Classification of holomorphic framed vertex operator algebras of central charge 24

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CLASSIFICATION OF HOLOMORPHIC FRAMED VERTEX OPERATOR ALGEBRAS OF CENTRAL CHARGE 24

By Ching Hung Lam and Hiroki Shimakura

Abstract. This article is a continuation of our work on the classification of holomorphic framed vertex operator algebras of central charge 24. We show that a holomorphic framed VOA of central charge 24 is uniquely determined by the Lie algebra structure of its weight one subspace. As a consequence, we completely classify all holomorphic framed vertex operator algebras of central charge 24 and show that there exist exactly 56 such vertex operator algebras, up to isomorphism.

1. Introduction. The classification of holomorphic vertex operator algebras (VOAs) of central charge 24 is one of the fundamental problems in vertex operator algebras and mathematical physics. In 1993 Schellekens [Sc93] obtained a partial classification by determining possible Lie algebra structures for the weight one subspaces of holomorphic VOAs of central charge 24. There are 71 cases in his list but only 39 of the 71 cases were known explicitly at that time. It is also an open question if the Lie algebra structure of the weight one subspace will determine the VOA structure uniquely when the central charge is 24. Recently, a special class of holomorphic VOAs, called framed VOAs, was studied in [La11, LS12]. Along with other results, 17 new examples were constructed. Moreover, it was shown in [La11, LS12] that there exist exactly 56 possible Lie algebras for holomorphic framed VOAs of central charge 24 and all cases can be constructed explicitly. In this article, we complete the classification of holomorphic framed VOAs of central charge 24. The main theorem is as follows:

Theorem 1.1. *The isomorphism class of a holomorphic framed VOA of central charge* 24 *is uniquely determined by the Lie algebra structure of its weight one subspace. In particular, there exist exactly* 56 *holomorphic framed VOAs of central charge* 24*, up to isomorphism.*

Remark 1.2. By our classification (see [LS12, Table 1]), we noticed that the levels of the representations of Lie algebra associated to the weight one subspace are powers of two for any holomorphic framed VOA of central charge 24. Conversely, by comparing with the list of Lie algebras in [Sc93], we found that except for one case where the Lie algebra has the type $E_{6,4}C_{2,1}A_{2,1}$, all other Lie algebras

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in [Sc93] can be obtained from holomorphic framed VOAs if the levels are powers of two.

First let us recall the results in [La11, LS12] and discuss our methods. It was shown in [LY08] that a code D of length divisible by 16 can be realized as a $1/16$-code of a holomorphic framed VOA if and only if D is triply even and the all-one vector $\mathbf{1} \in D$. Therefore, the classification of holomorphic framed VOAs of rank $8k$ can be reduced into the following 2 steps:

(1) classify all triply even codes D of length $16k$ such that $\mathbf{1} \in D$;

(2) determine all possible VOA structures with the $1/16$-code D for each triply even code D;

Notation 1.3. Let E be a doubly even code of length n and let $d : \mathbb{Z}_2^n \to \mathbb{Z}_2^{2n}$ be the linear map defined by $d(\alpha) = (\alpha, \alpha)$. The code $\mathcal{D}(E) = \langle d(E), (\mathbf{1}, 0) \rangle_{\mathbb{Z}_2}$ spanned by $d(E)$ and $(\mathbf{1}, 0)$ is called the *extended doubling* of E, where $\mathbf{1}$ is the all-one vector.

Let $\mathrm{RM}(1,4)$ be the first order Reed-Muller code of degree 4 and d_{16}^+ the unique indecomposable doubly even self-dual code of length 16. We also use $A \oplus B$ to denote the direct sum of two subcodes A and B.

Recently, triply even codes of length 48 were classified by Betsumiya-Munemasa [BM12, Theorem 29]: a maximal triply even code of length 48 is isomorphic to an extended doubling, a direct sum of extended doublings or an exceptional triply even code D^{ex} of dimension 9. By this result, the classification of holomorphic framed VOAs of central charge 24 can be divided into the following 4 cases. Let D be a $1/16$-code of a holomorphic framed VOA U of central charge 24. Then, up to equivalence,

(i) D is subcode of an extended doubling $\mathcal{D}(E)$ for some doubly even code E of length 24;

(ii) D is a subcode of $\mathrm{RM}(1,4)^{\oplus 3}$ but is not contained in an extended doubling;

(iii) D is a subcode of $\mathrm{RM}(1,4) \oplus \mathcal{D}(d_{16}^+)$ but is not contained in an extended doubling or $\mathrm{RM}(1,4)^{\oplus 3}$;

(iv) D is a subcode of the 9-dimensional exceptional triply even code D^{ex} of length 48 but is not contained in an extended doubling, $\mathrm{RM}(1,4)^{\oplus 3}$ or $\mathrm{RM}(1,4) \oplus \mathcal{D}(d_{16}^+)$.

The main idea is to enumerate all possible framed VOA structures in each case.

Case (i). If D is a subcode of an extended doubling, then it was shown [La11, Theorem 3.9] that U is isomorphic to a lattice VOA V_L or its $\mathbb{Z}_2$-orbifold $\tilde{V}_L$ associated to the -1-isometry of the lattice L. Conversely, any lattice VOA associated to an even unimodular lattice of rank 24 or its $\mathbb{Z}_2$-orbifold has a Virasoro frame whose $1/16$-code D satisfies (i). In this case, it was known [DGM96] that the VOA structure is determined by the Lie algebra structure of its weight one subspace.

PROPOSITION 1.4. [DGM96, Table 2, Proposition 6.5] *Let U be a holomorphic framed VOA of central charge* 24 *with a* $^1/_{16}$*-code satisfying (i). Then the isomorphism class of U is uniquely determined by the Lie algebra structure of U_1. In particular, there exist exactly* 39 *holomorphic framed VOAs of central charge* 24 *with* $^1/_{16}$*-codes satisfying (i), up to isomorphism.*

Case (ii). Suppose that D is a subcode of $\mathrm{RM}(1,4)^{\oplus 3}$. Then U is a simple current extension of $V^{\otimes 3}$, where $V = V^+_{\sqrt{2}E_8}$. This case was studied in [LS12, Section 5] and $U \cong \mathfrak{V}(\mathcal{S})$ for some maximal totally singular subspace $\mathcal{S}$ of $R(V)^3$. In particular, the following theorem was proved by the uniqueness of simple current extensions [DM04b]. (See Sections 4.1, 4.2, and 4.3 for the definition of $\mathcal{S}(k,m,n,\pm)$, $\mathfrak{V}(\mathcal{S})$ and $\mathfrak{g}(\Phi)$, respectively.)

PROPOSITION 1.5. [LS12, Theorem 5.46] *Let U be a holomorphic VOA of central charge* 24*. Assume that $U \cong \mathfrak{V}(\mathcal{S})$ for some maximal totally singular subspace $\mathcal{S}$ of $R(V)^3$.*

(1) *If U_1 is isomorphic to neither $\mathfrak{g}(C_8F_4^2)$ nor $\mathfrak{g}(A_7C_3^2A_3)$, then the isomorphism class of U is uniquely determined by the Lie algebra structure of U_1.*

(2) *If $U_1 \cong \mathfrak{g}(C_8F_4^2)$ then U is isomorphic to $\mathfrak{V}(\mathcal{S}(5,3,0,-))$ or $\mathfrak{V}(\mathcal{S}(5,3,2,+))$.*

(3) *If $U_1 \cong \mathfrak{g}(A_7C_3^2A_3)$ then U is isomorphic to $\mathfrak{V}(\mathcal{S}(5,2,1,+))$ or $\mathfrak{V}(\mathcal{S}(5,2,0))$.*

Hence, it remains to show that $\mathfrak{V}(\mathcal{S}(5,3,0,-)) \cong \mathfrak{V}(\mathcal{S}(5,3,2,+))$ and $\mathfrak{V}(\mathcal{S}(5,2,1,+)) \cong \mathfrak{V}(\mathcal{S}(5,2,0))$, which will be achieved in Section 4.4 (see Theorems 4.21 and 4.24). As a consequence, we obtain the following theorem:

THEOREM 1.6. *Let U be a holomorphic framed VOA of central charge* 24 *with a* $^1/_{16}$*-code satisfying (ii). Then the isomorphism class of U is uniquely determined by the Lie algebra structure of U_1. Excluding the VOAs in Proposition 1.4, there exist exactly* 10 *holomorphic framed VOAs of central charge* 24 *with* $^1/_{16}$*-codes satisfying (ii), up to isomorphism.*

Case (iii). If D is a subcode of $\mathrm{RM}(1,4) \oplus \mathcal{D}(d_{16}^+)$, then U is a simple current extension of $V^+_{\sqrt{2}E_8} \otimes V^+_{\sqrt{2}D_{16}^+}$ and this case was also studied in [LS12, Section 6]. Moreover, one has the following proposition by [LS12, Theorem 6.17], Theorem 1.6 and the uniqueness of simple current extensions [DM04b].

PROPOSITION 1.7. [LS12, Theorem 6.17] *Let U be a holomorphic framed VOA of central charge* 24 *with a* $^1/_{16}$*-code satisfying (iii). Then the isomorphism class of U is uniquely determined by the Lie algebra structure of U_1. Excluding the VOAs in Proposition 1.4 and Theorem 1.6, there exist exactly* 4 *holomorphic framed VOAs of central charge* 24 *with* $^1/_{16}$*-codes satisfying (iii), up to isomorphism.*

Therefore, no extra work is required for this case.

Case (iv). In [La11], holomorphic framed VOAs associated to the subcodes of D^{ex} have been studied and the Lie algebra structures of their weight one subspaces are determined. It was also shown that the Lie algebra structures of their weight one subspaces are uniquely determined by the $1/16$-codes [La11, Theorem 6.78].

Suppose that the $1/16$-code D satisfies (iv). Then by the classification [BM12] (see also http://www.st.hirosaki-u.ac.jp/∼betsumi/triply-even/), D is equivalent to $D^{ex} = D_{[10]}$, $D_{[8]}$ or $D_{[7]}$ (see Section 3.1 for the definition of $D_{[k]}$). Moreover, the Lie algebras of the VOAs associated to $D_{[10]}$, $D_{[8]}$ and $D_{[7]}$ are not included in Cases (i), (ii), and (iii) (see [LS12, Table 1]). Therefore, it remains to show that the VOA structure is uniquely determined by the $1/16$-code D if $D = D_{[10]}$, $D_{[8]}$ or $D_{[7]}$, which will be achieved in Corollary 3.14.

THEOREM 1.8. *Let U be a holomorphic framed VOA of central charge* 24 *with a $1/16$-code D satisfying (iv). Then the isomorphism class of U is uniquely determined by the Lie algebra structure of U_1. In particular, there exist exactly* 3 *holomorphic framed VOAs of central charge* 24 *with $1/16$-codes satisfying (iv), up to isomorphism.*

Our main theorem (Theorem 1.1) will then follow from Propositions 1.4 and 1.7 and Theorems 1.6 and 1.8.

The organization of the article is as follows. In Section 2, we recall some notions and basic facts about VOAs and framed VOAs. In Section 3, we study the framed VOA structures associated to a fixed $1/16$-code D. We show that the holomorphic framed VOA structure is uniquely determined by the $1/16$-code D if D is a subcode of the exceptional triply even code D^{ex}. In Section 4, the isomorphisms between holomorphic VOAs of central charge 24 associated to some maximal totally singular subspaces are discussed. We first recall a classification of maximal totally singular subspaces up to certain equivalence from [LS12]. The construction of a VOA $\mathfrak{V}(\mathcal{S})$ from a maximal totally singular subspace $\mathcal{S}$ is recalled. Some basic properties of the VOA $\mathfrak{V}(\mathcal{S})$ are also reviewed. In Section 4.3, the conjugacy classes of certain involutions in lattice VOAs are discussed. The results will then be used in Section 4.4 to establish the isomorphisms between some holomorphic VOAs associated to maximal totally singular subspaces. In Appendix A, certain ideals of the weight one subspaces of the VOAs $\mathfrak{V}(\mathcal{S})$ used in Section 4.4 are described explicitly.

2. Preliminaries.

Notations.

- $\langle\ ,\ \rangle$ the standard inner product in $\mathbb{Z}_2^n$, $\mathbb{R}^n$ or $(R(V)^3, q_V^3)$.
- $\mathbf{1}$ the all-one vector in $\mathbb{Z}_2^n$.
- $\mathbb{1}$ the vacuum vector of a VOA.
- $\boxtimes$ the fusion product for a VOA.

