Generalization of an Asymptotic Formula for the Smarandache *kn*-digital Sequence

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Abstract

The sequence {a(3,n)} is called the Smarandache 3n-digital sequence, if the digital of a(3,n) can be partitioned into two groups such that the second is 3 times of the first. Smarandache kn-digital sequence {a(k,n)} in the base p is defined similarly. This paper studies an asymptotic formula for Smarandache kn-digital sequence $\sum_{1\le n\le N} \ln a_n = 2N\ln N + O(N)$, $N\to+\infty$, $1\le k\le 9$ (defined in base ten) and generalizes the conclusion by proving that the asymptotic formula is true for any positive integer k and p (p>1). Furthermore, this paper proves some more precise asymptotic formulas for k=1,2,3,4,5,6,8,9,10,11 (defined in base ten) and for general positive integer k and p, and conjectures a more precise asymptotic formula for k=7.

Key words

Smarandache sequence;Asymptotic formula;Base

Contents

1 Introduction

For any arbitrary positive integer k, the sequences $\{a_n\}$ is called the Smarandache kn-digital sequence, if the digital of a_n can be partitioned into two groups such that the second is k times of the first. This sequence was defined by Smarandache, F. (1993, 2006, cited in Gou, S. 2010). There are a number of subsequent works.

Wu(2008:120-122) considered Zhang Wenpeng's conjecture that the Smarandache 3n-digital sequence does not contain any square number. Although this conjecture is not completely solved, Wu did prove the following results:

(1) a_n is not a square if n is square-free.

(2) a_n is not a square if n is a square.

(3) If a_n is a square, then $n = 2^{2\alpha_1} \cdot 3^{2\alpha_2} \cdot 5^{2\alpha_3} \cdot 11^{2\alpha_4} \cdot n_1$ holds, where $(n_1, 330) = 1$.

Lu, P.(2009:5-7, cited in Chen, J. 2012) considered whether there is a square number in the Smarandache 5n-digital sequence and got a negative answer when n equals some special values.

By using elementary method, Gou, S.(2010) proved that for any arbitrary positive integer N large enough, the asymptotic formula $\sum_{1\le n\le N} \ln a_n = 2N\ln N + O(N)$ holds when $k=3$. (When $k=3$, $\{a_n\} = \{13, 26, 39, 412, 515, 618, 721, 824, \cdots\}$.)

Chen, J.(2012:9-14) pointed out that equation $\sum_{1\le n\le N} \ln a_n = 2N\ln N + O(N)$ still holds under the condition of $1 \le k \le 9$ when $N \to +\infty$.

This paper generalizes the above asymptotic formula $\sum_{1\le n\le N} \ln a_n = 2N\ln N + O(N)$ to arbitrary k and arbitrary base p, and improves the estimates of the error term.

2 Theoretical Discussions

2.1 Lemmas and Simple Corollaries

2.1.1 Taylor series with the Peano form of the remainder

Let $f\left(x\right)$ be n times differentiable at x_0, then there must be a neighborhood of x_0, for any x in this neighborhood, the following holds:

$$f\left(x\right)=f\left(x_0\right)+f'\left(x_0\right)\left(x-x_0\right)+\frac{f''\left(x_0\right)}{2!}\left(x-x_0\right)^2+\cdots+\frac{f^{(n)}\left(x_0\right)}{n!}\left(x-x_0\right)^n+r_n\left(x\right) \quad ,$$

In the equation above, the remainder $r_n\left(x\right)$ equals $o\left(\left(x-x_0\right)^n\right)$. When $x_0=0$, the above equation is called the Maclaurin series.

2.1.2 Lemma: $\ln\left(x+1\right)=x+O\left(x^2\right)$, when $x\to 0$

Because $\ln\left(x+1\right)'=\frac{1}{x+1}$ and $\ln\left(x+1\right)''=-\frac{1}{\left(x+1\right)^2}$, using 2.1.1 (let $x_0=0$, $n=2$), we get the target equation as follows.

$$\begin{aligned}\ln\left(x+1\right)&=\ln\left(0+1\right)+\frac{1}{0+1}\left(x-0\right)+\frac{-\frac{1}{\left(0+1\right)^2}}{2!}\left(x-0\right)^2+o\left(\left(x-0\right)^2\right)\\&=x-\frac{1}{2}x^2+o\left(x^2\right)=x+O\left(x^2\right)\end{aligned}$$

2.1.3 Stirling's approximation

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n\left(1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{51840n^3}+\cdots\right)$$

2.1.4 Lemma: $\lim\limits_{N\to+\infty}\frac{(\ln N)^2}{N}=\lim\limits_{N\to+\infty}\frac{\ln N}{N}=0$

Using L'Hospital's rule:

$$\lim_{N\to+\infty}\frac{(\ln N)^2}{N}=\lim_{N\to+\infty}\frac{2\ln N\cdot\frac{1}{N}}{1}=\lim_{N\to+\infty}\frac{2\ln N}{N}=\lim_{N\to+\infty}\frac{\frac{2}{N}}{1}=0$$

$$\lim_{N\to+\infty}\frac{\ln N}{N}=\lim_{N\to+\infty}\frac{\frac{1}{N}}{1}=0$$

This means that $(\ln N)^2$ and $\ln N$ are the lower order infinities of N.

2.1.5 A computational result of dislocation subtraction

Conclusion: $\sum_{t=1}^{M}t\cdot p^t=\frac{1}{p-1}M\cdot p^{M+1}-\frac{p}{(p-1)^2}\left(p^M-1\right)$

We denote that $S=\sum_{t=1}^{M}t\cdot p^t$,有 $p\cdot S=\sum_{t=1}^{M}t\cdot p^{t+1}=\sum_{k=2}^{M+1}(t-1)\cdot p^t$,

and we have: $(p-1)S=p\cdot S-S=M\cdot p^{M+1}-\frac{p}{p-1}\left(p^M-1\right)$,

which means that $S=\frac{1}{p-1}M\cdot p^{M+1}-\frac{p}{(p-1)^2}\left(p^M-1\right)$.

Specifically, when $p=10$, we have $S=\frac{1}{9}M\cdot 10^{M+1}-\frac{10}{81}\left(10^M-1\right)$. We will directly use the computational result hereafter.

2.2 Proof When $k=3$ in Base 10

2.2.1 Identical deformation of the target equation

Let a_n be in the sequence, and assume that $3n$ has t digits ($n\epsilon\mathbb{Z}^+,t\epsilon\mathbb{Z}^+$), then $\frac{10^{t-1}}{3}\le n<\frac{10^t}{3}$. Because of the definition of the sequence $\{a_n\}$, we know that $a_n=n\cdot\left(10^t+3\right)$. When N is large enough, there exists a unique $M\in\mathbb{Z}^+$ such that

$\frac{10^M}{3} \le N < \frac{10^{M+1}}{3}$. This is because the intervals $J_t = \left(\frac{10^t}{3}, \frac{10^{t+1}}{3}\right], t = 0,1,2,\cdots$ are pair-wise disjoint, and their union is $\left(\frac{1}{3}, +\infty\right)$, which includes all positive integers, so N must be included in one of these intervals, which means that there must be a unique M . We will use the uniqueness of M directly hereafter. Assume that $3N$ has $(M+1)$ digits , so $a_N = N \cdot \left(10^{M+1}+3\right)$. Now we have the following identical equation:

$$\prod_{1\le n\le N} a_n = \prod_{n=1}^{3} a_n \cdot \prod_{n=4}^{33} a_n \cdots \prod_{n=\frac{1}{3}\left(10^{M-1}-1\right)+1}^{\frac{1}{3}\left(10^M-1\right)} a_n \cdot \prod_{n=\frac{1}{3}\left(10^M-1\right)+1}^{N} a_n$$

$$= N! \cdot (10+3)^3 \cdot (100+3)^{30} \cdots \left(10^M+3\right)^{3\cdot 10^{M-1}} \cdot \left(10^{M+1}+3\right)^{N-\frac{1}{3}\left(10^M-1\right)}$$

Take the natural logarithm of the both sides, and the equation becomes:

$$\sum_{1\le n\le N} \ln a_n = \ln N! + \sum_{t=1}^{M} \ln\left(10^t+3\right)^{3\cdot 10^{t-1}} + \ln\left(10^{M+1}+3\right)^{N-\frac{1}{3}\left(10^M-1\right)}$$

$$= \ln N! + 3\cdot \sum_{t=1}^{M} 10^{t-1} \cdot \ln\left(10^t+3\right) + \left[N - \frac{1}{3}\left(10^M-1\right)\right] \ln\left(10^{M+1}+3\right) \cdots (1)$$

2.2.2 Estimation of $N!$

Using Stirling's approximation (Lemma 2.1.3):

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \cdots\right)$$

According to 2.1.2, we take the natural logarithm of the both sides, and we get the equation as follows.

$$\ln N! = \ln\sqrt{2\pi N} + N\ln N - N + \ln\left(1 + \frac{1}{12N} + \frac{1}{288N^2} - \frac{139}{51840N^3} + \cdots\right)$$

$$= \left(N + \frac{1}{2}\right)\ln N - N + \ln\sqrt{2\pi} + O\left(\frac{1}{N}\right)$$

$$= \left(N + \frac{1}{2}\right)\ln N - N + O(1) \cdots\cdots (2)$$

We will use equation (2) directly hereafter.

2.2.3 Estimation of $3\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln\left(10^{t}+3\right)$

When $x\to 0$, $\ln\left(x+1\right)=x+O\left(x^{2}\right)$ (Lemma 2.1.2). According to 2.1.5, we get the following equation.

$$3\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln\left(10^{t}+3\right)=3\cdot\sum_{t=1}^{M}10^{t-1}\cdot\left[\ln\left(10^{t}\right)+\ln\left(1+\frac{3}{10^{t}}\right)\right]$$
$$=3\cdot\sum_{t=1}^{M}10^{t-1}\cdot\left[t\cdot\ln 10+\frac{3}{10^{t}}+O\left(\frac{1}{10^{2t}}\right)\right]$$
$$=3\ln 10\cdot\sum_{t=1}^{M}t\cdot 10^{t-1}+\frac{9}{10}M+O\left(1\right)$$
$$=\frac{\ln 10}{3}M\cdot 10^{M}-\frac{\ln 10}{27}\left(10^{M}-1\right)+\frac{9}{10}M+O\left(1\right)\cdots\cdots\left(3\right)$$

2.2.4 Estimation of $\left[N-\frac{1}{3}\left(10^{M}-1\right)\right]\ln\left(10^{M+1}+3\right)$

Note that when $x\to 0$, $\ln\left(x+1\right)=x+O\left(x^{2}\right)$ (Lemma 2.1.2). Therefore,

$$\left[N-\frac{1}{3}\left(10^{M}-1\right)\right]\ln\left(10^{M+1}+3\right)=\left[N-\frac{1}{3}\left(10^{M}-1\right)\right]\left[\ln\left(1+\frac{3}{10^{M+1}}\right)+\ln\left(10^{M+1}\right)\right]$$
$$=\left[N-\frac{1}{3}\left(10^{M}-1\right)\right]\left[\frac{3}{10^{M+1}}+O\left(\frac{1}{10^{2M}}\right)+\left(M+1\right)\cdot\ln 10\right],$$

where $\left[N-\frac{1}{3}\left(10^{M}-1\right)\right]=O\left(N\right)$, $\frac{3}{10^{M+1}}+O\left(\frac{1}{10^{2M}}\right)=O\left(\frac{1}{N}\right)$, and $O\left(N\right)\cdot O\left(\frac{1}{N}\right)=O\left(1\right)$.

This means that:

$$\left[N-\frac{1}{3}\left(10^M-1\right)\right]\ln(10^{M+1}+3)$$
$$=\left[N-\frac{1}{3}\left(10^M-1\right)\right]\cdot\left[\frac{3}{10^{M+1}}+O\left(\frac{1}{10^{2M}}\right)+\left(M+1\right)\cdot\ln10\right]$$
$$=\left[N-\frac{1}{3}\left(10^M-1\right)\right]\cdot\left(M+1\right)\cdot\ln10+O\left(1\right)$$
$$=\ln10\cdot\left(MN-\frac{1}{3}M\cdot10^M+\frac{1}{3}M+N-\frac{1}{3}\cdot10^M+\frac{1}{3}\right)+O\left(1\right)$$
$$=\ln10\cdot\left(MN-\frac{1}{3}M\cdot10^M+\frac{1}{3}M+N-\frac{1}{3}\cdot10^M\right)+O\left(1\right)\cdots\cdots\cdots\cdots\cdots\cdots(4)$$

2.2.5 Summate and analyze the error terms

Finaly we substitute $(2)(3)(4)$ into (1):

$$\sum_{1\le n\le N}\ln a_n=\left[\left(N+\frac{1}{2}\right)\ln N-N+O\left(1\right)\right]$$
$$+\left[\frac{\ln10}{3}M\cdot10^M-\frac{\ln10}{27}\left(10^M-1\right)+\frac{9}{10}M+O\left(1\right)\right]$$
$$+\left[\ln10\cdot\left(MN-\frac{1}{3}M\cdot10^M+\frac{1}{3}M+N-\frac{1}{3}\cdot10^M\right)+O\left(1\right)\right]$$
$$=N\ln N+\frac{1}{2}\ln N-N+\frac{\ln10}{3}\cdot M\cdot10^M-\frac{\ln10}{27}\cdot10^M+\frac{9}{10}M,$$
$$+\ln10\cdot MN-\frac{\ln10}{3}\cdot M\cdot10^M+\frac{\ln10}{3}\cdot M+\ln10\cdot N-\frac{\ln10}{3}\cdot10^M+O\left(1\right)$$
$$=\left(N\ln N+MN\cdot\ln10\right)+\frac{1}{2}\ln N+\left(\ln10-1\right)N-\frac{10\ln10}{27}\cdot10^M$$
$$+\left(\frac{\ln10}{3}+\frac{9}{10}\right)M+O\left(1\right)$$

then:

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln10\right)+\left[\left(\ln10-1\right)N-\frac{10\ln10}{27}\cdot10^M\right]$$
$$+\left[\left(\frac{\ln10}{3}+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O\left(1\right)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(5).$$

This is the asymptotic formula with an $O\left(1\right)$ error term when $k=3$.

Because the quotients of M over $\ln N$ and 10^M over N are bounded, the following equations hold:

$$\lim_{N\to+\infty}\frac{N\ln N+MN\ln 10}{N\ln N}=2,$$

$$\lim_{N\to+\infty}\frac{\frac{1}{2}\ln N+(\ln 10-1)N-\frac{10\ln 10}{27}10^M+\left(\frac{\ln 10}{3}+\frac{9}{10}\right)M}{N}=O(1).$$

This means that: $\sum_{1\le n\le N}\ln a_n=2N\ln N+O(N)$.

