A physical method for the analysis of the Riemann zeta function and the L-function

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Abstract

A physical method is developed to analyze the Riemann zeta function and the *L*-function. First, a physical model of an elastic bar fixed at both ends and subjected to symmetric axial loads with respect to the mid-span is constructed. The axial force of the bar at the left end is calculated with two methods: the equilibrium of forces and the principle of minimum potential energy. With the help of the orthogonality of $\{\sin ix\}$ $(i = 1, 2, 3, \cdots)$, the Parseval identity and the Bessel inequality of the physical model are obtained. Further, it is proven that, in all the possible displacements which satisfy the boundary conditions, the real one minimizes the total potential energy of the bar. With the equivalence of the two methods, an identity of a type of infinite series is derived. Based on the identity, $L(1)$, $L(3)$, a recurrence formula of $L(2k-1)$ $(k \geq 3)$, $\zeta(2)$, $\zeta(4)$, and a relationship between $\zeta(2k)$ $(k \geq 2)$ and $L(2i-1)$ $(i = 1, 2, ..., k)$ are then deduced with power axial load functions. The upper and lower bounds for the Riemann zeta function $\zeta(2k-1)$ and the *L*-function $L(2k)$ are given. The numerical results show that the upper and lower bounds are in excellent agreement with the accurate value. Finally, two improvement methods are proposed for estimating the circumference ratio. The numerical results show that the two methods can estimate the circumference ratio with high accuracy and that the L-function is more efficient than the Riemann zeta function in estimating the circumference ratio.

Keywords: physical method, Riemann zeta function, *L*-functionm, identity of a type of infinite series, upper and lower bounds, circumference ratio.

1 Introduction

In 1990, the great mathematician Hilbert presented 23 problems in the second International Congress of Mathematicians, including the famous Riemann hypothesis. The conjecture states that the analytic continuation of the zeta function has infinitely many nonreal roots which all lie on the critical line $x = 1/2$ in the complex plane. However, it still remains open[1]. In spite of this, researchers at home and abroad have achieved fruitful results in the aspect of the Riemann zeta function and the L-function. In 1735, Euler obtained the sum of the reciprocals of all positive integers, i.e. the Riemann zeta function $\zeta(2)$, using series expansion[2]. Subsquently, the method was further extended and used to evaluate the Riemann zeta function $\zeta(2k)$ [3, 4]. Yang also evaluated the Riemann zeta function $\zeta(2k)$ and other infinite series using the Parseval identity[5]. Lao and Zhao analyzed the Riemann zeta function $\zeta(2k)$ and the L-function $L(2k-1)$ based on a simply supported beam[6]. In 1979, Apéry proved that $\zeta(3)$ is irrational, but no similar results are known for other odd numbers[1]. Guan used the probability distribution of random variables to give the upper bound of the Riemann zeta function $\zeta(2k-1)$ [7]. Dancs and He obtained an Euler-type formula for $\zeta(2k-1)$ [8].

Based on previous studies, an identity of a type of infinite series is derived with a physical model of an elastic bar. The power axial load function is then chosen to evaluate the Riemann zeta function $\zeta(2k)$ and the L-function $L(2k-1)$. The upper and lower bounds for the Riemann zeta function $\zeta(2k-1)$ and the L-function $L(2k)$ are given. Finally, two improvement methods are proposed for estimating the circumference ratio.

2 Identity of a type of infinite series

An elastic bar with uniform cross-section and fixed at both ends is considered, as shown in Fig. 1. For the convenience of analysis, the elastic modulus E, cross-sectional area A, and length l are taken as

$$E = 1,\ A = 1,\ l = \pi \tag{1}$$

Obviously, the boundary conditions are as follows

$$u(x)\big|_{x=0} = u(x)\big|_{x=\pi} = 0 \tag{2}$$

where $u(x)$ is the displacement of the bar at any point x. According to mechanics of materials[9], the axial force $N(x)$ of the bar is positive when its direction is the same as the outward normal direction of the cross-section. If reversed, it is negative. When the bar is subjected to a symmetric axial load $p(x)$ with respect to the mid-span $x = \pi / 2$, i.e.,

