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Sharp affine isoperimetric inequalities for the Minkowski first mixed volume

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Abstract

The sharp bounds for the affine invariant ratio of the mixed cone-volume functional to the Minkowski first mixed volume are obtained, and therefore new affine isoperimetric inequalities for the Minkowski first mixed volume are established.

1. *Introduction*

The setting of this article is the n-dimensional Euclidean space, $\mathbb{R}^n$. A *convex body* (that is, a compact convex subset with nonempty interior) K in $\mathbb{R}^n$, is uniquely determined by its *support function* $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$, $h_K(u) = \max\{u \cdot x : x \in K\}$, where $\mathbb{S}^{n-1}$ is the unit sphere and $u \cdot x$ denotes the standard inner product of u and x. The *projection body* ΠK of K is a convex body with support function $h_{\Pi K}(u) = \mathrm{vol}_{n-1}(K|u^\perp)$, $u \in \mathbb{S}^{n-1}$, where vol_{n-1} denotes the $(n-1)$-dimensional volume and $K|u^\perp$ denotes the image of orthogonal projection of K onto the codimension 1 subspace orthogonal to u. The support function of ΠK can also be represented as

$$h_{\Pi K}(u) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |u \cdot v| dS_K(v), \qquad u \in \mathbb{S}^{n-1}, \tag{1.1}$$

where S_K is the *surface area measure* of convex body K. Formula (1.1) follows from the Cauchy projection formula; see, for example, [**19**, p. 569] for details.

The projection body is one of the most important objects in convex geometry, and has been intensively investigated during the past three decades; see, for example, [**1**, **5**–**7**, **13**–**15**, **23**, **24**], etc. It is centro-affine invariant, that is, for $T \in \mathrm{SL}(n)$, $\Pi(TK) = T^{-t}(\Pi K)$, where T^{-t} denotes the inverse of the transpose of T. It is worth mentioning that there stands a celebrated unsolved problem regarding projection bodies, called the Schneider projection problem: as K ranges over the class of origin-symmetric convex bodies in $\mathbb{R}^n$, what is the least upper bound of the affine-invariant ratio

$$[V(\Pi K)/V(K)^{n-1}]^{1/n},$$

where V denotes the n-dimensional volume; see, for example, [**18**, **20**] for details. The lower bound for this affine-invariant ratio is also unknown, Petty [**17**] conjectured that the minimum of this quantity is attained precisely by ellipsoids.

An effective tool to study Schneider's projection problem is the *cone-volume functional* U, which was introduced by Lutwak, Yang and Zhang (LYZ) [**16**]: If P is a convex polytope in $\mathbb{R}^n$ that contains the origin o in its interior, then $U(P)$ is defined by

$$U(P)^n = \frac{1}{n^n} \sum_{u_{i_1}\wedge\cdots\wedge u_{i_n}\neq 0} h_{i_1}\cdots h_{i_n} a_{i_1}\cdots a_{i_n},$$

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where $u_1, \ldots, u_N$ are the outer normal unit vectors to the facets of P, $h_1, \ldots, h_N$ are the corresponding distances from the origin to facets, and $a_1, \ldots, a_N$ are the corresponding areas of the facets.

Let $V_i = a_i h_i/n,\ i = 1, \ldots, N$. Then,

$$U(P)^n = \sum_{u_{i_1} \wedge \cdots \wedge u_{i_n} \neq 0} V_{i_1} \cdots V_{i_n}.$$

Since $V(P)^n = (\sum_{i=1}^N V_i)^n$, it follows that $U(P) < V(P)$. U is centro-affine invariant, that is, $U(TP) = U(P)$, for $T \in \mathrm{SL}(n)$. It is interesting that using this functional U, LYZ [**16**] presented an affirmative answer to the modified Schneider projection problem: If P is an origin-symmetric polytope in $\mathbb{R}^n$, then

$$\frac{V(\Pi P)}{U(P)^{n/2} V(P)^{(n/2)-1}} \leqslant 2^n \left(\frac{n^n}{n!}\right)^{1/2}, \tag{1.2}$$

with equality if and only if P is a parallelotope.

That showing the lower bound of functional U in terms of volume V makes an interesting story. LYZ [**16**] conjectured that if the centroid of polytope P in $\mathbb{R}^n$ is at the origin, then

$$U(P) \geqslant \frac{(n!)^{1/n}}{n} V(P), \tag{1.3}$$

with equality if and only if P is a parallelotope.

It took more than one dozen years to completely settle this conjecture. In [**9**], He, Leng and Li proved (1.3) for origin-symmetric polytopes, including its equality condition. In [**22**], the second author of this article gave a simplified proof for symmetric polytopes and proved (1.3), including the equality case, for two- and three-dimensional polytopes. A complete and final solution to this conjecture was attributed to Henk and Linke [**10**].

In 2015, Böröczky and LYZ [**3**] extended the domain of functional U from the class of polytopes to the set of convex bodies in $\mathbb{R}^n$ with origin in their interiors, $\mathcal{K}_o^n$, and defined

$$U(K)^n = \frac{1}{n^n} \int_{u_1 \wedge \cdots \wedge u_n \neq 0} h_K(u_1) \cdots h_K(u_n) dS_K(u_1) \cdots dS_K(u_n).$$

Since $V(K)^n = (\frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K dS_K)^n$, it follows that $U(K) \leqslant V(K)$. It is still centro-affine invariant, that is, $U(TK) = U(K)$, for $T \in \mathrm{SL}(n)$. $U(K)$ is positive homogeneity of degree n, that is, $U(tK) = t^n U(K)$, for $t > 0$. In 2016, Böröczky and Henk [**2**] further proved that the LYZ conjecture is also affirmative for convex bodies with centroid at the origin. Very recently, Hu and Xiong [**11**] extended the inequality (1.2) and the Minkowski first mixed volume was involved.