2.3 For Some Specific k ($k=1,2,4,5,6,8,9,10,11$, in Base 10)

The following proof is similar to $k=3$, but only different in the classification of n according to how many digits $k\cdot n$ has in the base 10. Chen, J.(2012:9-14) proved the asymptotic formula to be true when $1\le k\le 9$, but in fact, the asymptotic formula is true even when $k=10,11$.

2.3.1 $k=1$

Assume n has t digits ($n\epsilon\mathbb{Z}^+$, $t\epsilon\mathbb{Z}^+$), then $10^{t-1}\le n<10^t$. Because of the definition of the sequence $\{a_n\}$, we have $a_n=n\cdot(10^t+1)$. For any N that is large enough, there exists a unique $M\in\mathbb{Z}^+$ such that $10^M\le N<10^{M+1}$.

By the same argument:

$$\prod_{1\le n\le N}a_n=\prod_{n=1}^{9}a_n\cdot\prod_{n=10}^{99}a_n\cdots\prod_{n=10^{M-1}}^{10^M-1}a_n\cdot\prod_{n=10^M}^{N}a_n$$

$$=N!\cdot(10+1)^9\cdot(100+1)^{90}\cdots(10^M+1)^{9\cdot 10^{M-1}}\cdot(10^{M+1}+1)^{N-(10^M-1)}$$

$$\sum_{1\le n\le N}\ln a_n=\ln N!+\sum_{t=1}^{M}\ln(10^t+1)^{9\cdot 10^{t-1}}+\ln(10^{M+1}+1)^{N-(10^M-1)}$$

$$=\ln N!+9\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln(10^t+1)+\left[N-(10^M-1)\right]\ln(10^{M+1}+1)$$

We have: $\ln N!=\left(N+\frac{1}{2}\right)\ln N-N+O(1)$

When $x \to 0$, $\ln(x+1) = x + O(x^2)$, which means that:

$$\begin{aligned}
9\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln\left(10^t+1\right) &= 9\cdot\sum_{t=1}^{M}10^{t-1}\cdot\left[\ln\left(10^t\right)+\ln\left(1+\frac{1}{10^t}\right)\right] \\
&= 9\cdot\sum_{t=1}^{M}10^{t-1}\cdot\left[t\cdot\ln 10+\frac{1}{10^t}+O\left(\frac{1}{10^{2t}}\right)\right] \\
&= 9\cdot\ln 10\cdot\sum_{t=1}^{M}t\cdot 10^{t-1}+\frac{9}{10}M+O(1) \\
&= \ln 10\cdot M\cdot 10^M-\frac{\ln 10}{9}\left(10^M-1\right)+\frac{9}{10}M+O(1)
\end{aligned},$$

and

$$\begin{aligned}
\left[N-\left(10^M-1\right)\right]\ln\left(10^{M+1}+1\right) &= \left[N-\left(10^M-1\right)\right]\left[\ln\left(1+\frac{1}{10^{M+1}}\right)+\ln\left(10^{M+1}\right)\right] \\
&= \left[N-\left(10^M-1\right)\right]\left[\frac{1}{10^{M+1}}+O\left(\frac{1}{10^{2M}}\right)+(M+1)\cdot\ln 10\right] \\
&= \ln 10\cdot(\mathrm{M}+1)\left[N-\left(10^M-1\right)\right]+O(1) \\
&= \ln 10\cdot\left(MN-M\cdot 10^M+M+N-10^M+1\right)+O(1) \\
&= \ln 10\cdot\left(MN-M\cdot 10^M+M+N-10^M\right)+O(1)
\end{aligned}.$$

At last we have:

$$\begin{aligned}
\sum_{1\le n\le N}\ln a_n &= \left[\left(N+\frac{1}{2}\right)\ln N-N+O(1)\right] \\
&\quad+\left[\ln 10\cdot M\cdot 10^M-\frac{\ln 10}{9}\cdot\left(10^M-1\right)+\frac{9}{10}M+O(1)\right] \\
&\quad+\left[\ln 10\cdot\left(MN-M\cdot 10^M+M+N-10^M\right)+O(1)\right] \\
&= N\ln N+\frac{1}{2}\ln N-N+\ln 10\cdot M\cdot 10^M-\frac{\ln 10}{9}\cdot 10^M+\frac{9}{10}M \\
&\quad+\ln 10\cdot MN-\ln 10\cdot M\cdot 10^M+\ln 10\cdot M+\ln 10\cdot N-\ln 10\cdot 10^M+O(1) \\
&= \left(N\ln N+MN\cdot\ln 10\right)+\frac{1}{2}\ln N+(\ln 10-1)N-\frac{10\ln 10}{9}\cdot 10^M \\
&\quad+\left(\ln 10+\frac{9}{10}\right)M+O(1)
\end{aligned},$$

which means that:

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{10\ln 10}{9}\cdot 10^M\right]$$
$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)\cdots\cdots(6)$$

This is the asymptotic formula with an $O(1)$ error term when $k=1$.

Because the quotients of M over $\ln N$ and 10^M over N are bounded, the following equations hold:

$$\lim_{N\to+\infty}\frac{N\ln N+MN\ln 10}{N\ln N}=2,$$

$$\lim_{N\to+\infty}\frac{\left(\ln 10-1\right)N-\frac{10\ln 10}{9}10^M+\frac{1}{2}\ln N+\left(\ln 10+\frac{9}{10}\right)M}{N}=O(1),$$

which means that: $\sum_{1\le n\le N}\ln a_n=2N\ln N+O(N)$.

2.3.2 $k=2,4,5,6,8,9,10,11$

Because the proof for $k=2,4,5,6,8,9,10,11$ is tedious and highly similar to the proof of $k=1$ and $k=3$, the detailed proof is presented in **'5 Appendix'** and here only the results are presented below. **(Equations $(7)\sim(17)$ are also in '5 Appendix'.)**

For $k=2,4,5,6,8,9,10,11$, the asymptotic formula $\sum_{1\le n\le N}\ln a_n=2N\ln N+O(N)$ is true.

Furthermore, we get the following asymptotic formulas with an $O(1)$ error term, in which $M=\lfloor\log_{10}kN\rfloor$. ($\lfloor x\rfloor$ is the floor function of x)

For $k=2$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{5\ln 10}{9}\cdot 10^M\right]$$
$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)$$

For $k=4$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{5\ln 10}{18}\cdot 10^M\right]$$
$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)$$

For $k=5$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{2\ln 10}{9}\cdot 10^M\right]$$
$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)$$

For $k=6$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{5\ln 10}{27}\cdot 10^M\right]$$
$$+\left[\left(\frac{3\ln 10}{2}+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)$$

For $k=8$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{5\ln 10}{36}\cdot 10^M\right]$$
$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)$$

For $k=9$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{10\ln 10}{81}\cdot 10^M\right]$$
$$+\left[\left(\frac{\ln 10}{9}+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)$$

For $k=10$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{\ln 10}{9}\cdot 10^M\right]$$
$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)$$

For $k=11$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+\ln 10\cdot MN\right)+\left[\left(\ln 10-1\right)N-\frac{10\ln 10}{99}\cdot 10^M\right]$$
$$+\left[\frac{1}{2}\ln N+\left(\frac{9}{10}+\frac{\ln 10}{2}\right)M\right]+O(1)$$

According the detailed proof of $k=11$, the following conjecture arises.

For $k=7$,

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+\ln 10\cdot MN\right)+\left[\left(\ln 10-1\right)N-\frac{10\ln 10}{63}\cdot 10^{M}\right]$$
$$+\left[\frac{1}{2}\ln N+\left(\frac{9}{10}+\frac{\ln 10}{2}\right)M\right]+O\left(1\right)$$

2.4 For General k (in Base 10)

For general k, if the prime factors of k are good enough, we can still get the values of the exponential of $\left(10^{t}+k\right)$ accurately, thus giving a more precise asymptotic formula.

2.4.1 If there exists $\alpha\in\square$ such that $k=10^{\alpha}$

Let $10^{M}\le k\cdot N<10^{M+1}$, we will have the following equation.

$$\prod_{1\le n\le N}a_n=N!\cdot\left(10^{\alpha+1}+k\right)^{9}\cdot\left(10^{\alpha+2}+k\right)^{90}\cdots\left(10^{M}+k\right)^{9\cdot 10^{M-\alpha-1}}\cdot\left(10^{M+1}+k\right)^{N-\left(10^{M-\alpha}-1\right)}$$

$$\sum_{1\le n\le N}\ln a_n=\ln N!+\sum_{t=1}^{M-\alpha}\ln\left(10^{\alpha+t}+k\right)^{9\cdot 10^{t-1}}+\ln\left(10^{M+1}+1\right)k^{N-\left(10^{M-\alpha}-1\right)}$$
$$=\ln N!+9\cdot\sum_{t=1}^{M-\alpha}10^{t-1}\cdot\ln\left(10^{t+\alpha}+k\right)+\left[N-\left(10^{M-\alpha}-1\right)\right]\ln\left(10^{M+1}+k\right)$$

It comes out that $\ln N!=\left(N+\frac{1}{2}\right)\ln N-N+O\left(1\right)$ and

$$9\cdot\sum_{t=1}^{M-\alpha}10^{t-1}\cdot\ln\left(10^{t+\alpha}+k\right)=9\cdot\sum_{t=1}^{M-\alpha}10^{t-1}\cdot\left[\left(t+\alpha\right)\cdot\ln 10+\frac{k}{10^{t+\alpha}}+O\left(\frac{1}{10^{2t}}\right)\right]$$
$$=9\ln 10\cdot\sum_{t=1}^{M-\alpha}\left(t+\alpha\right)\cdot 10^{t-1}+\frac{9k}{10^{\alpha+1}}\left(M-\alpha\right)+O\left(1\right)$$
$$=\ln 10\cdot\left(M-\frac{1}{9}\right)\cdot 10^{M-\alpha}+\frac{9}{10}M+O\left(1\right)$$

and

$$\left[N-\left(10^{M-\alpha}-1\right)\right]\ln\left(10^{M+1}+1\right)=\left[N-\left(10^{M-\alpha}-1\right)\right]\left[\left(M+1\right)\cdot\ln 10+\frac{1}{10^{M+1}}+O\left(\frac{1}{10^{2M}}\right)\right]$$
$$=\ln 10\cdot(\mathrm{M}+1)\cdot\left[N-\left(10^{M-\alpha}-1\right)\right]+O(1) .$$
$$=\ln 10\cdot\left(MN-M\cdot 10^{M-\alpha}+M+N-10^{M-\alpha}\right)+O(1)$$

Finally we have:

$$\sum_{1\le n\le N}\ln a_n=\left[\left(N+\frac{1}{2}\right)\ln N-N+O(1)\right]$$
$$+\left[\ln 10\cdot\left(M-\frac{1}{9}\right)\cdot 10^{M-\alpha}+\frac{9}{10}M+O(1)\right]$$
$$+\left[\ln 10\cdot\left(MN-M\cdot 10^{M-\alpha}+M+N-10^{M-\alpha}\right)+O(1)\right]$$
$$=N\ln N+\frac{1}{2}\ln N-N+\ln 10\cdot M\cdot 10^{M-\alpha}-\frac{\ln 10}{9}\cdot 10^{M-\alpha}+\frac{9}{10}M$$
$$+\ln 10\cdot MN-\ln 10\cdot M\cdot 10^{M-\alpha}+\ln 10\cdot M+\ln 10\cdot N-\ln 10\cdot 10^{M-\alpha}+O(1)$$
$$=\left(N\ln N+MN\cdot\ln 10\right)+\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{10\ln 10}{9}\cdot 10^{M-\alpha}$$
$$+\left(\ln 10+\frac{9}{10}\right)M+O(1)$$

which gives:

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{10\ln 10}{9\cdot 10^{\alpha}}\cdot 10^{M}\right]$$
$$+\left[\frac{1}{2}\ln N+\left(\ln 10+\frac{9}{10}\right)M\right]+O(1)\cdots\cdots\cdots\cdots\cdots\cdots(18)$$

This is the asymptotic formula with an $O(1)$ error term when $\mathrm{k}=10^{\alpha}$.

Because the quotients of M over $\ln N$ and 10^{M} over N are bounded, $\lim\limits_{N\to+\infty}\frac{N\ln N+\ln 10\cdot MN}{N\ln N}=2$.
The orders of the other terms are no larger than N, which means that $\sum_{1\le n\le N}\ln a_n=2N\ln N+O(N)$.

2.4.2 If there exists $\alpha,\beta\in\mathbb{N}$ such that $k=2^{\alpha}\cdot 5^{\beta}$

Let $10^{M}\le k\cdot N<10^{M+1}$, we will have:

$$\prod_{1\le n\le N} a_n = N!\cdot\left(10^1+k\right)^{\frac{9}{k}}\cdot\left(10^2+k\right)^{\frac{9\cdot 10}{k}}\cdots\left(10^M+k\right)^{\frac{9\cdot 10^{M-1}}{k}}$$

$$\cdot\left(10^{M+1}+1\right)^{N-\frac{10^M}{k}+1}\cdot O(1)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(19)$$

Equation (19) holds because:

①: When $N\to+\infty, M\to+\infty$, we can find $m\ge\max\{\alpha,\beta\}, m\in\square^+$, then $\frac{10^m}{k}\in\square$,

which means that for m that is large enough, the exponential power $\frac{9\cdot 10^m}{k}$ in the above equation is a positive integer.

②: For a given positive integer t, we wonder what kind of integer n exists such that $a_n = n\cdot\left(10^t+1\right)$ holds, which is that kn has t digits , namely $10^{t-1}\le kn<10^t$. For t large enough, we can count the number of such n, which is exactly $\frac{10^t-10^{t-1}}{k}=\frac{9\cdot 10^{t-1}}{k}$.

③: From ①②, for a given k, the exponentials on the right-hand side of equation (19) can be replaced by $\frac{9\cdot 10^m}{k}$ for m that is large enough, except some finite terms at the first place. We still replace these exponentials by $\frac{9\cdot 10^m}{k}$ for m that is not large enough. The values of these finite terms is determined, which means that we might over-multiply the right-hand side by a value that is finite, so we can simply multiply the right-hand side by $O(1)$ to make the equation correct.

