A Study on p-Adic Valuation of Binomial Coefficients

二项式系数的素数方次数的研究

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摘要：在本工作中，我们从杨辉三角模 2 的代数模式与谢尔宾斯基三角形的集合属性具等价性的观察出发，针对二项式系数的素数方次数进行了系统研究，得到了丰富新颖的结果。

首先，我们用数学软件 *Mathematica* 计算了一些方次数序列。从结果序列中我们观察到了一些模式，并据此提出一系列关于二项式系数素数方次数序列性质的猜想，包括方次数序列中的周期性规律以及局部等中分的性质。我们利用 Kummer 定理严格证明了这些猜想。此外，我们详细讨论了方次数序列计算。我们发现，当 $k \le p$ 时，序列内每一个方次数都可以被精确算出来，但是当 $k > p$ 时，只有少数条件满足时才可以精确计算出方次数值。最后，我们还讨论了方次数的取值范围。通过定义 p 幂最小组合数和 p 幂最大组合数，我们详细研究了 $n=(p^{r+1-\alpha}-1)p^{\alpha}$ 和一般 n 的情况，并给出了对应的组合数计算公式。

本研究结果能够帮助提升相关计算效率，在大数据等相关应用领域有潜在应用。

关键词：二项式系数；方次数；Mathematica 实验；p 幂最小（大）组合数；计数公式。

A Study on p-Adic Valuation of Binomial Coefficients

Abstract: Based on the observation of the relation between Yang Hui's triangle and Sierpinski triangle, the p-adic valuation of binomial coefficients has been systematically studied, which leads to plenty of interesting and innovative results.

Our study is initiated from a number of experiments using the software *Mathematica* for generating the sequence $\left\{v_p(\binom{n}{k})\right\}$, from which some patterns can be observed. Based on the observation, a series of conjectures on the property of the p-adic valuation of the binomial coefficients is then proposed, including that the sequence $\left\{v_p(\binom{n}{k})\right\}$ has some periodic patterns and local properties. With the help of Kummer's theorem, the proposed conjectures are proved rigorously. Moreover, the calculation of $\left\{v_p(\binom{n}{k})\right\}$ are discussed in detail, and it is found that in the case $k \le p$, any element in the sequence can be evaluated, while in the case $k > p$, the value $v_p(\binom{n}{k})$ can be obtained only in two situations, i.e., $n \equiv 0, k-1 (\bmod p^a)$.

Finally, we further consider the range of $v_p(\binom{n}{k})$. After defining the minimum and maximum numbers of combinations of the power of *p*, respectively, we discuss two numbers for a specific $n = (p^{r+1-\alpha} - 1)p^{\alpha}$ and for a general *n*. As a result, two formulas are successfully proposed for the evaluations.

The results obtained from this work can effectively simplify the related calculations, and there are potential applications in a variety of areas such as big data.

Keywords: Binomial coefficient; p-adic valuation; Mathematica experiment minimum and maximum combination of power *p*; Enumeration formula.

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1. Introduction

Even it is elementary, the study on binomial has still been playing an important role in both mathematical theory and practical applications.

In application, for example, binomial heap serves as an implementation of the mergeable heap abstract data type in the computer science. One of its features is that it supports quick merging of two heaps, and it has applications on discrete event simulation and priority queues. In finance, the binomial options pricing model was proposed in 1979 [28], which used a discrete time strategy to study the varying price over time. In fact, the binomial model can be seen as a discrete time approximation to the continuous behavior underlying Black-Scholes model, which is the most famous model in finance. It is worth mentioning that currently there are still many works on the binomial option pricing model and its variations, see [29-31]. Even in social science, we have binomial voting system, which served in the parliamentary elections of Chile from 1989 to 2013 [32]. Besides the above applications, binomial has also been applied in other areas such as biology, linguistics.

In mathematical theory, the study on binomial can be traced back to 4th century B.C. when the binomial theorem for exponent 2 was mentioned by Greek mathematician Euclid. In China, the study on binomial coefficients started from late-Song dynasty (around 1200 AC) Chinese mathematician Yang Hui (杨辉 in Chinese) who developed the famous Yang Hui’s triangle for the binomial coefficients. In Europe, Yang Hui’s triangle is also called the “Pascal’s triangle” and people there preferred to recognize that the triangle was devised by Pascal in 1654. Nevertheless, the discovery of the Yang Hui’s triangle in China should be at least 300 years earlier than the discovery in Europe. Although it has a long history on the development of the study on binomial, this research area is still an active one, see [1-6].

In summary, from both theory and application points of view, it deserves to further study binomial, and explore more applications. In this work, we focus on the p-adic valuation of the coefficients to systematically study the binomial, and obtain plenty of interesting results.

Our study is initiated from a number of experiments using the software *Mathematica* for generating the sequence $\left\{v_p(\binom{n}{k})\right\}$, from which some patterns can be observed. Based on the observation, we then proposed a series of conjectures on the property of the prime power of the binomial coefficients, including that the sequence $\left\{v_p(\binom{n}{k})\right\}$ has some periodic patterns and sub-sub-nature locally. With the help of Kummer’s theorem, the proposed conjectures are proved rigorously.

Moreover, the calculation of $\left\{v_p(\binom{n}{k})\right\}$ are discussed in detail, and it is found that in the case $k \le p$, any element in the sequence can be evaluated, while in the case $k > p$, the value $v_p(\binom{n}{k})$ can be obtained only in two situations, i.e., $n \equiv 0, k-1 (\bmod p^{\alpha})$.

Finally, we further consider the range of $v_p(\binom{n}{k})$. After defining the minimum and maximum numbers of combinations of the power of p, respectively, we discuss two numbers for a specific $n = (p^{r+1-\alpha} - 1)p^{\alpha}$ and for a general n. As a result, two formulas are successfully proposed for the evaluations.

The results obtained from this work can effectively simplify the related calculations, and there are potential applications in a variety of areas such as big data.

The study is arranged as follow. In this Section, we briefly introduce some preliminaries on the definition and lemma, then our mathematical experiments are introduced in detail in Section 2, as well as the conjectures from observations from the experimental data, and related analysis. In Section 3, we further discuss the range and enumeration of p-adic valuation. In Section 4, the conclusion as well as future work are given.

Definition 1. 1 [22−23] For a prime number *p* and a non-zero integer *n, r,* is said to be the *p*-adic valuation of *n*, denoted by $v_p(n) = r$, if $p^r \mid n, p^{r+1} \nmid n$.

For example, since $2 \mid 6, 2^2 \nmid 6$, we have $v_2(6) = 1$; furthermore,

$v_p(mn) = v_p(m) + v_p(n)$.

Definition 1. 2 Assume $n = \sum_{i=0}^{r} n_i p^i$, $n_i \in \mathbf{N}$, then p -adic number of n is defined as

$$n = (n_r n_{r-1} \cdots n_1 n_0)_p .$$

Lemma 1. 1 [1−2] The p -adic valuation of $n!$ is given by

$$v_p(n!) = \sum_{i \ge 1} \left[\frac{n}{p^i} \right].$$

Lemma 1. 2 The p -adic valuation of $\binom{n}{k} (n \ge k \ge 0)$ is

$$v_p(\binom{n}{k}) = \frac{g_p(k)+g_p(n-k)-g_p(n)}{p-1},$$

where $g_p(n)$ is the sum of the digits in the p-adic number of n.

Proof Assume the p-adic number of n be $n=(n_r n_{r-1}\cdots n_1 n_0)_p$, then

$$\frac{n}{p^i} = (n_r n_{r-1}\cdots n_i . n_{i-1}\cdots n_1 n_0)_p, \quad \left[\frac{n}{p^i}\right] = (n_r n_{r-1}\cdots n_i)_p .$$

From Lemma 1. 1 we have

$$v_p(n!) = \sum_{i\ge 1}\left[\frac{n}{p^i}\right] = \sum_{i\ge 1}(n_r n_{r-1}\cdots n_i)_p = \sum_{i=1}^{r}\sum_{j=i}^{r} n_j \cdot p^{j-i} = \sum_{j=1}^{r}\sum_{i=1}^{j} n_j \cdot p^{j-i} = \sum_{j=1}^{r} n_j \sum_{i=1}^{j} p^{j-i}$$

$$= \sum_{j=1}^{r} n_j \sum_{i=0}^{j-1} p^i = \sum_{j=1}^{r} n_j \frac{p^j-1}{p-1} = \frac{\sum_{j=1}^{r} n_j p^j - \sum_{j=1}^{r} n_j}{p-1} = \frac{n-g_p(n)}{p-1}.$$

As a result

$$v_p(\binom{n}{k}) = v_p(\frac{n!}{k!(n-k)!}) = v_p(n!) - v_p(k!) - v_p((n-k)!) = \frac{g_p(k)+g_p(n-k)-g_p(n)}{p-1},$$

the proof is completed then.

The following two theorems are famous for describing the properties for the p-adic valuation and congruence of the binominal coefficients.