$$p(x) = p(\pi - x),\ 0 \le x \le \pi / 2 \tag{3}$$

the axial force and the displacement are antisymmetric and symmetric with respect to $x = \pi / 2$, respectively. This suggests that the axial force at the mid-span is equal to zero, i.e.,

$$N(x)\big|_{x=\pi/2} = 0 \tag{4}$$

It should be pointed out that, when the bar is subjected to a concentrated load at $x = \pi / 2$, the axial force at $x = \pi / 2$ is defined as the average of those of the left and right cross-sections at this point. The equilibrium of the forces acting on the left half of the bar shown in Fig. 1 immediately gives the axial force N_l at the left end

$$N_l = \int_0^{\pi/2} p(x) dx \tag{5}$$

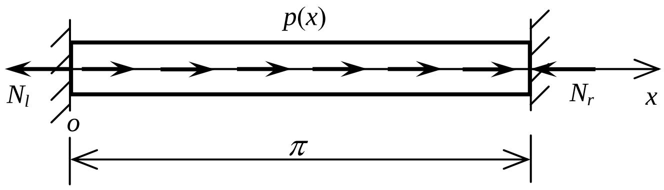

Fig. 1 A bar with uniform cross-section and subjected to axial loads

On the other hand, N_l can also be obtained from the principle of minimum potential energy as follows[10]. According to mechanics of materials[9], the strain and stress in the bar can be expressed as

$$\varepsilon(x) = \frac{du(x)}{dx} \tag{6}$$

$$\sigma(x) = E\varepsilon(x) = \frac{du(x)}{dx} \tag{7}$$

The strain energy E_i stored in the bar and the external potential energy E_e associated with the applied loads are given by[10]

$$E_i = \frac{A}{2}\int_0^{\pi} \sigma(x)\varepsilon(x) dx = \frac{1}{2}\int_0^{\pi} \left(\frac{du(x)}{dx}\right)^2 dx \tag{8}$$

$$E_e = -\int_0^{\pi} p(x)u(x)dx \tag{9}$$

Thus, the total potential energy E_t is equal to

$$E_t = E_i + E_e = \frac{1}{2}\int_0^{\pi}\left(\frac{du(x)}{dx}\right)^2 dx - \int_0^{\pi} p(x)u(x)dx \tag{10}$$

In view of the boundary conditions of Eq. (2), the displacement of the bar is of the form[10]

$$u(x) = \sum_{i=1}^{\infty} a_i \sin ix \tag{11}$$

where a_i $(i = 1, 2, 3, \cdots)$ are the coefficients to be determined.

To find the coefficients a_i $(i = 1, 2, 3, \cdots)$ with the principle of minimum potential energy, it is essential to analyze the orthogonality and completeness of the trigonometric function system $\{\sin ix\}$ $(i = 1, 2, 3, \cdots)$. For two different positive integers i and j, it is easily shown that

$$\int_0^{\pi} \sin ix \sin jx dx$$

$$= \frac{1}{2}\left[\frac{\sin(i-j)x}{i-j} - \frac{\sin(i+j)x}{i+j}\right]\Bigg|_0^{\pi} = 0 \tag{12}$$

When i is equal to j, we have

$$\int_0^{\pi} \sin ix \sin jx dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2ix}{2i}\right]\Bigg|_0^{\pi} = \frac{\pi}{2} \tag{13}$$

Eqs. (12) and (13) demonstrates that $\{\sin ix\}$ $(i = 1, 2, 3, \cdots)$ is orthogonal on $[0, \pi]$. The same is true for $\{\cos ix\}$ $(i = 0, 1, 2, \cdots)$ on $[0, \pi]$.

With the help of the orthogonality of $\{\sin ix\}$ $(i = 1, 2, 3, \cdots)$ on $[0, \pi]$, we have

$$\int_0^{\pi} u^2(x)dx$$

$$= \int_0^{\pi}\left(\sum_{i=1}^{\infty} a_i \sin ix\right)^2 dx$$

$$= \frac{\pi}{2}\sum_{i=1}^{\infty} a_i^2 \tag{14}$$

Eq. (14) is called the Parseval identity.

