). Then we have

$$1 \le \frac{\frac{1}{\sin\theta_2} + \frac{1}{\sin\theta_1}}{\frac{1}{\sin\theta_1} + \frac{1}{\sin\theta_3}} = \frac{\sigma_3^{\alpha-1}}{\sigma_2^{\alpha-1}} = (\frac{\sigma_3}{\sigma_2})^{\alpha-1} \le 2 \qquad (8.13)$$

Note since $\theta_3 \ge \frac{2\pi}{3}$, $\theta_2 \le \frac{\pi}{3}$, we have $\cos\theta_2 - \cos\theta_3 \ge 1$. Then $\sigma_2 = 3 + 2\cos\theta_1 - 2(\cos\theta_2 - \cos\theta_3) \le 3$. Furthermore

$$\sigma_3 - \sigma_2 = 4(\cos\theta_2 - \cos\theta_3) \ge 4. \qquad (8.14)$$

Then

$$\frac{\sigma_3}{\sigma_2} = 1 + \frac{\sigma_3 - \sigma_2}{\sigma_2} \ge 1 + \frac{4}{\sigma_2} \ge 1 + \frac{4}{3} = \frac{7}{3}.$$

$(\frac{\sigma_3}{\sigma_2})^{\alpha-1} \le 2$ implies $(\frac{7}{3})^{\alpha-1} \le 2$, which implies $\alpha < 2$. A contradiction.

□

Lemma 8.7. *If Γ_6 is in B, and $\theta_3 \ge \frac{2}{3}\pi$, then $\Gamma_n \ne \Gamma_n^-$ is not a critical point.*

Proof. In this case, we have $\theta_1 = 0$ and $\theta_2 + \theta_3 = \pi$, and therefore $\sin\theta_2 = \sin\theta_3 \ne 0$. Fix $\theta_1 = 0$, then $d\theta_1 = 0$. If Γ_6 is a critical point, from $dE_3(\alpha) = \Pi_2 d\theta_2 + \Pi_3 d\theta_3$ and $d\theta_2 + d\theta_3 = 0$, we have $\Pi_2 = \Pi_3$, which implies that

$$2\alpha\sin\theta_2(\sigma_2^{\alpha-1} - \sigma_3^{\alpha-1} - \sigma_1^{\alpha-1}) = 2\alpha\sin\theta_3(\sigma_3^{\alpha-1} - \sigma_1^{\alpha-1} - \sigma_2^{\alpha-1}),$$

which implies

$$(\sigma_2^{\alpha-1} - \sigma_3^{\alpha-1} - \sigma_1^{\alpha-1}) = (\sigma_3^{\alpha-1} - \sigma_1^{\alpha-1} - \sigma_2^{\alpha-1}),$$

which implies $\sigma_3 = \sigma_2$. But since $\theta_3 \ge \frac{2\pi}{3}$, we have $\sigma_3 - \sigma_2 \ge 4$ by (8.14), a contradiction. □

Lemma 8.8. *When $\theta_3 \le \frac{2}{3}\pi$, $E_\alpha(\Theta) < E_{6,3}^{2\alpha}(\Gamma_n^-)$ for $\alpha \ge 3$.*

Proof. Note first $\frac{1}{3}\pi \le \theta_3 \le \frac{2}{3}\pi$ implies $|\cos\theta_3| \le \frac{1}{2}$. Since $\theta_1 \le \theta_2 \le \theta_3$, $\cos\theta_1 \ge \cos\theta_2 \ge \cos\theta_3$. So

$$\sigma_1 = 3 - 2\cos\theta_1 + 2\cos\theta_2 + 2\cos\theta_3 \le 3 + 2\cos\theta_3 \le 4. \qquad (8.15)$$

Similarly

$$\sigma_2 = 3 - 2\cos\theta_2 + 2\cos\theta_3 + 2\cos\theta_1 \le 3 + 2\cos\theta_1 \le 5. \qquad (8.16)$$

$$\sigma_3 = 3 - 2\cos\theta_3 + 2\cos\theta_1 + 2\cos\theta_2 \le 4 + 2\cos\theta_1 + 2\cos\theta_2 \le 8. \qquad (8.17)$$

Then with (8.15), (8.16) and (8.17), we can determine α such that

$$E_\alpha(\Theta) = \sigma_1^\alpha + \sigma_2^\alpha + \sigma_3^\alpha \le 4^\alpha + 5^\alpha + 8^\alpha \le 9^\alpha + 2 = E_{6,3}^{2\alpha}(\Gamma_n^-). \qquad (8.18)$$

For $\alpha = 3$, we have $64+125+512 = 701 < 729+2$. So (8.18) holds for any $\alpha \ge 3$. □

Now we prove Proposition 8.4: Suppose $\alpha \geq 3$. The maximum point of $E_\alpha(\Theta)$ must be a local maximum of $E_\alpha(\Theta)$. In

$$\{\Gamma_6 | \Gamma_6 \text{ is central symmetry}, \max\{\theta_1, \theta_2, \theta_3\} \geq 2\pi/3\},$$

the only local maximum point of $E_\alpha(\Theta)$ is Γ_n^- by Lemmas 8.5, 8.6, 8.7; in

$$\{\Gamma_6 | \Gamma_6 \text{ is central symmetry}, \max\{\theta_1, \theta_2, \theta_3\} < 2\pi/3\},$$

$E_\alpha(\Theta) < E_{6,3}^{2\alpha}(\Gamma_n^-)$ by Lemma 8.8. So Γ_6^- is the maximum point of $E_\alpha(\Theta)$ among all central symmetry Γ_6.