Lemma 1. 3 (Kummer) [11] For given integers $n \ge m \ge 0$ and a prime number p, the p-adic valuation $v_p(\binom{n}{k})$ is equal to the number of carries when k is added to $n-k$ in base p.

Proof Let $n = k + m$, considering the p-adic numbers $n=(n_r\cdots n_i\cdots n_0)_p (n_r \ne 0)$, $k=(k_r\cdots k_i\cdots k_0)_p$, and $m=(m_r\cdots m_i\cdots m_0)_p$, then by defining ε_j as follows:

$$\varepsilon_j = \begin{cases} 1 & \text{when } k_j + \mathrm{m}_j + \varepsilon_{j-1} \ge p \\ 0 & \text{o.w.} \end{cases}.$$

we have $\varepsilon_r = 0, n_0 = k_0 + m_0 - p\varepsilon_0, n_j = k_j + m_j + \varepsilon_{j-1} - p\varepsilon_j (j \ge 1)$. From Lemma 1. 1. 2,

$$v_p(\binom{n}{k}) = \frac{g_p(k)+g_p(n-k)-g_p(n)}{p-1}$$

$$=\sum_{j=0}^{r}\frac{k_j+m_j-n_j}{p-1}$$

$$=\frac{p\varepsilon_0+\sum_{j=1}^{r}(p\varepsilon_j-\varepsilon_{j-1})}{p-1}$$

$$=\sum_{j=0}^{r-1}\varepsilon_j .$$

Corollary 1. 1 For given integers $n \ge m \ge 0$ and a prime number p, the p-adic valuation $v_p(\binom{n}{k})$ is equal to the number of borrows when k is subtracted to n in base p.

Lemma 1. 4[11,21,25] (Lucas, 1878) Assume $n=\sum_{i\ge 0} n_i p^i$, $k=\sum_{i\ge 0} k_i p^i$, then

$$\binom{n}{k} \equiv \prod_{i\ge 0}\binom{n_i}{k_i} (\bmod\, p).$$

2. Properties of the p-adic valuation sequence $\left\{ v_p(\binom{n}{k}) \right\}$

2. 1 Mathematica experiments

To find the law of the p-adic valuation sequence $\left\{ v_p(\binom{n}{k}) \right\}$, we first use the software Mathematica to produce some examples, and then we would like to generate some conjectures from these examples. Finally, rigorous proofs to these conjectures are expected to be provided.

Example 2. 1. 1 Evaluate the Pascal's Triangle (2-adic valuation). The Mathematica code is

```
(Print@@ Flatten[Riffle[#, "\t"]])   &/@Table[IntegerExponent[n!/ (n-i)   !/i!, 2],
{n, 0, 99}, {i, 0, n}];
```

Please refer to Appendix I to find the results.

Example 2. 1. 2 Evaluate the Pascal's Triangle (3-adic valuation). The Mathematica code is

```
(Print@@Flatten[Riffle[#, "\t"]])   &/@Table[IntegerExponent[n!/ (n-i)   !/i!, 3],
{n, 0, 99}, {i, 0, n}];
```

Please refer to Appendix I to find the results.

Example 2. 1. 3 Evaluate the first 198 terms of sequence $\left\{ v_2(\binom{n}{2}) \right\}$. The Mathematica code is

```
(Print@@Flatten[Riffle[#, "\t"]])  &/@Table[IntegerExponent[n!/ (n- i)  !/i!, 2], {n, 2, 200},
{i, 2, 2}];
```

The result is listed as below

0 0 1 1 0 0 2 2 0 0 1 1 0 0 **3 3** 0 0 1 1 0 0 2 2 0 0 1 1 0 0 **4 4** 0 0 1 1 0 0 2 2 0 0 1 1 0 0 **3 3** 0 0 1 1 0 0 2 2 0 0 1 1 0 0 **5 5** 0 0 1 1 0 0 2 2 0 0 1 1 0 0 **3 3** 0 0 1 1 0 0 2 2 0 0 1 1 0 0 **4 4** 0 0 1 1 0 0 2 2 0 0 1 1 0 0 **3 3** 0 0 1 1 0 0 2 2 0 0 1 1 0 0 **6 6** 0 0 1 1 0 0 2 2 0 0 1 1 0 0 **3 3** 0 0 1 1 0 0 2 2 0 0 1 1 0 0 **4 4** 0 0 1 1 0 0 2 2 0 0 1 1 0 0 **3 3** 0 0 1 1 0 0 2 2 0 0 1 1 0 0 **5 5** 0 0 1 1 0 0 2.

We can find that the numbers vary periodically with the format “0 0 1 1 0 0 2 2 0 0 1 1 0 0 ΔΔ”, where the first 14 terms are kept unchanged and the last two terms are changed with some certain law. From the observations, we conclude the conjecture Theorem 1. 2. 1 (1) .

To find how “ΔΔ” changes, we check the values only at the corresponding positions.

Example 2. 1. 4 Evaluate the values of the sequence $\left\{ v_2(\binom{n}{2}) \right\}$, $n \equiv 0 (\bmod 16)$. The Mathematica code is

```
(Print @@ Flatten[Riffle[#,  "\t"]])   & /@ Table[IntegerExponent[n!/ (n - i)  !/i!,
2],  {n, 16,  200, 16},  {i,  2,  2}];
```

The results are listed in the following table：

n	16	32	48	64	80	96	112	128	144	160	176	192
$v_2(n)$	4	5	4	6	4	5	4	7	4	5	4	6
$v_2(\binom{n}{2})$	3	4	3	5	3	4	3	6	3	4	3	5

From the table, we can identify the links between $v_2(\binom{n}{2})$ and $v_2(n)$, i.e. $v_2(\binom{n}{2}) = v_2(n) - 1$, when $v_2(n) \geq 1$. With the same strategy, we can find that the relationship also holds for the first 14 unchanged numbers “0 0 1 1 0 0 2 2 0 0 1 1 0 0”. Moreover, $v_2(\binom{n}{2}) = v_2(\binom{n+1}{2}) = v_2(n) - 1$. In summary, we can conclude the Theorem 2. 2. 1.

Similarly, we can obtain Theorem 2. 2. 2 by computing values for sequences $\left\{ v_3(\binom{n}{2}) \right\}$、

$\left\{v_5(\binom{n}{2})\right\}$.

2. 2 Properties for $\left\{v_p(\binom{n}{k})\right\}, k \le p$

Theorem 2. 2. 1 If $v_2(n) \ge 1$, then

$$v_2(\binom{n}{2}) = v_2(\binom{n+1}{2}) = v_2(n) - 1.$$

Proof Let $v_2(n) = \alpha \ge 1$, then

$$n = (n_r n_{r-1} \cdots n_\alpha \underbrace{00\cdots0}_{\alpha个0})_2,\ \ n+1 = (n_r n_{r-1} \cdots n_\alpha \underbrace{00\cdots0}_{e-1个0}1)_2,\ \ 2 = (10)_2.$$

Hence, $\alpha - 1$ borrows will be generated when 2 is subtracted to n or $n+1$. By Kummer's Theorem Corollary 1. 1, we have

$$v_2(\binom{n}{2}) = v_2(\binom{n+1}{2}) = \alpha - 1 = v_2(n) - 1.$$

Theorem 2. 2. 2 Given a prime $p \ge 3$,

(1) If $n \equiv 2, 3, \cdots, p-1 \pmod p$, then $v_p(\binom{n}{2}) = 0$;

(2) If $v_p(n) \ge 1$, then

$$v_p(\binom{n}{2}) = v_p(\binom{n+1}{2}) = v_p(n).$$

Proof (1) Method I: If $n \equiv 2, 3, \cdots, p-1 \pmod p$, let's say $n = m_i p + i,\ i = 2, 3, \cdots, p-1$,

then by Lemma 1. 4 (Lucas Theorem), we have $\binom{n}{2} = \binom{m_i p + i}{2} \equiv \binom{i}{2} \not\equiv 0 \pmod p$, thus

$v_p(\binom{n}{2}) = 0$.