Let $K \in \mathcal{K}_o^n$ and L be a convex body in $\mathbb{R}^n$. In [**12**], Lu, Sun and Xiong calculated the variation of U functional and showed that

$$\begin{aligned} U_1(K, L) &= \frac{1}{n} \lim_{\lambda \to 0^+} \frac{U(K + \lambda L) - U(K)}{\lambda} \\ &= \frac{(n-1) \int_{u_1 \wedge u_2 \wedge \cdots \wedge u_n \neq 0} \frac{1}{n} h_K(u_1)\, dS_1(K, L, u_1) dV_K(u_2) \cdots dV_K(u_n)}{n U(K)^{n-1}} \\ &\quad + \frac{\int_{u_1 \wedge u_2 \wedge \cdots \wedge u_n \neq 0} \frac{1}{n} h_L(u_1)\, dS_K(u_1) dV_K(u_2) \cdots dV_K(u_n)}{n U(K)^{n-1}}, \end{aligned} \tag{1.4}$$

where $S_1(K, L, \cdot)$ is the *first mixed area measure*; see equation (2.2) in Section 2 for more details.

In view of the fact that the *Minkowski first mixed volume* $V_1(K, L)$, the most important one among $V_i(K, L), i = 1, 2, \ldots, n$, is arising from the variation of the *volume* functional, we naturally call $U_1(K,L)$ the *mixed cone-volume functional* of K and L. Note that $U_1(K,K)=U(K)$. $U_1(K,L)$ is centro-affine invariant, that is, $U_1(TK,TL)=U_1(K,L)$, for $T\in \mathrm{SL}(n)$. It is striking that when K is strictly convex, $U_1(K,L)=V_1(K,L)$. So, in some sense, the functional U_1 gives a *new* connection to the Minkowski first mixed volume $V_1(K,L)$. One can refer to [**12**] for details.

In [**12**], the authors investigated the *affine invariant* ratio $U_1(K,L)/V_1(K,L)$, and proved that if the centroid of K is at the origin and L contains the origin in its interior, then

$$\frac{n!}{n^n}\left(\frac{V(K)}{U(K)}\right)^{n-1}\leqslant \frac{U_1(K,L)}{V_1(K,L)}\leqslant \left(\frac{V(K)}{U(K)}\right)^{n-1}.$$

Let $L=K$. The inequality on the left side, including its equality conditions, yields LYZ's conjectured inequality (1.3) directly.

In this article, we further study the extremal problem for the affine invariant ratio $\frac{U_1(K,L)}{V_1(K,L)}$ over the polytopes in $\mathbb{R}^n$, and established the following *sharp* affine isoperimetric inequalities. Particularly, the *parallelogram* is uniquely characterized.

THEOREM 1.1. *Let K be a parallelotope in $\mathbb{R}^n$ with the origin in its interior, and L be a convex body in $\mathbb{R}^n$ with the origin in its interior. Then*

$$\frac{\sqrt[n]{n!}}{n}\leqslant \frac{U_1(K,L)}{V_1(K,L)}\leqslant \frac{n^2-n+1}{n}\,\frac{\sqrt[n]{n!}}{n},$$

with equality on the left if and only if $\mathrm{supp}S_1(K,L)\subseteq \mathrm{supp}S_K$*; with equality on the right if and only if* $S_1(K,L,\mathrm{span}\{v_{i_1},\dots,v_{i_{n-1}}\}\cap\mathbb{S}^{n-1})=0$*, for each* $\{v_{i_1},\dots,v_{i_{n-1}}\}\subseteq\{v_1,\dots,v_n\}$*, where* $\{\pm v_1,\dots,\pm v_n\}$ *is the set of the outer normal unit vectors of* K.

THEOREM 1.2. *Let K and L be origin-symmetric polygons in $\mathbb{R}^2$. Then*

$$\frac{U_1(K,L)}{V_1(K,L)}\geqslant \frac{\sqrt{2}}{2},$$

with equality if and only if K and L are parallel parallelograms.

Here parallel parallelograms refer to a pair of parallelograms with the same set of outer normal unit vectors.

This article is organized as follows. For quick later reference, we collect some basic facts on convex bodies in Section 2. One can refer to excellent books by Gardner [**4**], Gruber [**8**], Schneider [**19**] and Thompson [**21**]. To prove Theorems 1.1 and 1.2, we provide several preparatory lemmas in Section 3. It is worth mentioning that Lemma 3.4, which is essentially proved by using the method of Lagrange multiplier, is crucial. The main results are proved in Section 4.

2. *Preliminaries*

Write $\mathcal{K}^n$ for the set of convex bodies in $\mathbb{R}^n$. A *polytope* in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$. Write $\mathcal{H}^k$ for the k-dimensional Hausdorff measure.

The *Minkowski combination* of $K,L\in\mathcal{K}^n$ is defined by

$$tK+sL=\{tx+sy: x\in K, y\in L\},\quad t,s\geqslant 0.$$

From the definition of support function, it follows that

$$h_{tK+sL}(u)=th_K(u)+sh_L(u),\quad u\in\mathbb{S}^{n-1}. \tag{2.1}$$

The *support set* of $K \in \mathcal{K}^n$ in the direction u, $u \in \mathbb{S}^{n-1}$, is defined by

$$F_K(u) = \{x \in K : x \cdot u = h_K(u)\}.$$

K is said to be *strictly convex*, if for each $u \in \mathbb{S}^{n-1}$ the set $F_K(u)$ contains only one point.

The *surface area measure* S_K of $K \in \mathcal{K}^n$ is a finite Borel measure on $\mathbb{S}^{n-1}$, defined for Borel $\omega \subseteq \mathbb{S}^{n-1}$ by $S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega))$, where $\nu_K : \partial' K \to \mathbb{S}^{n-1}$ is the Gauss map of K, defined on $\partial' K$, the set of points of ∂K that have a unique outer unit normal. Recall that $\mathcal{H}^{n-1}(\partial K \setminus \partial' K) = 0$; see, for example, [**19**, p. 84] for details.

By the definition of support set, it follows that $S_K(\{u\}) = \mathcal{H}^{n-1}(F_K(u))$, for $u \in \mathbb{S}^{n-1}$.