$\langle A\rangle_{\mathbb{F}}$	the subspace spanned by A over a field $\mathbb{F}$.
$\mathrm{Aut}\,X$	the automorphism group of X.
$\alpha\cdot\beta$	the coordinatewise product of $\alpha,\beta\in\mathbb{Z}_2^n$.
$D\cdot D$	the code $\langle\beta\cdot\beta' \mid \beta,\beta'\in D\rangle_{\mathbb{Z}_2}$, where D is a binary code.
D^{ex}	the exceptional triply even code of length 48.
$g\circ M$	the conjugate of a module M for a VOA by an automorphism g.
$\mathfrak{g}(\Phi)$	the semisimple Lie algebra with the root system Φ.
$[M]$	the isomorphism class of a module M for a VOA.
$M_C(\alpha,\beta)$	the irreducible module for V_C parametrized by $\alpha\in C^\perp$, $\beta\in\mathbb{Z}_2^n$.
$N(\Phi)$	the even unimodular lattice of rank 24 whose root system is Φ.
$O(R(V),q_V)$	the orthogonal group of the quadratic space $(R(V),q_V)$.
$\mathcal{Q}_D$	$\{\delta : D\to\mathbb{Z}_2^n/D^\perp \mid \delta$ is $\mathbb{Z}_2$-linear and $(\delta(\beta),\mathbf{1}+\beta)=0$ for all $\beta\in D\}$.
$\mathcal{S}(m,k_1,k_2,\varepsilon)$	the maximal totally singular subspace of R^3 defined in Theorem 4.1.
$\mathcal{S}(m,k_1,k_2)$	the maximal totally singular subspace of R^3 defined in Theorem 4.3.
$\mathrm{supp}(c)$	the support of $c=(c_i)\in\mathbb{Z}_2^n$, that is, the set $\{i \mid c_i\neq 0\}$.
Sym_n	the symmetric group of degree n.
$R(U)$	the set of all isomorphism classes of irreducible modules for a VOA U.
$(R(V),q_V)$	the 10-dimensional quadratic space $R(V)$ associated to $V=V_{\sqrt{2}E_8}^+$.
$L(\Phi)$	the root lattice with root system Φ.
V_C	the code VOA associated to binary code C.
V_L	the lattice VOA associated with even lattice L.
V_L^+	the fixed point subVOA of V_L with respect to a lift of the -1-isometry of L.
$\tilde{V}_L$	the $\mathbb{Z}_2$-orbifold of V_L associated to the -1-isometry of L.
$\mathfrak{V}(\mathcal{S})$	the holomorphic VOA associated to a maximal totally singular subspace $\mathcal{S}$.

2.1. Vertex operator algebras. Throughout this article, all VOAs are defined over the field $\mathbb{C}$ of complex numbers. We recall the notion of vertex operator algebras (VOAs) and modules from [Bo86, FLM88, FHL93].

A *vertex operator algebra* (VOA) $(V,Y,\mathbb{1},\omega)$ is a $\mathbb{Z}_{\geq 0}$-graded vector space $V=\bigoplus_{m\in\mathbb{Z}_{\geq 0}}V_m$ equipped with a linear map

$$Y(a,z)=\sum_{i\in\mathbb{Z}}a_{(i)}z^{-i-1}\in(\mathrm{End}(V))[[z,z^{-1}]],\quad a\in V$$

and the *vacuum vector* $\mathbb{1}$ and the *conformal element* ω satisfying a number of conditions [Bo86, FLM88]. We often denote it by V or (V,Y).

Two VOAs $(V,Y,\mathbb{1},\omega)$ and $(V',Y',\mathbb{1}',\omega')$ are said to be *isomorphic* if there exists a linear isomorphism g from V to V' such that

$$g\omega=\omega'\quad\text{and}\quad gY(v,z)=Y'(gv,z)g\quad\text{for all } v\in V.$$

When $V = V'$, such a linear isomorphism is called an *automorphism*. The group of all automorphisms of V is called the *automorphism group* of V and is denoted by $\mathrm{Aut}\,V$.

A *vertex operator subalgebra* (or a *subVOA*) is a graded subspace of V which has a structure of a VOA such that the operations and its grading agree with the restriction of those of V and that they share the vacuum vector. When they also share the conformal element, we will call it a *full subVOA*.

An (ordinary) module (M, Y_M) for a VOA V is a $\mathbb{C}$-graded vector space $M = \bigoplus_{m\in\mathbb{C}} M_m$ equipped with a linear map

$$Y_M(a,z) = \sum_{i\in\mathbb{Z}} a_{(i)} z^{-i-1} \in (\mathrm{End}(M))[[z,z^{-1}]], \quad a \in V$$

satisfying a number of conditions [FHL93]. We often denote it by M and its isomorphism class by $[M]$. The *weight* of a homogeneous vector $v \in M_k$ is k. A VOA is said to be *rational* if any module is completely reducible. A rational VOA is said to be *holomorphic* if itself is the only irreducible module up to isomorphism. A VOA is said to be *of CFT type* if $V_0 = \mathbb{C}\mathbb{1}$, and is said to be *$C_2$-cofinite* if $\dim V/\langle u_{(-2)}v \mid u,v \in V\rangle_{\mathbb{C}} < \infty$.

Let M be a module for a VOA V and let g be an automorphism of V. Then the module $g \circ M$ is defined by $(M, Y_{g\circ M})$, where $Y_{g\circ M}(v,z) = Y_M(g^{-1}(v),z)$, $v \in V$. Note that if M is irreducible then so is $g \circ M$.

Let V be a VOA of CFT type. Then the 0-th product gives a Lie algebra structure on V_1. Moreover, the operators $v_{(n)}$, $v \in V_1$, $n \in \mathbb{Z}$, define a representation of the affine Lie algebra associated to V_1. Note that $\mathrm{Aut}\,V$ acts on the Lie algebra V_1 as an automorphism group.

2.2. Fusion products and simple current extensions. Let V^0 be a simple rational C_2-cofinite VOA of CFT type and let W^1 and W^2 be V-modules. It was shown in [HL95] that the V^0-module $W^1 \boxtimes_{V^0} W^2$, called the *fusion product*, exists. A V^0-module M is called a *simple current* if for any irreducible V^0-module X, the fusion product $M \boxtimes_{V^0} X$ is also irreducible.

Let $\{V^\alpha \mid \alpha \in D\}$ be a set of inequivalent irreducible V^0-modules indexed by an abelian group D. A simple VOA $V_D = \bigoplus_{\alpha\in D} V^\alpha$ is called a *simple current extension* of V^0 if it carries a D-grading and every V^α is a simple current. Note that $V^\alpha \boxtimes_{V^0} V^\beta \cong V^{\alpha+\beta}$.

PROPOSITION 2.1. [DM04b, Proposition 5.3] *Let V^0 be a simple rational C_2-cofinite VOA of CFT type and let $V_D = \bigoplus_{\alpha\in D} V^\alpha$ and $\tilde{V}_D = \bigoplus_{\alpha\in D} \tilde{V}^\alpha$ be simple current extensions of V^0. If $V^\alpha \cong \tilde{V}^\alpha$ as V^0-modules for all $\alpha \in D$, then V_D and $\tilde{V}_D$ are isomorphic VOAs.*

2.3. Lattice VOAs and $\mathbb{Z}_2$-orbifolds. Let L be an even unimodular lattice and let V_L be the lattice VOA associated with L [Bo86, FLM88]. Then V_L is holomorphic [Do93]. Let $\theta \in \operatorname{Aut} V_L$ be a lift of $-1 \in \operatorname{Aut} L$ and let V_L^+ denote the subVOA of V_L consisting of vectors in V_L fixed by θ. Let V_L^T be a unique irreducible θ-twisted module for V_L and let $V_L^{T,+}$ be the irreducible V_L^+-submodule of V_L^T with integral weights. Set

$$\tilde{V}_L = V_L^+ \oplus V_L^{T,+}.$$

Then $\tilde{V}_L$ has a unique holomorphic VOA structure by extending its V_L^+-module structure, up to isomorphism [FLM88, DGM96]. The VOA $\tilde{V}_L$ is often called the $\mathbb{Z}_2$-*orbifold* of V_L. More generally, for an involution g in $\operatorname{Aut} V_L$, we can consider the same procedure. If we obtain a VOA as a simple current extension of the subVOA V_L^g of V_L fixed by g, we call it the $\mathbb{Z}_2$-orbifold of V_L associated to g.

2.4. Code VOAs and framed VOAs. In this section, we review the notion of code VOAs and framed VOAs from [Mi96, Mi98, DGH98, Mi04].

Let $\mathrm{Vir} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}\mathbf{c}$ be the Virasoro algebra. For any $c, h \in \mathbb{C}$, we denote by $L(c,h)$ the irreducible highest weight module of Vir with central charge c and highest weight h. It was shown in [FZ92] that $L(c,0)$ has a natural VOA structure. We call it the *simple Virasoro VOA* with central charge c.

Definition 2.2. Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a VOA. An element $e \in V_2$ is called an *Ising vector* if the subalgebra $\mathrm{Vir}(e)$ generated by e is isomorphic to $L(1/2, 0)$ and e is the conformal element of $\mathrm{Vir}(e)$. Two Ising vectors $u, v \in V$ are said to be *orthogonal* if $[Y(u,z_1), Y(v,z_2)] = 0$.

Remark 2.3. It is well-known that $L(1/2,0)$ is rational and has only three inequivalent irreducible modules $L(1/2,0)$, $L(1/2,1/2)$ and $L(1/2,1/16)$. The fusion products of $L(1/2,0)$-modules are computed in [DMZ94]:

$$\begin{gathered} L(1/2,1/2) \boxtimes L(1/2,1/2) = L(1/2,0), \quad L(1/2,1/2) \boxtimes L(1/2,1/16) = L(1/2,1/16), \\ L(1/2,1/16) \boxtimes L(1/2,1/16) = L(1/2,0) \oplus L(1/2,1/2). \end{gathered} \tag{2.1}$$

Definition 2.4. [DGH98] A simple VOA V is said to be *framed* if there exists a set $\{e^1, \dots, e^n\}$ of mutually orthogonal Ising vectors of V such that their sum $e^1 + \cdots + e^n$ is equal to the conformal element of V. The subVOA T_n generated by $e^1, \dots, e^n$ is thus isomorphic to $L(1/2,0)^{\otimes n}$ and is called a *Virasoro frame* of V.

THEOREM 2.5. [DGH98] *Any framed VOA is rational, C_2-cofinite, and of CFT type.*

Given a framed VOA V with a Virasoro frame T_n, one can associate two binary codes C and D of length n to V and T_n as follows: Since $T_n = L(1/2,0)^{\otimes n}$ is rational, V is a completely reducible T_n-module. That is,

$$V \cong \bigoplus_{h_i \in \{0, 1/2, 1/16\}} m_{h_1,\dots,h_n} L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n),$$

where the non-negative integer $m_{h_1,\dots,h_n}$ is the multiplicity of $L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n)$ in V. It was shown in [DMZ94] that all the multiplicities are finite and that $m_{h_1,\dots,h_n}$ is at most 1 if all h_i are different from $1/16$.

Let $U \cong L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n)$ be an irreducible module for T_n. Let $\tau(U)$ denote the binary word $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_2^n$ such that

$$\beta_i = \begin{cases} 0 & \text{if } h_i = 0 \text{ or } 1/2, \\ 1 & \text{if } h_i = 1/16. \end{cases} \tag{2.2}$$

For any $\beta \in \mathbb{Z}_2^n$, denote by V^β the sum of all irreducible submodules U of V such that $\tau(U) = \beta$. Set $D := \{\beta \in \mathbb{Z}_2^n \mid V^\beta \neq 0\}$. Then D becomes a binary code of length n. We call D the *$1/16$-code* with respect to T_n. Note that V can be written as a sum

$$V = \bigoplus_{\beta \in D} V^\beta.$$

For any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_2^n$, let M_α denote the T_n-submodule $m_{h_1,\dots,h_n} L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n)$ of V, where $h_i = 1/2$ if $\alpha_i = 1$ and $h_i = 0$ elsewhere. Note that $m_{h_1,\dots,h_n} \leq 1$ since $h_i \neq 1/16$. Set $C := \{\alpha \in \mathbb{Z}_2^n \mid M_\alpha \neq 0\}$. Then C also forms a binary code and $V^0 = \bigoplus_{\alpha \in C} M_\alpha$. The code VOA V_C associated to a binary code C was defined in [Mi96].

Definition 2.6. [Mi96] A framed VOA V is called a *code VOA* if $D = 0$, equivalently, $V = V^0$.