Now we have: $\prod_{1\le n\le N} a_n = N!\cdot\prod_{t=1}^{M}\left(10^t+k\right)^{\frac{9\cdot 10^{t-1}}{k}}\cdot\left(10^{M+1}+1\right)^{N-\frac{10^M}{k}+1}\cdot O(1)$

$$\sum_{1\le n\le N}\ln a_n = \ln N!+\sum_{t=1}^{M}\ln\left[\left(10^t+k\right)^{\frac{9\cdot 10^{t-1}}{k}}\right]+\ln\left[\left(10^{M+1}+k\right)^{N-\frac{10^M}{k}+1}\right]+O(1)$$

$$=\ln N!+\frac{9}{k}\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln\left(10^t+k\right)+\left(N-\frac{10^M}{k}+1\right)\ln\left(10^{M+1}+k\right)+O(1)$$

$$\ln N! = \left(N+\frac{1}{2}\right)\ln N - N + O(1)$$

$$\frac{9}{k}\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln\left(10^{t}+k\right) = \frac{9}{k}\cdot\sum_{t=1}^{M}10^{t-1}\cdot\left[t\cdot\ln 10+\frac{k}{10^{t}}+O\left(\frac{1}{10^{2t}}\right)\right]$$
$$= \frac{9\ln 10}{k}\cdot\sum_{t=1}^{M}t\cdot 10^{t-1}+\frac{9}{10}M+O(1)$$
$$= \frac{\ln 10}{k}\cdot\left(M-\frac{1}{9}\right)\cdot 10^{M}+\frac{9}{10}M+O(1)$$

and

$$\left(N-\frac{10^{M}}{k}+1\right)\ln\left(10^{M+1}+k\right) = \left(N-\frac{10^{M}}{k}+1\right)\left[(M+1)\cdot\ln 10+\frac{k}{10^{M+1}}+O\left(\frac{1}{10^{2M}}\right)\right]$$
$$= \left(N-\frac{10^{M}}{k}+1\right)\left[(M+1)\cdot\ln 10+O\left(\frac{1}{10^{M}}\right)\right]$$
$$= \ln 10\cdot MN+\ln 10\cdot M+\ln 10\cdot N-\frac{\ln 10}{k}\cdot M\cdot 10^{M}-\frac{\ln 10}{k}\cdot 10^{M}+O(1)$$

At last we have

$$\sum_{1\le n\le N}\ln a_n = \left[\left(N+\frac{1}{2}\right)\ln N - N\right]+\left[\frac{\ln 10}{k}\cdot\left(M-\frac{1}{9}\right)\cdot 10^{M}+\frac{9}{10}M\right]$$
$$+\left[\ln 10\cdot MN+\ln 10\cdot M+\ln 10\cdot N-\frac{\ln 10}{k}\cdot M\cdot 10^{M}-\frac{\ln 10}{k}\cdot 10^{M}\right]+O(1)$$
$$= N\ln N+\frac{1}{2}\ln N - N+\frac{\ln 10}{k}\cdot M\cdot 10^{M}-\frac{\ln 10}{9k}\cdot 10^{M}+\frac{9}{10}M$$
$$+\ln 10\cdot MN+\ln 10\cdot M+\ln 10\cdot N-\frac{\ln 10}{k}\cdot M\cdot 10^{M}-\frac{\ln 10}{k}\cdot 10^{M}+O(1)$$
$$= \left(N\ln N+MN\cdot\ln 10\right)+\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{10\ln 10}{9k}\cdot 10^{M}$$
$$+\left(\ln 10+\frac{9}{10}\right)M+O(1)$$
,

which means that

$$\sum_{1\le n\le N}\ln a_n = \left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{10\ln 10}{9k}\cdot 10^{M}\right]$$
$$+\left[\frac{1}{2}\ln N+\left(\ln 10+\frac{9}{10}\right)M\right]+O(1)\cdots\cdots\cdots\cdots(20)$$
.

This is the asymptotic formula with an $O(1)$ error term when $k = 2^{\alpha}\cdot 5^{\beta}$.

Because the quotients of M over $\ln N$ and 10^M over N are bounded, $\lim\limits_{N\to+\infty}\dfrac{N\ln N+\ln 10\cdot MN}{N\ln N}=2$.

The orders of the other terms are no larger than N, which means that $\sum\limits_{1\le n\le N}\ln a_n = 2N\ln N + O(N)$ holds.

2.4.3 If there is no $\alpha,\beta\in\mathbb{N}$ such that $k=2^\alpha\cdot 5^\beta$

If there is no $\alpha,\beta\in\mathbb{N}$ such that $k=2^\alpha\cdot 5^\beta$, then $k>1$ holds, so there exists a prime factor q of k such that $q\notin\{2,5\}$. Let $10^M\le k\cdot N<10^{M+1}$ and we have the following equation.

$$\prod_{1\le n\le N} a_n = N!\cdot\left(10^1+k\right)^{\frac{9}{k}+b_1}\cdot\left(10^2+k\right)^{\frac{9\cdot 10}{k}+b_2}\cdots\left(10^M+k\right)^{\frac{9\cdot 10^{M-1}}{k}+b_M},$$
$$\cdot\left(10^{M+1}+1\right)^{N-\frac{10^M}{k}+c}\cdots\cdots(21)$$

where: $b_t=\left\lfloor\dfrac{10^t}{k}\right\rfloor-\left\lfloor\dfrac{10^{t-1}}{k}\right\rfloor-\dfrac{9\cdot 10^{t-1}}{k}$ ($t=1,2,3,\cdots,M$), and $|b_t|<1$, $|c|<1$. ($\lfloor x\rfloor$ is the floor function of x)

Equation (21) holds because:

①: For any given positive integer t, we wonder what kind of n exists such that $a_n=n\cdot\left(10^t+1\right)$ holds, which is that kn has t digits (namely $10^{t-1}\le kn<10^t$). We can count the number of such n, which is exactly $\left\lfloor\dfrac{10^t}{k}\right\rfloor-\left\lfloor\dfrac{10^{t-1}}{k}\right\rfloor$. Therefore, $b_t=\left\lfloor\dfrac{10^t}{k}\right\rfloor-\left\lfloor\dfrac{10^{t-1}}{k}\right\rfloor-\dfrac{9\cdot 10^{t-1}}{k}$.

②: The value of the exponential power of $\left(p^{M+1}+1\right)$ should be $N-\left\lfloor\dfrac{p^M}{k}\right\rfloor$ exactly, which can be denoted as $N-\dfrac{p^M}{k}+c$, and thereby $|c|<1$. Therefore we have:

$$\prod_{1\le n\le N} a_n = N!\cdot\prod_{t=1}^{M}\left(10^t+k\right)^{\frac{9\cdot 10^{t-1}}{k}+b_t}\cdot\left(10^{M+1}+1\right)^{N-\frac{10^M}{k}+c}$$

$$\sum_{1\le n\le N}\ln a_n = \ln N!+\sum_{t=1}^{M}\ln\left[\left(10^t+k\right)^{\frac{9\cdot 10^{t-1}}{k}+b_t}\right]+\ln\left[\left(10^{M+1}+k\right)^{N-\frac{10^M}{k}+c}\right]$$

$$=\ln N!+\frac{9}{k}\cdot\sum_{t=1}^{M}\left(10^{t-1}+\frac{k}{9}\cdot b_t\right)\cdot\left(t\cdot\ln 10+\frac{k}{10^t}+O\left(\frac{1}{10^{2t}}\right)\right)$$

$$+\left(N-\frac{10^M}{k}+c\right)\cdot\left[\left(M+1\right)\cdot\ln 10+O\left(\frac{1}{10^M}\right)\right]\cdots\cdots\cdots\cdots\cdots\cdots\cdots\left(22\right)$$

Because b_t and c are bounded, we can replace b_t and c by $O\left(1\right)$.

$$\sum_{1\le n\le N}\ln a_n = \ln N!+\sum_{t=1}^{M}\ln\left[\left(10^t+k\right)^{\frac{9\cdot 10^{t-1}}{k}+O(1)}\right]+\ln\left[\left(10^{M+1}+k\right)^{N-\frac{10^M}{k}+O(1)}\right]$$

$$=\ln N!+\frac{9}{k}\cdot\sum_{t=1}^{M}\left(10^{t-1}+O\left(1\right)\right)\cdot\ln\left(10^t+k\right)$$

$$+\left(N-\frac{10^M}{k}+O\left(1\right)\right)\ln\left(10^{M+1}+k\right)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\left(23\right)$$

We deal with the first addend in equation $\left(23\right)$.

$$\ln N!=\left(N+\frac{1}{2}\right)\ln N-N+O\left(1\right)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\left(24\right)$$

We deal with the second addend in equation $\left(23\right)$.

$$\frac{9}{k}\cdot\sum_{t=1}^{M}\left(10^{t-1}+O\left(1\right)\right)\cdot\ln\left(10^t+k\right)=\frac{9}{k}\cdot\sum_{t=1}^{M}\left(10^{t-1}+O\left(1\right)\right)\cdot\left[t\cdot\ln 10+\frac{k}{10^t}+O\left(\frac{1}{10^{2t}}\right)\right]$$

$$=\frac{9\ln 10}{k}\cdot\sum_{t=1}^{M}t\cdot 10^{t-1}+\frac{9}{10}M+\frac{9}{k}\cdot\sum_{t=1}^{M}O\left(1\right)\cdot\left[t\cdot\ln 10+\frac{k}{10^t}+O\left(\frac{1}{10^{2t}}\right)\right]+O\left(1\right)\cdots\cdots\cdots\left(25\right)$$

We deal with the third addend in equation $\left(23\right)$.

$$\left(N-\frac{10^M}{k}+O\left(1\right)\right)\ln\left(10^{M+1}+k\right)=\left(N-\frac{10^M}{k}+O\left(1\right)\right)\left[\left(M+1\right)\cdot\ln 10+O\left(\frac{1}{10^M}\right)\right]$$

$$=\ln 10\cdot MN+\ln 10\cdot N-\frac{\ln 10}{k}\cdot M\cdot 10^M-\frac{\ln 10}{k}\cdot 10^M+O\left(M\right)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\left(26\right)$$

We substitute the equations $\left(24\right)\left(25\right)\left(26\right)$ into equation $\left(23\right)$ and sum up.

$$\sum_{1\le n\le N}\ln a_n=\left[\left(N+\frac{1}{2}\right)\ln N-N\right]+\left[\frac{\ln 10}{k}\cdot\left(M-\frac{1}{9}\right)\cdot 10^M\right]$$
$$+\left[\ln 10\cdot MN+\ln 10\cdot N-\frac{\ln 10}{k}\cdot M\cdot 10^M-\frac{\ln 10}{k}\cdot 10^M\right]+O\left(M^2\right)$$
$$=N\ln N+\frac{1}{2}\ln N-N+\frac{\ln 10}{k}\cdot M\cdot 10^M-\frac{\ln 10}{9k}\cdot 10^M$$
$$+\ln 10\cdot MN+\ln 10\cdot N-\frac{\ln 10}{k}\cdot M\cdot 10^M-\frac{\ln 10}{k}\cdot 10^M+O\left(M^2\right)$$
$$=\left(N\ln N+MN\cdot\ln 10\right)+\left(\ln 10-1\right)N-\frac{10\ln 10}{9k}\cdot 10^M+O\left(M^2\right)$$

The error term $O\left(M^2\right)$ has the same order of $\left(\ln N\right)^2$. Because the quotients of M over $\ln N$ and 10^M over N are bounded, we know that $\lim\limits_{N\to+\infty}\frac{N\ln N+MN\ln 10}{N\ln N}=2$ and $\lim\limits_{N\to+\infty}\frac{\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{10\ln 10}{9k}\cdot 10^M}{N}=O(1)$, which leads to $\sum_{1\le n\le N}\ln a_n=2N\ln N+O\left(N\right)$.

Go back to $(24)(25)(26)$. Because the quotients of M over $\ln N$ and 10^M over N are bounded, in the sum $\sum_{1\le n\le N}\ln a_n$, only the second addend $\frac{9}{k}\cdot\sum_{t=1}^{M}O(1)\cdot\left(t\cdot\ln 10\right)$ contains the expression that is of the order $\left(\ln N\right)^2$. Go back to equation (21) and we find out that the expression is $A=\frac{9}{k}\cdot\sum_{t=1}^{M}\left(\frac{k}{9}\cdot b_t\right)\cdot\left(t\cdot\ln 10\right)$.

Now we prove that $A=O\left(\ln N\right)$, and thus proving that the error term has the same order of $\ln N$ rather than $\left(\ln N\right)^2$. To complete the proof, we only have to make sure that $A'=\sum_{t=1}^{M}t\cdot b_t=O\left(\ln N\right)$. Now we divide k out of all the prime factors 2 and 5 to get a positive integer k_0. It is obvious that $\left(k_0,10\right)=1$. Let the order of 10 modulo k_0 be δ. When the positive integer s is large enough, we can compute $B=\sum_{t=s+1}^{s+\delta}b_t$.

$B=\sum_{t=s+1}^{s+\delta}\left(\left[\frac{10^{t}}{k}\right]-\left[\frac{10^{t-1}}{k}\right]-\frac{9\cdot 10^{t-1}}{k}\right)=\left[\frac{10^{s+\delta}}{k}\right]-\left[\frac{10^{s}}{k}\right]-\frac{10^{s+\delta}}{k}+\frac{10^{s}}{k}$, where $\frac{10^{s+\delta}}{k}-\frac{10^{s}}{k}=\frac{10^{s}\cdot\left(10^{\delta}-1\right)}{k}=\frac{10^{s}}{r}\cdot\frac{10^{\delta}-1}{k_0}\in\mathbb{Z}$. This is because of the definition of the order of 10 modulo k_0 and the order being large enough, where $r=\frac{k}{k_0}$ and r only contains the prime factors 2 and 5). We know that the difference between two numbers with the same decimal part equals the difference between their integer parts, so $\left[\frac{10^{s+\delta}}{k}\right]-\left[\frac{10^{s}}{k}\right]=\frac{10^{s+\delta}}{k}-\frac{10^{s}}{k}$, which means that $B=0$. When the positive integer s is not large enough, the sum of these terms is infinite and does not produce a number of the order $\left(\ln N\right)^{2}$. Now $B=0$ means that after a finite number of terms, the sum of δ consecutive terms of the sequence $\{b_n\}$ is 0, which means that $\{b_n\}$ is of period δ after a finite number of terms, namely $b_t=b_{t+\delta}$ for t large enough. Therefore,

$$\begin{aligned}A'&=\sum_{t=1}^{M}t\cdot b_t=\left[\frac{M}{\delta}\right]\cdot\sum_{t=1}^{\delta}t\cdot b_t+\sum_{t=1}^{\left\{\frac{M}{\delta}\right\}\cdot\delta}t\cdot b_t=\left[\frac{M}{\delta}\right]\cdot\sum_{t=1}^{\delta}t\cdot b_t+O\left(1\right)=\left[\frac{M}{\delta}\right]\cdot O\left(1\right)+O\left(1\right)\\&=O\left(M\right)\cdot O\left(1\right)+O\left(1\right)=O\left(\ln N\right)\end{aligned}.$$

After we complete the proof, we can make sure that:

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left(\ln 10-1\right)N-\frac{10\ln 10}{9k}\cdot 10^{M}+O\left(\ln N\right)\cdots\left(27\right)$$

(by replacing $O\left(M^{2}\right)$ by $O\left(\ln N\right)$)

Now the estimate is more precise. ($k\neq 2^{\alpha}\cdot 5^{\beta}$)

2.5 New Smarandache kn-digital Sequence Defined Similarly in Base p

For a positive integer $p\ge 2$, we can define a new Smarandache kn-digital sequence in a similar way. The digital of any number in the sequence can be partitioned into two groups in base p such that the second is k times of the first. For example, when $p=8, k=3$, $\{a_n\}=\{13, 26, 311, 414, 517, 622, 725, 1030, 1133\cdots\}$.