The *mixed area measure* $S(K_1, \ldots, K_{n-1}, \cdot)$ of $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$ is defined by

$$S(K_1, \ldots, K_{n-1}, \cdot) = \frac{1}{(n-1)!} \sum_{k=1}^{n-1} (-1)^{n+k-1} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} S_{K_{i_1} + \cdots + K_{i_k}}(\cdot).$$

For brevity, let

$$S_i(K, L) = S(\underbrace{K, \ldots, K}_{n-1-i}, \underbrace{L, \ldots, L}_{i}, \cdot), \quad i = 0, \ldots, n-1. \tag{2.2}$$

The *cone-volume measure* V_K of $K \in \mathcal{K}_o^n$ is a finite Borel measure on $\mathbb{S}^{n-1}$, defined for Borel $\omega \subseteq \mathbb{S}^{n-1}$ by

$$V_K(\omega) = \frac{1}{n} \int_\omega h_K(u)\, dS_K(u).$$

In particular,

$$V_K(\mathbb{S}^{n-1}) = V(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(u)\, dS_K(u).$$

The *mixed cone-volume measure* $V_{K,L}$ of $K, L \in \mathcal{K}_o^n$, originally defined by Hu and Xiong in [**11**], is

$$V_{K,L}(\omega) = \frac{1}{n} \int_\omega h_L(u)\, dS_K(u), \quad \text{for Borel set } \omega \subseteq \mathbb{S}^{n-1}.$$

Observe that $V_{K,L}(\mathbb{S}^{n-1}) = V_1(K, L)$, $V_{K,B} = \frac{1}{n} S_K$ and $V_{K,K} = V_K$. Thus, $V_{K,L}$ contains two most fundamental measures in convex geometry: the surface area measure S_K and the cone-volume measure V_K. For its properties and applications, one can refer to [**11**].

The *Minkowski first mixed volume* $V_1(K, L)$ of $K, L \in \mathcal{K}^n$ is defined by

$$V_1(K, L) = \frac{1}{n} \lim_{\lambda \to 0^+} \frac{V(K + \lambda L) - V(K)}{\lambda},$$

which has the following integral formula:

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u)\, dS_K(u) = \frac{1}{n} \int_{S^{n-1}} h_K(u)\, dS_1(K, L, u). \tag{2.3}$$

Since $V_1(K, L_1 + L_2) = V_1(K, L_1) + V_1(K, L_2)$, it follows that

$$S_1(K, L_1 + L_2, \cdot) = S_1(K, L_1, \cdot) + S_1(K, L_2, \cdot). \tag{2.4}$$

By (2.3), it yields that for $n = 2$,

$$V_1(K, L) = V_1(L, K). \tag{2.5}$$

When $K \in \mathcal{K}_o^2$ and L be a convex set in $\mathbb{R}^2$, by (1.4), we have

$$U_1(K,L) = \frac{\int_{u_1\wedge u_2\neq 0} \frac{1}{2} h_K(u_1)\, dS_L(u_1) dV_K(u_2) + \int_{u_1\wedge u_2\neq 0} \frac{1}{2} h_L(u_1)\, dS_K(u_1) dV_K(u_2)}{2U(K)}. \tag{2.6}$$

3. *Preparatory lemmas*

Lemma 3.1. *Let $K \in \mathcal{K}_o^n$, and L_1, L_2 be compact convex sets in $\mathbb{R}^n$ containing the origin. Then, $U_1(K, L_1 + L_2) = U_1(K, L_1) + U_1(K, L_2)$.*

Proof. From the representation (1.4) of $U_1(K,L)$, (2.4) and (2.1), it follows that

$$\begin{aligned}
&nU(K)^{n-1}U_1(K, L_1+L_2)\\
&= (n-1)\int_{u_1\wedge u_2\wedge\cdots\wedge u_n\neq 0} \frac{1}{n} h_K(u_1)\, dS_1(K, L_1+L_2, u_1) dV_K(u_2)\cdots dV_K(u_n)\\
&\quad + \int_{u_1\wedge u_2\wedge\cdots\wedge u_n\neq 0} \frac{1}{n} h_{L_1+L_2}(u_1)\, dS_K(u_1) dV_K(u_2)\cdots dV_K(u_n)\\
&= (n-1)\int_{u_1\wedge u_2\wedge\cdots\wedge u_n\neq 0} \frac{1}{n} h_K(u_1)\, dS_1(K, L_1, u_1) dV_K(u_2)\cdots dV_K(u_n)\\
&\quad + (n-1)\int_{u_1\wedge u_2\wedge\cdots\wedge u_n\neq 0} \frac{1}{n} h_K(u_1)\, dS_1(K, L_2, u_1) dV_K(u_2)\cdots dV_K(u_n)\\
&\quad + \int_{u_1\wedge u_2\wedge\cdots\wedge u_n\neq 0} \frac{1}{n} h_{L_1}(u_1)\, dS_K(u_1) dV_K(u_2)\cdots dV_K(u_n)\\
&\quad + \int_{u_1\wedge u_2\wedge\cdots\wedge u_n\neq 0} \frac{1}{n} h_{L_2}(u_1)\, dS_K(u_1) dV_K(u_2)\cdots dV_K(u_n)\\
&= \{ (n-1)\int_{u_1\wedge u_2\wedge\cdots\wedge u_n\neq 0} \frac{1}{n} h_K(u_1)\, dS_1(K, L_1, u_1) dV_K(u_2)\cdots dV_K(u_n)\\
&\quad + \int_{u_1\wedge u_2\wedge\cdots\wedge u_n\neq 0} \frac{1}{n} h_{L_1}(u_1)\, dS_K(u_1) dV_K(u_2)\cdots dV_K(u_n)\}\\
&\quad + \{ (n-1)\int_{u_1\wedge u_2\wedge\cdots\wedge u_n\neq 0} \frac{1}{n} h_K(u_1)\, dS_1(K, L_2, u_1) dV_K(u_2)\cdots dV_K(u_n)\\
&\quad + \int_{u_1\wedge u_2\wedge\cdots\wedge u_n\neq 0} \frac{1}{n} h_{L_2}(u_1)\, dS_K(u_1) dV_K(u_2)\cdots dV_K(u_n)\}\\
&= nU(K)^{n-1}(U_1(K, L_1) + U_1(K, L_2)).
\end{aligned}$$

That is,

$$U_1(K, L_1+L_2) = U_1(K, L_1) + U_1(K, L_2),$$

which is as desired. □

LEMMA 3.2. *If nonnegative real a_i are not all zeros, and $b_i, c_i > 0$, $i = 1, \ldots, N$, then*

$$\frac{\sum_{i=1}^N a_i b_i}{\sum_{i=1}^N a_i c_i} \geqslant \min_{1\leqslant i\leqslant N} \frac{b_i}{c_i}.$$

Proof.

$$\frac{\sum_{i=1}^N a_i b_i}{\sum_{i=1}^N a_i c_i} = \frac{\sum_{i=1}^N a_i c_i \cdot \frac{b_i}{c_i}}{\sum_{i=1}^N a_i c_i} \geqslant \frac{\min_{1\leqslant i\leqslant N} \frac{b_i}{c_i} \sum_{i=1}^N a_i c_i}{\sum_{i=1}^N a_i c_i} = \min_{1\leqslant i\leqslant N} \frac{b_i}{c_i},$$

which is as desired. □

One can refer to [**22**, p. 3222] for the proof of the following lemma.