Method II: If $n \equiv 2, 3, \cdots, p-1 \pmod p$, say $n = (n_r n_{r-1} \cdots n_1 n_0)_p$,

$n_0 = 2, 3, \cdots, p-1$, then $n - 2 = (n_r n_{r-1} \cdots n_1 n_0')_p$, $n_0' = n_0 - 2$. By Lemma 1. 2, we obtain

$$v_p(\binom{n}{2}) = \frac{g_p(2)+g_p(n-2)-g_p(n)}{p-1} = [2+(\sum_{k=1}^{r} n_k + n_0') - (\sum_{k=1}^{r} n_k + n_0)]/(p-1) = 0 .$$

(2) If $v_p(n)=\alpha \geq 1$, then

$$n = (n_r n_{r-1} \cdots n_\alpha \underbrace{00\cdots0}_{\alpha个0})_p,\quad n+1 = (n_r n_{r-1} \cdots n_\alpha \underbrace{00\cdots01}_{\alpha-1个0})_p,\quad 2=(2)_p .$$

Thus $\alpha - 1$ borrows will be generated when 2 is subtracted to n or $n+1$. By Kummer's Theorem Corollary 1. 1, we have

$$v_p(\binom{n}{2}) = v_p(\binom{n+1}{2}) = \alpha = v_p(n) .$$

Theorem 2. 2. 3 For given $n \in \mathbf{N}$, $n \geq 3$, if $n \equiv i(\bmod 4)$, $i = -1,0,1,2$, let $m=n-i$, then

$$v_2(\binom{m}{3}) = v_2(\binom{m+2}{3}) = v_2(m),$$

$$v_2(\binom{m+1}{3}) = v_2(m) - 1,$$

$$v_2(\binom{m-1}{3}) = 0 .$$

Proof Assume $v_2(m)=\alpha$, $m = (m_r m_{r-1} \cdots m_\alpha \underbrace{00\cdots0}_{\alpha个0})_2$, then

$$m+2 = (m_r m_{r-1} \cdots m_\alpha \underbrace{00\cdots010}_{\alpha-2个0})_2,\quad 3=(11)_2$$

$$m+1 = (m_r m_{r-1} \cdots m_\alpha \underbrace{00\cdots01}_{\alpha-1个0})_2,\quad m-1 = (m_r m_{r-1} \cdots m_1 0\underbrace{11\cdots1}_{\alpha个1})_2 .$$

Thus α borrows will be generated when 3 is subtracted to m or $m+2$, $\alpha - 1$ borrows will be generated when 3 is subtracted to $m+1$, while no borrows will be generated when 3 is subtracted to m. By Kummer's Theorem Corollary 1. 1,

$$v_2(\binom{m}{3}) = v_2(\binom{m+2}{3}) = v_2(m),$$

$$v_2(\binom{m+1}{3}) = v_2(m) - 1,$$

$$v_2(\binom{m-1}{3}) = 0 .$$

Theorem 2. 2. 4 For given $n \in \mathbf{N}$, $n \geq 3$,

(1) If $n \equiv 3,4,5,6,7,8(\bmod 9)$, $v_3(\binom{n}{3}) = 0$;

(2) If $n \equiv i(\bmod 9)$, $i = 0,1,2$, let $m = n - i$, then

$$v_3(\binom{m+i}{3}) = v_3(m) - 1.$$

Proof (1) If $n \equiv 3,4,5,6,7,8(\bmod 9)$, say $n = (n_r n_{r-1} \cdots n_2 n_1 n_0)_3 (1 \leq n_1 \leq 2)$, $3 = (10)_3$, then 0 borrow will be generated when 3 is subtracted to *n* in 3-adic system. By Kummer's Theorem Corollary 1. 1 we have $v_3(\binom{n}{3}) = 0$.

(2) Let $v_3(m) = \alpha$, $m = (m_r m_{r-1} \cdots m_\alpha \underbrace{00\cdots 0}_{\alpha个0})_3$, then $m + i = (m_r m_{r-1} \cdots m_\alpha \underbrace{00\cdots 0}_{\alpha-1个0} i)_3$, as a result, $\alpha - 1$ borrows are obtained when 3 is subtracted to $m+i$ in 3-adic system. By Kummer's Theorem Corollary 1. 1, we have $v_3(\binom{m+i}{3}) = v_3(m) - 1$.

Theorem 2. 2. 5 Given $p \geq 5$ and $n \geq 3$,

(1) If $n \equiv 3,4,\cdots,p-1(\bmod p)$, then $v_p(\binom{n}{3}) = 0$;

(2) If $n \equiv i(\bmod p)$, $i = 0,1,2$, let $m = n - i$, then

$$v_p(\binom{m+i}{3}) = v_p(m).$$

Proof (1) If $n \equiv 3,4,\cdots,p-1(\bmod p)$, let $n = (n_r n_{r-1} \cdots n_1 n_0)_p (3 \leq n_0 \leq p-1)$, $3 = (3)_p$, then 0 borrows will be generated when 3 is subtracted to *n* in *p*-adic system. By Kummer's Theorem Corollary 1. 1, we have $v_p(\binom{n}{3}) = 0$.

(2) Let $v_p(m) = \alpha$, $m = (m_r m_{r-1} \cdots m_\alpha \underbrace{00\cdots 0}_{\alpha个0})_p$, then

$$m + i = (m_r m_{r-1} \cdots m_\alpha \underbrace{00\cdots 0}_{\alpha-1个0} i)_p,\ i = 0,1,2,$$

then α borrows will be generated if 3 is substracted to m in p -adic system. By Kummer's Theorem Corollary 1. 1, we have $v_p(\binom{m+i}{3}) = v_p(m)$.

Example 2. 2. 5 Evaluate $v_{11}(\binom{11^6+1}{3})$.

Solution By Theorem 2. 2. 5, we have $v_{11}(\binom{11^6+1}{3}) = v_{11}(\binom{11^6}{3}) = v_{11}(11^6) = 6$.

Now consider the case $v_p(\binom{n}{4})$.

Theorem 2. 2. 6 Given $n \geq 3$,

(1) If $n \equiv 4,5,6,7 \pmod 8$, then $v_2(\binom{n}{4}) = 0$;

(2) If $n \equiv i \pmod p$, $i = 0,1,2,3$, let $m=n-i$, then

$$v_2(\binom{m+i}{4}) = v_2(m) - 2.$$

Proof (1) If $n \equiv 4,5,6,7 \pmod 8$, let $n = (n_r n_{r-1} \cdots n_3 1 n_1 n_0)_2$, then $4 = (100)_2$, no borrow will be generated when 4 is subtracted to n in binary system. By Kummer's Theorem Corollary 1. 1, we have $v_2(\binom{n}{4}) = 0$.

(2) Let $v_2(m)=\alpha$, $m = (m_r m_{r-1} \cdots m_\alpha \underbrace{00\cdots0}_{\alpha个})_2$, then

$i = (i_1 i_0)_2$, $i = 0,1,2,3$, $m+i = (a_r a_{r-1} \cdots a_1 1 \underbrace{00\cdots0}_{\alpha-2个} i_1 i_0)_2$, $4=(100)_2$.

$\alpha - 2$ borrows will be generated when 4 is subtracted to $m+i$ in binary system. By Kummer's Theorem Corollary 1. 1, we have $v_2(\binom{m+i}{4}) = v_2(m) - 2$.

Theorem 2. 2. 7 For sequence $\left\{v_p(\binom{n}{k})\right\}$, $p > k$,

(1) If $n \equiv k, k+1, \cdots, p-1 \pmod p$, then $v_p(\binom{n}{k}) = 0$;

(2) If $n \equiv i(\bmod p)$, $i = 0,1,2,\cdots,k-1$, let $m=n-i$, then

$$v_p(\binom{m+i}{k}) = v_p(m).$$

Proof (1) If $n \equiv k, k+1, \cdots, p-1(\bmod p)$, let $n=(n_r n_{r-1}\cdots n_1 n_0)_p (k \le n_0 \le p-1)$, $k=(k)_p$, then no borrow will be generated when k is subtracted to n in p-adic system. By Kummer's Theorem Corollary 1. 1, we have

$$v_p(\binom{n}{k}) = 0.$$

(2) Let $v_p(m)=\alpha$, $m=(m_r m_{r-1}\cdots m_\alpha \underbrace{00\cdots0}_{\alpha个0})_p$, $m+i=(m_r m_{r-1}\cdots m_1 m_0 \underbrace{00\cdots0}_{\alpha-1个0} i)_p$, $i = 0,1,2,\cdots,k-1$, then α borrows will be generated when k is subtracted to $m+i$ in p-adic system. By Kummer's Theorem Corollary 1. 1, we have $v_p(\binom{m+i}{k}) = v_p(m)$.

Example 2. 2. 6 In sequence $\left\{v_{11}(\binom{n}{10})\right\}$, if $n=121$, then $n \equiv 0(\bmod 11)$, $v_{11}(121)=2$, by Theorem 2. 2. 7, we obtain that $v_{11}(\binom{121}{10}) = v_{11}(121) = 2$.

Example 2. 2. 7 In sequence $\left\{v_{37}(\binom{n}{20})\right\}$, if $n = 37^2$, then $n \equiv 0(\bmod 37)$, $v_{37}(37^2)=2$, by Theorem 2. 2. 7, we obtain that $v_{37}(\binom{37^2}{20}) = v_{37}(37^2) = 2$.

Theorem 2. 2. 8 For sequence $v_p(\binom{n}{p})$,

(1) If $n \equiv p, p+1, \cdots, p^2-1(\bmod p^2)$, then $v_p(\binom{n}{p}) = 0$;

(2) If $n \equiv i(\bmod p^2)$, $i = 0,1,2,\cdots,p-1$, let $m=n-i$, then

$$v_p(\binom{m+i}{p}) = v_p(m)-1.$$

Proof (1) If $n \equiv p, p+1, \cdots, p^2-1 \pmod{p^2}$, let

$$n=(n_r n_{r-1}\cdots n_2 n_1 n_0)_3 (1 \le n_1 \le p-1),\ p=(10)_p,$$

then no borrow will be generated when p is subtracted to n in p-adic system. By Kummer's Theorem Corollary 1. 1, we have $v_p(\binom{n}{p})=0$.