For any $u(x)$ which is not identically zero on $[0, \pi]$, ε^2 is defined as

$$\varepsilon^2 = \int_0^\pi [u(x) - u_n(x)]^2 dx \tag{15}$$

where $u_n(x)$ is taken as the sum of the first n terms of Eq. (11), i.e.,

$$u_n(x) = \sum_{i=1}^{n} a_i \sin ix \tag{16}$$

Substituting Eq. (16) into Eq. (15) and noting the orthogonality of $\{\sin ix\}$ $(i = 1, 2, 3, \cdots)$ on $[0, \pi]$, we have

$$\begin{aligned}\varepsilon^2 &= \int_0^\pi u^2(x)dx - 2\int_0^\pi u(x)(\sum_{i=1}^{n} a_i \sin ix)dx + \int_0^\pi (\sum_{i=1}^{n} a_i \sin ix)^2 dx \\ &= \int_0^\pi u^2(x)dx - 2\int_0^\pi u(x)(\sum_{i=1}^{n} a_i \sin ix)dx + \frac{\pi}{2}\sum_{i=1}^{n} a_i^2 \end{aligned} \tag{17}$$

To minimize ε^2, the following conditions are specified

$$\frac{\partial \varepsilon^2}{\partial a_i} = 0,\ (i = 1, 2, 3, \cdots, n) \tag{18}$$

Substitution of Eq. (17) into Eq. (18) gives

$$a_i = \frac{2}{\pi}\int_0^\pi u(x)\sin ix dx,\ (i = 1, 2, 3, \cdots, n) \tag{19}$$

Thus, Eq. (17) can be rewritten as

$$\varepsilon^2 = \int_0^\pi u^2(x)dx - \frac{\pi}{2}\sum_{i=1}^{n} a_i^2 \geq 0 \tag{20}$$

Eq. (20) is called the Bessel inequality. It can be seen from Eq. (20) that, as n increases, ε^2 decreases monotonically. Further, when n approaches infinity, the Bessel inequality becomes the Parseval identity. Thus, we have

$$\lim_{n\to\infty}\int_0^\pi [u(x) - u_n(x)]^2 dx = 0 \tag{21}$$

Therefore, $\{\sin ix\}$ $(i = 1, 2, 3, \cdots)$ is complete on $[0, \pi]$.

Since the axial displacement $u(x)$ is symmetric but $\sin 2ix$ is antisymmetric with respect to $x = \pi / 2$, the coefficients a_{2i} $(i = 1, 2, 3, \cdots)$ in Eq. (11) are equal to zero. Thus, Eq. (11) reduces to

$$u(x) = \sum_{i=1}^{\infty} a_{2i-1} \sin(2i - 1)x \tag{22}$$

Substituting Eq. (22) into Eq. (10) and noting the orthogonality of $\{\cos(2i - 1)x\}$ $(i = 1, 2, 3, \cdots)$ on $[0, \pi]$ and the symmetry of $p(x)$ with respect to $x = \pi / 2$, we have

$$E_t = \frac{\pi}{4}\sum_{i=1}^{\infty}(2i-1)^2 a_{2i-1}^2 - 2\sum_{i=1}^{\infty} a_{2i-1}\int_0^{\pi/2} p(x)\sin(2i-1)x dx \tag{23}$$

According to Long and Bao[10], the principle of minimum potential energy results in

$$\frac{\partial E_t}{\partial a_{2i-1}} = 0\,,\ (i = 1, 2, 3, \cdots) \tag{24}$$

Substitution of Eq. (23) into Eq. (24) gives

$$a_{2i-1} = \frac{4}{\pi(2i-1)^2}\int_0^{\pi/2} p(x)\sin(2i-1)x dx\,,\ (i = 1, 2, 3, \cdots) \tag{25}$$