LEMMA 3.3. *Let P be a polytope in $\mathbb{R}^n$ with its centroid at the origin and outer normal unit vectors $u_1, u_2, \ldots, u_N$. Then*

$$V_P(\{\pm u_i\}) \leqslant \frac{1}{n} V(P), \quad i = 1, \ldots, N. \tag{3.1}$$

If P is a parallelotope, then the equality of (3.1) holds. Conversely, if the equalities of (3.1) hold for all i simultaneously, then P is a parallelotope.

LEMMA 3.4. *Let $2 \leqslant N \in \mathbb{N}$ and*

$$f(x_1, x_2, \ldots, x_N; y_2, y_3, \ldots, y_N) = \frac{\left(1 - \frac{1}{2}x_1 - \sum_{j=2}^N x_j y_j\right)^2}{1 - \sum_{i=1}^N x_i^2},$$

where $x_i, y_j \in [0, \frac{1}{2}], i = 1, 2, \ldots, N; j = 2, \ldots, N$. Then,

$$\min f \geqslant \frac{1}{2} \quad \textit{subject to} \quad \begin{cases} \displaystyle\sum_{i=1}^N x_i = 1 \\ \displaystyle\sum_{j=2}^N y_j = \frac{1}{2} \end{cases}.$$

Moreover, $f = \frac{1}{2}$ if and only if there exists a $j_0 \in \{2, \ldots, N\}$, such that $x_1 = x_{j_0} = y_{j_0} = \frac{1}{2}$.

Proof. Since f is continuous on the compact set

$$A = \left\{ (x_1, \ldots, x_N, y_2, \ldots, y_N) \in \left[0, \frac{1}{2}\right]^{2N-1} : \sum_{i=1}^N x_i = 1, \sum_{j=2}^N y_i = \frac{1}{2} \right\},$$

f attains its minimum on the set A.

Assume that f attains its minimum at $\bar{X} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N; \bar{y}_2, \bar{y}_3, \ldots, \bar{y}_N)$, and

$$\bar{y}_j > 0, \quad j = 2, \ldots, l; \quad \bar{y}_j = 0, \quad j = l + 1, \ldots, N,$$

where $l \in \{2, \ldots, N\}$. In the following, we recognize the extremal point $\bar{X}$ by two cases.

Case 1. Assume that there exists a $j_0 \in \{2, \ldots, l\}$, say $j_0 = 2$, such that $\bar{y}_{j_0} = \frac{1}{2}$. Then

$$f(\bar{X}) = \frac{\left(1 - \frac{1}{2}\bar{x}_1 - \frac{1}{2}\bar{x}_2\right)^2}{1 - \sum_{i=1}^N \bar{x}_i^2} \geqslant \frac{\left(1 - \frac{1}{2}\bar{x}_1 - \frac{1}{2}\bar{x}_2\right)^2}{1 - (\bar{x}_1^2 + \bar{x}_2^2)}, \tag{3.2}$$

with equality if and only if $\bar{x}_3 = \cdots = \bar{x}_N = 0$; by $\sum_{i=1}^{N} \bar{x}_i = 1$, with equality if and only if $\bar{x}_1 + \bar{x}_2 = 1$. In terms of $\bar{x}_1, \bar{x}_2 \in [0, \frac{1}{2}]$, so the equality holds if and only if $\bar{x}_1 = \bar{x}_2 = \frac{1}{2}$.

Let $g(x_1, x_2) = \frac{(1-\frac{1}{2}x_1-\frac{1}{2}x_2)^2}{1-(x_1^2+x_2^2)}$. For $x_1, x_2 \in (0, \frac{1}{2})$,

$$\begin{aligned}
\frac{\partial g}{\partial x_1} &= \frac{-(1-\frac{1}{2}x_1-\frac{1}{2}x_2)(1-x_1^2-x_2^2)+2x_1\left(1-\frac{1}{2}x_1-\frac{1}{2}x_2\right)^2}{(1-x_1^2-x_2^2)^2} \\
&= \frac{(1-\frac{1}{2}x_1-\frac{1}{2}x_2)[-1+x_1^2+x_2^2+2x_1(1-\frac{1}{2}x_1-\frac{1}{2}x_2)]}{(1-x_1^2-x_2^2)^2} \\
&= \frac{(1-\frac{1}{2}x_1-\frac{1}{2}x_2)[-1+x_2^2+2x_1(1-\frac{1}{2}x_2)]}{(1-x_1^2-x_2^2)^2} \\
&< \frac{(1-\frac{1}{2}x_1-\frac{1}{2}x_2)[-1+x_2^2+(1-\frac{1}{2}x_2)]}{(1-x_1^2-x_2^2)^2} \\
&= \frac{x_2(1-\frac{1}{2}x_1-\frac{1}{2}x_2)(x_2-\frac{1}{2})}{(1-x_1^2-x_2^2)^2} \\
&< 0.
\end{aligned}$$

By the symmetry of the function g in x_1 and x_2, we also have $\frac{\partial g}{\partial x_2} < 0,\ x_1, x_2 \in (0, \frac{1}{2})$. Thus, for $x_1, x_2 \in [0, \frac{1}{2}]$, we have

$$g(x_1, x_2) \geqslant g\left(\frac{1}{2}, x_2\right) \geqslant g\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}, \tag{3.3}$$

with equality if and only if $x_1 = x_2 = \frac{1}{2}$. Combining (3.2) with (3.3), we obtain $f(\bar{X}) \geqslant \frac{1}{2}$, with equality if and only if $\bar{x}_1 = \bar{x}_2 = \frac{1}{2}$. By the assumption that $\bar{X}$ is a minimal point, we conclude that $f(\bar{X}) = \frac{1}{2}$, and $\bar{x}_1 = \bar{x}_2 = \frac{1}{2}$.