PROPOSITION 2.7. [Mi96, Theorem 4.3], [Mi98, Theorem 4.5], and [DGH98, Proposition 2.16] *For any even code C, there exists the unique code VOA isomorphic to $\bigoplus_{\alpha \in C} M_\alpha$, up to isomorphism.*

Summarizing, there exists a pair of binary codes (C, D) such that

$$V = \bigoplus_{\beta \in D} V^\beta \quad \text{and} \quad V^0 = \bigoplus_{\alpha \in C} M_\alpha.$$

Note that all V^β, $\beta \in D$, are irreducible V^0-modules.

Since V is a VOA, its weights are integers and we have the lemma.

LEMMA 2.8. (1) *The code D is triply even, i.e.,* $\mathrm{wt}(\beta) \equiv 0 \mod 8$ *for all $\beta \in D$.*

(2) *The code C is even.*

The following theorems are well-known.

THEOREM 2.9. [DGH98, Theorem 2.9] and [Mi04, Theorem 6.1] *Let V be a framed VOA with binary codes (C,D). Then, V is holomorphic if and only if $C = D^{\perp}$.*

THEOREM 2.10. [LY08, Theorem7] *Let $V = \bigoplus_{\beta\in D} V^{\beta}$ be a framed VOA. Then V is a D-graded simple current extension of V^0.*

2.5. Representation theory of code VOAs. In this section, we review representation theory of code VOAs from [Mi98, Mi04, DGL07, LY08].

Let C be an even binary code of length n and V_C the code VOA associated to C. Let us recall a parametrization of irreducible V_C-modules by codewords from [LY08, Section 4.2]. Let $\beta \in C^{\perp}$ and $\gamma \in \mathbb{Z}_2^n$. We define a weight vector $h_{\beta,\gamma} = (h^1_{\beta,\gamma},\dots,h^n_{\beta,\gamma})$, $h^i_{\beta,\gamma} \in \{0, 1/2, 1/16\}$ by

$$h^i_{\beta,\gamma} := \begin{cases} \dfrac{1}{16} & \text{if } \beta_i = 1, \\[2ex] \dfrac{\gamma_i}{2} & \text{if } \beta_i = 0. \end{cases}$$

Let

$$L(h_{\beta,\gamma}) := L(1/2, h^1_{\beta,\gamma}) \otimes \cdots \otimes L(1/2, h^n_{\beta,\gamma})$$

be the irreducible $L(1/2,0)^{\otimes n}$-module with the weight $h_{\beta,\gamma}$. Let H be a maximal self-orthogonal subcode of $C_\beta = \{\alpha \in C \mid \mathrm{supp}(\alpha) \subset \mathrm{supp}(\beta)\}$. Then there exists an irreducible character $\tilde{\chi}_\gamma$ of the central extension of H such that $L(h_{\beta,\gamma}) \otimes \tilde{\chi}_\gamma$ is an irreducible V_H-module. Moreover, we obtain an irreducible V_C-module $M_C(\beta,\gamma)$ as its induced module.

THEOREM 2.11. [Mi98, Theorem 5.3] *Every irreducible V_C-module is isomorphic to an induced module $M_C(\beta,\gamma)$ and its module structure is uniquely determined by the structure of a V_H-submodule.*

Next let us review some basic properties of $M_C(\beta,\gamma)$.

LEMMA 2.12. [DGL07, Lemma 5.8] and [LY08, Lemma 3] *Let $\beta,\beta' \in C^{\perp}$ and $\gamma,\gamma' \in \mathbb{Z}_2^n$. Then the irreducible V_C-modules $M_C(\beta,\gamma)$ and $M_C(\beta',\gamma')$ are isomorphic if and only if*

$$\beta = \beta' \quad \textit{and} \quad \gamma + \gamma' \in C + H^{\perp_\beta},$$

where $H^{\perp_\beta} = \{\alpha \in \mathbb{Z}_2^n \mid \mathrm{supp}(\alpha) \subset \mathrm{supp}(\beta)$ and $\langle \alpha,\delta\rangle = 0$ for all $\delta \in H\}$.

Remark 2.13. [LY08, Remark 6] If C is even, $n \equiv 0 \pmod{16}$, and $C^{\perp}$ is triply even, then $H^{\perp_\beta} \subset C$ in Lemma 2.12.

LEMMA 2.14. [LY08, Lemma 7] *Let* $\alpha, \beta, \gamma \in \mathbb{Z}_2^n$ *with* $\beta \in C^\perp$. *Then*

$$M_C(0,\alpha) \underset{V_C}{\boxtimes} M_C(\beta,\gamma) \cong M_C(\beta,\alpha+\gamma).$$

Moreover, the difference between the top weight of $M_C(\beta,\gamma)$ *and that of* $M_C(\beta,\alpha+\gamma)$ *is congruent to* $\langle\alpha,\alpha+\beta\rangle/2$ *modulo* $\mathbb{Z}$.

Definition 2.15. Let C be an even code and $\alpha \in \mathbb{Z}_2^n$. Define the map $\sigma_\alpha : V_C \to V_C$ by

$$\sigma_\alpha(u) = (-1)^{\langle\alpha,\beta\rangle} u \quad \text{for } u \in M_\beta,\ \beta \in C.$$

It is known [Mi96] that σ_α is an automorphism of V_C.

Next lemma plays an important role in Section 3.

LEMMA 2.16. *Let* C *be an even code of length* n *and let* $\beta \in C^\perp$, $\alpha, \gamma \in \mathbb{Z}_2^n$. *Then*

$$\sigma_\alpha \circ M_C(\beta,\gamma) \cong M_C(0,\alpha\cdot\beta) \underset{V_C}{\boxtimes} M_C(\beta,\gamma),$$

where $\alpha = (\alpha_i), \beta = (\beta_i) \in \mathbb{F}_2^n$, $\alpha\cdot\beta = (\alpha_i\beta_i) \in \mathbb{F}_2^n$.

Proof. Let e_i be the vector in $\mathbb{F}_2^n$ which is 1 in the i-th coordinate and 0 in the other coordinates. Then $\sigma_\alpha = \prod_{i\in \mathrm{supp}(\alpha)} \sigma_{e_i}$. By Lemma 2.14, it suffices to show that

$$\sigma_{e_i} \circ M_C(\beta,\gamma) \cong \begin{cases} M_C(0,e_i) \boxtimes_{V_C} M_C(\beta,\gamma) & \text{if } i \in \mathrm{supp}(\beta), \\ M_C(\beta,\gamma) & \text{if } i \notin \mathrm{supp}(\beta). \end{cases}$$

Let H be a maximal self-orthogonal subcode of C_β. Then by Theorem 2.11, the V_C-module structure is uniquely determined by a V_H-submodule structure.

If $i \notin \mathrm{supp}(\beta)$, then σ_{e_i} is trivial on V_H. Hence $\sigma_{e_i} \circ \big(L(h_{\beta,\gamma}) \otimes \tilde{\chi}_\gamma\big) \cong L(h_{\beta,\gamma}) \otimes \tilde{\chi}_\gamma$ as V_H-modules, and we have $\sigma_{e_i} \circ M_C(\beta,\gamma) \cong M_C(\beta,\gamma)$ as V_C-modules.

Assume $i \in \mathrm{supp}(\beta)$. Then $h_{\beta,\gamma} = h_{\beta,\gamma+e_i}$. Let $c = (c_i) \in H$. Then σ_{e_i} acts on the submodule $\otimes_{i=1}^n L(1/2, c_i/2)$ of V_H by the scalar $(-1)^{\langle e_i,c\rangle}$. Therefore,

$$\sigma_{e_i} \circ \big(L(h_{\beta,\gamma}) \otimes \tilde{\chi}_\gamma\big) \cong L(h_{\beta,\gamma}) \otimes \chi,$$

where $\chi(c) = (-1)^{\langle e_i,c\rangle}\tilde{\chi}_\gamma(c) = \tilde{\chi}_{\gamma+e_i}(c)$ for all $c \in H$, which proves $\sigma_{e_i} \circ M_C(\beta,\gamma) \cong M_C(\beta,e_i+\gamma)$. The desired result follows from $M_C(\beta,e_i+\gamma) \cong M_C(0,e_i) \boxtimes_{V_C} M_C(\beta,\gamma)$ (Lemma 2.14). □

3. Uniqueness of framed VOAs associated to subcodes of D^{ex}.

In this section, we will show that the isomorphism class of a framed VOA is uniquely determined by the $^1/_{16}$-code D if D is a subcode of the 9-dimensional exceptional triply even code D^{ex} of length 48.

3.1. Exceptional triply even code of length 48.

First we recall the properties of the 9-dimensional exceptional triply even code D^{ex} of length 48 given by [BM12]. (See also [La11].)

Let $X = \{1,2,\dots,10\}$ be a set of 10 elements and let

$$\Omega := \binom{X}{2} = \big\{\{i,j\} \mid \{i,j\} \subset X\big\}$$

be the set of all 2-element subsets of X. Then $|\Omega| = \binom{10}{2} = 45$. The triangular graph on X is a graph whose vertex set is Ω and two vertices $S, S' \in \Omega$ are joined by an edge if and only if $|S \cap S'| = 1$. We will denote by $\mathcal{T}_{10}$ the binary code generated by the row vectors of the incidence matrix of the triangular graph on X. Note that $\dim \mathcal{T}_{10} = 8$.

Notation 3.1. For $\{i,j\} \in \Omega$, let $\gamma_{\{i,j\}}$ be the binary word supported at $\{\{k,\ell\} \mid |\{i,j\} \cap \{k,\ell\}| = 1\}$, i.e., the set of all vertices joining to $\{i,j\}$. Note that

$$\text{supp}(\gamma_{\{i,j\}}) = \big\{\{i,k\} \mid k \in X \setminus \{i,j\}\big\} \cup \big\{\{j,k\} \mid k \in X \setminus \{i,j\}\big\} \tag{3.1}$$

and $\text{wt}(\gamma_{\{i,j\}}) = 16$. For convenience, we often identify $\gamma_{\{i,j\}}$ with its support.

Now let $\iota : \mathbb{Z}_2^{45} \to \mathbb{Z}_2^{48}$ be the map defined by $\iota(\alpha) = (\alpha, 0, 0, 0)$. Then we can embed $\mathcal{T}_{10}$ into $\mathbb{Z}_2^{48}$ using ι.

Definition 3.2. Denote by D^{ex} the binary code generated by $\iota(\mathcal{T}_{10})$ and the all-one vector $\mathbf{1}$ in $\mathbb{Z}_2^{48}$. Clearly, $\dim D^{ex} = 9$.

Notation 3.3. For $\alpha = (\alpha_1,\dots,\alpha_n), \beta = (\beta_1,\dots,\beta_n) \in \mathbb{Z}_2^n$, we will denote by $\alpha \cdot \beta$ the coordinatewise product of α and β, i.e., $\alpha \cdot \beta = (\alpha_1\beta_1,\dots,\alpha_n\beta_n)$. For a binary code D, we also denote the code $\langle \beta \cdot \beta' \mid \beta, \beta' \in D\rangle_{\mathbb{Z}_2}$ by $D \cdot D$.

LEMMA 3.4. [BM12, Lemma 16] *For any* $2 \le i < j \le 10$, *denote* $\beta_{i,j} = \gamma_{\{1,i\}} \cdot \gamma_{\{1,j\}}$. *Then the set* $\{\iota(\beta_{i,j}) \mid 2 \le i < j \le 10\} \cup \{\mathbf{1}\} \cup \{(0^{45},1,1,0),(0^{45},1,0,1)\}$ *is a basis of* $(D^{ex})^{\perp}$.

PROPOSITION 3.5. *Let D be a d-dimensional subcode of D^{ex} containing $\mathbf{1}$ and let $B = \{\mathbf{1},\beta_1,\dots,\beta_{d-1}\}$ be a basis of D. Then the set $\mathcal{B} = \{\mathbf{1}\} \cup \{\beta \cdot \beta' \mid \beta, \beta' \in B, \beta \neq \beta'\}$ is linearly independent. In particular,* $\dim(D \cdot D) = \binom{d}{2} + 1$.

Proof. It suffices to consider the case where $D = D^{ex}$. Note that $d = 9$. Since $|\mathcal{B}| \le \binom{9}{2} + 1 = 37$ and $\langle \mathcal{B}\rangle_{\mathbb{Z}_2} = D \cdot D$, we have $\dim(D \cdot D) \le 37$. Therefore, it suffices to show that $\dim(D \cdot D) \ge 37$.