2.5.1 Any prime factor of k can divide p exactly (including $k=1$)

If any prime factor of k can divide p exactly (including $k=1$), then there exists a positive integer r that is large enough such that $\frac{p^r}{k}\in\mathbb{Z}$.

Let $p^M \le k\cdot N < p^{M+1}$, we will have:

$$\prod_{1\le n\le N} a_n = N!\cdot\left(p^1+k\right)^{\frac{(p-1)}{k}}\cdot\left(p^2+k\right)^{\frac{(p-1)\cdot p}{k}}\cdots\left(p^M+k\right)^{\frac{(p-1)\cdot p^{M-1}}{k}}$$
$$\cdot\left(p^{M+1}+1\right)^{N-\frac{p^M}{k}+1}\cdot O(1)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(28)$$

Equation (28) holds because:

①:When $N\to+\infty, M\to+\infty$, as long as $m\ge r, m\in\mathbb{N}^+$, we have $\frac{p^m}{k}\in\mathbb{Z}$, for which the exponential power $\frac{(p-1)\cdot p^m}{k}$ in the equation (28) is integer for m large enough.

②:For any given positive integer t, if there is an n such that $a_n=n\cdot\left(p^t+1\right)$ holds, kn must have t digits, which means that $p^{t-1}\le kn<p^t$. For t large enough, the number of such n is exactly $\frac{p^t-p^{t-1}}{k}=\frac{(p-1)\cdot p^{t-1}}{k}$.

③:According to ① and ②, for any given k, the exponentials on the right-hand side of equation (28) can be replaced by $\frac{(p-1)\cdot p^m}{k}$ for m that is large enough, except some finite terms at the first place. We still replace these exponentials by $\frac{(p-1)\cdot p^m}{k}$ for m that is not large enough. The values of these finite terms is determined, which means that we might over-multiply the right-hand side by a value that is finite, so we can simply multiply the right-hand side by $O(1)$ to make the equation correct.

We have: $\prod_{1\le n\le N} a_n = N!\cdot \prod_{t=1}^{M}\left(p^t+k\right)^{\frac{(p-1)\cdot p^{t-1}}{k}}\cdot\left(p^{M+1}+1\right)^{N-\frac{p^M}{k}+1}\cdot O(1)$

$$\sum_{1\le n\le N}\ln a_n = \ln N! + \sum_{t=1}^{M}\ln\left[\left(p^t+k\right)^{\frac{(p-1)\cdot p^{t-1}}{k}}\right] + \ln\left[\left(p^{M+1}+k\right)^{N-\frac{p^M}{k}+1}\right] + O(1)$$

$$= \ln N! + \frac{(p-1)}{k}\cdot\sum_{t=1}^{M} p^{t-1}\cdot\ln\left(p^t+k\right) + \left(N-\frac{p^M}{k}+1\right)\ln\left(p^{M+1}+k\right) + O(1)$$

we have: $\ln N! = \left(N+\frac{1}{2}\right)\ln N - N + O(1)$

$$\frac{(p-1)}{k}\cdot\sum_{t=1}^{M} p^{t-1}\cdot\ln\left(p^t+k\right) = \frac{(p-1)}{k}\cdot\sum_{t=1}^{M} p^{t-1}\cdot\left[t\cdot\ln p + \frac{k}{p^t} + O\left(\frac{1}{p^{2t}}\right)\right]$$

$$= \frac{(p-1)\ln p}{k}\cdot\sum_{t=1}^{M} t\cdot p^{t-1} + \frac{(p-1)}{p}M + O(1)$$

$$= \frac{\ln 10}{k}\cdot\left(M-\frac{1}{p-1}\right)\cdot p^M + \frac{(p-1)}{p}M + O(1)$$

and

$$\left(N-\frac{p^M}{k}+1\right)\ln\left(p^{M+1}+k\right) = \left(N-\frac{p^M}{k}+1\right)\left[(M+1)\cdot\ln p + \frac{k}{p^{M+1}} + O\left(\frac{1}{p^{2M}}\right)\right]$$

$$= \left(N-\frac{p^M}{k}+1\right)\left[(M+1)\cdot\ln p + O\left(\frac{1}{p^M}\right)\right].$$

$$= \ln p\cdot MN + \ln p\cdot M + \ln p\cdot N - \frac{\ln p}{k}\cdot M\cdot p^M - \frac{\ln p}{k}\cdot p^M + O(1)$$

At last we have:

$$\sum_{1\le n\le N}\ln a_n=\left[\left(N+\frac{1}{2}\right)\ln N-N\right]+\left[\frac{\ln p}{k}\cdot\left(M-\frac{1}{p-1}\right)\cdot p^M+\frac{p-1}{p}M\right]$$
$$+\left[\ln p\cdot MN+\ln p\cdot M+\ln p\cdot N-\frac{\ln p}{k}\cdot M\cdot p^M-\frac{\ln p}{k}\cdot p^M\right]+O(1)$$
$$=N\ln N+\frac{1}{2}\ln N-N+\frac{\ln p}{k}\cdot M\cdot p^M-\frac{\ln p}{(p-1)k}\cdot p^M+\frac{p-1}{p}M$$
$$+\ln p\cdot MN+\ln p\cdot M+\ln p\cdot N-\frac{\ln p}{k}\cdot M\cdot p^M-\frac{\ln p}{k}\cdot p^M+O(1)$$
$$=(N\ln N+MN\cdot\ln p)+\frac{1}{2}\ln N+(\ln p-1)N-\frac{p\ln p}{(p-1)k}\cdot p^M$$
$$+\left(\ln p+\frac{p-1}{p}\right)M+O(1)$$

Because the quotients of M over $\ln N$ and p^M over N are bounded, we have: $\sum_{1\le n\le N}\ln a_n=2N\ln N+O(N)$. More precisely,

$$\sum_{1\le n\le N}\ln a_n=(N\ln N+MN\cdot\ln p)+\left[(\ln p-1)N-\frac{p\ln p}{(p-1)k}\cdot p^M\right]$$
$$+\left[\left(\ln p+\frac{p-1}{p}\right)M+\frac{1}{2}\ln N\right]+O(1)\cdots\cdots(29),$$

where $M=\lfloor\log_p kN\rfloor$.

2.5.2 Not all prime factors of k can divide p exactly ($k\ne1$)

If not all prime factors of k can divide p exactly ($k\ne1$), there exists a prime factor q of k such that q cannot divide p exactly. Let $p^M\le k\cdot N<p^{M+1}$, and we have:

$$\prod_{1\le n\le N}a_n=N!\cdot\left(p^1+k\right)^{\frac{p-1}{k}+b_1}\cdot\left(p^2+k\right)^{\frac{(p-1)\cdot p}{k}+b_2}\cdots\left(p^M+k\right)^{\frac{(p-1)\cdot p^{M-1}}{k}+b_M}$$
$$\cdot\left(p^{M+1}+1\right)^{N-\frac{p^M}{k}+c}\cdots\cdots(30),$$

where $b_t=\left[\frac{p^t}{k}\right]-\left[\frac{p^{t-1}}{k}\right]-\frac{(p-1)\cdot p^{t-1}}{k}$ ($t=1,2,3,\cdots,M$) and $|b_t|<1$, $|c|<1$.

Equation (30) holds because:

①: For any given positive integer t, if there is an n such that $a_n = n\cdot\left(p^t+1\right)$ holds, kn must have t digits, namely $p^{t-1} \le kn < p^t$. We can count the number of such n, which is exactly $\left[\frac{p^t}{k}\right]-\left[\frac{p^{t-1}}{k}\right]$ ($[x]$ is the floor function of x). Therefore, $b_t = \left[\frac{p^t}{k}\right]-\left[\frac{p^{t-1}}{k}\right]-\frac{(p-1)\cdot p^{t-1}}{k}$.

②: The value of the exponential power of $\left(p^{M+1}+1\right)$ should be $N-\left[\frac{p^M}{k}\right]$, denoted as $N-\frac{p^M}{k}+c, |c|<1$, so we have:

$$\prod_{1\le n\le N} a_n = N!\cdot\prod_{t=1}^{M}\left(p^t+k\right)^{\frac{(p-1)\cdot p^{t-1}}{k}+b_t}\cdot\left(p^{M+1}+1\right)^{N-\frac{p^M}{k}+c}$$

$$\begin{aligned}\sum_{1\le n\le N}\ln a_n &= \ln N!+\sum_{t=1}^{M}\ln\left[\left(p^t+k\right)^{\frac{(p-1)\cdot p^{t-1}}{k}+b_t}\right]+\ln\left[\left(p^{M+1}+k\right)^{N-\frac{p^M}{k}+c}\right]\\ &= \ln N!+\frac{p-1}{k}\cdot\sum_{t=1}^{M}\left(p^{t-1}+\frac{k}{p-1}\cdot b_t\right)\cdot\left(t\cdot\ln p+\frac{k}{p^t}+O\left(\frac{1}{p^{2t}}\right)\right)\\ &\quad+\left(N-\frac{p^M}{k}+c\right)\cdot\left[(M+1)\cdot\ln p+O\left(\frac{1}{p^M}\right)\right]\cdots\cdots\cdots\cdots(31)\end{aligned}$$

Because b_t and c are bounded, we replace b_t and c in the equation (31) by $O(1)$ and find out that:

$$\begin{aligned}\sum_{1\le n\le N}\ln a_n &= \ln N!+\sum_{t=1}^{M}\ln\left[\left(p^t+k\right)^{\frac{(p-1)\cdot p^{t-1}}{k}+O(1)}\right]+\ln\left[\left(p^{M+1}+k\right)^{N-\frac{p^M}{k}+O(1)}\right]\\ &= \ln N!+\frac{p-1}{k}\cdot\sum_{t=1}^{M}\left(p^{t-1}+O(1)\right)\cdot\ln\left(p^t+k\right)\\ &\quad+\left(N-\frac{p^M}{k}+O(1)\right)\ln\left(p^{M+1}+k\right)\cdots\cdots\cdots\cdots(32)\end{aligned}$$

We deal with the first addend in equation (32).

$$\ln N! = \left(N+\frac{1}{2}\right)\ln N - N + O(1)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(33)$$

We deal with the second addend in equation (32).

$$\frac{p-1}{k}\cdot\sum_{t=1}^{M}\left(p^{t-1}+O(1)\right)\cdot\ln\left(p^{t}+k\right)=\frac{p-1}{k}\cdot\sum_{t=1}^{M}\left(p^{t-1}+O(1)\right)\cdot\left[t\cdot\ln p+\frac{k}{p^{t}}+O\left(\frac{1}{p^{2t}}\right)\right]$$

$$=\frac{(p-1)\ln p}{k}\cdot\sum_{t=1}^{M}t\cdot p^{t-1}+\frac{p-1}{p}M+\frac{p-1}{k}\cdot\sum_{t=1}^{M}O(1)\cdot\left[t\cdot\ln p+\frac{k}{p^{t}}+O\left(\frac{1}{p^{2t}}\right)\right]+O(1)\cdots(34)$$

We deal with the third addend in equation (32).

$$\left(N-\frac{p^{M}}{k}+O(1)\right)\ln\left(p^{M+1}+k\right)=\left(N-\frac{p^{M}}{k}+O(1)\right)\left[(M+1)\cdot\ln p+O\left(\frac{1}{p^{M}}\right)\right]$$

$$=\ln p\cdot MN+\ln p\cdot N-\frac{\ln p}{k}\cdot M\cdot p^{M}-\frac{\ln p}{k}\cdot p^{M}+O(M)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(35)$$

We substitute $(33)(34)(35)$ into (32) and make the summation.

$$\sum_{1\le n\le N}\ln a_n=\left[\left(N+\frac{1}{2}\right)\ln N-N\right]+\left[\frac{\ln p}{k}\cdot\left(M-\frac{1}{p-1}\right)\cdot p^{M}\right]$$

$$+\left[\ln p\cdot MN+\ln p\cdot N-\frac{\ln p}{k}\cdot M\cdot p^{M}-\frac{\ln p}{k}\cdot p^{M}\right]+O\left(M^{2}\right)$$

$$=N\ln N+\frac{1}{2}\ln N-N+\frac{\ln p}{k}\cdot M\cdot p^{M}-\frac{\ln p}{(p-1)k}\cdot p^{M}$$

$$+\ln p\cdot MN+\ln p\cdot N-\frac{\ln p}{k}\cdot M\cdot p^{M}-\frac{\ln p}{k}\cdot p^{M}+O\left(M^{2}\right)$$

$$=\left(N\ln N+MN\cdot\ln p\right)+(\ln p-1)N-\frac{p\ln p}{(p-1)k}\cdot p^{M}+O\left(M^{2}\right)$$

Now the error term $O\left(M^{2}\right)$ has the same order as $(\ln N)^{2}$. Because the quotients of M over $\ln N$ and p^{M} over N are bounded, $\lim_{N\to+\infty}\frac{N\ln N+MN\ln p}{N\ln N}=2$ and $\lim_{N\to+\infty}\frac{\frac{1}{2}\ln N+(\ln p-1)N-\frac{p\ln p}{(p-1)k}\cdot p^{M}}{N}=O(1)$ hold, which mean that $\sum_{1\le n\le N}\ln a_n=2N\ln N+O(N)$ is true.

Now go back to the equations $(33)(34)(35)$. Because the quotients of M over

$\ln N$ and p^M over N are bounded, in the summation $\sum_{1\le n\le N} \ln a_n$, only the second addend $\frac{p-1}{k}\cdot\sum_{t=1}^{M} O(1)\cdot(t\cdot\ln p)$ contains the expression that is of the order $(\ln N)^2$. Go back to equation (31) and we find out that the expression is $A=\frac{p-1}{k}\cdot\sum_{t=1}^{M}\left(\frac{k}{p-1}\cdot b_t\right)\cdot(t\cdot\ln p)$. Now we prove that $A=O(\ln N)$, thus proving that the error term is of the order $\ln N$ rather than $(\ln N)^2$. To complete the proof, we only have to make sure that $A'=\sum_{t=1}^{M} t\cdot b_t = O(\ln N)$. Let k_0 be a number of k divided out of all the prime factors of p. It is obvious that $(k_0, p)=1$. Let the order of p modulo k_0 be δ. When the positive integer s is large enough, we can compute $B=\sum_{t=s+1}^{s+\delta} b_t$.