Recall we use $E_\alpha(\Theta)$ to denote $E_{6,3}^{2\alpha}(\Gamma_6)$. So Γ_6^- is the maximum point of $E_{6,3}^{\alpha}(\Gamma_6)$ among all central symmetry Γ_6 when $\alpha > 6$. □

8.2 A physics meaning of $E_n^2(\Gamma_n)$

Remark 8.9. If we consider each vertex A_i of Γ_n has unit mass, and there no mass on the curve Γ. Then $E_n^2(\Gamma_n)$ is the the moment of inertia of Γ_n about its mass center, up to a constant n.

Proof. We choose the mass center of Γ_n be the origin O. Then by definition

$$\sum_{i=1}^{n} A_i = 0$$

Now

$$\begin{aligned} E_n^2(\Gamma_n) &= \sum_{i<j} |A_i - A_j|^2 = \sum_{i<j} \langle A_i - A_j, A_i - A_j \rangle \\ &= \frac{1}{2} \sum_{i,j} \langle A_i - A_j, A_i - A_j \rangle \\ &= \frac{1}{2} \sum_{i,j} (\langle A_i, A_i \rangle - \langle A_i, A_j \rangle - \langle A_i, A_j \rangle + \langle A_j, A_j \rangle) \\ &= \frac{1}{2} \sum_{i,j} (\langle A_i, A_i \rangle - 2 \langle A_i, A_j \rangle + \langle A_j, A_j \rangle) \\ &= n(\sum_i |A_i|^2 - \sum_{i,j} \langle A_i, A_j \rangle) \end{aligned}$$

On the other hand

$$\sum_{i,j} \langle A_i, A_j \rangle = \left\langle \sum_{i=1}^{n} A_i, \sum_{i=1}^{n} A_i \right\rangle = \langle 0, 0 \rangle = 0$$

So we have

$$E_n^2(\Gamma_n) = n \sum_i |A_i|^2.$$

That is to say, $\sum_i |A_i|^2$ is the moment of inertia of Γ_n about its mass center. □

9 文献

References

[AHM] C. Audet, P. Hansen, F. Messine, *Extremal problems for convex polygons.* J. Global Optim. 38 (2007), no. 2, 163–179.

[Ar] M.A. Armstrong, *Basic Topology.* McGraw-Hill, 1979.

[AP] E. L. Altschuler, A. Pérez-Garrido, *Global minimum for Thomson's problem of charges on a sphere* PHYSICAL REVIEW E 71, 047703 (2005)

[BH] J. Berman, K. Hanes, *Optimizing the arrangement of points on the unit sphere.* Math. Comp. 31 (1977), no. 140, 1006-1008.

[BKOS] M. de Berg, M. Kreveld, M. Overmars, O. Schwarzkopf, *Computational geometry. Algorithms and applications.* Springer-Verlag, Berlin, 2000.

[HLP] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities.* 2d ed. Cambridge, at the University Press, 1952.

[LZ] Z. Li, J.Y. Zhou, *Advanced Mathematics,* (Chinese) Peking University Press (2004)

[St] Gilbert Strang, *Calculus.* Wellesley-Cambridge Press, (1991)

[PB] B. Plestenjak, V. Batagelj, *Optimal arrangements of n points on a sphere and in a circle.* Proceedings (Preddvor), 83-87, Slov. Soc. Inform., Ljubljana (2001)

[Sm] S. Smale, *Mathematical Problems for the Next Century.* In Mathematics: Frontiers and Perspectives, edited by Arnold, Atiyah, Lax, and Mazur. Providence, RI: Amer. Math. Society, (2000).

[Th] J. J. Thomson, Philos. Mag. 7, 237 (1904).

[To] L. F. Tóth, *Über eine Punktverteilung auf der Kugel*, Acta Math. Hungar. 10 (1959) 13–19.

[Yo] C. Y. You, *Basic Topology.* (Chinese) Peking University Press (1997)

10 队员简介

王中子，男，汉族。

2001 年生于北京。

2012 年进入北京市人大附中早培班。

2016 年进入人大附中高一 24 班 (最好的理科班)。

喜欢思考数学和物理问题，学过一些大学的数学和物理。

数学方面做过几个小的研究习作。

业余爱好包括音乐和围棋 (业余五段)。