Below we further show that the total potential energy does achieve the minimum value at the solution of Eq. (25). For this purpose, any possible axial displacement which satisfies the boundary conditions of Eq. (2) and the symmetry with respect to $x = \pi/2$ can be constructed as

$$\widetilde{u}(x) = \sum_{i=1}^{\infty}(a_{2i-1} + \Delta a_{2i-1})\sin(2i-1)x \tag{26}$$

where Δa_{2i-1} $(i = 1, 2, 3, \cdots)$ is a set of arbitrary real numbers. Substitution of Eq. (26) into Eq. (10) gives the corresponding total potential energy

$$\widetilde{E}_t = \frac{\pi}{4}\sum_{i=1}^{\infty}(2i-1)^2(a_{2i-1} + \Delta a_{2i-1})^2 - 2\sum_{i=1}^{\infty}(a_{2i-1} + \Delta a_{2i-1})\int_0^{\pi/2} p(x)\sin(2i-1)x dx \tag{27}$$

It follows from Eqs. (23) and (27) that

$$\widetilde{E}_t - E_t = \frac{\pi}{4}\sum_{i=1}^{\infty}(2i-1)^2[2a_{2i-1}\Delta a_{2i-1} + (\Delta a_{2i-1})^2] - 2\sum_{i=1}^{\infty}\Delta a_{2i-1}\int_0^{\pi/2} p(x)\sin(2i-1)x dx$$

$$= \frac{\pi}{4}\sum_{i=1}^{\infty}(2i-1)^2(\Delta a_{2i-1})^2 + \frac{\pi}{2}\sum_{i=1}^{\infty}(2i-1)^2\Delta a_{2i-1}\left[a_{2i-1} - \frac{4}{\pi(2i-1)^2}\int_0^{\pi/2} p(x)\sin(2i-1)x dx\right] \tag{28}$$

Substitution of Eq. (25) into Eq. (28) yields

$$\widetilde{E}_t - E_t = \frac{\pi}{4}\sum_{i=1}^{\infty}(2i-1)^2(\Delta a_{2i-1})^2 \geq 0 \tag{29}$$

In Eq. (29), the equality is valid when and only when Δa_{2i-1} $(i = 1, 2, 3, \cdots)$ are all equal to zero. This proves that, in all the possible displacements which satisfy the boundary conditions of Eq. (2), the real one minimizes the total potential energy.

With the coefficients a_{2i-1} $(i = 1, 2, 3, \cdots)$ known, the axial force of the bar at the left end is given by

$$N_l = EA\left.\frac{du(x)}{dx}\right|_{x=0}$$

$$= \sum_{i=1}^{\infty} \frac{4}{\pi(2i-1)} \int_0^{\pi/2} p(x)\sin(2i-1)x dx \tag{30}$$

According to Long and Bao[10], N_l of Eq. (5) deduced from the equilibrium of forces is entirely equivalent to that of Eq. (30) derived from the principle of minimum potential energy, which gives the following identity of a type of infinite series

$$\sum_{i=1}^{\infty} \frac{4}{\pi(2i-1)} \int_0^{\pi/2} p(x)\sin(2i-1)x dx = \int_0^{\pi/2} p(x)dx \tag{31}$$

For problems in practical engineering, loads acting on a bar may be concentrated forces and/or a continuous or discontinuous distributed force. From the viewpoint of mechanics, the magnitude of each concentrated load and the resultant of the distributed load must be limited to guarantee the validity of the solutions of Eqs. (5) and (30). Therefore, the load function $p(x)$ should satisfy the following condition

$$\int_0^{\pi/2} |p(x)| dx < +\infty \tag{32}$$

In what follows, we will use Eq. (31) to analyze the Riemann zeta function and the L-function by choosing appropriate load functions.