$B=\sum_{t=s+1}^{s+\delta}\left(\left[\frac{p^t}{k}\right]-\left[\frac{p^{t-1}}{k}\right]-\frac{(p-1)\cdot p^{t-1}}{k}\right)=\left[\frac{p^{s+\delta}}{k}\right]-\left[\frac{p^s}{k}\right]-\frac{p^{s+\delta}}{k}+\frac{p^s}{k}$, where $\frac{p^{s+\delta}}{k}-\frac{p^s}{k}=\frac{p^s\cdot(p^\delta-1)}{k}=\frac{p^s}{r}\cdot\frac{p^\delta-1}{k_0}\in\mathbb{Z}$. This is because of the definition of the order of p modulo k_0 and the exponential power being large enough, where $r=\frac{k}{k_0}$, and it only contains the prime factors of p. The difference between two numbers which have the same decimal part equals the difference of their integer parts, so $\left[\frac{p^{s+\delta}}{k}\right]-\left[\frac{p^s}{k}\right]=\frac{p^{s+\delta}}{k}-\frac{p^s}{k}$, which means that $B=0$. When the positive integer s is not large enough, the sum of these terms is infinite, and does not produce a number of the order $(\ln N)^2$. Now $B=0$ means that after a finite number of terms, the sum of δ consecutive terms of the sequence $\{b_n\}$ is 0, which means that $\{b_n\}$ is of period δ after a finite number of terms, namely $b_t=b_{t+\delta}$ for t large enough.

Therefore,

$$A' = \sum_{t=1}^{M} t \cdot b_t = \left[\frac{M}{\delta}\right] \cdot \sum_{t=1}^{\delta} t \cdot b_t + \sum_{t=1}^{\left\{\frac{M}{\delta}\right\}\cdot\delta} t \cdot b_t = \left[\frac{M}{\delta}\right] \cdot \sum_{t=1}^{\delta} t \cdot b_t + O(1) = \left[\frac{M}{\delta}\right] \cdot O(1) + O(1).$$
$$= O(M) \cdot O(1) + O(1) = O(\ln N)$$

After we complete the proof, we can make sure that:

$$\sum_{1 \le n \le N} \ln a_n = (N \ln N + MN \cdot \ln p) + (\ln p - 1)N - \frac{p \ln p}{(p-1)k} \cdot p^M + O(\ln N) \cdots (36)$$

(replace $O(M^2)$ by $O(\ln N)$)

Now the estimate is more precise.

3 Conclusion

Let $\{a_n\}$ be a Smarandache kn-digital sequence in base p ($\forall k, p \in \mathbb{N}^+, p \ge 2$), then the equation $\sum_{1 \le n \le N} \ln a_n = 2N \ln N + O(N)$ holds when $N \to +\infty$.

More precisely,

(1) If all prime factors of k can divide p exactly (including $k = 1$), then

$$\sum_{1 \le n \le N} \ln a_n = (N \ln N + MN \cdot \ln p) + \left[(\ln p - 1)N - \frac{p \ln p}{(p-1)k} \cdot p^M\right] + \left[\left(\ln p + \frac{p-1}{p}\right)M + \frac{1}{2}\ln N\right] + O(1),$$ where

$M = \lfloor \log_p kN \rfloor$, $N \to +\infty$.

(2) If not all prime factors of k divide p exactly ($k = 1$ is excluded in this case), then $\sum_{1 \le n \le N} \ln a_n = (N \ln N + MN \cdot \ln p) + (\ln p - 1)N - \frac{p \ln p}{(p-1)k} \cdot p^M + O(\ln N)$ ($N \to +\infty$).

For some specific k (for example, $k = 3, 6, 9, 11$) and $p = 10$, we prove the asymptotic formulas, each of which has an $O(1)$ error term. (equation

$(5)(10)(12)(16)$). In the following formulas, $M = \lfloor \log_{10} kN \rfloor$.

When $k=3$,

$$\sum_{1\le n\le N} \ln a_n = \left(N\ln N + MN\cdot\ln 10\right) + \left[\left(\ln 10 - 1\right)N - \frac{10\ln 10}{27}\cdot 10^M\right] + \left[\left(\frac{\ln 10}{3}+\frac{9}{10}\right)M + \frac{1}{2}\ln N\right] + O(1).$$

When $k=6$,

$$\sum_{1\le n\le N} \ln a_n = \left(N\ln N + MN\cdot\ln 10\right) + \left[\left(\ln 10 - 1\right)N - \frac{5\ln 10}{27}\cdot 10^M\right] + \left[\left(\frac{3\ln 10}{2}+\frac{9}{10}\right)M + \frac{1}{2}\ln N\right] + O(1).$$

When $k=9$,

$$\sum_{1\le n\le N} \ln a_n = \left(N\ln N + MN\cdot\ln 10\right) + \left[\left(\ln 10 - 1\right)N - \frac{10\ln 10}{81}\cdot 10^M\right] + \left[\left(\frac{\ln 10}{9}+\frac{9}{10}\right)M + \frac{1}{2}\ln N\right] + O(1).$$

When $k=11$,

$$\sum_{1\le n\le N} \ln a_n = \left(N\ln N + \ln 10\cdot MN\right) + \left[\left(\ln 10 - 1\right)N - \frac{10\ln 10}{99}\cdot 10^M\right] + \left[\left(\frac{9}{10}+\frac{\ln 10}{2}\right)M + \frac{1}{2}\ln N\right] + O(1).$$

A Conjecture:

When $k=7$, if $10^M \le 7N < 10^{M+1}$,

$$\sum_{1\le n\le N} \ln a_n = \left(N\ln N + \ln 10\cdot MN\right) + \left[\left(\ln 10 - 1\right)N - \frac{10\ln 10}{63}\cdot 10^M\right] + \left[\frac{1}{2}\ln N + \left(\frac{9}{10}+\frac{\ln 10}{2}\right)M\right] + O(1)$$

(That is equation (17))

4 Further Discussions

When any prime factors of k can divide p exactly, we can get an asymptotic formula with an $O(1)$ error term. When not all prime factors of k divide p exactly, we can only get an asymptotic formula with an $O(\ln N)$ error term for general k and p. But for some specific k in base 10, for example, when $k=3,6,9,11$, we can get an asymptotic formula with an $O(1)$ error term, so for general k and p, when not all prime factors of k divide p exactly, it will be necessary to get the asymptotic formula with an $O(1)$ error term. We can even preserve more small terms, find out the constant, and give the asymptotic formula with an $O\left(\frac{1}{N}\right)$ error term. After all, the Stirling's approximation is very precise.

5 Appendix

For $k=2,4,5,6,8,9,10,11$, detailed proof is presented below. **(Equations $(7)\sim(17)$ are in this part.)**

5.1 $k=2$

Let $M\in\mathbb{Z}^{+}$, $\frac{1}{2}\cdot 10^{M}\le N<\frac{1}{2}\cdot 10^{M+1}$, and by the same argument we can get the following identity:

$$\prod_{1\le n\le N} a_n=\prod_{n=1}^{4}a_n\cdot\prod_{n=5}^{49}a_n\cdots\prod_{n=\frac{1}{2}\cdot 10^{M-1}}^{\frac{1}{2}\cdot 10^{M}-1}a_n\cdot\prod_{n=\frac{1}{2}\cdot 10^{M}}^{N}a_n$$
$$=N!\cdot(10+2)^{4}\cdot(100+2)^{45}\cdots\left(10^{M}+2\right)^{\frac{1}{2}\cdot 9\cdot 10^{M-1}}\cdot\left(10^{M+1}+2\right)^{N-\left(\frac{1}{2}\cdot 10^{M}-1\right)},$$

and:

$$\sum_{1\le n\le N}\ln a_n=\ln N!+\sum_{t=1}^{M}\ln\left(10^t+2\right)^{\frac{1}{2}\cdot 9\cdot 10^{t-1}}-\frac{\ln 12}{2}+\ln\left(10^{M+1}+2\right)^{N-\left(\frac{1}{2}\cdot 10^M-1\right)}$$
$$=\ln N!+\frac{1}{2}\cdot 9\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln\left(10^t+2\right)+\left[N-\left(\frac{1}{2}\cdot 10^M-1\right)\right]\ln\left(10^{M+1}+2\right)-\frac{\ln 12}{2}.$$

In the equation above, the subtraction of $\frac{\ln 12}{2}$ is because we replace the exponentials by $\frac{1}{2}\cdot 9\cdot 10^{t-1}$, and only in the first term $\left(10+2\right)^4$, we cannot replace 4 by $\frac{1}{2}\cdot 9\cdot 10^{1-1}=4.5$. Thereby, to make the equation correct, we have to subtract $\ln\frac{\left(10^1+2\right)^{4.5}}{\left(10^1+2\right)^4}=\frac{\ln 12}{2}$ from the left-hand side. We will use $O(1)$ to substitute this finite difference.

Therefore, we have: $\ln N!=\left(N+\frac{1}{2}\right)\ln N-N+O(1)$.

When $x\to 0$, $\ln(x+1)=x+O\left(x^2\right)$, which means that:

$$\frac{1}{2}\cdot 9\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln\left(10^t+2\right)=\frac{1}{2}\cdot 9\cdot\sum_{t=1}^{M}10^{t-1}\cdot\left[t\cdot\ln 10+\frac{2}{10^t}+O\left(\frac{1}{10^{2t}}\right)\right]$$
$$=\frac{1}{2}\cdot 9\cdot\ln 10\cdot\sum_{k=1}^{M}t\cdot 10^{t-1}+\frac{9}{10}M+O(1),$$
$$=\frac{\ln 10}{2}\cdot M\cdot 10^M-\frac{\ln 10}{18}\left(10^M-1\right)+\frac{9}{10}M+O(1)$$

and:

$$\left[N-\left(\frac{1}{2}\cdot 10^M-1\right)\right]\ln\left(10^{M+1}+2\right)=\left[N-\left(\frac{1}{2}\cdot 10^M-1\right)\right]\cdot\left[\frac{2}{10^{M+1}}+O\left(\frac{1}{10^{2M}}\right)+(M+1)\cdot\ln 10\right]$$
$$=\ln 10\cdot(\mathrm{M}+1)\cdot\left[N-\left(\frac{1}{2}\cdot 10^M-1\right)\right]+O(1).$$
$$=\ln 10\cdot\left(MN-\frac{1}{2}\cdot M\cdot 10^M+M+N-\frac{1}{2}\cdot 10^M\right)+O(1)$$

At last we have:

$$\sum_{1\le n\le N}\ln a_n=\left[\left(N+\frac{1}{2}\right)\ln N-N+O(1)\right]$$
$$+\left[\frac{\ln 10}{2}\cdot M\cdot 10^M-\frac{\ln 10}{18}\left(10^M-1\right)+\frac{9}{10}M+O(1)\right]$$
$$+\left[\ln 10\cdot\left(MN-\frac{1}{2}\cdot M\cdot 10^M+M+N-\frac{1}{2}\cdot 10^M\right)+O(1)\right]$$
$$=N\ln N+\frac{1}{2}\ln N-N+\frac{\ln 10}{2}\cdot M\cdot 10^M-\frac{\ln 10}{18}\cdot 10^M+\frac{9}{10}M,$$
$$+\ln 10\cdot MN-\frac{\ln 10}{2}\cdot M\cdot 10^M+\ln 10\cdot M+\ln 10\cdot N-\frac{\ln 10}{2}\cdot 10^M+O(1)$$
$$=\left(N\ln N+MN\cdot\ln 10\right)+\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{5\ln 10}{9}\cdot 10^M$$
$$+\left(\ln 10+\frac{9}{10}\right)M+O(1)$$

which means that:

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{5\ln 10}{9}\cdot 10^M\right]$$
$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)\cdots\cdots\cdots\cdots(7).$$

This is the asymptotic formula with an $O(1)$ error term when $\mathrm{k}=2$.

Because the quotients of M over $\ln N$ and 10^M over N are bounded, the following equations hold:

$$\lim_{N\to+\infty}\frac{N\ln N+MN\ln 10}{N\ln N}=2,\text{ and}$$

$$\lim_{N\to+\infty}\frac{\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{5\ln 10}{9}10^M+\left(\ln 10+\frac{9}{10}\right)M}{N}=O(1),$$

which means that: $\sum_{1\le n\le N}\ln a_n=2N\ln N+O(N)$.

5.2 $k=4$

When N is large enough, there exists a unique $M\in\mathbb{Z}^+$ such that $\frac{1}{4}\cdot 10^M\le N<\frac{1}{4}\cdot 10^{M+1}$.

By the same argument we can get the following identity:

$$\prod_{1\le n\le N} a_n = \prod_{n=1}^{2} a_n \cdot \prod_{n=3}^{24} a_n \cdot \prod_{n=25}^{249} a_n \cdots \prod_{n=\frac{1}{4}\cdot 10^{M-1}}^{\frac{1}{4}\cdot 10^{M}-1} a_n \cdot \prod_{n=\frac{1}{4}\cdot 10^{M}}^{N} a_n$$

$$= N!\cdot (10+4)^2 \cdot (100+4)^{22} \cdot (1000+4)^{225} \cdots \left(10^M+4\right)^{\frac{1}{4}\cdot 9\cdot 10^{M-1}} ,$$

$$\cdot\left(10^{M+1}+4\right)^{N-\left(\frac{1}{4}\cdot 10^{M}-1\right)}$$

and:

$$\sum_{1\le n\le N} \ln a_n = \ln N! + \sum_{t=1}^{M} \ln\left(10^t+4\right)^{\frac{1}{4}\cdot 9\cdot 10^{t-1}} + \ln\left(10^{M+1}+4\right)^{N-\left(\frac{1}{4}\cdot 10^{M}-1\right)} + O(1)$$

$$= \ln N! + \frac{1}{4}\cdot 9\cdot \sum_{t=1}^{M} 10^{t-1}\cdot \ln\left(10^t+4\right) + \left[N-\left(\frac{1}{4}\cdot 10^M-1\right)\right]\ln\left(10^{M+1}+4\right)+O(1) .$$