Let $\beta_{i,j} = \gamma_{\{1,i\}} \cdot \gamma_{\{1,j\}}$ be defined as in Lemma 3.4. Then $\iota(\beta_{i,j}) \in D \cdot D$ for all i,j. By Lemma 3.4, $\{\mathbf{1}\} \cup \{\iota(\beta_{i,j}) \mid 2 \le i < j \le 10\}$ is a linearly independent subset of $D \cdot D$ with 37 vectors. Hence, $\dim(D \cdot D) \ge 37$, and thus $\dim(D \cdot D) = 37$ as desired. □

Next we recall a notation for denoting subcodes of D^{ex} from [La11].

Notation 3.6. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of 10. Let $X_1, \ldots, X_m$ be subsets of X such that $X = \bigcup_{i=1}^m X_i$ and $|X_i \cap X_j| = \lambda_i \delta_{i,j}$ for $1 \le i,j \le m$. Let $D_{[\lambda_1,\ldots,\lambda_m]}$ denote the code of length 48 generated by the all-one vector $\mathbf{1}$ and $\{\gamma_{\{i,j\}} \mid \{i,j\} \subset X_k, 1 \le k \le m\}$. For convenience, we often omit the 1's in the partition. For example, $D_{[8]} = D_{[8,1,1]}$ and $D_{[7]} = D_{[7,1,1,1]}$. Note also that $D^{ex} = D_{[10]}$.

Remark 3.7. It is clear by the definition that the code $D_{[\lambda_1,\ldots,\lambda_m]}$ is uniquely determined by the shape of the partition $(\lambda_1, \ldots, \lambda_m)$ up to the action of Sym_{10}.

3.2. Framed VOA structures associated with a certain $^1/_{16}$-code. In this section, we show that the holomorphic framed VOA structure is uniquely determined by the $^1/_{16}$-code under certain assumptions on the $^1/_{16}$-code. As a corollary, we prove that the framed VOA structure is uniquely determined if the $^1/_{16}$-code is a subcode of the exceptional triply even code of length 48.

Definition 3.8. Let D be a triply even code of length n divisible by 16. Define

$$\mathcal{Q}_D = \{\delta : D \to \mathbb{Z}_2^n/D^\perp \mid \delta \text{ is } \mathbb{Z}_2\text{-linear and } \langle \delta(\beta), \mathbf{1}+\beta \rangle = 0 \text{ for all } \beta \in D\}.$$

Note that $\mathcal{Q}_D$ is a linear subspace of $\mathrm{Hom}_{\mathbb{Z}_2}(D, \mathbb{Z}_2^n/D^\perp)$.

LEMMA 3.9. *Let D be a d-dimensional triply even code of length n divisible by* 16. *Assume that D contains the all-one vector* $\mathbf{1}$.

(1) *Let $B = \{\mathbf{1}, \beta_1, \ldots, \beta_{d-1}\}$ be a basis of D and let $\delta \in \mathrm{Hom}_{\mathbb{Z}_2}(D, \mathbb{Z}_2^n/D^\perp)$. Then $\delta \in \mathcal{Q}_D$ if and only if both* (a) $\langle \delta(\beta), \mathbf{1}+\beta \rangle = 0$ *and* (b) $\langle \delta(\beta), \beta' \rangle = \langle \delta(\beta'), \beta \rangle$ *hold for all $\beta, \beta' \in B$.*

(2) $\dim \mathcal{Q}_D = 1 + \binom{d}{2}$.

Proof. (1) Assume $\delta \in \mathcal{Q}_D$. By the definition of $\mathcal{Q}_D$, (a) holds. Moreover, by the definition of $\mathcal{Q}_D$ and the $\mathbb{Z}_2$-linearity of δ, we have $\langle \delta(\beta+\beta'), \mathbf{1}+\beta+\beta' \rangle = \langle \delta(\beta), \beta' \rangle + \langle \delta(\beta'), \beta \rangle = 0$ for all $\beta, \beta' \in B$. Hence (b) holds.

Conversely, we assume (a) and (b). Then $\delta \in \mathcal{Q}_D$ since for $\sum_{\beta \in B} c_\beta \beta \in D$,

$$\left\langle \delta\left(\sum_{\beta \in B} c_\beta \beta \right), \mathbf{1} + \sum_{\beta \in B} c_\beta \beta \right\rangle = \sum_{\substack{\beta,\beta' \in B \\ \beta \ne \beta'}} c_\beta c_{\beta'} \langle \delta(\beta), \beta' \rangle + \sum_{\beta \in B} c_\beta \langle \delta(\beta), \mathbf{1}+\beta \rangle = 0.$$

(2) By (1), in order to determine $\dim \mathcal{Q}_D$, it suffices to count the possibilities of the images of elements in B satisfying (a) and (b). Note that for $\beta = \mathbf{1}$, (a) is automatically satisfied. For $\beta \neq \mathbf{1}$, the subspace of $\mathbb{Z}_2^n/D^\perp$ that satisfies (a) has dimension $d-1$. Therefore, to obtain a subset $\{\delta(\mathbf{1}), \delta(\beta_1), \dots, \delta(\beta_{d-1})\}$ satisfying (a) and (b), we have 2^d choices for $\delta(\mathbf{1})$ and $2^{(d-1)-1}$ choices for $\delta(\beta_1)$ that satisfies (a) and $\langle \delta(\beta_1), \mathbf{1}\rangle = \langle \delta(\mathbf{1}), \beta_1\rangle$. Similarly, we have $2^{(d-1)-i}$ choices for δ_{β_i} for $i = 2, \dots, d-1$. Hence we have

$$|\mathcal{Q}_D| = 2^d \cdot 2^{(d-2)} \cdot 2^{d-3} \cdots 2^1 \cdot 2^0 = 2^{d+(d-2)+\cdots+1} = 2^{1+\binom{d}{2}}$$

and $\dim \mathcal{Q}_D = 1 + \binom{d}{2}$ as desired. □

LEMMA 3.10. *For $\gamma \in \mathbb{Z}_2^n$, the map $\eta(\gamma) : D \to \mathbb{Z}_2^n/D^\perp$, $\beta \mapsto \gamma \cdot \beta + D^\perp$ belongs to $\mathcal{Q}_D$.*

Proof. Since the coordinatewise product $\cdot$ is $\mathbb{Z}_2$-linear, so is $\eta(\gamma)$. For $\beta \in D$, $\langle \eta(\gamma)(\beta), \mathbf{1} + \beta\rangle = \langle \gamma \cdot \beta, \mathbf{1} + \beta\rangle = 0$. Hence $\eta(\gamma) \in \mathcal{Q}_D$. □

LEMMA 3.11. *Let D be a d-dimensional triply even code of length n divisible by 16. Assume that D contains $\mathbf{1}$ and that $\dim(D \cdot D) = \binom{d}{2} + 1$. Then, for $\delta \in \mathcal{Q}_D$, there exists $\gamma \in \mathbb{Z}_2^n$ such that $\delta(\beta) = \gamma \cdot \beta \mod D^\perp$ for any $\beta \in D$.*

Proof. Set $\mathrm{Im}(\eta) = \{\eta(\gamma) \mid \gamma \in \mathbb{Z}_2^n\}$ and $\mathrm{Ker}(\eta) = \{\gamma \in \mathbb{Z}_2^n \mid \gamma \cdot \beta \in C \text{ for all } \beta \in D\}$. By Lemma 3.10, it suffices to prove that $\dim \mathcal{Q}_D = \dim \mathrm{Im}(\eta)$. Since

$$\begin{aligned} \gamma \in \mathrm{Ker}(\eta) &\iff \langle \gamma \cdot \beta, \beta'\rangle = 0 \quad \text{for all } \beta, \beta' \in D \\ &\iff \langle \gamma, \beta \cdot \beta'\rangle = 0 \quad \text{for all } \beta, \beta' \in D, \end{aligned}$$

we have $D \cdot D = \mathrm{Ker}(\eta)^\perp$. By the assumption, we have

$$\dim \mathrm{Im}(\eta) = n - \dim \mathrm{Ker}(\eta) = \dim \mathrm{Ker}(\eta)^\perp = 1 + \binom{d}{2}.$$

Therefore by Lemma 3.9, $\dim \mathrm{Im}(\eta) = \dim \mathcal{Q}_D$. □

LEMMA 3.12. *Let $V = \bigoplus_{\beta \in D} V^\beta$ and $U = \bigoplus_{\beta \in D} U^\beta$ be holomorphic framed VOAs with the same $1/16$-code D. Let $C = D^\perp$. Then there exists a unique $\delta \in \mathcal{Q}_D$ such that, as V_C-modules,*

$$U^\beta \cong M_C(0, \delta(\beta)) \boxtimes_{V_C} V^\beta \quad \text{for all } \beta \in D.$$

Proof. Recall that $V^0 \cong U^0 \cong V_C$. Let $\beta \in D$. Then by Theorems 2.10 and 2.11, Lemma 2.12 and Remark 2.13, there exist unique $\gamma_{\beta,V}, \gamma_{\beta,U} \in \mathbb{Z}_2^n/C$ such that $U^\beta \cong M_C(\beta, \gamma_{\beta,U})$ and $V^\beta \cong M_C(\beta, \gamma_{\beta,V})$ as V_C-modules. Let δ be the map from D to $\mathbb{Z}_2^n/C$ defined by $\delta(\beta) = \gamma_{\beta,U} + \gamma_{\beta,V}$. Then by Lemma 2.14

$$M_C(0,\delta(\beta)) \underset{V_C}{\boxtimes} V^\beta \cong U^\beta.$$

Let us show that $\delta \in \mathcal{Q}_D$. Since both U and V are simple current extensions (Theorem 2.10), we have $U^\beta \boxtimes_{V_C} U^{\beta'} \cong U^{\beta+\beta'}$ and $V^\beta \boxtimes_{V_C} V^{\beta'} \cong V^{\beta+\beta'}$ for all $\beta, \beta' \in D$. Hence

$$\delta(\beta) + \delta(\beta') = \delta(\beta+\beta') \quad \text{for all } \beta, \beta' \in D,$$

that is, the map $\delta : D \to \mathbb{Z}_2^n/C$ is $\mathbb{Z}_2$-linear. Moreover, U^β and V^β have integral weights for all $\beta \in D$. By Lemma 2.14, the difference of their top weights is $\langle \delta(\beta), \delta(\beta)+\beta\rangle/2 = \langle \delta(\beta), \mathbf{1}+\beta\rangle/2 \mod \mathbb{Z}$. Hence $\langle \delta(\beta), \mathbf{1}+\beta\rangle = 0$ for all $\beta \in D$. Thus $\delta \in \mathcal{Q}_D$. □

THEOREM 3.13. *Let D be a d-dimensional triply even code of length n divisible by* 16*. Assume that D contains* $\mathbf{1}$ *and that* $\dim(D\cdot D) = \binom{d}{2}+1$*. Let U and V be holomorphic framed VOAs with the same* $1/16$*-code D. Then $U \cong V$ as VOAs.*

Proof. Set $C = D^\perp$. Let $V = \bigoplus_{\beta\in D} V^\beta$ and $U = \bigoplus_{\beta\in D} U^\beta$. Note that $V^0 \cong U^0 \cong V_C$. By Lemma 3.12, there exists $\delta \in \mathcal{Q}_D$ such that

$$U^\beta \cong M_C(0,\delta(\beta)) \underset{V_C}{\boxtimes} V^\beta \quad \text{for all } \beta \in D$$

as V_C-modules. By Lemma 3.11, there exists $\gamma \in \mathbb{Z}_2^n$ such that $\delta(\beta) = \gamma\cdot\beta \mod C$ for all $\beta \in D$. By Lemma 2.16 we have

$$U^\beta \cong M_C(0,\gamma\cdot\beta) \underset{V_C}{\boxtimes} V^\beta \cong \sigma_\gamma \circ V^\beta$$

as V_C-modules for all $\beta \in D$. Hence $\sigma_\gamma \circ V \cong U$ as V_C-modules. By the uniqueness of simple current extensions (Proposition 2.1), $\sigma_\gamma \circ V \cong U$ as VOAs. The theorem follows since $V \cong \sigma_\gamma \circ V$ as VOAs. □

Combining Proposition 3.5 and Theorem 3.13, we obtain the following corollary:

COROLLARY 3.14. *For a subcode D of the exceptional triply even code D^{ex} of length* 48*, the isomorphism class of a framed VOA of central charge* 24 *with the* $1/16$*-code D is uniquely determined.*

4. Isomorphisms of holomorphic framed VOAs of central charge 24 associated to quadratic spaces. In this section, we discuss isomorphisms between holomorphic VOAs of central charge 24 associated to some maximal totally singular subspaces.