3 Evaluation of Riemann zeta function $\zeta(2k)$ and L-function $L(2k-1)$

The Riemann zeta function $\zeta(s)$ is defined as[11]

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{i^s},\ R(s) > 1 \tag{33}$$

In this paper, s is limited to positive integers k larger than or equal to 2. In this case, it is easily shown that $\zeta(k)$ is convergent and satisfies the following relationship

$$\zeta(k) = \frac{1}{1-2^{-k}} \sum_{i=1}^{\infty} \frac{1}{(2i-1)^k} \tag{34}$$

The L-function $L(s)$ is defined as[11]

$$L(s) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{(2i-1)^s},\ R(s) > 0 \tag{35}$$

Likewise, s is limited to positive integers k larger than or equal to 1. In this case, the alternating series $L(k)$ is convergent. Below we evaluate the Riemann zete function $\zeta(2k)$ and the L-function $L(2k-1)$ with power axial load functions:

(1) When the bar shown in Fig. 1 is subjected to a concentrated axial load at the mid-

span, the load function can be expressed, in terms of the Dirac delta function[12], as

$$p(x) = p_0\delta(x-\pi/2),\ 0 \le x \le \pi/2 \tag{36}$$

Here and thereafter p_0 denotes a non-zero real number. Substituting Eq. (36) into Eq. (31) and applying the integral properties of the Dirac delta function[12]

$$\int_0^{\pi/2} p_0\delta(x-\pi/2)\sin(2i-1)x dx = \frac{p_0}{2}\sin\frac{(2i-1)\pi}{2} = \frac{(-1)^{i-1}p_0}{2} \tag{37a}$$

$$\int_0^{\pi/2} p_0\delta(x-\pi/2) dx = \frac{p_0}{2} \tag{37b}$$

we have

$$L(1) = \frac{\pi}{4} \tag{38}$$

(2) When the bar shown in Fig. 1 is subjected to a uniform axial load, the load function can be expressed as

$$p(x) = p_0,\ 0 \le x \le \pi/2 \tag{39}$$

Substituting Eq. (39) into Eq. (31) and applying the following integral formulae[13]

$$\int \sin(2i-1)x dx = -\frac{\cos(2i-1)x}{(2i-1)} + C \tag{40a}$$

$$\int dx = x + C \tag{40b}$$

we have

$$\sum_{i=1}^{\infty}\frac{1}{(2i-1)^2} = \frac{\pi^2}{8} \tag{41}$$

With the help of the relationship of Eq. (34), $\zeta(2)$ is obtained as

$$\zeta(2) = \frac{\pi^2}{6} \tag{42}$$

(3) In a similar manner, when the bar shown in Fig. 1 is subjected to a symmetric linear and quadratic axial load, respectively, $L(3)$ and $\zeta(4)$ are evaluated as

$$L(3) = \frac{\pi^3}{32} \tag{43}$$

$$\zeta(4) = \frac{\pi^4}{90} \tag{44}$$

(4) From the table of integrals[13], the following integral formula is valid for any positive integer m

$$\int x^m \sin(2i-1)x dx = \cos(2i-1)x \sum_{r=0}^{[m/2]} (-1)^{r+1} \frac{m! x^{m-2r}}{(m-2r)!(2i-1)^{2r+1}}$$

$$+\sin(2i-1)x \sum_{r=0}^{[(m-1)/2]} (-1)^r \frac{m! x^{m-2r-1}}{(m-2r-1)!(2i-1)^{2r+2}} + C \tag{45}$$

When the bar shown in Fig. 1 is subjected to a $(2k-1)$-th power axial load, the load function can be expressed as

$$p(x) = p_0 x^{2k-1}, k \geq 2 \text{ and } 0 \leq x \leq \pi/2 \tag{46}$$

Substituting Eq. (46) into Eq. (31) and noting Eq. (45), we have

$$\sum_{i=1}^{\infty}\sum_{r=0}^{k-1} \frac{(-1)^{i+r-1} 2^{2r}}{\pi^{2r}(2k-2r-2)!(2i-1)^{2r+3}} = \frac{\pi^3}{32k(2k-1)!} \tag{47}$$

According to the definition of the L-function of Eq. (35), the recurrence formula of $L(2k+1)$ is obtained as

$$L(2k+1) = \frac{(-1)^{k-1}\pi^{2k+1}}{2^{2k+3}k(2k-1)!} - \sum_{r=0}^{k-2} \frac{(-1)^{k+r-1}\pi^{2k-2r-2}L(2r+3)}{2^{2k-2r-2}(2k-2r-2)!} \tag{48}$$