Case 2. Assume that $\bar{y}_j < \frac{1}{2}$, for each $j \in \{2, \ldots, l\}$. Under the constrained condition that $\sum_{j=2}^{l} y_j = \frac{1}{2}$, the function $f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N, y_2, y_3, \ldots, y_l, 0, \ldots, 0)$ in $y_2, y_3, \ldots, y_l$ has to attain its minimum at the interior point of the compact set $[0, \frac{1}{2}]^{l-1}$, say $(\bar{y}_2, \bar{y}_3, \ldots, \bar{y}_l)$. Thus, we can use the Lagrange multipliers method.

Let $F(y_2, y_3, \ldots, y_l, \lambda) = f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N, y_2, y_3, \ldots, y_l, 0, \ldots, 0) + \lambda(\frac{1}{2} - \sum_{j=2}^{l} y_j)$. Then,

$$\left.\frac{\partial F}{\partial y_k}\right|_{(\bar{y}_2, \bar{y}_3, \ldots, \bar{y}_l)} = \frac{-2\bar{x}_k\left(1-\frac{1}{2}\bar{x}_1-\sum_{j=2}^{l}\bar{x}_j\bar{y}_j\right)}{1-\sum_{i=1}^{N}\bar{x}_i^2} - \lambda = 0, \quad k = 2, \ldots, l. \tag{3.4}$$

By solving the above equations, we obtain

$$\bar{x}_2 = -\frac{1-\sum_{i=1}^{N}\bar{x}_i^2}{2\left(1-\frac{1}{2}\bar{x}_1-\sum_{j=2}^{l}\bar{x}_j\bar{y}_j\right)}\lambda = \bar{x}_3 = \cdots = \bar{x}_l.$$

So,

$$f(\bar{X}) = \frac{\left(1-\frac{1}{2}\bar{x}_1-\sum_{j=2}^{l}\bar{x}_j\bar{y}_j\right)^2}{1-\sum_{i=1}^{N}\bar{x}_i^2}$$

$$= \frac{\left(1 - \frac{1}{2}\bar{x}_1 - \bar{x}_2 \sum_{j=2}^{l} \bar{y}_j\right)^2}{1 - \sum_{i=1}^{N} \bar{x}_i^2}$$

$$= \frac{\left(1 - \frac{1}{2}\bar{x}_1 - \frac{1}{2}\bar{x}_2\right)^2}{1 - \sum_{i=1}^{N} \bar{x}_i^2}.$$

Observe that when $\bar{Y} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N, \frac{1}{2}, 0, \ldots, 0)$,

$$f(\bar{Y}) = \frac{\left(1 - \frac{1}{2}\bar{x}_1 - \frac{1}{2}\bar{x}_2\right)^2}{1 - \sum_{i=1}^{N} \bar{x}_i^2} = f(\bar{X}),$$

which implies that $\bar{Y}$ is also a minimal point of f. Since $\bar{Y}$ satisfies the condition in Case 1, it follows that $\bar{x}_1 = \bar{x}_2 = \frac{1}{2}$, $\quad f(\bar{Y}) = \frac{1}{2}$. So, $f(\bar{X}) = f(\bar{Y}) = \frac{1}{2}$. However,

$$f(\bar{X}) = \frac{\left(1 - \frac{1}{4} - \frac{1}{2}\bar{y}_2\right)^2}{1 - \frac{1}{4} - \frac{1}{4}} > \frac{\left(1 - \frac{1}{4} - \frac{1}{4}\right)^2}{1 - \frac{1}{4} - \frac{1}{4}} = \frac{1}{2},$$

which is a contradiction. Hence, Case 2 does not exist.

Consequently, f attains its minimum $\frac{1}{2}$ if and only if there exists a $j_0 \in \{2, \ldots, N\}$, such that $x_1 = x_{j_0} = y_{j_0} = \frac{1}{2}$. □

4. *Proofs of the main results*

Now, we finish the proofs of the main results.

Proof of Theorem 1.1. Since K is a parallelotope, it follows that for any $\{v_{i_1}, \ldots, v_{i_k}\} \subseteq \{v_1, \ldots, v_n\}$, $k = 1, 2, \ldots, n-1$,

$$V_K(\mathrm{span}\{v_{i_1}, \ldots, v_{i_k}\} \cap \mathbb{S}^{n-1}) = \sum_{j=1}^{k} \frac{1}{n} S_K(\{v_{i_j}\})(h_K(v_{i_j}) + h_K(-v_{i_j})) = \frac{k}{n} V(K). \tag{4.1}$$

So, by (4.1) and (1.3), if follows that

$$U(K)^n = \frac{n!}{n^n} V(K)^n \tag{4.2}$$

and

$$\int_{u_1 \wedge u_2 \wedge \cdots \wedge u_n \neq 0} \frac{1}{n} h_L(u_1)\, dS_K(u_1) dV_K(u_2) \ldots dV_K(u_n) = \frac{n!}{n^n} V(K)^{n-1} V_1(K, L). \tag{4.3}$$

Since $v_1, v_2, \ldots, v_n$ are the outer normal unit vectors of the parallelotope K, it follows that $v_1, v_2, \ldots, v_n$ consist of a basis of $\mathbb{R}^n$. So, for any k-dimensional subspace ξ of $\mathbb{R}^n$, $1 \leqslant k \leqslant n-1$, there are at most k pairs of vectors from $\{\pm v_1, v_2, \ldots, \pm v_n\}$ lying in ξ. Hence, by (4.1), it yields that

$$V_K(\xi \cap \mathbb{S}^{n-1}) = V_K(\xi \cap \{\pm v_1, \ldots, \pm v_n\}) = \sum_{v_i \in \xi} V_K(\{\pm v_i\}) \leqslant \frac{k}{n} V(K).$$

Therefore,

$$\int_{u \notin \xi} dV_K(u) = V(K) - V_K(\xi \cap \mathbb{S}^{n-1}) \geqslant \frac{n-k}{n} V(K), \tag{4.4}$$

with equality if and only if ξ precisely contains k pairs of vectors from $\{\pm v_1, v_2, \ldots, \pm v_n\}$.