(2) Let $v_p(m)=\alpha$, $m=(m_r m_{r-1}\cdots m_\alpha \underbrace{00\cdots0}_{\alpha个0})_p$, then $m+i=(m_r m_{r-1}\cdots m_\alpha \underbrace{00\cdots0 i}_{\alpha-1个0})_p$,

then $\alpha-1$ borrows will be generated when p is subtracted to $m+i$ in p-adic system. By Kummer's Theorem Corollary 1. 1, we have $v_p(\binom{m+i}{p})=v_p(m)-1$.

2. 3 Properties for $\left\{v_p(\binom{n}{k})\right\}, k>p$

We have figured out the properties for sequence $\left\{v_p(\binom{n}{k})\right\}$ when $k \le p$, and now the properties for the sequence when $k>p$ are discussed in the following.

It is noted that center-division property is found in one period for $\left\{v_p(\binom{n}{k})\right\}$, which can be described as the following theorem:

Theorem 2. 3. 1 Let $p^{\alpha-1} \le k < p^\alpha$, if $n+m \equiv k-1 \pmod{p^\alpha}$, $|m-n|<p^\alpha$, then

$$v_p(\binom{n}{k})=v_p(\binom{m}{k}).$$

Proof Since $p^{\alpha-1} \le k < p^\alpha$, $n+m \equiv k-1 \pmod{p^\alpha}$, let

$$k=(k_{\alpha-1}k_{\alpha-2}\cdots k_1 k_0)_p,\quad n=((t)_p n_{\alpha-1}n_{\alpha-2}\cdots n_1 n_0)_p,\quad n-k=((b_\alpha)_p b_{\alpha-1}b_{\alpha-2}\cdots b_1 b_0)_p,$$

$n+m=up^\alpha+k-1$, $u=2t$ or $2t+1$, then

$$up^\alpha=((u)_p \underbrace{00\cdots0}_{\alpha个})_p,\ up^\alpha-1=((u')_p \underbrace{qq\cdots q}_{\alpha个})_p,\ u'=u-1,\ q=p-1,$$

$$up^\alpha-1-n=((t')_p c_{\alpha-1}c_{\alpha-2}\cdots c_0)_p,\ up^\alpha-1-(n-k)=((b_\alpha')_p d_{\alpha-1}d_{\alpha-2}\cdots d_0)_p,$$

$$t'=u'-t,\quad b_\alpha{}'=u'-b_\alpha,\ c_i=q-n_i,\ d_i=q-b_i,\ i=0,1,\cdots,\alpha-1.$$

Thus by Lemma 1. 1. 2, we obtain that

$$(p-1)[v_p(\binom{n}{k})-v_p(\binom{m}{k})]=[g_p(k)+g_p(n-k)-g_p(n)]-[g_p(k)+g_p(m-k)-g_p(m)]$$

$$=g_p(n-k)-g_p(n)-g_p(up^\alpha-1-n)+g_p(up^\alpha-1-(n-k))$$

$$=g_p(b_\alpha)+\sum_{i=0}^{\alpha-1}b_\alpha-g_p(t)-\sum_{i=0}^{\alpha-1}n_\alpha-g_p(t')-\sum_{i=0}^{\alpha-1}c_\alpha+g_p(b_\alpha{}')+\sum_{i=0}^{\alpha-1}d_\alpha$$

$$=g_p(b_\alpha)-g_p(t)-g_p(t')+g_p(b_\alpha{}')$$

$$=g_p(b_\alpha)-g_p(t)-g_p(u-1-t)+g_p(u-1-b_\alpha).$$

1) If $u=2t$, by $|m-n|<p^\alpha$ we have $0\le n-tp^\alpha\le k-1<k$, thus $b_\alpha=t-1$, consequently

$$(p-1)[v_p(\binom{n}{k})-v_p(\binom{m}{k})]=g_p(b_\alpha)-g_p(t)-g_p(u-1-t)+g_p(u-1-b_\alpha)$$

$$=g_p(t-1)-g_p(t)-g_p(t-1)+g_p(2t-1-t+1)$$

$$=0.$$

2) If $u=2t+1$, by $|m-n|<p^\alpha$ we have $k\le n-tp^\alpha\le p^{\alpha-1}-1$, thus $b_\alpha=t$, therefore

$$(p-1)[v_p(\binom{n}{k})-v_p(\binom{m}{k})]=g_p(b_\alpha)-g_p(t)-g_p(u-1-t)+g_p(u-1-b_\alpha)$$

$$=g_p(t)-g_p(t)-g_p(t)+g_p(t)$$

$$=0.$$

In summary, $(p-1)[v_p(\binom{n}{k})-v_p(\binom{m}{k})]=0$, hence $v_p(\binom{n}{k})=v_p(\binom{m}{k})$.

Furthermore, the first part in one period is kept unchanged, which can be described by the following Theorem.

Theorem 2. 3. 2 Let $p^{\alpha-1}\le k<p^\alpha$, if $n+m\equiv k-1(\bmod p^\alpha)$, $n\equiv i(\bmod p^\alpha)$ and $k\le i\le p^\alpha-1$, then

$$v_p(\binom{n}{k})=v_p(\binom{m}{k}).$$

Proof Since $p^{\alpha-1}\le k<p^\alpha$, $n+m\equiv k-1(\bmod p^\alpha)$, let $k=(k_{\alpha-1}k_{\alpha-2}\cdots k_1k_0)_p$, $n=((t)_p n_{\alpha-1}n_{\alpha-2}\cdots n_1n_0)_p$, $n-k=((b_\alpha)_p b_{\alpha-1}b_{\alpha-2}\cdots b_1b_0)_p$, $n+m=up^\alpha+k-1$, then

$$up^{\alpha} = ((u)_p \underbrace{00\cdots0}_{\alpha个})_p,\ up^{\alpha}-1 = ((u')_p \underbrace{qq\cdots q}_{\alpha个})_p,\quad u' = u-1,\ q = p-1,$$

$$up^{\alpha}-1-n = ((t')_p c_{\alpha-1}c_{\alpha-2}\cdots c_0)_p,\ up^{\alpha}-1-(n-k) = ((b_{\alpha}{}')_p d_{\alpha-1}d_{\alpha-2}\cdots d_0)_p,$$

$$t' = u'-t,\quad b_{\alpha}{}' = u'-b_{\alpha},\ c_i = q-n_i,\ d_i = q-b_i,\ i = 0,1,\cdots,\alpha-1.$$

Note that $n \equiv i(\bmod p^{\alpha})$ and $k \le i \le p^{\alpha}-1$, thus $n_{\alpha-1} \ge k_{\alpha-1}$, $b_{\alpha}=t$. By Lemma 1. 1. 2, we obtain that

$$\begin{aligned}(p-1)[v_p(\binom{n}{k})-v_p(\binom{m}{k})] &= [g_p(k)+g_p(n-k)-g_p(n)]-[g_p(k)+g_p(m-k)-g_p(m)]\\
&= g_p(n-k)-g_p(n)-g_p(up^{\alpha}-1-n)+g_p(up^{\alpha}-1-(n-k))\\
&= g_p(b_{\alpha})+\sum_{i=0}^{\alpha-1}b_{\alpha}-g_p(t)-\sum_{i=0}^{\alpha-1}n_{\alpha}-g_p(t')-\sum_{i=0}^{\alpha-1}c_{\alpha}+g_p(b_{\alpha}{}')+\sum_{i=0}^{\alpha-1}d_{\alpha}\\
&= g_p(b_{\alpha})-g_p(t)-g_p(t')+g_p(b_{\alpha}{}')\\
&= g_p(b_{\alpha})-g_p(t)-g_p(u-1-t)+g_p(u-1-b_{\alpha})\\
&= g_p(t)-g_p(t)-g_p(u-1-t)+g_p(u-1-t)\end{aligned}$$

$= 0$, therefore $v_p(\binom{n}{k}) = v_p(\binom{m}{k})$.

Theorem 2. 3. 3 Let $p^{\alpha-1} \le k < p^{\alpha}$, If $n \equiv m \equiv i(\bmod p^{\alpha})$, $i = k, k+1, \cdots, p^{\alpha}-1$, then

$$v_p(\binom{n}{k}) = v_p(\binom{m}{k}).$$

Proof Let $p^{\alpha-1} \le k < p^{\alpha}$, by Theorem 2. 3. 2 we have that if $n+m' \equiv k-1(\bmod p^{\alpha})$, $n \equiv i(\bmod p^{\alpha})$ and $k \le i \le p^{\alpha}-1$, then

$$v_p(\binom{n}{k}) = v_p(\binom{m'}{k}).$$

If $m+m' \equiv k-1(\bmod p^{\alpha})$, $m \equiv i(\bmod p^{\alpha})$ and $k \le i \le p^{\alpha}-1$, then

$$v_p(\binom{m}{k}) = v_p(\binom{m'}{k}).$$

As a result, $v_p(\binom{n}{k}) = v_p(\binom{m}{k})$.