4.1. Quadratic spaces and maximal totally singular subspaces. First, we review a classification of maximal totally singular subspaces up to certain equivalence from [LS12]. For the notation and the detail, see [LS12, Section 4].

Let (R,q) be a $2m$-dimensional quadratic space of plus type over $\mathbb{F}_2$. Then (R^3,q^3) is a $6m$-dimensional quadratic space of plus type over $\mathbb{F}_2$, where $q^3 : R^3 \to \mathbb{F}_2$, $q^3(v_1,v_2,v_3) = \sum_{i=1}^{3} q(v_i)$.

Consider the following condition on maximal totally singular subspaces $\mathcal{S}$ of R^3:

$$(a_1,a_2,0),(0,a_2,a_3) \in \mathcal{S} \quad \text{for some } a_i \in R\setminus\{0\} \text{ with } q(a_i)=0. \tag{4.1}$$

We will recall the construction of certain maximal totally singular subspaces of R^3 not satisfying (4.1) from [LS12].

THEOREM 4.1. [LS12, Theorem 4.6] *Let S_1 be a k_1-dimensional totally singular subspace of R and let S_2 be a k_2-dimensional totally singular subspace of S_1. Assume that $m-k_1-k_2$ is even. Let P be an $(m-k_1-k_2)$-dimensional non-singular subspace of $S_1^{\perp}$ of ε type, where $\varepsilon \in \{\pm\}$. Let Q and T be complementary subspaces of S_1 and of S_2 in $(S_1 \perp P)^{\perp}$ and in $(S_2 \perp P)^{\perp}$, respectively. Then the following hold:*

(1) *T and $Q^{\perp}$ are non-singular isomorphic quadratic spaces;*

(2) *Let φ be an isomorphism of quadratic spaces from T to $Q^{\perp}$ and set*

$$\begin{aligned}&\mathcal{S}(S_1,S_2,P,Q,T,\varphi)\\&\quad=\{(s_1+p+q, s_2+p+t, q+\varphi(t)) \mid s_i \in S_i,\ p\in P,\ q\in Q,\ t\in T\}.\end{aligned}$$

Then $\mathcal{S}(S_1,S_2,P,Q,T,\varphi)$ is a maximal totally singular subspace of R^3;

(3) *$\mathcal{S}(S_1,S_2,P,Q,T,\varphi)$ depends only on k_1,k_2 and ε up to $O(R,q)\wr \mathrm{Sym}_3$.*

Notation 4.2. By (3), we denote $\mathcal{S}(S_1,S_2,P,Q,T,\varphi)$ by $\mathcal{S}(m,k_1,k_2,\varepsilon)$.

THEOREM 4.3. [LS12, Theorem 4.8] *Let S_1 be a k_1-dimensional totally singular subspace of R and let S_2 be a k_2-dimensional totally singular subspace of S_1. Assume that $m-k_1-k_2$ is odd. Let P and Q be $(m-k_1-k_2-1)$-dimensional and $(m-k_1+k_2-1)$-dimensional non-singular subspaces of $S_1^{\perp}$ and of $(S_1 \perp P)^{\perp}$ of plus type, respectively. Let B and T be complementary subspaces of S_1 and of S_2 in $(S_1 \perp P \perp Q)^{\perp}$ and in $(S_2 \perp P \perp B)^{\perp}$, respectively. Let $U = (Q \perp B)^{\perp}$. Then the following hold:*

(1) *B is a 2-dimensional non-singular subspace of plus type;*

(2) *T and U are isomorphic non-singular quadratic spaces of plus type;*

(3) *Let y be the non-singular vector in B and let z be a non-zero singular vector in B. Let φ be an isomorphism of quadratic spaces from T to U and set*

$$\begin{aligned}&\mathcal{S}(S_1,S_2,P,Q,B,T,z,\varphi)\\&\quad=\big\langle(s_1+p+q,s_2+p+t,q+\varphi(t)),(y,y,0),(y,0,y),\\&\qquad\qquad(z,z,z)\mid s_i\in S_i,\ p\in P,\ q\in Q,\ t\in T\big\rangle_{\mathbb{F}_2}.\end{aligned}$$

Then $\mathcal{S}(S_1,S_2,P,Q,B,T,z,\varphi)$ is a maximal totally singular subspace of R^3;

(4) *$\mathcal{S}(S_1,S_2,P,Q,B,T,z,\varphi)$ depends only on k_1,k_2 up to $O(R,q)\wr\mathrm{Sym}_3$.*

Notation 4.4. By (4), we denote $\mathcal{S}(S_1,S_2,P,Q,B,T,z,\varphi)$ by $\mathcal{S}(m,k_1,k_2)$.

In [LS12], maximal totally singular subspaces of R^3 were classified.

THEOREM 4.5. [LS12, Theorem 5.11] *Let $\mathcal{S}$ be a maximal totally singular subspace of R^3. Then, up to $O(R,q)\wr\mathrm{Sym}_3$, one of the following holds:*

(1) *$\mathcal{S}$ satisfies (4.1);*

(2) *$\mathcal{S}$ is conjugate to $\mathcal{S}(S_1,S_2,P,Q,T,\varphi)$ defined as in Theorem 4.1;*

(3) *$\mathcal{S}$ is conjugate to $\mathcal{S}(S_1,S_2,P,Q,B,T,z,\varphi)$ defined as in Theorem 4.3.*

4.2. Holomorphic VOAs $\mathfrak{V}(\mathcal{S})$. Next we review some facts about the VOA $\mathfrak{V}(\mathcal{S})$ defined in [Sh11, LS12]. Throughout this subsection, V denotes the VOA $V^+_{\sqrt{2}E_8}$.

Let $R(V)$ be the set of isomorphism classes of irreducible V-modules. Then under the fusion rules, $R(V)$ forms an elementary abelian 2-group of order 2^{10} [ADL05, Sh04]. Consider the map $q_V : R(V)\to\mathbb{F}_2$ defined by setting $q_V([M])=0$ and 1 if M is $\mathbb{Z}$-graded and is $(\mathbb{Z}+1/2)$-graded, respectively. Then $(R(V),q_V)$ is a 10-dimensional quadratic space of plus type over $\mathbb{F}_2$ [Sh04, Theorem 3.8] and $(R(V)^3,q_V^3)$ is a 30-dimensional quadratic space of plus type over $\mathbb{F}_2$.

Notation 4.6. Let $\mathcal{T}$ be a subset of $R(V)^3$. We set $\mathfrak{V}(\mathcal{T})=\bigoplus_{[M]\in\mathcal{T}}M$ and often view it as a $V^{\otimes 3}$-module by identifying $R(V^{\otimes 3})$ with $R(V)^3$ (cf. [FHL93, Section 4.7]).

PROPOSITION 4.7. [Sh11, Proposition 4.4] *Let $\mathcal{T}$ be a subset of $R(V)^3$. Then the $V^{\otimes 3}$-module $\mathfrak{V}(\mathcal{T})=\bigoplus_{[M]\in\mathcal{T}}M$ has a simple VOA structure of central charge 24 by extending its $V^{\otimes 3}$-module structure if and only if $\mathcal{T}$ is a totally singular subspace of $R(V)^3$. Moreover, $\mathfrak{V}(\mathcal{T})$ is holomorphic if and only if $\mathcal{T}$ is maximal.*

Remark 4.8. [LS12, Section 5] (1) A VOA is isomorphic to $\mathfrak{V}(\mathcal{T})$ for some totally singular subspace $\mathcal{T}$ of $R(V)^3$ if and only if it contains a full subVOA isomorphic to $V^{\otimes 3}$.

(2) If totally singular subspaces $\mathcal{T}_1$ and $\mathcal{T}_2$ of $R(V)^3$ are conjugate under $O(R(V),q_V)\wr\mathrm{Sym}_3$, then the VOAs $\mathfrak{V}(\mathcal{T}_1)$ and $\mathfrak{V}(\mathcal{T}_2)$ are isomorphic.

LEMMA 4.9. [LS12, Lemma 5.4] *Let $\mathcal{S}$ be a maximal totally singular subspace of $R(V)^3$. If $\mathcal{S}$ satisfies (4.1), then $\mathfrak{V}(\mathcal{S})$ is isomorphic to V_L or its $\mathbb{Z}_2$-orbifold $\tilde{V}_L$ for some even unimodular lattice L of rank 24 containing $(\sqrt{2}E_8)^{\oplus 3}$. Moreover, if $\mathcal{S}$ contains non-zero vectors $(a_1,0,0)$, $(0,a_2,0)$ and $(0,0,a_3)$ then $\mathfrak{V}(\mathcal{S}) \cong V_L$ for a lattice L with the same properties.*

4.3. Conjugacy classes of involutions in the automorphism group of V_L. In this section, we discuss the conjugacy classes of certain involutions in $\operatorname{Aut} V_L$ when L is the Niemeier lattice $N(A_{15}D_9)$ or $N(A_7^2D_5^2)$. Throughout this subsection, let $L(\Phi)$ denote the root lattice of a root system Φ.

First, we summarize a few facts about lattices.

LEMMA 4.10. *Let s be a root in D_5 and let $2\beta + L(D_5)$ be the order 2 element in $L(D_5)^*/L(D_5)$. Then $s + 4\beta + 2L(D_5)$ is conjugate to $s + 2L(D_5)$ under the Weyl group of D_5.*

Proof. Let $\{e_i \mid 1 \le i \le 5\}$ be an orthonormal basis of $\mathbb{R}^5$. Then $\{\pm(e_i + e_j), \pm(e_i - e_j) \mid 1 \le i < j \le 5\}$ is a root system of type D_5, and $2\beta + L(D_5) = e_1 + L(D_5)$. Hence one can easily prove this lemma. □

By [CS99, p 438, XVII], we obtain the following lemma.

LEMMA 4.11. *Let $N = N(A_7^2D_5^2)$ and $R = L(A_7^2D_5^2)$. Let τ be a diagram automorphism of $L(A_7)$ of order 2.*

(1) *There exist generators $\alpha \in L(A_7)^*/L(A_7)$ and $\beta \in L(D_5)^*/L(D_5)$ such that $N = \langle s,t,R\rangle_{\mathbb{Z}}$, where $s = (3\alpha,\alpha,\beta,0)$ and $t = (2\alpha,0,-\beta,\beta)$;*

(2) *The automorphism $(x_1,x_2,x_3,x_4) \mapsto (\tau(x_1),\tau(x_2),x_3,x_4)$ of R^* does not preserve N.*

Next, we recall the following from [Ka90, Proposition 8.1, Exercise 10 in Chapter 8]:

LEMMA 4.12. *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra and let g and h be automorphisms of $\mathfrak{g}$ of order 2. Assume that the fixed point subalgebras of $\mathfrak{g}$ for g and h are isomorphic. Then there exists an inner automorphism x of $\mathfrak{g}$ such that $xgx^{-1} = h$.*

The next two lemmas follow from explicit calculations based on [Ka90, Chapter 8]. For a root system Φ, let $\mathfrak{g}(\Phi)$ denote the semi-simple Lie algebra of type Φ.

LEMMA 4.13. (1) *Let s be a root in D_5 and let $f = \exp(\operatorname{ad}(\pi\sqrt{-1}s))$, where we view s as a vector in the Cartan subalgebra. Then $\mathfrak{g}(D_5)^f \cong \mathfrak{g}(A_3A_1^2)$.*

(2) *Let $g \in \operatorname{Aut}\mathfrak{g}(D_5)$ be an involution which is a lift of the -1-isometry of D_5. Then $\mathfrak{g}(D_5)^g \cong \mathfrak{g}(B_2^2)$.*

LEMMA 4.14. *Let* $\mathfrak{g}_1 \cong \mathfrak{g}_2 \cong \mathfrak{g}(A_7)$ *and* $\mathfrak{g}_3 \cong \mathfrak{g}_4 \cong \mathfrak{g}(D_5)$ *and set* $\mathfrak{g} = \oplus_{i=1}^4 \mathfrak{g}_i$. *Let* f *be an involution in* $\operatorname{Aut}\mathfrak{g}$ *such that* $\mathfrak{g}^f \cong \mathfrak{g}(A_7A_3B_2^2A_1^2)$. *Then the following hold:*

(1) $f(\mathfrak{g}_1) = \mathfrak{g}_2$, *and* $f(\mathfrak{g}_i) = \mathfrak{g}_i$ *for* $i = 3,4$.