(5) When the bar shown in Fig. 1 is subjected to a $(2k)$-th power axial load, the load function can be expressed as

$$p(x) = p_0 x^{2k}, k \geq 2 \text{ and } 0 \leq x \leq \pi/2 \tag{49}$$

Substituting Eq. (49) into Eq. (31) and noting Eq. (45), we have

$$\sum_{i=1}^{\infty}\left[\frac{(-1)^k}{(2i-1)^{2k+2}} + \sum_{r=0}^{k-1} \frac{(-1)^{i+r-1}\pi^{2k-2r-1}}{2^{2k-2r-1}(2k-2r-1)!(2i-1)^{2r+3}}\right] = \frac{\pi^{2k+2}}{2^{2k+3}(2k+1)(2k)!} \tag{50}$$

According to the definitions of the Riemann zeta function and the L-function of Eqs. (33) and (35), respectively, and the relationship of Eq. (34), Eq. (50) can be rewritten as

$$(-1)^k(1-2^{-2k-2})\zeta(2k+2) + \sum_{r=0}^{k-1} \frac{(-1)^r \pi^{2k-2r-1}L(2r+3)}{2^{2k-2r-1}(2k-2r-1)!} = \frac{\pi^{2k+2}}{2^{2k+3}(2k+1)(2k)!} \tag{51}$$

Rearrangement of Eq. (51) gives the relationship between the Riemann zeta function $\zeta(2k+2)$ and the L-functions $L(2r+3)$ $(r = 0, 1, \cdots, k-1)$ as follows

$$\zeta(2k+2) = \frac{(-1)^k 2^{2k+2}}{(2^{2k+2}-1)}\left[\frac{\pi^{2k+2}}{2^{2k+3}(2k+1)(2k)!} - \sum_{r=0}^{k-1} \frac{(-1)^r \pi^{2k-2r-1}L(2r+3)}{2^{2k-2r-1}(2k-2r-1)!}\right] \tag{52}$$

Thus, $L(2k-1)$ is evaluated from Eqs. (38), (43), and (48). With $L(2k-1)$ known, $\zeta(2k)$ can be obtained from Eq. (52). As an example, besides $L(1)$, $L(3)$, $\zeta(2)$, and $\zeta(4)$ given above, $L(5)$, $L(7)$, $\zeta(6)$, and $\zeta(8)$ are evaluated as $5\pi^5/1536$, $61\pi^7/184320$, $\pi^6/945$, and $\pi^8/9450$, respectively, which are the same as those in [6].

4 Upper and lower bounds of Riemann zeta function $\zeta(2k-1)$ and L-function $L(2k)$

Until now no evaluation on the Riemann zeta function $\zeta(2k-1)$ and the L-function $L(2k)$ has been reported in the literature. Therefore, it is essential to give their upper and lower bounds[7]. When $k \ge 2$, it follows from the definition of the Riemann zeta function of Eq. (33) that

$$\begin{aligned}\zeta(2k-1) &= \sum_{i=1}^{\infty}\frac{1}{i^{2k-1}} \\ &= \sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k-1}} + \sum_{i=1}^{\infty}\frac{1}{(2i)^{2k-1}} \\ &= \sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{(2i-1)^{2k-1}} + 2\sum_{i=1}^{\infty}\frac{1}{(4i-1)^{2k-1}} + \frac{1}{2^{2k-1}}\zeta(2k-1) \\ &= L(2k-1) + \frac{2}{3^{2k-1}} + 2\sum_{i=1}^{\infty}\frac{1}{(4i+3)^{2k-1}} + \frac{1}{2^{2k-1}}\zeta(2k-1) \end{aligned} \tag{53}$$