Theorem 2. 3. 4 Let $p^{\alpha-1} \le k < p^{\alpha}$, if $n \equiv i(\bmod p^{\alpha})$, $m \equiv j(\bmod p^{\alpha})$, $i, j = 0,1,\cdots,k-1$, $v_p(n-i)=e, v_p(m-j)=f$, then

$$v_p(\binom{n-i+t}{k}) - v_p(\binom{m-j+t}{k}) = e-f, \quad t = 0,1,\cdots,k-1.$$

Proof Since $p^{\alpha-1} \le k < p^{\alpha}$, $n \equiv i(\bmod p^{\alpha})$, $m \equiv j(\bmod p^{\alpha})$, $v_p(n-i)=e,$,

$v_p(m-j)=f$, thus

$$n-i=(\cdots n_e \underbrace{00\cdots0}_{e个})_p,\ m-j=(\cdots m_f \underbrace{00\cdots0}_{f个})_p,\ n_e \ge 1,\ m_f \ge 1,\ k=(k_{\alpha-1}k_{\alpha-2}\cdots k_1k_0)_p,$$

then for any $t = 0,1,\cdots,k-1$, the difference between the borrows generated by $n-i+t$ subtracting k and the borrows generated by $m-i$ subtracting k in p-adic system will be constant. By Kummer's Theorem Corollary 1. 1, we have

$$v_p(\binom{n-i+t}{k}) - v_p(\binom{m-j+t}{k}) = e-f.$$

It is desirable if we can evaluate $v_p(\binom{n}{k})$ for any $\binom{n}{k}$ and prime p. This is achievable when $k \le p$, which is given by Theorem 2. 2. 7 and Theorem 2. 2. 8. However, the general formula for the evaluation of $v_p(\binom{n}{k})$ is still unknown when $k > p$. Fortunately, when $n \equiv 0, k-1(\bmod p^{\alpha})$, $v_p(\binom{n}{k})$ can be obtained through the following theorems:

Theorem 2. 3. 5 If $2^{\alpha-1} \le k < 2^{\alpha}$, $n \equiv 0, k-1(\bmod 2^{\alpha})$, then

$$v_2(\binom{n}{k}) = v_2(n) - v_2(k).$$

Proof Since $2^{\alpha-1} \le k < 2^{\alpha}$, let $v_2(k) = b$, $n \equiv 0(\bmod 2^{\alpha})$, $v_2(n) = e \ge \alpha > b$, let

$$k = (k_{\alpha-1}k_{\alpha-2}\cdots k_b \underbrace{00\cdots0}_{b个})_2,\ n = (n_r n_{r-1}\cdots n_0 \underbrace{00\cdots0}_{e个})_2,\ n_0 \ge 1.$$

Hence $e-b$ borrows will be generated when k is subtracted to n in binary system. By Kummer's Theorem Corollary 1. 1, we have

$$v_2(\binom{n}{k}) = e-b$$

Note that if $m \equiv k-1 \pmod{2^\alpha}$, $n+m \equiv k-1 \pmod{2^\alpha}$, by Theorem 2. 3. 1 we have

$v_2(\binom{m}{k}) = v_2(\binom{n}{k}) = e-b$.

Therefore $v_2(\binom{n}{k}) = v_2(n) - v_2(k)$.

Theorem 2. 3. 6 If $3^{\alpha-1} \le k < 3^\alpha$, $n \equiv 0, k-1 \pmod{3^\alpha}$, then

$$v_3(\binom{n}{k}) = v_3(n) - v_3(k).$$

Proof Since $3^{\alpha-1} \le k < 3^\alpha$, let $v_3(k) = b$, $n \equiv 0 \pmod{3^\alpha}$, $v_3(n) = e \ge \alpha > b$, let

$$k = (k_{\alpha-1}k_{\alpha-2}\cdots k_b \underbrace{00\cdots0}_{b个})_3,\ n = (n_r n_{r-1}\cdots n_0 \underbrace{00\cdots0}_{e个})_3,\ n_0 \ge 1.$$

Thus $e-b$ borrows will be generated when k is subtracted to n in 3-adic system. By Kummer's Theorem Corollary 1. 1, we have

$$v_3(\binom{n}{k}) = e-b$$

And if $m \equiv k-1 \pmod{3^\alpha}$, $n+m \equiv k-1 \pmod{3^\alpha}$, by Theorem 2. 3. 1 we have

$$v_3(\binom{m}{k}) = v_3(\binom{n}{k}) = e-b.$$

Thus $v_3(\binom{n}{k}) = v_3(n) - v_3(k)$.

Now extend Theorem 2. 3. 6 to case $v_p(\binom{n}{k})$.

Theorem 2. 3. 7 If $p^{\alpha-1} \le k < p^\alpha$, $n \equiv 0, k-1 \pmod{p^\alpha}$, then

$$v_p(\binom{n}{k}) = v_p(n) - v_p(k).$$

Proof Since $p^{\alpha-1} \le k < p^{\alpha}$, let $v_p(k) = b$, $n \equiv 0 (\bmod p^{\alpha})$, $v_p(n) = e \ge \alpha > b$, let

$$k = (k_{\alpha-1}k_{\alpha-2}\cdots k_b \underbrace{00\cdots0}_{b个})_p,\ n = (n_r n_{r-1}\cdots n_0 \underbrace{00\cdots0}_{e个})_p,\ n_0 \ge 1.$$

Thus $e-b$ borrows will be generated when k is subtracted to n in p -adic system. By Kummer's Theorem Corollary 1. 1 we have

$$v_p(\binom{n}{k}) = e - b$$

And if $m \equiv k - 1(\bmod p^{\alpha})$, $n + m \equiv k - 1(\bmod p^{\alpha})$, from Theorem 2. 3. 1 we have

$$v_p(\binom{m}{k}) = v_p(\binom{n}{k}) = e - b .$$

Hence, $v_p(\binom{n}{k}) = v_p(n) - v_p(k)$.

Now we have established the formula to evaluate $v_p(\binom{n}{k})$ for the case when $p^{\alpha-1} \le k < p^{\alpha}$, $n \equiv 0, k-1(\bmod p^{\alpha})$.

Example 2. 2. 8 Let $v_7(n)=3$, evaluate $v_7(\binom{n}{9})$.

Solution Since $v_7(n)=3$, $7 \le 9 < 7^2$, $n \equiv 0(\bmod 49)$, thus from Theorem 2. 3. 7, we obtain that

$$v_7(\binom{n}{9}) = v_7(\binom{n}{9}) - v_7(9) = 3 .$$

Example 2. 2. 9 Let $v_7(n)=3$, evaluate $v_7(\binom{n}{56})$.

Solution Since $v_7(n)=3$, $7^2 \le 56 < 7^3$, $n \equiv 0(\bmod 343)$, thus by Theorem 2. 3. 7, we obtain that

$$v_7(\binom{n}{56}) = v_7(n) - v_7(56) = 3 - 1 = 2 .$$

Although we do not have a formula to evaluate $v_p(\binom{n}{k})$ for general $k > p$ case yet, some useful results can be derived for certain n and k.

Theorem 2. 3. 8 Given $n \equiv i_0 (\bmod p),\ 0 \le i_0 < p,\ k \equiv i (\bmod p), 0 \le i < p$, if $i > i_0$, then

$$v_p(\binom{n}{k}) \ge 1.$$

Proof Let $n = (\cdots n_r n_{r-1} \cdots n_1 i_0)_p,\ k = (\cdots k_r k_{r-1} \cdots k_1 i)_p$. Since $i > i_0$, there must be one borrows when k is subtracted to n. Then from Kummer's Theorem we have $v_p(\binom{n}{k}) \ge 1$.

The research mentioned above can not only be applied to evaluate the exponent, but also to the division theorems such as the application introduced below.