By using (4.4), it follows that

$$\begin{aligned}
&\int_{u_1\wedge u_2\wedge\cdots\wedge u_n\neq 0}\frac{1}{n}h_K(u_1)\,dS_1(K,L,u_1)dV_K(u_2)\ldots dV_K(u_n)\\
&\quad=\int_{u_1\wedge u_2\wedge\cdots\wedge u_{n-1}\neq 0}\int_{u_n\notin \mathrm{span}\{u_1,u_2,\ldots,u_{n-1}\}} dV_K(u_n)\frac{1}{n}h_K(u_1)\,dS_1(K,L,u_1)dV_K(u_2)\\
&\qquad\cdots dV_K(u_n)\\
&\quad\geqslant \frac{1}{n}V(K)\int_{u_1\wedge u_2\wedge\cdots\wedge u_{n-1}\neq 0}\frac{1}{n}h_K(u_1)\,dS_1(K,L,u_1)dV_K(u_2)\cdots dV_K(u_n)\\
&\quad\ldots\\
&\quad\geqslant \frac{(n-2)!}{n^{n-2}}V(K)^{n-2}\int_{u_1\in\mathbb{S}^{n-1}}\int_{u_2\neq\pm u_1}\frac{1}{n}h_K(u_1)\,dS_1(K,L,u_1)dV_K(u_2)\\
&\quad\geqslant \frac{n!}{n^n}V(K)^{n-1}V_1(K,L). \qquad (4.5)
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{U_1(K,L)}{V_1(K,L)} &\geqslant \frac{(n-1)\frac{n!}{n^n}V(K)^{n-1}V_1(K,L)+\frac{n!}{n^n}V(K)^{n-1}V_1(K,L)}{nU(K)^{n-1}V_1(K,L)}\\
&=\frac{n!}{n^n}\frac{V(K)^n}{U(K)^n}\frac{U(K)}{V(K)}=\frac{\sqrt[n]{n!}}{n}. \qquad (4.6)
\end{aligned}$$

Assume that $\mathrm{supp}S_1(K,L)\subseteq \mathrm{supp}S_K$. By the equality condition of (4.4), it follows that $V_K(\mathrm{span}\{u_1,\ldots,u_k\}\cap\mathbb{S}^{n-1})=\frac{k}{n}V(K)$, for any $u_1,\ldots,u_k\in\mathrm{supp}S_K$ with $u_1\wedge\cdots\wedge u_k\neq 0$; or for any $u_1\in\mathrm{supp}S_1(K,L)\subseteq\mathrm{supp}S_K$ and $u_2,\ldots,u_k\in\mathrm{supp}S_K$ with $u_1\wedge\cdots\wedge u_k\neq 0$. So, each equality in (4.5) holds. Therefore, the equality in (4.6) holds.

Conversely, from the equality condition of the last inequality in (4.5), it follows that $V_K(\mathrm{span}\{u\})=\frac{1}{n}V(K)$, for any $u\in\mathrm{supp}S_1(K,L)$. Added that K is a parallelotope, it yields that $\mathrm{supp}S_1(K,L)\subseteq\mathrm{supp}S_K$.

For the upper bound we observe that

$$\begin{aligned}
&\int_{u_1\wedge u_2\wedge\cdots\wedge u_n\neq 0}\frac{1}{n}h_K(u_1)\,dS_1(K,L,u_1)dV_K(u_2)\cdots dV_K(u_n)\\
&\quad=\int_{u_2\wedge\cdots\wedge u_n\neq 0}\int_{u_1\notin\mathrm{span}\{u_2,\ldots,u_n\}}\frac{1}{n}h_K(u_1)\,dS_1(K,L,u_1)dV_K(u_2)\cdots dV_K(u_n)\\
&\quad=\int_{u_2\wedge\cdots\wedge u_n\neq 0}\left(V_1(K,L)-\int_{u_1\in\mathrm{span}\{u_2,\ldots,u_n\}}\frac{1}{n}h_K(u_1)\,dS_1(K,L,u_1)\right)dV_K(u_2)\cdots dV_K(u_n)\\
&\quad\leqslant\int_{u_2\wedge\cdots\wedge u_n\neq 0}V_1(K,L)\,dV_K(u_2)\cdots dV_K(u_n)\\
&\quad=V_1(K,L)\int_{u_2\wedge\cdots\wedge u_{n-1}\neq 0}\big(V(K)-V_K(\mathrm{span}\{u_2,\ldots,u_{n-1}\}\cap\mathbb{S}^{n-1})\big)\,dV_K(u_2)\cdots dV_K(u_{n-1})
\end{aligned}$$

$$= \frac{2}{n} V_1(K,L)V(K) \int_{u_2 \wedge \cdots \wedge u_{n-1} \neq 0} dV_K(u_2) \cdots dV_K(u_{n-1})$$

$$\cdots$$

$$= \frac{2}{n}\frac{3}{n} \cdots \frac{n-1}{n} V_1(K,L)V(K)^{n-1}$$

$$= \frac{n!}{n^{n-1}} V_1(K,L)V(K)^{n-1}.$$

Thus,

$$\frac{U_1(K,L)}{V_1(K,L)} \leqslant \frac{(n-1)\frac{n!}{n^{n-1}}V(K)^{n-1} + \frac{n!}{n^n}V(K)^{n-1}}{nU(K)^{n-1}}$$

$$= \frac{n(n-1)+1}{n} \frac{n!}{n^n} \frac{V(K)^n}{U(K)^n} \frac{U(K)}{V(K)} = \frac{n^2-n+1}{n} \frac{\sqrt[n]{n!}}{n},$$

with equality if and only if $S_1(K, L, \operatorname{span}\{v_{i_1}, \ldots, v_{i_{n-1}}\} \cap \mathbb{S}^{n-1}) = 0$, for any $\{v_{i_1}, \ldots, v_{i_{n-1}}\} \subseteq \{v_1, \ldots, v_n\}$. $\square$

When $n = 2$, the equality condition of Theorem 1.1 becomes simple.

Corollary 4.1. *Let K be a parallelogram in $\mathbb{R}^2$ with the origin in its interior, and L be a convex set in $\mathbb{R}^2$ with the origin in its interior. Then,*

$$\frac{\sqrt{2}}{2} \leqslant \frac{U_1(K,L)}{V_1(K,L)} \leqslant \frac{3\sqrt{2}}{4},$$

with equality on the left if and only if K and L are parallel parallelograms; with equality on the right if and only if $S_L(\{\pm v_1, \pm v_2\}) = 0$, where $\{\pm v_1, \pm v_2\}$ is the set of the outer normal unit vectors of K.