(2) *As sets of isomorphism classes,* $\{\mathfrak{g}_3^f, \mathfrak{g}_4^f\} = \{\mathfrak{g}(B_2^2), \mathfrak{g}(A_3A_1^2)\}$.

In the following, we will show that the conjugacy classes of some involutions in V_L are uniquely determined by the isomorphism class of the fixed point Lie subalgebra of $(V_L)_1$ for $L \cong N(A_{15}D_9)$ and $N(A_7^2D_5^2)$. For a Lie algebra $\mathfrak{g}$, let $\operatorname{Inn}\mathfrak{g}$ denote the inner automorphism group of $\mathfrak{g}$. Since $\operatorname{Inn}(V_L)_1$ can be extended to an automorphism group of V_L, we view it as a subgroup of $\operatorname{Aut}V_L$.

THEOREM 4.15. *There exists exactly one conjugacy class of involutions* g *in* $\operatorname{Aut}V_{N(A_{15}D_9)}$ *such that the fixed point Lie subalgebra* $(V_{N(A_{15}D_9)}^g)_1$ *is isomorphic to* $\mathfrak{g}(C_8B_4^2)$.

Proof. Set $V = V_{N(A_{15}D_9)}$ and $\mathfrak{g} = V_1$. Let g and h be involutions in $\operatorname{Aut}V$ satisfying the assumption. Since Cartan subalgebras of arbitrary Lie algebra are conjugate under inner automorphisms and any automorphism of finite order preserves a Cartan subalgebra [Ka90, Lemma 8.1], we may assume that both g and h preserve the Cartan subalgebra $\mathbb{C}\otimes_{\mathbb{Z}} L(A_{15}D_9)$ of $\mathfrak{g}$. It follows from $\mathfrak{g} \cong \mathfrak{g}(A_{15}) \oplus \mathfrak{g}(D_9)$ that both g and h preserve each ideal. By Lemma 4.12, there exists $x \in \operatorname{Inn}\mathfrak{g} \subset \operatorname{Aut}V$ such that $g = xhx^{-1}$ on $\mathfrak{g}$.

Set $k = xhx^{-1}g^{-1}$. Then k is trivial on $\mathfrak{g}$. Set $N = N(A_{15}D_9)$ and $R = L(A_{15}D_9)$. Since V_R is generated by $\mathfrak{g}$ as a VOA, k is also trivial on V_R. By Schur's lemma k acts on each irreducible V_R-submodule $V_{\lambda+R}$ of V by a scalar. Hence there exists $v \in 2R^*/2N$ such that $k = \exp(\pi\sqrt{-1}v_{(0)})$. By [CS99, p439, XIX], we may assume that N/R is generated by $(2\alpha,\beta)$, where $L(A_{15})^*/L(A_{15}) = \langle\alpha\rangle$ and $L(D_9)^*/L(D_9) = \langle\beta\rangle$. Then the group $2R^*/2N$ is generated by $(2\alpha,0)$. We now consider the action of g on $R^* \subset \mathbb{C}\otimes_{\mathbb{Z}} R$. It follows from $\mathfrak{g}^g \cong \mathfrak{g}(C_8B_4^2)$ that $g(\alpha) = -\alpha$ and $g(\beta) = -\beta$ (cf. [Ka90, Proposition 8]). Hence $g(v) = -v$, and $gk^{1/2}g^{-1} = k^{-1/2}$, where $k^{1/2} = \exp(\pi\sqrt{-1}v_{(0)}/2) \in \operatorname{Aut}V$. Thus we obtain

$$k^{-1/2}xhx^{-1}k^{1/2} = k^{-1/2}kgk^{1/2} = g,$$

which proves the theorem. □

THEOREM 4.16. *There exists exactly one conjugacy class of involutions* g *in* $\operatorname{Aut}V_{N(A_7^2D_5^2)}$ *such that the fixed point Lie subalgebra* $(V_{N(A_7^2D_5^2)}^g)_1$ *is isomorphic to* $\mathfrak{g}(A_7A_3B_2^2A_1^2)$.

Proof. Set $V = V_{N(A_7^2 D_5^2)}$ and $\mathfrak{g} = V_1$. Then $\mathfrak{g} \cong \mathfrak{g}(A_7^2 D_5^2)$, and let $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, \mathfrak{g}_4$ be ideals of $\mathfrak{g}$ such that

$$\mathfrak{g} = \bigoplus_{i=1}^{4} \mathfrak{g}_i, \quad \mathfrak{g}_1 \cong \mathfrak{g}_2 \cong \mathfrak{g}(A_7), \quad \mathfrak{g}_3 \cong \mathfrak{g}_4 \cong \mathfrak{g}(D_5).$$

Let g and h be involutions in $\mathrm{Aut}\, V$ satisfying the assumption. Since Cartan subalgebras of arbitrary Lie algebra are conjugate under inner automorphisms and any automorphism of finite order preserves a Cartan subalgebra [Ka90, Lemma 8.1], we may assume that g and h preserve the Cartan subalgebra $\mathbb{C} \otimes_{\mathbb{Z}} L(A_7^2 D_5^2)$ of $\mathfrak{g}$. By Lemma 4.14, we may assume that

$$g(\mathfrak{g}_1) = h(\mathfrak{g}_1) = \mathfrak{g}_2, \quad \mathfrak{g}_3^g \cong \mathfrak{g}_3^h \cong \mathfrak{g}(B_2^2), \quad \mathfrak{g}_4^g \cong \mathfrak{g}_4^h \cong \mathfrak{g}(A_3 A_1^2).$$

By Lemma 4.12, there exists $x \in \mathrm{Inn}(\mathfrak{g}_3 \oplus \mathfrak{g}_4) \subset \mathrm{Aut}\, V$ such that $xhx^{-1} = g$ on $\mathfrak{g}_3 \oplus \mathfrak{g}_4$.

Set $h' = xhx^{-1}$. Let us consider the actions of g and h' on $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. By Lemma 4.14(1), $h'g^{-1}$ preserves both $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Set $a_i = (h'g^{-1})_{|\mathfrak{g}_i}$ for $i = 1,2$. Then $h' = a_1 a_2 g$ on $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Since the order of h' is 2, we have $a_2 = g a_1^{-1} g^{-1}$. Hence $h' = a_1 g a_1^{-1}$ and $a_1^{-1} h' a_1 = g$ on $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Suppose that a_1 is not inner. Then there exists $c \in \mathrm{Inn}\, \mathfrak{g}_1$ such that $c^{-1} a_1$ acts on the Cartan subalgebra $\mathbb{C} \otimes_{\mathbb{Z}} L(A_7)$ of $\mathfrak{g}_1$ as a diagram automorphism. Hence $(c^{-1}a_1)((c^{-1}a_1)^{-1})^g = c^{-1}h'cg^{-1}$ acts on the Cartan subalgebra $\mathbb{C} \otimes_{\mathbb{Z}} L(A_7^2 D_5^2)$ of $\mathfrak{g}$ as $(x_1, x_2, x_3, x_4) \mapsto (\tau(x_1), \tau(x_2), x_3, x_4)$. Since $c^{-1}h'cg^{-1} \in \mathrm{Aut}\, V$, its restriction on $\mathbb{C} \otimes_{\mathbb{Z}} L(A_7^2 D_5^2)$ preserves $N(A_7^2 D_5^2)$, which contradicts Lemma 4.11(2). Thus a_1 is inner, and it can be extended to $a \in \mathrm{Aut}\, V$. Note that $ah'a^{-1} = g$ on $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Since x is trivial on $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and a is trivial on $\mathfrak{g}_3 \oplus \mathfrak{g}_4$, we have $(ax)h(ax)^{-1} = g$ on $\mathfrak{g}$.

Set $h'' = (ax)h(ax)^{-1}$ and $k = h''g^{-1}$. Set $N = N(A_7^2 D_5^2)$ and $R = L(A_7^2 D_5^2)$. Then k is trivial on $\mathfrak{g}$, and so is on V_R. By Schur's lemma k acts on each irreducible V_R-submodule $V_{\lambda+R}$ of V by a scalar. Hence $k = \exp(\pi\sqrt{-1} v_0)$ for some $v \in 2R^*/2N$. Let $\alpha \in L(A_7)^*/L(A_7)$ and $\beta \in L(D_5)^*/L(D_5)$ given in Lemma 4.11(1). Then $2R^*/2N$ is generated by u and v, where $u = (0, 2\alpha, 0, 0)$, $v = (0,0,0,2\beta)$. Note that the orders of u and v are 8 and 4 in $2R^*/2N$, respectively. We now consider the action of g on $R^* \subset \mathbb{C} \otimes_{\mathbb{Z}} R$. It follows from $\mathfrak{g}^g = \mathfrak{g}(A_7 A_3 B_2^2 A_1^2)$ that $g(v_1, v_2, v_3, v_4) = (v_2, v_1, -v_3, v_4)$ (cf. [Ka90, Proposition 8.1]). Hence

$$g(u) = 3u - v, \quad g(v) = v \mod 2N. \tag{4.2}$$

Let $n, m \in \mathbb{Z}$ such that $k = \exp(\pi\sqrt{-1}(nu + mv)_{(0)})$. Since $h'' = kg$ is of order 2, we have

$$(kg)^2 = kgkg = \exp(\pi\sqrt{-1}(4nu + (-n+2m)v)_{(0)}) = \mathrm{Id}.$$

Hence $n \in 2\mathbb{Z}$. By (4.2) and $h'' = kg$, we have

$$\exp\left(\pi\sqrt{-1}(-nu_{(0)}/2)\right)^{-1} h'' \exp\left(\pi\sqrt{-1}(-nu_{(0)}/2)\right) = \exp\left(\pi\sqrt{-1}\left(\left(m+\frac{n}{2}\right)v_{(0)}\right)\right)g.$$

Hence we may assume that $n = 0$ and $k = \exp(\pi\sqrt{-1}mv_{(0)})$.

In order to complete the proof, it suffices to show that the involutions $\exp(\pi\sqrt{-1}mv_{(0)})g$ and g are conjugate. Since the order of $\exp(\pi\sqrt{-1}mv_{(0)})g$ is 2, we have $m \equiv 0 \pmod 2$ by (4.2). Hence we may assume $m = 2$. By Lemmas 4.13, g acts on $\mathfrak{g}_4$ as $\exp(\pi\sqrt{-1}s_{(0)})$ for some root $s \in L(D_5)$ up to conjugation. Then by Lemma 4.10, $\exp(\pi\sqrt{-1}s_{(0)})$ is conjugate to $\exp(\pi\sqrt{-1}(2v+s)_{(0)})$, which completes this theorem. □

4.4. Isomorphisms of the VOAs $\mathfrak{V}(\mathcal{S})$. In this section, we establish the isomorphisms between certain VOAs $\mathfrak{V}(\mathcal{S})$. Throughout this section, V denotes $V_{\sqrt{2}E_8}^+$.

Let $\mathcal{S}$ be a maximal totally singular subspace of $R(V)^3$. We now recall the $\mathbb{Z}_2$-orbifolds of $\mathfrak{V}(\mathcal{S})$ from [LS12, Section 4.7]. Let $W \in R(V)^3 \setminus \mathcal{S}$ with $q_V^3(W) = 0$. Let $\chi_W : \mathcal{S} \to \mathbb{F}_2$ be the linear character of $\mathcal{S}$ defined by $\chi_W(W') = \langle W, W' \rangle$. Then χ_W induces an automorphism g_W of $\mathfrak{V}(\mathcal{S})$ of order 2 acting on M' by $(-1)^{\chi_W(W')}$ for $W' = [M'] \in \mathcal{S}$. The fixed point subspace and the $\mathbb{Z}_2$-orbifold associated to g_W are given as follows:

PROPOSITION 4.17. [LS12, Proposition 4.4] *The fixed point subspace of $\mathfrak{V}(\mathcal{S})$ with respect to g_W is $\mathfrak{V}(\mathcal{S} \cap W^\perp)$, and the $\mathbb{Z}_2$-orbifold of $\mathfrak{V}(\mathcal{S})$ associated to g_W is given by $\mathfrak{V}(\langle W, \mathcal{S} \cap W^\perp \rangle_{\mathbb{F}_2})$.*

Remark 4.18. The $\mathbb{Z}_2$-orbifold of $\mathfrak{V}(\mathcal{S})$ associated to g_W exists and the VOA structure is uniquely determined. Hence if $g \in \operatorname{Aut}\mathfrak{V}(\mathcal{S})$ is conjugate to g_W, then the $\mathbb{Z}_2$-orbifolds of $\mathfrak{V}(\mathcal{S})$ associated to g and g_W are isomorphic.