Since

$$\frac{1}{4i+4} < \frac{1}{4i+3} < \frac{1}{4i+2} \tag{54}$$

we have

$$\sum_{i=1}^{\infty}\frac{1}{(4i+3)^{2k-1}} > \sum_{i=1}^{\infty}\frac{1}{(4i+4)^{2k-1}} = \frac{1}{2^{4k-2}}[\zeta(2k-1)-1] \tag{55}$$

$$\sum_{i=1}^{\infty}\frac{1}{(4i+3)^{2k-1}} < \sum_{i=1}^{\infty}\frac{1}{(4i+2)^{2k-1}} = \left(\frac{1}{2^{2k-1}} - \frac{1}{2^{4k-2}}\right)\zeta(2k-1) - \frac{1}{2^{2k-1}} \tag{56}$$

Substitution of Eqs. (55) and (56) into Eq. (53) gives the estimate of $\zeta(2k-1)$ as follows

$$\frac{L(2k-1) + \dfrac{2}{3^{2k-1}} - \dfrac{1}{2^{4k-3}}}{1 - \dfrac{1}{2^{2k-1}} - \dfrac{1}{2^{4k-3}}} < \zeta(2k-1) < \frac{L(2k-1) + \dfrac{2}{3^{2k-1}} - \dfrac{1}{2^{2k-2}}}{1 - \dfrac{3}{2^{2k-1}} + \dfrac{1}{2^{4k-3}}} \tag{57}$$

According to the definition of the L-function of Eq. (35), $L(2k)$ $(k \ge 1)$ is written as

$$\begin{aligned} L(2k) &= \sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{(2i-1)^{2k}} \\ &= \sum_{i=1}^{\infty}\frac{1}{(2i-1)^{2k}} - 2\sum_{i=1}^{\infty}\frac{1}{(4i-1)^{2k}} \end{aligned}$$

$$=\left(1-\frac{1}{2^{2k}}\right)\zeta(2k)-\frac{2}{3^{2k}}-2\sum_{i=1}^{\infty}\frac{1}{(4i+3)^{2k}} \tag{58}$$

In a similar manner, it follows from Eq. (54) that

$$\sum_{i=1}^{\infty}\frac{1}{(4i+3)^{2k}}>\sum_{i=1}^{\infty}\frac{1}{(4i+4)^{2k}}=\frac{1}{2^{4k}}[\zeta(2k)-1] \tag{59}$$

$$\sum_{i=1}^{\infty}\frac{1}{(4i+3)^{2k}}<\sum_{i=1}^{\infty}\frac{1}{(4i+2)^{2k}}=\left(\frac{1}{2^{2k}}-\frac{1}{2^{4k}}\right)\zeta(2k)-\frac{1}{2^{2k}} \tag{60}$$

Substitution of Eqs. (59) and (60) into Eq. (58) gives the estimate of $L(2k)$ as follows

$$\left(1-\frac{3}{2^{2k}}+\frac{1}{2^{4k-1}}\right)\zeta(2k)-\frac{2}{3^{2k}}+\frac{1}{2^{2k-1}}<L(2k)<\left(1-\frac{1}{2^{2k}}-\frac{1}{2^{4k-1}}\right)\zeta(2k)-\frac{2}{3^{2k}}+\frac{1}{2^{4k-1}} \tag{61}$$

As a numerical example, $\zeta(3)$, $\zeta(5)$, $\zeta(7)$, $L(2)$, $L(4)$, $L(6)$, and $L(8)$ are considered. Their accurate values, lower bounds, and upper bounds are listed in Tables 1 to 4, where the accurate values are computed from the sum of the first 100 terms of these series. At the same time, the upper bounds of $\zeta(3)$, $\zeta(5)$, and $\zeta(7)$ given by Guan[7] are also listed in Table 2. It can be seen from Tables 1 to 4 that the upper and lower bounds are in excellent agreement with the accurate value and that the computational accuracy increases rapidly as k increases. Table 2 also shows that the upper bound given in this paper is much more accurate than that given by Guan[7].