Z. W. Sun and R. Tauraso discussed the summation of binomial coefficients in 2006**:**

$$\begin{bmatrix} n \\ r \end{bmatrix}_m = \sum_{\substack{0 \le k \le n \\ k \equiv r (\bmod m)}} \binom{n}{k}.$$

Prof. Jin Yuan of Northwest University and her group studied the summation of certain power of the binomial coefficients in 2008:

$$\begin{bmatrix} n \\ r \end{bmatrix}_m^{(s)} = \sum_{\substack{0 \le k \le n \\ k \equiv r (\bmod m)}} \binom{n}{k}^s, 0 \le r \le m-1.$$

One of their results is stated as follows:

Lemma 2. 3. 1[27] Let $n \in \mathbf{N}^*, n = l_0 p + r_0$, r_0 is the nonnegative least residue in module p. If $r > r_0$, then

$$\begin{bmatrix} n \\ r \end{bmatrix}_p^{(s)} \equiv 0 (\bmod p).$$

Vandermonde's identity $\binom{n}{k} = \sum_{i=0}^{k} \binom{l_0 p}{i} \binom{r_0}{k-i}$ and Lemma 3. 3 introduced in the next section are used in the proof of this result in Yuan's work. A simpler proof is presented below by using our results on the p-adic valuation of the binomial coefficients:

Proof By $n = l_0 p + r_0$ we have $n \equiv r_0 (\bmod p),\ 0 \le r_0 < p,\ k \equiv r (\bmod p), 1 \le i < p$. Since $r > r_0$, by Theorem 2. 3. 8, we obtain that $v_p(\binom{n}{k}) \ge 1$, i.e., $\binom{n}{k} \equiv 0 (\bmod p)$. Thus

$$\begin{bmatrix} n \\ r \end{bmatrix}_p^{(s)} = \sum_{\substack{0 \le k \le n \\ k \equiv r \pmod p}} \binom{n}{k}^s \equiv 0 \pmod p .$$

With the above discussions, we would like to consider the range of $v_p(\binom{n}{k})$. The exact range of $v_p(\binom{n}{k})$ is given in the next section. Furthermore, we studied the applications of the range to the division theorems and the evaluation of the minimum and the maximum number of combinations of the power of p.

3. The range and enumeration of the p-adic valuation $v_p(\binom{n}{k})$

Theorem 3. 1 Let $n \ge k \ge 1, n, k \in \mathbf{N}, v_p(n) = \alpha, v_p(k) = \beta, v_p(\binom{n}{k}) = x$, and assuming that the p -system of n has $r+1$ ($r \ge 0$) digits.

(1) If $\alpha \ge \beta$, then $\alpha - \beta \le x \le r - \beta$;

(2) If $\alpha < \beta$, then $0 \le x \le r - \beta$.

Proof Let $n = (n_r n_{r-1} \cdots n_\alpha \underbrace{00\cdots0}_{\alpha})_p$, $k = (k_r k_{r-1} \cdots k_\beta \underbrace{00\cdots0}_{\beta})_p$.

(1) If $\alpha \ge \beta$, since $k_\beta \ne 0$ hence the subtraction of n and k has $\alpha - \beta$ borrows. Since we want x to be minimum, then $n_{\alpha+1}, \cdots, n_{r-1}, n_r$ will not generate borrows, i.e., $x \ge \alpha - \beta$. If x is the maximum, then $n_{\alpha+1}, \cdots, n_{r-1}, n_r$ will generate borrows, that is, $x \le (\alpha - \beta) + (r - \alpha) = r - \beta$. Therefore, $\alpha - \beta \le x \le r - \beta$.

(2) If $\alpha < \beta$, since $n \ge k$, therefore the subtraction between n and k will not generate borrows, hence $x \ge 0$. When x is a maximum, then $n_{\beta+1}, \cdots, n_{r-1}, n_r$ will generate borrows, hence $x \le r - \beta$. Therefore, $0 \le x \le r - \beta$.

We can utilize the above theorems to obtain some commonly encountered division conclusions, from which we can understand them clearer.

Lemma 3. 1[1] If p is a prime number, then $p \,\Big|\, \binom{p}{k}, 1 \le k < p$.

Lemma 3.2[2] If p is a prime number, then $p \,\Big|\, \binom{p^m}{k}, 1 \le k < p^m$.

Lemma 3.3[3,4] If p is a prime number, and $0 \le j < k, 1 \le i < p$, then $p \left| \binom{kp}{jp+i} \right.$.

We will give some generalizations of the above lemmas in the following.

Corollary 3.1 If p is a prime number, then $v_p(\binom{p}{k}) = 1, 1 \le k < p$.

Corollary 3.2 If p is a prime number, then $1 \le v_p(\binom{p^m}{k}) \le m, 1 \le k < p^m$.

Corollary 3. 3 If p is a prime number , and $0 \le j < k, 1 \le i < p$, then

$1 \le v_p(\binom{kp}{jp+i}) \le s+1$, where $s+1$ is the number of digits in the p-system of k.

We can obtain furthermore conclusions with similar ideas.

Theorem 3. 2 If p is a prime number, a has $t+1$ digits in the p-system, and $k = p^m a, (p,a) = 1$, $0 \le j < a$, $1 \le i < p^{m+1}$, then

$$1 \le v_p(\binom{kp}{jp^{m+1}+i}) \le m+t+1.$$

Proof By the assumptions, we let $a = (a_t a_{t-1} \cdots a_0)_p$, $kp = (a_t a_{t-1} \cdots a_0 \underbrace{00\cdots0}_{m+1个0})_p$, then by Theorem 3.1, we have $r = m+t+1, \alpha = m+1, 0 \le \beta \le m, \alpha > \beta$, $1 \le v_p(\binom{kp}{jp^{m+1}+i}) \le m+t+1$.

Definition 3. 1 Let n,k be positive integers, and $n \ge k$, n has $r+1$ digits in the p-system, and $v_p(n) = \alpha, v_p(k) = \beta$,

$v_p(\binom{n}{k}) = x$. $\binom{n}{k}$ is referred to as the minimum combination of power of p, if $x = \alpha - \beta$ ($\alpha \ge \beta$) or $x = 0$ ($\alpha < \beta$); $\binom{n}{k}$ is called the maximum combination of the power of p, if $x = r - \beta$.

Let p be a given prime number, and let $\theta_j(n) = \theta_j(p,n)$ denote the number of coefficient of $\binom{n}{k}(k=0,1,2,\cdots n)$ which can be divided by p^j, but not for p^{j+1}.

N. J. Fine has proven that

$$\theta_0(p,n)=(c_0+1)(c_1+1)\cdots(c_r+1)$$

L. Carlitz proved that

$$\theta_1(p,n)=\sum_{k=0}^{r-1}(c_0+1)\cdots(c_{k-1}+1)(p-c_k-1)c_{k+1}(c_{k+2}+1)\cdots(c_r+1)$$

For the above two situations, the condition $n=\sum_{k=0}^{r}c_i p^i (0\le c_i\le p-1)$ [11] has been assumed.

Let $q=p-1$, n be $r+1$ p-system, $n=(qq\cdots q\underbrace{00\cdots0}_{\alpha个0})_p$ (i.e., $n=(p^{r+1-\alpha}-1)p^{\alpha}$), $k=(k_r k_{r-1}\cdots k_{\beta}\underbrace{00\cdots0}_{\beta个0})_p$, if $\alpha\ge\beta$, then $\binom{n}{k}$ is the minimum combination of the power of p. A natural problem is that, for a given n, how many minimum combinations of the power of p?

We can start with the simple cases, and then obtain the general results.

Theorem 3.3 Given the $r+1$ digits *p-Adic* number $n=(p^{r+1-\alpha}-1)p^{\alpha}$, $v_p(k)=\beta$, $\alpha>\beta$, then the number of minimum combination of power of p of $\binom{n}{k}$ is $(p-1)^2 p^{r-\beta-1}$.

Proof Let $q=p-1$, $n=(qq\cdots q\underbrace{00\cdots0}_{\alpha})_p$, $k=(k_r k_{r-1}\cdots k_{\alpha+1}k_{\alpha}k_{\alpha-1}\cdots k_{\beta}\underbrace{00\cdots0}_{\beta})_p$. By Kummer's Theorem and Theorem 3.1, we find that the power of p is minimum is equivalent to the subtraction of $n-k$ in p-system has $\alpha-\beta$ borrows, in this case, $v_p(\binom{n}{k})=\alpha-\beta$. k_{β} can be chosen from $1,2,\cdots,q$, so it has choices; $k_{\beta+1},\cdots,k_{\alpha-1}$ can be chosen from $0,1,2,\cdots,q$, each of them has p choices; k_{α} can be chosen from $0,1,2,\cdots,q-1$, so it has $p-1$ choices; $k_{\alpha+1},\cdots,k_r$ can be chosen from $0,1,2,\cdots,q$, each of them has p choices. By the multiplication principle, the number of choices of k is

$$(p-1)p^{\alpha-\beta-1}(p-1)p^{r-\alpha}=(p-1)^2 p^{r-\beta-1},$$

The number of the minimum combination of power of p of $\binom{n}{k}$ is $(p-1)^2 p^{r-\beta-1}$.

Theorem 3. 4 Given the $r+1$ digits p-adic number $n=(p^{r+1-\alpha}-1)p^{\alpha}$, $v_p(k)=\alpha$, then the number of minimum combination of the power of p of $\binom{n}{k}$ is $(p-1)p^{r-\alpha}$.