4.4.1. Holomorphic VOAs with Lie algebra $\mathfrak{g}(C_8F_4^2)$. The aim of this section is to show that the VOAs $\mathfrak{V}(\mathcal{S}(5,3,0,-))$ and $\mathfrak{V}(\mathcal{S}(5,3,2,+))$ are obtained as the $\mathbb{Z}_2$-orbifolds of $V_{N(A_{15}D_9)}$ associated to conjugated involutions. For the descriptions of $\mathcal{S}(5,k_1,k_2)$ and $\mathcal{S}(5,k_1,k_2,\varepsilon)$, see Theorems 4.1 and 4.3, respectively. For the calculations in the Lie algebra $\mathfrak{V}(\mathcal{S})_1$, see [LS12, Section 5].

PROPOSITION 4.19. *Let $\mathcal{S} = \mathcal{S}(5,4,0)$. Let b and d be non-singular vectors in $B^\perp$ and in T. Set $W = (b, d+z, 0)$ and $\mathcal{T} = \langle \mathcal{S} \cap W^\perp, W \rangle_{\mathbb{F}_2}$.*

(1) *The subspace $\mathcal{T}$ is conjugate to $\mathcal{S}(5,3,0,-)$ under $O(R(V), q_V) \wr \mathrm{Sym}_3$.*

(2) *The Lie algebra $\mathfrak{V}(\mathcal{S} \cap W^\perp)_1$ is isomorphic to $\mathfrak{g}(C_8B_4^2)$.*

Proof. Let a and c be non-zero singular vectors in S_1 and in T such that $\langle a,b\rangle = \langle c,d\rangle = 1$, respectively. By the description of $\mathcal{S}(5,4,0)$, we have

$$\mathcal{S}\cap W^{\perp} = \big\langle (s,0,0),(a+y,y,0),(y,0,y),(z,z,z),(0,y+t',y+\varphi(t')), \\ (0,t,\varphi(t)) \mid s\in S_1\cap b^{\perp},\ t\in T\cap d^{\perp},\ t'\in c+T\cap d^{\perp}\big\rangle_{\mathbb{F}_2}.$$

Since W is singular, $\mathcal{T}$ is maximal totally singular. Moreover, $\mathcal{T}$ does not satisfy (4.1). By $\dim(\mathcal{T}\cap\{(r,0,0)\mid r\in R(V)\}) = 3$, $\dim(\mathcal{T}\cap\{(0,r,0)\mid r\in R(V)\}) = 0$ and Theorem 4.1, $\mathcal{T}$ is conjugate to $\mathcal{S}(5,3,0,\varepsilon)$. Since the image of the first coordinate projection $\mathcal{T}\to R(V)$ is $(S_1\cap b^{\perp})\perp\langle a+y,b\rangle_{\mathbb{F}_2}$ and both $a+y$ and b are non-singular, we have $\varepsilon = -$. Thus we obtain (1).

Set $\mathcal{U} = \mathcal{S}\cap W^{\perp}$ and $\mathcal{X}^{(1)} = \mathcal{X}\cap\{(0,u,v)\mid u,v\in R(V)\}$ for $\mathcal{X} = \mathcal{T},\mathcal{U}$. Then by [LS12, Proposition 5.31] $\mathfrak{V}(\mathcal{T}^{(1)})_1$ is an ideal of $\mathfrak{V}(\mathcal{T})_1$ and $\mathfrak{V}(\mathcal{T}^{(1)})_1\cong\mathfrak{g}(C_8)$. It follows from $\mathcal{T}^{(1)} = \mathcal{U}^{(1)}$ that $\mathfrak{V}(\mathcal{U}^{(1)})_1$ is an ideal of $\mathfrak{V}(\mathcal{U})_1$ isomorphic to $\mathfrak{g}(C_8)$. Set

$$\mathcal{U}' = \big\langle (s,0,0),(a+y,y,0),(y,0,y),(z,z,z)\mid s\in S_1\cap b^{\perp}\big\rangle_{\mathbb{F}_2}.$$

Then $\mathfrak{V}(\mathcal{U})_1 = \mathfrak{V}(\mathcal{U}^{(1)})_1\oplus\mathfrak{V}(\mathcal{U}')_1$, and $\mathfrak{V}(\mathcal{U}')_1$ is an ideal. By [LS12, Proposition 5.30], $\mathfrak{V}(\mathcal{S}(5,3,0,+))_1\cong\mathfrak{g}(B_4^2D_8)$. One can see that $\mathfrak{V}(\mathcal{U}')_1$ is isomorphic to the ideal $\mathfrak{g}(B_4^2)$ of $\mathfrak{V}(\mathcal{S}(5,3,0,+))_1$. Hence (2) holds. □

PROPOSITION 4.20. *Let S_1,S_2,S_3 be totally singular subspaces of $R(V)$ such that $S_3\subset S_2\subset S_1$ and $\dim S_1 = 4$, $\dim S_2 = 2$ and $\dim S_3 = 1$. Let Q and T be complementary subspaces of S_1 and of S_2 in $S_1^{\perp}$ and in $S_2^{\perp}$, respectively. Set $U = (S_3\perp Q)^{\perp}$. Let φ be an isomorphism from T to U. Let*

$$\mathcal{S} = \big\{(s_1+q,s_2+t,s_3+q+\varphi(t))\mid s_i\in S_i,\ q\in Q,\ t\in T\big\}.$$

Let $b\in(Q\perp U)^{\perp}$ be a non-zero singular vector such that $b\notin S_3^{\perp}$. Set $W = (b,0,b)$ and $\mathcal{T} = \langle\mathcal{S}\cap W^{\perp},W\rangle_{\mathbb{F}_2}$.

(1) *The subspace $\mathcal{S}$ of $R(V)^3$ is maximal totally singular.*

(2) *The VOA $\mathfrak{V}(\mathcal{S})$ is isomorphic to the lattice VOA $V_{N(A_{15}D_9)}$.*

(3) *The subspace $\mathcal{T}$ of $R(V)^3$ is conjugate to $\mathcal{S}(5,3,2,+)$ under $O(R(V),q_V)\wr$ Sym$_3$.*

(4) *The Lie algebra $\mathfrak{V}(\mathcal{S}\cap W^{\perp})_1$ is isomorphic to $\mathfrak{g}(C_8B_4^2)$.*

Proof. Since $\dim Q = 2$ and $\dim T = 6$, we have $\dim\mathcal{S} = 15$. By the definition of $\mathcal{S}$, it is totally singular. Hence we have (1).

Take non-zero singular vectors h_i in S_i for $i = 1,2,3$. Then $(h_1,0,0),(0,h_2,0),(0,0,h_3)\in\mathcal{S}$. By Lemma 4.9, $\mathfrak{V}(\mathcal{S})$ is a lattice VOA. By $\dim\mathfrak{V}(\mathcal{S})_1 = 408$ (cf. [LS12, Proposition 5.17]), we have $\mathfrak{V}(\mathcal{S})\cong V_{N(A_{15}D_9)}$. Hence (2) holds.

By the direct calculation, we have

$$\mathcal{S}\cap W^{\perp} = \big\{(s_1+s_3+q, s_2+t, s_3+q+\varphi(t)) \mid s_1\in S_1\cap b^{\perp},\ s_2\in S_2,\ s_3\in S_3,\ q\in Q,\ t\in T\big\}. \tag{4.3}$$

Since W is singular, $\mathcal{T}$ is maximal totally singular. Moreover $\mathcal{T}$ does not satisfy (4.1). Set $\mathcal{T}^{(ij)} = \{(r_1,r_2,r_3)\in\mathcal{T} \mid r_i = r_j = 0\}$. Then $\dim\mathcal{T}^{(23)} = 3$, $\dim\mathcal{T}^{(13)} = 2$, $\mathcal{T}^{(12)} = 0$, and by Theorem 4.1, $\mathcal{T}$ is conjugate to $\mathcal{S}(5,3,2,+)$, which proves (3).

By (4.3), $\mathfrak{V}(\mathcal{S}\cap W^{\perp})_1$ is the direct sum of two ideals

$$\mathfrak{V}(\{(s_1+s_3+q,0,s_3+q) \mid q\in Q,\ s_1\in S_1\cap b^{\perp},\ s_3\in S_3\})_1,$$
$$\mathfrak{V}(\{(0,s_2+t,\varphi(t)) \mid s_2\in S_2,\ t\in T\})_1,$$

and their dimensions are 136 and 72, respectively. The former is also an ideal of $\mathfrak{V}(\mathcal{T})_1$, and hence it is isomorphic to $\mathfrak{g}(C_8)$. One can see that the latter is isomorphic to the ideal $\mathfrak{g}(B_4^2)$ of $\mathfrak{V}(\mathcal{S}(5,3,0,+))_1$ (cf. [LS12, Proposition 5.30]). Hence (4) holds. □

It was shown in [LS12, Proposition 5.40] that $\mathfrak{V}(\mathcal{S}(5,4,0)) \cong V_{N(A_{15}D_9)}$. Combining Remark 4.18, Theorem 4.15, Propositions 4.17, 4.19, and 4.20, we obtain the following theorem:

THEOREM 4.21. *The VOAs $\mathfrak{V}(\mathcal{S}(5,3,0,-))$ and $\mathfrak{V}(\mathcal{S}(5,3,2,+))$ are isomorphic.*

4.4.2. Holomorphic VOAs with Lie algebra $\mathfrak{g}(A_7C_3^2A_3)$. The aim of this subsection is to show that the VOAs $\mathfrak{V}(\mathcal{S}(5,2,0))$ and $\mathfrak{V}(\mathcal{S}(5,2,1,+))$ are obtained as the $\mathbb{Z}_2$-orbifolds of $V_{N(A_7^2D_5^2)}$ associated to conjugated involutions. For the descriptions of $\mathcal{S}(5,k_1,k_2)$ and $\mathcal{S}(5,k_1,k_2,\varepsilon)$, see Theorems 4.1 and 4.3, respectively. For the calculations in the Lie algebra $\mathfrak{V}(\mathcal{S})_1$, see [LS12, Section 5].

PROPOSITION 4.22. *Let $\mathcal{S} = \mathcal{S}(5,2,0)$. Let a and b be non-zero singular vectors in P and Q, respectively. Set $W = (a+b,0,0)$ and $\mathcal{T} = \langle\mathcal{S}\cap W^{\perp}, W\rangle_{\mathbb{F}_2}$.*
(1) *The VOA $\mathfrak{V}(\mathcal{T})$ is isomorphic to $V_{N(A_7^2D_5^2)}$.*
(2) *The Lie algebra $\mathfrak{V}(\mathcal{S}\cap W^{\perp})_1$ is isomorphic to $\mathfrak{g}(A_7A_3B_2^2A_1^2)$.*

Proof. Let $c\in P$ and $d\in Q$ be non-singular vectors satisfying $\langle a,c\rangle = \langle b,d\rangle = 1$. Then

$$\mathcal{T} = \big\langle (s,t,\varphi(t)), (a,a,0), (b,0,b), (c+d,c,d),\ (y,0,y), (0,y,y) \mid s\in\langle S_1, a+b\rangle_{\mathbb{F}_2},\ t\in T\big\rangle_{\mathbb{F}_2}.$$

Since $\mathcal{T}$ contains $(a,a,0)$ and $(a,0,b)$, the VOA $\mathfrak{V}(\mathcal{T})$ satisfies (4.1). Hence by Lemma 4.9 it is isomorphic to a lattice VOA or its $\mathbb{Z}_2$-orbifold. It follows from $\dim \mathfrak{V}(\mathcal{T})_1 = 216$ (cf. [LS12, Proposition 5.17]) that $\mathfrak{V}(\mathcal{T}) \cong V_{N(A_7^2D_5^2)}$ or $\tilde{V}_{N(A_{17}E_7)}$. Note that $(V_{N(A_7^2D_5^2)})_1 \cong \mathfrak{g}(A_7^2D_5^2)$ and $(\tilde{V}_{N(A_{17}E_7)})_1 \cong \mathfrak{g}(D_9A_7)$. Since the subspace

$$\mathfrak{V}\Big(\big\langle(0,a,b),(0,y,y),(0,t,\varphi(t)) \mid t\in T\big\rangle_{\mathbb{F}_2}\setminus\big\langle(0,y,y),(0,y+a,y+b)\big\rangle_{\mathbb{F}_2}\Big)_1$$

is a 126-dimensional ideal, we have $\mathfrak{V}(\mathcal{T})_1 \cong \mathfrak{g}(A_7^2D_5^2)$ and $\mathfrak{V}(\mathcal{T}) \cong V_{N(A_7^2D_5^2)}$. Hence (1) holds.