Proof. The inequalities are derived from Theorem 1.1 directly. In the following, we show the equality conditions. Observe that when $n = 2$, $S_1(K, L, \cdot) = S_L(\cdot)$. By Theorem 1.1, the equality on the left holds if and only if $\operatorname{supp} S_L \subseteq \operatorname{supp} S_K$. Assume that $\operatorname{supp} S_L \subsetneqq \operatorname{supp} S_K$. In light of K is a parallelogram, it follows that $\operatorname{supp} S_L$ must concentrate on a closed hemisphere, which is impossible. Hence, $\operatorname{supp} S_L = \operatorname{supp} S_K$, which implies that L and K are parallel parallelograms.

By Theorem 1.1, the equality on the right holds if and only if $S_L(\operatorname{span}\{v\} \cap \mathbb{S}^1) = 0$, for each $v \in \{\pm v_1, \pm v_2\}$. Hence, $S_L(\{\pm v_1, \pm v_2\}) = 0$. $\square$

Now we present the proof of Theorem 1.2.

Proof of Theorem 1.2. The crucial point of the proof is to show that $\frac{U_1(K,L)}{V_1(K,L)} \geqslant \frac{U_1(K,L')}{V_1(K,L')} \geqslant \frac{U_1(K,L'')}{V_1(K,L'')}$, where L' and L'' are constructed from L.

Step 1. Assume that $\operatorname{supp} S_K = \{\pm v_1, \ldots, \pm v_N\}$. Let

$$L' = \{x \in \mathbb{R}^2 : |x \cdot v_i| \leqslant h_L(v_i),\ i = 1, \ldots, N\}.$$

Then, $L \subseteq L'$, $\operatorname{supp} S_{L'} \subseteq \operatorname{supp} S_K$, and $h_{L'}(\pm v_i) \leqslant h_L(\pm v_i)$, $i = 1, \ldots, N$. In light of $L \subseteq L'$, it follows that $h_{L'} \geqslant h_L$, and therefore $h_{L'}(\pm v_i) = h_L(\pm v_i)$, $i = 1, \ldots, N$. Hence,

$$V_1(K, L') = V_1(K, L) \tag{4.7}$$

and

$$\int_{u_1\wedge u_2=0} \frac{1}{2} h_{L'}(u_1)\, dS_K(u_1)dV_K(u_2) = \int_{u_1\wedge u_2=0} \frac{1}{2} h_L(u_1)\, dS_K(u_1)dV_K(u_2). \tag{4.8}$$

In the following, we aim to show that $U_1(K,L') \leqslant U_1(K,L)$.

By (2.6) and (2.5), we have

$$\begin{aligned}
&2U(K)U_1(K,L')\\
&= \int_{u_1\wedge u_2\neq 0} \frac{1}{2} h_K(u_1)\, dS_{L'}(u_1)dV_K(u_2) + \int_{u_1\wedge u_2\neq 0} \frac{1}{2} h_{L'}(u_1)\, dS_K(u_1)dV_K(u_2)\\
&= V(K)V_1(L',K) - \int_{u_1\wedge u_2=0} \frac{1}{2} h_K(u_1)\, dS_{L'}(u_1)dV_K(u_2)\\
&\quad + V(K)V_1(K,L') - \int_{u_1\wedge u_2=0} \frac{1}{2} h_{L'}(u_1)\, dS_K(u_1)dV_K(u_2)\\
&= 2V(K)V_1(K,L') - \int_{u_1\wedge u_2=0} \frac{1}{2} h_K(u_1)\, dS_{L'}(u_1)dV_K(u_2)\\
&\quad - \int_{u_1\wedge u_2=0} \frac{1}{2} h_{L'}(u_1)\, dS_K(u_1)dV_K(u_2).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
2U(K)U_1(K,L) = 2V(K)V_1(K,L) &- \int_{u_1\wedge u_2=0} \frac{1}{2} h_K(u_1)\, dS_L(u_1)dV_K(u_2)\\
&- \int_{u_1\wedge u_2=0} \frac{1}{2} h_L(u_1)\, dS_K(u_1)dV_K(u_2).
\end{aligned}$$

Combining the above two equalities with (4.8), we obtain

$$\begin{aligned}
&2U(K)(U_1(K,L') - U_1(K,L))\\
&= \int_{u_1\wedge u_2=0} \frac{1}{2} h_K(u_1)\, (dS_L(u_1) - dS_{L'}(u_1))dV_K(u_2)\\
&= \int_{u_1\in \operatorname{supp} S_K} \int_{u_2=\pm u_1} \frac{1}{2} h_K(u_1)(dS_L(u_1) - dS_{L'}(u_1))dV_K(u_2).
\end{aligned}$$

So, it suffices to show that $S_{L'}(\{u\}) \geqslant S_L(\{u\})$, for any $u \in \operatorname{supp} S_K$.

For $u \in \operatorname{supp} S_K$, let $H_u = \{x \in \mathbb{R}^2 : x\cdot u = h_L(u)\}$. Since $h_{L'}(u) = h_L(u)$, it follows that H_u is the support hyperplane of L' with outer normal u. So,

$$S_{L'}(\{u\}) = vol_1(L'\cap H_u), \quad S_L(\{u\}) = vol_1(L\cap H_u),$$

here vol_1 is the one-dimensional Lebesgue measure. From $L \subseteq L'$, it follows that $L\cap H_u \subseteq L'\cap H_u$. So, $S_{L'}(\{u\}) \geqslant S_L(\{u\})$. Hence, $U_1(K,L') \leqslant U_1(K,L)$. By (4.7), it yields that

$$\frac{U_1(K,L)}{V_1(K,L)} \geqslant \frac{U_1(K,L')}{V_1(K,L')}. \tag{4.9}$$

Step 2. By the definition of L', we can rewrite $L' = \sum_{i=1}^N a_i T[-v_i, v_i]$, here T is the orthogonal transform by anticlockwise rotation $\frac{\pi}{2}$; $a_i \geqslant 0$ are not all zeros, $i = 1, \dots, N$. From Lemmas 3.1 and 3.2, it follows that

$$\frac{U_1(K, L')}{V_1(K, L')} = \frac{\sum_{i=1}^N a_i U_1(K, T[-v_i, v_i])}{\sum_{i=1}^N a_i V_1(K, T[-v_i, v_i])} \geqslant \min_{1 \leqslant i \leqslant N} \frac{U_1(K, T[-v_i, v_i])}{V_1(K, T[-v_i, v_i])}. \tag{4.10}$$