Proof Let $q=p-1$, $n=(qq\cdots q\underbrace{00\cdots0}_{\alpha})_p$, $k=(k_r k_{r-1}\cdots k_{\alpha+1}k_\alpha\underbrace{00\cdots0}_{\alpha})_p$. By Theorem 3.1 and the Corollary 1.1 of Kummer's Theorem, we obtain that p is minimum is equivalent to the fact that the subtraction of $n-k$ in the p-system has no borrows, in this case, $v_p(\binom{n}{k})=\alpha-\beta=0$. Therefore, k_α can be chosen from $1,2,\cdots,q$ and it has choices; $k_{\alpha+1},\cdots,k_r$ can be chosen from $0,1,2,\cdots,q$, each of them has choices. By the multiply principle, the number of the choices for k is

$$(p-1)p^{r-\alpha}=(p-1)p^{r-\alpha},$$

the number of minimum combination of the power of p of $\binom{n}{k}$ is $(p-1)p^{r-\alpha}$.

Theorem 3.5 Given the $r+1$ digits p-adic number $n=(p^{r+1-\alpha}-1)p^{\alpha}$, $\alpha\geq v_p(k)$, then the number of minimum combination of the power of p of $\binom{n}{k}$ is $(p-1)p^{r}$.

Proof Let $v_p(k)=\beta(\beta=0,1,\cdots,\alpha)$, by Theorem 3. 3、3. 4, we obtain that if $\alpha>0$, the number is

$$\sum_{\beta=0}^{\alpha-1}(p-1)^2p^{r-\beta-1}+(p-1)p^{r-\alpha}=(p-1)^2\sum_{\beta=0}^{\alpha-1}p^{r-\beta-1}+(p-1)p^{r-\alpha}$$

$$=(p-1)^2\frac{p^{r-\alpha}(1-p^{\alpha})}{1-p}+(p-1)p^{r-\alpha}$$

$$=(p-1)p^{r-\alpha}(p^{\alpha}-1)+(p-1)p^{r-\alpha}$$

$$=(p-1)p^{r}\quad.$$

If $\alpha=0$, by Theorem 3.4, the number is $(p-1)p^{r-0}=(p-1)p^{r}$.

Example 3.1 $n=(1100)_2, p=2, r=3, \alpha=2,$ we can obtain following table

k	1	2	3	4	5	6	7	8	9	10	11	12
β	0	1	0	2	0	1	0	3	0	1	0	2
$\binom{n}{k}$	12	66	220	495	792	924	792	495	220	66	12	1
$v_p(\binom{n}{k})=\alpha-\beta$	√	√	√	√	×	×	×	×	√	√	√	√

By Theorem 3. 5, we have $(p-1)p^r=(2-1)2^3=8$, the results agree with above table.

Theorem 3. 6 Given $r+1$ digits p-adic number $n=(p^{r+1-\alpha}-1)p^{\alpha}$, $\alpha<v_p(k)$, then the number of minimum combination of the power of p of $\binom{n}{k}$ is $p^{r-\alpha}-1$.

Proof Let $q=p-1$, $n=(qq\cdots q\underbrace{00\cdots0}_{\alpha})_p$, $v_p(k)=\beta(\beta=\alpha+1,\cdots,r)$, $k=(k_rk_{r-1}\cdots k_\beta\underbrace{00\cdots0}_{\beta})_p$. By the Corollary 1.1 of Kummer's Theorem and Theorem 3.1, we can observe that the fact that the power of p is minimum is equivalent to the fact that the subtraction of $n-k$ in the p-system has no borrows , in this case $v_p(\binom{n}{k})=0$. For each β, v_β can be chosen from $1,2,\cdots,q$, and has $p-1$ choices; $v_{\beta+1},\cdots,v_r$ can take values in $0,1,2,\cdots,q$, and each of them has p choices, by the multiplication principle, the number of choices for k is $(p-1)p^{r-\beta}$.

Therefore, by using the addition principle, the total number of choices for k is

$$\sum_{\beta=\alpha+1}^{r}(p-1)p^{r-\beta}=(p-1)\sum_{\beta=\alpha+1}^{r}p^{r-\beta}$$

$$=(p-1)\frac{1-p^{r-\alpha}}{1-p}$$

$$=p^{r-\alpha}-1.$$

hence the number of minimum combination of the power of p of $\binom{n}{k}$ is $p^{r-\alpha}-1$.

By Theorem 3.5, Theorem 3.6 and the addition principle, we find that the number of minimum combination of the power of p of $\binom{n}{k}$ is $(p-1)p^r + p^{r-\alpha} - 1$, hence we can obtain the following theorem:

Theorem 3. 7 Given the $r+1$ bits p-adic number $n = (p^{r+1-\alpha} - 1)p^{\alpha}$, then the number of minimum combination pf the power of p of $\binom{n}{k}$ is $(p-1)p^r + p^{r-\alpha} - 1$.

Furthermore, for a more general number n, we can also the number of minimum combination of the power of *p* of $\binom{n}{k}$.

Theorem 3. 8 Given the $r+1$ digits *p*-adic number $n = (n_r \cdots n_\alpha \underbrace{0\cdots0}_{\alpha})_p$, $\alpha \geq v_p(k)$, $\alpha \in \mathbf{N}$, then the number of minimum power of the power of p of $\binom{n}{k}$ is

$$p^{\alpha} n_\alpha (n_{\alpha+1}+1)\cdots(n_r+1).$$

Proof Let $v_p(k) = \beta$, $k = (k_r k_{r-1} \cdots k_\beta \underbrace{00\cdots0}_{\beta})_p$.

(1) If $\alpha > \beta$, then $\alpha \geq 1$. By Kummer's Theorem and Theorem 3.1, p is minimum is equivalent to the fact that the subtraction of $n-k$ in the p-system has no borrows , in this case, $v_p(\binom{n}{k}) = \alpha - \beta$. For each β, k_β can take values in $1,2,\cdots,q$, and has $p-1$ choices; $k_{\beta+1},\cdots,k_{\alpha-1}$ can take values in $0,1,2,\cdots,q$, and each of them has choices; k_α can take values in $0,1,2,\cdots,n_\alpha - 1$, and has n_α choices; $k_{\alpha+1}$ can take values in $0,1,2,\cdots,n_{\alpha+1}$, and has $n_{\alpha+1}+1$ choices ; …; k_r can take values in $0,1,2,\cdots,n_r$, and has n_r+1 choices ; By using the multiplication principle, the number of choices for k is $(p-1)p^{\alpha-\beta-1} n_\alpha (n_{\alpha+1}+1)\cdots(n_r+1)$.

(2) If $\alpha = \beta$, then $k = (k_r k_{r-1} \cdots k_\alpha \underbrace{00\cdots0}_{\alpha})_p$. By Kummer's Theorem and Theorem 3.1, the power of p is a minimum is equivalent to the subtraction of $n-k$ in the p-system has no borrows, in this case, $v_p(\binom{n}{k}) = 0$. k_α can take values in $1,2,\cdots,n_\alpha$, and has n_α choices; $k_{\alpha+1}$ can take values in $0,1,2,\cdots,n_{\alpha+1}$, and each of them has $n_{\alpha+1}+1$ choices; …;

k_r can take values in $0,1,2,\cdots,n_r$, and has n_r+1 choices; By the multiplication principle, the number of choices for k is $n_\alpha(n_{\alpha+1}+1)\cdots(n_r+1)$.

Therefore, by using the addition principle, the total number of choices for k is

$$\sum_{\beta=0}^{\alpha-1}(p-1)p^{\alpha-\beta-1}n_\alpha(n_{\alpha+1}+1)\cdots(n_r+1)+n_\alpha(n_{\alpha+1}+1)\cdots(n_r+1)$$

$$=n_\alpha(n_{\alpha+1}+1)\cdots(n_r+1)[(p-1)\frac{1-p^\alpha}{1-p}+1]$$

$$=p^\alpha n_\alpha(n_{\alpha+1}+1)\cdots(n_r+1).$$

Theorem 3. 9 Given the $r+1$ bits p-adic number $n=(n_r\cdots n_\alpha\underbrace{0\cdots0}_{\alpha\uparrow 0})_p$, $\alpha<v_p(k)$, $\alpha\in\mathbf{N}$, then the number of minimum combination of the power of p of $\binom{n}{k}$ is

$$(n_{\alpha+1}+1)\cdots(n_r+1).$$

Proof Let $v_p(k)=\beta(\beta=\alpha+1,\cdots,r)$, $k=(k_rk_{r-1}\cdots k_\beta\underbrace{00\cdots0}_{\beta})_p$. By Kummer's Theorem and Theorem 3.1, it can be found that the power of p is a minimum is equivalent to the fact that the subtraction of $n-k$ in the p-system has no borrows, in this case $v_p(\binom{n}{k})=0$.