Let us determine the Lie algebra structure of $\mathfrak{g} = \mathfrak{V}(\mathcal{S}\cap W^{\perp})_1$. It is easy to see that

$$\begin{aligned}\mathcal{S}\cap W^{\perp} = \big\langle &(s,t,\varphi(t)),(a,a,0),(b,0,b),(c+d,c,d),\\ &(y,0,y),(0,y,y) \mid s\in S_1,\ t\in T\big\rangle_{\mathbb{F}_2}.\end{aligned}$$

Then the subspace

$$\mathfrak{V}\Big(\big\langle(0,y,y),(0,t,\varphi(t)) \mid t\in T\big\rangle_{\mathbb{F}_2}\setminus\{(0,y,y)\}\Big)_1$$

is a 63-dimensional ideal of $\mathfrak{g}$, and it is also an ideal of $\mathfrak{V}(\mathcal{S}(5,2,0))_1$ isomorphic to $\mathfrak{g}(A_7)$. One can see that the other 41-dimensional ideal

$$\mathfrak{V}\Big(\big\langle(s,0,0),(a,a,0),(b,0,b),(c+d,c,d),(y,y,0),(y,0,y) \mid s\in S_1\big\rangle_{\mathbb{F}_2}\Big)_1. \tag{4.4}$$

is isomorphic to $\mathfrak{g}(A_3B_2^2A_1^2)$. For the detail, see Appendix A.1. Hence (2) holds. □

PROPOSITION 4.23. *Let $\mathcal{S} = \mathcal{S}(5,2,1,+)$ and let a be a non-zero singular vector in Q. Set $W = (a,0,0)$ and $\mathcal{T} = \langle \mathcal{S}\cap W^{\perp}, W\rangle_{\mathbb{F}_2}$.*

(1) *The VOA $\mathfrak{V}(\mathcal{T})$ is isomorphic to $V_{N(A_7^2D_5^2)}$.*

(2) *The Lie algebra $\mathfrak{V}(\mathcal{S}\cap W^{\perp})_1$ is isomorphic to $\mathfrak{g}(A_7A_3B_2^2A_1^2)$.*

Proof. Set $Q' = Q\cap a^{\perp}$. Then

$$\begin{aligned}\mathcal{T} = \big\{(s_1+p+q, s_2+p+t, s_3+q+\varphi(t)) \mid\ & s_1\in\langle S_1,a\rangle_{\mathbb{F}_2},\\ & s_2\in S_2,\ s_3\in\langle a\rangle_{\mathbb{F}_2},\ p\in P,\ q\in Q',\ t\in T\big\}.\end{aligned}$$

Take a non-zero singular vector $h_2\in S_2$. Then it follows from $(a,0,0),(0,h_2,0),(0,0,a)\in\mathcal{T}$ and Lemma 4.9 that $\mathfrak{V}(\mathcal{T})$ is a lattice VOA. By $\dim\mathfrak{V}(\mathcal{T})_1 = 216$ (cf. [LS12, Proposition 5.17]), we have $\mathfrak{V}(\mathcal{T}) \cong V_{N(A_7^2D_5^2)}$. Hence (1) holds.

By direct calculation, we have

$$\mathcal{S}\cap W^{\perp} = \big\langle(s_1+p+q, s_2+p+t, q+\varphi(t)) \mid s_i\in S_i,\ p\in P,\ q\in Q',\ t\in T\big\rangle_{\mathbb{F}_2}.$$

Let us determine the Lie algebra structure of $\mathfrak{g} = \mathfrak{V}(\mathcal{S} \cap W^{\perp})_1$. Take non-zero singular vectors $h_1 \in S_1$ and $h_2 \in S_2$. Then by [LS12, Lemma 5.19(2)], $\mathfrak{V}(\{(h_1,0,0),(0,h_2,0)\})_1$ is a Cartan subalgebra of $\mathfrak{g}$. Consider the root space decomposition of $\mathfrak{g}$ with respect to the Cartan subalgebra. Then it is easy to see that

$$\mathfrak{V}(\langle (s_1+p+q, s_2+p, q) \mid s_i \in S_i,\ p \in P,\ q \in Q' \rangle_{\mathbb{F}_2} \setminus \{(h_1,0,0),(0,h_2,0)\})_1, \tag{4.5}$$

$$\mathfrak{V}(\{(0, s_2+t, \varphi(t)) \mid t \in T,\ s_2 \in S_2\} \setminus \{(0,h_2,0)\})_1 \tag{4.6}$$

are mutually orthogonal root spaces and their dimensions are 32 and 56. Since (4.6) is contained in $\mathfrak{V}(\mathcal{S}(5,2,1,+))_1$, it is a root space of $\mathfrak{g}(A_7)$. One can see that (4.5) is a root space of $\mathfrak{g}(A_3B_2^2A_1^2)$. For the detail, see Appendix A.2. Hence (2) holds. □

Combining Remark 4.18, Theorem 4.16, Propositions 4.17, 4.22, and 4.23, we obtain the following theorem:

THEOREM 4.24. *The VOAs* $\mathfrak{V}(\mathcal{S}(5,2,0))$ *and* $\mathfrak{V}(\mathcal{S}(5,2,1,+))$ *are isomorphic.*

Appendix A. Explicit descriptions of ideals in Section 4.4. In this appendix, we describe the ideals defined in (4.4) and (4.5) as a direct sum of simple ideals. Let $e_1, e_2, \ldots, e_8$ be an orthogonal basis of $\mathbb{R}^8$ such that $\langle e_i, e_j \rangle = 2\delta_{ij}$. Then

$$E = \sum_{1 \le i,j \le 8} \mathbb{Z}(e_i + e_j) + \mathbb{Z}\frac{1}{2}\sum_{i=1}^{8} e_i$$

is isomorphic to $\sqrt{2}E_8$. Note that $E^* = E/2$.

A.1. Explicit description for the ideal in (4.4). Set

$$\mathcal{U} = \langle (s,0,0),(a,a,0),(b,0,b),(c+d,c,d),(y,y,0),(y,0,y) \mid s \in S_1 \rangle_{\mathbb{F}_2}.$$

Then $\dim \mathfrak{V}(\mathcal{U})_1 = 41$. The aim of this subsection is to see $\mathfrak{V}(\mathcal{U})_1 \cong \mathfrak{g}(A_3B_2^2A_1^2)$.

Up to conjugation, we may assume that

$$S_1 = \langle [V_E^-],[V_{e_1+E}^+] \rangle_{\mathbb{F}_2}, \quad y = [V_{(e_1+e_2)/2+E}^+], \quad a = [V_{(e_1+e_2+e_3+e_4)/2+E}^+],$$
$$b = [V_{(e_1+e_2+e_5+e_6)/2+E}^+].$$

For the detail of irreducible V_E^+-modules, see [FLM88]. Note that

$$\mathfrak{V}(\{(s,0,0) \mid s \in S_1\})_1 = \langle e_i(-1), x(e_i)^{\pm} \mid 1 \le i \le 8 \rangle_{\mathbb{C}},$$

where $x(e_i)^{\pm} = e^{e_i} \pm \theta(e^{e_i}) \in V^{\pm}_{e_1+E}$. Then $\mathfrak{V}(\mathcal{U})_1$ is a direct sum of the following simple ideals:

$$\begin{aligned}
&\mathbb{C}e_1(-1) \oplus \mathbb{C}e_2(-1) \oplus \bigoplus_{\varepsilon\in\{\pm\},\, i=1,2} \mathbb{C}x(e_i)^{\varepsilon} \\
&\qquad \oplus \mathfrak{V}(\{(y+s,y,0),(y+s,0,y),(0,y,y) \mid s \in S_1\})_1, \\
&\mathbb{C}e_3(-1) \oplus \mathbb{C}e_4(-1) \oplus \bigoplus_{\varepsilon\in\{\pm\},\, i=3,4} \mathbb{C}x(e_i)^{\varepsilon} \oplus \mathfrak{V}(\{(y+a+s,y+a,0) \mid s \in S_1\})_1, \\
&\mathbb{C}e_5(-1) \oplus \mathbb{C}e_6(-1) \oplus \bigoplus_{\varepsilon\in\{\pm\},\, i=5,6} \mathbb{C}x(e_i)^{\varepsilon} \oplus \mathfrak{V}(\{(y+b+s,0,y+b) \mid s \in S_1\})_1, \\
&\mathbb{C}e_7(-1) \oplus \mathbb{C}x(e_7)^{+} \oplus \mathbb{C}x(e_7)^{-}, \\
&\mathbb{C}e_8(-1) \oplus \mathbb{C}x(e_8)^{+} \oplus \mathbb{C}x(e_8)^{-}.
\end{aligned}$$

Since their dimensions are 15, 10, 10, 3, and 3, we have $\mathfrak{V}(\mathcal{U})_1 \cong \mathfrak{g}(A_3B_2^2A_1^2)$.

A.2. Explicit description for the ideal in (4.5). By the arguments in the proof of Lemma 4.23, the 64-dimensional subalgebra $\mathfrak{V}(\{(0,s+t,\varphi(t)) \mid s \in S_2,\ t \in T\})_1$ contains the 63-dimensional ideal. Let H' be its 1-dimensional ideal. Then by (4.6), $H' \subset \mathfrak{V}(\{(0,h_2,0)\})_1$, where h_2 is the non-zero vector in S_2. Set

$$\mathcal{U} = \big\{(s_1+p+q,p+s_2,q) \mid s_i \in S_i,\ p \in P,\ q \in Q'\big\} \setminus \big\{(0,h_2,0)\big\}.$$

In this subsection, we show that $\mathfrak{V}(\mathcal{U})_1 \oplus H' \cong \mathfrak{g}(A_3B_2^2A_1^2)$. Note that its dimension is 41. Let p_0 be the non-singular vector in P. Take a non-singular vector $q_0 \in Q'$. Then the set of all non-singular vectors in Q' is $\{q_0, q_0+a\}$. Up to conjugation, we may assume that $S_1 = \langle [V_E^-], [V^+_{e_1+E}]\rangle_{\mathbb{F}_2}$, $S_2 = \{[V_E^{\varepsilon}] \mid \varepsilon \in \{\pm\}\}$, $p_0 = [V^+_{(e_1+e_2)/2+E}]$, $q_0 = [V^+_{(e_3+e_4)/2+E}]$, $a = [V^+_{(e_3+e_4+e_5+e_6)/2+E}]$. Then $\mathfrak{V}(\{(s,0,0) \mid s \in S_1\})_1 = \langle e_i(-1), x(e_i)^{\pm} \mid 1 \le i \le 8\rangle_{\mathbb{C}}$, where $x(e_i)^{\pm} = e^{e_i} \pm \theta(e^{e_i}) \in V^{\pm}_{e_i+E}$. One can see that $\mathfrak{V}(\mathcal{U})_1$ is a direct sum of the following simple ideals:

$$\begin{aligned}
&\mathbb{C}e_1(-1) \oplus \mathbb{C}e_2(-1) \oplus \bigoplus_{\varepsilon\in\{\pm\},\, i=1,2} \mathbb{C}x(e_i)^{\varepsilon} \\
&\qquad \oplus \mathfrak{V}(\{(p_0+s_1,p_0+s_2,0) \mid s_i \in S_i\})_1 \oplus H', \\
&\mathbb{C}e_3(-1) \oplus \mathbb{C}e_4(-1) \oplus \bigoplus_{\varepsilon\in\{\pm\},\, i=3,4} \mathbb{C}x(e_i)^{\varepsilon} \\
&\qquad \oplus \mathfrak{V}(\{(q_0+s_1,0,q_0) \mid s_1 \in S_1\})_1, \\
&\mathbb{C}e_5(-1) \oplus \mathbb{C}e_6(-1) \oplus \bigoplus_{\varepsilon\in\{\pm\},\, i=5,6} \mathbb{C}x(e_i)^{\varepsilon} \\
&\qquad \oplus \mathfrak{V}(\{(q_0+a+s_1,0,q_0+a) \mid s_1 \in S_1\})_1,
\end{aligned}$$

$$\mathbb{C}e_7(-1) \oplus \mathbb{C}x(e_7)^+ \oplus \mathbb{C}x(e_7)^-,$$
$$\mathbb{C}e_8(-1) \oplus \mathbb{C}x(e_8)^+ \oplus \mathbb{C}x(e_8)^-.$$

Since their dimensions are 15, 10, 10, 3, and 3, we have $\mathfrak{V}(\mathcal{U})_1 \oplus H' \cong \mathfrak{g}(A_3B_2^2A_1^2)$.

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