Assume that $\min_{1 \leqslant i \leqslant N} \frac{U_1(K,T[-v_i,v_i])}{V_1(K,T[-v_i,v_i])} = \frac{U_1(K,T[-v_1,v_1])}{V_1(K,T[-v_1,v_1])}$. Let $L'' = T[-v_1, v_1]$. Then,

$$\text{supp} S_{L''} = \{\pm v_1\}, \quad h_{L''}(u) = |u \cdot Tv_1|, \quad \forall u \in \mathbb{S}^1.$$

So,

$$\begin{aligned} \int_{u_1 \wedge u_2 = 0} \frac{1}{2} h_K(u_1)\, dS_{L''}(u_1) dV_K(u_2) &= \int_{u_1 \in \{\pm v_1\}} V_K(\{\pm u_1\}) \frac{1}{2} h_K(u_1)\, dS_{L''}(u_1) \\ &= h_K(v_1) S_{L''}(\{v_1\}) V_K(\{\pm v_1\}) \\ &= V_1(L'', K) V_K(\{\pm v_1\}) \\ &= V_1(K, L'') V_K(\{\pm v_1\}). \end{aligned}$$

In light of the fact that $v_1 \cdot Tv_1 = 0$, it yields that

$$\begin{aligned} \int_{u_1 \wedge u_2 = 0} \frac{1}{2} h_{L''}(u_1)\, dS_K(u_1) dV_K(u_2) &= \int_{u_1 \in \text{supp}\, S_K} V_K(\{\pm u_1\}) \frac{1}{2} |u_1 \cdot Tv_1|\, dS_K(u_1) \\ &= 2 \sum_{i=1}^N \frac{1}{2} |v_i \cdot Tv_1| S_K(\{v_i\}) V_K(\{\pm v_i\}) \\ &= \sum_{j=2}^N h_{L''}(v_j) S_K(\{v_j\}) V_K(\{\pm v_j\}). \end{aligned}$$

Thus,

$$U_1(K, L'') = \frac{2V(K) V_1(K, L'') - V_1(K, L'') V_K(\{\pm v_1\}) - \sum_{j=2}^N h_{L''}(v_j) S_K(v_j) V_K(\{\pm v_j\})}{2U(K)}.$$

Since $U(K)^2 = V(K)^2 - \int_{u_1 \wedge u_2 = 0} dV_K(u_1) dV_K(u_2) = V(K)^2 - \sum_{i=1}^N V_K(\{\pm v_i\})^2$, it follows that

$$\begin{aligned} \frac{U_1(K, L'')}{V_1(K, L'')} &= \frac{V(K) - \frac{1}{2} V_K(\{\pm v_1\}) - \sum_{j=2}^N \frac{h_{L''}(v_j) S_K(v_j)}{2V_1(K, L'')} V_K(\{\pm v_j\})}{\sqrt{V(K)^2 - \sum_{i=1}^N V_K(\{\pm v_i\})^2}} \\ &= \frac{1 - \frac{1}{2} \frac{V_K(\{\pm v_1\})}{V(K)} - \sum_{j=2}^N \frac{h_{L''}(v_j) S_K(v_j)}{2V_1(K, L'')} \frac{V_K(\{\pm v_j\})}{V(K)}}{\sqrt{1 - \sum_{i=1}^N \left(\frac{V_K(\{\pm v_i\})}{V(K)} \right)^2}}. \end{aligned}$$

By Lemma 3.3, it follows that $\frac{V_K(\{\pm v_i\})}{V(K)} \leqslant \frac{1}{2}$; by $\sum_{i=1}^N \frac{V_K(\{\pm v_i\})}{V(K)} = 1$, it yields that $\sum_{j=2}^N \frac{h_{L''}(v_j) S_K(v_j)}{2V_1(K, L'')} = \frac{1}{2}$. Therefore, by Lemma 3.4, it follows that

$$\frac{U_1(K, L'')}{V_1(K, L'')} \geqslant \frac{\sqrt{2}}{2}. \tag{4.11}$$

Consequently, combining (4.9), (4.10) with (4.11), it follows that

$$\frac{U_1(K,L)}{V_1(K,L)} \geqslant \frac{U_1(K,L')}{V_1(K,L')} \geqslant \frac{U_1(K,L'')}{V_1(K,L'')} \geqslant \frac{\sqrt{2}}{2}.$$

Now, we show the equality conditions.

On one hand, assume the equality holds. By the equality conditions in Lemma 3.4, there exists a $j_0 \in \{2,\dots,N\}$ such that $V_K(\{\pm v_1\}) = V_K(\{\pm v_{j_0}\}) = \frac{1}{2}V(K)$. By Lemma 3.3, it follows that K is a parallelogram. By Corollary 4.1, it follows that K and L are parallel parallelograms.

Assume that K and L are parallelograms. By Corollary 4.1, then the equality holds. □

Let $K = L$, we immediately obtain the following LYZ's conjectured inequality for $n = 2$.

Corollary 4.2. *Let K be an origin-symmetric polygon in $\mathbb{R}^2$. Then,*

$$\frac{U(K)}{V(K)} \geqslant \frac{\sqrt{2}}{2},$$

with equality if and only if K is a parallelogram.

From the Minkowski inequality, we immediately obtain the following.

Corollary 4.3. *Let K, L be origin-symmetric polygons in $\mathbb{R}^2$. Then*

$$U_1(K,L)^2 \geqslant \tfrac{1}{2}V(K)V(L),$$

with equality if and only if K and L are dilated parallelograms.

An obvious question regarding the functionals V_1 and U_1 begs to be asked.

Conjecture. Let K, L be convex bodies in $\mathbb{R}^n$ with centroid at the origin. Then

$$\frac{U_1(K,L)}{V_1(K,L)} \geqslant \frac{\sqrt[n]{n!}}{n},$$

with equality if and only if K is a parallelotope and supp $S_1(K,L) \subseteq$ supp S_K.

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