If $\alpha+1\le\beta\le r-1$, k_β can take values in $1,2,\cdots,n_\beta$, and has n_β choices; $k_{\beta+1}$ takes values in $0,1,2,\cdots,n_{\beta+1}$, and has $n_{\beta+1}+1$ choices; …; k_r can take values in $0,1,2,\cdots,n_r$, and it has n_r+1 choices; By using the multiplication principle, it can be found that the number of choices for k is $n_\beta(n_{\beta+1}+1)\cdots(n_r+1)$. If $\beta=r$, the number of choices is n_r+1. By using the addition principle, the number of minimum combination of the power of p of $\binom{n}{k}$ is

$$\sum_{\beta=\alpha+1}^{r-1}n_\beta(n_{\beta+1}+1)\cdots(n_r+1)+(n_r+1)=\sum_{\beta=\alpha+1}^{r-2}n_\beta(n_{\beta+1}+1)\cdots(n_r+1)+n_{r-1}(n_r+1)+(n_r+1)$$

$$=\sum_{\beta=\alpha+1}^{r-2}n_\beta(n_{\beta+1}+1)\cdots(n_r+1)+(n_{r-1}+1)(n_r+1)$$

$$=\cdots$$

$$=(n_{\alpha+1}+1)\cdots(n_r+1).$$

It is worthwhile to noting that in the Theorem 3. 8 and Theorem 3. 9, the equality $v_p(\binom{n}{k})=0$ holds, by adding the results of the above two theorems when $\alpha=0$, we obtain that

$$n_0(n_1+1)\cdots(n_r+1)+(n_1+1)\cdots(n_r+1)=(n_0+1)(n_1+1)\cdots(n_r+1).$$

The above conclusion agrees with the result of N.J. Fine, i.e., $\theta_0(p,n)=(c_0+1)(c_1+1)\cdots(c_r+1)$, and the conclusion of this paper is intended for providing a more accurate computation formula for different $\alpha(v(n)=\alpha)$.

Similarly, we can also compute the number of minimum combination of the power of p of $\binom{n}{k}$.

Theorem 3.10 Given the $r+1$ digits p-adic number $n=(n_r\cdots n_\alpha\underbrace{0\cdots0}_{\alpha})_p$, $\alpha\geq v_p(k)$, $\alpha\in\mathbf{N}$, then the number of minimum combination of the power of p of $\binom{n}{k}$ is

$$(p-n_{\alpha+1})\cdots(p-n_{r-1})n_r[(p-n_\alpha)p^\alpha-1].$$

Proof Let $q=p-1, v_p(k)=\beta$, $k=(k_rk_{r-1}\cdots k_\beta\underbrace{00\cdots0}_{\beta})_p$.

(1) If $\alpha>\beta$, then $\alpha\geq1$. By Kummer's Theorem and Theorem 3.1, it can be found that the power of p is a maximum is equivalent to the fact that the subtraction of $n-k$ in the p-system has no borrows , in this case, $v_p(\binom{n}{k})=r-\beta$. For each β, k_β can take values in $1,2,\cdots,q$, and has $p-1$ choices ; $k_{\beta+1},\cdots,k_{\alpha-1}$ can take values in $0,1,2,\cdots,q$, each of them has choices; k_α can take values in $n_\alpha,n_\alpha+1,\cdots,q$, and it has $p-n_\alpha$ choices; …; k_{r-1} can take values in $n_{r-1},n_{r-1}+1,\cdots,q$, and it has choices; k_r can take values in $0,1,2,\cdots,n_r-1$, and it has n_r choices, By using the multiplication principle, it can be found that the choices for k is $(p-1)p^{\alpha-\beta-1}(p-n_\alpha)\cdots(p-n_{r-1})n_r$.

(2) If $\alpha=\beta$, then $k=(k_r k_{r-1}\cdots k_\alpha \underbrace{00\cdots0}_{\alpha})_p$. By Kummer's Theorem and Theorem 3.1, the power of p s a maximum is equivalent to the fact that the subtraction of $n-k$ in the p-system has $r-\alpha$ borrows, in this case, $v_p(\binom{n}{k})=r-\alpha$. k_α can take values in $n_\alpha+1, n_\alpha+2,\cdots,q$, and it has choices; $k_{\alpha+1}$ can take values in $n_{\alpha+1}, n_{\alpha+1}+1,\cdots,q$, and has $p-n_{\alpha+1}$ choices; …; k_{r-1} can take values from $n_{r-1}, n_{r-1}+1,\cdots,q$, and it has choices; k_r can take values from $0,1,2,\cdots,n_r-1$, and has choices, by using the multiply principle, it can be found that the number of choices for k is $(p-1-n_\alpha)(p-n_{\alpha+1})\cdots(p-n_{r-1})n_r$.

Therefore, by using the addition principle, the number of maximum combination of the power of p of $\binom{n}{k}$ is

$$\sum_{\beta=0}^{\alpha-1}[(p-1)p^{\alpha-\beta-1}(p-n_\alpha)\cdots(p-n_{r-1})n_r]+(p-1-n_\alpha)(p-n_{\alpha+1})\cdots(p-n_{r-1})n_r$$

$$\begin{aligned}
&=(p-n_{\alpha+1})\cdots(p-n_{r-1})n_r[(p-1)(p-n_\alpha)\sum_{\beta=0}^{\alpha-1}p^{\alpha-\beta-1}+(p-1-n_\alpha)]\\
&=(p-n_{\alpha+1})\cdots(p-n_{r-1})n_r[(p-1)(p-n_\alpha)\frac{1-p^\alpha}{1-p}+(p-1-n_\alpha)]\\
&=(p-n_{\alpha+1})\cdots(p-n_{r-1})n_r[(p-n_\alpha)(p^\alpha-1)+(p-1-n_\alpha)]\\
&=(p-n_{\alpha+1})\cdots(p-n_{r-1})n_r[(p-n_\alpha)p^\alpha-1]
\end{aligned}$$

If $\alpha=\beta=0$, the number of maximum combination of the power of p of $\binom{n}{k}$ is $(p-1-n_0)(p-n_1)\cdots(p-n_{r-1})n_r$.

Theorem 3.11 Given the $r+1$ digits p-adic number $n=(n_r\cdots n_\alpha \underbrace{0\cdots0}_{\alpha})_p$, $\alpha<v_p(k)$, $\alpha\in\mathbf{N}$, then the number of maximum combination of the power of p of $\binom{n}{k}$ is

$$\sum_{\beta=\alpha+1}^{r}(p-1-n_\beta)(p-n_{\beta+1})\cdots(p-n_{r-1})n_r.$$

Proof Let $v_p(k)=\beta(\beta=\alpha+1,\cdots,r)$, $k=(k_r k_{r-1}\cdots k_\beta \underbrace{00\cdots0}_{\beta})_p$. By Kummer's

Theorem and Theorem 3.1, it can be found that the power of p is a maximum is equivalent to the fact that the subtraction of $n-k$ in the p-system has $r-\beta$ borrows, in this case, $v_p(\binom{n}{k})=r-\beta$. k_β can take values in $n_\beta+1, n_\beta+2\cdots,q$, and it has choices; $k_{\beta+1}$ can take values from $n_{\beta+1},n_{\beta+1}+1,\cdots,q$, and it has $p-n_{\beta+1}$ choices; …; k_{r-1} can be chosen from $n_{r-1},n_{r-1}+1,\cdots,q$, and it has $p-n_{r-1}$ choices; k_r can take values from $0,1,2,\cdots,n_r-1$, and it has choices; By using the multiplication principle, it can be found that the choices for k is $(p-1-n_\beta)(p-n_{\beta+1})\cdots(p-n_{r-1})n_r$. By using the addition principle, it can be found that the number of minimum combination of the power of p of $\binom{n}{k}$ is

$$\sum_{\beta=\alpha+1}^{r}(p-1-n_\beta)(p-n_{\beta+1})\cdots(p-n_{r-1})n_r.$$

4. Conclusion

In this work, we study the property and the enumeration problem of $v_p(\binom{n}{k})$, and obtain a series of conclusions. Our study was initiated from a number of experiments using the software *Mathematica* for generating the sequence $\left\{v_p(\binom{n}{k})\right\}$, from which some patterns could be observed. Based on the observation, we then proposed a series of conjectures on the property of the prime power of the binomial coefficients, including that the sequence $\left\{v_p(\binom{n}{k})\right\}$ has some periodic patterns and sub-sub-nature locally. With the help of Kummer's theorem, the proposed conjectures had been proved rigorously. Moreover, the calculation of $\left\{v_p(\binom{n}{k})\right\}$ were discussed in detail, and it was found that in the case $k\le p$, any element in the sequence could be evaluated, while in the case $k>p$, the value $v_p(\binom{n}{k})$ could be obtained only in two situations, i.e.,

$n \equiv 0, k-1 (\bmod p^{\alpha})$. Furthermore, we considered the range of $v_p(\binom{n}{k})$. After defining the minimum and maximum numbers of combinations of the power of p, respectively, we discussed two numbers for a specific $n = (p^{r+1-\alpha} - 1)p^{\alpha}$ and for a general n. As a result, two formulas were successfully proposed for the evaluations.

The results obtained from this work can effectively simplify the related calculations, and there are potential applications in a variety of areas such as big data. In addition, the conclusion of this paper can be extended from the p-adic valuation number of prime number to p-adic valuation number of composite number. The Gaussian coefficients $G_k^t (t = 0, 1, 2, \cdots, k)$ have many similar properties with the binomial coefficients, hence the present work can also be extended to study the Gaussian coefficients.

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