When $x\to 0$, $\ln N! = \left(N+\frac{1}{2}\right)\ln N - N + O(1)$, so $\ln(x+1) = x + O\left(x^2\right)$, which means that:

$$\frac{1}{4}\cdot 9\cdot \sum_{t=1}^{M} 10^{t-1}\cdot \ln\left(10^t+4\right) = \frac{1}{4}\cdot 9\cdot \sum_{t=1}^{M} 10^{t-1}\cdot\left[t\cdot \ln 10 + \frac{4}{10^t} + O\left(\frac{1}{10^{2t}}\right)\right]$$

$$= \frac{1}{4}\cdot 9\cdot \ln 10\cdot \sum_{t=1}^{M} t\cdot 10^{t-1} + \frac{9}{10}M + O(1)$$, and:

$$= \frac{\ln 10}{4}\cdot M\cdot 10^M - \frac{\ln 10}{36}\left(10^M-1\right) + \frac{9}{10}M + O(1)$$

$$\left[N-\left(\frac{1}{4}\cdot 10^M-1\right)\right]\ln\left(10^{M+1}+4\right) = \left[N-\left(\frac{1}{4}\cdot 10^M-1\right)\right]\left[\frac{4}{10^{M+1}} + O\left(\frac{1}{10^{2M}}\right) + (M+1)\cdot \ln 10\right]$$

$$= \ln 10\cdot (\mathrm{M+1})\left[N-\left(\frac{1}{4}\cdot 10^M-1\right)\right] + O(1) .$$

$$= \ln 10\cdot\left(MN - \frac{1}{4}\cdot M\cdot 10^M + M + N - \frac{1}{4}\cdot 10^M\right) + O(1)$$

At last we have:

$$\sum_{1\le n\le N}\ln a_n=\left[\left(N+\frac{1}{2}\right)\ln N-N+O(1)\right]$$

$$+\left[\frac{\ln 10}{4}\cdot M\cdot 10^M-\frac{\ln 10}{36}\cdot\left(10^M-1\right)+\frac{9}{10}M+O(1)\right]$$

$$+\left[\ln 10\cdot\left(MN-\frac{1}{4}\cdot M\cdot 10^M+M+N-\frac{1}{4}\cdot 10^M\right)+O(1)\right]$$

$$=N\ln N+\frac{1}{2}\ln N-N+\frac{\ln 10}{4}\cdot M\cdot 10^M-\frac{\ln 10}{36}\cdot 10^M+\frac{9}{10}M,$$

$$+\ln 10\cdot MN-\frac{\ln 10}{4}\cdot M\cdot 10^M+\ln 10\cdot M+\ln 10\cdot N-\frac{\ln 10}{4}\cdot 10^M+O(1)$$

$$=\left(N\ln N+MN\cdot\ln 10\right)+\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{5\ln 10}{18}\cdot 10^M$$

$$+\left(\ln 10+\frac{9}{10}\right)M+O(1)$$

which means that:

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{5\ln 10}{18}\cdot 10^M\right]$$

$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)\cdots\cdots\cdots(8).$$

This is the asymptotic formula with an $O(1)$ error term when $k=4$.

Because the quotients of M over $\ln N$ and 10^M over N are bounded, the following equations hold:

$\lim_{N\to+\infty}\frac{N\ln N+MN\ln 10}{N\ln N}=2$, and

$\lim_{N\to+\infty}\frac{\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{5\ln 10}{18}10^M+\left(\ln 10+\frac{9}{10}\right)M}{N}=O(1)$, which means that:

$\sum_{1\le n\le N}\ln a_n=2N\ln N+O(N)$.

5.3 $k=5$

When N is large enough, there exists a unique $M\in\mathbb{Z}^+$ such that

$\frac{1}{5}\cdot 10^M\le N<\frac{1}{5}\cdot 10^{M+1}$.

By the same argument we can get the following identity.

$$\prod_{1\le n\le N} a_n = \prod_{n=1}^{1} a_n \cdot \prod_{n=2}^{19} a_n \cdot \prod_{n=20}^{199} a_n \cdots \prod_{n=\frac{1}{5}\cdot 10^{M-1}}^{\frac{1}{5}\cdot 10^{M}-1} a_n \cdot \prod_{n=\frac{1}{5}\cdot 10^{M}}^{N} a_n ,$$

$$= N!\cdot(10+5)^{1}\cdot(100+5)^{18}\cdot(1000+5)^{180}\cdots\left(10^{M}+5\right)^{\frac{1}{5}\cdot 9\cdot 10^{M-1}}\cdot\left(10^{M+1}+5\right)^{N-\left(\frac{1}{5}\cdot 10^{M}-1\right)}$$

and:

$$\sum_{1\le n\le N}\ln a_n = \ln N! + \sum_{t=1}^{M}\ln\left(10^{t}+5\right)^{\frac{1}{5}\cdot 9\cdot 10^{t-1}} + \ln\left(10^{M+1}+5\right)^{N-\left(\frac{1}{5}\cdot 10^{M}-1\right)} + O(1)$$

$$= \ln N! + \frac{1}{5}\cdot 9\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln\left(10^{t}+5\right) + \left[N-\left(\frac{1}{5}\cdot 10^{M}-1\right)\right]\ln\left(10^{M+1}+5\right) + O(1).$$

When $x\to 0$, $\ln N! = \left(N+\frac{1}{2}\right)\ln N - N + O(1)$, so $\ln(x+1) = x + O\left(x^{2}\right)$, which means that:

$$\frac{1}{5}\cdot 9\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln\left(10^{t}+5\right) = \frac{1}{5}\cdot 9\cdot\sum_{t=1}^{M}10^{t-1}\cdot\left[t\cdot\ln 10 + \frac{5}{10^{t}} + O\left(\frac{1}{10^{2t}}\right)\right]$$

$$= \frac{1}{5}\cdot 9\cdot\ln 10\cdot\sum_{t=1}^{M} t\cdot 10^{t-1} + \frac{9}{10}M + O(1) ,$$

$$= \frac{\ln 10}{5}\cdot M\cdot 10^{M} - \frac{\ln 10}{45}\left(10^{M}-1\right) + \frac{9}{10}M + O(1)$$

and:

$$\left[N-\left(\frac{1}{5}\cdot 10^{M}-1\right)\right]\ln\left(10^{M+1}+5\right) = \left[N-\left(\frac{1}{5}\cdot 10^{M}-1\right)\right]\left[\frac{5}{10^{M+1}} + O\left(\frac{1}{10^{2M}}\right) + (M+1)\cdot\ln 10\right]$$

$$= \ln 10\cdot(\mathrm{M}+1)\left[N-\left(\frac{1}{5}\cdot 10^{M}-1\right)\right] + O(1).$$

$$= \ln 10\cdot\left(MN - \frac{1}{5}\cdot M\cdot 10^{M} + M + N - \frac{1}{5}\cdot 10^{M}\right) + O(1)$$

At last we have:

$$\sum_{1\le n\le N}\ln a_n=\left[\left(N+\frac{1}{2}\right)\ln N-N+O(1)\right]$$
$$+\left[\frac{\ln 10}{5}\cdot M\cdot 10^M-\frac{\ln 10}{45}\left(10^M-1\right)+\frac{9}{10}M+O(1)\right]$$
$$+\left[\ln 10\cdot\left(MN-\frac{1}{5}\cdot M\cdot 10^M+M+N-\frac{1}{5}\cdot 10^M\right)+O(1)\right]$$
$$=N\ln N+\frac{1}{2}\ln N-N+\frac{\ln 10}{5}\cdot M\cdot 10^M-\frac{\ln 10}{45}\cdot 10^M+\frac{9}{10}M$$
$$+\ln 10\cdot MN-\frac{\ln 10}{5}\cdot M\cdot 10^M+\ln 10\cdot M+\ln 10\cdot N-\frac{\ln 10}{5}\cdot 10^M+O(1)$$
$$=\left(N\ln N+MN\cdot\ln 10\right)+\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{2\ln 10}{9}\cdot 10^M$$
$$+\left(\ln 10+\frac{9}{10}\right)M+O(1)$$

which means that:

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{2\ln 10}{9}\cdot 10^M\right]$$
$$+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)\cdots\cdots(9)$$

This is the asymptotic formula with an $O(1)$ error term when $\mathrm{k}=5$.

Because the quotients of M over $\ln N$ and 10^M over N are bounded, the following equations hold:

$$\lim_{N\to+\infty}\frac{N\ln N+MN\ln 10}{N\ln N}=2\text{, and:}$$

$$\lim_{N\to+\infty}\frac{\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{2\ln 10}{9}10^M+\left(\ln 10+\frac{9}{10}\right)M}{N}=O(1),$$

which means that: $\sum_{1\le n\le N}\ln a_n=2N\ln N+O(N)$.

5.4 $k=6$

When N is large enough, there exists a unique $M\in\mathbb{Z}^+$ such that

$$\frac{10^M}{6}\le N<\frac{10^{M+1}}{6}.$$

By the same argument we can get the following identity:

$$\prod_{1\le n\le N} a_n = \prod_{n=1}^{1} a_n \cdot \prod_{n=2}^{16} a_n \cdot \prod_{n=17}^{166} a_n \cdot \prod_{n=167}^{1666} a_n \cdots \prod_{n=\frac{10^{M-1}+2}{6}}^{\frac{10^M+2}{6}-1} a_n \cdot \prod_{n=\frac{10^M+2}{6}}^{N} a_n$$

$$= N!\cdot(10+6)^1\cdot(100+6)^{15}\cdot(1000+6)^{150}\cdots\left(10^M+6\right)^{\frac{3}{2}\cdot 10^{M-1}}\cdot\left(10^{M+1}+6\right)^{N-\left(\frac{10^M+2}{6}-1\right)},$$

and:

$$\sum_{1\le n\le N}\ln a_n = \ln N! + \sum_{t=1}^{M}\ln\left(10^t+6\right)^{\frac{3}{2}\cdot 10^{t-1}} + \ln\left(10^{M+1}+6\right)^{N-\left(\frac{10^M+2}{6}-1\right)} + O(1)$$

$$= \ln N! + \frac{3}{2}\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln\left(10^t+6\right) + \left[N-\left(\frac{10^M+2}{6}-1\right)\right]\ln\left(10^{M+1}+6\right) + O(1).$$

We have: $\ln N! = \left(N+\frac{1}{2}\right)\ln N - N + O(1)$.

When $x\to 0$, $\ln(x+1) = x + O\left(x^2\right)$, which means that:

$$\frac{3}{2}\cdot\sum_{t=1}^{M}10^{t-1}\cdot\ln\left(10^t+6\right) = \frac{3}{2}\cdot\sum_{t=1}^{M}10^{t-1}\cdot\left[t\cdot\ln 10 + \frac{6}{10^t} + O\left(\frac{1}{10^{2t}}\right)\right]$$

$$= \frac{3}{2}\cdot\ln 10\cdot\sum_{t=1}^{M} t\cdot 10^{t-1} + \frac{9}{10}M + O(1),$$

$$= \frac{\ln 10}{6}\cdot M\cdot 10^M - \frac{\ln 10}{54}\left(10^M-1\right) + \frac{9}{10}M + O(1)$$

and:

$$\left[N-\left(\frac{10^M+2}{6}-1\right)\right]\ln\left(10^{M+1}+6\right) = \left[N-\left(\frac{10^M+2}{6}-1\right)\right]\left[\frac{6}{10^{M+1}} + O\left(\frac{1}{10^{2M}}\right) + (M+1)\cdot\ln 10\right]$$

$$= \ln 10\cdot(\mathrm{M}+1)\cdot\left[N-\left(\frac{10^M-4}{6}\right)\right] + O(1).$$

$$= \ln 10\cdot\left(MN - \frac{1}{6}\cdot M\cdot 10^M + \frac{3}{2}M + N - \frac{1}{6}\cdot 10^M\right) + O(1)$$

At last we have:

$$\sum_{1\le n\le N}\ln a_n=\left[\left(N+\frac{1}{2}\right)\ln N-N+O(1)\right]$$
$$+\left[\frac{\ln 10}{6}\cdot M\cdot 10^M-\frac{\ln 10}{54}\left(10^M-1\right)+\frac{9}{10}M+O(1)\right]$$
$$+\left[\ln 10\cdot\left(MN-\frac{1}{6}\cdot M\cdot 10^M+\frac{3}{2}M+N-\frac{1}{6}\cdot 10^M\right)+O(1)\right]$$
$$=N\ln N+\frac{1}{2}\ln N-N+\frac{\ln 10}{6}\cdot M\cdot 10^M-\frac{\ln 10}{54}\cdot 10^M+\frac{9}{10}M$$
$$+\ln 10\cdot MN-\frac{\ln 10}{6}\cdot M\cdot 10^M+\frac{3\ln 10}{2}\cdot M+\ln 10\cdot N-\frac{\ln 10}{6}\cdot 10^M+O(1)$$
$$=\left(N\ln N+MN\cdot\ln 10\right)+\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{5\ln 10}{27}\cdot 10^M$$
$$+\left(\frac{3\ln 10}{2}+\frac{9}{10}\right)M+O(1)$$

, which means that:

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{5\ln 10}{27}\cdot 10^M\right]$$
$$+\left[\left(\frac{3\ln 10}{2}+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)\cdots\cdots(10)$$
.

This is the asymptotic formula with an $O(1)$ error term when $k=6$.

Because the quotients of M over $\ln N$ and 10^M over N are bounded, the following equations hold:

$$\lim_{N\to+\infty}\frac{N\ln N+MN\ln 10}{N\ln N}=2,$$

$$\lim_{N\to+\infty}\frac{\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{5\ln 10}{27}10^M+\left(\frac{3\ln 10}{2}+\frac{9}{10}\right)M}{N}=O(1),$$

which means that: $\sum_{1\le n\le N}\ln a_n=2N\ln N+O(N)$.

5.5 $k=8$

When N is large enough, there exists a unique $M\in\mathbb{Z}^+$ such that $\frac{1}{8}\cdot 10^M\le N<\frac{1}{8}\cdot 10^{M+1}$.

By the same argument we can get the following identity:

$$\prod_{1\le n\le N} a_n = \prod_{n=1}^{1} a_n \cdot \prod_{n=2}^{12} a_n \cdot \prod_{n=13}^{124} a_n \cdot \prod_{n=125}^{1249} a_n \cdots \prod_{n=\frac{1}{8}\cdot 10^{M-1}}^{\frac{1}{8}\cdot 10^{M}-1} a_n \cdot \prod_{n=\frac{1}{8}\cdot 10^{M}}^{N} a_n$$

$$= N!\cdot (10+8)^{1} \cdot (100+8)^{11} \cdot (1000+8)^{112} \cdots (10^{M}+8)^{\frac{1}{8}\cdot 9\cdot 10^{M-1}}$$

$$\cdot (10^{M+1}+8)^{N-\left(\frac{1}{8}\cdot 10^{M}-1\right)}$$

and:

$$\sum_{1\le n\le N} \ln a_n = \ln N! + \sum_{t=1}^{M} \ln (10^{t}+8)^{\frac{1}{8}\cdot 9\cdot 10^{t-1}} + \ln (10^{M+1}+8)^{N-\left(\frac{1}{8}\cdot 10^{M}-1\right)} + O(1)$$
$$= \ln N! + \frac{1}{8}\cdot 9\cdot \sum_{t=1}^{M} 10^{t-1}\cdot \ln (10^{t}+8) + \left[N-\left(\frac{1}{8}\cdot 10^{M}-1\right)\right] \ln (10^{M+1}+8) + O(1).$$

We have: $\ln N! = \left(N+\frac{1}{2}\right)\ln N - N + O(1)$.