Table 1 Accurate value and lower bound of $\zeta(2k-1)$

$2k-1$	Accurate value	Lower bound	Relative error/%
3	1.202007401	1.199135171	0.239
5	1.036927753	1.036862365	0.000631
7	1.008349277	1.008347934	0.0000133

Table 2 Accurate value and upper bound of $\zeta(2k-1)$

$2k-1$	Accurate value	This paper		Guan[7]	
		Upper bound	Relative error/%	Upper bound	Relative error/%
3	1.202007401	1.208411887	0.533	1.334297702	11.0
5	1.036927753	1.037090044	0.00157	1.049330278	1.20
7	1.008349277	1.008354413	0.0000509	1.010688445	0.232

Table 3 Accurate value and lower bound of $L(2k)$

$2k$	Accurate value	Lower bound	Relative error/%
2	0.9159530951	0.8946280871	2.33
4	0.9889445514	0.9881520185	0.00801
6	0.9986852222	0.9986585333	0.000267
8	0.9998499902	0.9998493560	0.00000634

Table 4 Accurate value and upper bound of $L(2k)$

$2k$	Accurate value	Upper bound	Relative error/%
2	0.9159530951	0.9308616268	1.63
4	0.9889445514	0.9893436353	0.00404
6	0.9986852222	0.9986952912	0.000101
8	0.9998499902	0.9998504457	0.00000456

5 Estimate of circumference ratio

The circumference ratio is an important constant and can be estimated with various methods. From the viewpoint of infinite series, the estimate is mainly based on the Euler formula, i.e., $\zeta(2)$. Since the recurrence formulae of $\zeta(2k)$ and $L(2k-1)$ are obtained, they can used to improve the estimate of the circumference ratio.

As a numerical example, $\zeta(2)$, $\zeta(4)$, $\zeta(6)$, $L(1)$, $L(3)$, $L(5)$, and $L(7)$ are considered. When these series are approximated by the sum of the first ten terms, the numerical results are listed in Tables 5 and 6. It can be seen from Table 5 that, when $\zeta(2)$ is adopted, the relative error is 2.94%. When $\zeta(4)$ and $\zeta(6)$ are adopted, the relative error rapidly reduces to 0.00662% and 0.0000254%, respectively, which greatly increases the computational accuracy. It can be seen from Table 6, compared with the Riemann zeta function, the L-function is more efficient in estimating the circumference ratio. When $L(1)$, $L(3)$, $L(5)$, and $L(7)$ are adopted, the relative error is equal to 3.18%, 0.00212%, 0.00000306%, and 0.0000000318%, respectively.

Table 5 Estimated circumference ratio based on $\zeta(2k)$

$2k$	Accurate π	Estimated π	Relative error/%
2	3.141592654	3.049361636	2.94
4	3.141592654	3.141384623	0.00662
6	3.141592654	3.141591856	0.0000254

Table 6 Estimated circumference ratio based on $L(2k-1)$

$2k$ - 1	Accurate π	Estimated π	Relative error/%
1	3.141592654	3.041839619	3.18
3	3.141592654	3.141526088	0.00212
5	3.141592654	3.141592558	0.00000306
7	3.141592654	3.141592653	0.0000000318

6 Conclusions

From the analysis of the Riemann zeta function and the L -function, the main conclusions are made as follows.

(1) A physical model of an elastic bar fixed at both ends and subjected to symmetric axial loads with respect to the mid-span has been constructed. The equilibrium of forces and the priciple of minimum potential energy have been used to determine the axial force of the bar at the left end. It has been proven that, in all the possible displacements which satisfy the boundary conditions, the real one minimizes the total potential energy. From the entire equivalence of the two methods, an identity of a type of infinite series has been derived.

(2) Based on the identity of infinite series, the Riemann zeta function $\zeta(2k)$ and the L-function $L(2k-1)$ have been evaluated with power axial load functions.

(3) The upper and lower bounds of the Riemann zeta function $\zeta(2k-1)$ and the L-function $L(2k)$ have been given. Based on the numerical results, it has been shown that the upper and lower bounds are in excellent agreement with the accurate value.

(4) Two improvement methods have been proposed for estimating the circumference ratio. Numerical results show that, the larger the value of k is, the smaller the relative error of the circumference ratio is. Compared with the Riemann zeta function, the L-function is more efficient in estimating the circumference ratio.

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