When $x\to 0$, $\ln(x+1) = x + O(x^2)$,

which means that:

$$\frac{1}{8}\cdot 9\cdot \sum_{t=1}^{M} 10^{t-1}\cdot \ln(10^{t}+8) = \frac{1}{8}\cdot 9\cdot \sum_{t=1}^{M} 10^{t-1}\cdot \left[t\cdot \ln 10 + \frac{8}{10^{t}} + O\left(\frac{1}{10^{2t}}\right)\right]$$
$$= \frac{1}{8}\cdot 9\cdot \ln 10\cdot \sum_{t=1}^{M} t\cdot 10^{t-1} + \frac{9}{10} M + O(1)$$
$$= \frac{\ln 10}{8}\cdot M\cdot 10^{M} - \frac{\ln 10}{72}(10^{M}-1) + \frac{9}{10} M + O(1)$$

and:

$$\left[N-\left(\frac{1}{8}\cdot 10^{M}-1\right)\right]\ln(10^{M+1}+8) = \left[N-\left(\frac{1}{8}\cdot 10^{M}-1\right)\right]\left[\frac{8}{10^{M+1}} + O\left(\frac{1}{10^{2M}}\right) + (M+1)\cdot \ln 10\right]$$
$$= \ln 10\cdot (\mathrm{M}+1)\left[N-\left(\frac{1}{8}\cdot 10^{M}-1\right)\right] + O(1).$$
$$= \ln 10\cdot \left(MN - \frac{1}{8}\cdot M\cdot 10^{M} + M + N - \frac{1}{8}\cdot 10^{M}\right) + O(1)$$

Finally we have:

$$\begin{aligned}\sum_{1\le n\le N}\ln a_n &= \left[\left(N+\frac{1}{2}\right)\ln N - N + O(1)\right] \\ &+\left[\frac{\ln 10}{8}\cdot M\cdot 10^M - \frac{\ln 10}{72}\left(10^M-1\right)+\frac{9}{10}M+O(1)\right] \\ &+\left[\ln 10\cdot\left(MN-\frac{1}{8}\cdot M\cdot 10^M + M + N - \frac{1}{8}\cdot 10^M\right)+O(1)\right] \\ &= N\ln N+\frac{1}{2}\ln N - N+\frac{\ln 10}{8}\cdot M\cdot 10^M-\frac{\ln 10}{72}\cdot 10^M+\frac{9}{10}M \\ &+\ln 10\cdot MN-\frac{\ln 10}{8}\cdot M\cdot 10^M+\ln 10\cdot M+\ln 10\cdot N-\frac{\ln 10}{8}\cdot 10^M+O(1) \\ &= \left(N\ln N+MN\cdot\ln 10\right)+\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{5\ln 10}{36}\cdot 10^M \\ &+\left(\ln 10+\frac{9}{10}\right)M+O(1)\end{aligned}$$

, which means that:

$$\begin{aligned}\sum_{1\le n\le N}\ln a_n &= \left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{5\ln 10}{36}\cdot 10^M\right] \\ &+\left[\left(\ln 10+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)\cdots\cdots\cdots\cdots\cdots\cdots(11)\end{aligned}$$

.

This is the asymptotic formula with an $O(1)$ error term when $\mathrm{k}=8$.

Because the quotients of M over $\ln N$ and 10^M over N are bounded, the following equations hold:

$$\lim_{N\to+\infty}\frac{N\ln N+MN\ln 10}{N\ln N}=2,$$

$$\lim_{N\to+\infty}\frac{\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{5\ln 10}{36}10^M+\left(\ln 10+\frac{9}{10}\right)M}{N}=O(1),$$

which means that: $\sum_{1\le n\le N}\ln a_n = 2N\ln N+O(N)$.

5.6 $k=9$

When N is large enough, there exists a unique $M\in\mathbb{Z}^+$ such that $\frac{10^M+8}{9}\le N<\frac{10^{M+1}+8}{9}$.

By the same argument we can get the following identity:

$$\prod_{1\le n\le N} a_n = \prod_{n=1}^{1} a_n \cdot \prod_{n=2}^{11} a_n \cdot \prod_{n=12}^{111} a_n \cdot \prod_{n=112}^{1111} a_n \cdots \prod_{n=\frac{10^{M-1}+8}{9}}^{\frac{10^M+8}{9}-1} a_n \cdot \prod_{n=\frac{10^M+8}{9}}^{N} a_n$$

$$= N!\cdot(10+9)^1\cdot(100+9)^{10}\cdot(1000+9)^{100}\cdots(10^M+9)^{10^{M-1}}$$

$$\cdot(10^{M+1}+9)^{N-\left(\frac{10^M+8}{9}-1\right)}$$

and:

$$\sum_{1\le n\le N}\ln a_n = \ln N! + \sum_{t=1}^{M}\ln(10^t+9)^{10^{t-1}} + \ln(10^{M+1}+9)^{N-\left(\frac{10^M+8}{9}-1\right)} + O(1)$$

$$= \ln N! + \sum_{t=1}^{M}10^{t-1}\cdot\ln(10^t+9) + \left[N-\left(\frac{10^M+8}{9}-1\right)\right]\ln(10^{M+1}+9) + O(1)$$

We have: $\ln N! = \left(N+\frac{1}{2}\right)\ln N - N + O(1)$.

When $x\to 0$, $\ln(x+1) = x + O(x^2)$, which means that:

$$\sum_{t=1}^{M}10^{t-1}\cdot\ln(10^t+9) = \sum_{t=1}^{M}10^{t-1}\cdot\left[t\cdot\ln 10 + \frac{9}{10^t} + O\left(\frac{1}{10^{2t}}\right)\right]$$

$$= \ln 10\cdot\sum_{t=1}^{M}t\cdot 10^{t-1} + \frac{9}{10}M + O(1)$$

$$= \frac{\ln 10}{9}\cdot M\cdot 10^M - \frac{\ln 10}{81}(10^M-1) + \frac{9}{10}M + O(1)$$

and:

$$\left[N-\left(\frac{10^M+8}{9}-1\right)\right]\ln(10^{M+1}+9) = \left[N-\left(\frac{10^M+8}{9}-1\right)\right]\left[\frac{9}{10^{M+1}} + O\left(\frac{1}{10^{2M}}\right) + (M+1)\cdot\ln 10\right]$$

$$=\ln 10\cdot(\mathrm{M}+1)\cdot\left[N-\left(\frac{10^M-1}{9}\right)\right] + O(1)$$

$$= \ln 10\cdot\left(MN - \frac{1}{9}\cdot M\cdot 10^M + \frac{1}{9}M + N - \frac{1}{9}\cdot 10^M\right) + O(1)$$

Finally we have:

$$\sum_{1\le n\le N}\ln a_n=\left[\left(N+\frac{1}{2}\right)\ln N-N+O(1)\right]$$
$$+\left[\frac{\ln 10}{9}\cdot M\cdot 10^M-\frac{\ln 10}{81}\left(10^M-1\right)+\frac{9}{10}M+O(1)\right]$$
$$+\left[\ln 10\cdot\left(MN-\frac{1}{9}\cdot M\cdot 10^M+\frac{1}{9}M+N-\frac{1}{9}\cdot 10^M\right)+O(1)\right]$$
$$=N\ln N+\frac{1}{2}\ln N-N+\frac{\ln 10}{9}\cdot M\cdot 10^M-\frac{\ln 10}{81}\cdot 10^M+\frac{9}{10}M$$
$$+\ln 10\cdot MN-\frac{\ln 10}{9}\cdot M\cdot 10^M+\frac{\ln 10}{9}\cdot M+\ln 10\cdot N-\frac{\ln 10}{9}\cdot 10^M+O(1),$$
$$=\left(N\ln N+MN\cdot\ln 10\right)+\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{10\ln 10}{81}\cdot 10^M$$
$$+\left(\frac{\ln 10}{9}+\frac{9}{10}\right)M+O(1)$$

which means that:

$$\sum_{1\le n\le N}\ln a_n=\left(N\ln N+MN\cdot\ln 10\right)+\left[\left(\ln 10-1\right)N-\frac{10\ln 10}{81}\cdot 10^M\right]$$
$$+\left[\left(\frac{\ln 10}{9}+\frac{9}{10}\right)M+\frac{1}{2}\ln N\right]+O(1)\cdots\cdots(12).$$

This is the asymptotic formula with an $O(1)$ error term when $k=9$.

Because the quotients of M over $\ln N$ and 10^M over N are bounded, the following equations hold:

$$\lim_{N\to+\infty}\frac{N\ln N+MN\ln 10}{N\ln N}=2,$$

$$\lim_{N\to+\infty}\frac{\frac{1}{2}\ln N+\left(\ln 10-1\right)N-\frac{10\ln 10}{81}10^M+\left(\frac{\ln 10}{9}+\frac{9}{10}\right)M}{N}=O(1),$$

which means that: $\sum_{1\le n\le N}\ln a_n=2N\ln N+O(N)$.

5.7 $k=10$

When N is large enough, there exists a unique $M\in\mathbb{Z}^+$ such that $10^M\le 10N<10^{M+1}$.

By the same argument we can get the following identity:

$$\prod_{1\le n\le N} a_n = N!\cdot\left(10^2+10\right)^9\cdots\left(10^M+10\right)^{9\cdot 10^{M-2}}\cdot\left(10^{M+1}+10\right)^{N-10^{M-1}+1}$$

and:

$$\begin{aligned}\sum_{1\le n\le N}\ln a_n &= \ln N! + \sum_{t=1}^{M}\ln\left(10^t+10\right)^{9\cdot 10^{t-2}} + \ln(10^{M+1}+10)^{N-10^{M-1}+1} + O(1)\\ &= \ln N! + \sum_{t=1}^{M} 9\cdot 10^{t-2}\cdot\ln\left(10^t+10\right) + \left(N-10^{M-1}+1\right)\ln(10^{M+1}+10) + O(1)\end{aligned}.$$

We have: $\ln N! = N\ln N - N + \frac{1}{2}\ln N + O(1)$,

$$\begin{aligned}&\sum_{t=1}^{M} 9\cdot 10^{t-2}\cdot\ln\left(10^t+10\right) = \sum_{t=1}^{M} 9\cdot 10^{t-2}\cdot\left[t\cdot\ln 10 + \frac{1}{10^{t-1}} + O\left(\frac{1}{10^{2t}}\right)\right]\\ &= 9\ln 10\cdot\sum_{t=1}^{M} t\cdot 10^{t-2} + \frac{9}{10}M + O(1)\\ &= 9\ln 10\cdot\left[\frac{1}{9}\left(M-\frac{1}{9}\right)\cdot 10^{M-1} + \frac{1}{810}\right] + \frac{9}{10}M + O(1)\\ &= \ln 10\cdot M\cdot 10^{M-1} - \frac{\ln 10}{9}\cdot 10^{M-1} + \frac{9}{10}M + O(1)\end{aligned},$$

and:

$$\begin{aligned}&\left(N-10^{M-1}+1\right)\ln(10^{M+1}+10) = \left(N-10^{M-1}+1\right)\cdot\left[\left(M+1\right)\cdot\ln 10 + \frac{1}{10^M} + O\left(\frac{1}{10^{2M}}\right)\right]\\ &= \ln 10\cdot\left(MN + N - M\cdot 10^{M-1} - 10^{M-1} + M\right) + O(1)\end{aligned}.$$

Finally we have:

$$\begin{aligned}\sum_{1\le n\le N}\ln a_n &= \left[N\ln N - N + \frac{1}{2}\ln N + O(1)\right]\\ &+\left[\ln 10\cdot M\cdot 10^{M-1} - \frac{\ln 10}{9}\cdot 10^{M-1} + \frac{9}{10}M + O(1)\right]\\ &+\left[\ln 10\cdot\left(MN + N - M\cdot 10^{M-1} - 10^{M-1} + M\right) + O(1)\right]\\ &= N\ln N + \frac{1}{2}\ln N - N + \ln 10\cdot M\cdot 10^{M-1} - \frac{\ln 10}{9}\cdot 10^{M-1} + \frac{9}{10}M\\ &+\ln 10\cdot MN - \ln 10\cdot M\cdot 10^{M-1} + \ln 10\cdot M + \ln 10\cdot N - \ln 10\cdot 10^{M-1} + O(1)\\ &= \left(N\ln N + MN\cdot\ln 10\right) + \frac{1}{2}\ln N + \left(\ln 10 - 1\right)N - \frac{10\ln 10}{9}\cdot 10^{M-1}\\ &+\left(\ln 10 + \frac{9}{10}\right)M + O(1)\end{aligned}$$

which means that:

$$\sum_{1\le n\le N} \ln a_n = \left(N\ln N + MN\cdot\ln 10\right) + \left[\left(\ln 10 - 1\right)N - \frac{\ln 10}{9}\cdot 10^M\right]$$
$$+\left[\left(\ln 10 + \frac{9}{10}\right)M + \frac{1}{2}\ln N\right] + O(1)\cdots\cdots\cdots\cdots\cdots\cdots(13)$$

This is the asymptotic formula with an $O(1)$ error term when $k=10$.

Because the quotients of M over $\ln N$ and 10^M over N are bounded, the following equations hold:

$$\lim_{N\to+\infty}\frac{N\ln N + MN\ln 10}{N\ln N} = 2,$$

$$\lim_{N\to+\infty}\frac{\frac{1}{2}\ln N + \left(\ln 10 - 1\right)N - \frac{\ln 10}{9}10^M + \left(\ln 10 + \frac{9}{10}\right)M}{N} = O(1),$$

which means that: $\sum_{1\le n\le N}\ln a_n = 2N\ln N + O(N)$.

5.8 $k=11$

When $k=11$ and when N is large enough, we can also find a unique $M\in\mathbb{Z}^+$ such that $10^M \le 11N < 10^{M+1}$. But the order of 10 modulo 11 is 2 (that is $\delta_{11}(10)=2$.) , so we have to consider two cases. They are that M is odd and that M is even.

5.8.1 Case 1 for $M\in\mathbb{Z}^+$ such that $10^{2M}\le 11N < 10^{2M+1}$

If there exists an M such that $M\in\mathbb{Z}^+$ and $10^{2M}\le 11N<10^{2M+1}$, then $a_N = N\cdot\left(10^{2M+1}+11\right)$, we have:

$$\prod_{1\le n\le N} a_n = N!\cdot\left(10^2+11\right)^9\cdot\left(10^3+11\right)^{81}\cdot\left(10^4+11\right)^{819}\cdots$$
$$\cdot\left(10^{2M-1}+11\right)^{\frac{9}{11}\left(10^{2M-2}-1\right)}\cdot\left(10^{2M}+11\right)^{\frac{9}{11}\left(10^{2M-1}+1\right)}\cdot\left(10^{2M+1}+11\right)^{N-\frac{10^{2M}-1}{11}}.$$
$$= N!\cdot\prod_{t=0}^{M-1}\left(10^{2t+2}+11\right)^{\frac{9}{11}\left(10^{2t+1}+1\right)}\cdot\prod_{t=1}^{M-1}\left(10^{2t+1}+11\right)^{\frac{9}{11}\left(10^{2t}-1\right)}\cdot\left(10^{2M+1}+11\right)^{N-\frac{10^{2M}-1}{11}}$$

Take the natural logarithm of the both sides, and the equation becomes:

$$\sum_{1\le n\le N}\ln a_n=\ln(N!)+\frac{9}{11}\cdot\sum_{t=0}^{M-1}\left(10^{2t+1}+1\right)\ln\left(10^{2t+2}+11\right)+\frac{9}{11}\cdot\sum_{t=1}^{M-1}\left(10^{2t}-1\right)\ln\left(10^{2t+1}+11\right)$$
$$+\left(N-\frac{10^{2M}-1}{11}\right)\ln\left(10^{2M+1}+11\right).$$

We deal with every addend in the equation above.

$$\sum_{t=0}^{M-1}\left(10^{2t+1}+1\right)\ln\left(10^{2t+2}+11\right)=\sum_{t=0}^{M-1}\left(10^{2t+1}+1\right)\left[(2t+2)\ln 10+\frac{11}{10^{2t+2}}+O\left(\frac{1}{10^{4t}}\right)\right].$$
$$=\sum_{t=0}^{M-1}\left((2t+2)\ln 10\cdot 10^{2t+1}+\frac{11}{10}+(2t+2)\ln 10\right)+O(1)$$

$$\sum_{t=1}^{M-1}\left(10^{2t}-1\right)\ln\left(10^{2t+1}+11\right)=\sum_{t=1}^{M-1}\left(10^{2t}-1\right)\left[(2t+1)\ln 10+\frac{11}{10^{2t+1}}+O\left(\frac{1}{10^{4t}}\right)\right].$$
$$=\sum_{t=1}^{M-1}\left((2t+1)\ln 10\cdot 10^{2t}+\frac{11}{10}-(2t+1)\ln 10\right)+O(1)$$

If we summate the above two equations and multiply the both sides by $\frac{9}{11}$, we get:

$$\frac{9}{11}\sum_{t=0}^{M-1}\left(10^{2t+1}+1\right)\ln\left(10^{2t+2}+11\right)+\frac{9}{11}\sum_{t=1}^{M-1}\left(10^{2t}-1\right)\ln\left(10^{2t+1}+11\right)$$
$$=\frac{9}{11}\left\{\ln 10\cdot\sum_{t=2}^{2M}t\cdot 10^{t-1}+\frac{11}{5}M+\ln 10\cdot\left[(2M)-(2M-1)+\cdots+4-3+2\right]\right\}+O(1)$$
$$=\frac{9}{11}\left\{\frac{\ln 10}{9}\cdot\left(2M-\frac{1}{9}\right)\cdot 10^{2M}+\left(\frac{11}{5}+\ln 10\right)M\right\}+O(1),$$
$$=\frac{2\ln 10}{11}\cdot M\cdot 10^{2M}-\frac{\ln 10}{99}\cdot 10^{2M}+\left(\frac{9}{5}+\frac{9\ln 10}{11}\right)M+O(1)$$

where $\sum_{t=2}^{2M}t\cdot 10^{t-1}=\frac{1}{9}\left(2M-\frac{1}{9}\right)\cdot 10^{2M}+\frac{10}{81}$.

In addition,

$$\left(N-\frac{10^{2M}-1}{11}\right)\ln\left(10^{2M+1}+11\right)$$
$$=\left(N-\frac{1}{11}\cdot 10^{2M}+\frac{1}{11}\right)\left[(2M+1)\ln 10+O\left(\frac{1}{10^{2M}}\right)\right]\qquad\text{, and}$$
$$=(2M+1)\cdot N\cdot\ln 10-\frac{\ln 10}{11}\cdot(2M+1)\cdot 10^{2M}+\frac{2\ln 10}{11}\cdot M+O(1)$$

$\ln N!=N\ln N-N+\frac{1}{2}\ln N+O(1)$, which means that:

$$\sum_{1\le n\le N}\ln a_n = N\ln N - N + \frac{1}{2}\ln N + 2\ln 10\cdot MN + \ln 10\cdot N - \frac{2\ln 10}{11}\cdot M\cdot 10^{2M} - \frac{\ln 10}{11}\cdot 10^{2M}$$
$$+\frac{2\ln 10}{11}\cdot M + \frac{2\ln 10}{11}\cdot M\cdot 10^{2M} - \frac{\ln 10}{99}\cdot 10^{2M} + \left(\frac{9}{5}+\frac{9\ln 10}{11}\right)M + O(1)$$
$$=\left(N\ln N + 2\ln 10\cdot MN\right) + \left[\left(\ln 10 - 1\right)N - \frac{10\ln 10}{99}\cdot 10^{2M}\right]$$
$$+\left[\frac{1}{2}\ln N + \left(\frac{9}{5}+\ln 10\right)M\right] + O(1)$$
$$=\left(N\ln N + \ln 10\cdot\left(2M\right)\cdot N\right) + \left[\left(\ln 10 - 1\right)N - \frac{10\ln 10}{99}\cdot 10^{2M}\right]$$
$$+\left[\frac{1}{2}\ln N + \left(\frac{9}{10}+\frac{\ln 10}{2}\right)\cdot\left(2M\right)\right] + O(1)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(14)$$

This is the asymptotic formula with an $O(1)$ error term when $\mathrm{k}=11$ in case 1.

Because the quotients of $2M$ over $\ln N$ and 10^{2M} over N are bounded, $\lim\limits_{N\to+\infty}\dfrac{N\ln N + 2\ln 10\cdot MN}{N\ln N} = 2$.
The orders of the other terms are no larger than N, which means that $\sum\limits_{1\le n\le N}\ln a_n = 2N\ln N + O(N)$.

5.8.2 Case 2 for $M\in\mathbb{Z}^{+}$ such that $10^{2M+1}\le 11N < 10^{2M+2}$

(just as case 1) If there exists an M such that $10^{2M+1}\le 11N < 10^{2M+2}$ and $M\in\mathbb{Z}^{+}$,

then $a_N = N\cdot\left(10^{2M+2}+11\right)$.

$$\prod_{1\le n\le N} a_n = N!\cdot\left(10^2+11\right)^{9}\cdot\left(10^3+11\right)^{81}\cdot\left(10^4+11\right)^{819}\cdots$$
$$\left(10^{2M}+11\right)^{\frac{9}{11}\left(10^{2M-1}+1\right)}\cdot\left(10^{2M+1}+11\right)^{\frac{9}{11}\left(10^{2M}-1\right)}\cdot\left(10^{2M+2}+11\right)^{N-\frac{10^{2M+1}-10}{11}}$$
$$=N!\cdot\prod_{t=1}^{M}\left(10^{2t}+11\right)^{\frac{9}{11}\cdot\left(10^{2t-1}+1\right)}\cdot\prod_{t=1}^{M}\left(10^{2t+1}+11\right)^{\frac{9}{11}\left(10^{2t}-1\right)}\cdot\left(10^{2M+2}+11\right)^{N-\frac{10^{2M+1}-10}{11}}$$

Take the natural logarithm of the both sides.

$$\sum_{1\le n\le N}\ln a_n = \ln N! + \frac{9}{11}\cdot\sum_{t=1}^{M}\left(10^{2t-1}+1\right)\ln\left(10^{2t}+11\right) + \frac{9}{11}\cdot\sum_{t=1}^{M}\left(10^{2t}-1\right)\ln\left(10^{2t+1}+11\right)$$
$$+\left(N - \frac{10^{2M+1}-10}{11}\right)\ln\left(10^{2M+2}+11\right),$$

where
$$\sum_{t=1}^{M}\left(10^{2t-1}+1\right)\ln\left(10^{2t}+11\right)=\sum_{t=1}^{M}\left(10^{2t-1}+1\right)\left[2t\ln 10+\frac{11}{10^{2t}}+O\left(\frac{1}{10^{4t}}\right)\right]$$
$$=\sum_{t=1}^{M}\left(2t\ln 10\cdot 10^{2t-1}+\frac{11}{10}+2t\cdot\ln 10\right)+O(1)$$

and
$$\sum_{t=1}^{M}\left(10^{2t}-1\right)\ln\left(10^{2t+1}+11\right)=\sum_{t=1}^{M}\left(10^{2t}-1\right)\left[(2t+1)\ln 10+\frac{11}{10^{2t+1}}+O\left(\frac{1}{10^{4t}}\right)\right].$$
$$=\sum_{t=1}^{M}\left((2t+1)\ln 10\cdot 10^{2t}+\frac{11}{10}-(2t+1)\ln 10\right)+O(1)$$

If we summate the above two equations and multiply the both sides by $\frac{9}{11}$, we get

$$\frac{9}{11}\cdot\sum_{t=1}^{M}\left(10^{2t-1}+1\right)\ln\left(10^{2t}+11\right)+\frac{9}{11}\cdot\sum_{t=1}^{M}\left(10^{2t}-1\right)\ln\left(10^{2t+1}+11\right)$$
$$=\frac{9}{11}\left\{\ln 10\cdot\sum_{t=2}^{2M+1}t\cdot 10^{t-1}+\frac{11}{5}M-\ln 10\cdot M\right\}+O(1)$$
$$=\frac{9}{11}\left\{\frac{\ln 10}{9}\cdot\left(2M+\frac{8}{9}\right)\cdot 10^{2M+1}+\left(\frac{11}{5}-\ln 10\right)M\right\}+O(1)$$
$$=\frac{2\ln 10}{11}\cdot M\cdot 10^{2M+1}+\frac{8\ln 10}{99}\cdot 10^{2M+1}+\left(\frac{9}{5}-\frac{9\ln 10}{11}\right)M+O(1)$$

and
$$\left(N-\frac{10^{2M+1}-10}{11}\right)\ln\left(10^{2M+2}+11\right)$$
$$=\left(N-\frac{1}{11}\cdot 10^{2M+1}+\frac{10}{11}\right)\left[(2M+2)\ln 10+O\left(\frac{1}{10^{2M}}\right)\right]$$
$$=(2M+2)\cdot N\cdot\ln 10-\frac{\ln 10}{11}\cdot(2M+2)\cdot 10^{2M+1}+\frac{20\ln 10}{11}\cdot M+O(1)$$
and

$$\ln N!=N\ln N-N+\frac{1}{2}\ln N+O(1),$$

which means that:

$$\sum_{1\le n\le N}\ln a_n = N\ln N - N + \frac{1}{2}\ln N + 2\ln 10\cdot MN + 2\ln 10\cdot N - \frac{2\ln 10}{11}\cdot M\cdot 10^{2M+1} - \frac{2\ln 10}{11}\cdot 10^{2M+1}$$

$$+\frac{20\ln 10}{11}M + \frac{2\ln 10}{11}\cdot M\cdot 10^{2M+1} + \frac{8\ln 10}{99}\cdot 10^{2M+1} + \left(\frac{9}{5}-\frac{9}{11}\ln 10\right)M + O(1)$$

$$=\left(N\ln N + 2\ln 10\cdot MN\right) + \left[\left(2\ln 10 - 1\right)N - \frac{10\ln 10}{99}\cdot 10^{2M+1}\right]$$

$$+\left[\frac{1}{2}\ln N + \left(\frac{9}{5}+\ln 10\right)M\right] + O(1)$$

$$=\left(N\ln N + \ln 10\cdot(2M+1)\cdot N\right) + \left[\left(\ln 10 - 1\right)N - \frac{10\ln 10}{99}\cdot 10^{2M+1}\right]$$

$$+\left[\frac{1}{2}\ln N + \left(\frac{9}{10}+\frac{\ln 10}{2}\right)\cdot(2M+1)\right] + O(1)\cdots\cdots(15)$$

This is the asymptotic formula with an $O(1)$ error term when $\mathrm{k}=11$ in case 2.

Because the quotients of $2M$ over $\ln N$ and 10^{2M+1} over N are bounded, $\lim_{N\to+\infty}\frac{N\ln N + 2\ln 10\cdot MN}{N\ln N}=2$.

The orders of the other terms are no larger than N ,which means that $\sum_{1\le n\le N}\ln a_n = 2N\ln N + O(N)$.

5.8.3 summary of the two cases when $k=11$

Now we compare the two equations $(14)(15)$ with each other.

$$\sum_{1\le n\le N}\ln a_n = \left(N\ln N + \ln 10\cdot(2M)\cdot N\right) + \left[\left(\ln 10 - 1\right)N - \frac{10\ln 10}{99}\cdot 10^{2M}\right]$$

$$+\left[\frac{1}{2}\ln N + \left(\frac{9}{10}+\frac{\ln 10}{2}\right)\cdot(2M)\right] + O(1)\cdots\cdots(14)$$

$$\sum_{1\le n\le N}\ln a_n = \left(N\ln N + \ln 10\cdot(2M+1)\cdot N\right) + \left[\left(\ln 10 - 1\right)N - \frac{10\ln 10}{99}\cdot 10^{2M+1}\right]$$

$$+\left[\frac{1}{2}\ln N + \left(\frac{9}{10}+\frac{\ln 10}{2}\right)\cdot(2M+1)\right] + O(1)\cdots\cdots(15)$$

We find out that if $10^M \le 11N < 10^{M+1}$, then the asymptotic formula has nothing to do with the parity of M . To sum up, the following asymptotic formula is true.

$$\sum_{1\le n\le N}\ln a_n = \left(N\ln N + \ln 10\cdot MN\right) + \left[\left(\ln 10 - 1\right)N - \frac{10\ln 10}{99}\cdot 10^M\right]$$
$$+\left[\frac{1}{2}\ln N + \left(\frac{9}{10} + \frac{\ln 10}{2}\right)M\right] + O\left(1\right)\cdots\cdots\left(16\right)$$

When $k=7$, by this means we will have to discuss 6 cases (the order of 10 modulo 7 is 6), which can be very tedious. But based on the computational results of the case when $k=11$, we guess that in the 6 cases, the 6 asymptotic formulas are similar in form. We now give the conjecture when $k=7$ directly without proof.

If $10^M \le 7N < 10^{M+1}$, then

$$\sum_{1\le n\le N}\ln a_n = \left(N\ln N + \ln 10\cdot MN\right) + \left[\left(\ln 10 - 1\right)N - \frac{10\ln 10}{63}\cdot 10^M\right]$$
$$+\left[\frac{1}{2}\ln N + \left(\frac{9}{10} + \frac{\ln 10}{2}\right)M\right] + O\left(1\right)\cdots\cdots\left(17\right)$$

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